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The existence of solutions for nonlinear elliptic equations: Simple proofs and extensions of a paper by Y. Shi ☆

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Abstract

The paper [39] uses the Craig-Wayne-Bourgain method to construct solutions of an elliptic problem involving parameters. The results of [39] include regularity assumptions on the perturbation and involve excluding parameters. The paper [39] also constructs response solutions to a quasi-periodically perturbed (ill-posed evolution) problem.

In this paper, we use several classical methods (freezing of coefficients, alternative methods for nonlinear elliptic equations) to extend the results of [39]. We weaken the regularity assumptions on the perturbation and we describe the phenomena that happens for all parameters. In the ill-posed problem, we use a recently developed time-dependent center manifold theorem which allows to reduce the problem to a finite-dimensional ODE with quasi-periodic dependence on time. The bounded and sufficiently small solutions of these ODE give solutions of the ill-posed PDE.

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1. Introduction

1.1. Previous results

The recent paper [39] considers the problem

$$-\Delta u - \mathfrak{m}u + \epsilon f(x,u) = 0,\ \ x \in \mathcal{D} := \mathbb{R}^d/(2\pi\beta_i\mathbb{Z})^d,\ \ d \in \mathbb{Z}_+, \tag{1.1}$$

where $\mathfrak{m} > 0$ (as we will see later, the case $\mathfrak{m} \le 0$ is easy), $\epsilon \ge 0$ and $\beta = (\beta_1, \cdots, \beta_d) \in [1/2, 1]^d$. The unknown function is $u : \mathcal{D} \to \mathbb{R}$, and data is the nonlinearity $f : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, which in [39] is assumed to be a polynomial in u with coefficients that are trigonometric polynomials in x.

The paper [39] uses the Craig-Wayne-Bourgain (CWB) method [18,19,8,17,9] to prove existence of analytic solutions of (1.1) when $(\beta_1^{-1}, \cdots, \beta_d^{-1}) \in [1, 2]^d$ lies in an appropriate set whose measure is estimated.

The paper [39] also considers the formal "evolution" problem

$$-u_{tt} - \Delta u - \mathfrak{m}u + \epsilon f(t, x, u) = 0,\ \ x \in \mathcal{D}, \tag{1.2}$$

where $\mathfrak{m} > 0, \epsilon \ge 0, \beta = (\beta_1, \beta_2, \cdots, \beta_d) \in [1/2, 1]^d$. $f : \mathbb{R} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is quasi-periodic with respect to time t with frequency vector $\omega \in \mathbb{R}^b$, $b \in \mathbb{Z}_+$. The paper [39] produces response solutions (i.e., quasi-periodic solutions with the same frequency as the forcing).

Note that the differential operator in (1.2) is also an elliptic operator, so considering it as an evolution equation leads to an ill-posed problem. Nevertheless, even if one cannot produce solutions for all initial conditions, it is possible to obtain interesting solutions. Indeed, the consideration of elliptic problems in cylindrical domains as "evolution" problems has been considered in several papers [28,32] and, more recently, [20,36,37,13].

1.2. The results in this paper

In this paper, we revisit and extend the results above using some classical methods (freezing of coefficients, alternative method) for the problem (1.1) or some more modern methods (reduction to center manifolds for ill-posed equations [32,20,13]) for the problem (1.2).

An outline of the main ideas is as follows: as for the treatment of (1.1) we distinguish whether the spectrum of $-\Delta - \mathfrak{m}$ is away from zero (we will call these cases *non-resonant*) or whether the spectrum of $-\Delta - \mathfrak{m}$ contains zero (we call these cases *resonant*).

1.2.1. Treatment of (1.1) *when* $-\Delta - \mathfrak{m}$ *is invertible: freezing of coefficients*

When the spectrum of $-\Delta - \mathfrak{m}$ is away from zero,[1] we transform (1.1) into a fixed point problem in an appropriate Banach space. This allows us to remove the assumption in [39] that the nonlinearity is polynomial in x, u and it also allows us to deal with nonlinearities that involve the derivatives of u up to order 2. That is, we allow nonlinearities $f(x, u, Du(x), D^2u(x))$ and even more general functions denoted by $\mathcal{F}[u]$. This extra generality includes several interesting cases, that have attracted attention in recent times such as fractional derivatives, $(-\Delta)^{\alpha}u$, the Kirkhoff terms $\int_{\mathbb{T}^d} |\nabla u|^2)\Delta u$ or the water wave terms $(-\Delta)^{1/2}\tanh((-\Delta)^{1/2})u$ ([35,33,21]). We will just require that the functional $\mathcal{F}$ is Lipschitz mapping from a space of differentiable functions to another space of differentiable functions (with two derivatives less). As it is well known from classical potential theory, the scale of spaces has to be carefully chosen so that the gain of regularity obtained by applying $(-\Delta - \mathfrak{m})^{-1}$ compensates the loss of regularity incurred by $\mathcal{F}$.

Remark 1. For the experts in classical elliptic regularity theory [6,3,4] [40, Chapter 15] we anticipate that the method is very similar to the classic "freezing of coefficients" but that in our case, we do not need to localize the problem, so that we do not need to use commutator estimates, which makes it possible to obtain analytic results for analytic f.

The spaces we work with are chosen so that they can be analytic functions for some values of the parameters and finite-differentiable functions for other parameter values.

The results on analytic (and finite-differentiable regularity) depend crucially on choosing a remarkable family (indexed by two parameters) of function spaces where to formulate the functional analysis problem.

This family of spaces has been used in the past, [11,10,41]. These spaces enjoy many remarkable properties (presented here in Appendix A) including that they are Banach algebras from some ranges of the parameters. In this paper we prove Lemma 37, which improves the range of parameters for the Banach algebra properties established in [11,10,41]. This immediately leads to improvements in the range of parameters in the above references. In the notation of the above references, the assumption $r > d$ can be weakened to $r > d/2$ using Lemma 37 in the present paper.

1.2.2. Treatment of (1.1) *when* $-\Delta - \mathfrak{m}$ *is not invertible: bifurcation theory*

When the spectrum of $-\Delta - \mathfrak{m}$ contains zero (a problem not considered in [39]), we note that zero is an eigenvalue of finite multiplicity so that one can apply the classical Cesari alternative method [14] (also called Lyapunov-Schmidt reduction [27]). In our case, there are some unusual properties such as the kernel having large dimension and the presence of symmetries, so that the calculations involve several algebraic surprises. The algebraic difficulty increases with the dimension, so we present a complete example in dimensions 1 and 2.

With some appropriate conditions on the nonlinearity, we can indeed obtain smooth branches of solutions.

Putting together the two results, we can obtain results for all the choices of $\nu, \mathfrak{m}$ provided that some explicit non-degeneracy conditions on the nonlinearity hold.

[1] In general, the spectrum depends on the space one is considering the operator acting. However, for elliptic operators in bounded domains, the spectrum is largely independent of the space. Later we will specify which spaces we are considering.

1.3. Treatment of (1.2): fixed point methods and reduction principles

As for the equation (1.2), it is natural to consider the equation acting on a space of quasi-periodic functions. The spectrum of the linearized system always contains semi-lines when the frequencies ω have dimension 2 or more. An elementary result along the lines of the previous result is obtained by assuming that the spectrum does not contain zero which happens for some $\beta, \mathfrak{m}, \omega$.

A more sophisticated method to study (1.2), which applies to all $\beta, \mathfrak{m}$ is to observe that we can apply the time-dependent center manifold theorem introduced in [13] to establish the existence of the time-dependent invariant manifold for the evolution equation (1.2). Using this center manifold, we can reduce the original problem to a finite-dimensional quasi-periodic ODE problem. The fixed point method presented here allows nonlinearities that loose two derivatives. The reduction principle of [13] allows nonlinearities that loose $(2-\kappa)$ derivatives. In this paper, we will present a detailed proof of the simpler case when there is only one derivative.

The study of solutions for quasi-periodic equation in finite dimensions is well developed and there are a large variety of techniques (which cannot be even reviewed) to produce interesting solutions [34,31]. These well studied solutions include indeed response solutions, but also subharmonic response solutions and many others.

If we find solutions of the problem which remain in a sufficiently small neighborhood of the origin, they will become solutions to the original problem (remember that the center manifold is locally invariant). Therefore, we can produce solutions of the original problem, just by producing solutions of a finite-dimensional problem. Such procedures are often called *reduction principles*.

It is interesting to mention that the reduction to finite dimensions in [13] does not require any assumption on the perturbing frequencies. Of course, the analysis of the resulting finite-dimensional system using KAM theory may require that the frequency satisfies some number theoretic properties. Other methods may have other assumptions, but we will not detail them here.

1.4. Organization of this paper

This paper is organized as follows: In Section 2, we present the main idea of proving the existence of solutions through the several classical methods we mentioned above. In Section 3, we introduce a convenient two parameter family of function spaces $H^{\rho,r}$. When $\rho > 0$, the space $H^{\rho,r}$ consists of analytic functions, but $H^{0,r}$ is the standard Sobolev space. In Sections 4-7, we give our several main results which are the simple proof of the results in [39] and the extensions. Precisely, in Section 4, we study the case when the spectrum of the operator $-\Delta - \mathfrak{m}$ of equation (1.1) is non-resonant. In Section 5, we introduce the Cesari alternative method to deal with the case that the spectrum of $-\Delta - \mathfrak{m}$ is resonant. For the ill-posed evolution equation (1.2), when the spectrum of $-\partial_{tt} - \Delta - \mathfrak{m}$ is non-resonant, we introduce our results in Section 6 and for the resonant case, we introduce the center manifold theorem to solve (1.2) in Section 7.

2. A preview of the results

In this section, we describe formally the methods we will use, ignoring for the moment questions of spaces, domains, etc. These will be taken care later. The precise definitions will be motivated by the desire to make the formal manipulations go through.

2.1. Elliptic theory away from resonances

We consider the variable x on $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. To rewrite (1.1) in a more convenient way, we denote by $\mathcal{L}_{\nu,\mathfrak{m}}$ the linear operator

$$\mathcal{L}_{\nu,\mathfrak{m}} := \sum_{i=1}^{d} \nu_i^2 \frac{\partial^2}{\partial x_i^2} + \mathfrak{m}, \tag{2.1}$$

where $\nu = (\nu_1, \cdots, \nu_d) = (\beta_1^{-1}, \cdots, \beta_d^{-1}) \in [1,2]^d$. Moreover, we allow that the nonlinearity f also depends on $Du(x)$, $D^2u(x)$. For convenience, we represent the nonlinearity by $\mathcal{F}(u)(x) := f(x, u, Du, D^2u)$, then it suffices to verify the abstract hypothesis for $\mathcal{F}$.

Then, the equation (1.1) becomes

$$(\mathcal{L}_{\nu,\mathfrak{m}} u)(x) = \epsilon \mathcal{F}(u)(x), \;\; x \in \mathbb{T}^d. \tag{2.2}$$

We notice that $\mathcal{L}_{\nu,\mathfrak{m}}$ is a diagonal operator in the Fourier basis. Precisely,

$$\mathcal{L}_{\nu,\mathfrak{m}}(\exp\{\mathrm{i}kx\}) = \Upsilon_{k,\nu,m} \exp\{\mathrm{i}kx\}$$

with $\Upsilon_{k,\nu,m} = \sum_{i=1}^{d} -\nu_i^2 k_i^2 + \mathfrak{m}$, $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$.

Remark 2. Since flipping the signs of components of k does not change the eigenvalue of the operator $\mathcal{L}_{\nu,\mathfrak{m}}$ (but it changes the eigenvector when some of the components of k are not zero), then the eigenvalues have always multiplicity at least $2^{\eta(k)}$ where $\eta(k)$ is the number of components of k which are not zero. Of course, we have $0 \le \eta(k) \le d$ and, for $\eta(k)$ outside of the coordinate hyperplanes $\eta(k) = d$, we need ν_i^2 have some rational relations.

If the ν_i^2 have some rational relations, the multiplicity could be higher[2] but this happens in a set of measure zero of ν.

For simplicity of the discussion, we will assume that the ν we consider have no rational relation. We will furthermore assume that $\eta(k) = d$. See Assumption 18. This assumption could be avoided with longer explicit calculations.

In reasonable spaces, for which exponentials will be a basis (for example in the spaces presented in Section 3) $\mathcal{L}_{\nu,\mathfrak{m}}$ will be a self-adjoint operator and the $\Upsilon_{k,\nu,m}$ will be its spectrum. Notice that this spectrum is a discrete set going to infinity and that the eigenvalues have finite multiplicity. (Again, we recall that, for the present operators, the spectrum is largely independent of the space we consider it. We will of course, make the spaces explicit later since the choice of spaces plays a big role in the treatment of nonlinear terms.)

Since we will use functional analysis, we will find it convenient to consider that the right hand side of (2.2) is written as $\mathcal{F}[u]$ and we will think of $\mathcal{F}$ as a mapping that maps a space of functions with a certain number of derivatives to another spaces of functions (which have possibly less derivatives).

[2] If an eigenvalue has multiplicity bigger than $2^{\eta(k)}$, the ν_i should satisfy some linear relations. Therefore, except for a set of measure zero of the ν_i, all the eigenvalues will have multiplicity exactly $2^{\eta(k)}$. Note that $\eta(k) \le d$, so in dimension 1 the kernel will have dimension 2.

We will prove our results under abstract assumptions on the operator $\mathcal{F}$. Afterwards we will show that if $\mathcal{F}[u]$ is given by

$$\mathcal{F}[u](x) = f(x, u(x), Du(x), D^2u(x)), \tag{2.3}$$

where f is a sufficiently smooth function of its finite-dimensional arguments, then the abstract hypotheses for $\mathcal{F}$ are satisfied.

If the parameters ν, $\mathfrak{m}$ are such that the operator $\mathcal{L}_{\nu,\mathfrak{m}}$ is boundedly invertible (we will indicate the explicit spaces later), we rewrite (2.2) as

$$u(x) = \epsilon \mathcal{L}_{\nu,\mathfrak{m}}{}^{-1}\mathcal{F}(u)(x) \equiv \mathcal{T}(u)(x). \tag{2.4}$$

2.2. *Remarks on spaces*

We see that, to apply the above program, it is useful to formulate the problem in spaces of functions that satisfy the following properties (there are links among these properties as we will see in the concrete examples in Section 3).

- Consist of analytic functions (or functions with a specified regularity).
- The norms can be read off from the Fourier coefficients.
- It is possible to give estimates of the composition on the right with f under regularity properties in f.
- It is possible to obtain Lipschitz estimates of the operator composing with f (such operators are often called "Nemitski operators", "left composition operators" or "nonlinear superposition operators" [7,26]).
- These spaces are Banach algebras under pointwise multiplication.
- These spaces are Hilbert spaces (so that we can take advantage of selfadjointness of some operators and use sharp results in spectral theory).
- The operators we consider that are diagonal with real eigenvalues are selfadjoint.

Some spaces that satisfy these conditions are introduced in Section 3. These spaces are inspired by the Bargman spaces used in quantum field theory and in complex analysis. They have already been used in other papers [11,10,41]. We note that, in comparison with the papers above, we present Lemma 37 showing that the good properties of these spaces are valid for a larger range of parameter values than those considered in [10,41]. Hence, the results in the above papers can be extended slightly.

2.3. *The alternative method used in the elliptic case*

If the parameters ν, $\mathfrak{m}$ are such that the operator $\mathcal{L}_{\nu,\mathfrak{m}}$ has zero eigenvalue, we use the classical alternative method of bifurcation theory. In the case that the eigenvalues are simple, this method was considered in [16]. In our case, the eigenvalues have always higher multiplicity, hence, we will follow [14,15,1,25,2].

For fixed $\mathfrak{m}_0 > 0$, we denote by

$$\mathcal{L}_{\nu,\mathfrak{m}} := \mathcal{L}_{\nu,\mathfrak{m}_0} + (\mathfrak{m} - \mathfrak{m}_0),$$

where

$$\mathcal{L}_{\nu,\mathfrak{m}_0} := \sum_{i=1}^{d} \nu_i^2 \frac{\partial^2}{\partial x_i^2} + \mathfrak{m}_0$$

has zero eigenvalue, and we call $(\mathfrak{m} - \mathfrak{m}_0)$ the bifurcation parameter.

We realize that, since $\mathcal{L}_{\nu,\mathfrak{m}_0}$ is self-adjoint (again, we will specify the appropriate spaces later), its kernel and the closure of its range are orthogonal. Since its spectrum is discrete, we can define spectral projections on the kernel and the range of $\mathcal{L}_{\nu,\mathfrak{m}_0}$. We call attention that we only use the operator for $\mathfrak{m} = \mathfrak{m}_0$.

We will denote by Π_K, Π_R the projections on the kernel and the closure of the range, respectively. These projections are complementary (i.e. $\Pi_K + \Pi_R = \mathrm{Id}$) and orthogonal.

Therefore, the equation (2.2) is equivalent to the system of equations obtained taking projections of (2.2) on the kernel and on the range. Introducing, furthermore, the notation

$$\hat{u} = \Pi_R u, \ \overline{u} = \Pi_K u$$

(so that $u = \hat{u} + \overline{u}$), then (2.2) can be rewritten as:

$$\begin{aligned} (\mathfrak{m} - \mathfrak{m}_0)\overline{u} &= \epsilon \Pi_K \mathcal{F}(\hat{u} + \overline{u}), \\ \Pi_R \mathcal{L}_{\nu,\mathfrak{m}_0}\hat{u} &= -(\mathfrak{m} - \mathfrak{m}_0)\hat{u} + \epsilon \Pi_R \mathcal{F}(\hat{u} + \overline{u}). \end{aligned} \tag{2.5}$$

Furthermore, when ${\mathcal{L}_{\nu,\mathfrak{m}_0}}^R \equiv \Pi_R \mathcal{L}_{\nu,\mathfrak{m}_0} \Pi_R$ is boundedly invertible as an operator on the closure of the range of $\mathcal{L}_{\nu,\mathfrak{m}_0}$, we have that (2.5) is equivalent to

$$\begin{aligned} (\mathfrak{m} - \mathfrak{m}_0)\overline{u} &= \epsilon \Pi_K \mathcal{F}(\hat{u} + \overline{u}), \\ \hat{u} &= ({\mathcal{L}_{\nu,\mathfrak{m}_0}}^R)^{-1}(-(\mathfrak{m} - \mathfrak{m}_0)\hat{u} + \epsilon \Pi_R \mathcal{F}(\hat{u} + \overline{u})). \end{aligned} \tag{2.6}$$

The system (2.6) is a system for the unknowns $\hat{u}, \overline{u}$.

The first equation in (2.6) is often called the *"bifurcation equation"* and the second one is called the *"range equation"*.

The classical method, which we will follow, to analyze (2.6) is to, for a given $\overline{u}$, find a $\hat{u}(\overline{u}, \epsilon)$ that solves the range equation. This will be an easy application of the contraction mapping. Once we have obtained such $\hat{u}(\overline{u}, \epsilon)$, the bifurcation equation becomes an equation for $\overline{u}$ alone, namely

$$(\mathfrak{m} - \mathfrak{m}_0)\overline{u} = \epsilon \Pi_K \mathcal{F}(\hat{u}(\overline{u}, \epsilon) + \overline{u}). \tag{2.7}$$

Since $\overline{u}$ is a finite-dimensional variable, the equation (2.7) is a finite-dimensional equation, which can be analyzed using the methods of singularity theory.

The interesting cases are when the nonlinearity is at least quadratic in the known. The linear terms can be absorbed in the linear part.

This equation (2.7) will, under some explicitly non-degeneracy conditions which depend only on the derivatives w.r.t. ϵ of the left hand side of (2.7), have several branches of solutions and require somewhat complicated non-linear analysis, but it is a finite-dimensional problem. The study of the branches etc. involves some assumptions on the nonlinearity f. To analyze the bifurcation equation, there are several methods in the literature.

a) Using the jets of the equation to apply a degenerate implicit function theorem.
b) Using some fixed point theorem based on index theory.

These methods, of course require some non-degeneracy assumptions but give very precise information on the detailed nonlinearities, which are affected by the symmetry etc. We refer to the references above for the rich mathematical results and applications of singularity theory and bifurcation theory.

In this paper, we will just discuss a very simple explicit nonlinearity and show that putting together bifurcation theory and the fixed point theory, for all small enough ϵ, we can analyze all the possible ranges of the parameters $\nu, \mathfrak{m}$ in (1.1). In this example, the bifurcation theory gives an explanation why the fixed point method breaks down. Indeed, when the parameters of the problem are close to the resonant values, there are several small solutions.

2.4. *The ill-posed evolution problem under nonresonance*

For the evolution equation (1.2), we are interested in finding quasi-periodic solutions of the form $u(t,x) = U(\omega t, x)$ with frequency $\omega \in \mathbb{R}^b$, where $U : \mathbb{T}^b \times \mathbb{T}^d \to \mathbb{R}$ is the hull function of the solution u.

We will present two different ways of analyzing the equation (1.2). A fixed point analysis and method based on reduction to time-dependent center manifolds.[3]

We denote by $\mathcal{Q}_{\omega,\nu,\mathfrak{m}}$ the linear operator

$$\mathcal{Q}_{\omega,\nu,\mathfrak{m}} := (\omega \cdot \partial_\theta)^2 + \sum_{i=1}^{d} \nu_i^2 \frac{\partial^2}{\partial x_i^2} + \mathfrak{m}, \tag{2.8}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in [1,2]^d$.

2.4.1. *A fixed point analysis*

In the fixed point method, we allow that the nonlinearity loose 2 derivatives with the following form:

$$\mathcal{N}(U)(\theta, x) := f(\theta, x, U(\theta, x), D_x U(\theta, x), D_x^2 U(\theta, x)),$$

then, (1.2) becomes

$$\mathcal{Q}_{\omega,\nu,\mathfrak{m}} U = \epsilon \mathcal{N}(U). \tag{2.9}$$

We notice that $\mathcal{Q}_{\omega,\nu,\mathfrak{m}}$ is a diagonal operator in the Fourier basis, i.e.

$$\mathcal{Q}_{\omega,\nu,\mathfrak{m}}\{\exp\{\mathrm{i}(l\theta + kx)\}\} = \Upsilon_{l,k}\{\exp\{\mathrm{i}(l\theta + kx)\}\}$$

with

[3] The fixed point analysis will allow nonlinearities that loose 2 derivatives, whereas the reduction to center manifolds allows to loss of $(2-\kappa)$ derivatives.

$$\Upsilon_{l,k} := -\langle \omega, l\rangle^2 - \sum_{i=1}^{d} \nu_i^2 k_i^2 + \mathfrak{m}. \tag{2.10}$$

The problem with the analysis of the multipliers (2.10) is that when b, the dimension of the frequencies, is bigger than 1, the set $\{\langle \omega, l\rangle\}_{l\in\mathbb{Z}^b}$ is dense on the reals. Hence, $\{\langle \omega, l\rangle^2\}_{l\in\mathbb{Z}^b}$ is dense on $\mathbb{R}_+$.

The hypotheses of the fixed point approach are as follows:

Suppose that the parameters ω, ν, $\mathfrak{m}$ meet one of the following hypotheses:

(H1) The value of $-\sum_{i=1}^{d} \nu_i^2 k_i^2 + \mathfrak{m}$ is negative;
(H2) The value of $-\sum_{i=1}^{d} \nu_i^2 k_i^2 + \mathfrak{m}$ is positive and ω is 1-dimensional & $\omega \in [1, 2] \subset \mathbb{R}^1$.

In these two cases, we can still use the freezing of coefficient method used for the elliptic case away from resonances to obtain the solutions. Otherwise, the freezing of coefficient method fails to solve (1.2) and we have to resort to the method described in the next Section.

2.4.2. Time-dependent center manifolds

A method of wider applicability (and which produces solutions more general than response solutions) is to apply a time-dependent center manifold theorem.

We will allow that the nonlinearity f depends on $D_x U$. More generally that the nonlinearity is given by a function which looses $(2 - \kappa)$ derivatives.

The recent paper [13] develops a time-dependent center manifold theory that applies to ill-posed equations. More precisely the methods of [13] require that $\mathcal{N}(U)$ is several times differentiable from a space of functions having r derivatives to a space of functions having $(r - 2 + \kappa)$ (for some $\kappa > 0$) derivatives.[4]

Note that, when the freezing of coefficient method applies, we do not need to include the κ, and we could obtain results for nonlinearities that loose 2 derivatives. Furthermore, the results on the elliptic case, require only that the nonlinearity is Lipschitz, but for the center manifold, we will need that the nonlinearity is several times differentiable.

Since the results for nonlinearities that loose $(2 - \kappa)$ derivatives can be obtained just directly from [13], in this paper we will present only the results for nonlinearities that loose 1 derivative and present full details in this case. As we will see, dealing with the case that the nonlinearity looses one derivative, is simpler than the case discussed in [13]. We hope that the present simple proof can be pedagogically motivating for these areas of results. Of course, in the classical problems in which the losses of derivatives are caused by applying differentials, the loss of derivatives are integers and, loosing one derivative is the best that one can do in this classical case.

We will show that the results of [13] apply to (1.2). Then, we conclude that even if (1.2) is ill-posed, there is a finite-dimensional manifold evolving quasiperiodically which is invariant under (1.2).

Once this center manifold is established, one can use finite-dimensional methods to obtain varieties of solutions: response subharmonics, (un)stable manifolds, etc. by a finite computation.

More precisely: By using a quasi-periodic parameterization of the center manifold, we are reduced to studying a finite-dimensional non-autonomous differential equation. We will provide the first terms in the expansion these manifold and recall that there are many results in the litera-

[4] We do not know whether the requirement of $\kappa > 0$ is really needed of it is a limitation of the method.

ture of finite-dimensional system which allow to conclude that, if the perturbations of the system satisfy some concrete non-degeneracy assumptions, the perturbed system admits interesting orbits.

For example, using KAM theory, one can get response solutions or solutions with external and inner frequencies. If these periodic solutions have positive Lyapunov exponents (which can be computed perturbatively), one can produce stable manifolds, using Melnikov theory, one can get subharmonic quasi-periodic orbits (and possibly their stable/unstable manifolds). There are many such results in the finite-dimensional theory that give precise conditions for the persistence of orbits of some kind if the perturbations satisfy different conditions.

We will not present these finite-dimensional results in detail since they are well established [34,31] and their methodology is rather different from the main thrust of this paper and its main use is applications to concrete models.

Remark 3. The paper [12] considers ill-posed time-independent manifolds and shows that there are infinite-dimensional manifolds of solutions that converge to them.

It seems likely that one can adapt the proofs of existence of stable manifolds for autonomous models to the non-autonomous cases considered here. If such adaptation was possible, besides the finite-dimensional families solutions produced here, one would get infinite-dimensional families asymptotic to them in the future (or in the past).

Of course, adapting the results of [12] to the time-dependent case in [13] would have several other applications.

3. Function spaces

Definition 4. Given $\rho > 0$, we introduce the complex torus $\mathbb{T}^d_\rho$:

$$\mathbb{T}^d_\rho := \{x \in \mathbb{C}^d/(2\pi\mathbb{Z})^d : \mathrm{Re}(x_j) \in \mathbb{T},\ \ |\mathrm{Im} x_j| \le \rho,\ \ j = 1, \cdots, d\}.$$

Note that $\mathbb{T}^d_\rho$ can be considered as a $2d$-dimensional real manifold with boundary.

For a function $u : \mathbb{T}^d_\rho \to \mathbb{C}$, we denote its Fourier expansion:

$$u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{\mathrm{i}k\cdot x},$$

where $k \cdot x = \sum_{i=1}^d k_i x_i$ and $\hat{u}_k$ are the Fourier coefficients of u.

If u is analytic and bounded on $\mathbb{T}^d_\rho$, the Fourier coefficients satisfy the Cauchy bounds:

$$|\hat{u}_k| \le e^{-|k|\rho} \max_{x \in \mathbb{T}^d_\rho} |u(x)|$$

with $|k| = \sum_{i=1}^d |k_i|$.

The spaces we will work with are:

Definition 5. For $\rho \geq 0$, $r \in \mathbb{Z}_+$, we denote by $H^{\rho,r}$

$$H^{\rho,r} := H^{\rho,r}(\mathbb{T}^d_\rho) = \left\{ u : \mathbb{T}^d_\rho \to \mathbb{C} \;\middle|\; \|u\|^2_{\rho,r} = \sum_{k\in\mathbb{Z}^d} |\hat{u}_k|^2 e^{2|k|\rho}(1+|k|^2)^r < +\infty \right\}.$$

Note that $(H^{\rho,r}, \|\cdot\|_{\rho,r})$ is a Hilbert space.

Remark 6. When $\rho = 0$, $H^r(\mathbb{T}^d) := H^{0,r}(\mathbb{T}^d)$ is the standard Sobolev space. According to the Sobolev embedding theorem, we have that the space $H^{r+\lambda}(\mathbb{T}^d)$ $(\lambda = 1, 2, \cdots)$ is continuously embedded into $C^\lambda(\mathbb{T}^d)$ for $r > d/2$ (see [40]).

When $\rho > 0$, the space $H^{\rho,r}$ is a closed space of standard Sobolev space $H^r(\mathbb{T}^d_\rho)$, which consists of complex analytic functions.

The spaces $H^{\rho,r}$ enjoy many remarkable properties. We have collected the ones we will use in Appendix A. The most important ones are the properties of the operator given by composition in the left. See Lemma 38. This will justify that the operator $\mathcal{F}(u) = f(x, u, Du, D^2u)$ satisfies the abstract properties when the function f is analytic (or sufficiently differentiable).

The following result is a straightforward consequence of the fact that the norms in the $H^{\rho,r}$ spaces are weighted sums of the Fourier coefficients.

Proposition 7. *([10]) If we have a linear operator $\mathcal{A}$ which is diagonal in the Fourier basis,*

$$\mathcal{A}\exp\{\mathrm{i}kx\} = \Upsilon_k \exp\{\mathrm{i}kx\},$$

for some suitable coefficients Υ_k, we get

$$\|\mathcal{A}\|_{H^{\rho,r}\to H^{\rho,r}} \leq \sup_k |\Upsilon_k|.$$

More generally, if

$$|\Upsilon_k| \leq C(1+|k|^2)^{-\lambda/2},$$

one has

$$\|\mathcal{A}\|_{H^{\rho,r}\to H^{\rho,r+\lambda}} \leq \sup_k |\Upsilon_k|(1+|k|^2)^{\lambda/2}.$$

4. Nonresonant case

In this section, we give a simple proof of the main result in [39] but we weaken the assumption that the nonlinearity $f(x, u)$ is trigonometric polynomial. We allow that the nonlinearity is $\mathcal{F}(u)(x) = f(x, u, Du, D^2u)$ with f analytic or finitely differentiable.

We now present our result for the model (2.2) in elliptic case far from resonances.

Theorem 8. *For fixed* $\mathfrak{m} > 0$ *and any* $0 < \delta \ll 1$*, given* $\rho \geq 0, r - 2 > d/2$*, let* $B_s(0) \subset H^{\rho,r}$ *be the closed ball around the origin with the radius* $s > 0$*.*

Assume that $\mathcal{F}$ *is Lipschitz from* $B_s(0)$ *into* $H^{\rho,r-2}$*.*

There exist $\epsilon_* > 0$ *depending on* $\nu, \mathfrak{m}, \delta, s, Lip(\mathcal{F})$ *and a set* $I \subset [1,2]^d$ *with Lebesgue measure* $\mathrm{mes}(I) = O(\delta)$*, such that when* $0 < \epsilon < \epsilon_*$*, for any* $\nu \in [1,2]^d \backslash I$*, the equation* (2.2) *admits a unique solution* $u \in B_s(0)$*.*

Remark 9. When $\rho > 0$, the solutions produced by Theorem 8 will be analytic as functions of their arguments. When $\rho = 0$, the solutions will be in the classical Sobolev space and finitely differentiable.

Remark 10. Since we are just applying the contraction mapping principle if $\mathcal{F}$ is differentiable (or analytic with respect to parameters), the solutions produced by Theorem 8 will depend differentiably (or analytic) in parameters. The analyticity with respect to parameters is very natural when we consider the nonlinearities of the form (2.3).

We will show in Lemma 38 that if f is analytic in a small ball of its arguments, $\mathcal{F}$ is indeed Lipschitz (analytic) on $B_s(0)$ of the spaces $H^{\rho,r}$ for any $\rho \geq 0$, $r - 2 > d/2$. If f depends analytically on the parameters, the function $\mathcal{F}$ is also analytic in the sense of analytic functions from one Banach space to another. See [24, Chapter III] for more details on the theory of analytic functions from a Banach space to another.

We also show in Lemma 38 that if f is C^{r+1}, the function $\mathcal{F}$ is Lipschitz from H^r to H^{r-2} for $r - 2 > d/2$.

To prove Theorem 8, we first prove that the operator $\mathcal{L}_{\nu,\mathfrak{m}}$ is boundedly invertible from $H^{\rho,r-2}$ to $H^{\rho,r}$. These are, of course, standard elliptic estimates showing that inverting the operator *gains two derivatives* [6,40].

We note that if $\rho > 0$ or $\rho = 0$, $r > d/2 + 2$, the solutions produced here satisfy the equation (1.1) in the classical sense.

4.1. Estimates on the inverse operator ${\mathcal{L}_{\nu,\mathfrak{m}}}^{-1}$

First, we give the measure of the parameter set in ν space which produces resonance term.

Lemma 11. *For sufficiently small* $\delta > 0$ *and* $d \in \mathbb{Z}_+$*, fixed* $\mathfrak{m} > 0$*, we define the following parameter set of* ν*:*

$$I = \left\{ \nu \in [1,2]^d \mid \exists\, k \in \mathbb{Z}^d,\ \text{such that} \left| -\sum_{i=1}^{d} k_i^2 \nu_i^2 + \mathfrak{m} \right| \leq \delta \right\}.$$

Then, we have $\mathrm{mes}(I) = O(\delta)$*, where* $O(\delta)$ *is the same order of* δ*.*

Proof. When $k = 0$, $\mathrm{mes}(I) = 0$. When $k \in \mathbb{Z}^d \backslash \{0\}$, for fixed $\mathfrak{m} > 0$, we define the set

$$K = \left\{ k^{(n)} \in \mathbb{Z}^d \setminus \{0\} \mid \sum_{i=1}^{d} (k_i^{(n)})^2 \nu_i^2 = \mathfrak{m},\ n = 1, \cdots, N \right\}.$$

We choose $k^{(n)} \in K$ and define the set of ν as

$$I_n = \left\{ \nu \in [1,2]^d \;\middle|\; |-F_{k^{(n)}}(\nu) + \mathfrak{m}| \le \delta \right\}$$

with $F_{k^{(n)}}(\nu) = \sum_{i=1}^d \left(k_i^{(n)}\right)^2 \nu_i^2$. Then,

$$\operatorname{mes}(I_n) \le \frac{2\delta}{\inf\{|\nabla F_{k^{(n)}}(\nu)|_2\}} \le \delta,$$

where $\nabla F_{k^{(n)}}(\nu) = \left(2(k_1^{(n)})^2\nu_1, 2(k_2^{(n)})^2\nu_2, \cdots, 2(k_d^{(n)})^2\nu_d\right)$, $|\cdot|_2$ is l^2-norm.

Thus, we have

$$\operatorname{mes}(I) = \operatorname{mes}\left(\bigcup_{n=1}^N I_n\right) \le \sum_{n=1}^N \operatorname{mes}(I_n) = O(\delta). \quad \square$$

According to the Lemma, we conclude that when $\nu \in [1,2]^d \setminus I$, the diagonal operator $\mathcal{L}_{\nu,\mathfrak{m}}$ does not have zero eigenvalues and that the absolute value of the eigenvalues is bounded from below by δ.

We furthermore observe that for large k, the eigenvalues are bounded from below by $C(1 + |k|^2)$. If the eigenvalues do not vanish, we have a bound $|\Upsilon_{k,\mathfrak{m}}| \ge C(1 + |k|^2)$. Therefore, by Proposition 7, one has the following proposition:

Proposition 12. *If $\nu \in [1,2]^d \setminus I$, we have*

$$\|\mathcal{L}_{\nu,\mathfrak{m}}{}^{-1}\|_{H^{\rho,r} \to H^{\rho,r}} \le \delta^{-1},$$
$$\|\mathcal{L}_{\nu,\mathfrak{m}}{}^{-1}\|_{H^{\rho,r-2} \to H^{\rho,r}} < C$$

for some constant C.

4.2. Existence of solutions

Recall the equation (2.4)

$$u(x) = \epsilon \mathcal{L}_{\nu,\mathfrak{m}}{}^{-1} \mathcal{F}(u)(x) \equiv \mathcal{T}(u)(x).$$

We will use the contraction mapping principle.

We assume that there is a closed ball $B_s(0)$ around the origin in $H^{\rho,r}$ with radius $s > 0$, where

$$B_s(0) = \{u \in H^{\rho,r} \;\big|\; \|u\|_{\rho,r} \le s\},$$

such that $\mathcal{F}$ is defined in $B_s(0)$ as an operator from $B_s(0) \subset H^{\rho,r}$ to $H^{\rho,r-2}$. Moreover, we assume that $\mathcal{F}$ is Lipschitz in $B_s(0)$ as an operator from the $H^{\rho,r}$ to $H^{\rho,r-2}$. That is

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{\rho,r-2} \le \operatorname{Lip}(\mathcal{F}) \|u_1 - u_2\|_{\rho,r}.$$

We will show that, for sufficiently small ϵ, the operator $\mathcal{T}$ maps the ball $B_s(0)$ into itself and is a contraction. Therefore, it has a unique fixed point in this ball.

Denote $\epsilon_* = \min\{1/(2C\text{Lip}(\mathcal{F})), s/(2C\|\mathcal{F}(0)\|_{\rho,r-2})\}$. When $0<\epsilon<\epsilon_*$, for any $u_1, u_2 \in B_s(0)$, one has

$$\begin{aligned}\|\mathcal{T}(u_1)-\mathcal{T}(u_2)\|_{\rho,r} &= \|\epsilon\mathcal{L}_{\nu,\mathrm{m}}{}^{-1}(\mathcal{F}(u_1)-\mathcal{F}(u_2))\|_{\rho,r}\\ &\le \epsilon\|\mathcal{L}_{\nu,\mathrm{m}}{}^{-1}\|_{H^{\rho,r-2}\to H^{\rho,r}}\|\mathcal{F}(u_1)-\mathcal{F}(u_2)\|_{\rho,r-2}\\ &\le \epsilon\|\mathcal{L}_{\nu,\mathrm{m}}{}^{-1}\|_{H^{\rho,r-2}\to H^{\rho,r}}\text{Lip}(\mathcal{F})\|u_1-u_2\|_{\rho,r}\\ &\le \frac{1}{2}\|u_1-u_2\|_{\rho,r}.\end{aligned}$$

For $u \in B_s(0)$, one has

$$\begin{aligned}\|\mathcal{T}(u)\|_{\rho,r} &= \|\mathcal{T}(0)+\mathcal{T}(u)-\mathcal{T}(0)\|_{\rho,r}\\ &\le \epsilon\|\mathcal{L}_{\nu,\mathrm{m}}{}^{-1}\|_{H^{\rho,r-2}\to H^{\rho,r}}\|\mathcal{F}(0)\|_{\rho,r-2}+\frac{1}{2}s\\ &\le s.\end{aligned}$$

It follows from the contraction principle that there exists a unique solution $u(x) \in B_s(0)$ belonging to $H^{\rho,r}$ for (2.2).

One could think of optimizing the choice of ϵ_* to obtain uniqueness in a larger ball. A different optimization is to locate the solution in a smaller ball.

5. Resonant case

In this section, we study the case when the operator $\mathcal{L}_{\nu,\mathrm{m}_0}$ has a nontrivial kernel (this case was not considered in [39]).

We will explain the general theory and give a concrete example when the nonlinearity is just $f(u)=u^2$. Note that, in this case, the forcing is identically zero at the origin, so that the solutions produced by Theorem 8 are just 0. We will show that, near the bifurcation points, there are some solutions besides those produced by Theorem 8.

The nonlinearity u^2 has been chosen for simplicity. As we will see in Remark 17, the same results apply for all linearities which vanish to second order in u. Notice that adding nonlinearities with a nonvanishing linear term in u is better dealt with by changing the linear operator we are considering.

We first rescale the original system to get a slow system. Let $v=\epsilon u$. Then the equation (1.2) becomes

$$\mathcal{L}_{\nu,\mathrm{m}}v(x)=v^2(x). \tag{5.1}$$

Equation (5.1) can be rewritten as

$$(\mathcal{L}_{\nu,\mathrm{m}_0}+(\mathrm{m}-\mathrm{m}_0))v(x)=v^2(x). \tag{5.2}$$

We will show that for value $\mathfrak{m}$ close to $\mathfrak{m}_0$, the problem (5.1) may have several small solutions. These solutions are functions of the *bifurcation parameter* $(\mathfrak{m} - \mathfrak{m}_0)$. Clearly, the existence of several small solutions establishes that one cannot apply the contraction mapping principle.

Remark 13. Notice that if $v : \mathbb{T}^d \to \mathbb{R}$ satisfies (5.1), then for any $x_* \in \mathbb{R}^d$, the functions

$$v_{x_*}(x) = v(x + x_*)$$

are also the solutions so that we always obtain d-dimensional families of solutions.

Notice also that the dimension of the kernel is expected to be 2^d (see Remark 2) and $2^d > d$. So that, when the dimension grows, the dimension of the kernel grows much faster than the dimension of the families.

We anticipate that the main difficulty is that the kernel will be high dimension and that, at the same time, there are symmetries. Since the dimension of the kernel grows exponentially with the dimension d and the dimension of the symmetry is d, we will restrict our study to $d = 1, 2$ and only make some remarks about $d \geq 3$.

We give the following result:

Theorem 14. *Consider the problem* (5.2)*.*

Assume that the dimension d of the space is either 1 *or* 2 *and that the parameter ν satisfies Assumption 18. Assume also that the parameter ν does not belong to another set of measure zero on which an explicit rational function vanishes.*

Let $\mathfrak{m}_0$ be such that the operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ has a nontrivial kernel, which by Assumption 18, has dimension 2^d and consists of exponentials of wave vectors which are obtained by changing signs in a vector.

Let σ be the sign of an explicit formula $A + B$ (given in (5.14) *in two dimensions).*

Then for $\mathfrak{m}$ sufficiently close to (not equal to) $\mathfrak{m}_0$, and with $\mathfrak{m} - \mathfrak{m}_0$ having the sign σ, the problem (5.2) *admits d-dimensional families of non-zero solutions.*

Moreover, these solutions are functions of the bifurcation parameter $(\mathfrak{m} - \mathfrak{m}_0)$ and are analytic in $|\mathfrak{m} - \mathfrak{m}_0|^{\frac{1}{2}}$.

Remark 15. Notice that the branches of (5.2) exist for both intervals $|\mathfrak{m} - \mathfrak{m}_0|$ small and satisfying that when $d = 1$, $\mathfrak{m} - \mathfrak{m}_0$ is positive, and when $d = 2$, $\mathfrak{m} - \mathfrak{m}_0$ having the same sign with $A + B$, which is denoted by σ in Theorem 14.

The cases when the extra branch appears for $\mathfrak{m} - \mathfrak{m}_0 < 0$ are called *subcritical bifurcation* and the cases when the extra branch of solutions appears for $\mathfrak{m} - \mathfrak{m}_0 > 0$ are called *supercritical bifurcation*.

Most of the classical bifurcation theory is concerned with existence of stationary solutions, but in this case, due to the symmetries of the problem, we obtain always families of equilibria.

The proof of Theorem 14 is based on the alternative method of bifurcation theory explained before. In this section, we give all the needed details for the specific nonlinearity $f(u) = u^2$.

From the analysis in Section 2, we know that equation (5.2) can be regarded as the *"bifurcation equation"* and the *"range equation"*, respectively:

$$(\mathfrak{m} - \mathfrak{m}_0)\overline{v} = \Pi_K(\hat{v} + \overline{v})^2, \tag{5.3}$$

$$\big(\mathcal{L}_{\nu,\mathfrak{m}_0} + (\mathfrak{m} - \mathfrak{m}_0)\big)\hat{v} = \Pi_R(\hat{v} + \overline{v})^2. \tag{5.4}$$

5.1. Some general results on the range equation

The range equation is much simpler to deal with than the bifurcation equation and admits a general theory. As we will see, it admits solutions which are analytic in $\overline{v}$ under rather general circumstances.

For notational convenience later, we will introduce some coordinates

$$\alpha = (\cdots, \alpha_j, \cdots), \ \ j = 1, \cdots, 2^d$$

in the kernel. So that $\overline{v} = \sum_j \alpha_j \exp(\mathrm{i}k^j \cdot x)$ for the k^j the wave numbers in the kernel of $\mathcal{L}_{\nu,\mathfrak{m}_0}$. Then, we will show that the $\hat{v}$ solving the range equation is analytic as the function of α.

Since we are going to do algebra of polynomials, it is convenient to think of α as complex numbers (even if the problem at hand is real). It also allows us to present the eigenfunctions as exponentials rather than as pair of sin / cos. This is useful when considering products.

The counting of dimensions is slightly delicate: If we want to have real solutions, we will need that if $k = -k$, the corresponding coefficients satisfy $\alpha = \alpha^*$. The complex dimension of the space of α is the dimension of the kernel. If we require that the solutions are real, the complex dimension will be half the dimension of the kernel, which corresponds to having a real dimension equal to the dimension of the kernel.

Consider the range equation (5.4) in the space $H^{\vartheta,\rho,r}$ with $0 < \vartheta \ll 1$, where

$$H^{\vartheta,\rho,r} = \Big\{v : \mathbb{C}^{2^d} \to H^{\rho,r} \ \Big| \ v(\alpha) = \sum_{j=0}^{\infty} v_j \alpha^j, \ \|v\|_{\vartheta,\rho,r} = \sum_{j=0}^{\infty} \|v_j\|_{\rho,r}\vartheta^j < \infty\Big\}$$

is a Banach algebra. Then we have the following results for any dimension d.

Lemma 16. *There exists a solution $\hat{v} \in H^{\vartheta,\rho,r}$, for the range equation* (5.4)*, which is analytic in $\alpha \in \mathbb{C}^{2^d}$.*

Proof. We assume $\mathfrak{m} - \mathfrak{m}_0 = O(\epsilon)$. Then, range equation (5.4) can be rewritten as:

$$\hat{v} = \mathcal{L}_{\nu,\mathfrak{m}_0}{}^{-1}\big(-\epsilon\hat{v} + \Pi_R(\hat{v} + \overline{v})^2\big) \equiv T(\hat{v}). \tag{5.5}$$

It is easy to see that the operator T defined in (5.5) maps the space $H^{\vartheta,\rho,r}$ into itself.

We choose a ball $\mathcal{B}_s(0) \subset H^{\vartheta,\rho,r}$. The remaining task is to verify that the operator T maps the ball into itself and it is a contraction in this ball.

Since $\overline{v} = \sum_{j=1}^{2^d} \alpha_j \exp \mathrm{i}k^j x$, belonging to the kernel space of $\mathcal{L}_{\nu,\mathfrak{m}_0}$, only contains finite terms, there exists a constant C such that

$$\|\overline{v}\|_{\vartheta,\rho,r} \leq C\vartheta.$$

Therefore, one has

$$\mathrm{Lip}(T) \leq C(\epsilon + \vartheta + s) \leq \frac{1}{10}$$

when we choose ϵ small enough and $s < \frac{1}{20C} - \vartheta$. This reveals that T is a contraction in the ball $\mathcal{B}_s(0)$. On the other hand, for $U \in \mathcal{B}_s(0)$ with s chosen above, one has

$$\|T(\hat{v})\|_{\vartheta,\rho,r} \leq \|T(0)\|_{\vartheta,\rho,r} + \|T(\hat{v}) - T(0)\|_{\vartheta,\rho,r} \leq C\vartheta + \frac{1}{10}s \leq s$$

by choosing the radius s satisfies $\frac{10C}{9}\vartheta \leq s < \frac{1}{20C} - \vartheta$.

In conclusion, by the fixed point theorem in the Banach space $H^{\vartheta,\rho,r}$, there exists a unique solution $\hat{v} \in H^{\vartheta,\rho,r}$ analytic in α for the equation (5.5). □

Remark 17. Note that the proof of Lemma 16 works even if the nonlinearity is an analytic function starting with quadratic terms.

There are also versions of the argument assuming only finite differentiability of the nonlinearity (acting on spaces of finite-differentiable functions). Note that, for subsequent use, we only need a finite number of derivatives. The aim of this paper is not to give a complete coverage, but to illustrate the possibilities in one example.

5.2. *Some preliminary analysis of the bifurcation equation*

The fact that $\hat{v}$ is analytic in α, shows that we can write the bifurcation equation (5.3) as

$$\epsilon\alpha = B(\alpha) \equiv \sum_{j \in \mathbb{Z}^{\mathrm{L}}} B_j \alpha^j, \tag{5.6}$$

where we are using multi-index notation for α^j, denoting by $\mathrm{L} := 2^d$ the dimension of the kernel of $\mathcal{L}_{\nu,\mathfrak{m}_0}$ and the B_j are (complex) vectors of length L.

In this section, we will present some general results about the bifurcation equation, which hold for all dimensions of the kernel.

Later on, we will present some complete results for the low dimensional cases and some remarks show that the higher dimensional cases are more complicated.

We will assume for all subsequent work:

Assumption 18. The wave numbers of the eigenfunctions in the kernel of $\mathcal{L}_{\nu,\mathfrak{m}_0}$ are obtained by changing signs of components of a vector.

As indicated before, for a set of full measure of ν, Assumption 18 holds for all the eigenvalues of $\mathcal{L}_{\nu,\mathfrak{m}_0}$.

An important observation is that, since all the eigenvalues are exponentials and the product of eigenvalues is also an exponential. The projections over the kernel and the range are very easy acting on exponentials. They either return the same exponential of zero. Similarly, we recall that $\mathcal{L}_{\nu,\mathfrak{m}_0}$ and $\mathcal{L}_{\nu,\mathfrak{m}_0}{}^{-1}$ acting on exponentials are just multiplying by a number.

Lemma 19. *With the notations of* (5.6) *and Assumption 18. If* $|j|$ *is even, then* $B_j = 0$.

Proof. We observe that the powers of $\overline{v}$ have products of $\alpha_j \exp(\mathrm{i}k^j \cdot x)$. So, they are monomials of the form:

$$B_j \alpha_1^{j_1} \cdots \alpha_L^{j_L} \exp(\mathrm{i}(k^1 j_1 + \cdots + k^L j_L) \cdot x) \tag{5.7}$$

with the k^j being wave numbers of functions in the kernel.

The following result is obvious.

Proposition 20. *We consider the class of functions $\mathcal{G}$ which are analytic functions of α and all the terms are of the form (5.7).*

If $v_1, v_2 \in \mathcal{G}$, the following belong to $\mathcal{G}$

$$v_1 + v_2,\, v_1 \cdot v_2,\, \Pi_K v_1,\, \Pi_R v_1,\, \mathcal{L}_{\nu,\mathfrak{m}_0}{}^{-1} \Pi_R v_1.$$

Hence, it follows that all the terms in $\Pi_K(\overline{v} + \hat{v})^2$ are of the form (5.7). The bifurcation equation (5.6) is obtained by equating the coefficients of the same exponential functions.

Therefore, the only terms that can appear in the bifurcation equation are terms in which $j_1 k^1 + \cdots + j_L k^L$ is one of the wave numbers in the kernel.

If we look at the first component, the components of the k^1 are $\pm a$. A necessary condition for the sum to be in the kernel is that the first component is $\pm a$.

We just observe that it is impossible to add an even number of $\pm a$ in such a way that the sum is either a or $-a$. □

5.2.1. 2-dimensional case

We present our main idea for 2-dimensional case under Assumption 18. We remark that the same analysis applies in higher dimensions when the wave numbers of eigenvalues in the kernel has only 2 nonzero components.

We denote by

$$\mathcal{L}_{\nu,\mathfrak{m}_0} \exp\{\mathrm{i}kx\} = \Upsilon_{k,\nu,m_0} \exp\{\mathrm{i}kx\},$$

where $\Upsilon_{k,\nu,m_0} = -\sum_{j=1}^{2} k_j^2 \nu_j^2 + \mathfrak{m}_0, k \in \mathbb{Z}^2$, are the eigenvalues of the operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ and $\exp\{\mathrm{i}kx\}$ are the eigenfunctions corresponding to Υ_{k,ν,m_0}. We observe that the null space of the operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ has complex dimension 4, and the eigenvectors are $\exp\{\mathrm{i}kx\}$, $k \in \mathcal{K}$, where the set $\mathcal{K}$ is defined as:

$$\begin{aligned} \mathcal{K} := \{k \in \mathbb{Z}^2 \,\big|\, k = \{(\pm a, \pm b)\},\ a, b \in \mathbb{N}/\{0\}, \\ \text{satisfying } \Upsilon_{k,\nu,m_0} = 0,\ \text{i.e. },\, -(a^2\nu_1^2 + b^2\nu_2^2) + \mathfrak{m}_0 = 0\}. \end{aligned} \tag{5.8}$$

In the following, we simplify the notation Υ_{k,ν,m_0} as Υ_k.

We denote the null space of operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ by

$$Ker := \{\overline{v}_\alpha : \mathbb{T}_\rho^2 \to \mathbb{C} \,\big|\, \overline{v}_\alpha(x) = \sum_{j=1}^{4} \alpha_j \exp\{\mathrm{i}k^j x\}, \alpha_j \in \mathbb{C}, k^j \in \mathcal{K}\}.$$

Note that for real function, without loss of generality, we suppose that $k^1 = -k^4$, $k^2 = -k^3$ then $\alpha_4 = \alpha_1^*$, $\alpha_3 = \alpha_2^*$. It suffices to determine α_1 and α_2. More precisely, we take $k^1 = (a, b)$, $k^2 = (a, -b)$, $k^3 = (-a, b)$, $k^4 = (-a, -b)$.

Now, we go back to the bifurcation equation

$$(\mathfrak{m}-\mathfrak{m}_0)\overline{v}_\alpha(x)=\Pi_K(\hat{v}_\alpha(x)+\overline{v}_\alpha(x))^2, \tag{5.9}$$

which can be rewritten as

$$\epsilon\alpha_l\exp(\mathrm{i}k^l x)=\sum_{|j|\ge 3}B^l_j\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4}\exp(\mathrm{i}(k^1 j_1+\cdots+k^4 j_4)\cdot x) \tag{5.10}$$

with $j=(j_1,\cdots,j_4)\in\mathbb{N}^4$, $|j|=|j_1|+\cdots|j_4|$ and the B^l_j are (complex) vectors of length 1 corresponding to l. It suffices to solve the equation for α:

$$\epsilon\alpha_l=\sum_{\substack{|j|\ge 3\\ k^1 j_1+\cdots+k^4 j_4=k^l}}B^l_j\alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4}:=B(\alpha),\ l=1,\cdots,4, \tag{5.11}$$

where $B(\alpha)$ satisfies the following proposition.

Proposition 21. *For 2-dimensional case, any term in $B(\alpha)$ defined in (5.11):*

(1) *contains a factor α_l;*
(2) *the other terms are powers of $|\alpha_1|^2,\ |\alpha_2|^2$.*

Proof. It follows from (5.8) that the wave vectors $k^l=(\pm a,\pm b)$ with $\Upsilon_{k^l}=0$. For simplicity, we denote k^l using the sign only, i.e. $k^1=(+,+)$, $k^2=(+,-)$, $k^3=(-,+)$, $k^4=(-,-)$. It suffices to prove the case of $k^1=(+,+)$ (by defining a,b to have the appropriate sign).

From $k^1 j_1+\cdots+k^4 j_4=(+,+)$, one has

$$\begin{cases}(j_1-j_4)+(j_2-j_3)=1,\\ (j_1-j_4)+(j_3-j_2)=1.\end{cases}$$

This indicates $j_1=j_4+1,\ j_2=j_3$. Therefore, when $l=1$,

$$B(\alpha)=\sum_{j_3+j_4\ge 1}B^1_j\alpha_1^{j_4+1}\alpha_2^{j_3}\alpha_3^{j_3}\alpha_4^{j_4}=\alpha_1\sum_{j_3+j_4\ge 1}B^1_j(|\alpha_1|^2)^{j_4}(|\alpha_2|^2)^{j_3}.$$

This concludes our results. □

Proposition 21 gives that the bifurcation equation (5.11) can be represented as:

$$\epsilon I=Mz+P(z),\ z=(|\alpha_1|^2,|\alpha_2|^2)^T,\ I=(1,1)^T, \tag{5.12}$$

where M is a 2×2 matrix (will be given later) and P is a homogeneous polynomial of degree 2 or higher and we have introduced the typographical simplification $\epsilon=\mathfrak{m}-\mathfrak{m}_0$.

Note that, introducing $\widetilde{z}=\epsilon z$, the equation (5.12) can be rewritten as

$$I=M\widetilde{z}+\epsilon\widetilde{P}(\widetilde{z},\epsilon), \tag{5.13}$$

where $\widetilde{P}(\widetilde{z}, \epsilon) = \epsilon^{-1} P(\epsilon\widetilde{z})$. Since P vanishes to order 2, we have that $\widetilde{P}$ is an analytic function.

The formulation of the equation as (5.13) makes it clear that we can use the implicit function theorem. We will show that, for a set of ν of full measure (see Proposition 22 in the following) we have that the matrix M is invertible, it follows from the implicit function theorem that there exists a solution $z = z(\epsilon)$ next to zero for the equation (5.12).

Note, however that the problem we have is not just to find solutions z of (5.12). Since the meaning of the components of z are squares of modulus, we need that both of them are positive. Of course, we could consider choosing the sign of ϵ, but it is non-trivial than choosing one sign of ϵ. We can ensure that both components of the solution have a positive sign.

Hence, we will show in Proposition 23 that the components of $M^{-1}I$ have the same sign. Precisely, when the components are positive, for $\epsilon > 0$, we interpret the solutions z for the equation (5.12) as the absolute values of two complex numbers. When the components are negative, we get solutions z for $\epsilon < 0$. We note that the condition is an explicit condition, on the vectors in the kernel, as well as $\nu, \mathfrak{m}$.

The remaining task is to give the formula of the matrix M and prove it is invertible in a set of ν of full measure and that the solutions of (5.12) have both components positive for small values ϵ.

By observation we find that when $d = 2$, the following two statements hold: for $p, q, j = 1, \cdots, 4$,

($S1$): If $\exp\{\mathrm{i}k^p x\}, \exp\{\mathrm{i}k^q x\} \in Ker$, then $\exp\{\mathrm{i}(k^p + k^q)\}$ is in the range space.

($S2$): If $\exp\{\mathrm{i}k^p x\}, \exp\{\mathrm{i}k^q x\}, \exp\{\mathrm{i}k^j x\} \in Ker$ and $\exp\{\mathrm{i}(k^p + k^q + k^j)x\} \in Ker$, then two of k^p, k^q, k^j are opposite.

Consider the range equation (5.4). It follows from the fact $\hat{v}$ should be a quadratic function of $\overline{v}$, i.e., $\hat{v} = O(\overline{v}^2)$ that the equation (5.4) becomes:

$$(\mathcal{L}_{\nu,\mathfrak{m}_0} + (\mathfrak{m} - \mathfrak{m}_0))\hat{v} = \overline{v}^2 + O(\overline{v}^3).$$

Since we assume $\mathfrak{m} - \mathfrak{m}_0 = O(\epsilon)$, we have

$$\mathcal{L}_{\nu,\mathfrak{m}_0}\hat{v} = \overline{v}^2 + O(\overline{v}^3).$$

As a consequence, $\hat{v} = O(\overline{v}^2)$ has the form:

$$\begin{aligned}
\hat{v} &= \sum_{p,q=1}^{4} \frac{\alpha_p\alpha_q}{\Upsilon_{k^p+k^q}} \exp\{\mathrm{i}(k^p + k^q)x\} + O(|\alpha_p| + |\alpha_q|)^3 \\
&= \frac{\alpha_1^2 \exp\{2\mathrm{i}k^1 x\}}{\Upsilon_{(2a,2b)}} + \frac{\alpha_2^2 \exp\{2\mathrm{i}k^2 x\}}{\Upsilon_{(2a,-2b)}} + \frac{\alpha_3^2 \exp\{2\mathrm{i}k^3 x\}}{\Upsilon_{(-2a,2b)}} + \frac{\alpha_4^2 \exp\{2\mathrm{i}k^4 x\}}{\Upsilon_{(-2a,-2b)}} \\
&\quad + \frac{2\alpha_1\alpha_2 \exp\{\mathrm{i}(k^1 + k^2)x\}}{\Upsilon_{(2a,0)}} + \frac{2\alpha_1\alpha_3 \exp\{\mathrm{i}(k^1 + k^3)x\}}{\Upsilon_{(0,2b)}} \\
&\quad + \frac{2\alpha_2\alpha_4 \exp\{\mathrm{i}(k^2 + k^4)x\}}{\Upsilon_{(0,-2b)}} + \frac{2\alpha_3\alpha_4 \exp\{\mathrm{i}(k^3 + k^4)x\}}{\Upsilon_{(-2a,0)}} + \frac{2(\alpha_1\alpha_4 + \alpha_2\alpha_3)}{\Upsilon_{(0,0)}},
\end{aligned}$$

which is well defined since $\Upsilon_{k^p+k^q} \neq 0$ according to $(S1)$.

Consider the bifurcation equation (5.3). Combining $(S1)$ and $(S2)$, one has

$$
\begin{aligned}
\epsilon\overline{v} &= \Pi_K(\overline{v}+\hat{v})^2 \\
&= \Pi_K(2\overline{v}\hat{v}) + O(|\alpha_1|+|\alpha_2|)^4 \\
&= \left(\frac{2\alpha_1^2\alpha_4}{\Upsilon_{(2a,2b)}} + \frac{4(\alpha_1\alpha_4+\alpha_2\alpha_3)\alpha_1}{\Upsilon_{(0,0)}} + 4\alpha_1\alpha_2\alpha_3(\frac{1}{\Upsilon_{(0,2b)}} + \frac{1}{\Upsilon_{(2a,0)}})\right)\exp\{\mathrm{i}k^1x\} \\
&\quad + \left(\frac{2\alpha_2^2\alpha_3}{\Upsilon_{(2a,-2b)}} + \frac{4(\alpha_1\alpha_4+\alpha_2\alpha_3)\alpha_2}{\Upsilon_{(0,0)}} + 4\alpha_1\alpha_2\alpha_4(\frac{1}{\Upsilon_{(0,-2b)}} + \frac{1}{\Upsilon_{(2a,0)}})\right)\exp\{\mathrm{i}k^2x\} \\
&\quad + \left(\frac{2\alpha_3^2\alpha_2}{\Upsilon_{(-2a,2b)}} + \frac{4(\alpha_1\alpha_4+\alpha_2\alpha_3)\alpha_3}{\Upsilon_{(0,0)}} + 4\alpha_1\alpha_3\alpha_4(\frac{1}{\Upsilon_{(0,2b)}} + \frac{1}{\Upsilon_{(-2a,0)}})\right)\exp\{\mathrm{i}k^3x\} \\
&\quad + \left(\frac{2\alpha_4^2\alpha_1}{\Upsilon_{(-2a,-2b)}} + \frac{4(\alpha_1\alpha_4+\alpha_2\alpha_3)\alpha_4}{\Upsilon_{(0,0)}} + 4\alpha_2\alpha_3\alpha_4(\frac{1}{\Upsilon_{(0,-2b)}} + \frac{1}{\Upsilon_{(-2a,0)}})\right)\exp\{\mathrm{i}k^4x\} \\
&\quad + O(|\alpha_1|+|\alpha_2|)^4 \\
&= \alpha_1\left(|\alpha_1|^2A + |\alpha_2|^2B\right)\exp\{\mathrm{i}k^1x\} + \alpha_2\left(|\alpha_2|^2A + |\alpha_1|^2B\right)\exp\{\mathrm{i}k^2x\} \\
&\quad + \alpha_3\left(|\alpha_3|^2A + |\alpha_4|^2B\right)\exp\{\mathrm{i}k^3x\} + \alpha_4\left(|\alpha_4|^2A + |\alpha_3|^2B\right)\exp\{\mathrm{i}k^4x\} \\
&\quad + O(|\alpha_1|+|\alpha_2|)^4,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{2}{\Upsilon_{(2a,2b)}} + \frac{4}{\Upsilon_{(0,0)}}, \\
B &= \frac{4}{\Upsilon_{(2a,0)}} + \frac{4}{\Upsilon_{(0,2b)}} + \frac{4}{\Upsilon_{(0,0)}}.
\end{aligned}
\tag{5.14}
$$

Then, the linear part of the equation (5.12) for the non-zero $|\alpha_1|^2$, $|\alpha_2|^2$, is the following factorized equation;

$$
\begin{cases}
\epsilon = |\alpha_1|^2A + |\alpha_2|^2B, \\
\epsilon = |\alpha_1|^2B + |\alpha_2|^2A.
\end{cases}
$$

We denote by

$$
M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{5.15}
$$

Indeed, M is invertible in a full measure set of ν. See the following Proposition.

Proposition 22. *The determinant of the matrix M defined in (5.15) is different from zero for a set of ν of full measure.*

Proof. Since

$$\det(M) = A^2 - B^2 = (A+B)\cdot(A-B),$$

it suffices to consider $A \pm B$. If we consider $A \pm B$ as a function of ν, it is a rational function.

It is not difficult to compute the numerators of $A \pm B$ and to check that they have a non-trivial term. So, both $A \pm B$ are non-trivial rational functions of ν. Therefore, they can vanish only on a set of ν of measure zero. This set is the set alluded to in the hypothesis of Theorem 14.

Note that the set of ν for which $A \pm B$ vanish depends on k and if we fix k this is the only set we need to exclude for this k. Since the set of k is countable, we can exclude a set of ν for all the k. □

Proposition 23. *With the notations above, we have that both components of $M^{-1}I$ have the same sign.*

Proof. We have

$$M^{-1}I = \frac{1}{\det(M)}\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} I = \frac{1}{A^2 - B^2}\begin{pmatrix} A-B \\ A-B \end{pmatrix} = \frac{1}{A+B}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Recall that we have included in our assumptions that the parameters ν are such that $A+B$ is not zero, so that the leading term of the solution has a definitive sign. Both components of the leading solution have the same sign (they are identical) and we can choose ϵ having the same sign with $A+B$ (this is the σ included in Theorem 14), so that the solutions for the equation (5.12) are positive.

Once we know that the leading approximation is positive, the implicit function theorem tells us we can choose a family z, which remains positive for ϵ such that $|\epsilon|$ is small enough and the sign of ϵ is σ. □

5.2.2. The bifurcation equation for $d = 1$

The case of $d = 1$ is much easier since Proposition 21 and Proposition 22 are easier to be proved for $d = 1$.

It is obvious that the kernel space of the operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ is 2-dimensional, which implies that the kernel space of the operator $\mathcal{L}_{\nu,\mathfrak{m}_0}$ can be represented as the following:

$$\begin{aligned} Ker := \{\overline{v}_\alpha : \mathbb{T}_\rho \to \mathbb{C} \,\big|\, \overline{v}_\alpha(x) = \alpha_1 \exp\{\mathrm{i}k^1 x\} + \alpha_2 \exp\{\mathrm{i}k^2 x\}, \alpha_1, \alpha_2 \in \mathbb{C}, \\ \text{and } k^j \in \mathbb{Z} \ \text{ satisfying } \Upsilon_{k^j} = 0, \ j = 1, 2.\} \end{aligned} \tag{5.16}$$

Note that $k^1 = -k^2$, then for real functions, one has $\alpha_1 = \alpha_2^*$.

First, by the range equation (5.4), $\hat{v} = O(\overline{v}^2)$ has the form of

$$\hat{v} = \alpha_1^2 \exp\{\mathrm{i}(2k^1 x)\}\Upsilon_{2k^1}^{-1} + \alpha_2^2 \exp\{\mathrm{i}(2k^2 x)\}\Upsilon_{2k^2}^{-1} + 2\alpha_1\alpha_2\Upsilon_0^{-1} + O(|\alpha_1| + |\alpha_2|)^3, \tag{5.17}$$

which is well defined since $\Upsilon_{2k^1}, \Upsilon_{2k^2}, \Upsilon_0 \neq 0$.

In order to obtain the coefficients α_1, α_2, we concentrate on the bifurcation equation (5.3). Combining with (5.16) and (5.17), one has

$$\begin{aligned}\epsilon\overline{v} &= \Pi_K(\overline{v}+\hat{v})^2\\ &= \Pi_K(2\overline{v}\hat{v}) + O(|\alpha_1|+|\alpha_2|)^4\\ &= (\frac{2\alpha_1^2\alpha_2}{\Upsilon_{2k^1}} + \frac{4\alpha_1^2\alpha_2}{\Upsilon_0})\exp\{ik^1x\} + (\frac{2\alpha_1\alpha_2^2}{\Upsilon_{2k^2}} + \frac{4\alpha_1\alpha_2^2}{\Upsilon_0})\exp\{ik^2x\} + O(|\alpha_1|+|\alpha_2|)^4\\ &= 2|\alpha_1|^2\alpha_1\left(\frac{1}{\Upsilon_{2k^1}}+\frac{2}{\Upsilon_0}\right)\exp\{ik^1x\} + 2|\alpha_1|^2\alpha_2\left(\frac{1}{\Upsilon_{2k^2}}+\frac{2}{\Upsilon_0}\right)\exp\{ik^2x\}\\ &\quad + O(|\alpha_1|+|\alpha_2|)^4.\end{aligned}$$

For nonzero α_1,

$$M = 2(\frac{1}{\Upsilon_{2k^1}} + \frac{2}{\Upsilon_0}) = \frac{5}{3m_0} > 0. \tag{5.18}$$

Using Lemma 19 and noting that Proposition 21 applies also to the case $d=1$, we obtain that the bifurcation equation (5.12) can be written as

$$\epsilon\alpha = \alpha P(|\alpha|^2)$$

with P an analytic function and $P(0)=0$.

In the previous analysis, we have computed $P'(0)=M$ and, in particular shown that $P'(0)\neq 0$ by (5.18). Hence, we can define a local inverse for P and the branches are given by $|\alpha|^2 = P^{-1}(\epsilon) = P'(0)^{-1}\epsilon + O(\epsilon^2)$. Note that, since $|\alpha|^2 \geq 0$, we only obtain solutions for ϵ with a fixed positive sign.

Notice that the fact that the bifurcation equations determine only $|\alpha|$ and not the phase. This is consistent with Remark 13.

5.2.3. *Some remarks about the case $d \geq 3$*

Unfortunately, when the dimension is bigger, the algebra becomes more complicated.

Notably the factorization of the bifurcation equation does not hold. Note

$$(a,b,c) = (a,b,-c) + (a,-b,c) + (-a,b,c)$$

so that the bifurcation equation for α_1 contains a term which does not have a factor α_1. There are many other examples.

Remark 24. In 3 or higher dimensions, the statement ($S2$) is not true and the only thing we can say, at the moment, about the bifurcation equations is that they have the form

$$\epsilon\alpha_n = P_n(\alpha),$$

where P_n is a homogeneous polynomial of degree 3 or higher order with real coefficients.

In dimension $d \geq 3$, the bifurcation equations include 2^d real variables (2^{d-1} complex variables). The symmetry in Remark 13 shows that solutions have to be related in d-dimensional families. When $d \geq 3$, one has $2^{d-1} > d$. So that the bifurcation equations are expected to give more branches. Also the branches are not just characterized by the absolute values since there are more variables than phases to adjust using Remark 13. Of course, it is possible that there are other symmetries beyond the ones pointed out in Remark 13.

6. Ill-posed evolution equations

6.1. A fixed point approach

A very similar approach in Section 4 can be applied to the problem of finding solutions for the nonlinear elliptic type evolution equations (1.2).

By the analysis in Section 2, the problem is equivalent to looking for solutions of the form $U : \mathbb{T}^b_\rho \times \mathbb{T}^d_\rho \to \mathbb{C}$ for (2.9):

$$\mathcal{Q}_{\omega,\nu,\mathfrak{m}} U = \epsilon \mathcal{N}(U), \tag{6.1}$$

where

$$\mathcal{N}(U)(\theta, x) := f(\theta, x, U(\theta, x), D_x U(\theta, x), D_x^2 U(\theta, x)). \tag{6.2}$$

Moreover, we assume the parameters $\omega, \nu, \mathfrak{m}$ meet the hypothesis (H1) or (H2) mentioned in Section 2.4.1.

As in the previous analysis, the key is to define an appropriate space.

For $\rho \geq 0$, $r, b, d \in \mathbb{Z}_+$, we define the following space of analytic functions U in $\mathbb{T}^{b+d}_\rho$ with finite norm:

$$\begin{aligned} H^{\rho,r}_* &:= H^{\rho,r}_*(\mathbb{T}^{b+d}_\rho) \\ &= \Bigg\{ U : \mathbb{T}^b_\rho \times \mathbb{T}^d_\rho \to \mathbb{C} \;\Bigg|\; U(\theta, x) = \sum_{(l,k)\in\mathbb{Z}^b\times\mathbb{Z}^d} \hat{U}_{l,k} e^{\mathrm{i}(l\theta + kx)}, \\ &\qquad \|U\|^2_{\rho,r} = \sum_{(l,k)\in\mathbb{Z}^b\times\mathbb{Z}^d} |\hat{U}_{l,k}|^2 e^{2\rho(|l|+|k|)} (1+|l|^2)^r (1+|k|^2)^r < \infty \Bigg\}. \end{aligned}$$

Remark 25. It is natural to think of $H^{\rho,r}_*$ as a space of functions from $\mathbb{T}^b_\rho$ into $H^{\rho,r}(\mathbb{T}^d_\rho)$. We think of $U(\omega t, \cdot)$ as a quasi-periodic function in the space $H^{\rho,r}$ of functions of $\mathbb{T}^d_\rho$. From this point of view, it would have been natural to include different parameters for the regularity in θ and the regularity in x, but we have decided not to include it to avoid creating more complexity.

Note that the norm is equivalent to the norm

$$\|U\|^2_{\rho,r} = \sum_{(l,k)\in\mathbb{Z}^b\times\mathbb{Z}^d} |\hat{U}_{l,k}|^2 e^{2\rho(|l|+|k|)} (1+|l|^2+|k|^2)^r < \infty.$$

Before giving the main result of the evolution equation, we need to introduce the following lemma about the measure estimates of the parameter sets, which will produce resonance, corresponding to the hypotheses (H1) and (H2) respectively.

Lemma 26. *Given fixed* $\mathfrak{m} > 0$ *and sufficiently small* $\delta > 0$*, we consider the following set of parameters* (ω, ν)*,*

$$\tilde{I} = \left\{(\omega, \nu) \in [1,2]^{1+d} \mid \exists\, l \in \mathbb{Z}^1,\ k \in \mathbb{Z}^d,\ \text{such that} \left| -\omega^2 l^2 - \sum_{i=1}^{d} k_i^2 \nu_i^2 + \mathfrak{m} \right| \le \delta \right\},$$

corresponding to the hypotheses (H2) *in the Section 2.4.1. Then, the set has the Lebesgue measure:*

$$\mathrm{mes}(\tilde{I}) = O(\delta).$$

In this case, we can regard the parameters set as the nonresonant elliptic case where the dimension of x increases by 1. Thus, the Lemma 26 can be obtained by adapting slightly the proof of Lemma 11, we omit it.

Using Proposition 7, we obtain the following estimates:

Proposition 27. *If the parameters* ω, ν *meet the hypotheses (H1) (or (H2)), then for all* $(\omega, \nu) \subset \mathbb{R}^b \times [1,2]^d$ *(or* $(\omega, \nu) \in [1,2]^{1+d} \setminus \tilde{I}$*), we have*

$$\|\mathcal{Q}_{\omega,\nu,\mathfrak{m}}{}^{-1}\|_{H_*^{\rho,r} \to H_*^{\rho,r}} \le \delta^{-1},$$
$$\|\mathcal{Q}_{\omega,\nu,\mathfrak{m}}{}^{-1}\|_{H_*^{\rho,r-2} \to H_*^{\rho,r}} < C,$$

where C *is a constant.*

Now, we give the following result for the evolution equations:

Theorem 28. *For fixed* $\mathfrak{m} > 0$ *and any* $0 < \delta \ll 1$*, given* $\rho \ge 0, r - 2 > d/2$*, let* $\mathbb{B}_s(0) \subset H_*^{\rho,r}$ *be closed ball around the origin with the radius* $s > 0$*. Suppose that the parameters* ω, ν *meet the hypotheses (H1) (or (H2)).*

Assume that $\mathcal{N}$ *defined in* (6.2) *is Lipschitz from* $\mathbb{B}_s(0) \subset H_*^{\rho,r}$ *into* $H_*^{\rho,r-2}$*.*

Then, there exists $\epsilon_* > 0$ *depending on* $\nu, \mathfrak{m}, \delta, s, Lip(\mathcal{N})$*, such that when* $0 < \epsilon < \epsilon_*$*, for any* $(\omega, \nu) \in \mathbb{R}^b \times [1,2]^d$ *(or* $(\omega, \nu) \in [1,2]^{1+d} \backslash \tilde{I}$*,* $\tilde{I} \subset [1,2]^{1+d}$ *with* $\mathrm{mes}(\tilde{I}) = O(\delta)$*), the equation* (6.1) *admits a unique solution* $u \in \mathbb{B}_s(0)$*.*

The proof of Theorem 28 is very similar to Theorem 8, we omit it here.

7. Time-dependent center manifold approach

We notice that the method in Section 6 can not solve all the cases when the parameters set leads to the center direction. Thus, we will introduce the center manifold theorem [20,13] which is a powerful tool to analyze the evolution equation.

These results construct a finite-dimensional quasi-periodic manifold (with boundary) inside a function space of solutions.

This quasi-periodic manifold in function space has the property that the PDE restricted to the manifold is equivalent to an ODE in the manifold. Therefore, the solutions of the ODE that do not reach the boundary the solutions of the PDE stay in the manifold for a short time. Hence, to analyze the behavior of the PDE, we can study the behavior of the finite-dimensional system given by the motion in this manifold. The solutions of the finite-dimensional system will correspond to solutions of the PDE.

Similar procedures (often called also *reduction principles*) have been used in PDE, including ill-posed PDE. Notably, in the case of elliptic PDE in cylindrical domains [28,32]. Once the existence of invariant manifolds is established, one can use standard methods of finite-dimensional dynamical systems to establish a variety of solutions [36,37]. The case of time-dependent manifolds, which is the most relevant for us was developed in [13]. The method of [13] gives information on the center manifold and the dynamics on it. Then, any finite result of finite-dimensional systems that gives computable conditions for the existence of an interesting solution, can be adapted to the PDE. The method presented here gives expressions for the dynamics in the manifold given the form of the PDE. Imposing that the dynamics in the manifold satisfies the conditions of the constructive theorems is ensured by explicit conditions on the PDE. We will not give explicit examples of this rather standard but long calculations. Some interesting examples appear in [23]. Note that the invariant quasi-periodic manifold will be only a finite-differentiable function in the space, even if the space itself consists of functions of analytic functions in space and, therefore, the solutions of the PDE are analytic in space and time. Even if each of the solutions are analytic, the finite differentiability refers to the way that these solutions are stacked together.

The strategy of [13], which we will implement in this section, consists in deriving a functional equation for the representation of a time-dependent locally invariant manifold as a graph, formulate an invariance equation and reduce it into a fixed point problem. It is quite remarkable that the method applies even when the equation is ill-posed. Many standard methods in invariant manifold theory such as the graph transform do not apply.

In this section, we will consider (1.2) with the frequency $\omega \in \mathbb{T}^b$, $b \in \mathbb{Z}_+$. We will allow that the forcing f depends on u, $D_x u$ and present a very explicit proof of Theorem 33.

As we will see in Remark 35 the methods of [13] allow to deal with forcing terms that depend on higher (fractional) derivatives but they cannot deal with nonlinearities depending on $D_x^2 u$. Since, for possible applications it will be important to obtain explicit formulas, we have decided to present full details in case simpler than another one with optimal regularity.

More precisely, we consider

$$u_{tt} + \sum_{i=1}^{d} \nu_i^2 \frac{\partial^2}{\partial x_i^2} u + \mathfrak{m} u = \epsilon f(\omega t, x, u, D_x u), \ \ x \in \mathbb{T}^d, \ \ t \in \mathbb{R}. \tag{7.1}$$

Setting $u_t = v, \ z = (u, v)^\top$, (7.1) can be rewritten as the system

$$\begin{cases} \dot{\theta} = \omega \\ \dot{z} = \mathcal{A} z + \epsilon \mathcal{N}(\theta, z) \end{cases}, \tag{7.2}$$

where

$$\mathcal{A}=\begin{pmatrix} 0 & 1 \\ -\sum_{i=1}^{d}\nu_i^2\frac{\partial^2}{\partial x_i^2}-\mathfrak{m} & 0 \end{pmatrix}, \qquad \mathcal{N}=\begin{pmatrix} 0 \\ f(\omega t, x, u, D_x u) \end{pmatrix}. \tag{7.3}$$

7.1. Choice of spaces

To construct the center manifold, we first need to choose the suitable Banach spaces which admit cut-off functions and such that the nonlinear operator is differentiable in them. (The paper [13] uses the two space approach of [22] and obtains results with weaker regularity. See Remark 35.)

For (7.2), we consider the analytic function u in $H^{\rho,r}$ which is a Hilbert space and admits a cut-off function. Then

$$z=(u,v)^\top \in X := H^{\rho,r}\times H^{\rho,r-1}.$$

We will assume that f is analytic. By Banach algebra and composition properties, we have that for $r > d/2+1$,

$$f(\omega t, x, u, D_x u) : \mathbb{T}^b \times \mathbb{T}^d \times H^{\rho,r}\times H^{\rho,r-1} \to H^{\rho,r-1}.$$

Thus, we have that the nonlinearity $\mathcal{N}$ is bounded from X to X.

7.2. Analysis of the linear term

To identify the basis of the stable and unstable spaces, we will analyze the linear operator $\mathcal{A}$.

Let $\Lambda := \{+1,-1\}$, one can check that $\Phi_k^\Lambda(x) = (e^{\mathrm{i}kx}, \lambda_k^\Lambda e^{\mathrm{i}kx})^\top$ is the eigenvector of $\mathcal{A}$ belonging to the eigenvalue

$$\mathrm{Spec}(\mathcal{A}) = \{\lambda_k^\Lambda\}_{k\in\mathbb{Z}^d} = \left\{\Lambda\Big(\sum_{i=1}^{d}k_i^2\nu_i^2-\mathfrak{m}\Big)^{1/2}\right\}_{k\in\mathbb{Z}^d}.$$

Thus, we take $\{\Phi_k^\Lambda(x)\}_{k\in\mathbb{Z}^d}$ as a basis of the space X.

Remark 29. The operator $\mathcal{A}$ has discrete spectrum in X. Furthermore, we have:

1) The center spectrum of $\mathcal{A}$ consists of a finite number of eigenvalues, since there is only a finite number of k satisfying:

$$\sum_{i=1}^{d}k_i^2\nu_i^2-\mathfrak{m}\le 0,\ \ k\in\mathbb{Z}^d,$$

for fixed $\mathfrak{m}>0$.

2) The hyperbolic spectrum is well separated from the center spectrum.

Thus, we know that the spectrum of the linear operator $\mathcal{A}$ satisfies the following Proposition.

Proposition 30. *For fixed* $\mathfrak{m} > 0$*, there exist* $\beta_1 > \beta_3^- \geq 0$*,* $\beta_2 > \beta_3^+ \geq 0$*, and a splitting of spectrum of linear operator* $\mathcal{A}$*, i.e.,*

$$\mathrm{Spec}(\mathcal{A}) = \sigma_s \cup \sigma_c \cup \sigma_u,$$

where

$$\begin{aligned}\sigma_s &= \{\lambda_k^\alpha \mid Re\lambda_k^\alpha < -\beta_1,\ k \in \mathbb{Z}^d\},\\ \sigma_u &= \{\lambda_k^\alpha \mid Re\lambda_k^\alpha > \beta_2,\ k \in \mathbb{Z}^d\},\\ \sigma_c &= \{\lambda_k^\alpha \mid \beta_3^- \leq Re\lambda_k^\alpha < \beta_3^+,\ k \in \mathbb{Z}^d\}.\end{aligned} \tag{7.4}$$

We note that σ_c *contains not only the center eigenvalues but also the eigenvalues with slow stability/unstability.*

As a conclusion, there is a decomposition

$$X = X_s \oplus X_c \oplus X_u, \tag{7.5}$$

where $X_{\widetilde{\sigma}}$ $(\widetilde{\sigma} = s, c, u)$ *are invariant for* $\mathcal{A}$*, i.e.,* $\mathcal{A}(D(\mathcal{A}) \cap X_{\widetilde{\sigma}}) \subset X_{\widetilde{\sigma}}$*. We denote by* $\Pi_{\widetilde{\sigma}}$ *the projection operator over* $X_{\widetilde{\sigma}}$*, which is bounded in* X*.*

Proof. We notice that the eigenvalues of $\mathcal{A}$ are discrete and $\lambda_k \to \infty$ when $k \to \infty$. Therefore, we can choose appropriate $\beta_1, \beta_2, \beta_3^+, \beta_3^-$, which can split the spectrum of $\mathcal{A}$ into $\sigma_s, \sigma_c, \sigma_u$, such that $\sigma_s, \sigma_c, \sigma_u$ are disjoint and cover all the eigenvalues. The existence of the decomposition is the point of the spectral theorem. □

In the dynamical systems theory, the conclusion of Proposition 30 is described as $\mathcal{A}$ has a trichotomy for the generator of the evolution. We note, however that the operator $\mathcal{A}$ does not generate an evolution. As we detail in Lemma 31, it generates semigroups in the future or in the past in subspaces.

Lemma 31. *Denote by* $\mathcal{A}_s, \mathcal{A}_u, \mathcal{A}_c$ *the restrictions of* $\mathcal{A}$ *to* X_s, X_u, X_c *respectively.*

Then, we can define the following (semi)groups

$$\begin{aligned}&\{\mathcal{A}_s(t) \equiv e^{t\mathcal{A}_s}\}_{t\geq 0}\\ &\{\mathcal{A}_u(t) \equiv e^{t\mathcal{A}_u}\}_{t\leq 0}\\ &\{\mathcal{A}_c(t) \equiv e^{t\mathcal{A}_c}\}_{t\in\mathbb{R}}\end{aligned}$$

defined in the spaces X_s, X_u, X_c *respectively.*

Moreover, the following estimates hold:

$$\begin{aligned}&\|\mathcal{A}_s(t)\|_{X_s,X} \leq e^{-\beta_1 t},\ t > 0;\\ &\|\mathcal{A}_u(t)\|_{X_u,X} \leq e^{-\beta_2|t|},\ t < 0;\\ &\|\mathcal{A}_c(t)\|_{X_c,X} \leq e^{\beta_3^-|t|},\ t \leq 0;\\ &\|\mathcal{A}_c(t)\|_{X_c,X} \leq e^{\beta_3^+|t|},\ t \geq 0.\end{aligned} \tag{7.6}$$

Proof. Suppose $z \in X$ has the following Fourier expansion

$$z = \sum_{k \in \mathbb{Z}^d, \Lambda \in \{-1,1\}} \hat{z}_k^\Lambda \Phi_k^\Lambda(x)$$

with norm in X (which is equivalent to the norm in $H^{\rho,r} \times H^{\rho,r-1}$)

$$\|z\|_X^2 = \sum_{k \in \mathbb{Z}^d, \Lambda \in \{-1,1\}} |\hat{z}_k^\Lambda|^2 e^{2\rho|k|}(1 + |k|^2)^r.$$

Then, for $z \in X_s$,

$$\mathcal{A}_s(t)z = \sum_{k \in \mathbb{Z}^d, \Lambda \in \{-1,1\}} e^{\lambda_k^\Lambda t} \hat{z}_k^\Lambda \Phi_k^\Lambda(x), \quad t > 0,$$

we have

$$\begin{aligned}
\|\mathcal{A}_s(t)z\|_X^2 &= \sum_{\substack{k \in \mathbb{Z}^d, \Lambda \in \{-1,1\} \\ \lambda_k^\Lambda \in \sigma_s}} |e^{\lambda_k^\Lambda t} \hat{z}_k^\Lambda|^2 e^{2\rho|k|}(1 + |k|^2)^r \\
&\le e^{-2\beta_1 t} \sum_{\substack{k \in \mathbb{Z}^d, \Lambda \in \{-1,1\} \\ \lambda_k^\Lambda \in \sigma_s}} |\hat{z}_k^\Lambda|^2 e^{2\rho|k|}(1 + |k|^2)^r \\
&= e^{-2\beta_1 t} \|z\|_X^2.
\end{aligned}$$

Therefore,

$$\|\mathcal{A}_s(t)\|_{X_s, X} \le e^{-\beta_1 t}, \quad t > 0.$$

Similarly, the remaining three inequalities hold. □

Remark 32. Since the center space X_c is finite-dimensional, X_c admits C^r cut-off function.

Since we only construct the evolutions of the equations with sufficiently small perturbations, we can not consider the equation (7.2) directly. We have to introduce the “prepared equation” ([29]).

For any $z \in X$, $\Pi_\sigma z = z_\sigma$. We consider the following prepared equation of (7.2):

$$\begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{dz_s}{dt} = \mathcal{A}_s z_s + \epsilon \mathcal{N}_s(\theta, x, z_s, z_c, z_u) \\ \dfrac{dz_c}{dt} = \mathcal{A}_c z_c + \epsilon \varphi(z_c) \mathcal{N}_c(\theta, x, z_s, z_c, z_u) \\ \dfrac{dz_u}{dt} = \mathcal{A}_u z_u + \epsilon \mathcal{N}_u(\theta, x, z_s, z_c, z_u) \end{cases}, \tag{7.7}$$

where $\varphi : X \to \mathbb{R}$ is a C^r cut-off function such that it is identically 1 in the ball of radius 1/2 centered at the origin and identically 0 outside the ball of radius 1, where $C^r(X, Y)$ is a set of functions from X to Y that have continuous derivatives of order less than or equal to r. Then, $\varphi(z_c)\mathcal{N}_c(\theta, x, z_s, z_c, z_u)$ is a uniformly C^r function, we can also arrange that the C^r norm of the ϵ is as small as we needed. We obtain the flow on X_c and denote it by $J_t^w(z_c(0))$. Denote by $\widetilde{\mathcal{N}}$ the nonlinearity

$$\widetilde{\mathcal{N}} = \epsilon \Big(\mathcal{N}_s(\theta, x, z_s, z_c, z_u),\ \varphi(z_c)\mathcal{N}_c(\theta, x, z_s, z_c, z_u),\ \mathcal{N}_u(\theta, x, z_s, z_c, z_u) \Big).$$

Our goal is to find a function $w : \mathbb{T}^{b+d} \times X_c \to X_s \oplus X_u$, and verify the graph of w which is denoted by

$$\mathcal{W} = \big\{ (\Theta, w_s(\Theta, x, J_t^w(z_c(0))), J_t^w(z_c(0)), w_u(\Theta, x, J_t^w(z_c(0)))) \big\}$$

is invariant under (7.7).

In conclusion, we have checked that the system (7.7) satisfies the following hypothesis:

- **H1)** The decomposition (7.5) of the space X is invariant under $\mathcal{A}$ and $\Pi_{\widetilde{\sigma}}(\widetilde{\sigma} = s, c, u)$ is bounded in X.
- **H2)** The operator $\mathcal{A}$ generates semi-groups $\mathcal{A}_{s,c,u}(t)$, with the quantitative assumption (7.6) on the contraction rates.
- **H3)** The nonlinearity $\widetilde{\mathcal{N}} : \mathbb{T}^{b+d} \times X \to \mathbb{T}^{b+d} \times X$ is C^r and $\|\widetilde{\mathcal{N}}\|_{C^r}$ is sufficiently small.

We will give a detailed and explicit proof for the following result Theorem 33, which is a particular case of Theorem 3.1 of [13].

To obtain Theorem 33 from Theorem 3.1 of [13], it suffices to take the two spaces X, Y used in [13] to be equal to X. We will present a detailed proof of Theorem 33. In Remark 35 we will discuss the results that are obtained using the full force of Theorem 3.1 of [13].

Theorem 33. *Assume in the space X, the linear operator $\mathcal{A}$ and the nonlinearity $\widetilde{\mathcal{N}}$ satisfy the assumptions* **H1**), **H2**), **H3**) *respectively.*

Then, there exists a $C^{r-1+Lip}$ function $w : \mathbb{T}^{b+d} \times X_c \to X_s \oplus X_u$, and $\mathcal{W}$, the graph of w, is globally invariant by (7.7). Furthermore, $\mathcal{W}$ is $C^{r-1+Lip}$ locally invariant by (7.2).

Remark 34. Note that even if f is C^ω, the cut-off function is only C^r. The center manifold obtained will be invariant for the cut-off equations but only locally invariant for the original equation.

We are going to consider the equation as an evolution in spaces of analytic functions, so that all the solutions of the PDE we consider, will be analytic in the space variable. As a consequence, they will be also analytic in time. Nevertheless, in spite of the fact that the solutions are analytic, the center manifold we construct will be only C^r for a finite r. Note however that it is a C^r manifold in a space of analytic functions (of the space variable).

Even if for every r, we can find a C^r manifold, it may be impossible to find a C^∞ manifold. This is because to increase the r we may need to have a stronger cut-off in the preparation so that we cannot take the limit. There are well known examples of this phenomenon even in finite-dimensional, polynomial ODE's [29].

7.3. The proof of Theorem 33

The proof of Theorem 33 is based on the contraction principle. We consider the center direction in (7.7) and have the following evolution equation for z_c:

$$\begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{dz_c}{dt} = \mathcal{A}_c z_c + \widetilde{\mathcal{N}}_c(\theta, x, z_s, z_c, z_u) \end{cases}$$

with initial value $(\theta_0, z_c(0))$. By the Duhamel principle, we obtain the solution of this equation:

$$\Theta = \theta_0 + \omega t,$$

$$J_t^w(z_c(0)) = e^{\mathcal{A}_c t} z_c(0) + \int_0^t e^{\mathcal{A}_c(t-\tau)} \widetilde{\mathcal{N}}_c(\Theta(\tau), x, J_\tau^w(z_c(0)), w(\Theta(\tau), J_\tau^w(z_c(0))))d\tau. \quad (7.8)$$

Moreover, one has

$$z_s(t) = e^{\mathcal{A}_s t} z_s(0) + \int_0^t e^{\mathcal{A}_s(t-\tau)} \widetilde{\mathcal{N}}_s(\Theta(\tau), x, J_\tau^w(z_c(0)), w(\Theta(\tau), J_\tau^w(z_c(0))))d\tau, \ t \geq 0,$$

$$z_u(t) = e^{\mathcal{A}_u t} z_u(0) + \int_0^t e^{\mathcal{A}_u(t-\tau)} \widetilde{\mathcal{N}}_u(\Theta(\tau), x, J_\tau^w(z_c(0)), w(\Theta(\tau), J_\tau^w(z_c(0))))d\tau, \ t \leq 0.$$

To verify the graph of w is invariant, we denote by

$$(z_s, z_u) = (w_s(\Theta, x, J_t^w(z_c(0))), w_u(\Theta, x, J_t^w(z_c(0)))).$$

From Duhamel principle and $(\Theta(t), J_t^w(z_c(0)))$ is invertible, do some variable transformation on t, we have, when $t \to \mp\infty$,

$$w_s(\theta_0, z_c(0)) = \int_{-\infty}^{0} e^{-\tau \mathcal{A}_s} \widetilde{\mathcal{N}}_s(\Theta(\tau), x, J_\tau^w(z_c(0)), w(\Theta(\tau), J_\tau^w(z_c(0))))d\tau, \tag{7.9}$$

$$w_u(\theta_0, z_c(0)) = -\int_{0}^{\infty} e^{-\tau \mathcal{A}_u} \widetilde{\mathcal{N}}_u(\Theta(\tau), x, J_\tau^w(z_c(0)), w(\Theta(\tau), J_\tau^w(z_c(0))))d\tau. \tag{7.10}$$

We denote by $\mathcal{T}_c, \mathcal{T}_s, \mathcal{T}_u$ the right hand side (RHS) of equations (7.8), (7.9), (7.10) respectively. Then we obtain a fixed point equation

$$(J_t^w(z_c(0)), w_s(\theta_0, z_c(0)), w_u(\theta_0, z_c(0))) \equiv \mathcal{T} = (\mathcal{T}_c, \mathcal{T}_s, \mathcal{T}_u).$$

The rest of the work is to construct solutions of (7.9) and (7.10).

There are fixed points of the operators that to $w_s(\theta_0, z_c(0))$ and $w_u(\theta_0, z_c(0))$ associate the RHS of (7.9) and (7.10), respectively. The proof of the existence of the fixed point is done in great detail in [20,13]. Both of them are based on a method from [29]. The basic idea is to show that there is a C^{r+1} ball that gets mapped onto itself by the operator (this is obtained using the estimates on composition of C^{r+1} functions, the estimates on derivatives of solutions of an ODE and the different rates). The second step is to prove that this operator is a contraction in a C^0 norm (this is done by applying systematically adding and subtracting so that only one term is modified at the time. The most difficult step is estimating the change of the solutions of the ODE when the coefficients are changed). By studying the properties of the solution, it is also shown that the solutions of the fixed point problem are a solution of the original problem.

Similar equations appear in the study of center manifolds. Note that the nonlinear perturbations we have considered are differentiable and that the linear parts generate reasonable evolutions. So that there is not much difference between the finite-dimensional proofs and the proof needed. For a treatment of a similar problem, we refer to [32]. The paper [20] deals with a more general situation.

Remark 35. The proof of [13] can deal with forcing nonlinearities that are more singular than first derivatives. The Theorem 3.1 of [13] applies to problems

$$u_{tt} = u_{xx} + \mathcal{F}(\omega t, u),$$

where $\mathcal{F}$ is a differentiable functional from the spaces indicated by

$$\mathcal{F} : \mathbb{T}^d \times H^{\rho,m} \to H^{\rho,m-2+\delta}, \quad \delta > 0.$$

The very interesting case $\delta = 0$ is not covered by the results.

The method of [13] goes through equations (7.9), (7.10) and it also uses the strategy of proving propagated bounds and C^0 contraction. The analysis, however, is more careful and takes advantage – following [22] of the fact that the operator $\mathcal{A}_s(t)$, $\mathcal{A}_u(t)$ are smoothing. They are bounded operators from $H^{\rho,m-2+\delta} \times H^{\rho,m-3+\delta}$ to $H^{\rho,m} \times H^{\rho,m-1}$ and the bounds are integrable.

Appendix A. Some properties of $H^{\rho,r}(\mathbb{T}^d_\rho)$

In this section, we collect a few lemmas about the properties on $H^{\rho,r}$, which play a crucial role in the proof. Similar contents have appeared in other papers. We note that Lemma 37 assumes only $r > d/2$ whereas in previous papers it was assumed $r > d$. This leads to similar improvements in the previous papers [11,10,41].

A small observation that can be found in the previous papers is that the $H^{\rho,r}$ norm is equivalent to the L^2 norm of derivatives up to order r. The derivatives can be taken to be either real derivatives or complex derivatives. That is,

$$\begin{aligned}\|u\|_{\rho,r} &\approx \|u\|_{L^2(\mathbb{T}^d_\rho)} + \|Du\|_{L^2(\mathbb{T}^d_\rho)} + \cdots \|D^r u\|_{L^2(\mathbb{T}^d_\rho)}\\ &\approx \|(1-\Delta)^{r/2}u\|_{L^2(\mathbb{T}^d_\rho)}\\ &\approx \|(1-\bar{\partial}_z\cdot\partial_z)^{r/2}u\|_{L^2(\mathbb{T}^d_\rho)}.\end{aligned}$$

The following result is elementary but crucial:

Proposition 36. *Assume that $r > d/2$, then*

$$\sup_{z\in\mathbb{T}^d_\rho} |u(z)| \le C_{r,d}\|u\|_{\rho,r}$$

for some constant $C_{r,d}$ depending on d, r.

Proof. Using triangle and Cauchy-Schwartz inequalities, we have

$$\begin{aligned}\sup_{z\in\mathbb{T}^d_\rho} |u(z)| &\le \sum_k |\hat{u}_k|e^{\rho|k|} = \sum_k |u_k|e^{\rho|k|}(1+|k|^2)^{r/2}(1+|k|^2)^{-r/2}\\ &\le \left(\sum_k |u_k|^2 e^{2\rho|k|}\cdot(1+|k|^2)^r\right)^{1/2}\left(\sum_k (1+|k|^2)^{-r}\right)^{1/2}\\ &= \|u\|_{\rho,r}C_{r,d}. \quad \square\end{aligned}$$

Notice that this inequality is better than the Sobolev inequality if we considered $\mathbb{T}^d_\rho$ as a $2d$-dimensional real manifold and $H^{\rho,r}$ as a closed space of the (real) Sobolev space $H^r(\mathbb{T}^d_\rho)$. Applying the real Sobolev embedding – as was done in [11] — requires $r > d$.

The reason is that, even if $\mathbb{T}^d_\rho$ is a $2d$-dimensional real manifold, due to the maximum principle for analytic functions, the sizes of the functions in $H^{\rho,r}$ are controlled by the H^r norm to the restriction to of the functions to the d-dimensional manifolds given by $\mathrm{Im}(z_i) = \pm\rho$. There are 2^d components each of which is a real d-dimensional torus.

As we will see immediately, similar results appear in the Banach algebra properties.

Note that we also get improved Sobolev embedding theorems. If $r = d/2 + \lambda$, we obtain

$$\|u\|_{C^\lambda(\mathbb{T}^d_\rho)} \le C\|u\|_{\rho,r}. \tag{A.1}$$

Of course, in (A.1), the regularity in the interior is not an issue (the functions are analytic) but we obtain quantitative bounds.

Lemma 37. *Banach algebra properties:*

(1) *Sobolev case: Let $\rho = 0, r > d/2$. Then there exists a positive constant $C_{r,d}$ depending on r, d, so that for any u_1, $u_2 \in H^r(\mathbb{T}^d, \mathbb{R})$, the product $u_1 \cdot u_2$ is in $H^r(\mathbb{T}^d, \mathbb{R})$, and*

$$\|u_1 \cdot u_2\|_r \le C_{r,d}\|u_1\|_r\|u_2\|_r.$$

(2) *Analytic case: Let $\rho > 0, r > d/2$. Then there exists a positive constant $C_{\rho,r,d}$ depending on ρ, r, d, so that for any $u_1, u_2 \in H^{\rho,r}(\mathbb{T}^d_\rho, \mathbb{C})$, the product $u_1 \cdot u_2$ is in $H^{\rho,r}(\mathbb{T}^d_\rho, \mathbb{C})$, and*

$$\|u_1 \cdot u_2\|_{\rho,r} \le C_{\rho,r,d}\|u_1\|_{\rho,r}\|u_2\|_{\rho,r}.$$

Proof. Part (1) is the classic result, see [40,5]. We prove the part (2).

Denote the set $S := \{\varsigma \mid \varsigma = \{1, -1\}^d\}$, where ς is a d-dimensional vector, and the i-th component is 1 or -1, $i = 1, \cdots, d$. We choose each component of $-\varsigma^*$ has the same sign with k, then

$$e^{-2k\varsigma\rho} \le e^{2|k|\rho} = e^{-2k\varsigma^*\rho} \le \sum_{\varsigma\in S} e^{-2k\varsigma\rho}, \quad for\ k \in \mathbb{Z}^d.$$

For any Fourier coefficient $\hat{u}_k$ of $u(x)$, we have

$$|\hat{u}_k|^2 e^{-2k\varsigma\rho} \le |\hat{u}_k|^2 e^{2|k|\rho} \le \sum_{\varsigma\in S} |\hat{u}_k|^2 e^{-2k\varsigma\rho}.$$

Multiplying the three sides of the above inequality by $(1 + |k|^2)^r$ and summing in k, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |\hat{u}_k|^2 e^{-2k\varsigma\rho}(1 + |k|^2)^r &\le \sum_{k\in\mathbb{Z}^d} |\hat{u}_k|^2 e^{2|k|\rho}(1 + |k|^2)^r \\ &\le \sum_{\varsigma\in S}\sum_{k\in\mathbb{Z}^d} |\hat{u}_k|^2 e^{-2k\varsigma\rho}(1 + |k|^2)^r.\end{aligned}$$

If we define the norm $\|\cdot\|_{\varsigma,\rho,r}$ as:

$$\|u\|^2_{\varsigma,\rho,r} := \sum_{k\in\mathbb{Z}^d} |\hat{u}_k|^2 e^{-2k\varsigma\rho}(1 + |k|^2)^r,$$

we obtain that the norm $\|u\|_{\rho,r}$ is equivalent to $(\sum_{\varsigma\in S}\|u\|^2_{\varsigma,\rho,r})^{1/2}$.

First, we verify that $\|\cdot\|_{\varsigma,\rho,r}$ is a Banach algebra in the d-dimensional manifold: $\{\theta \mid \mathrm{Re}(\theta) \in \mathbb{T}^d,\ \mathrm{Im}(\theta) = \varsigma\rho\}$. Denote $a = \mathrm{Re}(\theta)$, and the function $\Gamma_\varsigma u : \mathbb{T}^d \to \mathbb{C}$,

$$(\Gamma_\varsigma u)(a) = u(a + \mathrm{i}\varsigma\rho).$$

One has $\Gamma_\varsigma(u_1u_2)(a) = [\Gamma_\varsigma(u_1)\Gamma_\varsigma(u_2)](a)$. Thus, when $r > d/2$, we have

$$\|u_1u_2\|_{\varsigma,\rho,r} = \|\Gamma_\varsigma(u_1u_2)\|_r \le \|\Gamma_\varsigma(u_1)\|_r\|\Gamma_\varsigma(u_2)\|_r = \|u_1\|_{\varsigma,\rho,r}\|u_2\|_{\varsigma,\rho,r}.$$

Then,

$$\begin{aligned}\Big(\sum_{\varsigma\in S}\|u_1u_2\|^2_{\varsigma,\rho,r}\Big)^{\frac12} &\le \Big(\sum_{\varsigma\in S}(\|u_1\|^2_{\varsigma,\rho,r}\cdot\|u_2\|^2_{\varsigma,\rho,r})\Big)^{\frac12}\\ &\le \Big(\sum_{\varsigma\in S}\|u_1\|^2_{\varsigma,\rho,r}\cdot\sum_{\varsigma\in S}\|u_2\|^2_{\varsigma,\rho,r}\Big)^{\frac12}\\ &\le \Big(\sum_{\varsigma\in S}\|u_1\|^2_{\varsigma,\rho,r}\Big)^{\frac12}\cdot\Big(\sum_{\varsigma\in S}\|u_2\|^2_{\varsigma,\rho,r}\Big)^{\frac12}.\end{aligned}$$

That is to say, when $r > d/2$, $(\sum_{\varsigma\in S}\|u\|_{\varsigma,\rho,r})^{1/2}$ is a Banach algebra. By the equivalence of norms, we have completed the proof. □

Note that having Proposition 36, we could have followed also the standard proof using the Leibnitz formula.

For the purposes of this paper, the main issue is the study of the operator given by composition on the left.

Many other composition properties can be found in [10] the Proposition 3.9 in [40] for details. For more results, one can also refer to [7,26,30,38].

Lemma 38. *Composition properties:*

(1) *Sobolev case: Let $f \in C^r(\mathbb{R}^n,\mathbb{R}^n)$ and assume that $f(0) = 0$. Then, for $u \in H^r(\mathbb{T}^d,\mathbb{R}^n)\bigcap L^\infty(\mathbb{T}^d,\mathbb{R}^n)$, we have*

$$\|f(u)\|_r \le C_r(\|u\|_{L^\infty})(1+\|u\|_r),$$

where $C_r := C_r(\eta) = \sup_{|x|\le\eta,\,\alpha\le r}|D^\alpha f(x)|$. Particularly, when $r > d/2$, if $f \in C^{r+2}$ and $u, v, u+v \in H^r$, then

$$\|f\circ(u+v) - f\circ u - Df\circ u\cdot v\|_r \le C_{r,d}(\|u\|_{L^\infty})(1+\|u\|_r)\|f\|_{C^{r+2}}\|v\|_r^2, \tag{A.2}$$

for some $C_{r,d} > 0$ depending on the norm of u.

(2) *Analytic case: Let $f : B \to \mathbb{C}^n$ with B being an open ball around the origin in $\mathbb{C}^n$ and assume that f is analytic in B. Then, for $u \in H^{\rho,r}(\mathbb{T}^d_\rho,\mathbb{C}^n)\bigcap L^\infty(\mathbb{T}^d_\rho,\mathbb{C}^n)$ with $u(\mathbb{T}^d_\rho) \subset B$, we have*

$$\|f(u)\|_{\rho,r} \le C_{\rho,r}(\|u\|_{L^\infty})(1+\|u\|_{\rho,r}). \tag{A.3}$$

In the case of $r > d/2$, we have

$$\|f \circ (u+v) - f \circ u - Df \circ u \cdot v\|_{\rho,r} \leq C_{\rho,r,d}(\|u\|_{L^\infty})(1+\|u\|_{\rho,r})\|v\|^2_{\rho,r}. \tag{A.4}$$

As a corollary of (A.4)*, we obtain that, under the hypotheses of the Lemma, the operator* $u \to f \circ u$ *is differentiable.*
Since the Hilbert space $H^{\rho,r}$ *is a complex space, and the differentiability is in the complex sense, we conclude that the operator* $u \to f(u)$ *is analytic.*

Proof. The finite-differentiable case of Lemma 38 is a well known consequence of Gagliardo-Nirenberg-Moser composition estimates, for specific proof see Proposition 3.9 in [40].

Here we give the proof of (A.3) and (A.2). We notice that if u is bounded (in particular if $r > d/2$ by Lemma 36) and that the range of u is in the domain of f, we have that, by the chain rule, $f \circ u$ is complex differentiable in $\mathbb{T}^d_\rho$ and that $f \circ u$, $(Df) \circ u$ are bounded.

Since $D(f \circ u) = Df \circ u Du$, we obtain, computing $\int_{T^d_\rho} |Df \circ u Du|^2$, that $f \circ u \in H^{\rho,1}$ if $u \in H^{\rho,1}$. To get the result for arbitrary r, we can use the Faa-Di-Bruno formula for higher derivatives and use the bounds we already have from the previous stages.

Let $u, v \in H^r$, $\xi = u + \zeta v$ for some $\zeta \in [0,1]$, ξ is in the domain of f. By the fundamental theorem of calculus, we have

$$\begin{aligned} f(u+v) &= f(u) + \int_0^1 Df(\xi) \cdot v d\zeta \\ &= f(u) + Df(u) \cdot v + \int_0^1 \int_0^\zeta D^2 f(u+\zeta t v) v^2 dt d\zeta. \end{aligned}$$

The fact that the differentiable functions of a complex Banach space are analytic is proved in [24, Chapter III]. □

References

[1] Antonio Ambrosetti, David Arcoya, An Introduction to Nonlinear Functional Analysis and Elliptic Problems, Progress in Nonlinear Differential Equations and Their Applications, vol. 82, Birkhäuser Boston, Ltd., Boston, MA, 2011.
[2] V.I. Arnold, V.S. Afrajmovich, Yu.S. Il'yashenko, L.P. Shil'nikov, Bifurcation Theory and Catastrophe Theory, Springer-Verlag, Berlin, 1999, Translated from the 1986 Russian original by N.D. Kazarinoff, Reprint of the 1994 English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical Systems. V, Encyclopaedia Math. Sci., 5, Springer, Berlin, 1994; MR1287421 (95c:58058)].
[3] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Commun. Pure Appl. Math. 12 (1959) 623–727.
[4] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, Commun. Pure Appl. Math. 17 (1964) 35–92.
[5] Robert A. Adams, John J.F. Fournier, Sobolev Spaces, second edition, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
[6] Shmuel Agmon, Lectures on Elliptic Boundary Value Problems, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr., Van Nostrand Mathematical Studies, vol. 2, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.
[7] Jürgen Appell, Petr P. Zabrejko, Nonlinear Superposition Operators, Cambridge Tracts in Mathematics, vol. 95, Cambridge University Press, Cambridge, 1990.

[8] Jean Bourgain, Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE, Int. Math. Res. Not. 475 (11) (1994), 21 pp.
[9] J. Bourgain, Green's Function Estimates for Lattice Schrödinger Operators and Applications, Annals of Mathematics Studies, vol. 158, Princeton University Press, Princeton, NJ, 2005.
[10] Renato C. Calleja, Alessandra Celletti, Livia Corsi, Rafael de la Llave, Response solutions for quasi-periodically forced, dissipative wave equations, SIAM J. Math. Anal. 49 (4) (2017) 3161–3207.
[11] Renato C. Calleja, Alessandra Celletti, Rafael de la Llave, Construction of response functions in forced strongly dissipative systems, Discrete Contin. Dyn. Syst. 33 (10) (2013) 4411–4433.
[12] Hongyu Cheng, Rafael de la Llave, Stable manifolds to bounded solutions in possibly ill-posed PDEs, J. Differ. Equ. 268 (8) (2020) 4830–4899.
[13] Hongyu Cheng, Rafael de la Llave, Time dependent center manifold in PDEs, Discrete Contin. Dyn. Syst. 40 (12) (2020) 6709–6745.
[14] Lamberto Cesari, Alternative methods in nonlinear analysis, in: International Conference on Differential Equations, Proc., Univ. Southern California, Los Angeles, Calif., 1974, 1975, pp. 95–148.
[15] Shui Nee Chow, Jack K. Hale, Methods of Bifurcation Theory, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science), vol. 251, Springer-Verlag, New York-Berlin, 1982.
[16] Michael G. Crandall, Paul H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.
[17] Craig Walter, Problèmes de petits diviseurs dans les équations aux dérivées partielles, Panoramas et Synthèses (Panoramas and Syntheses), vol. 9, Société Mathématique de France, Paris, 2000.
[18] Craig Walter, C. Eugene Wayne, Newton's method and periodic solutions of nonlinear wave equations, Commun. Pure Appl. Math. 46 (11) (1993) 1409–1498.
[19] Craig Walter, C. Eugene Wayne, Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method, in: Hamiltonian Mechanics, Toruń, 1993, in: NATO Adv. Sci. Inst. Ser. B Phys., vol. 331, Plenum, New York, 1994, pp. 103–122.
[20] Rafael de la Llave, A smooth center manifold theorem which applies to some ill-posed partial differential equations with unbounded nonlinearities, J. Dyn. Differ. Equ. 21 (3) (2009) 371–415.
[21] Piero D'Ancona, Sergio Spagnolo, Nonlinear perturbations of the Kirchhoff equation, Commun. Pure Appl. Math. 47 (7) (1994) 1005–1029.
[22] Henry Daniel, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin-New York, 1981.
[23] Mariana Haragus, Gérard Iooss, Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems, Universitext, Springer-Verlag London, Ltd., London, 2011; EDP Sciences, Les Ulis.
[24] Einar Hille, Ralph S. Phillips, Functional Analysis and Semigroups, revised ed., vol. 31, American Mathematical Society (AMS), Providence, RI, 1957.
[25] Gérard Iooss, Daniel D. Joseph, Elementary Stability and Bifurcation Theory, second edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1990.
[26] H. Inci, T. Kappeler, P. Topalov, On the regularity of the composition of diffeomorphisms, Mem. Am. Math. Soc. 226 (1062) (2013) vi+60.
[27] Hansjörg Kielhöfer, Bifurcation theory, in: An Introduction with Applications to Partial Differential Equations, second edition, in: Applied Mathematical Sciences, vol. 156, Springer, New York, 2012.
[28] Klaus Kirchgässner, Jürgen Scheurle, On the bounded solutions of a semilinear elliptic equation in a strip, J. Differ. Equ. 32 (1) (1979) 119–148.
[29] Oscar E. Lanford, Bifurcation of periodic solutions into invariant tori: the work of Ruelle and Takens, in: In Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1973, pp. 159–192.
[30] Jerry Marsden, Applications of Global Analysis in Mathematical Physics, Mathematical Lecture Series, vol. 2, Publish or Perish, Inc., Boston, Mass, 1974.
[31] V.K. Melnikov, On the stability of a center for time-periodic perturbations, Tr. Mosk. Mat. Obŝ. 12 (1963) 3–52.
[32] Alexander Mielke, Hamiltonian and Lagrangian Flows on Center Manifolds, Lecture Notes in Mathematics, vol. 1489, Springer-Verlag, Berlin, 1991, With applications to elliptic variational problems.
[33] M. Milla Miranda, L.P. San Gil Jutuca, Existence and boundary stabilization of solutions for the Kirchhoff equation, Commun. Partial Differ. Equ. 24 (9–10) (1999) 1759–1800.
[34] Yu.A. Mitropolsky, A.M. Samoĭlenko, D.I. Martinyuk, Systems of Evolution Equations with Periodic and Quasiperiodic Coefficients, Mathematics and Its Applications (Soviet Series), vol. 87, Kluwer Academic Publishers Group, Dordrecht, 1993, Translated from the 1984 Russian original.
[35] Igor Podlubny, Fractional Differential Equations: an Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Elsevier, 1998.

[36] Peter Poláčik, Darío A. Valdebenito, Existence of quasiperiodic solutions of elliptic equations on $\mathbb{R}^{N+1}$ via center manifold and KAM theorems, J. Differ. Equ. 262 (12) (2017) 6109–6164.
[37] Peter Poláčik, Darío A. Valdebenito, Existence of quasiperiodic solutions of elliptic equations on the entire space with a quadratic nonlinearity, Discrete Contin. Dyn. Syst., Ser. S 13 (4) (2020) 1369–1393.
[38] Thomas Runst, Winfried Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.
[39] Yunfeng Shi, Analytic solutions of nonlinear elliptic equations on rectangular tori, J. Differ. Equ. 267 (9) (2019) 5576–5600.
[40] Michael E. Taylor, Partial differential equations. III, in: Nonlinear Equations, in: Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997, Corrected reprint of the 1996 original.
[41] Fenfen Wang, Rafael de la Llave, Response solutions to quasi-periodically forced systems, even to possibly ill-posed PDEs, with strong dissipation and any frequency vectors, SIAM J. Math. Anal. 52 (4) (2020) 3149–3191.