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KAM quasi-periodic solutions for the dissipative standard map

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ABSTRACT

We present results towards a constructive approach to show the existence of quasi-periodic solutions in non-perturbative regimes of some dissipative systems, called conformally symplectic systems. Finding a quasi-periodic solution of conformally symplectic systems with fixed frequency requires to choose a parameter, called the *drift parameter*.

The first step of the strategy is to establish a very explicit quantitative theorem in an a-posteriori format as in Calleja et al. (2013). A-posteriori theorems show that if we can find an approximate solution of an invariance equation, which is sufficiently approximate with respect to some condition numbers (algebraic expressions of derivatives of the approximate solution and estimates on the derivatives of the map), then there is a true solution.

The second step in the strategy is to produce numerically a very accurate solution of the invariance equation (discretizations with 2^{18} Fourier coefficients, each one computed with 100 digits of precision).

The third step is to compute in a concrete example, the dissipative standard map, the condition numbers and verify numerically the conditions of the theorem in the approximate solutions. For some families which have been studied numerically, the results agree with three figures with the best numerical values. We point out however that the numerical methods developed here work also in examples which have not been accessible to other more conventional methods.

The verification of the estimates presented here is not completely rigorous, since we do not control the round-off error, nor the truncation error of several operations in Fourier space. We hope that the positive step taken in this paper will stimulate the complete computer-assisted proof. Making explicit the condition numbers and verifying the conditions (even in an incomplete way) will be valuable for the computation close to breakdown.

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We make available the approximate solutions, the highly efficient algorithm (quadratic convergence, low storage requirements, low operation count per step) to compute them (incorporating high precision based on the MPFR library) and the routines used to verify the applicability of the theorem.

1. Introduction

The goal of this paper is to develop a methodology to compute efficiently and reliably quasi-periodic solutions in concrete systems and to provide an analytical estimate of their breakdown threshold (compare with [1–11]).

The KAM theory, started in [12–14], solved the outstanding problem of establishing the persistence of quasi-periodic orbits under small perturbations. An important motivation was represented by problems in celestial mechanics [15]. By now, KAM theory has developed into a very useful paradigm. Surveys of KAM theory and its applications are: [3,15–21].

At the beginning of the theory, the quantitative requirements for applicability led to unrealistic smallness estimates. In a well known calculation [22], M. Hénon made a preliminary study of the parameters required to apply the theorem to the three-body problem [13] and obtained that the small parameter (representing the Jupiter–Sun mass-ratio) should be smaller than 10^{-48}, whereas the real value is about 10^{-3}. Discouraged by this result, the often quoted conclusion of [22] was that[5]

"Ainsi, ces théorèmes, bien que d'un très grand intérêt théorique, ne semblent pas pouvoir ***en leur état actuel*** *être appliqués á des problèmes pratiques".*

The statement of [22] is perfectly correct as stated, but removing the words we have set in bold one obtains a statement invalid 50 years after the original statement.

The first attempts to study the problem numerically were disappointing. The persistence of quasi-periodic solutions indeed depends on rather high regularity of the perturbation (the smoothness requirements of some versions of KAM theory are optimal, [23–25]) and attempts based on low regularity discretizations such as finite elements were discouraging [26]. Furthermore, lacking a good theory, one can be misled by spurious solutions and it is hard to believe the true solutions.

By the late 70's it was folklore belief that the estimates of KAM theory were essentially optimal (the estimates for a step were optimal and it was expected that they could be saturated simultaneously). By now, the situation has changed drastically: general bounds based on different schemes [23,27–29] lead to substantially better bounds, than those coming from older methods.

More related to the present paper, in recent times there has been a rapid development in proofs of KAM theorems in the *"a-posteriori"* format common in numerical analysis; a general format of an a-posteriori theorem is given below.

Theorem Format 1. *Let $\mathcal{X}_1 \subset \mathcal{X}_0$ be Banach spaces and $\mathcal{U} \subset \mathcal{X}_0$ an open set. Consider the map*

$$\mathcal{F} : \mathcal{U} \to \mathcal{X}_0 \ ;$$

assume that there are functionals $m_1, \dots, m_n : \mathcal{U} \to \mathbb{R}^+$ and $x_0 \in \mathcal{U}$, such that:

(1) $\|\mathcal{F}(x_0)\|_{\mathcal{X}_0} < \varepsilon$ *for some* $\varepsilon \in \mathbb{R}$;
(2) $m_1(x_0) \le M_1, \dots, m_n(x_0) \le M_n$ *for some condition numbers* $M_1, \dots, M_n$;
(3) $\varepsilon \le \varepsilon^*(M_1, \dots, M_n)$, *where* ε^* *is an explicit function of the condition numbers.*

Then, there exists an $x^* \in \mathcal{X}_1$ such that $\mathcal{F}(x^*) = 0$ and $\|x_0 - x^*\|_{\mathcal{X}_1} \le C_{M_1,\dots,M_n}\varepsilon$ for some positive constant $C_{M_1,\dots,M_n}$.

One can formulate several classical KAM theorems in this format. One needs to choose an appropriate functional $\mathcal{F}$ whose zeros imply the existence of quasi-periodic solutions (in such applications x is an embedding that belongs to a suitable space of functions, see 2.1.1). Notice that we do not need that the system is close to integrable and we do not require any global assumption on the map, but only some functionals evaluated in the approximate solution.

Another important development is that some of the proofs in a-posteriori theorems have been established by studying the convergence of numerically efficient methods, which are usually based on developing a Newton method that uses the geometric properties to take advantage of remarkable cancellations. Note that to develop efficient algorithms, there are several rather stringent requirements that are not present in more theoretical treatments (for example, it is important to reduce the number of variables involved; the theoretically very powerful transformation theory is difficult to implement) and need to consider methods to discretize functions (note that KAM theory requires high regularity; hence, discretization methods, such as finite elements that do not represent well high derivatives, are not practical).

[5] "It does not seem that these theorems, though having a great theoretical interest, can be applied, **in their present state**, to practical problems" [22].

Such Newton-like algorithms are different from the standard Newton's method in that the inverse derivative is an unbounded operator. Hence the steps are forced to consider corrections which are less regular than the original function. The standard *interval Newton's method* [30] does not apply in this context, since it assumes that the iterative step maps elements of a function space into another function in the same space. Many KAM theorems (based in C^ω smoothing) are based on considering steps which are defined from one Banach space to another (these Banach spaces consist of analytic functions defined in decreasing domains). The proof of convergence depends crucially on the sequence of domains losses considered.

There are different geometric properties that lead to a KAM theory (see [31,32] for a discussion of the classical contexts – general, symplectic, volume preserving, reversible – formulated in a format which is not a-posteriori). Other more modern contexts are presymplectic [33], or closer to the goals of this paper, conformally symplectic [34].

A remarkable result on the existence of normally hyperbolic invariant tori carrying quasi-periodic motions of prefixed frequencies has been presented in [35,36], where smooth families of real-analytic maps are considered. The papers [35,36] are based on the theory of normally hyperbolic invariant manifolds [37,38] and (dissipative) KAM theory [31,32], but they do not assume the presence of any geometrical structure.

Notice that an a-posteriori theorem allows to validate the existence of an approximate solution, independently of how it has been obtained.

As it turns out, there exist computer science techniques (interval analysis, see [39–41]) which allow one to perform rigorous bounds mechanically. The coupling of an a-posteriori theorem with interval arithmetic has led to many *computer assisted proofs* of mathematically relevant problems (see, in particular, [42]) that are reduced to the existence of a fixed point[6].

Therefore, a way to prove the existence of a quasi-periodic solution has different stages, each of them requiring a different methodology.

(A) For a fixed geometric context, prove an a-posteriori KAM theorem.
(B) Make sure that the conditions of the a-posteriori theorem in part A are made explicit and computable.
(C) Produce approximate solutions.
(D) Verify the conditions given in B) on the approximate solutions produced in C).

Point A) of the above strategy was implemented for two dimensional symplectic mappings in [43] (which also established upper and lower bounds of Siegel radius) and, more recently, [7] (which gives a very innovative implementation of a-posteriori KAM estimates). The technique of [7] is successfully applied to the standard map, obtaining computer-assisted estimates in agreement of 99.996% with numerical upper bounds. The paper [7] has also considered applications to the non-twist standard map and to the Froeschlé map.

Part A) requires the traditional methods of analysis, but the goals should be an explicit formulation that makes efficient the other parts of the strategy. Notably, the functional equations should involve functions of as little variables as possible[7].

Many of the more modern proofs in Part A) are based in describing an iterative process and showing it converges when started on a sufficiently approximate solution. For our case, the proof presented in [34] is particularly well suited for numerical applications. It leads to a quadratically convergent algorithm that requires little storage and a small operation count per step. The algorithm can be used as the basis of a continuation method, and also a practical method to compute the breakdown.

Part B) is, in principle, straightforward given the theoretical work already done in [34]; a high quality implementation requires taking advantage of the cancellations and organizing the estimates very efficiently.

Part C) is very traditional in numerical analysis and can be accomplished in many ways, for example discretizing the invariance equation, but we stress that there are some interactions with the other parts. Notably, the high accuracy calculation is based on implementing the algorithm in [34]. Note that if we discretize the functions considered in N elements, the algorithm requires storage $O(N)$ and a (quadratically convergent) step requires $O(N\log(N))$ operations. This efficiency allows us to take $N = 2^{18}$ and use extended precision using only a today's desktop computer. Clearly, if storage was $O(N^2)$ as in standard "big matrix" Newton methods, storage would have been challenging in today's computers.

To have an effective part D), the discretization used has to be such that it allows the evaluation of the norms involved. As indicated above, the KAM theorem requires derivatives of rather high order, so it seems that a Fourier discretization could be effective if we consider norms that can be read from the Fourier coefficients.

Part D) requires a finite number of operations, but the number is too large to be done by hand. The most delicate estimates concern the error incurred by the initial approximation and the condition numbers. Finally, one needs to verify a much smaller (about a dozen) of inequalities that ensure that any solution with these initial error and condition numbers is the starting point of a convergent iterative method.

The goal of this paper is to present an implementation of a substantial part of this strategy for conformally symplectic mappings and obtain concrete results for an emblematic example that has been considered many times in the literature:

[6] We note, however, that, besides computer assisted proofs based on fixed point theorems, there are other computer assisted proofs which do not involve fixed points theorems, but which are based on other arguments (exclusion of matches, algebraic operations, etc.).

[7] The difficulty of dealing with functions grows very fast with the number of variables. This is known as the *curse of dimensionality*.

the dissipative standard map. We give explicit estimates on the numerical validation of the golden mean attractor, providing results for values of the parameters in agreement within 99.94% of the value obtained using numerical results [44]. We remark that the same strategy has been used in [45] (see also [46,47]), showing an application to a problem of interest in Celestial Mechanics: the spin–orbit problem with tidal torque.

Even if the present implementation to the standard map goes beyond the results in straightforward numerical computations, we are not claiming it is a complete computer-assisted proof. The caveats are that in part D), we have not provided rigorous estimates for the truncation of the evaluation of the error and we have not used interval arithmetic to control the round-off error. We have performed the calculations with more than 100 digits of precision and checked that changing the number of figures carried does not affect the estimates of the error.

Although this paper does not accomplish a complete proof, we hope that, by providing extremely fast and efficient algorithms, making explicit the condition numbers and the inequalities that need to be checked, it goes significantly beyond the usual numerical analysis practices. Of course, we hope to complete the proof or stimulate interest of others. Note that the paper contains as supplements the implementation of the calculation of the solutions in extended precision as well as the solutions themselves.

This paper is organized as follows. In Section 2, we present some standard preliminaries. In Section 3 we state a very explicit KAM theorem in an a-posteriori format, Theorem 10 which implements part A) of the strategy indicated above. The proof of Theorem 10 is given in Section 4, see also Appendix B. The KAM estimates for the standard map are presented in Section 5.

2. Preliminaries

In this Section, we collect several notions that play a role in our results. The material in Section 2.1 concerns standard properties of analytic functions and can be used mainly as a reference for the notation. In Section 2.2 we introduce conformally symplectic systems, which are our main geometric assumption. In Section 2.3 we introduce the concrete model we will study.

2.1. Norms and preliminary lemmas

In this Section we need to specify the norms (see Section 2.1.1), to estimate the composition of functions (see Section 2.1.2), to bound derivatives (see Section 2.1.3), to introduce Diophantine numbers (see Section 2.1.4), and to give estimates of a cohomology equation associated to the linearization of the invariance equation (see Section 2.1.5).

2.1.1. Norms

For a vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ and for a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2$, we define the norms as

$$\|v\| = |v_1| + |v_2| \,, \qquad \|A\| = \max\Big\{|a_{11}| + |a_{21}|,\ |a_{12}| + |a_{22}|\Big\} \,.$$

We start by introducing for $\rho > 0$ the following complex extensions of a torus $\mathbb{T}$, of a set B and of the manifold $\mathcal{M} = B \times \mathbb{T}$:

$$\begin{aligned} \mathbb{T}_\rho &\equiv \{z = x + iy \in \mathbb{C}/\mathbb{Z} :\ x \in \mathbb{T} \,,\ |y| \le \rho\} \,, \\ B_\rho &\equiv \{z = x + iy \in \mathbb{C} :\ x \in B \,,\quad |y| \le \rho\} \,, \\ \mathcal{M}_\rho &= B_\rho \times \mathbb{T}_\rho \,. \end{aligned}$$

We denote by $\mathcal{A}_\rho$ the set of functions which are analytic in $\mathrm{Int}(\mathbb{T}_\rho)$ and that extend continuously to the boundary of $\mathbb{T}_\rho$. We endow $\mathcal{A}_\rho$ with the norm

$$\|f\|_\rho = \sup_{z \in \mathbb{T}_\rho} |f(z)| \,,$$

that makes it into a Banach space.

For a domain $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}\backslash\mathbb{Z}$, let $F \in \mathcal{A}_\mathcal{C}$ be an analytic function on $\mathcal{C}$ and let

$$\|F\|_\mathcal{C} = \sup_{z \in \mathcal{C}} |F(z)| \,.$$

Then, for a vector valued function $f = (f_1, f_2, \dots, f_n)$, $n \ge 1$, we define the norm

$$\|f\|_\mathcal{C} = \sup_{z \in \mathcal{C}} (|f_1| + |f_2| + \cdots + |f_n|) \,. \tag{2.1}$$

We notice that, in practical applications, it is convenient to use the following upper bound, instead of (2.1):

$$\|f\|_\mathcal{C} \le \sup_{z \in \mathcal{C}} |f_1| + \sup_{z \in \mathcal{C}} |f_2| + \cdots + \sup_{z \in \mathcal{C}} |f_n| = \|f_1\|_\mathcal{C} + \|f_2\|_\mathcal{C} + \cdots + \|f_n\|_\mathcal{C} \,.$$

By the maximum principle, it suffices to compute the supremum over the boundary of the domain:

$$\|f\|_{\mathcal{C}} \leq \sup_{z\in\partial\mathcal{C}} |f_1| + \sup_{z\in\partial\mathcal{C}} |f_2| + \cdots + \sup_{z\in\partial\mathcal{C}} |f_n| \ .$$

For an $n_1 \times n_2$ matrix valued function F we define

$$\|F\|_{\mathcal{C}} = \sup_{z\in\mathcal{C}} |F(z)| \ ,$$

that we can bound as

$$\|F\|_{\mathcal{C}} \leq \max\left\{\sum_{i=1}^{n_1} \sup_{z\in\mathcal{C}} |F_{i1}(z)|, \sum_{i=1}^{n_1} \sup_{z\in\mathcal{C}} |F_{i2}(z)|, \ldots, \sum_{i=1}^{n_1} \sup_{z\in\mathcal{C}} |F_{in_2}(z)|\right\} \ ;$$

as before, we can use the maximum principle to compute the supremum on the boundary as

$$\|F\|_{\mathcal{C}} \leq \max\left\{\sum_{i=1}^{n_1} \sup_{z\in\partial\mathcal{C}} |F_{i1}(z)|, \sum_{i=1}^{n_1} \sup_{z\in\partial\mathcal{C}} |F_{i2}(z)|, \ldots, \sum_{i=1}^{n_1} \sup_{z\in\partial\mathcal{C}} |F_{in_2}(z)|\right\} \ .$$

Notice that if F is a matrix valued function and f is a vector valued function, then one has

$$\|F f\|_{\mathcal{C}} \leq \|F\|_{\mathcal{C}} \ \|f\|_{\mathcal{C}} \ .$$

If, instead of $\mathcal{A}_{\mathcal{C}}$, we have a function defined on $\mathcal{A}_\rho$, we introduce the corresponding norms as

$$\|F\|_\rho = \sup_{z\in\mathbb{T}_\rho} |F(z)|$$

for a function $F \in \mathcal{A}_\rho$ and

$$\|F\|_\rho \leq \max\left\{\sum_{i=1}^{n_1} \sup_{z\in\mathbb{T}_\rho} |F_{i1}(z)|, \sum_{i=1}^{n_1} \sup_{z\in\mathbb{T}_\rho} |F_{i2}(z)|, \ldots, \sum_{i=1}^{n_1} \sup_{z\in\mathbb{T}_\rho} |F_{in_2}(z)|\right\}$$

for an $n_1 \times n_2$ matrix valued function with components in $\mathcal{A}_\rho$.

2.1.2. Composition lemma

Composition of two functions is an important operation in dynamical systems, which enters our main functional equation, see (2.8) below.

Lemma 2. *Let $F \in \mathcal{A}_{\mathcal{C}}$ be an analytic function on a domain $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}/\mathbb{Z}$.*

Assume that the function g is such that $g(\mathbb{T}_\rho) \subset \mathcal{C}$ and that the components of g are in $\mathcal{A}_\rho$ with $\rho > 0$. Then, $F \circ g \in \mathcal{A}_\rho$ and

$$\|F \circ g\|_\rho \leq \|F\|_\rho \ .$$

If, furthermore, we have that $\mathrm{dist}(g(\mathcal{A}_\rho), \mathbb{C} \times \mathbb{C}\backslash\mathbb{Z} \setminus \mathcal{C}) = \eta > 0$, then we have:

(i) For all $h \in \mathcal{A}_\rho$ with $\|h\|_\rho < \eta/4$, we can define $F \circ (g + h)$.
(ii) We have:

$$\|F \circ (g+h) - F \circ g\|_\rho \leq \sup_{z,\mathrm{dist}(z,\mathcal{C})\leq\eta/4} (|DF(z)|) \ \|h\|_\rho \ ,$$

$$\|F \circ (g+h) - F \circ g - DF \circ g\, h\|_\rho \leq \frac{1}{2} \sup_{z,\mathrm{dist}(z,\mathcal{C})\leq\eta/4} (|D^2F(z)|) \ \|h\|_\rho^2 \ .$$

2.1.3. Cauchy estimates on the derivatives

Estimates on the derivatives will be needed throughout the whole proof of the main result (Theorem 10).

Lemma 3. *For a function $h \in \mathcal{A}_\rho$, we have the following estimate on the first derivative on a smaller domain:*

$$\|Dh\|_{\rho-\delta} \leq \delta^{-1} \ \|h\|_\rho \ ,$$

where $0 < \delta < \rho$. For the ℓ-th order derivatives with $\ell \geq 1$, one has:

$$\|D^\ell h\|_{\rho-\delta} \leq \ell! \ \delta^{-\ell} \ \|h\|_\rho \ .$$

2.1.4. Diophantine numbers

The following definition is standard in number theory and appears frequently in KAM theory.

Definition 4. Let $\omega \in \mathbb{R}$, $\tau \geq 1$, $\nu \geq 1$. We say that ω is Diophantine of class τ and constant ν, if the following inequality is satisfied:

$$|\omega k - q| \ \geq \ \nu |k|^{-\tau} \ , \qquad q \in \mathbb{Z} \ , \quad k \in \mathbb{Z} \backslash \{0\} \ . \tag{2.2}$$

The set of Diophantine numbers satisfying (2.2) is denoted by $\mathcal{D}(\nu, \tau)$. The union over $\nu > 0$ of the sets $\mathcal{D}(\nu, \tau)$ has full Lebesgue measure in $\mathbb{R}$.

2.1.5. Estimates on the cohomology equation

Given any real-analytic function η, we consider the following cohomology equation for $\lambda \in \mathbb{R}$:

$$\varphi(\theta + \omega) - \lambda \varphi(\theta) = \eta(\theta) \ , \qquad \theta \in \mathbb{T} \ . \tag{2.3}$$

The solution of an equation of the form (2.3) will be an essential ingredient of the proof, see e.g. (4.5) below. The two following Lemmas show that there is one real-analytic function φ, which is the solution of (2.3). Precisely, Lemma 5 applies for $|\lambda| \neq 1$, $\omega \in \mathbb{R}$ and it provides an estimate on the solution φ which is not uniform in λ, while Lemma 6 applies to any λ and any ω Diophantine, and it provides an estimate on the solution which is uniform in λ.

Lemma 5. *Assume $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, $\omega \in \mathbb{R}$. Then, given any real-analytic function η, there is one real-analytic function φ satisfying (2.3). Furthermore, the following estimate holds:*

$$\|\varphi\|_\rho \leq \Big| \, |\lambda| - 1 \Big|^{-1} \|\eta\|_\rho \ .$$

Moreover, one can bound the derivatives of φ with respect to λ as

$$\|D^j_\lambda \varphi\|_\rho \leq \frac{j!}{\Big| \, |\lambda| - 1 \Big|^{j+1}} \ \|\eta\|_\rho \ , \qquad j \geq 1 \ .$$

Lemma 6. *Consider (2.3) for $\lambda \in [A_0, A_0^{-1}]$ for some $0 < A_0 < 1$ and let $\omega \in \mathcal{D}(\nu, \tau)$. Assume that $\eta \in \mathcal{A}_\rho$, $\rho > 0$ and that*

$$\int_{\mathbb{T}} \eta(\theta) \, d\theta = 0 \ .$$

Then, there is one and only one solution of (2.3) with zero average: $\int_{\mathbb{T}} \varphi(\theta) \, d\theta = 0$. Furthermore, if $\varphi \in \mathcal{A}_{\rho-\delta}$ for $0 < \delta < \rho$, then we have

$$\|\varphi\|_{\rho-\delta} \leq C_0 \ \nu^{-1} \ \delta^{-\tau} \|\eta\|_\rho \ , \tag{2.4}$$

where

$$C_0 = \frac{1}{(2\pi)^\tau} \ \frac{\pi}{2^\tau (1 + \lambda)} \sqrt{\frac{\Gamma(2\tau + 1)}{3}} \ . \tag{2.5}$$

The proof of Lemma 5 is given in [34], while that of Lemma 6 with the constant C_0 as in (2.5) is given in Appendix A.

2.2. Conformally symplectic systems

In this Section we give the definition of conformally symplectic systems for two-dimensional maps. Indeed, the dissipative standard map that we will introduce in Section 2.3 and that we will consider throughout this paper, is a two-dimensional, conformally symplectic map. A more general definition of a conformally symplectic system in the n-dimensional case is provided in [34].

Definition 7. Let $\mathcal{M}$ be an analytic symplectic manifold with $\mathcal{M} \equiv B \times \mathbb{T}$, where $B \subseteq \mathbb{R}$ is an open, simply connected domain with a smooth boundary. Let Ω be the symplectic form associated to $\mathcal{M}$. Let f be a diffeomorphism defined on the phase space $\mathcal{M}$. The diffeomorphism f is conformally symplectic, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that

$$f^* \Omega = \lambda \Omega \ ,$$

where f^* denotes the pull-back of f.

We remark that when $n = 1$, then λ can be a function of the coordinates, while it is shown [48] that for $n \geq 2$, then λ has to be constant.

In the following discussion, we will always assume that λ is a constant, as in the model (2.6) below, which is the main goal of the present work.

2.3. A specific model

In this work we consider a specific 1-parameter family f_μ of two-dimensional, conformally symplectic maps, known as the *dissipative standard map:*

$$I' = \lambda I + \mu + \frac{\varepsilon}{2\pi}\sin(2\pi\varphi) ,$$
$$\varphi' = \varphi + I' , \tag{2.6}$$

where $I \in B \subseteq \mathbb{R}$ with B as in Definition 7, $\varphi \in \mathbb{T}$, $\varepsilon \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$, $\mu \in \mathbb{R}$. This model has been studied both numerically and theoretically in the literature. For example [49–51] consider the breakdown and conjecture universality properties; [44] studies the breakdown even for complex values of the parameters; [52] studies the invariant bundles near the circles and find scaling properties at breakdown; [53,54] study the domains of analyticity in the limit of small dissipation.

To fix some terminology, we shall refer to ε as the *perturbing parameter*, to λ as the *dissipative parameter*, and to μ as the *drift parameter*.

Notice that the Jacobian of the mapping (2.6) is equal to λ, so that the mapping is contractive for $\lambda < 1$, volume expanding for $\lambda > 1$ and it is symplectic for $\lambda = 1$.

We denote by $(\cdot, \cdot)$ the Euclidean scalar product. We remark that if $J = J(x)$ is the matrix representing Ω at x in the Euclidean metric, namely $\Omega_x(u, v) = (u, J(x)v)$ for any u, $v \in \mathbb{R}$, then for the mapping (2.6), J is the following constant matrix:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.7}$$

The map (2.6) is conformally symplectic of factor λ for the standard symplectic form Ω.

2.3.1. Formulation of the problem of an invariant attractor

We proceed to provide the definition of a KAM attractor with frequency ω.

Having fixed a value of the dissipative parameter, our goal will be to prove the persistence of invariant attractors associated to (2.6) for non-zero values of the perturbing parameter. To this end, we need to require that the frequency of the attractor, say $\omega \in \mathbb{R}$, is Diophantine according to Definition (2.2). We note that this will require adjusting the drift parameter μ.

Definition 8. Given a family of conformally symplectic maps $f_\mu : \mathcal{M} \to \mathcal{M}$, a KAM attractor with frequency ω is an invariant torus which can be described by an embedding $K : \mathbb{T} \to \mathcal{M}$, such that the following invariance equation is satisfied for all $\theta \in \mathbb{T}$:

$$f_\mu \circ K(\theta) = K(\theta + \omega) . \tag{2.8}$$

The Eq. (2.8) will be the key of our statements; it includes both the embedding K and the parameter μ as unknowns.

Remark 9. It is interesting to notice that for $\varepsilon = 0$ the embedding can be chosen as $K(\theta) = (\theta, \omega)$. In this case, the mapping (2.6) admits a natural attractor with frequency $\omega = \mu/(1-\lambda)$. This simple observation highlights the role of the drift μ and its relation to the frequency ω.

3. A KAM theorem

In this Section we state the main mathematical result, Theorem 10, which is a KAM result in the a-posteriori format described in Theorem Format 1. Theorem 10 is a constructive version of Theorem 20 in [34] and it specifies the condition numbers to be measured in the approximate solution as well as the inequalities that imply the existence of a KAM attractor. It shows that, if there is a function K_0 and a number μ_0 that, when substituted in (2.8), give a residual (measured in a norm that we specify) which is smaller than a function of the condition numbers, then there is a solution of (2.8) close (in some norm that we specify) to K_0, μ_0.

We also note that the method of proof, which is based on constructing an iterative procedure, leads to a very efficient algorithm. The focus of this paper will be in giving explicit estimates and showing that the hypotheses of the theorem are satisfied numerically in the example (2.6) for explicit values of ε, λ. In particular, we will verify numerically that the estimates of the theorem are satisfied taking a numerically computed solution as the approximate solution (see Section 5).

For an embedding $K_0 = K_0(\theta)$ and a frequency ω, we start by introducing some auxiliary quantities defined as follows:

$$M_0(\theta) \equiv [DK_0(\theta) \,|\, J^{-1}\, DK_0(\theta)N_0(\theta)] ,$$
$$S_0(\theta) \equiv ((DK_0N_0) \circ T_\omega)^\top(\theta)Df_{\mu_0} \circ K_0(\theta)J^{-1}DK_0(\theta)N_0(\theta) ,$$
$$N_0(\theta) \equiv (DK_0(\theta)^\top DK_0(\theta))^{-1} , \tag{3.1}$$

where the superscript $\top$ denotes the transposition and T_ω denotes the shift by ω: for a function $P = P(\theta)$, then $(P \circ T_\omega)(\theta) = P(\theta + \omega)$.

Theorem 10. *Consider a family $f_\mu : \mathcal{M} \to \mathcal{M}$ of conformally symplectic mappings with conformal factor $0 < \lambda < 1$, defined on the manifold $\mathcal{M} \equiv B \times \mathbb{T}$ with $B \subseteq \mathbb{R}$ an open, simply connected domain with a smooth boundary. Let the mappings f_μ be analytic on an open connected domain $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}/\mathbb{Z}$. Let the following assumptions be satisfied.*

H1 *Let $\omega \in \mathcal{D}(\nu, \tau)$ as in (2.2).*

H2 *There exists an approximate solution (K_0, μ_0) with $K_0 \in \mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$ and with $\mu_0 \in \Lambda$, $\Lambda \subset \mathbb{R}$ open. Let (K_0, μ_0) be such that (2.8) is satisfied up to an error function $E_0 = E_0(\theta)$, namely*

$$f_{\mu_0} \circ K_0(\theta) - K_0(\theta + \omega) = E_0(\theta) \ .$$

Let ε_0 denote the size of the error function, i.e.

$$\varepsilon_0 \equiv \|E_0\|_{\rho_0} \ .$$

H3 *Assume that the following non-degeneracy condition holds:*

$$\det \begin{pmatrix} \overline{S}_0 & \overline{S_0 (B_{b0})^0} + \overline{\widetilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\widetilde{A}_0^{(2)}} \end{pmatrix} \neq 0 \ ,$$

where S_0 is given in (3.1), $\widetilde{A}_0^{(1)}$, $\widetilde{A}_0^{(2)}$ denote the first and second elements of the vector $\widetilde{A}_0 \equiv M_0^{-1} \circ T_\omega D_\mu f_{\mu_0} \circ K_0$, $(B_{b0})^0$ is the solution (with zero average in the $\lambda = 1$ case) of the equation $\lambda (B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\widetilde{A}_0^{(2)})^0$, where $(\widetilde{A}_0^{(2)})^0$ denotes the zero average part of $\widetilde{A}_0^{(2)}$. Denote by $\mathcal{T}_0$ the twist constant *defined as*

$$\mathcal{T}_0 \equiv \left\| \begin{pmatrix} \overline{S}_0 & \overline{S_0 (B_{b0})^0} + \overline{\widetilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\widetilde{A}_0^{(2)}} \end{pmatrix}^{-1} \right\| \ .$$

H4 *Assume there exists $\zeta > 0$, so that*

$$\mathrm{dist}(\mu_0, \partial\Lambda) \geq \zeta \ , \qquad \mathrm{dist}(K_0(\mathbb{T}_{\rho_0}), \partial\mathcal{C}) \geq \zeta \ .$$

H5 *Let $0 < \delta_0 < \rho_0$. Let $\kappa_\mu \equiv 4C_{\sigma 0}$ with $C_{\sigma 0}$ constant (whose explicit expression is given in Appendix C). Let the quantities Q_0, $Q_{\mu 0}$, $Q_{z\mu 0}$, $Q_{\mu\mu 0}$, Q_{E0} be defined as*

$$\begin{aligned}
Q_0 &\equiv \sup_{z \in \mathcal{C}} |D f_{\mu_0}(z)| \ , \\
Q_{\mu 0} &\equiv \sup_{z \in \mathcal{C}} |D_\mu f_{\mu_0}(z)| \ , \\
Q_{z\mu 0} &\equiv \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D f_\mu(z)| \ , \\
Q_{\mu\mu 0} &\equiv \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu^2 f_\mu(z)| \ , \\
Q_{E0} &\equiv \frac{1}{2} \max\left\{ \|D^2 E_0\|_{\rho_0 - \delta_0}, \|D D_\mu E_0\|_{\rho_0 - \delta_0}, \|D_\mu^2 E_0\|_{\rho_0 - \delta_0} \right\} \ .
\end{aligned} \tag{3.2}$$

Assume that ε_0 satisfies the following smallness conditions for suitable real constants $C_{\eta 0}$, $C_{\mathcal{E}0}$, C_{d0}, $C_{\sigma 0}$, C_σ, C_{W0}, C_W, $C_{\mathcal{R}}$ (see Appendix C for their explicit expressions):

$$C_{\eta 0}\, \nu^{-1} \delta_0^{-\tau} \varepsilon_0 < \zeta \ , \tag{3.3}$$

$$2^{3\tau+4}\, C_{\mathcal{E}0}\, \nu^{-2}\, \delta_0^{-2\tau} \varepsilon_0 \leq 1 \ , \tag{3.4}$$

$$4 C_{d0} \nu^{-1} \delta_0^{-\tau} \varepsilon_0 < \zeta \ , \tag{3.5}$$

$$4 C_{\sigma 0} \varepsilon_0 < \zeta \ , \tag{3.6}$$

$$\|N_0\|_{\rho_0} \left(2\|DK_0\|_{\rho_0} + D_K\right) D_K < 1 \tag{3.7}$$

$$4 Q_{z\mu 0} C_{\sigma 0} \varepsilon_0 < Q_0 \ , \tag{3.8}$$

$$4 Q_{\mu\mu 0} C_{\sigma 0} \varepsilon_0 < Q_{\mu 0} \ , \tag{3.9}$$

$$C_\sigma\ D_K \leq C_{\sigma 0} \ , \tag{3.10}$$

$$D_K (C_{W0} + \|M_0\|_{\rho_0} C_W + C_W D_K) \leq C_{d0} \ , \tag{3.11}$$

$$D_K \left(C_W\ \nu \delta_0^{-1+\tau} + C_{\mathcal{R}} \right) \leq C_{\mathcal{E}0} \ , \tag{3.12}$$

where D_K is defined as

$$D_K \equiv 4C_{d0}\ \nu^{-1}\delta_0^{-\tau-1}\ \varepsilon_0\ . \tag{3.13}$$

Then, there exists an exact solution (K_e, μ_e) of (2.8) such that

$$f_{\mu_e} \circ K_e - K_e \circ T_\omega = 0\ .$$

The quantities K_e, μ_e are close to the approximate solution, since one has

$$\begin{aligned} \|K_e - K_0\|_{\rho_0-\delta_0} &\leq 4C_{d0}\ \nu^{-1}\ \delta_0^{-\tau}\ \|E_0\|_{\rho_0}\ , \\ |\mu_e - \mu_0| &\leq 4C_{\sigma 0}\ \|E_0\|_{\rho_0}\ . \end{aligned} \tag{3.14}$$

In [34] there are two versions of Theorem 10. One version which applies for any value $\lambda \in [A^{-1}, A]$ including the symplectic case $\lambda = 1$ (the uniform version); another version which works for a fixed value of $\lambda \in \mathbb{R}\backslash\{1\}$ (the non-uniform version), which is the one we have used as the basis of the results in this paper, where the constants involved depend on λ. Many KAM theorems (based on C^ω smoothing) are based on considering steps which are defined from one Banach space to another (these Banach spaces consist of analytic functions defined in decreasing domains). The proof of convergence depends crucially on the sequence of domain losses considered. The price that [34] pays to obtain the uniform version is that the small divisor estimates need to be used twice, rather than once. Hence the powers of δ, that appears in (3.14), are higher in the uniform case.

The explicit expressions of the constants entering in the conditions (3.3)-(3.12) are obtained by implementing constructively the KAM proof presented in [34]. In Section 5 the family f_μ will be taken as the dissipative standard map defined in (2.6); then, the explicit expressions for the constants – provided in Appendix C – will allow us to compute concrete values for ε_0, once we fix the frequency ω and the conformal factor λ. Therefore, the conditions (3.3)-(3.12) will ensure the existence of an invariant attractor with fixed frequency ω and for a given conformal factor λ.

Remark 11. For any value of λ with $|\lambda| < 1$, Theorem 10 also ensures that the quasi-periodic solution provided by the manifold $K_e(\mathbb{T})$ is a local attractor and that the dynamics on this attractor is analytically conjugated to a rigid rotation. We also mention that Theorem 10 implies regularity in the parameters, as already stated in [34], which discusses the Lipschitz dependence of the solution with respect to the drift (see Section 5.1.3 of [34]) and the differentiability of the drift with respect to parameters (see Section 10.3 of [34]).

Remark 12. An interesting question is how it is possible to use the computer to verify hypotheses that involve irrational numbers and indeed the Diophantine properties. After all, the standard computer numbers are only rational numbers.

The answer is that the a-posteriori theorem uses the Diophantine properties and that this theorem is indeed given a traditional proof. To verify the hypothesis, we compute numerically $\|f_\mu \circ K - K \circ T_{\omega_0}\|$ where ω_0 is indeed a rational number.

It is clear that for $\xi \in (\omega_0, \omega)$:

$$\begin{aligned} \|f_\mu \circ K - K \circ T_\omega\| &\leq \|f_\mu \circ K - K \circ T_{\omega_0}\| + \|K \circ T_{\omega_0} - K \circ T_\omega\| \\ &\leq \|f_\mu \circ K - K \circ T_{\omega_0}\| + \|DK \circ T_\xi\| |\omega - \omega_0|\ . \end{aligned}$$

In our case, we see that $|\omega - \omega_0| \leq 10^{-100}$ and that DK is a number of order 1. Hence, the last term does not affect too much the final result.

Of course, implementing interval arithmetic, one can also use an interval that contains the desired frequency and obtain estimates for the error of the invariance valid uniformly for all ω in this interval.

4. A constructive version of the proof of Theorem 10

We note that in the statement of Theorem 10 (and in the subsequent text) all the constants are given explicitly (see Appendix C). There are only a few dozen of conditions to check; all these conditions are easy algebraic expressions that can be checked with a computer.

The proof of Theorem 10 is presented in detail in [34]. However, in [34] the proof was given for a general case and no explicit estimates on the constants were provided, which are instead given in this Section.

We anticipate that it is easy to see that in the two-dimensional case of the mapping (2.6) all invariant curves are Lagrangian; this observation will simplify the proof presented in Section 4 with respect to that developed in [34].

4.1. Estimate on the error R_0

Let (K_0, μ_0) be an approximate solution of the invariance equation (2.8) and let $E_0 = E_0(\theta)$ be the associated error function. In coordinates, the Lagrangian condition $K^*\Omega = 0$ becomes

$$DK_0^\top(\theta)\, J\, DK_0(\theta) = 0\ ,$$

which shows that the tangent space can be decomposed as,

$$\mathrm{Range}\Big(DK_0(\theta)\Big) \oplus \mathrm{Range}\Big(J^{-1}DK_0(\theta)\Big).$$

Up to a remainder function $R_0 = R_0(\theta)$, the following identity is satisfied:

$$Df_{\mu_0} \circ K_0(\theta)\, M_0(\theta) = M_0(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S_0(\theta) \\ 0 & \lambda\, \mathrm{Id} \end{pmatrix} + R_0(\theta) \tag{4.1}$$

with M_0 and S_0 as in (3.1). Using that any torus associated to a two-dimensional map is always Lagrangian, one obtains

$$\|R_0\|_{\rho_0} = \|DE_0\|_{\rho_0} \ . \tag{4.2}$$

Using Cauchy estimates, a bound on R_0 in (4.2) is given by

$$\|R_0\|_{\rho_0-\delta_0} \le \delta_0^{-1}\ \|E_0\|_{\rho_0} \ . \tag{4.3}$$

4.2. Estimates for the increment in the steps

We proceed to find some corrections W_0 and σ_0 such that, setting $K_1 = K_0 + M_0 W_0$, $\mu_1 = \mu_0 + \sigma_0$, one has that the new approximation (K_1, μ_1) satisfies the following invariance equation:

$$f_{\mu_1} \circ K_1(\theta) - K_1(\theta+\omega) = E_1(\theta)$$

for some error function $E_1 = E_1(\theta)$. The requirement on E_1 is that its norm is quadratically smaller than the norm of the initial approximation E_0. This can be obtained provided that the following equation is satisfied:

$$Df_{\mu_0} \circ K_0(\theta)\, M_0(\theta) W_0(\theta) - M_0(\theta+\omega)\, W_0(\theta+\omega) + D_\mu f_{\mu_0} \circ K_0(\theta)\sigma_0 = -E_0(\theta)\ . \tag{4.4}$$

Using (4.1), (4.4) and neglecting higher order terms, one obtains two cohomology equations with constant coefficients for W_0 and σ_0. More precisely, writing W_0 in components as $W_0 = (W_0^{(1)}, W_0^{(2)})$, such cohomological equations are given by

$$\begin{aligned} W_0^{(1)}(\theta) - W_0^{(1)}(\theta+\omega) &= -\widetilde{E}_0^{(1)}(\theta) - S_0(\theta) W_0^{(2)}(\theta) - \widetilde{A}_0^{(1)}(\theta)\sigma_0\ , \\ \lambda W_0^{(2)}(\theta) - W_0^{(2)}(\theta+\omega) &= -\widetilde{E}_0^{(2)}(\theta) - \widetilde{A}_0^{(2)}(\theta)\sigma_0 \end{aligned} \tag{4.5}$$

with S_0 given in (3.1), while $\widetilde{E}_0$, $\widetilde{A}_0$ are defined as

$$\widetilde{E}_0 \equiv (\widetilde{E}_0^{(1)}, \widetilde{E}_0^{(2)}) \equiv M_0^{-1} \circ T_\omega E_0\ , \qquad \widetilde{A}_0 \equiv M_0^{-1} \circ T_\omega\, D_\mu f_{\mu_0} \circ K_0\ , \tag{4.6}$$

where we denote by $\widetilde{A}_0^{(1)}$, $\widetilde{A}_0^{(2)}$ the first and second elements of the vector $\widetilde{A}_0$.

We remark that the first equation in (4.5) involves small divisors. In fact, the Fourier expansion of the l.h.s. of the first equation in (4.5) is given by

$$W_0^{(1)}(\theta) - W_0^{(1)}(\theta+\omega) = \sum_{k\in\mathbb{Z}} \widehat{W}_{0,k}^{(1)}\, e^{2\pi i k\theta}(1 - e^{2\pi i k\omega})\ .$$

Then, we notice that for $k = 0$ there appears the zero factor $1 - e^{2\pi i k\omega} = 0$. On the other hand, the second equation in (4.5) is always solvable for any $|\lambda| \neq 1$ by a contraction mapping argument.

Let us split $W_0^{(2)}$ as $W_0^{(2)} = \overline{W_0^{(2)}} + (W_0^{(2)})^0$, where the first term denotes the average of $W_0^{(2)}$ and the second term the zero-average part. We remark that the average of $W_0^{(1)}$ can be set to zero without loss of generality. On the other hand, computing the averages of the cohomological equations (4.5), one can determine $\overline{W_0^{(2)}}$, σ_0 by solving the system of equations

$$\begin{pmatrix} \overline{S_0} & \overline{S_0 (B_{b0})^0} + \overline{\widetilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\widetilde{A}_0^{(2)}} \end{pmatrix} \begin{pmatrix} \overline{W_0^{(2)}} \\ \sigma_0 \end{pmatrix} = \begin{pmatrix} -\overline{S_0 (B_{a0})^0} - \overline{\widetilde{E}_0^{(1)}} \\ -\overline{\widetilde{E}_0^{(2)}} \end{pmatrix}\ , \tag{4.7}$$

where we have split $(W_0^{(2)})^0$ as $(W_0^{(2)})^0 = (B_{a0})^0 + \sigma_0 (B_{b0})^0$, where $(B_{a0})^0$, $(B_{b0})^0$ are the zero average solutions of

$$\begin{aligned} \lambda (B_{a0})^0 - (B_{a0})^0 \circ T_\omega &= -(\widetilde{E}_0^{(2)})^0\ , \\ \lambda (B_{b0})^0 - (B_{b0})^0 \circ T_\omega &= -(\widetilde{A}_0^{(2)})^0 \end{aligned} \tag{4.8}$$

with $(\widetilde{E}_0^{(2)})^0$, $(\widetilde{A}_0^{(2)})^0$ denoting the zero average parts of $\widetilde{E}_0^{(2)}$, $\widetilde{A}_0^{(2)}$. After solving (4.7), one can proceed to solve (4.5) for the zero average parts of $W_0^{(1)}$, $W_0^{(2)}$. Estimates on the corrections W_0 and σ_0 are given by the following result.

Lemma 13. *Let $K_0 \in \mathcal{A}_{\rho_0+\delta_0}$, $K_0(\mathbb{T}_{\rho_0}) \subset$ domain (f_μ), $\mathrm{dist}(K_0(\mathbb{T}_{\rho_0}), \partial($ domain$(f_\mu))) \geq \zeta > 0$ with ρ_0, δ_0, ζ as in Theorem 10. For any $|\lambda| \neq 1$ we have*

$$\begin{aligned} \|W_0\|_{\rho_0-\delta_0} &\leq C_{W0}\,\nu^{-1}\,\delta_0^{-\tau}\,\|E_0\|_{\rho_0}\ , \\ |\sigma_0| &\leq C_{\sigma 0}\,\|E_0\|_{\rho_0}\ , \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} C_{\sigma 0} &\equiv \mathcal{T}_0\,\Big[|\lambda-1|\,\Big(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\Big)+\|S_0\|_{\rho_0}\Big]\|M_0^{-1}\|_{\rho_0}\ , \\ \overline{C}_{W_20} &\equiv 2\,\mathcal{T}_0\,\Big(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\Big)\,Q_{\mu 0}\,\|M_0^{-1}\|^2_{\rho_0}\ , \\ C_{W_20} &\equiv \frac{1}{||\lambda|-1|}\Big(1+C_{\sigma 0}\,Q_{\mu 0}\Big)\|M_0^{-1}\|_{\rho_0}\ , \\ C_{W_10} &\equiv C_0\Big[\|S_0\|_{\rho_0}(C_{W_20}+\overline{C}_{W_20})+\|M_0^{-1}\|_{\rho_0}+Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_{\sigma 0}\Big]\ , \\ C_{W0} &\equiv C_{W_10}+(C_{W_20}+\overline{C}_{W_20})\nu\delta_0^{\tau}\ . \end{aligned} \tag{4.10}$$

Proof. Let $Q_{\mu 0}$ be an upper bound on the norm of $D_\mu f_{\mu_0}$ as in (3.2). Let $\widetilde{A}_0$ be defined as in (4.6); then, we have:

$$\|\widetilde{A}_0\|_{\rho_0} \leq Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}\ .$$

Recalling the definition of S_0 in (3.1), we obtain

$$\|S_0\|_{\rho_0} \leq J_e\ Q_0\ \|DK_0\|^2_{\rho_0}\|N_0\|^2_{\rho_0} \leq J_e\ \ Q_0\ \|K_0\|^2_{\rho_0+\delta_0}\|N_0\|^2_{\rho_0}\ \delta_0^{-2}\ ,$$

where we used the estimate $\|DK_0\|_{\rho_0} \leq \|K_0\|_{\rho_0+\delta_0}\ \delta_0^{-1}$ and where J_e denotes the norm of the symplectic matrix J in (2.7) (the norm of J^{-1} is again bounded by J_e); with the choice of the norms in Section 2.1.1 it is $J_e = 1$. We notice that, recalling the definition of S_0 and M_0 in (3.1), one can compute directly the functions and evaluate their norm.

For any $|\lambda| \neq 1$, we have the estimates given below, which follow from (4.6), (4.7), (4.8):

$$\begin{aligned} |\overline{W}_0^{(2)}| &\leq \mathcal{T}_0\Big(\|\overline{S_0(B_{b0})^0}+\overline{\widetilde{A}_0^{(1)}}\|_{\rho_0}\,\|\overline{\widetilde{E}_0^{(2)}}\|_{\rho_0}+\|\overline{S_0(B_{a0})^0}+\overline{\widetilde{E}_0^{(1)}}\|_{\rho_0}\,\|\overline{\widetilde{A}_0^{(2)}}\|_{\rho_0}\Big) \\ &\leq \mathcal{T}_0\Big[\Big(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\Big)\,Q_{\mu 0}\|M_0^{-1}\|^2_{\rho_0}\,\|E_0\|_{\rho_0} \\ &\quad+\Big(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\Big)\,\|M_0^{-1}\|^2_{\rho_0}\,\|E_0\|_{\rho_0}Q_{\mu 0}\Big] \\ &\leq \overline{C}_{W_20}\,\|E_0\|_{\rho_0}\ , \\ |\sigma_0| &\leq \mathcal{T}_0\Big(|\lambda-1|\,\|\overline{S_0(B_{a0})^0}+\overline{\widetilde{E}_0^{(1)}}\|_{\rho_0}+\|S_0\|_{\rho_0}\,\|\overline{\widetilde{E}_0^{(2)}}\|_{\rho_0}\Big) \\ &\leq C_{\sigma 0}\,\|E_0\|_{\rho_0} \end{aligned}$$

with $\overline{C}_{W_20}$, $C_{\sigma 0}$ as in (4.10). Then, using Lemma 5 and Lemma 6, we have:

$$\begin{aligned} &\|(W_0^{(2)})^0\|_{\rho_0} \leq C_{W_20}\,\|E_0\|_{\rho_0}\ , \\ &\|W_0^{(1)}\|_{\rho_0-\delta_0} \leq C_{W_10}\,\nu^{-1}\delta_0^{-\tau}\|E_0\|_{\rho_0} \end{aligned}$$

with C_{W_10}, C_{W_20} as in (4.10). In conclusion, recalling the definition of C_{W0} in (4.10), we obtain (4.9). □

4.3. Estimates for the convergence of the iterative step

Let us define the error functional

$$\mathcal{E}[K_0, \mu_0] \equiv f_{\mu_0}\circ K_0 - K_0\circ T_\omega\ .$$

Let

$$(\Delta_0, \sigma_0) = -\eta[K_0, \mu_0]E_0\ ,$$

where $\Delta_0 = -(\eta[K_0, \mu_0]E_0)_1$, $\sigma_0 = -(\eta[K_0, \mu_0]E_0)_2$. Then, using that $\Delta_0 = M_0W_0$, one has:

$$\|\eta[K_0, \mu_0]E_0\|_{\rho_0-\delta_0} \leq \|M_0\|_{\rho_0}\|W_0\|_{\rho_0-\delta_0}+|\sigma_0| \leq C_{\eta 0}\nu^{-1}\delta_0^{-\tau}\|E_0\|_{\rho_0}\ ,$$

where

$$C_{\eta 0} \equiv C_{W0}\|M_0\|_{\rho_0}+C_{\sigma 0}\nu\delta_0^{\tau}\ . \tag{4.11}$$

In this Section we give quadratic estimates on the norm of $D\mathcal{E}[K_0,\mu_0]\Delta_0+D_\mu\mathcal{E}[K_0,\mu_0]\sigma_0+E_0$ with $\Delta_0\equiv M_0W_0$; these estimates are needed to bound the error of the new approximate solution as it will be done in Section 4.4.

Lemma 14. *We have the following estimate:*

$$\|E_0+D\mathcal{E}[K_0,\mu_0]\Delta_0+D_\mu\mathcal{E}[K_0,\mu_0]\sigma_0\|_{\rho_0-\delta_0}\le C_{W0}\,\nu^{-1}\,\delta_0^{-1-\tau}\,\|E_0\|^2_{\rho_0}\ .$$

Proof. Taking into account that $W_0=M_0^{-1}\Delta_0$, from (7.15) in [34] we have that,

$$E_0+D\mathcal{E}[K_0,\mu_0]\Delta_0+D_\mu\mathcal{E}[K_0,\mu_0]\sigma_0=R_0W_0\ .$$

From Lemma 13 and (4.3), we obtain that

$$\|E_0+D\mathcal{E}[K_0,\mu_0]\Delta_0+D_\mu\mathcal{E}[K_0,\mu_0]\sigma_0\|_{\rho_0-\delta_0}\le C_{W0}\nu^{-1}\delta_0^{-1-\tau}\|E_0\|^2_{\rho_0}\ .\quad\square$$

4.4. Estimates for the error of the new solution

We proceed to bound the error corresponding to the new approximate solution.

Lemma 15. *Let $\eta[K_0,\mu_0]$ be as in Lemma 14 and let $\zeta>0$ be such that*

$$\mathrm{dist}(\mu_0,\partial\Lambda)\ge\zeta\ ,\qquad \|E\|_{\rho_0,k}\,\mathrm{dist}(K_0(\mathbb{T}_{\rho_0}),\partial\mathcal{C})\ge\zeta\ .$$

Assume that

$$C_{\eta0}\ \nu^{-1}\delta_0^{-\tau}\|E_0\|_{\rho_0}<\zeta<1 \tag{4.12}$$

with $C_{\eta0}$ as in (4.11). Then, we obtain the following estimate for the error:

$$\|\mathcal{E}[K_0+\Delta_0,\mu_0+\sigma_0]\|_{\rho_0-\delta_0}\le C_{\mathcal{E}0}\,\nu^{-2}\,\delta_0^{-2\tau}\,\|E_0\|^2_{\rho_0}\ ,$$

where

$$\begin{aligned}C_{\mathcal{R}0}&\equiv Q_{E0}(\|M_0\|^2_{\rho_0}C^2_{W0}+C^2_{\sigma0}\nu^2\delta_0^{2\tau})\ ,\\ C_{\mathcal{E}0}&\equiv C_{W0}\nu\delta_0^{-1+\tau}+C_{\mathcal{R}0}\ .\end{aligned}\tag{4.13}$$

Proof. We define the remainder of the Taylor series expansion as

$$\mathcal{R}[(K_0,\mu_0),(K_0',\mu_0')]\equiv\mathcal{E}[K_0',\mu_0']-\mathcal{E}[K_0,\mu_0]-D\mathcal{E}[K_0,\mu_0](K_0'-K_0)-D_\mu\mathcal{E}[K_0,\mu_0](\mu_0'-\mu_0)\ .$$

Then, we can write

$$\mathcal{E}[K_0+\Delta_0,\mu_0+\sigma_0]=E_0+D\mathcal{E}[K_0,\mu_0]\Delta_0+D_\mu\mathcal{E}[K_0,\mu_0]\sigma_0+\mathcal{R}[(K_0,\mu_0),(K_0+\Delta_0,\mu_0+\sigma_0)]\ .$$

From Lemma 13 and the definition of Q_{E0} in (3.2), we obtain

$$\|\mathcal{R}\|_{\rho_0-\delta_0}\le Q_{E0}\left(\|\Delta_0\|^2_{\rho_0-\delta_0}+|\sigma_0|^2\right)\equiv C_{\mathcal{R}0}\nu^{-2}\,\delta_0^{-2\tau}\,\|E_0\|^2_{\rho_0}$$

with $C_{\mathcal{R}0}$ as in (4.13). Then, from Lemma 14 we conclude that

$$\begin{aligned}\|\mathcal{E}[K_0+\Delta_0,\mu_0+\sigma_0]\|_{\rho_0-\delta_0}&\le C_{W0}\nu^{-1}\delta_0^{-1-\tau}\ \|E_0\|^2_{\rho_0}+C_{\mathcal{R}0}\nu^{-2}\delta_0^{-2\tau}\ \|E_0\|^2_{\rho_0}\\&\le C_{\mathcal{E}0}\ \nu^{-2}\delta_0^{-2\tau}\ \|E_0\|^2_{\rho_0}\end{aligned}$$

with $C_{\mathcal{E}0}$ as in (4.13). Notice that (4.12) guarantees that

$$\|\Delta_0\|_{\rho_0-\delta_0}<\zeta\ ,\qquad |\sigma_0|<\zeta\ .\quad\square$$

4.5. Analytic convergence

In this Section we prove that if we start with a small enough error, it is possible to repeat indefinitely the algorithm and that iterating the algorithm, we obtain a sequence of approximate solutions which converge to the true solution of the invariance equation (2.8).

Again, let (K_0,μ_0) be the initial approximate solution with $K_0\in\mathcal{A}_{\rho_0}$ for some $\rho_0>0$ as in Theorem 10 and define the sequence of parameters $\{\delta_h\}$, $\{\rho_h\}$, $h\ge0$, as

$$\delta_h\equiv\frac{\rho_0}{2^{h+2}}\ ,\qquad \rho_{h+1}\equiv\rho_h-\delta_h\ ,\qquad h\ge0\ .$$

With this choice of parameters the domain of analyticity where the true solution is defined will be a non-empty domain with size ρ_∞ given by

$$\rho_\infty = \rho_0 - \sum_{j=0}^{\infty} \frac{\rho_0}{2^{j+2}} = \rho_0 - \frac{\rho_0}{2} > 0 \ .$$

Let (K_h, μ_h), $h \geq 1$, be the approximate solution constructed by finding at each step the corrections (W_h, σ_h) solving the analogous of the cohomological equations (4.5) for $h = 0$. To make the notation precise, all quantities associated to (K_h, μ_h) will carry a subindex h, indicating the step of the algorithm. Define

$$\varepsilon_h \equiv \|\mathcal{E}(K_h, \mu_h)\|_{\rho_h} \ ,$$

and let us introduce the following quantities:

$$d_h \equiv \|\Delta_h\|_{\rho_{h+1}}, \quad v_h \equiv \|D\Delta_h\|_{\rho_{h+1}}, \quad s_h \equiv |\sigma_h| \ .$$

By Lemma 13 we have the following inequalities:

$$d_h \leq C_{dh} \nu^{-1} \delta_h^{-\tau} \ \varepsilon_h \ , \qquad v_h \leq C_{dh} \nu^{-1} \delta_h^{-\tau-1} \ \varepsilon_h \ , \qquad s_h \leq C_{\sigma h} \varepsilon_h \ ,$$

where

$$C_{dh} \equiv C_{Wh} \|M_h\|_{\rho_h} \tag{4.14}$$

where the quantities C_{Wh}, $C_{\sigma h}$ are obtained as follows:

$$\begin{aligned} C_{\sigma h} &\equiv \mathcal{T}_h \left[|\lambda - 1| \Big(\frac{1}{||\lambda| - 1|} \|S_h\|_{\rho_h} + 1 \Big) + \|S_h\|_{\rho_h} \right] \|M_h^{-1}\|_{\rho_h} \ , \\ \overline{C}_{W_2 h} &\equiv 2\mathcal{T}_h \Big(\frac{1}{||\lambda| - 1|} \|S_h\|_{\rho_h} + 1 \Big) Q_{\mu h} \|M_h^{-1}\|^2_{\rho_h} \ , \\ C_{W_2 h} &\equiv \frac{1}{||\lambda| - 1|} \Big(1 + C_{\sigma h} \ Q_{\mu h} \Big) \|M_h^{-1}\|_{\rho_h} \ , \\ C_{W_1 h} &\equiv C_0 \Big[\|S_h\|_{\rho_h} (C_{W_2 h} + \overline{C}_{W_2 h}) + \|M_h^{-1}\|_{\rho_h} + Q_{\mu h} \|M_h^{-1}\|_{\rho_h} \ C_{\sigma h} \Big] \ , \\ C_{Wh} &\equiv C_{W_1 h} + (C_{W_2 h} + \overline{C}_{W_2 h}) \nu \delta_h^{\tau} \ . \end{aligned} \tag{4.15}$$

Remark 16. By Lemma 15 one has

$$\varepsilon_{h+1} \leq C_{\mathcal{E} h} \nu^{-2} \delta_h^{-2\tau} \varepsilon_h^2$$

with

$$\begin{aligned} C_{\mathcal{R} h} &\equiv Q_{Eh} \ (\|M_h\|^2_{\rho_h} C^2_{Wh} + C^2_{\sigma h} \nu^2 \delta_h^{2\tau}) \\ Q_{Eh} &\equiv \frac{1}{2} \max \Big\{ \|D^2 E_h\|_{\rho_h - \delta_h}, \|D D_\mu E_h\|_{\rho_h - \delta_h}, \|D^2_\mu E_h\|_{\rho_h - \delta_h} \Big\} \\ C_{\mathcal{E} h} &\equiv C_{Wh} \nu \delta_h^{-1+\tau} + C_{\mathcal{R} h} \ . \end{aligned} \tag{4.16}$$

The results of Theorem 10 are based on the following proposition.

Proposition 17. *Let the constants C_{d0}, $C_{\sigma 0}$, $C_{\mathcal{E} 0}$ be as in (4.14), (4.15), (4.16) with $h = 0$. Define the following quantities:*

$$\kappa_K \equiv 4 C_{d0} \nu^{-1} \delta_0^{-\tau} \ , \qquad \kappa_\mu \equiv 4 C_{\sigma 0} \ , \qquad \kappa_0 \equiv 2^{2\tau+1} \ C_{\mathcal{E} 0} \nu^{-2} \delta_0^{-2\tau} \ . \tag{4.17}$$

Assume that the following conditions are satisfied:

$$2^{\tau+3} \kappa_0 \varepsilon_0 \leq 1 \ , \tag{4.18}$$

$$\kappa_K \varepsilon_0 < \zeta \ , \tag{4.19}$$

$$\kappa_\mu \varepsilon_0 < \zeta \ , \tag{4.20}$$

$$\|N_0\|_{\rho_0} \ (2\|DK_0\|_{\rho_0} + D_K) \ D_K \ < 1 \tag{4.21}$$

$$C_\sigma \ D_K \leq C_{\sigma 0} \ , \tag{4.22}$$

$$D_K \ \Big(C_W \nu \delta_0^{-1+\tau} + C_{\mathcal{R}} \Big) \leq C_{\mathcal{E} 0} \ , \tag{4.23}$$

$$D_K(C_{W0} + \|M_0\|_{\rho_0} C_W + C_W D_K) \le C_{d0} \ , \tag{4.24}$$

$$4Q_{z\mu 0} C_{\sigma 0} \varepsilon_0 < Q_0 \ , \tag{4.25}$$

$$4Q_{\mu\mu 0} C_{\sigma 0} \varepsilon_0 < Q_{\mu 0} \ , \tag{4.26}$$

where the constants C_σ, C_W, $C_{\mathcal{R}}$, C_{W0}, D_K *are defined in Appendix C. Then, for all integers* $h \ge 0$ *the following inequalities* $(p1; h)$, $(p2; h)$, $(p3; h)$ *hold:*

$(p1; h)$

$$\|K_h - K_0\|_{\rho_h} \le \kappa_K\, \varepsilon_0 < \zeta \ , \qquad |\mu_h - \mu_0| \le \kappa_\mu\, \varepsilon_0 < \zeta \ ; \tag{4.27}$$

$(p2; h)$

$$\varepsilon_h \le (\kappa_0 \varepsilon_0)^{2^h - 1} \varepsilon_0 \ ;$$

$(p3; h)$

$$C_{dh} \le 2C_{d0} \ , \qquad C_{\sigma h} \le 2C_{\sigma 0} \ , \qquad C_{\mathcal{E}h} \le 2C_{\mathcal{E}0} \ .$$

The proof of Proposition 17 is quite long (see Appendix B), but it is well structured and broken into small steps that can be easily verified. Proposition 17 allows to give the proof of Theorem 10 by *analytic smoothing:* at each step, the corrections (W_h, σ_h) yield increasingly approximate solutions, defined on smaller analyticity domains. The loss of domain is such that the exact solution is defined on a domain with positive radius of analyticity.

Proof (*of Theorem* 10)**.** The inequalities (3.14) follow directly from (4.27) and (4.17). The condition (3.3) follows from (4.12) of Lemma 15, while the conditions (3.4)-(3.12) follow from (4.18)-(4.26) of Proposition 17. □

We conclude by mentioning that the solution is locally unique. In fact, according to [34], if there exist two solutions (K_a, μ_a), (K_b, μ_b) close enough, then there exists $s \in \mathbb{R}$ such that for all $\theta \in \mathbb{T}$:

$$K_b(\theta) = K_a(\theta + s) \ , \qquad \mu_a = \mu_b \ .$$

5. KAM estimates for the standard map

In this Section we implement Theorem 10 to obtain explicit estimates on the numerical validation of the golden mean curve of the dissipative standard map (2.6); such estimates turn out to be close to the numerical breakdown value. We need to start with an approximate solution (K_0, μ_0), which satisfies the invariance equation (2.8) with an error term E_0, whose norm on a domain of radius $\rho_0 > 0$ was denoted as ε_0 in Theorem 10.

The construction of the approximate solution (K_0, μ_0) can be obtained by implementing the algorithm described in [34] and reviewed in Section 5.1 below. An estimate on the quantity ε_0 is obtained by imposing the list of conditions (3.3)-(3.12); explicit bounds are given in Section 5.2, using the definitions of the constants provided in Appendix C.

5.1. Construction of the approximate solution

To construct an approximate solution (K_0, μ_0) of the invariance equation (2.8), we make use of the fact that the a-posteriori format described in [34] provides an explicit algorithm, which can be implemented numerically in a very efficient way. Each step of the algorithm is denoted as follows: "$a \leftarrow b$" means that the quantity a is assigned by the quantity b.

Algorithm 18. Given $K_0 : \mathbb{T} \to \mathcal{M}$, $\mu_0 \in \mathbb{R}$, we denote by $\lambda \in \mathbb{R}$ the conformal factor for f_{μ_0}. We perform the following computations:

1) $E_0 \leftarrow f_{\mu_0} \circ K_0 - K_0 \circ T_\omega$
2) $\alpha \leftarrow DK_0$
3) $N_0 \leftarrow [\alpha^\top \alpha]^{-1}$
4) $M_0 \leftarrow [\alpha |\, J^{-1} \alpha N_0]$
5) $\beta \leftarrow M_0^{-1} \circ T_\omega$
6) $\widetilde{E}_0 \leftarrow \beta E_0$
7) $P_0 \leftarrow \alpha N_0$
   $S_0 \leftarrow (P_0 \circ T_\omega)^\top Df_{\mu_0} \circ K_0\, J^{-1}\, P_0$
   $\widetilde{A}_0 \leftarrow M_0^{-1} \circ T_\omega\, D_\mu f_{\mu_0} \circ K_0$

Table 1
The analytical estimate ε_{KAM} for the golden mean curve of (2.6) with $\lambda = 0.9$ for different values of the parameter ρ_0 measuring the width of the analyticity domain considered for K.

ρ_0	ε_{KAM}	Agreement with ε_c	μ
10^{-5}	0.97094171	99.89%	0.06139053
$2 \cdot 10^{-5}$	0.97136363	99.93%	0.06139054
$3 \cdot 10^{-5}$	0.97142178	99.94%	0.06139056
$4 \cdot 10^{-5}$	0.97136363	99.93%	0.06139060
$5 \cdot 10^{-5}$	0.97133318	99.93%	0.06139063
$6 \cdot 10^{-5}$	0.97127502	99.92%	0.06139068
$7 \cdot 10^{-5}$	0.97120503	99.92%	0.06139072
$8 \cdot 10^{-5}$	0.97114973	99.91%	0.06139075
$9 \cdot 10^{-5}$	0.97094171	99.89%	0.06139079
10^{-4}	0.97094171	99.89%	0.06139082
$2 \cdot 10^{-4}$	0.97011584	99.80%	0.06139146

8) $(B_{a0})^0$ solves $\quad \lambda(B_{a0})^0 - (B_{a0})^0 \circ T_\omega = -(\widetilde{E}_0^{(2)})^0$
$(B_{b0})^0$ solves $\quad \lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\widetilde{A}_0^{(2)})^0$

9) Find $\overline{W}_0^{(2)}$, σ_0 solving

$$0 = -\overline{S}_0\, \overline{W}_0^{(2)} - \overline{S_0(B_{a0})^0} - \overline{S_0(B_{b0})^0}\sigma_0 - \overline{\widetilde{E}_0^{(1)}} - \overline{\widetilde{A}_0^{(1)}}\sigma_0$$
$$(\lambda - 1)\overline{W}_0^{(2)} = -\overline{\widetilde{E}_0^{(2)}} - \overline{\widetilde{A}_0^{(2)}}\sigma_0\ .$$

10) $(W_0^{(2)})^0 = (B_{a0})^0 + \sigma_0(B_{b0})^0$

11) $W_0^{(2)} = (W_0^{(2)})^0 + \overline{W}_0^{(2)}$

12) $(W_0^{(1)})^0$ solves $(W_0^{(1)})^0 - (W_0^{(1)})^0 \circ T_\omega = -(S_0 W_0^{(2)})^0 - (\widetilde{E}_0^{(1)})^0 - (\widetilde{A}_0^{(1)})^0\sigma_0$

13) $K_1 \leftarrow K_0 + M_0 W_0$
$\mu_0 \leftarrow \mu_0 + \sigma_0$.

Remark 19. We call attention on the fact that steps 2), 8), 10), 11), 12) involve diagonal operations in the Fourier space. On the contrary, the other steps are diagonal in the real space (while steps 10), 11) are diagonal in both spaces). If we represent a function in discrete points or in Fourier space, then we can compute the other functions by applying the Fast Fourier Transform (FFT). This implies that if we use N Fourier modes to discretize the function, then we need $O(N)$ storage and $O(N\log(N))$ operations.

Next task is to translate the procedure described before into a numerical algorithm that computes invariant tori of (2.6). To this end, we fix the frequency equal to the golden ratio:

$$\omega = \frac{\sqrt{5}-1}{2}\ . \tag{5.1}$$

We remark that the golden ratio (5.1) satisfies the Diophantine condition (2.2) with constants $\nu = \frac{2}{3+\sqrt{5}}$, $\tau = 1$.

Then, we start from $(K_0, \mu_0) = (0, 0)$, implement Algorithm 5.1 using Fast Fourier Transforms and perform a continuation method to get an approximation of the invariant circle close to the breakdown value.

To get closer to breakdown, one needs to implement Algorithm 5.1 with a sufficient accuracy. The result described in Section 5.2 is obtained making all computations by means of the GNU MPFR Library using 115 significant digits. We use our own extended precision implementation of the classical radix-2 Cooley–Tukey in [55] by using GNU MPFR. We compute 2^{18} Fourier coefficients to discretize the invariant circle; we ask for a tolerance equal to 10^{-46} in the approximation of the analytic norm (2.1) and of the invariance equation (2.8) to have convergence.

We fix $\lambda = 0.9$ and (by trial and error to optimize the final result) we select the parameters measuring the size of the domain as $\rho_0 = 3 \cdot 10^{-5}$, $\delta_0 = \rho_0/4$. This choice of ρ_0 is taken to optimize the final result. We denote by ε_{KAM} the value of the parameter ε (appearing in (2.6)) after the algorithm has converged to an approximate solution (K, μ); all the estimates of Theorem 10 (precisely (3.3)-(3.4)-(3.5)-(3.6)-(3.7)-(3.8)-(3.9)-(3.10)-(3.11)-(3.12)) have been verified numerically for that approximate solution. Table 1 provides the value of ε_{KAM} obtained with 2^{18} Fourier coefficients for different values of ρ_0. We emphasize that the a-posteriori format of Theorem 10 verifies the solution and does not need to justify how the approximate solution is constructed.

As Table 2 shows, the higher the number of Fourier coefficients, the better is the result, although the execution time becomes longer. We also notice that the improvement is smaller as the number of Fourier coefficients increases; in particular, the results are very similar when taking 2^{17} and 2^{18} Fourier coefficients.

The output of the construction of the approximate solution via the MPRF program is represented by the analytic norms of the following quantities, which will be used to check the conditions (3.3)-(3.12), needed to implement Theorem 10.

Table 2
The analytical estimate ε_{KAM} for the golden mean curve of (2.6) with $\lambda = 0.9$, $\rho_0 = 3 \cdot 10^{-5}$, as the number of Fourier coefficients of the solution increases.

n. Fourier coefficients	ε_{KAM}	μ	Agreement with ε_c	Execution time (s)
2^{13}	0.95730400	0.06140120	98.49%	612.28
2^{14}	0.96512016	0.06139562	99.29%	2015.22
2^{15}	0.96807778	0.06139307	99.60%	3205.34
2^{16}	0.97011583	0.06139161	99.81%	8460.19
2^{17}	0.97094171	0.06139089	99.89%	13375.78
2^{18}	0.97142178	0.06139056	99.94%	38222.48

All quantities are given with 30 decimal digits:

$$
\begin{aligned}
\|M_0\|_{\rho_0} &= 44.927081199027441045214818426 7 \ , \\
\|M_0^{-1}\|_{\rho_0} &= 39.930678840711850152808576113 \ , \\
\|Df_{\mu_0}\|_{\rho_0} &= 5.075500117375219593476390324 33 \ , \\
\|D^2 f_{\mu_0}\|_{\rho_0} &= 12.207407719777848573255701888 3 \ , \\
\|S_0\|_{\rho_0} &= 215.247207629124637162864040 04 \ , \\
\|N_0\|_{\rho_0} &= 156.534312450915756580422752539 \ , \\
\|N_0^{-1}\|_{\rho_0} &= 591.408362768291837018626059244 \ , \\
\|DK_0\|_{\rho_0} &= 44.927081199027441045214818426 7 \ , \\
\|D^2 K_0\|_{\rho_0} &= 221591.876024617607481468301961 \ , \\
\mathcal{T}_0 &= 7.643426562237616735264957751 2 \ , \\
\|E_0\|_{\rho_0} &= 7.716503514518325668474908492 33\, 10^{-36} \ , \\
\|D^2 E_0\|_{\rho_0} &= 5.157630049285180696439553000 6\, 10^{-24} \ .
\end{aligned}
\tag{5.2}
$$

With reference to the quantities in (3.2), we notice that in the case of the dissipative standard map (2.6) we have $Q_{\mu 0} = 1$ and $Q_{z\mu 0} = Q_{\mu\mu 0} = 0$. The computation of such quantities requires instead a major effort in different models, like the dissipative spin–orbit problem, see [45] for details. We stress that the quantity which requires the hardest computation effort is the error E_0 and its derivatives.

5.2. Check of the conditions of Theorem 10 and results

We verify numerically the estimates of the theorem on the existence of the golden mean torus for the dissipative standard map described by equation (2.6) with frequency as in (5.1) and $\lambda = 0.9$. The corresponding breakdown threshold, as computed by means of the Sobolev's method used in [44], or equivalently by means of Greene's technique (see [44,56]), gives

$$\varepsilon_c = 0.97198 \ , \tag{5.3}$$

(compare with [44]). On the other hand, implementing the analytical estimates of Section 4, we obtain that the conditions (3.3)-(3.12), appearing in Theorem 10 are satisfied for a value of the perturbing parameter equal to

$$\varepsilon_{KAM} = 0.971421780429401935547661013138 \ . \tag{5.4}$$

The corresponding value of the drift parameter amounts to

$$\mu = 0.061390559555891469231218991051 \ . \tag{5.5}$$

The result is validated by running the program with different precision on a DELL Machine with an Intel Xeon Processor E5-2643 (Quad Core, 3.30 GHz Turbo, 10MB, 8.0 GT/s) and 16 GB RAM. Precisely, we provide in Table 3 the results with different significant digits.

The results shown in Table 3 suggest that the norms provided in (5.2) are robust and, even if we do not implement interval arithmetic, we can conjecture that the values provided in (5.2) are not affected by numerical errors. Below 50 digits of precision, the algorithm does not produce any result, since some quantities are so small that a precision less than 50 digits is not enough. This remark leads us to the following statement.

Verification of Theorem 10 for the dissipative standard map. Let us consider the map (2.6) with $\lambda = 0.9$. Let $\rho_0 = 3 \cdot 10^{-5}$, $\delta_0 = \rho_0/4$, $\zeta = 3 \cdot 10^{-5}$; let us fix the frequency as $\omega = \frac{\sqrt{5}-1}{2}$. Assume that the norms of M_0, M_0^{-1}, Df_{μ_0}, $D^2 f_{\mu_0}$, S_0, N_0, N_0^{-1}, DK_0, $D^2 K_0$, E_0, $D^2 E_0$ and the twist constant $\mathcal{T}_0$ are given by the values provided in (5.2). Then, there

Table 3
The analytical estimate ε_{KAM} for the golden mean curve of (2.6) with $\lambda = 0.9$, $\rho_0 = 3 \cdot 10^{-5}$, number of Fourier coefficients equal to 2^{18} and for different precision of the computation, obtained varying the number of digits as in the first column.

Digits	ε_{KAM}	Execution time (s)
50	0.97142178	27632.88
60	0.97142178	29027.68
70	0.97142178	30094.44
85	0.97142178	32685.89
100	0.97142178	35390.35
115	0.97142178	38222.48

exists an invariant attractor with frequency ω for $\varepsilon \approx \varepsilon_{KAM}$ with ε_{KAM} as in (5.4) and for a value of the drift parameter as in (5.5).

The result stated before verifies the estimates for ε_{KAM}, which is consistent within 99.94% of the numerical value ε_c given in (5.3).

Of course, the numerical value in (5.3) is based on indirect numerical methods and there is no theory to estimate its error. We also point out that the method used to obtain the approximate solution seems more widely applicable than the numerical method. The Greene's method is difficult to make work for standard maps with two or more frequencies, whereas the continuation method works without any problem.

This result shows that, beside a world-wide recognized theoretical interest, KAM theory can also provide a constructive effective algorithm to estimate the breakdown value with great accuracy. We also point out that computing close to the breakdown is not just a challenge for numerics. The breakdown of KAM tori is known to be the source of very interesting mathematical problems. Notably, the very deep renormalization group for quasi-periodic problems was discovered by numerical experiments [49,57].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Lemma 6

In this appendix, we include the proof of Lemma 6. In the proof we follow the construction of [58,59] to derive the constant C_0 in (2.5).

Proof. For the proof of the existence of the solution of (2.3) we refer to [34]; here we provide an explicit estimate of C_0. To this end, let us expand φ and η in Fourier series as

$$\varphi(\theta) = \sum_{k\in\mathbb{Z}\backslash\{0\}} \hat{\varphi}_k e^{2\pi ik\theta} \ , \qquad \eta(\theta) = \sum_{k\in\mathbb{Z}\backslash\{0\}} \hat{\eta}_k e^{2\pi ik\theta} \ ,$$

where $\hat{\varphi}_k$, $\hat{\eta}_k$ denote the Fourier coefficients. Note that, since we are assuming $\int \eta(\theta) = d\theta = 0$, we do not need to deal with averages. Then, equation (2.3) becomes

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}_k (e^{2\pi ik\omega} - \lambda) e^{2\pi ik\theta} = \sum_{k\in\mathbb{Z}} \hat{\eta}_k e^{2\pi ik\theta} \ ; \tag{A.1}$$

computing the coefficients $\hat{\varphi}_k$ from (A.1) and adding the Fourier terms, one obtains:

$$\varphi(\theta) = \sum_{k\in\mathbb{Z}} \frac{\hat{\eta}_k}{e^{2\pi ik\omega} - \lambda} \ e^{2\pi ik\theta} \ .$$

Let $Z_k \equiv \min_{q\in\mathbb{Z}} |\omega\, k - q|$; then, using $\sin(x)/x \geq 2/\pi$ for all $0 < x < \frac{\pi}{2}$, we have the following inequality

$$\begin{aligned} |e^{2\pi ik\omega} - \lambda|^2 &= (1-\lambda)^2 \cos^2(\pi k\omega) + (1+\lambda)^2 \sin^2(\pi k\omega) \\ &\geq (1+\lambda)^2 \sin^2(\pi k\omega) \geq 4(1+\lambda)^2 Z_k^2 \ . \end{aligned}$$

Therefore we obtain:

$$|e^{2\pi ik\omega} - \lambda| \geq 2(1+\lambda)|Z_k| \geq 2(1+\lambda)\nu|k|^{-\tau} \ .$$

Finally, using the estimate $\sum_{k\in\mathbb{Z}} |\hat{\eta}_k|^2 e^{4\pi\rho|k|} \le 2\|\eta\|_\rho^2$ (see [58]) and setting $F(\delta) \equiv 4 \sum_{k=1}^{\infty} \frac{e^{-4\pi\delta|k|}}{|e^{2\pi ik\omega}-\lambda|^2}$, one has

$$\begin{aligned}\|\varphi\|_{\rho-\delta} &\le \sum_{k\in\mathbb{Z}} |\hat{\eta}_k| e^{2\pi\rho|k|} \frac{e^{-2\pi\delta|k|}}{|e^{2\pi ik\omega}-\lambda|}\\ &\le \sqrt{\sum_{k\in\mathbb{Z}} |\hat{\eta}_k|^2 e^{4\pi\rho|k|}} \sqrt{\sum_{k\in\mathbb{Z}} \frac{e^{-4\pi\delta|k|}}{|e^{2\pi ik\omega}-\lambda|^2}}\\ &\le \|\eta\|_\rho \sqrt{F(\delta)} ,\end{aligned}$$

where in the last inequality we used Parseval's identity $\sum_{k\in\mathbb{Z}} |\hat{\eta}_k|^2 e^{\pm 4\pi\rho|k|} = \int_0^1 |\eta(\theta \mp i\rho)|^2 d\theta \le \|\eta\|_\rho^2$ and that the function is real $|\hat{\eta}_{-k}| = |\eta_k^*|$. Denoting by Γ the Euler gamma function, using the estimates of [59], one has that

$$F(\delta) \le \frac{\pi^2 \Gamma(2\tau+1)}{3\nu^2(1+\lambda)^2(2\delta)^{2\tau}(2\pi)^{2\tau}} ,$$

which leads to (2.4) with C_0 as in (2.5). □

Appendix B. Proof of Proposition 17

The proof of Proposition 17 proceeds by induction. We start by noticing that $(p1;0)$, $(p2;0)$, $(p3;0)$ are trivial. Let $H \in \mathbb{Z}_+$ and assume that $(p1;h)$, $(p2;h)$, $(p3;h)$ are true for $h = 1,\dots,H$. Then, by Lemma 15 we obtain the Taylor estimate

$$\varepsilon_h = \|\mathcal{E}(K_{h-1}+\Delta_{h-1}, \mu_{h-1}+\sigma_{h-1})\|_{\rho_h} \le 2C_{\mathcal{E}0}\nu^{-2}\delta_{h-1}^{-2\tau}\varepsilon_{h-1}^2 , \tag{B.1}$$

using $C_{\mathcal{E},h-1} \le 2C_{\mathcal{E}0}$ due to $(p3;h)$ for $h = 1,\dots,H$. The estimate (B.1) allows to have a bound of ε_h, $h = 1,\dots,H$, in terms of ε_0:

$$\begin{aligned}\varepsilon_h &\le 2C_{\mathcal{E}0}\ \nu^{-2}\delta_0^{-2\tau}\ 2^{2\tau(h-1)}\varepsilon_{h-1}^2\\ &\le (2C_{\mathcal{E}0}\nu^{-2}\delta_0^{-2\tau})^{1+2+\dots+2^{h-1}}\ 2^{2\tau((h-1)+2(h-2)+\dots+2^{h-2})}\varepsilon_0^{2^h}\\ &\le (2C_{\mathcal{E}0}\nu^{-2}\delta_0^{-2\tau})^{2^h-1}\ 2^{2\tau(2^h-(h+1))}\varepsilon_0^{2^h}\\ &\le (2C_{\mathcal{E}0}\nu^{-2}\delta_0^{-2\tau}2^{2\tau}\varepsilon_0)^{2^h-1}\varepsilon_0 .\end{aligned}$$

In Sections Appendix B.1, B.2, B.3, we will prove $(p1;H+1)$, $(p2;H+1)$, $(p3;H+1)$, under the induction assumptions $(p1;h)$, $(p2;h)$, $(p3;h)$ for $h = 1,\dots,H$. To get such result, we need the following Lemma.

Lemma 20. *Assume that $(p1;h)$, $(p2;h)$, $(p3;h)$ hold for $h = 1,\dots,H$. For $H \in \mathbb{Z}_+$ the following inequality holds:*

$$\|DK_{H+1} - DK_0\|_{\rho_{H+1}} \le D_K , \tag{B.2}$$

where D_K is as in (3.13), C_{d0} as in (4.14) and provided that

$$2^{\tau+1}\kappa_0\varepsilon_0 \le \frac{1}{2} \tag{B.3}$$

with κ_0 as in (4.17). Furthermore, under the inequality:

$$\|N_0\|_{\rho_0} (2\|DK_0\|_{\rho_0} + D_K)\, D_K < 1 , \tag{B.4}$$

the following relations hold for $0 \le h \le H+1$:

$$\|N_h - N_0\|_{\rho_h} \le C_N D_K , \tag{B.5}$$

$$\|M_h - M_0\|_{\rho_h} \le C_M D_K , \tag{B.6}$$

$$\|M_h^{-1} - M_0^{-1}\|_{\rho_h} \le C_{Minv} D_K , \tag{B.7}$$

where C_N, C_M, C_{Minv} are defined as follows:

$$\begin{aligned}C_N &\equiv \|N_0\|_{\rho_0}^2 \frac{2\|DK_0\|_{\rho_0} + D_K}{1 - \|N_0\|_{\rho_0} D_K (2\|DK_0\|_{\rho_0} + D_K)} ,\\ C_M &\equiv 1 + J_e\Big[C_N\Big(\|DK_0\|_{\rho_0} + D_K\Big) + \|N_0\|_{\rho_0}\Big] ,\\ C_{Minv} &\equiv C_N(\|DK_0\|_{\rho_0} + D_K) + \|N_0\|_{\rho_0} + J_e .\end{aligned} \tag{B.8}$$

Proof. We start by proving (B.2), under (B.3) and with D_K as in (3.13):

$$\|DK_{H+1} - DK_0\|_{\rho_{H+1}} \le \sum_{j=0}^{H} v_j \le \sum_{j=0}^{H} C_{dj}\nu^{-1}\delta_j^{-\tau-1}\varepsilon_j \le D_K \ . \tag{B.9}$$

The proof of (B.5) is obtained as follows. From the relations

$$DK_h = DK_0 + \widetilde{K}_h \ , \qquad DK_h^\top = DK_0^\top + \widetilde{K}_h^\top \ , \qquad \widetilde{K}_h = \sum_{j=0}^{h-1} D\Delta_j \ ,$$

we obtain

$$\begin{aligned} N_h &= (DK_h^\top DK_h)^{-1} = \Big((DK_0^\top + \widetilde{K}_h^\top)(DK_0 + \widetilde{K}_h)\Big)^{-1} \\ &= (DK_0^\top DK_0)^{-1}\Big(1 + (DK_0^\top DK_0)^{-1}(\widetilde{K}_h^\top DK_0 + DK_0^\top \widetilde{K}_h + \widetilde{K}_h^\top \widetilde{K}_h)\Big)^{-1} \\ &= N_0(1+\chi_h)^{-1} \ , \end{aligned}$$

having set $\chi_h \equiv N_0(\widetilde{K}_h^\top DK_0 + DK_0^\top \widetilde{K}_h + \widetilde{K}_h^\top \widetilde{K}_h)$. Under the inequality (B.4), ensuring that $\|\chi_h\|_{\rho_h} < 1$ and using (B.9), we have the following bound:

$$\|(1+\chi_h)^{-1} - 1\| \le \frac{\|\chi_h\|}{1 - \|\chi_h\|} \ ,$$

which leads to the following relation with C_N as in (B.8):

$$\|N_h - N_0\|_{\rho_h} \le \|N_0\|_{\rho_0}\ \|(1+\chi_h)^{-1} - 1\|_{\rho_h} \le \|N_0\|_{\rho_0}\ \frac{\|\chi_h\|_{\rho_h}}{1 - \|\chi_h\|_{\rho_h}} \le C_N\ D_K \ . \tag{B.10}$$

The proof of (B.6) is obtained starting from the identity

$$M_h - M_0 = (DK_h - DK_0 \mid J^{-1}\ DK_h\ N_h - J^{-1}\ DK_0\ N_0) \ .$$

Then, one has

$$\|M_h - M_0\|_{\rho_h} \le \|DK_h - DK_0\|_{\rho_h} + J_e\ \|DK_hN_h - DK_0N_0\|_{\rho_h} \ .$$

From

$$DK_hN_h - DK_0N_0 = DK_hN_h - DK_hN_0 + DK_hN_0 - DK_0N_0$$

and from (B.10), we obtain that

$$\|DK_hN_h - DK_0N_0\|_{\rho_h} \le (C_N\|DK_h\|_{\rho_h} + \|N_0\|_{\rho_0})\ D_K \ . \tag{B.11}$$

Bounding $\|DK_h\|_{\rho_h}$ as $\|DK_h\|_{\rho_h} \le \|DK_0\|_{\rho_0} + \|DK_h - DK_0\|_{\rho_h}$, we obtain (B.6).

The proof of (B.7) is obtained as follows. Given that the symplectic form is the standard one, the inverse of the matrix M_h is

$$M_h^{-1}(\theta) = \Big(DK_h\ N_h^\top \,|\, J^\top\ DK_h\Big)^\top = \begin{pmatrix} N_hDK_h^\top \\ DK_h^\top J \end{pmatrix} .$$

Indeed, one can verify that due to the Lagrangian character, $M_h^{-1}M_h = \mathrm{Id}$. By computing the inverse of M_0 in an analogous way, one has

$$M_h^{-1} - M_0^{-1} = \begin{pmatrix} N_hDK_h^\top - N_0DK_0^\top \\ DK_h^\top J - DK_0^\top J \end{pmatrix} \ .$$

The bound for the first row $N_hDK_h^\top - N_0DK_0^\top$ is obtained as in (B.11), while the second row is bounded by $J_e\ D_K$. This yields (B.7) with C_{Minv} as in (B.8). □

We are now in the position to continue with the proof of $(p1; H+1)$, $(p2; H+1)$, $(p3; H+1)$ to which we devote the rest of this Section.

B.1. *Proof of* $(p1; H+1)$

Using the inequality $j+1 \le 2^j$, one has

$$\|K_{H+1} - K_0\|_{\rho_{H+1}} \le \sum_{j=0}^{H} d_j \le \sum_{j=0}^{H}(C_{dj}\nu^{-1}\delta_j^{-\tau}\varepsilon_j) \le 4C_{d0}\nu^{-1}\delta_0^{-\tau}\varepsilon_0 \ ;$$

assuming that ε_0 satisfies (4.19), we obtain the first inequality in (4.27). Moreover, we have:

$$|\mu_{H+1}-\mu_0| \le \sum_{j=0}^{H} s_j \le \sum_{j=0}^{H} C_{\sigma_j}\varepsilon_j \le 2C_{\sigma 0}\ \sum_{j=0}^{H}(\kappa_0\varepsilon_0)^{2^j-1}\varepsilon_0\ ;$$

assuming that ε_0 satisfies (4.18) and (4.20), we obtain the second inequality in (4.27), hence we obtain $(p1; H+1)$.

B.2. Proof of $(p2; H+1)$

Having proven $(p1; H+1)$, we use the Taylor estimate (B.1) with $H+1$ in place of h to obtain $(p2; H+1)$:

$$\varepsilon_{H+1} \le (2C_{\mathcal{E}0}\nu^{-2}\delta_0^{-2\tau}2^{2\tau}\varepsilon_0)^{2^{H+1}-1}\varepsilon_0 = (\kappa_0\varepsilon_0)^{2^{H+1}-1}\varepsilon_0\ ,$$

due to the definition of κ_0 in (4.17).

B.3. Proof of $(p3; H+1)$

The proof of $(p3; H+1)$ is rather cumbersome and needs several auxiliary results. Given the inductive assumption, we want to prove that

$$C_{d,H+1} \le 2C_{d0}\ , \qquad C_{\sigma,H+1} \le 2C_{\sigma 0}\ , \qquad C_{\mathcal{E},H+1} \le 2C_{\mathcal{E}0}\ . \tag{B.12}$$

B.3.1. Estimate on $|\mathcal{T}_h - \mathcal{T}_0|$

We start with the following result.

Lemma 21. *Assume that* $(p1; h)$, $(p2; h)$, $(p3; h)$ *hold for* $h = 1, \dots, H$ *and that the condition* (B.4) *of Lemma 20 is valid together with*

$$4Q_{z\mu 0}C_{\sigma 0}\varepsilon_0 < Q_0\ , \qquad 4Q_{\mu\mu 0}C_{\sigma 0}\varepsilon_0 < Q_{\mu 0}\ .$$

Let τ_0 *and* τ_h *be defined as*

$$\tau_0 \equiv \begin{pmatrix} \overline{S}_0 & \overline{S_0(B_{b0})^0} + \overline{\widetilde{A}_0^{(1)}} \\ \lambda-1 & \overline{\widetilde{A}_0^{(2)}} \end{pmatrix}^{-1}$$

and

$$\tau_h \equiv \begin{pmatrix} \overline{S}_h & \overline{S_h(B_{bh})^0} + \overline{\widetilde{A}_h^{(1)}} \\ \lambda-1 & \overline{\widetilde{A}_h^{(2)}} \end{pmatrix}^{-1}$$

and let $\mathcal{T}_0 \equiv \|\tau_0\|_{\rho_0}$, $\mathcal{T}_h \equiv \|\tau_h\|_{\rho_h}$. *For* $h \in \mathbb{N}$, $h = 1, \dots, H$, *the following inequality holds:*

$$|\mathcal{T}_h - \mathcal{T}_0| \le C_T D_K\ , \tag{B.13}$$

where C_T *is defined as*

$$C_T \equiv \frac{\mathcal{T}_0^2}{1-\mathcal{T}_0 C_\tau}\ \max\Big\{C_S, C_{SB}+2C_{Minv}Q_{\mu 0}\Big\} \tag{B.14}$$

with

$$\begin{aligned}
C_S \ &\equiv 2J_e\ Q_0\ \Big\{\Big(\|N_0\|_{\rho_0}+C_N D_K\Big)\ \Big[D_K(\|N_0\|_{\rho_0}+C_N D_K) \\
&\quad +\ \|DK_0\|_{\rho_0}\|N_0\|_{\rho_0}+\|DK_0\|_{\rho_0}C_N D_K\Big] + C_N\ \|DK_0\|_{\rho_0}\ \Big[D_K(\|N_0\|_{\rho_0}+C_N D_K) \\
&\quad +\ \|DK_0\|_{\rho_0}\|N_0\|_{\rho_0}+\|DK_0\|_{\rho_0}C_N D_K\Big] + \|N_0\|_{\rho_0}\|DK_0\|_{\rho_0}(\|N_0\|_{\rho_0}+C_N D_K) \\
&\quad +\ C_N\|N_0\|_{\rho_0}\|DK_0\|^2_{\rho_0}\Big\}\ , \\
C_{SB} \ &\equiv \frac{1}{||\lambda|-1|}Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_S + 2J_e Q_0\ \|N_0\|^2_{\rho_0}\ \|DK_0\|^2_{\rho_0}\frac{1}{||\lambda|-1|}\ C_{Minv}\ Q_{\mu 0} \\
&\quad +\ 2C_S\ \frac{1}{||\lambda|-1|}\ C_{Minv}\ Q_{\mu 0}\ D_K\ , \\
C_\tau \ &\equiv \max\Big\{C_S, C_{SB}+2C_{Minv}Q_{\mu 0}\Big\}\ D_K\ .
\end{aligned} \tag{B.15}$$

Proof. From

$$\mathcal{T}_h \le \|\tau_0\|_{\rho_0} + \|\widetilde{\tau}_h\|_{\rho_h} = \mathcal{T}_0 + \|\widetilde{\tau}_h\|_{\rho_h} \ ,$$

where $\widetilde{\tau}_h \equiv \tau_h - \tau_0 = \tau_0^2 \left[\left(I + \tau_0(\tau_h^{-1} - \tau_0^{-1})\right)^{-1} (\tau_0^{-1} - \tau_h^{-1})\right]$, we have the estimates

$$|\mathcal{T}_h - \mathcal{T}_0| \le \|\widetilde{\tau}_h\|_{\rho_h} \le \frac{\mathcal{T}_0^2}{1 - \mathcal{T}_0 C_\tau} C_\tau \ , \tag{B.16}$$

where C_τ is a bound on $\tau_h^{-1} - \tau_0^{-1}$, say

$$\|\tau_h^{-1} - \tau_0^{-1}\|_{\rho_h} \equiv \left\| \begin{pmatrix} \overline{S}_h - \overline{S}_0 & \overline{S_h(B_{bh})^0} + \overline{\widetilde{A}_h^{(1)}} - (\overline{S_0(B_{b0})^0} + \overline{\widetilde{A}_0^{(1)}}) \\ 0 & \overline{\widetilde{A}_h^{(2)}} - \overline{\widetilde{A}_0^{(2)}} \end{pmatrix} \right\|_{\rho_h} \le C_\tau \ . \tag{B.17}$$

To obtain an expression for C_τ, we bound term by term the matrix appearing in (B.17).

We start with $\|\overline{S}_h - \overline{S}_0\|_{\rho_h}$. From (3.1) we have that S_h is defined by

$$S_h = N_h(\theta+\omega)^\top DK_h(\theta+\omega)^\top \ Df_{\mu_h} \circ K_h(\theta)\, J^{-1}\ DK_h(\theta) N_h(\theta) \ .$$

Then, we bound $Df_{\mu_h} \circ K_h$ as

$$\begin{aligned} \sup_{z\in\mathcal{C}} |Df_{\mu_h}(z)| &\le \sup_{z\in\mathcal{C}} |Df_{\mu_0}(z)| + \sup_{z\in\mathcal{C}} |Df_{\mu_h}(z) - Df_{\mu_0}(z)| \\ &\le Q_0 + \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu Df_\mu(z)|\ |\mu_h - \mu_0| \\ &\le Q_0 + 4Q_{z\mu 0}\ C_{\sigma 0}\varepsilon_0 \le 2Q_0 \ , \end{aligned}$$

if (4.25) holds. Notice that we have used $(p1; H+1)$ to bound $\mu_h - \mu_0$ for $h = 1, \dots, H+1$. Finally, we obtain

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} \le\ & 2Q_0\ \|\ N_h(\theta+\omega)^\top DK_h(\theta+\omega)^\top\ J_e^{-1}\ DK_h(\theta) N_h(\theta) \\ & - N_0(\theta+\omega)^\top DK_0(\theta+\omega)^\top\ J_e^{-1}\ DK_0(\theta) N_0(\theta)\ \|_{\rho_h} \ . \end{aligned}$$

Setting $\widetilde{N}_h = N_h - N_0$ and writing DK_h as $DK_h = DK_h - DK_0 + DK_0$, one obtains

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} \le\ & 2Q_0\ \|\ (N_0 + \widetilde{N}_h)(\theta+\omega)^\top (DK_h - DK_0 + DK_0)(\theta+\omega)^\top \\ & J_e^{-1}\ (DK_h - DK_0 + DK_0)(\theta)(N_0 + \widetilde{N}_h)(\theta) \\ & - N_0(\theta+\omega)^\top DK_0(\theta+\omega)^\top\ J_e^{-1}\ DK_0(\theta) N_0(\theta)\ \|_{\rho_h} \ . \end{aligned}$$

Let us bound $\widetilde{N}_h$ using (B.5). Then, using that J is a constant matrix, we have:

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} \le\ & 2Q_0\ \|\ [((N_0 + \widetilde{N}_h) \circ T_\omega)^\top\ ((DK_h - DK_0) \circ T_\omega)^\top \\ & + (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top + (\widetilde{N}_h \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top]\, J_e^{-1} \\ & [(DK_h - DK_0)(N_0 + \widetilde{N}_h) + DK_0 N_0 + DK_0 \widetilde{N}_h] \\ & - (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top\ J_e^{-1}\ DK_0 N_0\ \|_{\rho_h} \\ \le\ & 2J_e\ Q_0\ \Big\{ (\|N_0\|_{\rho_0} + \|\widetilde{N}_h\|_{\rho_h})\ \|DK_h - DK_0\|_{\rho_h} \\ & \Big[\|DK_h - DK_0\|_{\rho_h}(\|N_0\|_{\rho_0} + \|\widetilde{N}_h\|_{\rho_h}) + \|DK_0\|_{\rho_0}\|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0}\|\widetilde{N}_h\|_{\rho_h}\Big] \\ & + \|\widetilde{N}_h\|_{\rho_h}\|DK_0\|_{\rho_0} \Big[\ \|DK_h - DK_0\|_{\rho_h}(\|N_0\|_{\rho_0} + \|\widetilde{N}_h\|_{\rho_h}) \\ & + \|DK_0\|_{\rho_0}\|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0}\|\widetilde{N}_h\|_{\rho_h}\ \Big] \\ & + \|N_0\|_{\rho_0}\|DK_0\|_{\rho_0}\|DK_h - DK_0\|_{\rho_h}\ (\|N_0\|_{\rho_0} + \|\widetilde{N}_h\|_{\rho_h}) \\ & + \|N_0\|_{\rho_0}\|DK_0\|_{\rho_0}^2\|\widetilde{N}_h\|_{\rho_h}\ \Big\} \ . \end{aligned}$$

Taking into account (B.5), (B.8), we obtain:

$$\|S_h - S_0\|_{\rho_h} \le C_S\ D_K \tag{B.18}$$

with C_S as in (B.15). Now we bound the upper right element of the matrix appearing in (B.17). This computation will lead us also to bound the lower right element of the matrix in (B.17). We start from (see (4.6))

$$\widetilde{A}_h = M_h^{-1} \circ T_\omega\ D_\mu f_{\mu_h} \circ K_h \ , \qquad \widetilde{A}_0 = M_0^{-1} \circ T_\omega\ D_\mu f_{\mu_0} \circ K_0$$

and the following estimate that uses (4.26):

$$\sup_{z\in\mathcal{C}} |D_\mu f_{\mu_h}(z)| \le Q_{\mu 0} + 4Q_{\mu\mu 0}\, C_{\sigma_0}\varepsilon_0 \le 2Q_{\mu 0}\ .$$

Then, we have:

$$\|\widetilde{A}_h - \widetilde{A}_0\|_{\rho_h} \le 2Q_{\mu 0}\ \|M_h^{-1} - M_0^{-1}\|_{\rho_h} \le 2C_{Minv}\ Q_{\mu 0}\ D_K\ .$$

Next we estimate $\|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\|_{\rho_h}$; recall that from (4.8) we have that $(B_{bh})^0$ and $(B_{b0})^0$ are the solutions of

$$\lambda(B_{bh})^0 - (B_{bh})^0\circ T_\omega = -(\widetilde{A}_h^{(2)})^0\ ,\qquad \lambda(B_{b0})^0 - (B_{b0})^0\circ T_\omega = -(\widetilde{A}_0^{(2)})^0\ . \tag{B.19}$$

Expanding (B.19) in Fourier series and equating the coefficients, we obtain $(B_{bh})^0$ and $(B_{b0})^0$ leading to

$$(B_{bh})^0(\theta) - (B_{b0})^0(\theta) = -\sum_{j\in\mathbb{Z}} \frac{(\widehat{A}_h^{(2)})_j^0 - (\widehat{A}_0^{(2)})_j^0}{\lambda - e^{2\pi i j\omega}}\ e^{2\pi i j\theta}\ . \tag{B.20}$$

From (B.20), let us write $(B_{bh})^0$ as

$$(B_{bh})^0 = (B_{b0})^0 + \widetilde{B}_h\ ,$$

where

$$\widetilde{B}_h \equiv -\sum_{j\in\mathbb{Z}} \frac{(\widehat{A}_h^{(2)})_j^0 - (\widehat{A}_0^{(2)})_j^0}{\lambda - e^{2\pi i j\omega}}\ e^{2\pi i j\theta}\ .$$

Let us introduce $\widetilde{S}_h \equiv S_h - S_0$, whose norm can be bounded by (B.18). Then, we have:

$$\begin{aligned}
\|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\| &= \|\overline{(S_0+\widetilde{S}_h)\,((B_{b0})^0+\widetilde{B}_h)} - \overline{S_0(B_{b0})^0}\| \\
&\le \|(B_{b0})^0\|_{\rho_0}\|\widetilde{S}_h\|_{\rho_h} + \|S_0\|_{\rho_0}\|\widetilde{B}_h\|_{\rho_h} + \|\widetilde{S}_h\|_{\rho_h}\|\widetilde{B}_h\|_{\rho_h}\ ,
\end{aligned}$$

where

$$\begin{aligned}
\|S_0\|_{\rho_0} &\le J_e\ Q_0\|N_0\|^2_{\rho_0}\ \|DK_0\|^2_{\rho_0}\ ,\\
\|\widetilde{S}_h\|_{\rho_h} &\le C_S\ D_K\ ,\\
\|(B_{b0})^0\|_{\rho_0} &\le \frac{1}{||\lambda|-1|}\|\widetilde{A}_0^{(2)}\|_{\rho_0} \le \frac{1}{||\lambda|-1|}Q_{\mu 0}\ \|M_0^{-1}\|_{\rho_0}\ ,\\
\|\widetilde{B}_h\|_{\rho_h} &\le \frac{1}{||\lambda|-1|}\ 2C_{Minv}\ Q_{\mu 0}\ D_K\ .
\end{aligned}$$

Then, we obtain:

$$\|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\| \le C_{SB}\ D_K\ ,$$

where C_{SB} is as in (B.15). Recalling (B.17), we obtain

$$\|\tau_h^{-1} - \tau_0^{-1}\|_{\rho_h} \le \max\Big\{\|\bar{S}_h - \bar{S}_0\|_{\rho_h},\ \|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\|_{\rho_h} + \sum_{j=1}^{2}\|\overline{\widetilde{A}_h^{(j)}} - \overline{\widetilde{A}_0^{(j)}}\|_{\rho_h}\Big\} \equiv C_\tau\ ,$$

where C_τ is as in (B.15). From (B.16) we get (B.13). □

B.3.2. Proof of $C_{\sigma,H+1} \le 2C_{\sigma 0}$

We now prove (B.12) and we begin from the second inequality. We start with the following relations, which are a consequence of (4.15):

$$\begin{aligned}
C_{\sigma,H+1} &= \mathcal{T}_{H+1}\Big[|\lambda-1|\Big(\frac{1}{||\lambda|-1|}\|S_{H+1}\|_{\rho_{H+1}} + 1\Big) + \|S_{H+1}\|_{\rho_{H+1}}\Big]\ \|M_{H+1}^{-1}\|_{\rho_{H+1}}\ ,\\
C_{\sigma 0} &= \mathcal{T}_0\Big[|\lambda-1|\ \Big(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0} + 1\Big) + \|S_0\|_{\rho_0}\Big]\ \|M_0^{-1}\|_{\rho_0}
\end{aligned}$$

with

$$\|M_{H+1}^{-1}\|_{\rho_{H+1}} \le \|M_0^{-1}\|_{\rho_0} + \|M_{H+1}^{-1} - M_0^{-1}\|_{\rho_{H+1}} \le \|M_0^{-1}\|_{\rho_0} + C_{Minv}\ D_K$$

with C_{Minv} as in (B.8). We also have

$$\|S_{H+1}\|_{\rho_{H+1}} \le \|S_0\|_{\rho_0} + \|S_{H+1} - S_0\|_{\rho_{H+1}} \le \|S_0\|_{\rho_0} + C_S\ D_K$$

with C_S as in (B.15). From the relation

$$\mathcal{T}_{H+1} = \mathcal{T}_0 + (\mathcal{T}_{H+1} - \mathcal{T}_0) \leq \mathcal{T}_0 + C_T\ D_K$$

with C_T as in (B.14), we obtain:

$$\begin{aligned} C_{\sigma,H+1} &\leq (\mathcal{T}_0 + C_T\ D_K)\Big\{|\lambda - 1|\ \Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big] \\ &\quad + \Big(\|S_0\|_{\rho_0} + C_S D_K\Big)\Big\}\ \Big(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K\Big) \\ &= C_{\sigma 0} + C_\sigma D_K \leq 2C_{\sigma 0}\ , \end{aligned}$$

if (4.22) holds with

$$\begin{aligned} C_\sigma &\equiv C_T\ \Big\{|\lambda - 1|\ \Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big] \\ &\quad + \Big(\|S_0\|_{\rho_0} + C_S D_K\Big)\Big\}\ \Big(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K\Big) \\ &\quad + \mathcal{T}_0\ \Big\{|\lambda - 1|\Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big]C_{Minv} + C_{Minv}\|S_0\|_{\rho_0} \\ &\quad + |\lambda - 1|\ \frac{1}{||\lambda| - 1|}\ \|M_0^{-1}\|_{\rho_0}\, C_S + C_S\Big(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K\Big)\Big\}\ . \end{aligned} \tag{B.21}$$

B.3.3. Proof of $C_{\mathcal{E},H+1} \leq 2C_{\mathcal{E}0}$

Recall that $\delta_{H+1} = \frac{\delta_0}{2^{H+1}}$ and that from (4.16), one has

$$C_{\mathcal{E},H+1} \equiv C_{W,H+1}\nu\delta_{H+1}^{-1+\tau} + C_{\mathcal{R},H+1}\ .$$

First, it suffices to prove that

$$C_{W,H+1} \leq C_{W0} + C_W\ D_K \tag{B.22}$$

for a constant C_W as in (B.26) below. From (4.15), for $C_{W_2,H+1}$ we have:

$$C_{W_2,H+1} \leq \frac{1}{||\lambda| - 1|}\ \ \Big[1 + 2Q_{\mu 0}(C_{\sigma 0} + D_K C_\sigma)\Big](\|M_0^{-1}\|_{\rho_0} + D_K) \leq C_{W_2 0} + D_K\ C_{W_2}\ ,$$

where

$$C_{W_2} \equiv \frac{1}{||\lambda| - 1|}\ \Big[1 + 2Q_{\mu 0}\|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu 0} C_{\sigma 0} + 2Q_{\mu 0} C_\sigma D_K\Big]\ . \tag{B.23}$$

Concerning $\overline{C}_{W_2,H+1}$, we have:

$$\begin{aligned} \overline{C}_{W_2,H+1} &\leq 4(\mathcal{T}_0 + C_T D_K)\ \Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big]\ Q_{\mu 0}\ (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\ &= \overline{C}_{W_2 0} + \overline{C}_{W_2} D_K \end{aligned}$$

with

$$\begin{aligned} \overline{C}_{W_2} &\equiv 4C_T\ \Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big]\ Q_{\mu 0}(\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\ &\quad + 4\mathcal{T}_0 Q_{\mu 0}\ \frac{1}{||\lambda| - 1|} C_S\ (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\ &\quad + 4\mathcal{T}_0\ Q_{\mu 0}\Big[\frac{1}{||\lambda| - 1|}(\|S_0\|_{\rho_0} + C_S D_K) + 1\Big](D_K + 2\|M_0^{-1}\|_{\rho_0})\ . \end{aligned} \tag{B.24}$$

As for $C_{W_1,H+1}$ we have:

$$\begin{aligned} C_{W_1,H+1} &\leq C_0\ \Big[(\|S_0\|_{\rho_0} + C_S D_K)(C_{W_2 0} + C_{W_2} D_K + \overline{C}_{W_2 0} + \overline{C}_{W_2} D_K) + \|M_0^{-1}\|_{\rho_0} \\ &\quad + D_K + 2Q_{\mu 0}(\|M_0^{-1}\|_{\rho_0} + D_K)(C_{\sigma 0} + D_K C_\sigma)\Big] = C_{W_1 0} + D_K C_{W_1}\ , \end{aligned}$$

where

$$\begin{aligned} C_{W_1} &\equiv C_0\Big[\|S_0\|_{\rho_0} C_{W_2} + C_S C_{W_2 0} + C_S C_{W_2} D_K + \|S_0\|_{\rho_0}\overline{C}_{W_2} + C_S\overline{C}_{W_2 0} + C_S\overline{C}_{W_2} D_K + 1 \\ &\quad + 2Q_{\mu 0}\|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu 0} C_{\sigma 0} + 2Q_{\mu 0} C_\sigma D_K\Big]\ . \end{aligned} \tag{B.25}$$

In conclusion, from (4.15) we have:

$$\begin{aligned} C_{W,H+1} &\equiv (C_{W_10}+D_K C_{W_1})+(C_{W_20}+D_K C_{W_2}+\overline{C}_{W_20}+D_K\overline{C}_{W_2})\nu\delta_0^\tau 2^{-\tau(H+1)} \\ &\le C_{W0}+C_W D_K \end{aligned}$$

with

$$C_W \equiv C_{W_1}+C_{W_2}\nu\delta_0^\tau+\overline{C}_{W_2}\nu\delta_0^\tau\ . \tag{B.26}$$

In order to get $C_{\mathcal{E},H+1}$ as in (4.16), we estimate $C_{\mathcal{R},H+1}$. To this end, we use the following inequality:

$$Q_{E,H+1}\le Q_{E0}+C_Q D_{2K} \tag{B.27}$$

for a suitable constant C_Q that will be given later in (B.34) and for D_{2K} defined as

$$D_{2K}\equiv 4C_{d0}\ \nu^{-1}\delta_0^{-\tau-2}\ \varepsilon_0\ . \tag{B.28}$$

We postpone for a moment the proof of (B.27) and we rather stress that, as a consequence of (B.27), we obtain:

$$C_{\mathcal{R},H+1}\le Q_{E,H+1}\ (\|M_{H+1}\|^2_{\rho_{H+1}}\ C^2_{W,H+1}+C^2_{\sigma,H+1}\nu^2\delta^{2\tau}_{H+1})\le C_{\mathcal{R}0}+C_{\mathcal{R}}D_K\ ,$$

where

$$\begin{aligned} C_{\mathcal{R}} \equiv\ & Q_{E0}\Big[(2C_M\|M_0\|_{\rho_0}+C_M^2D_K)(C_{W0}+C_WD_K)^2+\|M_0\|^2_{\rho_0}(C_W^2D_K+2C_{W0}C_W) \\ &+(C_\sigma^2D_K+2C_{\sigma0}C_\sigma)\nu^2\delta_0^{2\tau}\Big]+C_Q\Big[(\|M_0\|_{\rho_0}+C_MD_K)^2(C_{W0}+C_WD_K)^2 \\ &+(C_{\sigma0}+C_\sigma D_K)^2\nu^2\delta_0^{2\tau}\Big]\ \delta_0^{-1}\ . \end{aligned} \tag{B.29}$$

We obtain that, under (4.23):

$$C_{\mathcal{E},H+1}\le (C_{W0}+C_WD_K)\nu\delta_0^{-1+\tau}2^{-(-1+\tau)(H+1)}+C_{\mathcal{R}0}+C_{\mathcal{R}}D_K\le 2C_{\mathcal{E}0}\ .$$

Let us conclude by proving (B.27) starting from the definition

$$Q_{E,H+1}\equiv\frac12\max\Big\{\|D^2E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}},\ \|DD_\mu E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}},\ \|D^2_\mu E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}}\Big\}\ .$$

We recall that

$$E_{H+1}=\mathcal{E}[K_{H+1},\mu_{H+1}]=f_{\mu_{H+1}}\circ K_{H+1}-K_{H+1}\circ T_\omega\ .$$

It is convenient to introduce Δ_H and Ξ_H such that

$$K_{H+1}=K_0+(K_{H+1}-K_0)\equiv K_0+\Delta_H\ ,\qquad \mu_{H+1}=\mu_0+\sum_{j=0}^{H}\sigma_j\equiv\mu_0+\Xi_H\ .$$

Then, we have the following bound on E_{H+1}:

$$\begin{aligned} \|E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}} =\ & \|\ (f_{\mu_0}\circ K_0-K_0\circ T_\omega)+f_{\mu_{H+1}}\circ K_{H+1}-f_{\mu_0}\circ K_0 \\ &-(K_{H+1}-K_0)\circ T_\omega\ \|_{\rho_{H+1}-\delta_{H+1}} \\ \le\ & \|E_0\|_{\rho_0}\ +(1+\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|Df_\mu(z)|)\ \kappa_K\varepsilon_0 \\ &+\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|D_\mu f_\mu(z)|\ \kappa_\mu\varepsilon_0\ . \end{aligned}$$

We now observe that the first and second derivatives of $f\circ K$ are given by

$$\begin{aligned} D(f\circ K) &= D(f(K(\theta)))=Df(K(\theta))\ DK(\theta) \\ D^2(f\circ K) &= D^2(f(K(\theta)))(DK(\theta))^2+Df(K(\theta))\ D^2K(\theta)\ . \end{aligned} \tag{B.30}$$

Then, one has

$$\begin{aligned} \|D^2E_{H+1}\|_{\rho_{H+1}} \le\ & \|D^2E_0\|_{\rho_0}+D_{2K}+\|\ D^2f_{\mu_0+\Xi_H}(K_0+\Delta_H)\ (DK_0+D\Delta_H) \\ &-D^2f_{\mu_0}(K_0)\ DK_0\ \|_{\rho_{H+1}}\ \ \|DK_0\|_{\rho_0} \\ &+\|Df_{\mu_0+\Xi_H}(K_0+\Delta_H)-Df_{\mu_0}(K_0)\|_{\rho_{H+1}}\ \|D^2K_0\|_{\rho_0} \\ &+\|D^2f_{\mu_0+\Xi_H}(K_0+\Delta_H)\ (DK_0+D\Delta_H)\ D\Delta_H\|_{\rho_{H+1}} \\ &+\|Df_{\mu_0+\Xi_H}(K_0+\Delta_H)\ D^2\Delta_H\|_{\rho_{H+1}} \end{aligned}$$

$$\begin{aligned}
\leq\quad & \|D^2E_0\|_{\rho_0} + D_{2K} + \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ \|DK_0\|^2_{\rho_0}\ \kappa_K\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ \|DK_0\|^2_{\rho_0}\ \kappa_\mu\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\ D_K \\
&+ \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\ \kappa_K\varepsilon_0\ D_K \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ \|DK_0\|_{\rho_0}\ \kappa_\mu\varepsilon_0\ D_K \\
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \|D^2K_0\|^2_{\rho_0}\ \kappa_K\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D f_\mu(z)|\ \|D^2K_0\|_{\rho_0}\ \kappa_\mu\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0} + D_K)\ D_K \\
&+ \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0} + D_K) D_K\ \kappa_K\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ (\|DK_0\|_{\rho_0} + D_K) D_K\ \kappa_\mu\varepsilon_0 \\
&+ \sup_{z\in\mathcal{C}} |D f_{\mu_0}(z)|\ D_{2K} + \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \kappa_K\varepsilon_0\ D_{2K} \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D f_\mu(z)|\ \kappa_\mu\varepsilon_0\ D_{2K}\ , \qquad\qquad (B.31)
\end{aligned}$$

with $\|DK_H\|_{\rho_H} \leq \|DK_0\|_{\rho_0} + D_K$, where D_K was estimated as in (3.13), $\|D^2K_H\|_{\rho_H} \leq \|D^2K_0\|_{\rho_0} + D_{2K}$, where D_{2K} is defined through the following inequalities and using (4.18):

$$\begin{aligned}
\|D^2K_{H+1} - D^2K_0\|_{\rho_{H+1}} &\leq \sum_{j=0}^{H} \|D^2\Delta_j\|_{\rho_j} \leq \sum_{j=0}^{H} \delta_j^{-1} v_j \\
&\leq \sum_{j=1}^{H} C_{dj}\nu^{-1}\delta_j^{-\tau-2}\varepsilon_j \leq 4C_{d0}\nu^{-1}\delta_0^{-\tau-2}\varepsilon_0 = D_{2K}\ .
\end{aligned}$$

In a similar way we obtain the following estimate. Given $f\circ K$, from (B.30) we have

$$DD_\mu(f\circ K) = DD_\mu(f(K(\theta)))DK(\theta)\ .$$

Then, we have

$$DD_\mu E_{H+1} = DD_\mu E_0 + DD_\mu f_{\mu_0}(K_0)\,(DK_{H+1} - DK_0) + DD_\mu(f_{\mu_{H+1}}(K_0 + \Delta_H) - f_{\mu_0}(K_0))\,DK_{H+1}\ ,$$

so that

$$\begin{aligned}
\|DD_\mu E_{H+1}\|_{\rho_{H+1}} &\leq \|DD_\mu E_0\|_{\rho_0} + \sup_{z\in\mathcal{C}} |DD_\mu f_{\mu_0}(z)|\ D_K \\
&+ \sup_{z\in\mathcal{C}} |D^2D_\mu f_{\mu_0}(z)|\ \|\Delta_H\|_{\rho_{H+1}}\ (\|DK_0\|_{\rho_0} + \|D\Delta_H\|_{\rho_{H+1}}) \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |DD^2_\mu f_{\mu_0}(z)|\ \|\Xi_H\|_{\rho_{H+1}}(\|DK_0\|_{\rho_0} + \|D\Delta_H\|_{\rho_{H+1}}) \\
&\leq \|DD_\mu E_0\|_{\rho_0} + \sup_{z\in\mathcal{C}} |DD_\mu f_{\mu_0}(z)|\ D_K \\
&+ \sup_{z\in\mathcal{C}} |D^2D_\mu f_{\mu_0}(z)|\ \kappa_K\varepsilon_0\ (\|DK_0\|_{\rho_0} + D_K) \\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |DD^2_\mu f_\mu(z)|\ \kappa_\mu\varepsilon_0\ (\|DK_0\|_{\rho_0} + D_K)\ . \qquad\qquad (B.32)
\end{aligned}$$

Finally, we have:

$$\|D^2_\mu E_{H+1}\|_{\rho_{H+1}} \leq \|D^2_\mu E_0\|_{\rho_0} + \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D^3_\mu f_\mu(z)|\ \kappa_\mu\varepsilon_0\ . \qquad (B.33)$$

Casting together (B.31), (B.32), (B.33), we obtain (B.27) with

$$C_Q \equiv \frac{1}{2}\ \max\Big\{\ 1 + \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ \|DK_0\|^2_{\rho_0}\delta_0^2$$

$$
\begin{aligned}
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ \|DK_0\|_{\rho_0}^2\ \frac{C_{\sigma 0}}{C_{d0}}\delta_0^{\tau+2}\\
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\delta_0\\
&+ \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\ 4C_{d0}\nu^{-1}\delta_0^{-\tau+1}\varepsilon_0\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ \|DK_0\|_{\rho_0}\ 4C_{\sigma 0}\delta_0\varepsilon_0
\end{aligned}
\tag{B.34}
$$

$$
\begin{aligned}
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \|D^2K_0\|_{\rho_0}^2\ \delta_0^2\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D f_\mu(z)|\ \|D^2K_0\|_{\rho_0}\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\\
&+ \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0}+D_K)\delta_0\\
&+ \sup_{z\in\mathcal{C}} |D^3 f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0}+D_K)4C_{d0}\nu^{-1}\delta_0^{-\tau+1}\varepsilon_0\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D^2 f_\mu(z)|\ (\|DK_0\|_{\rho_0}+D_K)\ 4C_{\sigma 0}\delta_0\varepsilon_0\\
&+ \sup_{z\in\mathcal{C}} |Df_{\mu_0}(z)| + \sup_{z\in\mathcal{C}} |D^2 f_{\mu_0}(z)|\ \kappa_K\varepsilon_0\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu D f_\mu(z)|\ \kappa_\mu\varepsilon_0\ ,\\
&\quad \sup_{z\in\mathcal{C}} |DD_\mu f_{\mu_0}(z)|\delta_0 + \sup_{z\in\mathcal{C}} |D^2D_\mu f_{\mu_0}(z)|\delta_0^2\ (\|DK_0\|_{\rho_0}+D_K)\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |DD_\mu^2 f_\mu(z)|\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\ (\|DK_0\|_{\rho_0}+D_K),\\
&\quad \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu^3 f_\mu(z)|\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\ \Big\}\ .
\end{aligned}
$$

B.3.4. Proof of $C_{d,H+1}\le 2C_{d0}$

From (4.14), (B.22), using (4.24) we have:

$$
C_{d,H+1} = \|M_{H+1}\|_{\rho_{H+1}}\ C_{W,H+1} \le (\|M_0\|_{\rho_0}+D_K)(C_{W0}+C_WD_K) \le 2C_{d0}\ .
$$

Appendix C. Constants of the KAM theorem

The constants entering in the conditions(3.3)-(3.12) of Theorem 10 are defined through the following (long) list. For fast reference, before each constant we provide the label of the formula where the constant was introduced. We note that the constants are given in an explicit format and evaluating them requires only a few lines of code.

(4.10) $C_{\sigma 0} \equiv \mathcal{T}_0\left[|\lambda-1|\left(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\right)+\|S_0\|_{\rho_0}\right]\|M_0^{-1}\|_{\rho_0}\ ,$

(4.10) $C_{W_20} \equiv \frac{1}{||\lambda|-1|}\Big(1+C_{\sigma 0}Q_{\mu 0}\Big)\|M_0^{-1}\|_{\rho_0}\ ,$

(4.10) $\overline{C}_{W_20} \equiv 2\mathcal{T}_0\left(\frac{1}{||\lambda|-1|}\|S_0\|_{\rho_0}+1\right)Q_{\mu 0}\ \|M_0^{-1}\|_{\rho_0}^2\ ,$

(4.10) $C_{W_10} \equiv C_0\Big(\|S_0\|_{\rho_0}(C_{W_20}+\overline{C}_{W_20})+\|M_0^{-1}\|_{\rho_0}+Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_{\sigma 0}\Big)\ ,$

(4.10) $C_{W0} \equiv C_{W_10}+(C_{W_20}+\overline{C}_{W_20})\nu\delta_0^{\tau}\ ,$

(4.11) $C_{\eta 0} \equiv C_{W0}\|M_0\|_{\rho_0}+C_{\sigma 0}\nu\delta_0^{\tau}\ ,$

(4.13) $C_{\mathcal{R}0} \equiv Q_{E0}(\|M_0\|_{\rho_0}^2C_{W0}^2+C_{\sigma 0}^2\nu^2\delta_0^{2\tau})\ ,$

(4.13) $C_{\mathcal{E}0} \equiv C_{W0}\nu\delta_0^{-1+\tau}+C_{\mathcal{R}0}\ ,$

(4.14) $C_{d0} \equiv C_{W0}\ \|M_0\|_{\rho_0}\ ,$

(4.17) $\kappa_0 \equiv 2^{2\tau+1}C_{\mathcal{E}0}\nu^{-2}\delta_0^{-2\tau}\ ,$

(4.17) $\kappa_K \equiv 4C_{d0}\ \nu^{-1}\delta_0^{-\tau}\ ,$

$$
\begin{aligned}
&(4.17) && \kappa_\mu \equiv 4C_{\sigma 0}\ ,\\
&(3.13) && D_K \equiv 4C_{d0}\nu^{-1}\delta_0^{-\tau-1}\ \varepsilon_0\ ,\\
&(B.28) && D_{2K} \equiv 4\ C_{d0}\nu^{-1}\delta_0^{-\tau-2}\ \varepsilon_0\ ,\\
&(B.8) && C_N \equiv \|N_0\|^2_{\rho_0}\ \frac{2\|DK_0\|_{\rho_0}+D_K}{1-\|N_0\|_{\rho_0}D_K(2\|DK_0\|_{\rho_0}+D_K)}\ ,\\
&(B.8) && C_M \equiv 1+J_e\Big[C_N(\|DK_0\|_{\rho_0}+D_K)+\|N_0\|_{\rho_0}\Big]\ ,\\
&(B.8) && C_{Minv} \equiv C_N(\|DK_0\|_{\rho_0}+D_K)+\|N_0\|_{\rho_0}+J_e\ ,\\
&(B.15) && C_\tau \equiv \max\Big\{C_S,\, C_{SB}+2C_{Minv}Q_{\mu 0}\Big\}\ D_K\ ,\\
&(B.14) && C_{\mathcal{T}} \equiv \frac{\mathcal{T}_0^2}{1-\mathcal{T}_0 C_\tau}\ \max\Big\{C_S,\, C_{SB}+2C_{Minv}Q_{\mu 0}\Big\}\ ,\\
&(B.15) && C_S \equiv 2J_eQ_0\ \Big\{(\|N_0\|_{\rho_0}+C_ND_K)\ \Big[D_K(\|N_0\|_{\rho_0}+C_ND_K)\\
&&&\quad +\ \|DK_0\|_{\rho_0}\|N_0\|_{\rho_0}+\|DK_0\|_{\rho_0}C_ND_K\Big]\\
&&&\quad +\ C_N\|DK_0\|_{\rho_0}\Big[D_K(\|N_0\|_{\rho_0}+C_ND_K)+\|DK_0\|_{\rho_0}\|N_0\|_{\rho_0}+\|DK_0\|_{\rho_0}C_ND_K\Big]\\
&&&\quad +\ \|N_0\|_{\rho_0}\|DK_0\|_{\rho_0}(\|N_0\|_{\rho_0}+C_ND_K)+C_N\|N_0\|_{\rho_0}\|DK_0\|^2_{\rho_0}\Big\}\ ,\\
&(B.15) && C_{SB} \equiv \frac{1}{||\lambda|-1|}Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_S+2J_eQ_0\ \|N_0\|^2_{\rho_0}\ \|DK_0\|^2_{\rho_0}\frac{1}{||\lambda|-1|}\ C_{Minv}\ Q_{\mu 0}\\
&&&\quad +\ 2C_S\ \frac{1}{||\lambda|-1|}\ C_{Minv}\ Q_{\mu 0}\ D_K\ ,\\
&(B.21) && C_\sigma \equiv C_{\mathcal{T}}\ \Big\{|\lambda-1|\ \Big[\frac{1}{||\lambda|-1|}(\|S_0\|_{\rho_0}+C_SD_K)+1\Big]\\
&&&\quad +\ \Big(\|S_0\|_{\rho_0}+C_SD_K\Big)\Big\}\ \Big(\|M_0^{-1}\|_{\rho_0}+C_{Minv}D_K\Big)\\
&&&\quad +\ \mathcal{T}_0\ \Big\{|\lambda-1|\ \Big[\frac{1}{||\lambda|-1|}(\|S_0\|_{\rho_0}+C_SD_K)+1\Big]C_{Minv}\\
&&&\quad +\ |\lambda-1|\ \frac{1}{||\lambda|-1|}\ \|M_0^{-1}\|_{\rho_0}C_S+C_S\Big(\|M_0^{-1}\|_{\rho_0}+C_{Minv}D_K\Big)\\
&&&\quad +\ C_{Minv}\|S_0\|_{\rho_0}\Big\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
&(B.34) && C_Q \equiv \frac{1}{2}\ \max\Big\{1+\sup_{z\in\mathcal{C}}|D^3f_{\mu_0}(z)|\ \|DK_0\|^2_{\rho_0}\delta_0^2\\
&&&\quad +\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|D_\mu D^2f_\mu(z)|\ \|DK_0\|^2_{\rho_0}\ \frac{C_{\sigma 0}}{C_{d0}}\delta_0^{\tau+2}\\
&&&\quad +\sup_{z\in\mathcal{C}}|D^2f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\delta_0\\
&&&\quad +\sup_{z\in\mathcal{C}}|D^3f_{\mu_0}(z)|\ \|DK_0\|_{\rho_0}\ 4C_{d0}\nu^{-1}\delta_0^{-\tau+1}\varepsilon_0\\
&&&\quad +\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|D_\mu D^2f_\mu(z)|\ \|DK_0\|_{\rho_0}\ 4C_{\sigma 0}\delta_0\varepsilon_0\\
&&&\quad +\sup_{z\in\mathcal{C}}|D^2f_{\mu_0}(z)|\ \|D^2K_0\|^2_{\rho_0}\delta_0^2\\
&&&\quad +\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|D_\mu Df_\mu(z)|\ \|D^2K_0\|_{\rho_0}\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\\
&&&\quad +\sup_{z\in\mathcal{C}}|D^2f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0}+D_K)\delta_0\\
&&&\quad +\sup_{z\in\mathcal{C}}|D^3f_{\mu_0}(z)|\ (\|DK_0\|_{\rho_0}+D_K)4C_{d0}\nu^{-1}\delta_0^{-\tau+1}\varepsilon_0\\
&&&\quad +\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0}|D_\mu D^2f_\mu(z)|\ (\|DK_0\|_{\rho_0}+D_K)\ 4C_{\sigma 0}\delta_0\varepsilon_0\\
&&&\quad +\sup_{z\in\mathcal{C}}|Df_{\mu_0}(z)|+\sup_{z\in\mathcal{C}}|D^2f_{\mu_0}(z)|\ \kappa_K\varepsilon_0
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu Df_\mu(z)|\ \kappa_\mu\varepsilon_0\ ,\\
&\sup_{z\in\mathcal{C}} |DD_\mu f_{\mu_0}(z)|\delta_0 + \sup_{z\in\mathcal{C}} |D^2 D_\mu f_{\mu_0}(z)|\delta_0^2\ (\|DK_0\|_{\rho_0}+D_K)\\
&+ \sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |DD_\mu^2 f_\mu(z)|\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\ (\|DK_0\|_{\rho_0}+D_K),\\
&\sup_{z\in\mathcal{C},\mu\in\Lambda,|\mu-\mu_0|<2\kappa_\mu\varepsilon_0} |D_\mu^3 f_\mu(z)|\ \frac{C_{\sigma 0}}{C_{d0}}\nu\delta_0^{\tau+2}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\text{(B.24)}\quad \overline{C}_{W_2} \equiv\ & 4C_T\left[\frac{1}{||\lambda|-1|}(\|S_0\|_{\rho_0}+C_S D_K)+1\right] Q_{\mu 0}(\|M_0^{-1}\|_{\rho_0}+D_K)^2\\
&+ 4\mathcal{T}_0 Q_{\mu 0}\ \frac{1}{||\lambda|-1|}C_S\ (\|M_0^{-1}\|_{\rho_0}+D_K)^2\\
&+ 4\mathcal{T}_0\ Q_{\mu 0}\left[\frac{1}{||\lambda|-1|}(\|S_0\|_{\rho_0}+C_S D_K)+1\right](D_K+2\|M_0^{-1}\|_{\rho_0})
\end{aligned}
$$

$$
\begin{aligned}
\text{(B.29)}\quad C_{\mathcal{R}} \equiv\ & Q_{E0}\Big[(2C_M\|M_0\|_{\rho_0}+C_M^2 D_K)(C_{W0}+C_W D_K)^2+\|M_0\|_{\rho_0}^2(C_W^2 D_K+2C_{W0}\ C_W)\\
&+ (C_\sigma^2 D_K+2C_{\sigma 0}C_\sigma)\nu^2\delta_0^{2\tau}\Big] + C_Q\ \Big[(\|M_0\|_{\rho_0}+C_M D_K)^2(C_{W0}+C_W D_K)^2\\
&+ (C_{\sigma 0}+C_\sigma D_K)^2\nu^2\delta_0^{2\tau}\Big]\delta_0^{-1}\ ,
\end{aligned}
$$

$$
\text{(B.23)}\quad C_{W_2} \equiv \frac{1}{||\lambda|-1|}\left[1+2Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_\sigma+2Q_{\mu 0}C_{\sigma 0}+2Q_{\mu 0}C_\sigma D_K\right],
$$

$$
\begin{aligned}
\text{(B.25)}\quad C_{W_1} \equiv\ & C_0\Big[\|S_0\|_{\rho_0}C_{W_2}+C_S C_{W_2 0}+C_S C_{W_2}D_K+\|S_0\|_{\rho_0}\overline{C}_{W_2}\\
&+ C_S\overline{C}_{W_2 0}+C_S\overline{C}_{W_2}D_K+1\\
&+ 2Q_{\mu 0}\|M_0^{-1}\|_{\rho_0}C_\sigma+2Q_{\mu 0}C_{\sigma 0}+2Q_{\mu 0}C_\sigma D_K\Big]\ ,
\end{aligned}
$$

$$
\text{(B.26)}\quad C_W \equiv C_{W_1}+C_{W_2}\nu\delta_0^\tau+\overline{C}_{W_2}\nu\delta_0^\tau\ .
$$

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