Commun. Math. Phys. 387, 793–863 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04202-y

Communications in
**Mathematical
Physics**

Rigorous Derivation of a Ternary Boltzmann Equation for a Classical System of Particles

Ioakeim Ampatzoglou[1], **Nataša Pavlović**[2]

[1] Courant Institute of Mathematical Sciences, New York University, New York, USA.
E-mail: ioakampa@cims.nyu.edu
[2] Department of Mathematics, The University of Texas at Austin, Austin, USA

Received: 8 October 2020 / Accepted: 18 August 2021
Published online: 28 August 2021 –

Abstract: In this paper, we present a rigorous derivation of a new kinetic equation describing the limiting behavior of a classical system of particles with three particle elastic instantaneous interactions, which are modeled using a non-symmetric version of a ternary distance. The ternary collisional operator we derive can be seen as the first step towards obtaining a toy model for a non-ideal gas where higher order interactions are taken into account.

Contents

1. Introduction

The Boltzmann equation [8–11] is the central equation of collisional kinetic theory. It is a nonlinear integro-differential equation giving the statistical description of a dilute gas in non-equilibrium in $\mathbb{R}^d$, for $d \geq 2$. It is given by

$$\partial_t f + v \cdot \nabla_x f = Q_2(f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.1}$$

where the unknown function $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ represents the probability density of finding a molecule of the gas in position $x \in \mathbb{R}^d$, with velocity $v \in \mathbb{R}^d$, at time $t \geq 0$. The expression $Q_2(f, f)$ on the right hand side of (1.1) is the collisional operator which is an appropriate quadratic integral operator acting on f, taking into account **binary** interactions of a pair of gas particles. Its exact form depends on the type of interaction between particles. Since the gas is assumed to be very dilute, interactions among three particles or higher order interactions are neglected due to much lower probability of occurring compared to binary.

However, when the gas is dense enough, higher order interactions are much more likely to happen, therefore they produce a significant effect to the evolution of the gas and one needs to take them into consideration. An example of such a situation is a colloid, which is a homogeneous non-crystalline substance consisting of either large molecules or ultramicroscopic particles of one substance dispersed through a second substance. As pointed out in [29], multi-interactions among particles significantly contribute to the grand potential of a colloidal gas and are modeled by a sum of higher order interaction terms. A surprising but very important result of [29] is that interactions among three particles actually depend on the sum of the distances between particles, as opposed to depending on different geometric configurations among interacting particles. This observation is apparently of invaluable computational importance since it significantly simplifies numerical calculations on three particle interactions. The results of [29] have been further verified experimentally e.g. [16] and numerically e.g. [25].

1.1. The program introduced and the goal of this paper. Motivated by the fact that the Boltzmann equation is valid only for very dilute gases and by the observations of [29] that multi-interactions among particles contribute to the colloidal gas (although in this paper we do not model colloids), we aim to introduce and rigorously derive (from a system of classical particles) a kinetic model which goes beyond binary interactions, by incorporating a sum of higher order interaction terms in (1.1). Such an equation, which could serve as a toy model for a non-ideal gas, would be of the form

$$\partial_t f + v \cdot \nabla_x f = \sum_{k=2}^{m} Q_k(\underbrace{f, f, \cdots, f}_{k\text{-times}}), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.2}$$

where for $k = 1, \dots, m$, the expression $Q_k(f, f, \dots, f)$ is the k-th order collisional operator and $m \in \mathbb{N}$ is the accuracy of the approximation. Notice that for $m = 2$, Eq. (1.2) reduces to the classical Boltzmann equation (1.1).

The task of rigorously deriving an equation of the form (1.2) from a classical many particle system, even for the case $m = 2$, is a challenging problem that has been settled for short times only in certain situations; for hard-sphere interactions, the analysis was pioneered by Lanford [27] and recently completed by Gallagher, Saint-Raymond, Texier [19], while for short-range potentials, it has been done in [19,26,28]. Up to our knowledge, the case $m = 3$ i.e. derivation of the equation

$$\partial_t f + v \cdot \nabla_x f = Q_2(f, f) + Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.3}$$

has not been studied at all. We refer to (1.3) as the binary–ternary Boltzmann equation. We mention that in a recent work with Gamba and Tasković [3] we proved global well-posedness of (1.3) for small initial data near vacuum.

In addition to understanding binary interactions and interactions among three particles, derivation of (1.3) requires careful analysis of their mutual interactions. This challenging task has been carried out in a subsequent work [4] since it requires a deep understanding of interactions between three particles and their connection to binary interactions. For this reason, in this paper, we focus on understanding interactions among three particles and rigorously deriving a purely ternary equation, which itself brings a lot of challenges due to combinatorial and configurational intricacies of evolving in time interactions among three particles. We derive an equation of the form

$$\partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.4}$$

where $Q_3(f, f, f)$ is the ternary collisional operator which is an integral operator of cubic order in f. We refer to (1.4) as the ternary Boltzmann equation. Global well-posedness for small initial data near vacuum holds as a special case of the results of [3].

Let us mention that Maxwell models with multiple particle interactions have been studied in [5,6] using Fourier transform methods.

Also, we note that attempts for generalization of the Boltzmann equation using formal density expansions were made by physicists in the past, see e.g. [12,13,22,23,30], but in a different context than ours. These attempts have not been further developed since since the fourth and higher order collisions, terms as well as the virial expansion of the solution, diverged as the number of particles increased. According to [14], the divergences originate from the desire to make a systematic expansion of the macroscopic properties of a large system consisting of many particles in terms of the properties of small (isolated) groups of 2, 3, 4 etc particles, i.e., from the basic idea of the virial expansion itself. This leads to formal expansions in terms of collision integrals containing the dynamics of an increasing number of particles. These integrals diverge, however, in general, due to long range dynamical correlations between successive collisions of these particles, introduced by the possibility of unrestricted free motion of particles between successive collisions.

1.2. Ternary interactions and their scaling. In a typical, dilute hard-sphere gas, the probability of a simultaneous contact of three hard-spheres is very small compared to e.g. the situation when one of the three particles is in simultaneous contact with the other two particles. Motivated by this observation and the fact that in some physical situations, such as when one considers colloids as in [29], interactions among three particles are determined by the sum of the distances of the interacting particles, we introduce the notion of an interaction of three particles based on a non-symmetric version of a ternary distance. More precisely, we introduce the ternary distance:

$$d(x_1; x_2, x_3) := \sqrt{|x_1 - x_2|^2 + |x_1 - x_3|^2}, \quad x_1, x_2, x_3 \in \mathbb{R}^d. \tag{1.5}$$

Having defined the ternary distance, we introduce the notion of a ternary interaction. Let $\epsilon > 0$ and consider three particles i, j, k with positions and velocities (x_i, v_i), (x_j, v_j), $(x_k, v_k) \in \mathbb{R}^{2d}$. We say that the particles i, j, k are in $(i; j, k)$ ternary ϵ-interaction[1] if the following geometric condition holds:

$$d^2(x_i; x_j, x_k) = |x_i - x_j|^2 + |x_i - x_k|^2 = 2\epsilon^2. \tag{1.6}$$

The parameter ϵ above is called interaction zone. The i-th particle is called the central collisional particle, while the particles j, k are called adjacent collisional particles.

Heuristically speaking, an $(i; j, k)$ interaction expresses the interaction of the central particle i with the pair of the uncorrelated adjacent particles (j, k) with respect to the interaction zone ϵ. By uncorrelated, we mean that particles j, k are not directly affected by each other. For example, Fig. 1 shows particles that are not in ternary interaction, while Fig. 2 offers two examples of particles which are in ternary interaction.

Let us now describe how velocities instantaneously transform when a ternary interaction happens. Consider an $(i; j, k)$ ternary ϵ-interaction. Let v_i^*, v_j^*, v_k^* denote the velocities of the interacting particles after the interaction. Assuming the particles are

[1] When not ambiguous, we will refer to $(i; j, k)$ ternary ϵ-interaction as $(i; j, k)$ interaction.

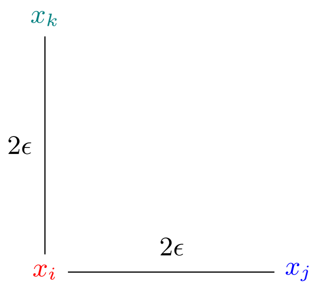

Fig. 1. No ternary interaction

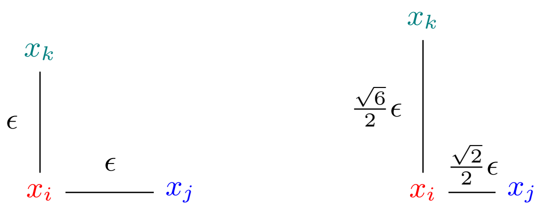

Fig. 2. Ternary interaction

of equal mass $m = 1$, we consider the interaction to be elastic i.e. the three particle momentum-energy conservation system is satisfied:

$$v_i^* + v_j^* + v_k^* = v_i + v_j + v_k, \tag{1.7}$$

$$|v_i^*|^2 + |v_j^*|^2 + |v_k^*|^2 = |v_i|^2 + |v_j|^2 + |v_k|^2. \tag{1.8}$$

Now we introduce the relative positions re-scaled vectors $(\widetilde{\omega}_1, \widetilde{\omega}_2) := \left(\frac{x_j - x_i}{\sqrt{2}\epsilon}, \frac{x_k - x_i}{\sqrt{2}\epsilon}\right)$. Notice that (1.6) implies $(\widetilde{\omega}_1, \widetilde{\omega}_2) \in \mathbb{S}_1^{2d-1}$ i.e. $|\widetilde{\omega}_1|^2 + |\widetilde{\omega}_2|^2 = 1$. We shall call the vectors $\widetilde{\omega}_1, \widetilde{\omega}_2$ impact directions of the interaction. Since the i particle interacts with the pair of uncorrelated particles (j, k), we assume the velocities v_j, v_k transform with respect to the impact directions unit vector i.e.

$$\begin{pmatrix} v_j^* \\ v_k^* \end{pmatrix} = \begin{pmatrix} v_j \\ v_k \end{pmatrix} - c \begin{pmatrix} \widetilde{\omega}_1 \\ \widetilde{\omega}_2 \end{pmatrix}, \tag{1.9}$$

for some $c \in \mathbb{R}$. We note that once we added condition[2] (1.9) to the system (1.7)–(1.8), the new system has a unique solution that algebraically characterizes the conservation of momentum and energy for the type of ternary interaction defined in (1.6). It is straightforward to verify that (1.7)–(1.9) yield that v_i^*, v_j^*, v_k^* are given by the collisional formulas

[2] We note that (1.9) is the ternary analogue of the condition that appears when one considers binary interactions, see e.g. [19].

$$
\begin{aligned}
v_i^* &= v_i + \frac{\langle \widetilde{\omega}_1, v_j - v_i \rangle + \langle \widetilde{\omega}_2, v_k - v_i \rangle}{1 + \langle \widetilde{\omega}_1, \widetilde{\omega}_2 \rangle} (\widetilde{\omega}_1 + \widetilde{\omega}_2), \\
v_j^* &= v_j - \frac{\langle \widetilde{\omega}_1, v_j - v_i \rangle + \langle \widetilde{\omega}_2, v_k - v_i \rangle}{1 + \langle \widetilde{\omega}_1, \widetilde{\omega}_2 \rangle} \widetilde{\omega}_1, \\
v_k^* &= v_k - \frac{\langle \widetilde{\omega}_1, v_j - v_i \rangle + \langle \widetilde{\omega}_2, v_k - v_i \rangle}{1 + \langle \widetilde{\omega}_1, \widetilde{\omega}_2 \rangle} \widetilde{\omega}_2.
\end{aligned}
\tag{1.10}
$$

1.3. Phase space and scaling of ternary interactions. Now we are ready to describe the evolution of a system of N-particles of ϵ-interaction zone. Recall that in this paper we pursue only ternary interactions analysis, thus the phase space will take into account only those.

Definition 1.1. Let $d \in \mathbb{N}$, with $d \geq 2$, $N \in \mathbb{N}$ and $\epsilon > 0$. The phase space of the N-particle system of ϵ-interaction zone is defined as:

$$
\mathcal{D}_{N,\epsilon} = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d^2(x_i; x_j, x_k) \geq 2\epsilon^2 \quad \forall 1 \leq i < j < k \leq N \right\}, \tag{1.11}
$$

where $d^2(x_i; x_j, x_k) = |x_i - x_j|^2 + |x_i - x_k|^2$, and $X_N = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$, $V_N = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, represent the positions and velocities of the N-particles.

In terms of scaling, one could interpret an $(i; j, k)$ of interaction zone ϵ as a special hard sphere interaction of radius $\sqrt{2}\epsilon$ in $\mathbb{R}^{2d}$, since expression (1.6) can be written as

$$
|\boldsymbol{x_i} - \boldsymbol{x_{j,k}}|_{2d} = \sqrt{2}\epsilon,
$$

where $\boldsymbol{x_{i,i}} = \begin{pmatrix} x_i \\ x_i \end{pmatrix}$ and $\boldsymbol{x_{j,k}} = \begin{pmatrix} x_j \\ x_k \end{pmatrix}$. Then a $2d$-particle with position $\boldsymbol{x_{i,i}}$ would span a volume of order ϵ^{2d-1} in a unit of time. In order to observe $O(1)$ interaction per unit of time, there are N^2 options for the $2d$-particle positioned at $\boldsymbol{x_{j,k}}$. We obtain that $N^2 \epsilon^{2d-1} = O(1)$ or equivalently

$$
N\epsilon^{d-1/2} = O(1). \tag{1.12}
$$

This is the new scaling in which we will observe this kind of ternary interactions, see Sect. 4 for the explicit appearance of this scaling in the calculations.

Remark 1.2. The phase space (1.11) will produce the kinetic equation (1.15), in which the tracked particle is always the central particle of the interactions occurring. Alternatively, by working on the phase space

$$
\widetilde{D}_{N,\epsilon} = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d_l^2(x_i, x_j, x_k) \geq 2\epsilon^2, \ \forall (i, j, k, l) \in \widetilde{\mathcal{I}}_N \right\}, \tag{1.13}
$$

where

$$
\widetilde{\mathcal{I}}_N = \{(i, j, k, l) : (i, j, k) \in \mathcal{I}_N \text{ and } l : \{i, j, k\} \to \{i, j, k\} \text{ is a permutation}\},
$$

$$
d_l(x_i, x_j, x_k) = \sqrt{|x_{l_i} - x_{l_j}|^2 + |x_{l_i} - x_{l_k}|^2},
$$

and using similar arguments as in this paper, one can derive a symmetrized version of (1.15), in which the tracked particle can be either central or adjacent. Moreover, it has been shown in [2], that the symmetrized ternary equation satisfies similar statistical and entropy production properties as the classical Boltzmann equation. In particular, it has a weak formulation which yields an $\mathcal{H}$-Theorem and local conservation of mass, momentum and energy. For simplicity, we opt to work with the phase space (1.11). However, we would like to mention that all the intermediate results needed for the derivation of the symmetrized ternary equation can be obtained after some minor changes, see [2] for more details.

1.4. Global existence of a flow and the Liouville equation. Let us now describe the evolution in time of a system of particles in the phase space (1.11). Consider an initial configuration $Z_N \in \mathcal{D}_{N,\epsilon}$. The motion is described as follows:

(I) Particles are assumed to perform rectilinear motion as long as there is no interaction i.e.

$$\dot{x}_i = v_i, \quad \dot{v}_i = 0, \quad \forall i \in \{1, \ldots, N\}.$$

(II) Assume now that an initial configuration $Z_N = (X_N, V_N)$ has evolved until time $t > 0$, reaching $Z_N(t) = (X_N(t), V_N(t))$, and there is an $(i; j, k)$ interaction at time t. Then the velocities $(v_i(t), v_j(t), v_k(t))$ instantaneously transform to $(v_i^*(t), v_j^*(t), v_k^*(t))$.

We remark that it is not at all obvious that (I)–(II) produce a well defined dynamics, since the evolution is not smooth in time, and the system can possibly run into pathological configurations. In the case of binary interactions, the analogous result has been established in the work of Alexander [1]. Our dynamics will be constructed in a similar spirit to [1]. However a distinction between ternary precollisional and postcollisional configurations as well as new geometric estimates are needed in order to control possible emergence of pathological trajectories.

We informally state the first main result of this paper, for a rigorous statement see Theorem 3.14.

Existence of a global flow: Let $m \in \mathbb{N}$ and $0 < \sigma << 1$. There is a global in time measure-preserving flow $(\Psi_m^t)_{t\in\mathbb{R}} : \mathcal{D}_{m,\sigma} \to \mathcal{D}_{m,\sigma}$ which preserves kinetic energy. This flow is called the σ-interaction zone flow of m-particles or simply the interaction flow.

The main difficulty in proving Theorem 3.14 is the elimination of configurations following pathological trajectories in time. In particular, in order to go from local to global in time flow we establish the following crucial fact—when an $(i; j, k)$ interaction happens, then the subsequent interaction cannot involve the same triplet of particles. This observation enables us to develop ellipsoidal coverings and new geometric estimates to control the measure of these pathological sets.

The global measure-preserving interaction flow established yields a Liouville equation (see (3.31)) for the evolution f_N of an initial N-particle of ϵ-interaction zone probability density $f_{N,0}$.

1.5. The ternary equation derived. Although Liouville's equation is a linear transport equation, efficiently solving it is almost impossible in case where the particle number N is very large. This is why an accurate statistical description is welcome, and to obtain it one wants to understand the limiting behavior of it as $N \to \infty$ and $\epsilon \to 0^+$, with the hope that qualitative properties will be revealed for a large but finite N. Letting the number of particles $N \to \infty$ and the interaction zone $\epsilon \to 0^+$ in the **new scaling**:

$$N\epsilon^{d-1/2} = 2^{1-d/2}, \tag{1.14}$$

we derive the ternary Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{1.15}$$

The expression $Q_3(f, f, f)$ is the ternary cubic order collisional operator, given by:

$$\begin{aligned} Q_3(f, f, f) = \int_{\mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}} & \frac{b_+(\omega_1, \omega_2, v_1 - v, v_2 - v)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \\ & \left(f^* f_1^* f_2^* - f f_1 f_2\right) d\omega_1 \, d\omega_2 \, dv_1 \, dv_2, \end{aligned} \tag{1.16}$$

where

$$\begin{aligned} & b(\omega_1, \omega_2, v_1 - v, v_2 - v) := \langle \omega_1, v_1 - v \rangle + \langle \omega_2, v_2 - v \rangle, \quad b_+ = \max\{b, 0\}, \\ & f^* = f(t, x, v^*), \, f = f(x, t, v), \, f_i^* = f_i^*(t, x, v_i^*), \, f_i = f(t, x, v_i) \text{ for } i \in \{1, 2\}. \end{aligned} \tag{1.17}$$

Remark 1.3. The ternary collisional operator could be written in a more general form as:

$$Q_3(f, f, f) = \int_{\mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}} B(\boldsymbol{u}, \boldsymbol{\omega}) \left(f^* f_1^* f_2^* - f f_1 f_2\right) d\omega_1 \, d\omega_2 \, dv_1 \, dv_2,$$

where $\boldsymbol{u} = \binom{v_1 - v}{v_2 - v} \in \mathbb{R}^{2d}$, $\boldsymbol{\omega} = \binom{\omega_1}{\omega_2} \in \mathbb{S}_1^{2d-1}$ are the vectors of relative velocities and scaled relative positions of the colliding particles. Of particular interest would be the power law potentials:

$$B(\boldsymbol{u}, \boldsymbol{\omega}) = |\boldsymbol{u}|^\gamma \widetilde{b}(\widehat{\boldsymbol{u}} \cdot \boldsymbol{\omega}, \langle \omega_1, \omega_2 \rangle),$$

where $\widetilde{b}$ is the differential cross-section and $\widehat{\boldsymbol{u}}$ is the unit vector in the direction of $\boldsymbol{u}$. In this paper, we derive Eq. (1.15) for the case

$$\gamma = 1, \quad \widetilde{b}(\widehat{\boldsymbol{u}} \cdot \boldsymbol{\omega}, \langle \omega_1, \omega_2 \rangle) = \frac{(\widehat{\boldsymbol{u}} \cdot \boldsymbol{\omega})_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}}.$$

For a study of the global well-posedness of (1.15) for power law potentials with $\gamma \in (-2d + 1, 1]$, see [3].

1.6. Strategy of the derivation and statement of the main result. Now the natural question is: how do we pass from the N-particle dynamics to the kinetic equation (1.15)? We implement the program pioneered by Lanford [27] and recently refined by Gallagher, Saint-Raymond, Texier [19] for deriving, for short times, the classical Boltzmann equation (1.1) for hard-spheres in the Boltzmann-Grad [20,21] scaling $N\epsilon^{d-1} \simeq 1$. This program has been implemented in the case of short range potentials too e.g. [19,26,28]. However, to the best of our knowledge, the program has not been explored outside of the context of binary interactions. By generalizing the program to allow consideration of ternary particle interactions, we illustrate that the program is universal enough. However to make it applicable to ternary interactions we follow evolution in time of ternary particle interactions, that inform new mathematical arguments described below.

We first derive a finite two-step[3] coupled hierarchy of equations for the marginals densities of the solution to the Liouville equation, which we call the BBGKY[4] hierarchy. We then formally let $N \to \infty$ and $\epsilon \to 0^+$ in the scaling (1.14) to obtain an infinite two-step coupled hierarchy of equations, which we call the Boltzmann hierarchy. It can be observed that for factorized initial data, the Boltzmann hierarchy reduces to the ternary Boltzmann equation (1.15). This observation connects the Boltzmann hierarchy with the ternary Boltzmann equation.

To make this argument rigorous, we first need to show that the BBGKY and Boltzmann hierarchy are well-posed, at least for short times, and then that if the BBGKY initial data converge to the Boltzmann hierarchy initial data, then this convergence propagates in time in the scaling (1.14). Local well-posedness is shown in Sect. 5, see Theorems 5.5 and 5.8. Showing convergence is a very challenging task and is the heart of our contribution. We informally state our main result here. For a rigorous statement of the result see Theorem 6.9.

Statement of the main result: Let F_0 be initial data for the Boltzmann hierarchy, and $F_{N,0}$ be some BBGKY hierarchy initial data which "approximate" F_0 as $N \to \infty$, $\epsilon \to 0^+$ under the scaling (1.14). Let $\boldsymbol{F_N}$ be the solution to the BBGKY hierarchy with initial data $F_{N,0}$, and $\boldsymbol{F}$ the solution to the Boltzmann hierarchy, with initial data F_0, up to short time $T > 0$. Then $\boldsymbol{F_N}$ converges in observables to $\boldsymbol{F}$ in $[0, T]$ as $N \to \infty$, $\epsilon \to 0^+$, under the scaling (1.14). In the case of Hölder continuous $C^{0,\gamma}$,$\gamma \in (0, 1]$ tensorized Boltzmann hierarchy initial data and approximation by conditioned BBGKY hierarchy initial data, we obtain convergence to the solution of the ternary Boltzmann equation (1.15) with a rate $O(\epsilon^r)$ for any $0 < r < \min\{1/2, \gamma\}$.

The proof of this result is achieved by repeatedly using Duhamel's formula for the finite and infinite hierarchy respectively and comparing the corresponding series expansions. However this a delicate point because of the divergence between the finite particle flow and the free flow, due to the ternary interactions of particles in the finite particle case. The problem of divergence is present in the derivation of the classical Boltzmann equation as well, see [19,27], but our case is significantly harder due the complexity of ternary interactions. To overcome this problem, we develop new geometric and combinatorial estimates, that help us extract small measure sets of initial data which lead to these diverging trajectories. In particular the main difficulty is to control post-collisional configurations and it requires completely new treatment. To achieve that, we need to explicitly calculate the Jacobian of ternary interactions with respect to impact directions, and estimate the surface measure of sets of the form $(K_\rho^d \times \mathbb{R}^d) \cap \mathcal{S}$, where K_ρ^d is a

[3] The two-step refers to the coupling between the k-th element of the hierarchy and the $(k+2)$-th element of the hierarchy.

[4] Bogoliubov, Born, Green, Kirkwood, Yvon.

d-dimensional solid cylinder of radius ρ and $\mathcal{S}$ is an appropriate ellipsoid in $\mathbb{R}^{2d}$. These results are thoroughly presented in Sect. 8.

1.7. Further discussion. While this paper models ternary interactions among particles via a concept of a ternary distance (namely when (1.6) holds), we note that a more physical way would be to employ a three-body potential of a small interaction zone. In particular, one could consider $\Phi : \mathbb{R}^{2d} \to \mathbb{R}$ non-negative, smooth and supported in the unit ball B_1^{2d}. Then one would work in the entire space with Newton's equations

$$\dot{x}_i = v_i \quad \dot{v}_i = -\frac{1}{\epsilon} \sum_{\substack{i,j,k\in\{1,\ldots,N\}\\ i\neq j\neq k}} \nabla\Phi\left(\frac{x_i - x_j}{\epsilon}, \frac{x_i - x_k}{\epsilon}\right).$$

Although we did not pursue analysis of this model, we expect that the relevant scaling (1.14) and the techniques introduced in this paper might be helpful in that context as well.

1.8. Notation. For convenience, we introduce some basic notation which will be used throughout the manuscript:

- We write $x \lesssim y$ if there exists $C_d > 0$ with $x \leq C_d y$.
- Given $n \in \mathbb{N}$, $\rho > 0$ and $w \in \mathbb{R}^n$, we write $B_\rho^n(w)$ for the n-closed ball of radius $\rho > 0$, centered at $w \in \mathbb{R}^n$. In particular, we write $B_\rho^n := B_\rho^n(0)$ for the ρ-ball centered at the origin.
- Given $n \in \mathbb{N}$ and $\rho > 0$, we write $\mathbb{S}_\rho^{n-1}$ for the $(n-1)$-sphere of radius $\rho > 0$.
- We write $x << y$, when $x < cy$ for some number $0 < c < 1$ small enough.

2. Collisional Transformation of Three Particles

In this section, we define the collisional transformation of three particles induced by a pair of impact directions, and investigate its properties.

For convenience, given $(\omega_1, \omega_2, v_1, v_2, v_3) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{3d}$, let us write

$$c_{\omega_1,\omega_2,v_1,v_2,v_3} = \frac{\langle\omega_1, v_2 - v_1\rangle + \langle\omega_2, v_3 - v_1\rangle}{1 + \langle\omega_1, \omega_2\rangle}. \tag{2.1}$$

Notice that $c_{\omega_1,\omega_2,v_1,v_2,v_3}$ is well-defined for all $(\omega_1, \omega_2, v_1, v_2, v_3) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{3d}$, since

$$1 + \langle\omega_1, \omega_2\rangle \geq 1 - |\omega_1||\omega_2| \geq 1 - \frac{1}{2}\left(|\omega_1|^2 + |\omega_2|^2\right) = \frac{1}{2}. \tag{2.2}$$

Definition 2.1. Consider impact directions $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$. We define the collisional transformation induced by $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$ as $T_{\omega_1,\omega_2} : (v_1, v_2, v_3) \in \mathbb{R}^{3d} \longrightarrow (v_1^*, v_2^*, v_3^*) \in \mathbb{R}^{3d}$, where

$$\begin{cases} v_1^* = v_1 + c_{\omega_1,\omega_2,v_1,v_2,v_3}(\omega_1 + \omega_2), \\ v_2^* = v_2 - c_{\omega_1,\omega_2,v_1,v_2,v_3}\omega_1, \\ v_3^* = v_3 - c_{\omega_1,\omega_2,v_1,v_2,v_3}\omega_2, \end{cases} \tag{2.3}$$

and $c_{\omega_1,\omega_2,v_1,v_2,v_3}$ is given by (2.1).

In the following definition, we introduce the notion of the cross-section which will have a prominent role in the rest of the paper.

Definition 2.2. We define the cross-section[5] $b : \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} \to \mathbb{R}$ as:

$$b(\omega_1, \omega_2, \nu_1, \nu_2) = \langle \omega_1, \nu_1 \rangle + \langle \omega_2, \nu_2 \rangle, \quad (\omega_1, \omega_2, \nu_1, \nu_2) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}. \tag{2.4}$$

Notice that by (2.1), (2.4) we have

$$b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) = (1 + \langle \omega_1, \omega_2 \rangle)\, c_{\omega_1,\omega_2,v_1,v_2,v_3}. \tag{2.5}$$

Direct algebraic calculations illustrate the main properties of the collisional tranformation.

Proposition 2.3. *Consider a pair of impact directions $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$. The induced collisional transformation T_{ω_1,ω_2} has the following properties:*

(i) Conservation of momentum

$$v_1^* + v_2^* + v_3^* = v_1 + v_2 + v_3. \tag{2.6}$$

(ii) Conservation of energy:

$$|v_1^*|^2 + |v_2^*|^2 + |v_3^*|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2. \tag{2.7}$$

(iii) Conservation of relative velocities magnitude:

$$|v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2 = |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2. \tag{2.8}$$

(iv) Micro-reversibility of the cross-section:

$$b(\omega_1, \omega_2, v_2^* - v_1^*, v_3^* - v_1^*) = -b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1). \tag{2.9}$$

(v) T_{ω_1,ω_2} is a linear involution i.e. T_{ω_1,ω_2} is linear, $T^{-1}_{\omega_1,\omega_2} = T_{\omega_1,\omega_2}$. In particular $|\det T_{\omega_1,\omega_2}| = 1$, thus T_{ω_1,ω_2} is measure-preserving.

Proof. *(i)* and *(ii)* are guaranteed by construction. *(iii)* comes immediately after combining *(i)* and *(ii)*. To prove *(iv)*, we use (2.3) to obtain

$$v_2^* - v_1^* = v_2 - v_1 - 2c\omega_1 - c\omega_2, \quad v_3^* - v_1^* = v_3 - v_1 - 2c\omega_2 - c\omega_1.$$

Using the fact that $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$, and recalling (2.5), we get

$$\begin{aligned} b^* = \langle \omega_1, v_2^* - v_1^* \rangle + \langle \omega_2, v_3^* - v_1^* \rangle &= \langle \omega_1, v_2 - v_1 \rangle + \langle \omega_2, v_3 - v_1 \rangle \\ &\quad - 2c_{\omega_1,\omega_2,v_1,v_2,v_3}\,(1 + \langle \omega_1, \omega_2 \rangle) = -b, \end{aligned}$$

where we use the notation $b := b(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1)$, $b^* := b(\omega_1, \omega_2, v_2^* - v_1^*, v_3 - v_1^*)$. To prove *(v)*, first notice that T_{ω_1,ω_2} is linear in velocities. Recalling notation from (2.5), *(iv)* implies that $c^* = -c$ where $c^* := c_{\omega_1,\omega_2,v_1^*,v_2^*,v_3^*}$, $c := c_{\omega_1,\omega_2,v_1,v_2,v_3}$. This observation and (2.3) directly imply that $T^{-1}_{\omega_1,\omega_2} = T_{\omega_1,\omega_2}$. Clearly $\left|\det T_{\omega_1,\omega_2}\right| = 1$ and T_{ω_1,ω_2} is measure-preserving. $\square$

[5] We

3. Dynamics of *m*-Particles

In this section we rigorously define the dynamics of m-particles of small interaction zone $0 < \sigma << 1$. Heuristically speaking particles perform free motion as long as they are not interacting, and instantaneously transform velocities according to the collisional transformation, defined in Sect. 2, when they interact. Intuitively, the dynamics is well-defined as long as we have well-separated in time interactions, such that each of those interactions involves only one triplet. Here, we show that a flow can be defined for almost all initial configurations.

Throughout this section we consider $m \in \mathbb{N}$ and $0 < \sigma << 1$. We assume $m \geq 3$ unless stated.

3.1. Phase space definitions. Consider the set $\mathcal{I}_m := \{(i,j,k) \in \{1,\ldots,m\}^3 : i < j < k\}$ of ordered triples in $\{1,\ldots,m\}$. We define the phase space of the m-particles of σ-interaction zone as

$$\mathcal{D}_{m,\sigma} := \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d^2(x_i; x_j, x_k) \geq 2\sigma^2, \quad \forall (i,j,k) \in \mathcal{I}_m \right\}, \tag{3.1}$$

where $X_m = (x_1, \ldots, x_m) \in \mathbb{R}^{dm}$, $V_m = (v_1, \ldots, v_m) \in \mathbb{R}^{dm}$ represent the positions and velocities of the m-particles respectively, and

$$d(x_i; x_j, x_k) = \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2}, \tag{3.2}$$

is the distance in positions of the particles i, j, k. Finally, we also define $\mathcal{D}_{1,\sigma} \equiv \mathbb{R}^{2d}$, $\mathcal{D}^X_{1,\sigma} \equiv \mathbb{R}^d$. Elements of $\mathcal{D}_{m,\sigma}$ are called phase space configurations.

The phase space $\mathcal{D}_{m,\sigma}$ decomposes to the interior and the boundary:

$$\mathring{\mathcal{D}}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d^2(x_i; x_j, x_k) > 2\sigma^2, \quad \forall (i,j,k) \in \mathcal{I}_m \right\}, \tag{3.3}$$

$$\partial\mathcal{D}_{m,\sigma} = \bigcup_{(i,j,k)\in\mathcal{I}_m} \Sigma_{ijk}, \;\; \Sigma_{ijk} := \left\{ Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma} : d^2(x_i; x_j, x_k) = 2\sigma^2 \right\}. \tag{3.4}$$

We further decompose the boundary to simple collisions and multiple collisions respectively:

$$\partial_{sc}\mathcal{D}_{m,\sigma} = \left\{ Z_m = (X_m, V_m) \in \partial\mathcal{D}_{m,\sigma} : \text{there is unique } (i,j,k) \in \mathcal{I}_m : Z_m \in \Sigma_{ijk} \right\}, \tag{3.5}$$

$$\partial_{mc}\mathcal{D}_{m,\sigma} = \big\{ Z_m = (X_m, V_m) \in \partial\mathcal{D}_{m,\sigma} : \text{there are } (i,j,k) \neq (i',j',k') \in \mathcal{I}_m : Z_m \in \Sigma_{ijk} \cap \Sigma_{i'j'k'} \big\}. \tag{3.6}$$

Notice that in the special case $m = 3$, we have $\partial_{mc}\mathcal{D}_{3,\sigma} = \emptyset$ and $\partial\mathcal{D}_{3,\sigma} = \partial_{sc}\mathcal{D}_{3,\sigma}$, i.e. there are no multiple collisions when we consider only three particles.

Definition 3.1. Consider $Z_m \in \partial_{sc}\mathcal{D}_{m,\sigma}$. Then there is a unique triplet $(i, j, k) \in \mathcal{I}_m$ such that $Z_m \in \Sigma_{ijk}$. In this case we will say that Z_m is an $(i; j, k)$ simple collision and we will write

$$\Sigma_{ijk}^{sc} := \left\{Z_m = (X_m, V_m) \in \partial_{sc}\mathcal{D}_{m,\sigma} : Z_m \text{ is } (i; j, k) \text{ simple collision}\right\}. \quad (3.7)$$

Remark 3.2. Notice that $\Sigma_{ijk}^{sc} \cap \Sigma_{i'j'k'}^{sc} = \emptyset, \quad \forall (i, j, k) \neq (i', j', k') \in \mathcal{I}_m$, and $\partial_{sc}\mathcal{D}_{m,\sigma}$ decomposes to $\partial_{sc}\mathcal{D}_{m,\sigma} = \bigcup_{(i,j,k)\in\mathcal{I}_m} \Sigma_{ijk}^{sc}$.

For the purposes of defining a global flow, throughout this section we use the following notation:

Definition 3.3. Let $(i, j, k) \in \mathcal{I}_m$ and $Z_m \in \Sigma_{ijk}^{sc}$. We introduce

$$(\widetilde{\omega}_1, \widetilde{\omega}_2) := \frac{1}{\sqrt{2}\sigma}\left(x_j - x_i, x_k - x_i\right) \in \mathbb{S}_1^{2d-1}. \quad (3.8)$$

Therefore, each $(i; j, k)$ simple collision naturally induces impact directions $(\widetilde{\omega}_1, \widetilde{\omega}_2) \in \mathbb{S}_1^{2d-1}$, and a collisional transformation $T_{\widetilde{\omega}_1,\widetilde{\omega}_2}$.

We also give the following definition:

Definition 3.4. Let $(i, j, k) \in \mathcal{I}_m$ and $Z_m = (X_m, V_m) \in \Sigma_{ijk}^{sc}$. We denote $Z_m^* = (X_m, V_m^*)$, where

$$V_m^* = (v_1, \dots, v_{i-1}, v_i^*, v_{i+1}, \dots, v_{j-1}, v_j^*, v_{j+1}, \dots, v_{k-1}, v_k^*, v_{k+1}, \dots, v_m),$$

and $(v_i^*, v_j^*, v_k^*) = T_{\widetilde{\omega}_1,\widetilde{\omega}_2}(v_i, v_j, v_k), \quad (\widetilde{\omega}_1, \widetilde{\omega}_2) \in \mathbb{S}_1^{2d-1}$ are given by (3.8).

3.2. Classification of simple collisions. Now, we classify simple collisions in order to eliminate collisions which graze under time evolution. Informally speaking, a simple collisional configuration will be precollisional when the three interacting particles have the velocities which led them to the interaction and postcollisional when the velocities have already changed by the collision according to the transformation (2.3). As we will see in Lemma 3.7, a simple collisional configuration can be characterized by the sign of the cross-section. More specifically, we introduce the following language:

Definition 3.5. Let $(i, j, k) \in \mathcal{I}_m$ and $Z_m \in \Sigma_{ijk}^{sc}$. The configuration Z_m is called:

(i) pre-collisional when $b(\widetilde{\omega}_1, \widetilde{\omega}_2, v_j - v_i, v_k - v_i) < 0$,
(ii) post-collisional when $b(\widetilde{\omega}_1, \widetilde{\omega}_2, v_j - v_i, v_k - v_i) > 0$,
(iii) grazing when $b(\widetilde{\omega}_1, \widetilde{\omega}_2, v_j - v_i, v_k - v_i) = 0$, where $(\widetilde{\omega}_1, \widetilde{\omega}_2) \in \mathbb{S}_1^{2d-1}$ is given by (3.8) and b is given by (2.4).

Remark 3.6. Let $(i, j, k) \in \mathcal{I}_m$ and $Z_m \in \Sigma_{ijk}^{sc}$. Using (2.9), we obtain the following:

(i) Z_m is pre-collisional iff Z_m^* is post-collisional.
(ii) Z_m is post-collisional iff Z_m^* is pre-collisional.
(iii) $Z_m = Z_m^*$ iff Z_m is grazing.

We consider the subset of the phase space: $\mathcal{D}_{m,\sigma}^* = \mathring{\mathcal{D}}_{m,\sigma} \cup \partial_{sc,ng}\mathcal{D}_{m,\sigma}$, where

$$\partial_{sc,ng}\mathcal{D}_{m,\sigma} = \left\{Z_m = (X_m, V_m) \in \partial_{sc}\mathcal{D}_{m,\sigma} : Z_m \text{ is non-grazing}\right\}.$$

Notice that $\mathcal{D}_{m,\sigma}^*$ is a full measure subset of $\mathcal{D}_{m,\sigma}$ and $\partial_{sc,ng}\mathcal{D}_{m,\sigma}$ is a full surface measure subset of $\partial\mathcal{D}_{m,\sigma}$.

3.3. Construction of the local flow. Here, we show that each $Z_m \in \mathcal{D}^*_{m,\sigma}$ follows a well-defined trajectory for short time. Next Lemma defines the flow for any initial configuration in $\mathcal{D}^*_{m,\sigma}$ up to the first collision time.

Lemma 3.7. *Consider $Z_m = (X_m, V_m) \in \mathcal{D}^*_{m,\sigma}$. Then there is a time $\tau^1_{Z_m} \in (0,\infty]$ such that defining $Z_m(\cdot) : (0, \tau^1_{Z_m}] \to \mathbb{R}^{2dm}$ by:*

$$\begin{aligned} Z_m(t) &= (X_m(t), V_m(t)) \\ &:= \begin{cases} (X_m + tV_m, V_m), & \text{if } Z_m \text{ is non-collisional or post-collisional,} \\ (X_m + tV_m^*, V_m^*), & \text{if } Z_m \text{ is pre-collisional,} \end{cases} \end{aligned}$$

the following hold:

(i) $Z_m(t) \in \mathring{\mathcal{D}}_{m,\sigma}, \quad \forall t \in (0, \tau^1_{Z_m})$,
(ii) if $\tau^1_{Z_m} < \infty$*, then* $Z_m(\tau^1_{Z_m}) \in \partial\mathcal{D}_{m,\sigma}$,
(iii) If $Z_m \in \Sigma^{sc}_{ijk}$ *for some* $(i,j,k) \in \mathcal{I}_m$*, and* $\tau^1_{Z_m} < \infty$*, then* $Z_m(\tau^1_{Z_m}) \notin \Sigma_{ijk}$,

The time $\tau^1_{Z_m}$ is called the first (forward) collision time of Z_m. The first negative collision time can be defined analogously.

Proof. Let us make the convention $\inf \emptyset = +\infty$. We define

$$\tau^1_{Z_m} = \begin{cases} \inf\left\{t > 0 : X_m + tV_m \in \partial\mathcal{D}_{m,\sigma}\right\}, & \text{if } Z_m \text{ is post-collisional,} \\ \inf\left\{t > 0 : X_m + tV_m^* \in \partial\mathcal{D}_{m,\sigma}\right\}, & \text{if } Z_m \text{ is pre-collisional.} \end{cases} \tag{3.9}$$

• Assume that $Z_m \in \mathring{\mathcal{D}}_{m,\sigma}$. Since $\mathring{\mathcal{D}}_{m,\sigma}$ is open and the free flow is continuous, we obtain $\tau^1_{Z_m} > 0$, and claims *(i)*-*(ii)* follow immediately from (3.9).
• Assume now that $Z_m \in \partial_{sc,ng}\mathcal{D}_{m,\sigma}$, hence Z_m is a simple non-grazing collision. Therefore we may distinguish the following cases:

(I) Z_m is an $(i; j, k)$ post-collisional configuration: For any $t > 0$, we have

$$d^2(x_i + tv_i; x_j + tv_j, x_k + tv_k) \geq 2\sigma^2 + 2tb(x_j - x_i, x_k - x_i, v_j - v_i, v_k - v_i) > 2\sigma^2, \tag{3.10}$$

since $b(\widetilde{\omega}_1, \widetilde{\omega}_2, v_j - v_i, v_k - v_i) > 0$. This inequality and the fact that Z_m is simple collision imply that $\tau^1_{Z_m} > 0$, and claim *(i)* holds. Claim *(ii)* follows from (3.9) and claim *(iii)* follows from (3.10).

(II) Z_m is $(i; j, k)$ pre-collisional: We use the same argument for Z_m^* which is $(i; j, k)$ post-collisional. □

Let us make an elementary, but crucial remark which will turn of fundamental importance when extending the flow globally in time.

Remark 3.8. For configurations with $\tau^1_{Z_m} = \infty$ the flow is globally defined as the free flow. In the case where $\tau^1_{Z_m} < \infty$ and $Z_m(\tau^1_{Z_m}) \in \partial_{sc,ng}\mathcal{D}_{m,\sigma}$, we may apply Lemma 3.7 once more, considering $Z_m(\tau^1_{Z_m})$ as initial point, and extend the flow up to the second collision time $\tau^2_{Z_m} := \tau^1_{Z_m(\tau^1_{Z_m})}$. Moreover, if $Z_m(\tau^1_{Z_m}) \in \Sigma^{sc}_{ijk}$ for some $(i,j,k) \in \mathcal{I}_m$, part *(iii)* of Lemma 3.7 implies that $Z_m(\tau^2_{Z_m}) \notin \Sigma_{ijk}$.

3.4. Extension to a global interaction flow. Now, we extract a null set from $\mathcal{D}^*_{m,\sigma}$ such that the flow is globally defined for positive times on the complement. For this purpose, we consider truncation parameters in the scaling:

$$0 < \delta R << \sigma << 1 < R < \rho. \tag{3.11}$$

We first assume initial positions are in B_ρ^{dm} and initial velocities in B_R^{dm}. We decompose $D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm})$ as follows:

$$\begin{aligned}
I_{free} &:= \Big\{ Z_m = (X_m, V_m) \in D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} > \delta \Big\},\\
I^1_{sc,ng} &:= \big\{ Z_m = (X_m, V_m) \in D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} < \delta,\\
&\qquad Z_m(\tau^1_{Z_m}) \in \partial_{sc,ng}\mathcal{D}_{m,\sigma}, \text{ and } \ \tau^2_{Z_m} > \delta \big\},\\
I^1_{sc,g} &:= \big\{ Z_m = (X_m, V_m) \in D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} < \delta,\\
&\qquad Z_m(\tau^1_{Z_m}) \in \partial_{sc}\mathcal{D}_{m,\sigma}, \text{ but } Z_m(\tau^1_{Z_m}) \text{ is grazing} \big\},\\
I^1_{mc} &:= \Big\{ Z_m = (X_m, V_m) \in D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} < \delta,\\
&\qquad Z_m(\tau^1_{Z_m}) \in \partial_{mc}\mathcal{D}_{m,\sigma} \Big\},\\
I^2_{sc,ng} &:= \big\{ Z_m = (X_m, V_m) \in D^*_{m,\sigma} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} < \delta,\\
&\qquad Z_m(\tau^1_{Z_m}) \in \partial_{sc,ng}\mathcal{D}_{m,\sigma}, \text{ but } \ \tau^2_{Z_m} \le \delta \big\}.
\end{aligned} \tag{3.12}$$

Notice that for $Z_m \in I_{free} \cup I^1_{sc,ng}$, thanks to Lemma 3.7, the flow is well defined up to time δ, and there occurs at most one simple non-grazing collision in $(0, \delta)$.

3.4.1. Covering arguments Now, we make an ellipsoid shell covering of the set $I^1_{mc} \cup I^2_{sc,ng}$ in a way that we can estimate the measure of the coverings.

Lemma 3.9. *For $m = 3$, there holds $I^1_{mc} = I^2_{sc,ng} = \emptyset$. For $m \ge 4$, the following inclusion holds:*

$$I^2_{sc,ng} \cup I^1_{mc} \subseteq \bigcup_{(i,j,k)\neq(i',j',k')\in\mathcal{I}_m} \left(U_{ijk} \cap U_{i'j'k'}\right), \tag{3.13}$$

$$U_{ijk} := \Big\{ Z_m = (X_m, V_m) \in B_\rho^{dm} \times B_R^{dm} : 2\sigma^2 \le d^2(x_i; x_j, x_k) \le (\sqrt{2}\sigma + 4\delta R)^2 \Big\}. \tag{3.14}$$

Proof. For $m = 3$, we have $\partial_{mc}\mathcal{D}_{3,\sigma} = \emptyset$, thus $I^1_{mc} = \emptyset$. Also, since $m = 3$, we obtain $\mathcal{I}_3 = \{(1, 2, 3)\}$, hence Remark 3.8 implies that $\tau^2_{Z_m} = \infty$ i.e. there is no other collision in the future, so $I^2_{sc,ng} = \emptyset$.

Let $m \ge 4$. We first assume that either $Z_m \in \mathring{\mathcal{D}}_{m,\sigma}$ or Z_m is post-collisional. We first prove the inclusion for $I^2_{sc,ng}$. Assuming that $Z_m(\tau^1_{Z_m}) \in I^2_{sc,ng}$ is an $(i; j, k)$ non-grazing collision, we have

$$d^2\left(x_i\left(\tau^1_{Z_m}\right); x_j\left(\tau^1_{Z_m}\right), x_k\left(\tau^1_{Z_m}\right)\right) = 2\sigma^2.$$

Since there is free motion up to $\tau^1_{Z_m}$ and $\tau^1_{Z_m} \leq \delta$, triangle inequality implies

$$|x_i - x_j| \leq |x_i(\tau^1_{Z_m}) - x_j(\tau^1_{Z_m})| + \delta|v_i - v_j| \leq |x_i(\tau^1_{Z_m}) - x_j(\tau^1_{Z_m})| + 2\delta R. \tag{3.15}$$

Since there is collision at $\tau^1_{Z_m}$, we have

$$|x_i(\tau^1_{Z_m}) - x_j(\tau^1_{Z_m})|^2 + |x_i(\tau^1_{Z_m}) - x_k(\tau^1_{Z_m})|^2 = 2\sigma^2 \Rightarrow |x_i(\tau^1_{Z_m}) - x_j(\tau^1_{Z_m})| \leq \sqrt{2}\sigma. \tag{3.16}$$

Combining (3.15)–(3.16), we obtain

$$|x_i - x_j|^2 \leq |x_i(\tau^1_{Z_m}) - x_j(\tau^1_{Z_m})|^2 + 4\sqrt{2}\sigma\delta R + 4\delta^2 R^2. \tag{3.17}$$

Using the same argument for the pair (i, k), adding, and recalling the fact that there is $(i; j, k)$ simple collision at $\tau^1_{Z_m}$, we obtain

$$2\sigma^2 \leq d^2(x_i; x_j, x_k) \leq 2\sigma^2 + 8\sqrt{2}\sigma R\delta + 8\delta R^2 \leq (\sqrt{2}\sigma + 4\delta R)^2, \tag{3.18}$$

where the lower inequality holds trivially since $Z_m \in \mathcal{D}_{m,\sigma}$. By (3.18), we obtain $Z_m \in U_{ijk}$.

Remark 3.8 guarantees that $Z_m(\tau^2_{Z_m}) \notin \Sigma_{ijk}$. So $Z_m(\tau^2_{Z_m}) \in \Sigma_{i'j'k'}$ for some $(i', j', k') \neq (i, j, k)$. Moreover, particles keep performing free motion in $[\tau^1_{Z_m}, \tau^2_{Z_m})$ except particles i, j, k whose velocities instantaneously transform because of the collision at $\tau^1_{Z_m}$. Recall we wish to prove as well:

$$Z_m \in U_{i'j'k'} \Leftrightarrow 2\sigma^2 \leq d^2(x_{i'}; x_{j'}, x_{k'}) \leq (\sqrt{2}\sigma + 4\delta R)^2. \tag{3.19}$$

The lower inequality trivially holds because of the phase space so it suffices to prove the upper inequality. Since $(i, j, k) \neq (i', j', k')$, it suffices to distinguish the following cases:

(I) $i', j', k' \notin \{i, j, k\}$: Since particles (i', j', k') perform free motion up to $\tau^2_{Z_m}$, a similar argument to the one we used to obtain (3.18) yields $Z_m \in U_{i'j'k'}$. The only difference is that we apply the argument up to time $\tau^2_{Z_m}$.

(II) At least one of i', j', k' belongs to $\{i, j, k\}$ but no more than two. The argument is similar to (I), the only difference being that velocities of the recolliding particles transform at $\tau^1_{Z_m}$. Since the argument is similar for all cases, let us provide the proof in detail only for one case, for instance $(i', j', k') = (i, k, k')$, for some $k' > k$. The fact that $V_m \in B^{dm}_R$, conservation of energy by the free flow and conservation of energy by the collision (2.7) imply $v^*_i\left(\tau^1_{Z_m}\right), v^*_j\left(\tau^1_{Z_m}\right), v^*_k\left(\tau^1_{Z_m}\right) \in B^d_R$. For the pair (i, k), we have

$$x_i(\tau^2_{Z_m}) = x_i(\tau^1_{Z_m}) + (\tau^2_{Z_m} - \tau^1_{Z_m})v^*_i\left(\tau^1_{Z_m}\right) = x_i + \tau^1_{Z_m}v_i + (\tau^2_{Z_m} - \tau^1_{Z_m})v^*_i\left(\tau^1_{Z_m}\right),$$
$$x_k(\tau^2_{Z_m}) = x_k(\tau^1_{Z_m}) + (\tau^2_{Z_m} - \tau^1_{Z_m})v^*_k\left(\tau^1_{Z_m}\right) = x_k + \tau^1_{Z_m}v_k + (\tau^2_{Z_m} - \tau^1_{Z_m})v^*_k\left(\tau^1_{Z_m}\right).$$

Therefore, triangle inequality implies

$$
\begin{aligned}
|x_i - x_k| &\leq |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + \tau^1_{Z_m}|v_i - v_k| + (\tau^2_{Z_m} - \tau^1_{Z_m})|v_i^*\left(\tau^1_{Z_m}\right) - v_k^*\left(\tau^1_{Z_m}\right)| \\
&\leq |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + 2\tau^1_{Z_m} R + 2(\tau^2_{Z_m} - \tau^1_{Z_m})R \\
&\leq |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + 2\delta R.
\end{aligned}
$$

Similarly, for the pair (i, k'), we obtain $|x_i - x_{k'}| \leq |x_i(\tau^2_{Z_m}) - x_{k'}(\tau^2_{Z_m})| + 2\delta R$. By an argument similar to (3.18), inequality (3.19) follows. Inclusion (3.13) is proved for $I^2_{sc,ng}$. The inclusion for I^1_{mc} follows similarly.

Assume now that Z_m is pre-collisional. By Remark 3.6, Z_m^* is post-collisional and by (2.7) $Z_m^* \in B_\rho^{dm} \times B_R^{dm}$. By a similar argument to the post-collisional case, we obtain the result. □

3.4.2. Measure estimates Now we estimate the measure of $I^1_\delta \cup I^1_{sc,g} \cup I^1_{mc} \cup I^2_{sc,ng}$ in order to show that outside of a small measure set we have a well defined flow up to small time δ. To estimate the measure of $I^1_{mc} \cup I^2_{sc,ng}$, we will strongly rely on the shell-like covering made in Lemma 3.9.

For this purpose, we first introduce some notation. Consider $(i, j, k) \in \mathcal{I}_m$, a permutation $\pi : \{i, j, k\} \to \{i, j, k\}$ and $(x_{\pi_j}, x_{\pi_k}) \in \mathbb{R}^{2d}$. We define the set

$$
S_{\pi_i}(x_{\pi_j}, x_{\pi_k}) = \{x_{\pi_i} \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk}\}. \tag{3.20}
$$

Lemma 3.10. *Let $(i, j, k) \in \mathcal{I}_m$, a permutation $\pi : \{i, j, k\} \to \{i, j, k\}$ and $(x_{\pi_j}, x_{\pi_k}) \in \mathbb{R}^{2d}$. Then*

$$
|S_{\pi_i}(x_{\pi_j}, x_{\pi_k})|_d \leq C_{d,R}\delta. \tag{3.21}
$$

Proof. By symmetry, it suffices to prove (3.21) for the permutations $\pi = (i, j, k)$ and $\pi = (k, i, j)$. For convenience, let us write $\sigma_0 = \sqrt{2}\sigma$, $\delta_0 = 4\delta R$. Scaling (3.11) implies $0 < \delta_0 << \sigma_0 << 1$.

The proof for $\pi = (k, i, j)$: Consider $(x_i, x_j) \in \mathbb{R}^{2d}$, and let us write $\alpha = |x_i - x_j|$. Recalling (3.20), we have $S_k(x_i, x_j) = \{x_k \in \mathbb{R}^d : \sigma_0^2 - \alpha^2 \leq |x_i - x_k|^2 \leq (\sigma_0 + \delta_0)^2 - \alpha^2\}$. We distinguish the following cases:

- $\alpha > \sigma_0$: We have $(\sigma_0 + \delta_0) - \alpha^2 < (\sigma_0 + \delta_0)^2 - \sigma_0^2 = \delta_0(2\sigma_0 + \delta_0) < \delta_0$, since $0 < \delta_0 << \sigma_0 << 1$. Thus $S_k(x_i, x_j) \subseteq \left\{x_k \in \mathbb{R}^d : |x_i - x_k| \leq \sqrt{\delta_0}\right\}$, so $|S_k(x_i, x_j)|_d \lesssim \delta_0^{d/2} \leq \delta_0 = 4R\delta$, since $\delta_0 < 1$ and $d \geq 2$.

- $\alpha \leq \sigma_0$: By (3.20), $S_k(x_i,x_j) = \{x_k \in \mathbb{R}^d : \sqrt{\sigma_0^2-\alpha^2} \leq |x_i - x_k| \leq \sqrt{(\sigma_0+\delta_0)^2-\alpha^2}\}$. Therefore

$$|S_k(x_i,x_j)|_d \simeq \left(\sqrt{(\sigma_0+\delta_0)^2-\alpha^2}\right)^d - \left(\sqrt{\sigma_0^2-\alpha^2}\right)^d \tag{3.22}$$

$$= \frac{\delta_0(2\sigma_0+\delta_0)}{\sqrt{(\sigma_0+\delta_0)^2-\alpha^2}+\sqrt{\sigma_0^2-\alpha^2}} \sum_{m=0}^{d-1} \left(\sqrt{(\sigma_0+\delta_0)^2-\alpha^2}\right)^{d-1-m} \left(\sqrt{\sigma_0^2-\alpha^2}\right)^m$$

$$\leq \frac{\delta_0}{\sqrt{(\sigma_0+\delta_0)^2-\alpha^2}+\sqrt{\sigma_0^2-\alpha^2}} \left(\sqrt{(\sigma_0+\delta_0)^2-\alpha^2} + (d-1)\sqrt{\sigma_0^2-\alpha^2}\right) \tag{3.23}$$

$$\leq (d-1)\delta_0 = 4(d-1)R\delta, \tag{3.24}$$

where to obtain (3.23) we use the fact that $0 < \delta_0 << \sigma_0 << 1$, and to obtain (3.24) we use the fact that $d \geq 2$. Estimate (3.21) is proved for the case (k,i,j).

The proof for $\pi = (i,j,k)$: Consider $(x_j,x_k) \in \mathbb{R}^{2d}$. Completing the square, one can see that

$$S_i(x_j,x_k) = \left\{x_i \in \mathbb{R}^d : \sigma_0^2 - \alpha^2 \leq \left|x_i - \frac{x_j+x_k}{2}\right|^2 \leq (\sigma+\delta_0)^2 - \alpha^2\right\},$$

where $\sigma_0 = \sigma$, $\delta_0 = \frac{4\delta R}{\sqrt{2}}$, $\alpha = \frac{1}{2}\sqrt{2(|x_j|^2+|x_k|^2) - |x_j+x_k|^2}$. Scaling (3.11) implies $0 < \delta_0 << \sigma_0 << 1$. The estimate follows by an argument identical to the the previous case. □

Lemma 3.11. *The following measure estimate holds:*

$$|I_{sc,g}^1 \cup I_{sc,ng}^2 \cup I_{mc}^1|_{2dm} \leq C_{m,d,R}\,\rho^{d(m-2)}\delta^2.$$

Proof. First, we notice that $I_{sc,g}^1$ has measure zero since it is covered by codimension-2 submanifolds of the phase space. For $m=3$, the result comes trivially from Lemma 3.9. Assume $m \geq 4$. By Lemma 3.9, it suffices to uniformly estimate the measure of $U_{ijk} \cap U_{i'j'k'}$, for all $(i,j,k) \neq (i',j',k') \in \mathcal{I}_m$. Consider $(i,j,k) \neq (i',j',k') \in \mathcal{I}_m$, and recall notation from (3.20). We will strongly rely on Lemma 3.10. We distinguish the following cases:

(I) $i',j',k' \notin \{i,j,k\}$: Fubini's Theorem and (3.21) imply

$$|U_{ijk} \cap U_{i'j'k'}|_{2dm} \lesssim R^{dm}\rho^{d(m-6)} \int_{B_\rho^{6d}} \mathbb{1}_{S_k(x_i,x_j)\cap S_{k'}(x_{i'},x_{j'})}\, dx_i\, dx_j\, dx_k\, dx_{i'}\, dx_{j'}\, dx_{k'}$$

$$\leq R^{dm}\rho^{d(m-6)} \left(\int_{B_\rho^d \times B_\rho^d} \int_{\mathbb{R}^d} \mathbb{1}_{S_k(x_i,x_j)}\, dx_k\, dx_j\, dx_i\right)$$

$$\left(\int_{B_\rho^d \times B_\rho^d} \int_{\mathbb{R}^d} \mathbb{1}_{S_{k'}(x_{i'},x_{j'})}\, dx_k'\, dx_j'\, dx_i'\right)$$

$$\leq C_{m,d,R}\,\rho^{d(m-2)}\delta^2.$$

(II) Exactly one of i', j', k' belongs to $\{i, j, k\}$: Without loss of generality, we consider the case $i' = i$, $j' \neq j$, $k \neq k'$. Fubini's Theorem and (3.21) imply

$$
\begin{aligned}
|U_{ijk} \cap U_{ij'k'}|_{2dm} &\lesssim R^{dm} \rho^{d(m-5)} \int_{B_\rho^{5d}} \mathbb{1}_{S_k(x_i,x_j) \cap S_{k'}(x_i,x_{j'})} \, dx_i \, dx_j \, dx_k \, dx_{j'} \, dx_{k'} \\
&\leq R^{dm} \rho^{d(m-5)} \int_{B_\rho^d} \left(\int_{B_\rho^d} \int_{\mathbb{R}^d} \mathbb{1}_{S_k(x_i,x_j)} \, dx_k \, dx_j \right) \\
&\quad \left(\int_{B_\rho^d} \int_{\mathbb{R}^d} \mathbb{1}_{S_{k'}(x_i,x_{j'})} \, dx_{k'} \, dx_{j'} \right) dx_i \\
&\leq C_{m,d,R} \rho^{d(m-2)} \delta^2.
\end{aligned}
$$

(III) Exactly two of i', j', k' belong to $\{i, j, k\}$: Without loss of generality, we consider the case $i' = i$, $j' = j$, $k \neq k'$. Fubini's Theorem and (3.21) imply

$$
\begin{aligned}
|U_{ijk} \cap U_{ijk'}|_{2dm} &\lesssim R^{dm} \rho^{d(m-4)} \int_{B_\rho^{4d}} \mathbb{1}_{S_k(x_i,x_j) \cap S_{k'}(x_i,x_{j'})} \, dx_i \, dx_j \, dx_k \, dx_{k'} \\
&\leq R^{dm} \rho^{d(m-4)} \int_{B_\rho^d \times B_\rho^d} \left(\int_{\mathbb{R}^d} \mathbb{1}_{S_k(x_i,x_j) \, dx_k} \right) \left(\int_{\mathbb{R}^d} \mathbb{1}_{S_{k'}(x_i,x_j) \, dx_{k'}} \right) dx_j \, dx_i \\
&\leq C_{m,d,R} \rho^{d(m-2)} \delta^2.
\end{aligned}
$$

□

Remark 3.12. For negative times, analogous results of Lemmas 3.9 and 3.11 follow similarly.

3.4.3. The global interaction flow We inductively use Lemma 3.11 to define a global flow which preserves energy for almost all configuration. For this purpose, given $Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}$, we define its kinetic energy as:

$$E_m(Z_m) := \frac{1}{2} \sum_{i=1}^{m} |v_i|^2. \tag{3.25}$$

For convenience, let us define the free flow of m-particles.

Definition 3.13. Let $m \in \mathbb{N}$. We define the free flow of m-particles as the family of maps $(\Phi_m^t)_{t \in \mathbb{R}} : \mathbb{R}^{2dm} \to \mathbb{R}^{2dm}$, given by $\Phi_m^t Z_m = \Phi_m^t(X_m, V_m) := (X_m + tV_m, V_m)$.

We establish the existence of σ-interaction zone flow of m-particles.

Theorem 3.14. (Existence of the interaction flow) *Let $m \in \mathbb{N}$ and $0 < \sigma << 1$. There exists a full measure G_δ-subset $\Gamma_{m,\sigma} \subseteq \mathcal{D}_{m,\sigma}^*$ and a measure-preserving family of diffeomorphisms $(\Psi_m^t)_{t \in \mathbb{R}} : \Gamma_{m,\sigma} \to \Gamma_{m,\sigma}$ such that*

$$\Psi_m^{t+s} Z_m = (\Psi_m^t \circ \Psi_m^s)(Z_m) = (\Psi_m^s \circ \Psi_m^t)(Z_m), \quad \forall Z_m \in \Gamma_{m,\sigma}, \quad \forall t, s \in \mathbb{R}, \tag{3.26}$$

$$E_m\left(\Psi_m^t Z_m\right) = E_m(Z_m), \quad \forall Z_m \in \Gamma_{m,\sigma}, \quad \forall t \in \mathbb{R}. \tag{3.27}$$

Moreover for $m \geq 3$ the flow is defined a.e. on $\Gamma_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}$ with respect to the induced measure $d\sigma$ and

$$\Psi_m^t Z_m^* = \Psi_m^t Z_m, \quad \sigma - a.e. \text{ on } \Gamma_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}, \quad \forall t \in \mathbb{R}. \tag{3.28}$$

This family of maps is called the σ-interaction zone flow of m-particles. For $m = 1, 2$, the flow coincides with the free flow.

Proof. Having established the bounds of Lemma 3.11, which are valid for both positive and negative collision times (by Remark 3.12), existence of the set Γ and (3.26)–(3.27) follow in the same spirit as in [1]. An outline of the proof can also be found in [19]. For details of the proof, see [2].

It remains to prove that the flow is a.e. defined on $\Gamma_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}$ and that (3.28) holds. We use an argument similar to [31]. By the definition of the flow, (3.28) holds on $\Gamma_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}$. Therefore, it suffices to prove $I_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}$ is a null subset of $\partial_{sc,ng}\mathcal{D}_{m,\sigma}$, where $I_{m,\sigma} := \mathcal{D}_{m,\sigma}^* \setminus \Gamma_{m,\sigma}$ is the set of configurations which run into pathological trajectories in finite time. Let $Z_m' \in \partial_{sc,ng}\mathcal{D}_{m,\sigma}$. Then by Lemma 3.7, the flow can be defined up to time $\tau_{Z_m'}^1 > 0$ and $\Psi_m^t Z_m' \in \mathring{\mathcal{D}}_{m,\sigma}$ for all $0 < t < \tau_{Z_m'}^1$. But since $I_{m,\sigma}$ is of measure zero and $\Gamma_{m,\sigma}$ is invariant under the flow, we have

$$0 = \int_{I_{m,\sigma} \cap \mathring{\mathcal{D}}_{m,\sigma}} dZ_m = \int_{I_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}} \int_0^{\tau_{Z_m'}^1} dt\, d\sigma(Z_m') = \int_{I_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}} \tau_{Z_m'}^1\, d\sigma(Z_m'),$$

which implies that $\sigma(I_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}) = 0$, since $\tau_{Z_m'}^1 > 0$. □

3.5. The Liouville equation. We introduce the flow operators used throughout the paper, and then derive the m-particle Liouville equation for $m \geq 3$.

Definition 3.15. For $t \in \mathbb{R}$, we define the σ-interaction zone flow of m-particles operator $T_m^t : L^\infty(\mathcal{D}_{m,\sigma}) \to L^\infty(\mathcal{D}_{m,\sigma})$ as

$$T_m^t g_m(Z_m) = g_m(\Psi_m^{-t} Z_m). \tag{3.29}$$

Definition 3.16. For $t \in \mathbb{R}$ and $m \in \mathbb{N}$, we define the free flow of m-particles operator $S_m^t : L^\infty(\mathbb{R}^{2dm}) \to L^\infty(\mathbb{R}^{2dm})$ as:

$$S_m^{\bar{t}} g_m(Z_m) = g_m(\Phi_m^{-t} Z_m) = g_m(X_m - tV_m, V_m). \tag{3.30}$$

Assume $m \geq 3$. Given a symmetric with respect to the particles initial probability density $f_{m,0}$ supported in $\mathcal{D}_{m,\sigma}$, we define its evolution as $f_m(t, Z_m) := T_m^t f_{m,0}$. Clearly, f_m is symmetric and supported in $\mathcal{D}_{m,\sigma}$. Theorem 3.14 implies that f_m formally satisfies the m-particle Liouville equation

$$\begin{cases} ll\partial_t f_m + \sum_{i=1}^m v_i \cdot \nabla_{x_i} f_m = 0, & (t, Z_m) \in (0,\infty) \times \mathring{\mathcal{D}}_{m,\sigma}, \\ f_m(t, Z_m^*) = f_m(t, Z_m), & (t, Z_m) \in [0,\infty) \times \partial_{sc}\mathcal{D}_{m,\sigma}, \\ f_m(0, Z_m) = f_{m,0}(Z_m), & Z_m \in \mathring{\mathcal{D}}_{m,\sigma}. \end{cases} \tag{3.31}$$

4. BBGKY Hierarchy, Boltzmann Hierarchy and the Ternary Boltzmann Equation

In this section we consider N-particles of ϵ-interaction zone, where $N \geq 3$ and $0 < \epsilon << 1$. We integrate the N-particle Liouville's equation to formally obtain a linear hierarchy of integro-differential equations satified by the marginals of its solution (BBGKY hierarchy). We then formally derive the limiting hierarchy (Boltzmann hierarchy) occuring under the appropriate scaling and formally show it reduces to a nonlinear integro-differential equation (the new ternary Boltzmann equation) for chaotic initial data.

4.1. The BBGKY hierarchy. Consider N-particles of interaction zone $0 < \epsilon << 1$, where $N \geq 3$. For $s \in \mathbb{N}$, we define the s-marginal of a symmetric probability density f_N, supported in $\mathcal{D}_{N,\epsilon}$, as

$$f_N^{(s)}(Z_s) = \begin{cases} \displaystyle\int_{\mathbb{R}^{2d(N-s)}} f_N(Z_N)\, dx_{s+1} \ldots dx_N\, dv_{s+1} \ldots dv_N, \ 1 \leq s < N, \\ f_N, \ s = N, \\ 0, \ s > N, \end{cases} \tag{4.1}$$

where for $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, we write $Z_N = (X_s, x_{s+1}, \ldots, x_N, V_s, v_{s+1}, \ldots, v_N)$. It is straightforward that, for all $1 \leq s \leq N$, the marginals $f_N^{(s)}$ are symmetric probability densities, supported in $\mathcal{D}_{s,\epsilon}$.

Assume now that f_N is formally the solution to the N-particle Liouville equation (3.31) with initial data $f_{N,0}$. We seek to formally find a hierarchy of equations satisfied by the marginals of f_N. The answer is obvious for $s \geq N$ since by definition $f_N^{(N)} = f_N$ and $f_N^{(s)} = 0$ for $s > N$.

Notice that $\partial \mathcal{D}_{N,\epsilon}$ is equivalent up to surface measure zero to $\Sigma^X \times \mathbb{R}^{dN}$, where $\Sigma^X := \bigcup_{(i,j,k)\in\mathcal{I}_N} \Sigma_{ijk}^{sc,X}$, and $\Sigma_{ijk}^{sc,X}$ are given by (3.7). Moreover, Σ^X is a pairwise disjoint union.

We proceed by integrating by parts the Liouville equation. Consider $1 \leq s \leq N - 1$. The boundary and initial conditions can be easily recovered integrating Liouville's equation boundary and initial conditions respectively i.e.

$$\begin{cases} ll f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s), & (t, Z_s) \in [0, \infty) \times \partial_{sc}\mathcal{D}_{s,\epsilon}, \quad s \geq 3, \\ f_N^{(s)}(0, Z_s) = f_{N,0}^{(s)}(Z_s), & Z_s \in \mathring{\mathcal{D}}_{s,\epsilon}. \end{cases} \tag{4.2}$$

Notice that for $s = 1, 2$ there is no boundary condition, since $\mathcal{D}_{s,\epsilon} = \mathbb{R}^{2ds}$ by definition.

Consider now a smooth test function ϕ_s compactly supported in $(0, \infty) \times \mathcal{D}_{s,\epsilon}$ such that whenever $(i, j, k) \in \mathcal{I}_N$ with $j \leq s$, the following holds:

$$\phi_s(t, p_s Z_N^*) = \phi_s(t, p_s Z_N) = \phi_s(t, Z_s), \quad \forall (t, Z_N) \in (0, \infty) \times \Sigma_{ijk}^{sc}, \tag{4.3}$$

where $p_s(Z_N) := Z_s$ is the natural projection in space and velocities.

Multiplying the Liouville equation by ϕ_s, and integrating , we obtain

$$\int_{(0,\infty)\times\mathcal{D}_{N,\epsilon}} \left(\partial_t f_N\,(t, Z_N) + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N\,(t, Z_N) \right) \phi_s(t, Z_s)\, dX_N\, dV_N\, dt = 0. \tag{4.4}$$

For the time derivative in (4.4), integration by parts in time, Fubini's Theorem and then again integration by parts in time imply

$$\int_{(0,\infty)\times\mathcal{D}_{N,\epsilon}} \partial_t f_N(t,Z_N)\phi_s(t,Z_s)\,dX_N\,dV_N\,dt$$
$$= \int_{(0,\infty)\times\mathcal{D}_{s,\epsilon}} \partial_t f_N^{(s)}(t,Z_s)\phi_s(t,Z_s)\,dX_s\,dV_s\,dt. \tag{4.5}$$

For the material derivative term in (4.4), the Divergence Theorem implies

$$\int_{\mathcal{D}_{N,\epsilon}} \sum_{i=1}^{N} v_i\cdot\nabla_{x_i} f_N\,(t,Z_N)\,\phi_s(t,Z_s)\,dX_N\,dV_N$$
$$= \int_{\mathcal{D}_{N,\epsilon}} \mathbf{div}_{X_N}\,[f_N\,(t,Z_N)\,V_N]\,\phi_s(t,Z_s)\,dX_N\,dV_N$$
$$= A_1 + A_2, \tag{4.6}$$

$$A_1 := -\int_{\mathcal{D}_{N,\epsilon}} V_N\cdot\nabla_{X_N}\phi_s(t,Z_s)f_N(t,Z_N)\,dX_N\,dV_N$$
$$A_2 := \int_{\Sigma^X\times\mathbb{R}^{dN}} \hat{n}\,(X_N)\cdot V_N f_N\,(t,Z_N)\,\phi_s\,(t,Z_s)\;dV_N\,d\sigma,$$

where $\hat{n}(X_N)$ is the outwards normal vector on Σ^X at $X_N\in\Sigma^X$, $d\sigma$ is the surface measure on Σ^X. Moreover, by the fact that f_N is supported in $\mathcal{D}_{N,\epsilon}$, the Divergence Theorem and the fact that ϕ_s is compactly supported, we obtain

$$A_1 = \int_{\mathbb{R}^{2dN}} V_s\cdot\nabla_{X_s}\phi_s(t,Z_s)f_N(t,Z_N)\,dX_N\,dV_N$$
$$= \int_{\mathbb{R}^{2ds}} V_s\cdot\nabla_{X_s}\phi_s(t,Z_s)f_N^{(s)}(t,Z_s)\,dX_s\,dV_s$$
$$= -\int_{\mathbb{R}^{2ds}} \mathbf{div}_{X_s}[f_N^{(s)}(t,Z_s)V_s]\phi_s(t,Z_s)\,dX_s\,dV_s$$
$$= -\int_{\mathcal{D}_{s,\epsilon}} \sum_{i=1}^{s} v_i\nabla_{x_i} f_N^{(s)}(t,Z_s)\phi_s(t,Z_s)\,dX_s\,dV_s, \tag{4.7}$$

Combining (4.4)–(4.6), (4.7), we obtain

$$\int_{(0,\infty)\times\mathcal{D}_{s,\epsilon}} \left(\partial_t f_N^{(s)}\,(t,Z_s) + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i} f_N^{(s)}\,(t,Z_s)\right)\phi_s\,(t,Z_s)\;dX_s\,dV_s\,dt$$
$$= \sum_{(i,j,k)\in\mathcal{I}_N} \int_0^\infty C_{ijk}(t)\,dt, \tag{4.8}$$
$$C_{ijk}(t) := -\int_{\Sigma_{i,j,k}^{sc,X}\times\mathbb{R}^{dN}} \hat{n}_{ijk}\,(X_N)\cdot V_N f_N\,(t,Z_N)\,\phi_s\,(t,Z_s)\;dV_N\,d\sigma_{ijk}, \tag{4.9}$$

and $\hat{n}_{ijk}(X_N)$ is the outwards normal vector on $\Sigma_{ijk}^{sc,X}$ at $X_N \in \Sigma_{ijk}^{sc,X}$, $d\sigma_{ijk}$ is the surface measure on $\Sigma_{ijk}^{sc,X}$. We easily calculate

$$-\hat{n}_{ijk}(X_N) \cdot V_N = (\sqrt{2})^{-1} \frac{\langle \frac{x_j - x_i}{\sqrt{2}\epsilon}, v_j - v_i \rangle + \langle \frac{x_k - x_i}{\sqrt{2}\epsilon}, v_k - v_i \rangle}{\sqrt{1 + \langle \frac{x_j - x_i}{\sqrt{2}\epsilon}, \frac{x_k - x_i}{\sqrt{2}\epsilon} \rangle}}. \tag{4.10}$$

Notice that since we are integrating over $\Sigma_{ijk}^{sc,X}$, we have $\left(\frac{x_j - x_i}{\sqrt{2}\epsilon}, \frac{x_k - x_i}{\sqrt{2}\epsilon}\right) \in \mathbb{S}_1^{2d-1}$. Making the change of variables $(v_i, v_j, v_k) \to (v_i^*, v_j^*, v_k^*)$, under the collisional transformation induced by $\left(\frac{x_j - x_i}{\sqrt{2}\epsilon}, \frac{x_k - x_i}{\sqrt{2}\epsilon}\right)$, using (4.10), Proposition 2.3 parts *(iv)*, *(v)* and the boundary condition of (3.31), we obtain

$$C_{ijk}(t) = -(\sqrt{2})^{-1} \int_{\Sigma_{ijk}^{sc,X} \mathbb{R}^{dN}} \frac{\langle \frac{x_j - x_i}{\sqrt{2}\epsilon}, v_j - v_i \rangle + \langle \frac{x_k - x_i}{\sqrt{2}\epsilon}, v_k - v_i \rangle}{\sqrt{1 + \langle \frac{x_j - x_i}{\sqrt{2}\epsilon}, \frac{x_k - x_i}{\sqrt{2}\epsilon} \rangle}}$$
$$f_N(t, Z_N^*)\phi_s(t, \pi_s Z_N^*)\, dV_N\, d\sigma_{ijk}. \tag{4.11}$$

Equations (4.9)–(4.11) and the test function condition (4.3) imply

$$\begin{aligned} &C_{ijk}(t) = 0, \quad \forall (i, j, k) \notin \widetilde{\mathcal{I}}_N, \quad \forall t > 0, \\ &\quad \text{where } \widetilde{\mathcal{I}}_N := \{(i, j, k) \in \mathcal{I}_N : 1 \leq i \leq s < j < k \leq N\}. \end{aligned} \tag{4.12}$$

Notice we immediately observe that the $(N-1)$- marginal satisfies the $(N-1)$-Liouville equation given in (3.31).

For $1 \leq s \leq N-2$ and $(i, j, k) \in \widetilde{\mathcal{I}}_N$, the $(dN-1)$-surface measure on $\Sigma_{ijk}^{sc,X}$ can be written as $d\sigma_{ijk}(X_N) = dS_{x_i}(x_j, x_k) \prod_{\substack{\ell=1 \\ \ell \neq j,k}}^{N} dx_\ell$, where, given $x_i \in \mathbb{R}^d$, dS_{x_i} is the surface measure on the sphere of center $(x_i, x_i) \in \mathbb{R}^{2d}$ and radius $\sqrt{2}\epsilon$. By this decomposition and the symmetry assumption on f_N we obtain $C_{ijk}(t) = C_{i,s+1,s+2}(t), \quad \forall (i, j, k) \in \widetilde{\mathcal{I}}_N, \quad \forall t > 0$. This observation and, (4.12) yield

$$\begin{aligned} \sum_{(i,j,k) \in \mathcal{I}_N} C_{ijk}(t) &= \sum_{i=1}^{s} \sum_{j=s+1}^{N-1} \sum_{k=j+1}^{N} C_{i,s+1,s+2}(t) \\ &= \sum_{i=1}^{s} \sum_{j=s+1}^{N-1} (N-j) C_{i,s+1,s+2}(t) = (1 + 2 + \ldots + N - s - 1) \\ &\quad \sum_{i=1}^{s} C_{i,s+1,s+2}(t) \\ &= \frac{1}{2}(N-s)(N-s-1) \sum_{i=1}^{s} C_{i,s+1,s+2}(t), \quad \forall t > 0. \end{aligned} \tag{4.13}$$

Fix $i \in \{1, \ldots, s\}$. Substituting $(\omega_1, \omega_2) = \left(\frac{x_{s+1} - x_i}{\sqrt{2}\epsilon}, \frac{x_{s+2} - x_i}{\sqrt{2}\epsilon}\right)$, and recalling the notation from (2.4), we obtain thanks to (4.9)–(4.10), (4.1) and the fact that supp $f_N^{(s+2)} \subseteq$

$\mathcal{D}_{s+2,\epsilon}$ that

$$\int_0^\infty C_{i,s+1,s+2}(t)\,dt = \int_{(0,\infty)\times\mathcal{D}_{s,\epsilon}} 2^{d-1}\epsilon^{2d-1} \int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}} \frac{b\left(\omega_1,\omega_2,v_{s+1}-v_i,v_{s+2}-v_i\right)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} \times f_N^{(s+2)}(t,X_s,x_i+\sqrt{2}\epsilon\omega_1,x_i+\sqrt{2}\epsilon\omega_2,V_s,v_{s+1},v_{s+2})\, d\omega_1\,d\omega_2\,dv_{s+1}\,dv_{s+2}\,dX_s\,dV_s\,dt. \tag{4.14}$$

Splitting the cross-section to positive and negative parts, followed by an application of the relevant boundary condition to the positive part, and substituting $(\omega_1,\omega_2)\to(-\omega_1,-\omega_2)$ for the negative part, the right hand side of (4.14) becomes:

$$\int_{(0,\infty)\times\mathcal{D}_{s,\epsilon}} 2^{d-1}\epsilon^{2d-1} \int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}} b_+\left(\omega_1,\omega_2,v_{s+1}-v_i,v_{s+2}-v_i\right) \times\left(f_N^{(s+2)}(t,Z_{s+2,\epsilon}^{i*})-f_N^{(s+2)}(t,Z_{s+2,\epsilon}^{i})\right) d\omega_1\,d\omega_2\,dv_{s+1}\,dv_{s+2}\,dX_s\,dV_s\,dt, \tag{4.15}$$

where given $i\in\{1,\dots,s\}$, we denote

$$\begin{aligned} Z_{s+2,\epsilon}^{i} &= (x_1,\dots,x_i,\dots,x_s,x_i-\sqrt{2}\epsilon\omega_1,x_i-\sqrt{2}\\ &\qquad \epsilon\omega_2,v_1,\dots v_{i-1},v_i,v_{i+1},\dots,v_s,v_{s+1},v_{s+2}),\\ Z_{s+2,\epsilon}^{i*} &= (x_1,\dots,x_i,\dots,x_s,x_i+\sqrt{2}\epsilon\omega_1,x_i+\sqrt{2}\\ &\qquad \epsilon\omega_2,v_1,\dots v_{i-1},v_i^*,v_{i+1},\dots,v_s,v_{s+1}^*,v_{s+2}^*). \end{aligned}$$

Finally, combining (4.8), (4.13)–(4.15), we formally obtain the BBGKY hierarchy for $s\in\mathbb{N}$:

$$\begin{cases} \partial_t f_N^{(s)}+\sum_{i=1}^s v_i\cdot\nabla_{x_i}f_N^{(s)}=\mathcal{C}_{s,s+2}^N f_N^{(s+2)}, & (t,Z_s)\in(0,\infty)\times\mathring{\mathcal{D}}_{s,\epsilon},\\ f_N^{(s)}(t,Z_s^*)=f_N^{(s)}(t,Z_s), & (t,Z_s)\in[0,\infty)\times\partial_{sc}\mathcal{D}_{s,\epsilon},\ \text{whenever } s\geq 3,\\ f_N^{(s)}(0,Z_s)=f_{N,0}^{(s)}(Z_s), & Z_s\in\mathring{\mathcal{D}}_{s,\epsilon}, \end{cases} \tag{4.16}$$

where

$$\mathcal{C}_{s,s+2}^N=\mathcal{C}_{s,s+2}^{N,+}-\mathcal{C}_{s,s+2}^{N,-}, \tag{4.17}$$

for $1\leq s\leq N-2$ we denote

$$\mathcal{C}_{s,s+2}^{N,+}f_N^{(s+2)}(t,Z_s)=A_{N,\epsilon,s}\sum_{i=1}^s\int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}}\frac{b_+}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} f_N^{(s+2)}\left(t,Z_{s+2,\epsilon}^{i*},\right)d\omega_1\,d\omega_2\,dv_{s+1}\,dv_{s+2}, \tag{4.18}$$

$$\mathcal{C}_{s,s+2}^{N,-}f_N^{(s+2)}(t,Z_s)=A_{N,\epsilon,s}\sum_{i=1}^s\int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}}\frac{b_+}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} f_N^{(s+2)}\left(t,Z_{s+2,\epsilon}^{i}\right)d\omega_1\,d\omega_2\,dv_{s+1}\,dv_{s+2}, \tag{4.19}$$

and we use the notation

$$
\begin{aligned}
&A_{N,\epsilon,s} = 2^{d-2}(N-s)(N-s-1)\epsilon^{2d-1},\\
&\quad b = b(\omega_1, \omega_2, v_{s+1}-v_i, v_{s+2}-v_i), \quad b_+ = \max\{b, 0\},\\
&Z^i_{s+2,\epsilon} = (x_1, \dots, x_i, \dots, x_s, x_i - \sqrt{2}\epsilon\omega_1, x_i\\
&\quad - \sqrt{2}\epsilon\omega_2, v_1, \dots v_{i-1}, v_i, v_{i+1}, \dots, v_s, v_{s+1}, v_{s+2}),\\
&Z^{i*}_{s+2,\epsilon} = (x_1, \dots, x_i, \dots, x_s, x_i + \sqrt{2}\epsilon\omega_1, x_i\\
&\quad + \sqrt{2}\epsilon\omega_2, v_1, \dots v_{i-1}, v_i^*, v_{i+1}, \dots, v_s, v_{s+1}^*, v_{s+2}^*).
\end{aligned}
\tag{4.20}
$$

For $s \geq N-1$ we trivially define $\mathcal{C}^{N,+}_{s,s+2} \equiv \mathcal{C}^{N,-}_{s,s+2} \equiv 0$.

Duhamel's formula implies that the BBGKY hierarchy can be written in mild form as follows

$$
f_N^{(s)}(t, Z_s) = T_s^t f_{N,0}^{(s)}(Z_s) + \int_0^t T_s^{t-\tau} \mathcal{C}^N_{s,s+2} f_N^{(s+2)}(\tau, Z_s)\, d\tau, \quad s \in \mathbb{N}, \tag{4.21}
$$

where T_s^t is the ϵ-interaction zone flow of s-particles operator given in (3.29). See Remark 5.3 for the validity of (4.21) in L^∞.

4.2. The Boltzmann hierarchy. We will now derive the Boltzmann hierarchy as the formal limit of the BBGKY hierarchy as $N \to \infty$ and $\epsilon \to 0^+$ under the scaling

$$
N\epsilon^{d-1/2} = 2^{1-d/2}. \tag{4.22}
$$

This scaling guarantees that for a fixed $s \in \mathbb{N}$, we have $A_{N,\epsilon,s} \longrightarrow 1$, as $N \to \infty$ and $\epsilon \to 0^+$ in the scaling (4.22). Formally taking the limit under the scaling imposed we may define the following collisional operator:

$$
\mathcal{C}^\infty_{s,s+2} = \mathcal{C}^{\infty,+}_{s,s+2} - \mathcal{C}^{\infty,-}_{s,s+2}, \tag{4.23}
$$

$$
\begin{aligned}
\mathcal{C}^{\infty,+}_{s,s+2} f^{(s+2)}(t, Z_s) = \sum_{i=1}^s \int_{(\mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d})} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} f^{(s+2)}\\
\left(t, Z^{i*}_{s+2}\right) d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2},
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
\mathcal{C}^{\infty,-}_{s,s+2} f^{(s+2)}(t, Z_s) = \sum_{i=1}^s \int_{(\mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d})} \frac{b_+}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \times f^{(s+2)}\\
\left(t, Z^{i}_{s+2}\right) d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2},
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
&b = b(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2}, v_i), \quad b_+ = \max\{b, 0\},\\
&Z^i_{s+2} = (x_1, \dots, x_i, \dots, x_s, x_i, x_i, v_1, \dots v_{i-1}, v_i, v_{i+1}, \dots, v_s, v_{s+1}, v_{s+2}),\\
&Z^{i*}_{s+2} = (x_1, \dots, x_i, \dots, x_s, x_i, x_i, v_1, \dots v_{i-1}, v_i^*, v_{i+1}, \dots, v_s, v_{s+1}^*, v_{s+2}^*).
\end{aligned}
\tag{4.26}
$$

Now we are ready to introduce the Boltzmann hierarchy. More precisely, given an initial data $f_0^{(s)}$, the Boltzmann hierarchy for $s \in \mathbb{N}$ is given by:

$$\begin{cases} ll\partial_t f^{(s)} + \sum_{i=1}^{s} v_i \nabla_{x_i} f^{(s)} = \mathcal{C}_{s,s+2}^{\infty} f^{(s+2)}, & (t, Z_s) \in (0,\infty) \times \mathbb{R}^{2ds}, \\ f^{(s)}(0, Z_s) = f_0^{(s)}(Z_s), & \forall Z_s \in \mathbb{R}^{2ds}. \end{cases} \tag{4.27}$$

Duhamel's formula implies that the Boltzmann hierarchy can be written in mild form as follows

$$f^{(s)}(t, Z_s) = S_s^t f_0^{(s)}(Z_s) + \int_0^t S_s^{t-\tau} \mathcal{C}_{s,s+2}^{\infty} f^{(s+2)}(\tau, Z_s)\, d\tau, \quad s \in \mathbb{N}. \tag{4.28}$$

where S_s^t denotes free flow of s-particles operator given in (3.30). See Remark 5.7 for the validity of (4.28) in L^∞.

4.3. The ternary Boltzmann equation. A situation of particular physical interest is when particles are initially independently distributed. This translates to factorized Boltzmann hierarchy initial data i.e.

$$f_0^{(s)}(Z_s) = f_0^{\otimes s}(Z_s) = \prod_{i=1}^{s} f_0(x_i, v_i), \quad s \in \mathbb{N}, \tag{4.29}$$

where $f_0 : \mathbb{R}^{2d} \to \mathbb{R}$ is a given function. One can easily verify that the anszatz

$$f^{(s)}(t, Z_s) = f^{\otimes s}(t, Z_s) = \prod_{i=1}^{s} f(t, x_i, v_i), \quad s \in \mathbb{N}, \tag{4.30}$$

solves the Boltzmann hierarchy with initial data given by (4.29), if $f : [0,\infty) \times \mathbb{R}^{2d} \to \mathbb{R}$ satisfies the ternary Boltzmann equation

$$\begin{cases} ll\partial_t f + v \cdot \nabla_x f = Q_3(f, f, f), & (t, x, v) \in (0,\infty) \times \mathbb{R}^{2d}, \\ f(0, x, v) = f_0(x, v), & (x, v) \in \mathbb{R}^{2d}, \end{cases} \tag{4.31}$$

where, using the notation from (1.17), the ternary collisional operator Q_3 is given by (1.16)–(1.17). Duhamel's formula implies the ternary Boltzmann equation can be written in mild form as follows

$$f(t, x, v) = S_1^t f_0(x, v) + \int_0^t S_1^{t-\tau} Q_3(f, f, f)(\tau, x, v)\, d\tau. \tag{4.32}$$

See Remark 5.10 for the validity of (4.32) in L^∞.

5. Local Well-Posedness

In this section we address local well-posedness (LWP) for the BBGKY and Boltzmann hierarchies and the ternary Boltzmann equation. As expected, these well-posedness proofs are closely related, and they rely on defining appropriate functional spaces and establishing appropriate a-priori bounds. For this reason we provide the proofs only for the BBGKY case (for more details see [2]). The functional spaces we introduce to address well-posedness are inspired by the spaces used in [19,27].

5.1. LWP for the BBGKY hierarchy. Consider (N, ϵ) in the scaling (4.22). For $1 \leq s \leq N$ and $\beta > 0$ we define the Banach spaces

$$X_{N,\beta,s} := \left\{ g_{N,s} \in L^\infty(\mathcal{D}_{s,\epsilon}) : |g_{N,s}|_{N,\beta,s} := \operatorname*{ess\,sup}_{Z_s \in \mathbb{R}^{2ds}} |g_{N,s}(Z_s)| e^{\beta E_s(Z_s)} < \infty \right\},$$

where $E_s(Z_s)$ is the kinetic energy of s-particles given by (3.25). For $s > N$ we trivially define $X_{N,\beta,s} := \{0\}$.

Consider $\mu \in \mathbb{R}$. We define the Banach space

$$\begin{aligned} X_{N,\beta,\mu} := \Big\{ G_N = (g_{N,s})_{s\in\mathbb{N}} : g_{N,s} \in X_{N,\beta,s},\ \forall s \in \mathbb{N} \text{ and } \|G_N\|_{N,\beta,\mu} \\ := \sup_{s\in\mathbb{N}} e^{\mu s} |g_{N,s}|_{N,\beta,s} < \infty \Big\}. \end{aligned}$$

Finally, given $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$, we define the Banach space

$$\begin{aligned} \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}} &:= C^0\left([0,T], X_{N,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\right), \text{ with norm } |||\boldsymbol{G_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \\ &= \sup_{t\in[0,T]} \|\boldsymbol{G_N}(t)\|_{N,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}. \end{aligned}$$

Now, given $m \in \mathbb{N}$, we prove an important continuity estimate for the operator $\mathcal{C}^N_{m,m+2}$.

Lemma 5.1. *Let $m \in \mathbb{N}$, $\beta > 0$ and $g_{N,m+2} \in X_{N,m+2,\beta}$. Then, the following continuity estimate holds for any*

$$\begin{aligned} \left|\mathcal{C}^N_{m,m+2} g_{N,m+2}(Z_m)\right| &\lesssim \beta^{-d} \left(m\beta^{-1/2} + \sum_{i=1}^m |v_i| \right) e^{-\beta E_m(Z_m)} |g_{N,m+2}|_{N,\beta,m+2}, \\ \forall Z_m &\in \mathcal{D}_{m,\epsilon}. \end{aligned}$$

Proof. Let $g_{N,m+2} \in X_{N,m+2,\beta}$ and $Z_m = (X_m, V_m) \in \mathbb{N}$. If $m \geq N-1$ both sides vanish, so we may assume that $m \leq N-2$. Notice that conservation of energy (2.7) implies

$$E_{m+2}(Z^{i,*}_{m+2,\epsilon}) = E_{m+2}(Z^i_{m+2,\epsilon}), \quad \forall i = 1, \ldots, m. \tag{5.1}$$

Moreover, (2.2), Cauchy–Schwarz inequality and triangle inequality yield

$$\begin{aligned} &\frac{b_+(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} \leq 2\sqrt{2}\left(|v_1| + |v_2| + |v_3|\right), \\ &\forall (\omega_1, \omega_2, v_1, v_2, v_3) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{3d}. \end{aligned} \tag{5.2}$$

Therefore, by (5.1)–(5.2), the definition of the norm and scaling (4.22)

$$\begin{aligned} \left|\mathcal{C}^N_{m,m+2} g_{N,m+2}(Z_m)\right| &\lesssim e^{-\beta E_m(Z_m)} |g_{N,m+2}|_{N,\beta,m+2} \\ &\times \sum_{i=1}^m \int_{\mathbb{R}^{2d}} \left(|v_i| + |v_{m+1}| + |v_{m+2}|\right) e^{-\frac{\beta}{2}(|v_{m+1}|^2 + |v_{m+2}|^2)}\, dv_{m+1}\, dv_{m+2}. \end{aligned}$$

Using Fubini's theorem and the elementary integrals

$$\int_0^\infty e^{-\frac{\beta}{2}x^2}dx \simeq \beta^{-1/2}, \int_0^\infty xe^{-\frac{\beta}{2}x^2}dx \simeq \beta^{-1},$$

we obtain the required estimate. □

Now we define a mild solution of the BBGKY hierarchy in the scaling (4.22) as follows:

Definition 5.2. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $G_{N,0} = \left(g_{N,s,0}\right) \in X_{N,\beta_0,\mu_0}$. A map $\boldsymbol{G_N} = \left(g_{N,s}\right)_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution of the BBGKY hierarchy (4.16) in $[0, T]$ if it satisfies

$$\boldsymbol{G_N}(t) = \mathcal{T}^t G_{N,0} + \int_0^t \mathcal{T}^{t-\tau}\mathcal{C}_N \boldsymbol{G_N}(\tau)\,d\tau, \tag{5.3}$$

where $\mathcal{C}_N G_N = \left(\mathcal{C}_{s,s+2}^N g_{N,s+2}\right)_{s\in\mathbb{N}}$ and $\mathcal{T}^t = (T_s^t)_{s\in\mathbb{N}}$, where T_s^t is given by (3.29).

Remark 5.3. We note that the above collision operators $\mathcal{C}_{s,s+2}^N$ are ill-defined on L^∞ since they involve integration over a set of measure zero (the sphere $\mathbb{S}_1^{d-1}$). However, by filtering our BBGKY hierarchy by the flow T_s^{-t}, we may obtain a well defined operator on $\boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$. This is done in detail in the erratum of Chapter 5 of [19] and does not affect the energy estimates or local well-posedness of the hierarchy. This filtering process can be adapted to our context. Hence, we will abuse the notation and continue to work with (5.3). See also [31] for a different approach which avoids this issue by working with measures on the phase space.

We will address well-posedness of the BBGKY hierarchy by a fixed point argument. For this purpose, we state an important estimate.

Lemma 5.4. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\boldsymbol{\beta}_\lambda, \boldsymbol{\mu}_\lambda : [0, T] \to \mathbb{R}$ given by*

$$\boldsymbol{\beta}_\lambda(t) = \beta_0 - \lambda t, \quad \boldsymbol{\mu}_\lambda(t) = \mu_0 - \lambda t. \tag{5.4}$$

Then for any $\mathcal{F}(t) \subseteq [0, t]$ measurable, $s \in \mathbb{N}$ and $\boldsymbol{G_N} = \left(g_{N,s}\right)_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}$ the following bound holds:

$$\begin{aligned}&\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau}\boldsymbol{\mathcal{C}_N G_N}(\tau)\,d\tau\right|\right|\right|_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} \leq C(d, \beta_0, \mu_0, T, \lambda)|||\boldsymbol{G_N}|||_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda},\\ &C(d, \beta_0, \mu_0, T, \lambda) \simeq \lambda^{-1}e^{-2\boldsymbol{\mu}(T)}\boldsymbol{\beta}_\lambda(T)^{-d}\left(1 + \boldsymbol{\beta}_\lambda\,(T)^{-1/2}\right).\end{aligned} \tag{5.5}$$

Proof. Since energy is conserved by the flow and we have the continuity estimate of Lemma 5.1 for the collisional operator, the proof follows similarly to the proof of Lemma 5.3.1. in [19]. □

Choosing $\lambda = \beta_0/2T$, and $T = T(\beta_0, \mu_0)$ small enough, Lemma 5.4 implies local well-posedness of the BBGKY hierarchy via a fixed point argument.

Theorem 5.5. *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial datum $F_{N,0} = (f_{N,0}^{(s)})_{s\in\mathbb{N}} \in X_{N,\beta_0,\mu_0}$ there is unique mild solution $\boldsymbol{F_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ of the BBGKY hierarchy (4.16) in $[0, T]$ for the functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ given by*

$$\boldsymbol{\beta}(t) = \beta_0 - \frac{\beta_0}{2T}t, \quad \boldsymbol{\mu}(t) = \mu_0 - \frac{\beta_0}{2T}t. \tag{5.6}$$

Moreover, for any $\mathcal{F}(t) \subseteq [0, t]$ measurable, the following bounds hold:

$$\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau} \mathcal{C}_N \boldsymbol{G_N}(\tau)\, d\tau\right|\right|\right|_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \le \frac{1}{8}|||\boldsymbol{G_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}}, \quad \forall \boldsymbol{G_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}, \tag{5.7}$$

$$|||\boldsymbol{F_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \le 2\|F_{N,0}\|_{N,\beta_0,\mu_0}. \tag{5.8}$$

5.2. LWP for the Boltzmann hierarchy. For the Boltzmann hierarchy analogous estimates follow in a similar manner as for the BBGKY hierarchy in the appropriate functional spaces.

Given $\beta > 0$ and $s \in \mathbb{N}$ we define the Banach space

$$X_{\infty,\beta,s} := \left\{ g_s \in L^\infty(\mathbb{R}^{2ds}) : |g_s|_{\infty,\beta,s} := \operatorname*{ess\,sup}_{Z_s\in\mathbb{R}^{2ds}} |g_s(Z_s)|e^{\beta E_s(Z_s)} < \infty \right\}.$$

Consider as well $\mu \in \mathbb{R}$. We define the Banach space

$$X_{\infty,\beta,\mu} := \left\{ G = (g_s)_{s\in\mathbb{N}} : \|G\|_{\infty,\beta,\mu} := \sup_{s\in\mathbb{N}} e^{\mu s}|g_s|_{\infty,\beta,s} < \infty \right\}.$$

Finally, for $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(T) > 0$ we define the Banach space

$$\boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} = C^0\left([0, T], X_{\infty,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\right), \text{ with norm } |||\boldsymbol{G}||| := \sup_{t\in[0,T]} \|\boldsymbol{G}(t)\|_{\infty,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}.$$

We define a mild solution of the Boltzmann hierarchy as follows.

Definition 5.6. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $G_0 = \left(g_{s,0}\right) \in X_{\infty,\beta_0,\mu_0}$. A map $\boldsymbol{G} = (g_s)_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution of the Boltzmann hierarchy (4.27) in $[0, T]$, with initial data G_0, if it satisfies:

$$\boldsymbol{G}(t) = \mathcal{S}^t G_0 + \int_0^t \mathcal{S}^{t-\tau} \mathcal{C}_\infty \boldsymbol{G}(\tau)\, d\tau, \tag{5.9}$$

where $\mathcal{C}_\infty G = \left(\mathcal{C}_{s,s+2}^\infty g_{s+2}\right)_{s\in\mathbb{N}}$, and $\mathcal{S}^t G = (S_s^t g_s)_{s\in\mathbb{N}}$, where S_s^t is given by (3.30).

Remark 5.7. As noted in Remark 5.3, the operators $\mathcal{C}_{s,s+2}^\infty$ are ill defined on L^∞ due to the integration over the lower dimension manifold $\mathbb{S}_1^{d-1}$. As in the BBGKY case, one can filter the infinite hierarchy by S_s^{-t} to obtain a well defined mild formulation of the hierarchy. However, for simplicity, we will abuse notation and continue to use (5.9)

Now we state the well-posedness result for the Boltzmann hierarchy.

Theorem 5.8. *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is*[6] $T = T(d, \beta_0, \mu_0) > 0$ *such that for any initial datum* $F_0 = (f_0^{(s)})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$ *there is unique mild solution* $\boldsymbol{F} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ *of the Boltzmann hierarchy* (4.27) *in* $[0, T]$ *for the functions* $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ *given by* (5.6).

Moreover, for any $\mathcal{F}(t) \subseteq [0, t]$ *measurable, the following estimates hold:*

$$\left|\left|\left| \int_{\mathcal{F}(t)} \mathcal{S}^{t-\tau} C_\infty \boldsymbol{G}(\tau)\, d\tau \right|\right|\right|_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{8} |||\boldsymbol{G}|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}, \quad \forall \boldsymbol{G} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}, \tag{5.10}$$

$$|||\boldsymbol{F}|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|F_0\|_{\infty,\beta_0,\mu_0}. \tag{5.11}$$

5.3. LWP for the ternary Boltzmann equation and propagation of chaos. Here, we first present local well-posedness for the ternary Boltzmann equation. The proofs are nonlinear analogues of the arguments used in the BBGKY case (for details see [2]). Furthermore, we show that for chaotic initial data their tensorized product produces the unique mild solution of the Boltzmann hierarchy, hence chaos is propagated.

For $\beta > 0$ let us define the Banach space

$$X_{\beta,\mu} := \left\{ g \in L^\infty(\mathbb{R}^{2d}) : |g|_{\beta,\mu} := \operatorname*{ess\,sup}_{(x,v)\in\mathbb{R}^{2d}} |g(x, v)| e^{\mu + \frac{\beta}{2}|v|^2} < \infty \right\}.$$

Consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$ and $\boldsymbol{\mu}(0) = \mu_0$. We define the Banach space

$$\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}} := C^0\left([0, T], X_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\right), \text{ with norm } \|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} = \operatorname*{ess\,sup}_{t\in[0,T]} |\boldsymbol{g}(t)|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}.$$

We define mild solutions to the ternary Boltzmann equation as follows:

Definition 5.9. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ decreasing functions of time, with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $g_0 \in X_{\beta_0,\mu_0}$. A map $\boldsymbol{g} \in \boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution of the ternary Boltzmann equation (4.31) in $[0, T]$, with initial data $g_0 \in X_{\beta_0,\mu_0}$, if it satisfies

$$\boldsymbol{g}(t) = S_1^t g_0 + \int_0^t S_1^{t-\tau} Q_3(\boldsymbol{g}, \boldsymbol{g}, \boldsymbol{g})(\tau)\, d\tau. \tag{5.12}$$

where S_1^t denotes the free flow of 1-particle given in (3.30).

Remark 5.10. As in Remarks 5.3 and 5.7, the operators Q_3 can be filtered by the free flow S_1^{-t} in order to define the above equation on L^∞. Hence, we will abuse notation and continue to work with (5.12).

Let us write $B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ for the unit ball of $\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$. Then the following well-posedness result holds

Theorem 5.11. *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is*[7] $T = T(d, \beta_0, \mu_0) > 0$ *such that for any initial data* $f_0 \in X_{\beta_0,\mu_0}$, *with* $|f_0|_{\beta_0,\mu_0} \leq \frac{1}{2}$, *there is a unique mild solution* $\boldsymbol{f} \in B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ *to the ternary Boltzmann equation in* $[0, T]$ *with initial data* f_0, *where* $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ *are the functions given by* (5.6).

[6] The time of existence is the same as in Theorem 5.5.

[7] The time of existence is the same as in Theorem 5.5.

Remark 5.12. The smallness assumption on the initial data is needed in order to produce a solution up to the time of existence of solutions to the BBGKY and Boltzmann hierarchy obtained in Theorems 5.5 and 5.8 respectively. One can produce a solution for general initial data, as was done for the Boltzmann equation in [27], but the time of existence would be smaller due to the nonlinearity of (4.31).

We can now prove that chaos is propagated by the Boltzmann hierarchy.

Theorem 5.13. (Propagation of chaos) *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ the time obtained by Theorem 5.11 and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ the functions defined by (5.6). Consider $f_0 \in X_{\beta_0,\mu_0}$ with $|f_0|_{\beta_0,\mu_0} \leq \frac{1}{2}$. Assume $\boldsymbol{f} \in B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ is the corresponding mild solution of the ternary Boltzmann equation in $[0, T]$, with initial data f_0 given by Theorem 5.11. Then the following hold:*

(i) $F_0 = (f_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$.
(ii) $\boldsymbol{F} = (\boldsymbol{f}^{\otimes s})_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$.
(iii) *$\boldsymbol{F}$ is the unique mild solution of the Boltzmann hierarchy in $[0, T]$, with initial data F_0.*

Proof. *(i)* is verified by the bound on the initial data and the definition of the norms. By the the same bound again, we may apply Theorem 5.11 to obtain the unique mild solution $\boldsymbol{f} \in B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ of the corresponding ternary Boltzmann equation. Since $\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq 1$, the definition of the norms directly imply *(ii)*. It is also staightforward to verify that $\boldsymbol{F}$ is a mild solution of the Boltzmann hierarchy in $[0, T]$, with initial data F_0. Uniqueness of the mild solution to the Boltzmann hierarchy, obtained by Theorem 5.8, implies that $\boldsymbol{F}$ is the unique mild solution. □

6. Convergence Statement

In this section, we define an appropriate notion of convergence, namely convergence in observables, and we state the main result of this paper. While our convergence result is valid for a general type of Boltzmann initial data and approximation by BBGKY hierarchy initial data (see Definition 6.1), we also provide a rate of convergence in the case of chaotic Boltzmann initial data and initial approximation by conditioned BBGKY hierarchy initial data (introduced in Definition 6.4).

Throughout this section, we consider (N, ϵ) in the scaling (4.22). We will also use the phase space $\mathcal{D}_{m,\epsilon}$ of m-particles of ϵ-interaction zone given by (3.1) and the functional spaces of Sect. 5.

6.1. Approximation of Boltzmann initial data. This Subsection focuses on introducing relevant types of initial data. First, we define the general notion of BBGKY hierarchy sequences approximating Boltzmann hierarchy initial data. Then we show that chaotic initial data produced by tensorized probability densities are approximated by conditioned BBGKY hierarchy sequences in the scaling (4.22).

Definition 6.1. Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $G_0 = (g_{s,0})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$. A sequence $G_{N,0} = (g_{N,s,0})_{s\in\mathbb{N}} \in X_{N,\beta_0,\mu_0}$ is called a BBGKY hierarchy sequence approximating G_0 if the following conditions hold:

(i) $\sup_{N\in\mathbb{N}} \|G_{N,0}\|_{N,\beta_0,\mu_0} < \infty$.
(ii) For any $s \in \mathbb{N}$ there holds $\lim_{N\to\infty} \|g_{N,s,0} - g_{s,0}\|_{L^\infty(\mathcal{D}_{s,\epsilon})} = 0$.

Remark 6.2. Every $G_0 = (g_{s,0})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$ has a BBGKY hierarchy approximating sequence. Indeed, it is straightforward to verify that the sequence $G_{N,0} = (g_{N,s,0})_{s\in\mathbb{N}}$ given by $g_{N,s,0} = \mathbb{1}_{\mathcal{D}_{s,\epsilon}} g_{s,0}$ satisfies the properties stated above in the scaling (4.22).

Especially meaningful initial data, corresponding to initial independence between particles, are given below:

Remark 6.3. Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density i.e. $g_0 > 0$ a.e. and $\int_{\mathbb{R}^{2d}} g_0(x,v)\,dx\,dv = 1$ and assume that $\|g_0\|_{\beta_0,\mu_0+1} \leq 1$. Then one can easily see that the chaotic configuration $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1} \subseteq X_{\infty,\beta_0,\mu_0}$. This type of initial data, corresponding to tensorized initial measures, will lead to the ternary Boltzmann equation (4.31). In fact, we will see that one can approximate tensorized initial data in the scaling (4.22) by conditioned BBGKY hierarchy initial data which are defined below.

Definition 6.4. Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density and denote $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1}$. We define the conditioned BBGKY hierarchy sequence $G_{N,0} = (g_{N,0}^{(s)})_{s\in\mathbb{N}}$ of G_0 as:

$$g_{N,0}^{(s)}(X_s,V_s) = \begin{cases} \mathcal{Z}_N^{-1} \displaystyle\int_{\mathbb{R}^{2d(N-s)}} \mathbb{1}_{\mathcal{D}_{N,\epsilon}} g_0^{\otimes N}(X_s, x_{s+1},\ldots,x_N, V_s, v_{s+1},\ldots,v_N) \\ \quad dx_{s+1}\,dv_{s+1}\ldots dx_N\,dv_N, \quad 1 \leq s < N \\ \mathcal{Z}_N^{-1} \mathbb{1}_{\mathcal{D}_{N,\epsilon}} g_0^{\otimes N}(Z_N), \quad s = N, \\ 0, \quad s > N. \end{cases} \tag{6.1}$$

where the normalization is preserved by the introduction of the partition function:

$$\mathcal{Z}_m = \int_{\mathbb{R}^{2dm}} \mathbb{1}_{\mathcal{D}_{m,\epsilon}} g_0^{\otimes m}(X_m,V_m)\,dX_m\,dV_m, \quad m \in \mathbb{N}.$$

Notice that since g_0 is a.e. positive and integrates to 1, we have $0 < Z_m < 1$ for all $m \in \mathbb{N}$.

Let us now prove that the conditioned BBGKY hierarchy sequence of tensorized initial data is an approximating sequence (according to Definition 6.1). This will be a crucial tool to obtain rate of convergence to the solution of the ternary Boltzmann equation (4.31) (see Corollary 6.11 for more details). We will need the following auxiliary estimate on the partition functions.

Lemma 6.5. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $g_0 \in L_x^\infty L_v^1(\mathbb{R}^{2d})$ be a positive probability density. Then for all (N,ϵ) in the scaling (4.22) with $2C_d\epsilon^{1/2}\|g_0\|_{L_x^\infty L_v^1} < 1$, where C_d is a positive constant, and all $m \in \mathbb{N}$ with $m < N$, there holds*

$$1 \leq \mathcal{Z}_N^{-1}\mathcal{Z}_{N-m} \leq (1 - C_d\|g_0\|_{L_x^\infty L_v^1}\epsilon^{1/2})^{-m},$$

for some constant $C_d > 0$.

Proof. The left hand side inequality is immediate from the definition of the phase space (3.1). To prove the right hand side consider $k \in \mathbb{N}$ with $k \leq N$. Notice that for any $Z_{k+1} = (X_{k+1},V_{k+1}) \in \mathbb{R}^{2d(k+1)}$, we have

$$\mathbb{1}_{\mathcal{D}_{k+1,\epsilon}}(X_{k+1},V_{k+1}) \geq \mathbb{1}_{\mathcal{D}_{k,\epsilon}}(X_k,V_k)\prod_{i=1}^k \mathbb{1}_{|x_i - x_{k+1}| > \sqrt{2}\epsilon}(x_i),$$

by the definition of the phase space (3.1). Let us note that the above inequality applies specifically to the ternary interactions we consider. Then we can proceed in a similar manner as in the proof of Lemma 6.1.2 in [19], using the ternary scaling (4.22) instead. More specifically, the previous inequality and Fubini's Theorem imply

$$\begin{aligned}\mathcal{Z}_{k+1} &= \int_{\mathbb{R}^{2d(k+1)}} \mathbb{1}_{\mathcal{D}_{k+1,\epsilon}} g_0^{\otimes(k+1)}(X_{k+1}, V_{k+1})\, dX_{k+1}\, dV_{k+1} \\ &\geq \int_{\mathbb{R}^{2dk}} \left(\int_{\mathbb{R}^{2d}} \prod_{i=1}^{k} \mathbb{1}_{|x_i - x| > \sqrt{2}\epsilon}(x_i) g_0(x, v)\, dx\, dv \right) \\ &\quad \mathbb{1}_{\mathcal{D}_{k,\epsilon}}(X_k, V_k) g_0^{\otimes m}(X_k, V_k)\, dX_k\, dV_k.\end{aligned}$$

But since g_0 integrates to 1, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \prod_{i=1}^{k} \mathbb{1}_{|x_i - x| > \sqrt{2}\epsilon}(x_i) g_0(x, v)\, dx\, dv \geq 1 \\ &- \sum_{i=1}^{k} \int_{\mathbb{R}^{2d}} \mathbb{1}_{|x_i - x| \leq \sqrt{2}\epsilon}(x_i) g_0(x, v)\, dx\, dv \geq 1 - k C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^d,\end{aligned}$$

upon integrating on a d-ball of radius $\sqrt{2}\epsilon$. Hence

$$\begin{aligned}\mathcal{Z}_{k+1} &\geq (1 - k C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^d) \mathcal{Z}_k \geq (1 - N C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^d) \\ \mathcal{Z}_k &\simeq (1 - C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2}) \mathcal{Z}_k,\end{aligned} \tag{6.2}$$

due to scaling (4.22). For $2C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2} < 1$, we may apply inductively (6.2) for $k = m, \ldots, N-1$, and the claim follows. □

Proposition 6.6. *Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density with $|g_0|_{\beta_0,\mu_0+1} \leq 1$ and $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1} \subseteq X_{\infty,\beta_0,\mu_0}$. Let $G_{N,0} = (g_{N,0}^{(s)})_{s\in\mathbb{N}}$ be the conditioned BBGKY hierarchy sequence of the tensorized initial data G_0 given in Definition 6.4. Then $G_{N,0}$ is a BBGKY hierarchy sequence approximating G_0 (in the sense of Definition 6.1) in the scaling (4.22). In particular for all (N, ϵ) in the scaling (4.22) with N large enough (or equivalently ϵ small enough), there holds the estimate*

$$\|g_{N,0}^{(s)} - g_0^{\otimes s}\|_{L^\infty(\mathcal{D}_{s,\epsilon})} \leq C_{d,s,\beta_0,\mu_0} \epsilon^{1/2} \|G_0\|_{\infty,\beta_0,\mu_0}. \tag{6.3}$$

Proof. By definition of the phase space (3.1), for any $s \in \mathbb{N}$, with $s < N$ and $Z_N \in \mathcal{D}_{N,\epsilon}$ we can write

$$\begin{aligned}\mathbb{1}_{\mathcal{D}_{N,\epsilon}}(Z_N) =& \mathbb{1}_{\mathcal{D}_{s,\epsilon}}(Z_s) \prod_{1\leq i<j\leq s<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon^2}(x_i, x_j, x_k) \\ &\prod_{1\leq i\leq s<j<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon^2}(x_i, x_j, x_k) \\ &\prod_{s+1\leq i<j<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon^2}(x_i, x_j, x_k).\end{aligned}$$

Again this decomposition of the phase space is due to the ternary interactions we consider and is necessary to track all the cases arising from ternary interactions. Moreover, by symmetry, for $s < N$ we can also write

$$\mathcal{Z}_{N-s} = \int_{\mathbb{R}^{2d(N-s)}} \prod_{s+1\leq \ell_1<\ell_2<\ell_3\leq N} \mathbb{1}_{|x_{\ell_1}-x_{\ell_2}|^2+|x_{\ell_1}-x_{\ell_3}|^2>2\epsilon^2}(x_{\ell_1}, x_{\ell_2}, x_{\ell_3}) \prod_{\ell=s+1}^{N} g_0(x_\ell, v_\ell)\, dZ_{(s+1,N)},$$

where $dZ_{(s+1,N)} := dx_{s+1}\dots dx_N\, dv_{s+1}\dots dv_N$. Therefore, given $Z_s \in \mathbb{R}^{2ds}$, an elementary calculation gives

$$g_{N,0}^{(s)}(Z_s) = \mathcal{Z}_N^{-1} \mathbb{1}_{\mathcal{D}_{s,\epsilon}}(Z_s) g_0^{\otimes s}(Z_s)\left(\mathcal{Z}_{N-s} - \mathcal{R}_{s+1,N}(Z_s)\right), \tag{6.4}$$

where the error term $\mathcal{R}_{s+1,N}(Z_s) > 0$ is given by

$$\begin{aligned}
&\mathcal{R}_{s+1,N}(Z_s)\\
&= \int_{\mathbb{R}^{2d(N-s)}} \Bigg(1 - \prod_{1\leq i<j\leq s<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon^2}(x_k)\\
&\quad \prod_{1\leq i\leq s<j<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon^2}(x_j, x_k)\Bigg)\\
&\quad \prod_{s+1\leq \ell_1<\ell_2<\ell_3\leq N} \mathbb{1}_{|x_{\ell_1}-x_{\ell_2}|^2+|x_{\ell_1}-x_{\ell_3}|^2>2\epsilon^2}(x_{\ell_1}, x_{\ell_2}, x_{\ell_3}) \prod_{\ell=s+1}^{N} g_0(x_\ell, v_\ell)\, dZ_{(s+1,N)}\\
&\leq \int_{\mathbb{R}^{2d(N-s)}} \Bigg(\sum_{1\leq i<j\leq s<k\leq N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2\leq 2\epsilon^2}(x_k)\\
&\quad + \sum_{1\leq i\leq s<j<k\leq N} \int_{\mathbb{R}^{2d}} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2\leq 2\epsilon^2}(x_j, x_k)\Bigg)\\
&\quad \prod_{s+1\leq \ell_1<\ell_2<\ell_3\leq N} \mathbb{1}_{|x_{\ell_1}-x_{\ell_2}|^2+|x_{\ell_1}-x_{\ell_3}|^2>2\epsilon^2}(x_{\ell_1}, x_{\ell_2}, x_{\ell_3}) \prod_{\ell=s+1}^{N} g_0(x_\ell, v_\ell)\, dZ_{(s+1,N)}\\
&:= I_1 + I_2.
\end{aligned} \tag{6.5}$$

By (6.4), and the fact that $\mathcal{Z}_{N-s} \leq 1$ since g_0 integrates to 1, by definition of the norms, we have

$$\|G_{N,0}\|_{N,\beta_0,\mu_0} \leq \mathcal{Z}_N^{-1} \|G_0\|_{\infty,\beta_0,\mu_0} < \infty,$$

so $G_{N,0} \in X_{N,\beta_0,\mu_0}$ for all $N \in \mathbb{N}$. Moreover, since

$$\|g_0\|_{L^\infty_x L^1_v} \leq C_d \beta^{-1/2} e^{-\mu_0} |g_0|_{\beta_0,\mu_0} < \infty, \tag{6.6}$$

for $2C_d\epsilon^{1/2}\|g_0\|_{L^\infty_x L^1 v} < 1$ (or equivalently for N large enough), Lemma 6.5 gives

$$g_{N,0}^{(s)}(Z_s) \leq (1-C_d\|g_0\|_{L^\infty_x L^1_v}\epsilon^{1/2})^{-s} g_0^{\otimes s}(Z_s) \leq e^{2sC_d\|g_0\|_{L^\infty_x L^1_v}\epsilon^{1/2}} g_0^{\otimes s}(Z_s) \leq e^s g_0^{\otimes s}(Z_s),$$

where we used the inequality $2x - \ln(1-x) \geq 0$, $x \in [0, 1/2]$. This clearly implies

$$\|G_{N,0}\|_{N,\beta_0,\mu_0} \leq \|G_0\|_{\infty,\beta_0,\mu_0+1} < \infty,$$

for N large enough, thus $\sup_{N\in\mathbb{N}} \|G_{N,0}\|_{N,\beta_0,\mu_0} < \infty$.

To prove convergence, by (6.4) and the definition of the norms we take

$$\left|\mathbb{1}_{\mathcal{D}_{s,\epsilon}}(g_0^{\otimes s} - g_{N,0}^{(s)})(Z_s)\right| \leq \left(\left|1-\mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\right| + \mathcal{Z}_N^{-1}\mathcal{R}_{s+1,N}(Z_s)\right) e^{-s\mu_0}\|G_0\|_{\infty,\beta_0,\mu_0}. \tag{6.7}$$

Let us estimate each term on (6.7) separately. By Lemma 6.5 and the inequality $2x - \ln(1-x) \geq 0$, $x \in [0, 1/2]$, for $2\epsilon^{1/2}C_d\|g_0\|_{L^\infty_x L^1_v} < 1$, we have

$$|1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}| \leq e^{2s\epsilon^{1/2}C_d\|g_0\|_{L^\infty_x L^1_v}} - 1 \leq 2es\epsilon^{1/2}C_d\|g_0\|_{L^\infty_x L^1_v}, \tag{6.8}$$

by the Mean Value Theorem.

For the term $\mathcal{Z}_N^{-1}\mathcal{R}_{s+1,N}$, we estimate each of the terms I_1, I_2 in (6.5). For the term I_1, fix $1 \leq i < j \leq s < k \leq N$. Notice the inequality

$$\mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2\leq 2\epsilon^2}(x_k) \leq \mathbb{1}_{|x_i-x_k|\leq\sqrt{2}\epsilon}(x_k).$$

Then, by symmetry, the term corresponding to i, j, k is estimated by

$$\begin{aligned}&\int_{\mathbb{R}^{2d(N-s-1)}}\left(\int_{\mathbb{R}^{2d}} \mathbb{1}_{|x_i-x_{s+1}|<\sqrt{2}\epsilon}(x_{s+1})g_0(x_{s+1}, v_{s+1})\,dx_{s+1}\,dv_{s+1}\right)\\ &\qquad \prod_{s+2\leq\ell_1<\ell_2<\ell_3\leq N} \mathbb{1}_{|x_{\ell_1}-x_{\ell_2}|^2+|x_{\ell_1}-x_{\ell_3}|^2>2\epsilon^2}(x_{\ell_1}, x_{\ell_2}, x_{\ell_3}) \prod_{\ell=s+2}^{N} g_0(x_\ell, v_\ell)\,dZ_{(s+2,N)}\\ &\quad \leq C_d\|g_0\|_{L^\infty_x L^1_v}\epsilon^d\mathcal{Z}_{N-s-1},\end{aligned}$$

after integrating in a d-ball of radius $\sqrt{2}\epsilon$ centered at x_i. Adding for $1 \leq i < j \leq s < k \leq N$ we obtain

$$I_1 \leq s^2NC_d\|g_0\|_{L^\infty_x L^1_v}\epsilon^d\mathcal{Z}_{N-s-1} \simeq C_d s^2\epsilon^{1/2}\|g_0\|_{L^\infty_x L^1_v}\mathcal{Z}_{N-s-1}, \tag{6.9}$$

due to (4.22). For the term I_2, fix $1 \leq 1 \leq s < j < k \leq N$. By symmetry again the corresponding term is estimated by

$$\begin{aligned}&\int_{\mathbb{R}^{2d(N-s-2)}}\left(\int_{\mathbb{R}^{2d}} \mathbb{1}_{|x_i-x_{s+1}|^2+|x_i-x_{s+2}|^2\leq 2\epsilon^2}(x_{s+1}, x_{s+2})\right.\\ &\qquad \left. g_0(x_{s+1}, v_{s+1})g_0(x_{s+2}, v_{s+2})\,dx_{s+1}\,dx_{s+2}\,dv_{s+1}\,dv_{s+2}\right)\\ &\qquad \prod_{s+3\leq\ell_1<\ell_2<\ell_3\leq N} \mathbb{1}_{|x_{\ell_1}-x_{\ell_2}|^2+|x_{\ell_1}-x_{\ell_3}|^2>2\epsilon^2}(x_{\ell_1}, x_{\ell_2}, x_{\ell_3}) \prod_{\ell=s+3}^{N} g_0(x_\ell, v_\ell)\,dZ_{(s+3,N)}\\ &\quad \leq C_d\|g_0\|^2_{L^\infty_x L^1_v}\epsilon^{2d}\mathcal{Z}_{N-s-2}.\end{aligned}$$

after integrating in a $2d$-ball of radius ϵ centered at $\binom{x_i}{x_i}$. Adding for $1 \leq i \leq s < j < k \leq N$ we obtain

$$I_2 \leq sN^2 C_d \|g_0\|^2_{L^\infty_x L^1_v} \epsilon^{2d} \mathcal{Z}_{N-s-2} \simeq sC_d \|g_0\|^2_{L^\infty_x L^1_v} \epsilon \mathcal{Z}_{N-s-2}, \tag{6.10}$$

Using (6.5)–(6.10) and Lemma 6.5 (applied for $m = s + 1$ and $m = s + 2$, we obtain

$$\begin{aligned} \mathcal{Z}_N^{-1} \mathcal{R}_{s+1,N}(Z_s) &\lesssim s^2 C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2} (1 - C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2})^{-(s+1)} \\ &\quad + sC_d \|g_0\|^2_{L^\infty_x L^1_v} \epsilon (1 - C_d \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2})^{-(s+2)} \\ &\lesssim C_{d,s} \|g_0\|_{L^\infty_x L^1_v} \epsilon^{1/2}, \end{aligned} \tag{6.11}$$

since $2C_d \epsilon^{1/2} \|g_0\|_{L^\infty_x L^1_v} < 1$. Combining (6.7)–(6.8), (6.11), and (6.6), we obtain estimate (6.3) and the required convergence follows. □

6.2. Convergence in observables. Now, we define the convergence in observables. Given $s \in \mathbb{N}$, we use the space of test continuous and compactly supported functions in velocities $C_c(\mathbb{R}^{ds})$.

Definition 6.7. Consider $T > 0$, $s \in \mathbb{N}$ and $g_s \in C^0\left([0, T], L^\infty\left(\mathbb{R}^{2ds}\right)\right)$. Given a test function $\phi_s \in C_c(\mathbb{R}^{ds})$, we define the s-observable functional as: $I_{\phi_s} g_s(t)(X_s) = \displaystyle\int_{\mathbb{R}^{ds}} \phi_s(V_s) g_s(t, X_s, V_s)\, dV_s$.

Before giving the definition of convergence in observables, we start with some definitions on the configurations we are using. Given $m \in \mathbb{N}$ and $\sigma > 0$, we define the set of well-separated spatial configurations

$$\Delta_m^X(\sigma) = \{\widetilde{X}_m \in \mathbb{R}^{dm} : |\widetilde{x}_i - \widetilde{x}_j| > \sigma, \quad \forall 1 \leq i < j \leq m\}, \quad m \geq 2, \quad \Delta_1^X(\sigma) = \mathbb{R}^{2d}, \tag{6.12}$$

and the set of well separated configurations

$$\Delta_m(\sigma) = \Delta_m^X(\sigma) \times \mathbb{R}^{dm}. \tag{6.13}$$

Definition 6.8. Let $T > 0$. For each $N \in \mathbb{N}$, consider $\boldsymbol{G_N} = (g_{N,s})_{s\in\mathbb{N}} \in \prod_{s=1}^\infty C^0\left([0, T], L^\infty\left(\mathbb{R}^{2ds}\right)\right)$ and $\boldsymbol{G} = (g_s)_{s\in\mathbb{N}} \in \prod_{s=1}^\infty C^0\left([0, T], L^\infty\left(\mathbb{R}^{2ds}\right)\right)$. We say that the sequence $(\boldsymbol{G_N})_{N\in\mathbb{N}}$ converges in observables to $\boldsymbol{G}$, and write

$$\boldsymbol{G_N} \overset{\sim}{\longrightarrow} \boldsymbol{G},$$

if for any $\sigma > 0$, $s \in \mathbb{N}$, and $\phi_s \in C_c(\mathbb{R}^{ds})$, we have

$$\lim_{N\to\infty} \|I_{\phi_s} g_{N,s}(t) - I_{\phi_s} g_s(t)\|_{L^\infty(\Delta_s^X(\sigma))} = 0, \quad \text{uniformly in } [0, T].$$

6.3. Statement of the main result. We are now in the position to state our main result.

Theorem 6.9. (Convergence) *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and consider Boltzmann hierarchy initial data $F_0 = (f_0^{(s)})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$. Let $\left(F_{N,0}\right)_{N\in\mathbb{N}}$ be a BBGKY hierarchy sequence approximating F_0 . Assume that:*

- *For each N, $\boldsymbol{F_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ is the mild solution of the BBGKY hierarchy (4.16) with initial data $F_{N,0}$ in $[0, T]$.*
- *$\boldsymbol{F} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ is the mild solution of the Boltzmann hierarchy (4.27) with initial data F_0 in $[0, T]$.*
- *F_0 satisfies the following uniform continuity condition: There exists $C > 0$ such that, for any $\zeta > 0$, there is $q = q(\zeta) > 0$ such that for all $s \in \mathbb{N}$, and for all $Z_s, Z_s' \in \mathbb{R}^{2ds}$ with $|Z_s - Z_s'| < q$, we have*

$$|f_0^{(s)}(Z_s) - f_0^{(s)}(Z_s')| < C^{s-1}\zeta. \tag{6.14}$$

Then $\boldsymbol{F_N} \overset{\sim}{\longrightarrow} \boldsymbol{F}$.

Remark 6.10. To prove Theorem 6.9 it suffices to prove

$$\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\sigma))} \overset{N\to\infty}{\longrightarrow} 0, \text{ uniformly in } [0, T],$$

for any $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ and $\sigma > 0$, where

$$I_s^N(t)(X_s) := I_{\phi_s} f_N^{(s)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_N^{(s)}(t, X_s, V_s)\, dV_s, \tag{6.15}$$

$$I_s^\infty(t)(X_s) := I_{\phi_s} f^{(s)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s)}(t, X_s, V_s)\, dV_s. \tag{6.16}$$

The following Corollary of Theorem 6.9 justifies the derivation of our ternary Boltzmann equation from finitely many particle systems.

Corollary 6.11. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $f_0 \in X_{\beta_0,\mu_0+1}$ be a Hölder continuous $C^{0,\gamma}$, $\gamma \in (0, 1]$ probability density with $|f_0|_{\beta_0,\mu_0+1} \le 1/2$. Let us write $F_0 = (f_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1}$ and let $F_{N,0} = (f_{N,0}^{(s)})_{s\in\mathbb{N}}$ be the conditioned BBGKY hierarchy sequence given in Definition 6.4 approximating the tensorized data F_0. Then for any $\sigma > 0$, $s \in \mathbb{N}$ and $\phi_s \in C_c(\mathbb{R}^{ds})$, we have the rate of convergence*

$$\|I_{\phi_s} f_N^{(s)}(t) - I_{\phi_s} f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\sigma))} = O(\epsilon^r), \quad \textit{uniformly in } [0, T], \tag{6.17}$$

for any $0 < r < \min\{1/2, \gamma\}$, where $\boldsymbol{F_N} = (f_N^{(s)})_{s\in\mathbb{N}} \in \boldsymbol{X_{N,\beta,\mu}}$ is the mild solution of the BBGKY hierarchy (4.16) in $[0, T]$ with initial data $F_{N,0}$ and f is the mild solution to the ternary Boltzmann equation (4.31) in $[0, T]$, with initial data f_0.

7. Reduction to Term by Term Convergence

Now, we reduce the proof of Theorem 6.9 to term by term convergence by truncating the observables. Throughout this section, we consider $\beta_0 > 0, \mu_0 \in \mathbb{R}, T = T(d, \beta_0, \mu_0) > 0$ be the time given by Theorems 5.5 and 5.8, the functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ defined by (5.6), (N, ϵ) in the scaling (4.22) and initial data $F_{N,0} \in X_{N,\beta_0,\mu_0}$, $F_0 \in X_{\infty,\beta_0,\mu_0}$. Let $\boldsymbol{F_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$, $\boldsymbol{F} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ be the mild solutions of the corresponding BBGKY hierarchy and Boltzmann hierarchy in $[0, T]$, given by Theorems 5.5 and 5.8.

7.1. Series expansion. Let us fix $s \in \mathbb{N}$. Using iteratively the Duhamel's formula for the mild solution of the BBGKY hierarchy, given by (5.3), we get the following expansion:

$$f_N^{(s)}(t, Z_s) = \sum_{k=0}^{n} f_N^{(s,k)}(Z_s) + R_N^{(n+1)}(t, Z_s), \tag{7.1}$$

where for $k \in \mathbb{N}$, we define

$$\begin{aligned} f_N^{(s,k)}(t, Z_s) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} & T_s^{t-t_1} \mathcal{C}_{s,s+2}^N T_{s+2}^{t_1-t_2} \dots T_{s+2k-2}^{t_{k-1}-t_k} \\ & \mathcal{C}_{s+2k-2,s+2k}^N T_{s+2k}^{t_k} f_{N,0}^{(s+2k)}(Z_s)\, dt_k \dots dt_1, \end{aligned} \tag{7.2}$$

for $k = 0$, we define $f_N^{(s,0)}(t, Z_s) := T_s^t f_{N,0}^{(s)}(Z_s)$, and for the remainder we write

$$\begin{aligned} R_N^{(s,n+1)}(t, Z_s) := \int_0^t \int_0^{t_1} \dots \int_0^{t_n} & T_s^{t-t_1} \mathcal{C}_{s,s+2}^N T_{s+2}^{t_1-t_2} \dots T_{s+2n-2}^{t_{n-1}-t_n} \\ & \mathcal{C}_{s+2n-2,s+2n}^N T_{s+2n}^{t_n-t_{n+1}} f_N^{(s+2n+2)}(t_{n+1}, Z_s) \\ & dt_{n+1} \dots dt_1. \end{aligned} \tag{7.3}$$

Similarly, using iteratively Duhamel's formula for the solution of the Boltzmann hierarchy, one gets

$$f^{(s)}(t, Z_s) = \sum_{k=0}^{n} f^{(s,k)}(Z_s) + R^{(n+1)}(t, Z_s) \tag{7.4}$$

where for $k \in \mathbb{N}$, we define

$$\begin{aligned} f^{(s,k)}(t, Z_s) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} & S_s^{t-t_1} \mathcal{C}_{s,s+2}^\infty S_{s+2}^{t_1-t_2} \dots S_{s+2k-2}^{t_{k-1}-t_k} \\ & \mathcal{C}_{s+2k-2,s+2k}^\infty S_{s+2k}^{t_k} f_0^{(s+2k)}(Z_s)\, dt_k \dots dt_1, \end{aligned} \tag{7.5}$$

for $k = 0$, we define $f^{(s,0)}(t, Z_s) := S_s^t f_0^{(s)}(Z_s)$, and for the remainder we write

$$\begin{aligned} R^{(s,n+1)}(t, Z_s) := \int_0^t \int_0^{t_1} \dots \int_0^{t_n} & S_s^{t-t_1} \mathcal{C}_{s,s+2}^\infty S_{s+2}^{t_1-t_2} \dots S_{s+2n-2}^{t_{n-1}-t_n} \\ & \mathcal{C}_{s+2n-2,s+2n}^\infty S_{s+2n}^{t_n-t_{n+1}} f^{(s+2n+2)}(t_{n+1}, Z_s) \\ & dt_{n+1} \dots dt_1. \end{aligned} \tag{7.6}$$

7.2. Reduction to term by term convergence. Here we reduce the convergence proof to term by term convergence of bounded energy and separated collision times observables.

Recalling (3.25), given $R > 0$, $\ell \in \mathbb{N}$, we define the energy truncated operators

$$\mathcal{C}_{\ell,\ell+2}^{N,R} g_{N,\ell+2} := \mathcal{C}_{\ell,\ell+2}^N \left(g_{N,\ell+2} \mathbb{1}_{[E_{\ell+2} \le R^2]} \right), \quad \mathcal{C}_{\ell,\ell+2}^{\infty,R} g_{\ell+2} := \mathcal{C}_{\ell,\ell+2}^\infty \left(g_{\ell+2} \mathbb{1}_{[E_{\ell+2} \le R^2]} \right). \tag{7.7}$$

Consider $\delta > 0$. Given $t \geq 0$ and $k \in \mathbb{N}$, we define the separated collision times

$$\mathcal{T}_{k,\delta}(t) := \{(t_1, \ldots, t_k) \in \mathcal{T}_k(t) : 0 \leq t_{i+1} \leq t_i - \delta,\ d\forall i \in [0, k]\},\ t_{k+1} := 0,\ t_0 := t. \tag{7.8}$$

For the BBGKY hierarchy, we define for $k \in \mathbb{N}$:

$$f_{N,R,\delta}^{(s,k)}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} T_s^{t-t_1} \mathcal{C}_{s,s+2}^{N,R} T_{s+2}^{t_1-t_2} \ldots T_{s+2k-2}^{t_{k-1}-t_k} \mathcal{C}_{s+2k-2,s+2k}^{N,R} T_{s+2k}^{t_k} f_{N,0}^{(s+2k)}(Z_s)\, dt_k \ldots dt_1, \tag{7.9}$$

and for $k = 0$, we define $f_{N,R,\delta}^{(s,0)}(t, Z_s) := T_s^t \left(f_{N,0} \mathbb{1}_{[E_s \leq R^2]}\right)(Z_s)$.

For the Boltzmann hierarchy, we define for $k \in \mathbb{N}$:

$$f_{R,\delta}^{(s,k)}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1} \mathcal{C}_{s,s+2}^{\infty,R} S_{s+2}^{t_1-t_2} \ldots S_{s+2k-2}^{t_{k-1}-t_k} \mathcal{C}_{s+2k-2,s+2k}^{\infty,R} S_{s+2k}^{t_m} f_0^{(s+2k)}(Z_s)\, dt_k \ldots dt_1, \tag{7.10}$$

and for $k = 0$, we define $f_{R,\delta}^{(s,0)}(t, Z_s) := S_s^t \left(f_0 \mathbb{1}_{[E_s \leq R^2]}\right)(Z_s)$.

Given $\phi_s \in C_c(\mathbb{R}^{ds})$ and $k \in \mathbb{N} \cup \{0\}$, let us write

$$I_{s,k,R,\delta}^N(t)(X_s) := I_{\phi_s} f_{N,R,\delta}^{(s,k)}(t)(X_s) = \int_{B_R^{ds}} \phi_s(V_s) f_{N,R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s, \tag{7.11}$$

$$I_{s,k,R,\delta}^\infty(t)(X_s) := I_{\phi_s} f_{R,\delta}^{(s,k)}(t)(X_s) = \int_{B_R^{ds}} \phi_s(V_s) f_{R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s. \tag{7.12}$$

Recalling the observables I_s^N, I_s^∞ defined in (6.15)–(6.16), the following estimates hold

Proposition 7.1. *For any $s, n \in \mathbb{N}$, $R > 1$, $\delta > 0$ and $t \in [0, T]$, the following estimates hold:*

$$\|I_s^N(t) - \sum_{k=0}^n I_{s,k,R,\delta}^N(t)\|_{L_{X_s}^\infty} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\|I_s^\infty(t) - \sum_{k=0}^n I_{s,k,R,\delta}^\infty(t)\|_{L_{X_s}^\infty} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proof. For the proof, one needs to successively perform the reductions described above using the a-priori bounds of Sect. 5 and connect them through the triangle inequality. For the reduction to finitely many terms and for the energy truncation see Propositions 7.1.1., 7.2.1. in [19], and for the time separation part see [2]. □

Proposition 7.1 and triangle inequality imply that the convergence proof reduces to controlling the differences $I_{s,k,R,\delta}^N(t) - I_{s,k,R,\delta}^\infty(t)$. However obtaining such a control requires some delicate analysis because of possible recollisions of the backwards interaction flow.

8. Geometric Estimates

In this section we provide the crucial geometric estimates, many of them novel, which will be of fundamental importance in eliminating recollisions of the backwards interaction flow in Sects. 9 and 10.

Let us introduce some notation which we will be using from now on. For $w \in \mathbb{R}^d$, $y \in \mathbb{R}^d \setminus \{0\}$ and $\rho > 0$, we write $K_\rho^d(w, y)$ for the closed d-dimensional cylinder of center w, direction y and radius ρ. In case we do not need to specify the center and direction we will just be writing K_ρ^d for convenience.

8.1. Spherical estimates. Here, we derive the spherical estimates which will enable us to control pre-collisional configurations. We will strongly rely on the following estimate, see Lemma 4 in [15] for the proof.

Lemma 8.1. *Given $\rho, r > 0$ the following estimate holds for the d-spherical measure of radius $r > 0$:*

$$\left|\mathbb{S}_r^{d-1} \cap K_\rho^d\right|_{\mathbb{S}_r^{d-1}} \lesssim r^{d-1} \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}.$$

Integrating this estimate we obtain the following result, which will be used in Sect. 9:

Proposition 8.2. *Given $0 < \rho \leq 1 \leq R$, the following estimate holds:*

$$|B_R^d \cap K_\rho^d|_d \lesssim R^d \rho^{\frac{d-1}{2}}.$$

Proof. Using Lemma 8.1, we obtain

$$\begin{aligned} |B_R^d \cap K_\rho^d|_d &\simeq \int_0^R |\mathbb{S}_r^{d-1} \cap K_\rho^d|_{\mathbb{S}_r^{d-1}}\, dr \lesssim \int_0^R r^{d-1} \min\left\{1, (\frac{\rho}{r})^{\frac{d-1}{2}}\right\} dr \\ &\leq \int_0^\rho r^{d-1}\, dr + \rho^{\frac{d-1}{2}} \int_0^R r^{\frac{d-1}{2}}\, dr \simeq \rho^d + \rho^{\frac{d-1}{2}} R^{\frac{d+1}{2}}, \quad \text{since } d \geq 2 \\ &\leq R^d \rho^{\frac{d-1}{2}}, \quad \text{since } 0 < \rho \leq 1 \leq R. \end{aligned} \tag{8.1}$$

□

We now obtain new geometric estimates which will be essential to derive the ellipsoidal estimates, enabling us to control post-collisional configurations. To achieve those estimates we strongly rely on the following representation of $\mathbb{S}_1^{2d-1}$:

$$\mathbb{S}_1^{2d-1} = \left\{(\omega_1, \omega_2) \in \mathbb{R}^d \times B_1^d : \omega_1 \in \mathbb{S}^{d-1}_{\sqrt{1-|\omega_2|^2}}\right\}. \tag{8.2}$$

Lemma 8.3. *For any $r, \rho > 0$, the following estimates hold for the $(2d-1)$-spherical measure*

$$\left|\mathbb{S}_r^{2d-1} \cap \left(K_\rho^d \times \mathbb{R}^d\right)\right|_{\mathbb{S}_r^{2d-1}}, \quad \left|\mathbb{S}_r^{2d-1} \cap \left(\mathbb{R}^d \times K_\rho^d\right)\right|_{\mathbb{S}_r^{2d-1}} \lesssim r^{2d-1} \min\left\{1, (\frac{\rho}{r})^{\frac{d-1}{2}}\right\}.$$

Proof. By symmetry it suffices to prove the estimate when intersecting the sphere with $K_\rho^d \times \mathbb{R}^d$. Also, after rescaling we may assume $r = 1$. The idea is to integrate Lemma 8.1 using the representation (8.2). In particular by (8.2) and Lemma 8.1, we have

$$\left|\mathbb{S}_1^{2d-1} \cap \left(K_\rho^d \times \mathbb{R}^d\right)\right|_{\mathbb{S}_1^{2d-1}} = \int_{B_1^d} \left|\mathbb{S}^{d-1}_{\sqrt{1-|\omega_2|^2}} \cap K_\rho^d\right|_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_2|^2}}} \, d\omega_2$$
$$\lesssim \int_{B_1^d} (1-|\omega_2|^2)^{\frac{d-1}{2}} \min\left\{1, \left(\frac{\rho}{\sqrt{1-|\omega_2|^2}}\right)^{\frac{d-1}{2}}\right\} d\omega_2$$
$$\lesssim \int_0^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \min\left\{1, \left(\frac{\rho}{\sqrt{1-s^2}}\right)^{\frac{d-1}{2}}\right\} ds. \tag{8.3}$$

Let us write $I(\rho) := \int_0^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \min\left\{1, \left(\frac{\rho}{\sqrt{1-s^2}}\right)^{\frac{d-1}{2}}\right\} ds$. In the case where $\rho \geq 1$, we have

$$I(\rho) \lesssim \int_0^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \, ds \simeq 1. \tag{8.4}$$

Assume now $0 < \rho < 1$. Then, we may decompose $I(\rho)$ as follows:

$$I(\rho) = \int_0^{\sqrt{1-\rho^2}} s^{d-1}(1-s^2)^{\frac{d-1}{2}} \left(\frac{\rho}{\sqrt{1-s^2}}\right)^{\frac{d-1}{2}} ds$$
$$+ \int_{\sqrt{1-\rho^2}}^1 s^{d-1}(1-s^2)^{\frac{d-1}{2}} \, ds. \tag{8.5}$$

Performing the change of variables $u = 1 - s^2$, equation (8.5) can be written as:

$$I(\rho) = \frac{1}{2}\rho^{\frac{d-1}{2}} \int_{\rho^2}^1 (1-u)^{\frac{d-2}{2}} u^{\frac{d-1}{4}} \, du + \frac{1}{2}\int_0^{\rho^2} (1-u)^{\frac{d-2}{2}} u^{\frac{d-1}{2}} \, du$$
$$\overset{(d\geq 2)}{\lesssim} \rho^{\frac{d-1}{2}} \int_{\rho^2}^1 u^{\frac{d-1}{4}} \, du + \int_0^{\rho^2} u^{\frac{d-1}{2}} \, du \simeq \rho^{\frac{d-1}{2}} \left(1 - \rho^{\frac{d+3}{2}}\right) + \rho^{d+1} \lesssim \rho^{\frac{d-1}{2}}, \tag{8.6}$$

since $\rho < 1$. Combining (8.3)–(8.4) and (8.6), we obtain the result. □

In the same spirit as in Lemma 8.3, we obtain the following estimate for the intersection of $\mathbb{S}_1^{2d-1}$ with the strip:

$$W^{2d}_{\rho,\mu,\lambda} := \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\mu\omega_1 - \lambda\omega_2| \leq \rho\}, \text{ where } \mu, \lambda \neq 0. \tag{8.7}$$

Lemma 8.4. *For any $r, \rho > 0$ the following estimate holds for the $(2d-1)$-spherical measure:*

$$\left|\mathbb{S}_r^{2d-1} \cap W^{2d}_{\rho,\mu,\lambda}\right|_{\mathbb{S}_r^{2d-1}} \lesssim r^{2d-1} \min\left\{1, \left(\frac{\rho}{|\mu| r}\right)^{\frac{d-1}{2}}, \left(\frac{\rho}{|\lambda| r}\right)^{\frac{d-1}{2}}\right\}.$$

Proof. The proof follows the same steps as the proof of Lemma 8.3 after noticing that

$$\begin{aligned} W^{2d}_{\rho,\mu,\lambda} &= \{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \omega_1\in B^d_{\rho/|\mu|}(\lambda\mu^{-1}\omega_2)\}\\ &\subseteq \{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \omega_1\in K^d_{\rho/|\mu|}(\lambda\mu^{-1}\omega_2)\},\\ W^{2d}_{\rho,\mu,\lambda} &= \{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \omega_1\in B^d_{\rho/|\mu|}(\mu\lambda^{-1}\omega_2)\}\\ &\subseteq \{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \omega_1\in K^d_{\rho/|\lambda|}(\mu\lambda^{-1}\omega_2)\}, \end{aligned}$$

where given $\omega_2\in\mathbb{R}^d$, $K^d_{\rho/|\mu|}(\lambda\mu^{-1}\omega_2)$, $K^d_{\rho/|\lambda|}(\mu\lambda^{-1}\omega_2)$ are any cylinders of radius $\rho/|\mu|$, $\rho/|\lambda|$ centered at $\lambda\mu^{-1}\omega_2$, $\mu\lambda^{-1}\omega_2$ respectively. □

8.2. The transition map. Now, we construct a transition map which will allow us to control post-collisional configurations using some appropriate ellipsoidal estimates developed in Sect. 8.3. We first introduce some notation. Given $v_1, v_2, v_3\in\mathbb{R}^{2d}$, we define

$$\Omega = \{\boldsymbol{\omega}=(\omega_1,\omega_2)\in\mathbb{R}^{2d} : |\omega_1|^2+|\omega_2|^2<\frac{3}{2} \text{ and } b(\omega_1,\omega_2,v_2-v_1,v_3-v_1)>0\},$$

where $b(\omega_1,\omega_2,v_2-v_1,v_3-v_1)$ is the cross-section given in (2.4), and

$$\mathcal{S}^+_{v_1,v_2,v_3} := \mathbb{S}^{2d-1}_1\cap\Omega = \{\boldsymbol{\omega}=(\omega_1,\omega_2)\in\mathbb{S}^{2d-1}_1 : b(\omega_1,\omega_2,v_2-v_1,v_3-v_1)>0\}. \tag{8.8}$$

We also define the smooth map $\Psi(\nu_1,\nu_2)=|\nu_1|^2+|\nu_2|^2+|\nu_1-\nu_2|^2$ and the $(2d-1)$-ellipsoid

$$\mathbb{E}^{2d-1}_1 := [\Psi=1] = \left\{(\nu_1,\nu_2)\in\mathbb{R}^{2d} : |\nu_1|^2+|\nu_2|^2+|\nu_1-\nu_2|^2=1\right\}. \tag{8.9}$$

Proposition 8.5. *Consider $v_1, v_2, v_3\in\mathbb{R}^d$ and $r>0$ such that*

$$|v_1-v_2|^2+|v_1-v_3|^2+|v_2-v_3|^2=r^2. \tag{8.10}$$

We define the transition map $\mathcal{J}_{v_1,v_2,v_3}:\Omega\to\mathbb{R}^{2d}\setminus\left\{r^{-1}\begin{pmatrix}v_1-v_2\\v_1-v_3\end{pmatrix}\right\}$ by[8]

$$\boldsymbol{\nu}=\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix}=\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}) := \frac{1}{r}\begin{pmatrix}v_1^*-v_2^*\\v_1^*-v_3^*\end{pmatrix},\quad \boldsymbol{\omega}=(\omega_1,\omega_2)\in\Omega. \tag{8.11}$$

(i) $\mathcal{J}_{v_1,v_2,v_3}$ is smooth in Ω with bounded derivative uniformly in r i.e.

$$\|D\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})\|_\infty\le C_d,\quad \forall\boldsymbol{\omega}\in\Omega. \tag{8.12}$$

[8] By a small abuse of notation we extend the collisional operator T_{ω_1,ω_2} for $(\omega_1,\omega_2)\in\Omega$, see Sect. 2.

(ii) The Jacobian of $\mathcal{J}_{v_1,v_2,v_3}$ is given by:

$$\operatorname{Jac}(\mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega}) \simeq r^{-2d}\frac{b^{2d}(\omega_1,\omega_2,v_2-v_1,v_3-v_1)}{(1+\langle\omega_1,\omega_2\rangle)^{2d+1}} > 0,$$
$$\forall\boldsymbol{\omega}=(\omega_1,\omega_2)\in\Omega. \tag{8.13}$$

Moreover, for any $\boldsymbol{\omega}=(\omega_1,\omega_2)\in\Omega$, there holds the estimate:

$$\operatorname{Jac}(\mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega}) \approx r^{-2d}b^{2d}(\omega_1,\omega_2,v_2-v_1,v_3,v_1). \tag{8.14}$$

(iii) The map $\mathcal{J}_{v_1,v_2,v_3}:\mathcal{S}^+_{v_1,v_2,v_3}\to\mathbb{E}_1^{2d-1}\setminus\left\{r^{-1}\begin{pmatrix}v_1-v_2\\v_1-v_3\end{pmatrix}\right\}$ is bijective. Morever, there holds

$$\mathcal{S}^+_{v_1,v_2,v_3}=[\Psi\circ\mathcal{J}_{v_1,v_2,v_3}=1]. \tag{8.15}$$

(iv) For any measurable $g:\mathbb{R}^{2d}\to[0+\infty]$, there holds the estimate

$$\int_{\mathcal{S}^+_{v_1,v_2,v_3}}(g\circ\mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega})|\operatorname{Jac}\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})|\,d\boldsymbol{\omega}\lesssim\int_{\mathbb{E}_1^{2d-1}}g(\boldsymbol{\nu})\,d\boldsymbol{\nu}. \tag{8.16}$$

Proof. For convenience, let us use the notation[9]:

$$\boldsymbol{\nu}=\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix},\quad \boldsymbol{v}=\begin{pmatrix}v_1-v_2\\v_1-v_3\end{pmatrix},\quad \boldsymbol{\omega}=\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix},$$
$$\pi(\boldsymbol{\omega})=\langle\omega_1,\omega_2\rangle,\quad c:-\frac{\langle\boldsymbol{\omega},\boldsymbol{v}\rangle}{1+\pi(\boldsymbol{\omega})}. \tag{8.17}$$

By (8.11) and (2.3), we have

$$\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})=r^{-1}\left(\boldsymbol{v}+c\boldsymbol{A}\boldsymbol{\omega}\right),\quad\text{where }\boldsymbol{A}=\begin{pmatrix}2I_d & I_d\\ I_d & 2I_d\end{pmatrix}. \tag{8.18}$$

Notice that $\mathcal{J}_{v_1,v_2,v_3}$ maps in $\mathbb{R}^{2d}\setminus\{r^{-1}\boldsymbol{v}\}$. Indeed, assume that $\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})=r^{-1}\boldsymbol{v}$ for some $\boldsymbol{\omega}\in\Omega$. Since $\boldsymbol{A}$ is invertible and $\boldsymbol{\omega}\neq 0$, (8.18) implies $c=0\Rightarrow\langle\boldsymbol{\omega},\boldsymbol{v}\rangle=0$, which is a contradiction, since $\boldsymbol{\omega}\in\Omega$.

(i): Let us calculate the derivative of $\mathcal{J}_{v_1,v_2,v_3}$. Using (8.18), we obtain

$$D\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})=r^{-1}\boldsymbol{A}\left(c\boldsymbol{I_{2d}}+\boldsymbol{\omega}\nabla_{\boldsymbol{\omega}}^Tc\right). \tag{8.19}$$

Using notation from (8.17), we obtain

$$\nabla_{\boldsymbol{\omega}}c=-\frac{\boldsymbol{v}}{1+\pi(\boldsymbol{\omega})}+\frac{\langle\boldsymbol{\omega},\boldsymbol{v}\rangle\tilde{\boldsymbol{\omega}}}{(1+\pi(\boldsymbol{\omega}))^2}, \tag{8.20}$$

where $\tilde{\boldsymbol{\omega}}:=\nabla_{\boldsymbol{\omega}}\pi(\boldsymbol{\omega})=\begin{pmatrix}\omega_2\\\omega_1\end{pmatrix}$. Combining (8.19)–(8.20), we obtain

$$D\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})=r^{-1}\left(-\frac{\langle\boldsymbol{\omega},\boldsymbol{v}\rangle\boldsymbol{A}}{1+\pi(\boldsymbol{\omega})}-\frac{\boldsymbol{A}\boldsymbol{\omega}\boldsymbol{v}^T}{1+\pi(\boldsymbol{\omega})}+\frac{\langle\boldsymbol{\omega},\boldsymbol{v}\rangle\boldsymbol{A}\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}^T}{(1+\pi(\boldsymbol{\omega}))^2}\right). \tag{8.21}$$

[9] By a small abuse of notation we write $\langle\cdot\,,\cdot\rangle$ for the inner product in $\mathbb{R}^{2d}$ as well.

Recall we have assumed $\boldsymbol{\omega} \in \Omega \Rightarrow |\omega_1|^2 + |\omega_2|^2 < \dfrac{3}{2}$, so Cauchy–Schwartz inequality implies

$$\frac{1}{4} < 1 + \pi(\boldsymbol{\omega}) < \frac{7}{4}, \tag{8.22}$$

therefore $\mathcal{J}_{v_1,v_2,v_3}$ is differentiable in Ω. It is clear from (8.21)–(8.22) that $\mathcal{J}_{v_1,v_2,v_3}$ is in fact smooth. Moreover using (8.21), bound (8.12) follows after using Cauchy–Schwartz inequality, the fact that $\boldsymbol{\omega} \in \Omega$, (8.10), (8.22) and (8.10).

(ii): To calculate the Jacobian, we use (8.19) and apply Lemma A.1 (see "Appendix"), to obtain

$$\operatorname{Jac}(\mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega}) = \det(r^{-1}\boldsymbol{A}) \det(c\boldsymbol{I_{2d}} + \boldsymbol{\omega}\nabla_{\boldsymbol{\omega}}^T c) \simeq r^{-2d} c^{2d} \left(1 + c^{-1}\langle \boldsymbol{\omega}, \nabla_{\boldsymbol{\omega}} c\rangle\right). \tag{8.23}$$

Recalling $\tilde{\boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}}\pi(\boldsymbol{\omega}) = \begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix}$, we obtain $c^{-1}\langle \boldsymbol{\omega}, \nabla_{\boldsymbol{\omega}} c\rangle = \left(1 - \dfrac{2\pi(\boldsymbol{\omega})}{1+\pi(\boldsymbol{\omega})}\right)$. Hence (8.23) and (2.5) imply (8.13). To obtain (8.14), we combine (8.13) and estimate (8.22).

(iii): Let us first show that $\mathcal{J}_{v_1,v_2,v_3} : \mathcal{S}^+_{v_1,v_2,v_3} \to \mathbb{E}_1^{2d-1}\setminus\{r^{-1}\boldsymbol{v}\}$. Fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathcal{S}^+_{v_1,v_2,v_3}$. Using conservation of relative velocities (2.8) and (8.10), we get

$$|\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = \frac{|v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2}{r^2} = 1,$$

thus $\mathcal{J}_{v_1,v_2,v_3} : \mathcal{S}^+_{v_1,v_2,v_3} \to \mathbb{E}_1^{2d-1}\setminus\{r^{-1}\boldsymbol{v}\}$. To prove injectivity, let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{S}^+_{v_1,v_2,v_3}$ with $\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}) = \mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}')$. Since $\boldsymbol{A}$ is invertible, (8.18) implies $c\omega = c'\omega'$ where $c' = c_{\omega_1',\omega_2',v_1,v_2,v_3}$. Since $\omega, \omega' \in \Omega$, we have $c, c' \neq 0$ thus $\boldsymbol{\omega} = c^{-1}c'\boldsymbol{\omega}'$. Since $\boldsymbol{\omega}, \boldsymbol{\omega} \in \mathcal{S}^+_{v_1,v_1,v_3}$, we obtain $c = c'$, thus $\boldsymbol{\omega} = \boldsymbol{\omega}'$.

To prove surjectivity, consider $\boldsymbol{\nu} \in \mathbb{E}_1^{2d-1}\setminus\{r^{-1}\boldsymbol{v}\}$. and define

$$\boldsymbol{\omega} := \frac{-\operatorname{sgn}(\langle A^{-1}(\boldsymbol{v} - r\boldsymbol{\nu}), \boldsymbol{v}\rangle)}{\sqrt{\langle \boldsymbol{A}^{-1}(\boldsymbol{v} - r\boldsymbol{\nu}), \boldsymbol{v}\rangle - \langle \boldsymbol{A}_1^{-1}(\boldsymbol{v} - r\boldsymbol{\nu}), \boldsymbol{A}_2^{-1}(\boldsymbol{v} - r\boldsymbol{\nu})\rangle}} \boldsymbol{A}^{-1}(\boldsymbol{v} - r\boldsymbol{\nu}).$$

By (8.10) and the fact that $\boldsymbol{\nu} \in \mathbb{E}_1^{2d-1}\setminus\{r^{-1}\boldsymbol{v}\}$, we have that $\boldsymbol{\omega} \in \mathcal{S}^+_{v_1,v_2,v_3}$ is the unique solution in Ω of $\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}) = \boldsymbol{\nu}$. Relation (8.15) follows from the fact $\mathcal{J}_{v_1,v_2,v_3} : \Omega \to \mathbb{R}^{2d}\setminus\{r^{-1}\boldsymbol{v}\}$ and the previous consideration.

(iv): We easily calculate $4\Psi(\boldsymbol{\nu}) \leq |\nabla\Psi(\boldsymbol{\nu})|^2 \leq 16\Psi(\boldsymbol{\nu})$, for all $\boldsymbol{\nu} \in \mathbb{R}^{2d}$, so $\nabla\Psi(\boldsymbol{\nu}) \neq 0$, for all $\boldsymbol{\nu} \in [\frac{1}{2} < \Psi < \frac{3}{2}]$. To prove the estimate we will rely on Lemma A.2 (see Appendix). We have

$$\begin{aligned}
&\int_{\mathcal{S}^+_{v_1,v_2,v_3}} (g \circ \mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega}) |\operatorname{Jac}\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})| \frac{|\nabla\Psi(\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}))|}{|\nabla(\Psi \circ \mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega})|}\, d\boldsymbol{\omega}\\
&= \int_{[\Psi\circ\mathcal{J}_{v_1,v_2,v_3}=1]} (g \circ \mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega}) |\operatorname{Jac}\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})| \frac{|\nabla\Psi(\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}))|}{|\nabla(\Psi \circ \mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega})|}\, d\boldsymbol{\omega}
\end{aligned} \tag{8.24}$$

$$= \int_{[\Psi=1]} g(\boldsymbol{\nu}) \mathcal{N}_{\mathcal{J}_{v_1,v_2,v_3}}(\boldsymbol{\nu}, [\Psi \circ \mathcal{J}_{v_1,v_2,v_3} = 1])\, d\boldsymbol{\nu} \tag{8.25}$$

$$= \int_{\mathbb{E}_1^{2d-1}} g(\boldsymbol{\nu}) \mathcal{N}_{\mathcal{J}_{v_1,v_2,v_3}}(\boldsymbol{\nu}, \mathcal{S}^+_{v_1,v_2,v_3})\, d\boldsymbol{\nu}, \tag{8.26}$$

where to obtain (8.24) we use (8.15), to obtain (8.25) we use Lemma A.2, to obtain (8.26) we use (8.9) and (8.15). Moreover, by the chain rule and (8.12), we obtain

$$
\begin{aligned}
\frac{|\nabla(\Psi\circ\mathcal{J}_{v_1,v_2,v_3})(\boldsymbol{\omega})|}{|\nabla\Psi(\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}))|} &= \frac{|D^T\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})\nabla\Psi(\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}))|}{|\nabla\Psi(\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega}))|}\\
&= C_d\|D\mathcal{J}_{v_1,v_2,v_3}(\boldsymbol{\omega})\|_\infty \le C_d,
\end{aligned}
$$

and (8.16) follows, since $g \ge 0$. □

8.3. Ellipsoidal estimates. Now, we derive the ellipsoidal estimates which will enable us to control post-collisional configurations.

Lemma 8.6. *Let $v_1, v_2, v_3 \in \mathbb{R}^d$ and $r > 0$ satisfying $|v_1-v_2|^2+|v_1-v_3|^2+|v_2-v_3|^2 = r^2$. Denoting $(\nu_1,\nu_2) = \mathcal{J}_{v_1,v_2,v_3}(\omega_1,\omega_2)$ and considering $\rho > 0$, the following holds:*

$$
\begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \in \boldsymbol{K}_\rho \quad\Leftrightarrow \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \boldsymbol{S}_{\mathbf{12}}^{-\mathbf{1}}\bar{\boldsymbol{K}}_{\rho/r},\quad \begin{pmatrix} v_1^* \\ v_3^* \end{pmatrix} \in \boldsymbol{K}_{\boldsymbol{\rho}} \quad\Leftrightarrow \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \boldsymbol{S}_{\mathbf{13}}{}^{-1}\bar{\boldsymbol{K}}_{\rho/r},
$$

$$
\boldsymbol{S}_{\mathbf{12}} = \begin{pmatrix} I_d & I_d \\ -2I_d & I_d \end{pmatrix},\quad \boldsymbol{S}_{\mathbf{13}} = \begin{pmatrix} I_d & I_d \\ I_d & -2I_d \end{pmatrix}, \tag{8.27}
$$

and $\boldsymbol{K}_{\boldsymbol{\rho}}$ is either of the form $K_\rho^d\times\mathbb{R}^d$ or $\mathbb{R}^d\times K_\rho^d$ while $\bar{\boldsymbol{K}}_{\rho/r}$ is either of the form $\bar{K}_{\rho/r}^d\times\mathbb{R}^d$ or $\mathbb{R}^d\times\bar{K}_{\rho/r}^d$ respectively, and K_ρ^d, $\bar{K}_{\rho/r}^d$ are d-cylinders or radius ρ and ρ/r respectively.

Proof. Using (8.11) to eliminate $c\omega_1, c\omega_2$ from (2.3), we obtain

$$
\begin{aligned}
v_1^* &= \frac{v_1+v_2+v_3}{3} + \frac{r}{3}(\nu_1+\nu_2),\\
v_2^* &= \frac{v_1+v_2+v_3}{3} + \frac{r}{3}(-2\nu_1+\nu_2),\\
v_3^* &= \frac{v_1+v_2+v_3}{3} + \frac{r}{3}(\nu_1-2\nu_2).
\end{aligned}
$$

The conclusion is immediate after a translation and a dilation. □

Recalling $\mathbb{E}_1^{2d-1}$ from (8.9), one can see that $\boldsymbol{S}_{\mathbf{12}}(\mathbb{E}_1^{2d-1}) = \boldsymbol{S}_{\mathbf{13}}(\mathbb{E}_1^{2d-1})$. We will denote

$$
\begin{aligned}
\mathcal{S} &:= \boldsymbol{S}_{\mathbf{12}}(\mathbb{E}_1^{2d-1}) = \boldsymbol{S}_{\mathbf{13}}(\mathbb{E}_1^{2d-1})\\
&= \left\{(y_1,y_2)\in\mathbb{R}^{2d} : |y_1|^2+|y_2|^2+\langle y_1,y_2\rangle = \frac{3}{2}\right\}.
\end{aligned} \tag{8.28}
$$

The following result will allow us to derive the ellipsoidal estimates from the spherical estimates.

Lemma 8.7. *There exist linear bijections $T_1, T_2, P_1, P_2 : \mathbb{R}^{2d}\to\mathbb{R}^{2d}$ and $c > 0$, with the following properties:*

(i) $T_1(\mathcal{S}) = \mathbb{S}_1^{2d-1}$ and for any $\rho > 0$, there holds $T_1(\bar{K}_\rho^d\times\mathbb{R}^d) \subseteq \widetilde{K}_{c\rho}^d\times\mathbb{R}^d$,
(ii) $T_2(\mathcal{S}) = \mathbb{S}_1^{2d-1}$ and for any $\rho > 0$, there holds: $T_2(\mathbb{R}^d\times\bar{K}_\rho^d) \subseteq \widetilde{K}_{c\rho}^d\times\mathbb{R}^d$,
(iii) $P_1(\mathbb{E}_1^{2d-1}) = \mathbb{S}_1^{2d-1}$ and for any $\rho > 0$, there holds: $P_1(\bar{K}_\rho^d\times\mathbb{R}^d) \subseteq \widetilde{K}_{c\rho}^d\times\mathbb{R}^d$,

(iv) $P_2(\mathbb{E}_1^{2d-1}) = \mathbb{S}_1^{2d-1}$ *and for any* $\rho > 0$*, there holds:* $P_2(\mathbb{R}^d \times \bar{K}_\rho^d) \subseteq \widetilde{K}_{c\rho}^d \times \mathbb{R}^d$*, where* $\bar{K}_\rho^d$ *is any* d*-cylinder of radius* ρ *and* $\widetilde{K}_{c\rho}^d$ *is a* d*-cylinder of radius* $c\rho$ *and same direction as* $\bar{K}_\rho^d$*.*

Proof. A direct algebraic calculation shows that the maps given by:

$$T_1 = \begin{pmatrix} -\frac{\sqrt{2}}{2} I_d & 0 \\ \frac{\sqrt{6}}{6} I_d & \frac{\sqrt{6}}{3} I_d \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} I_d \\ \frac{\sqrt{6}}{3} I_d & \frac{\sqrt{6}}{6} I_d \end{pmatrix}, P_1 = \begin{pmatrix} \frac{\sqrt{6}}{2} I_d & 0 \\ -\frac{\sqrt{2}}{2} I_d & \sqrt{2} I_d \end{pmatrix}, P_2 = \begin{pmatrix} 0 & \frac{\sqrt{6}}{2} I_d \\ \sqrt{2} I_d & -\frac{\sqrt{2}}{2} I_d \end{pmatrix}, \tag{8.29}$$

satisfy the properties listed above. □

Now we are ready to apply the results of Sect. 8.1 to obtain ellipsoidal estimates. Recalling from (8.7) the strip $W_{\rho,1,1}^{2d}$, we obtain the following ellipsoidal estimates:

Proposition 8.8. *For any* $r, \rho > 0$*, the following estimates hold:*

(i) $\left|\mathcal{S} \cap \left(\bar{K}_{\rho/r}^d \times \mathbb{R}^d\right)\right|_{\mathcal{S}} \lesssim \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}$.

(ii) $\left|\mathcal{S} \cap \left(\mathbb{R}^d \times \bar{K}_{\rho/r}^d\right)\right|_{\mathcal{S}} \lesssim \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}$.

(iii) $\left|\mathbb{E}_1^{2d-1} \cap \left(B_{\rho/r}^d \times \mathbb{R}^d\right)\right|_{\mathbb{E}_1^{2d-1}} \lesssim \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}$.

(iv) $\left|\mathbb{E}_1^{2d-1} \cap \left(\mathbb{R}^d \times B_{\rho/r}^d\right)\right|_{\mathbb{E}_1^{2d-1}} \lesssim \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}$.

(v) $\left|\mathbb{E}_1^{2d-1} \cap W_{\rho/r,1,1}^{2d}\right|_{\mathbb{E}_1^{2d-1}} \lesssim \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}$.

Proof. Let us first provide the proof of *(i)*. Lemma 8.7 asserts $T_1 : \mathcal{S} \to \mathbb{S}_1^{2d-1}$ is a linear bijection such that $T_1(\bar{K}_{\rho/r}^d \times \mathbb{R}^d) \subseteq \widetilde{K}_{c\rho/r} \times \mathbb{R}^d$, thus substituting $\boldsymbol{\theta} = T_1\boldsymbol{\omega}$, we have

$$\begin{aligned}\int_{\mathcal{S}} \mathbb{1}_{\bar{K}_{\rho/r}^d \times \mathbb{R}^d}(\boldsymbol{\omega})\, d\boldsymbol{\omega} &= \int_{\mathcal{S}} \mathbb{1}_{T_1(\bar{K}_{\rho/r}^d \times \mathbb{R}^d)}(T_1\boldsymbol{\omega})\, d\boldsymbol{\omega} \simeq \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{T_1(\bar{K}_{\rho/r}^d \times \mathbb{R}^d)}(\boldsymbol{\theta})\, d\boldsymbol{\theta} \\ &\lesssim \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{\widetilde{K}_{c\rho/r}^d \times \mathbb{R}^d}(\boldsymbol{\theta})\, d\boldsymbol{\theta} \lesssim \min\left\{1, (\frac{\rho}{r})^{\frac{d-1}{2}}\right\},\end{aligned}$$

by Lemma 8.3. The proof for *(ii)* is identical using bijection T_2 instead. For estimates *(iii)* and *(iv)* we use in a similar way bijections P_1, P_2 and the fact that ball $B_{\rho/r}^d$ embeds in a cylinder of the form $\bar{K}_{\rho/r}^d$. For estimate *(v)*, recalling notation from (8.7), notice that $P_1(W_{\eta/r,1,1}^{2d}) = W_{\eta/r,\mu,\lambda}^{2d}$, for $\mu = (3\sqrt{2}+\sqrt{6})/6$ and $\lambda = -\sqrt{6}/3$. Then the claim comes with a similar argument using Lemma 8.4 instead of Lemma 8.3. □

9. Good Configurations and Stability

In this section we define good configurations and study their stability properties under the adjunction of a collisional pair of particles. Heuristically speaking, given $m \in \mathbb{N}$, a configuration $Z_m \in \mathbb{R}^{2dm}$ is called good configuration if the backwards interaction flow coincides with the backwards free flow. The aim of this section is to investigate conditions under which a given good configuration Z_m remains a good configuration after adding a pair of particles. This is possible on the complement of a small measure set of particles which is constructed in Proposition 9.2. Proposition 9.4 uses the geometric tools developed in Sect. 8 to derive a measure estimate for this pathological set.

This section is the heart of our contribution, since we will strongly rely on Propositions 9.2 and 9.4 when we use them inductively to control the differences of the BBGKY hierarchy truncated observable, given in (7.11), and the Boltzmann hierarchy truncated observable, given in (7.12).

We recall the cylinder notation introduced at the beginning of Sect. 8.

9.1. Adjunction of new particles. We start with some definitions on the configurations we are using. Given $m \in \mathbb{N}$ and $\sigma > 0$, recall from (6.12)–(6.13) the set of well-separated spatial configurations

$$\Delta_m^X(\sigma) = \{\widetilde{X}_m \in \mathbb{R}^{dm} : |\widetilde{x}_i - \widetilde{x}_j| > \sigma, \quad \forall 1 \le i < j \le m\}, \quad m \ge 2, \quad \Delta_1^X(\sigma) = \mathbb{R}^{2d},$$

and the set of well separated configurations

$$\Delta_m(\sigma) = \Delta_m^X(\sigma) \times \mathbb{R}^{dm}.$$

Given $\sigma > 0$, $t_0 > 0$, we define the set of good configurations as:

$$G_m(\sigma, t_0) = \left\{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\sigma), \quad \forall t \ge t_0\right\}, \tag{9.1}$$

where $Z_m(t) = (X_m - tV_m, V_m)$ denotes the backwards in time free flow of $Z_m = (X_m, V_m)$. From now on, we consider parameters $R >> 1$ and $0 < \delta, \eta, \epsilon_0, \alpha << 1$ satisfying:

$$\alpha << \epsilon_0 << \eta\delta, \quad R\alpha << \eta\epsilon_0. \tag{9.2}$$

For convenience we choose the parameters in (9.2) in the very end of the paper, see (11.21).

The following result, see Lemma 12.2.1 in [19] for the proof, is useful for the adjunction of particles to a given configuration.

Lemma 9.1. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2) and $\epsilon << \alpha$. Let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}^d$, with $|\bar{y}_1 - \bar{y}_2| > \epsilon_0$ and $v_1 \in B_R^d$. Then there is a d-cylinder $K_\eta^d \subseteq \mathbb{R}^d$, such that for any $Z_2 = (y_1, y_2, v_1, v_2)$ with $y_1 \in B_\alpha^d(\bar{y}_1)$, $y_2 \in B_\alpha^d(\bar{y}_2)$ and $v_2 \in B_R^d \setminus K_\eta^d$, we have $Z_2 \in G_2(\sqrt{2}\epsilon, 0) \cap G_2(\epsilon_0, \delta)$.*

9.2. Stability of good configurations under adjunction of collisional pair. We prove a statement and a measure estimate regarding the stability of good configurations under the adjunction of a collisional pair of particles to any of the initial configurations.

Recalling the cross-section b given in (2.4), given $v \in \mathbb{R}^d$, we denote

$$\begin{aligned}\left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right)^+(v) = \big\{&(\omega_1, \omega_2, v_1, v_2) \in \mathbb{S}_1^{2d-1} \\ &\times B_R^{2d} : b(\omega_1, \omega_2, v_1 - v, v_2 - v) > 0\big\}.\end{aligned} \tag{9.3}$$

We prove the following Proposition, which will be the inductive step of the convergence proof. We then provide the corresponding measure estimate.

Recall that given $m \in \mathbb{N}$ and $Z_m \in \mathbb{R}^{2dm}$ we denote as $Z_m(t) = (X_m(t), V_m(t)) = (X_m - tV_m, V_m)$ the backwards evolution in time of Z_m. In particular, $Z_m(0) = Z_m$. Recall also the notation from (3.3)

$$\mathring{\mathcal{D}}_{m+2,\epsilon} = \Big\{Z_{m+2} = (X_{m+2}, V_{m+2}) \in \mathbb{R}^{2d(m+2)} : d^2(x_i; x_j, x_k) > 2\epsilon^2, \\ \forall i < j < k \in \{1, \dots, m+2\}\Big\}.$$

Proposition 9.2. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2) and $\epsilon << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \dots, m\}$ and $X_m \in B^{dm}_{\alpha/2}(\bar{X}_m)$. Then there is a subset $\mathcal{B}_\ell(\bar{Z}_m) \subseteq (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_\ell)$ such that:*

(i) For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_\ell)\setminus\mathcal{B}_\ell(\bar{Z}_m)$, one has:

$$Z_{m+2}(t) \in \mathring{\mathcal{D}}_{m+2,\epsilon}, \quad \forall t \geq 0, \tag{9.4}$$
$$Z_{m+2} \in G_{m+2}(\epsilon_0/2, \delta), \tag{9.5}$$
$$\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta), \tag{9.6}$$

where

$$\begin{aligned} Z_{m+2} &= (x_1, \dots, x_\ell, \dots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \dots, \bar{v}_\ell, \dots, \bar{v}_m, v_{m+1}, v_{m+2}),\\ x_{m+1} &= x_\ell - \sqrt{2}\epsilon\omega_1, \quad x_{m+2} = x_\ell - \sqrt{2}\epsilon\omega_2,\\ \bar{Z}_{m+2} &= (\bar{x}_1, \dots, \bar{x}_\ell, \dots, \bar{x}_m, \bar{x}_\ell, \bar{x}_\ell, \bar{v}_1, \dots, \bar{v}_\ell, \dots, \bar{v}_m, v_{m+1}, v_{m+2}). \end{aligned} \tag{9.7}$$

(ii) For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_\ell)\setminus\mathcal{B}_\ell(\bar{Z}_m)$, one has:

$$Z^*_{m+2}(t) \in \mathring{\mathcal{D}}_{m+2,\epsilon}, \quad \forall t \geq 0, \tag{9.8}$$
$$Z^*_{m+2} \in G_{m+2}(\epsilon_0/2, \delta), \tag{9.9}$$
$$\bar{Z}^*_{m+2} \in G_{m+2}(\epsilon_0, \delta), \tag{9.10}$$

where

$$\begin{aligned} Z^*_{m+2} &= (x_1, \dots, x_\ell, \dots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \dots, \bar{v}^*_\ell, \dots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}),\\ x_{m+1} &= x_\ell + \sqrt{2}\epsilon\omega_1, \quad x_{m+2} = x_\ell + \sqrt{2}\epsilon\omega_2,\\ \bar{Z}^*_{m+2} &= (\bar{x}_1, \dots, \bar{x}_\ell, \dots, \bar{x}_m, \bar{x}_\ell, \bar{x}_\ell, \bar{v}_1, \dots, \bar{v}^*_\ell, \dots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}),\\ &(\bar{v}^*_\ell, v^*_{m+1}, v^*_{m+2}) = T_{\omega_1,\omega_2}(\bar{v}_\ell, v_{m+1}, v_{m+2}). \end{aligned} \tag{9.11}$$

Proof. By symmetry, we may assume without loss of generality that $\ell = m$. For convenience, let us define the set of indices:

$$\mathcal{F}_{m+2} = \{(i, j) \in \{1, \dots, m+2\} \times \{1, \dots, m+2\} : \ i < \min\{j, m\}\}.$$

Proof of *(i)* Here we use the notation from (9.7). We start by formulating the following claim, which will imply (9.4).

Lemma 9.3. *Under the hypothesis of Proposition 9.2, there is a set $\mathcal{B}_m^{0,-}(\bar{Z}_m) \subseteq \mathbb{S}_1^{2d-1} \times B_R^{2d}$ such that for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right)^+ (\bar{v}_m) \setminus \mathcal{B}_m^{0,-}(\bar{Z}_m)$, there holds:*

$$|x_i(t) - x_j(t)| > \sqrt{2}\epsilon, \quad \forall t \geq 0, \quad \forall (i, j) \in \mathcal{F}_{m+2}, \tag{9.12}$$

$$d^2\left(x_m(t); x_{m+1}(t), x_{m+2}(t)\right) > 2\epsilon^2, \quad \forall t \geq 0. \tag{9.13}$$

We observe that (9.12)–(9.13) imply (9.4).

Proof of Lemma 9.3. Step 1—the proof of (9.12): Fix $(i, j) \in \mathcal{F}_{m+2}$. We distinguish the following cases:

- $j \leq m$: Since $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $i < j \leq m$, we have $|\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| > \epsilon_0$ for all $t \geq 0$. Hence, triangle inequality implies

$$\begin{aligned} |x_i(t) - x_j(t)| = |x_i - x_j - t(\bar{v}_i - \bar{v}_j)| &\geq |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| \\ - \alpha \geq \epsilon_0 - \alpha &\geq \frac{\epsilon_0}{2} > \sqrt{2}\epsilon, \end{aligned} \tag{9.14}$$

  since $\epsilon << \alpha << \epsilon_0$. Therefore, (9.12) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d}$.
- $j = m+1$: Since $(i, j) \in \mathcal{F}_{m+2}$ we have $i \leq m-1$. Then for $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $X_m \in B_{\alpha/2}^{dm}(\bar{X}_m)$, we conclude

$$\begin{aligned} &|\bar{x}_i - \bar{x}_m| > \epsilon_0, \quad |x_{m+1} - \bar{x}_m| \leq |x_m - \bar{x}_m| + |x_{m+1} - x_m| \leq \frac{\alpha}{2} + \sqrt{2}\epsilon|\omega_1| \\ &\leq \frac{\alpha}{2} + \sqrt{2}\epsilon < \alpha. \end{aligned}$$

  Applying part *(i)* of Lemma 9.1 with $\bar{y}_1 = \bar{x}_i$, $\bar{y}_2 = \bar{x}_m$, $y_1 = x_i$, $y_2 = x_{m+1}$, we can find a cylinder $K_\eta^{d,i}$ such that for any $v_{m+1} \in B_R^d \setminus K_\eta^{d,i}$ we have: $|x_i(t) - x_{m+1}(t)| > \sqrt{2}\epsilon$, for all $t \geq 0$. Hence (9.12) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus U_{m+1}^i$, where

$$U_{m+1}^i = \mathbb{S}_1^{2d-1} \times K_\eta^{d,i} \times \mathbb{R}^d. \tag{9.15}$$

- $j = m+2$: Since $(i, j) \in \mathcal{F}_{m+2}$, we obtain $i < m$. Hence, a similar argument to the previous case yields that (9.12) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus U_{m+2}^i$, where

$$U_{m+2}^i = \mathbb{S}_1^{2d-1} \times \mathbb{R}^d \times K_\eta^{d,i}. \tag{9.16}$$

  We conclude that (9.12) holds for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \bigcup_{i=1}^{m-1} (U_{m+1}^i \cup U_{m+2}^i)$.
  Step 2—the proof of (9.13): Let us recall notation from (9.3). Fixing $t \geq 0$ and considering $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right)^+ (\bar{v}_m)$, we have

$$\begin{aligned} &d^2\left(x_m(t); x_{m+1}(t), x_{m+2}(t)\right) \\ &= |\sqrt{2}\epsilon\omega_1 + t(v_{m+1} - \bar{v}_m)|^2 + |\sqrt{2}\epsilon\omega_2 + t(v_{m+2} - \bar{v}_m)|^2 \\ &\geq 2\epsilon^2(|\omega_1|^2 + |\omega_2|^2) + 2\sqrt{2}\epsilon t b(\omega_1, \omega_2, v_{m+1} - \bar{v}_m, v_{m+2} - \bar{v}_m) \\ &> 2\epsilon^2, \end{aligned} \tag{9.17}$$

where to obtain (9.17) we use the fact that $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m)$. Defining the set $\mathcal{B}_m^{0,-}(\bar{Z}_m) = \bigcup_{i=1}^{m-1}(U_{m+1}^i \cup U_{m+2}^i)$, Lemma 9.3 is proved, and (9.4) follows.

Let us now find a set $\mathcal{B}_m^{\delta,-}(\bar{Z}_m) \subseteq \mathbb{S}_1^{2d-1} \times B_R^{2d}$ such that (9.5) holds in the complement. We distinguish the following cases

- $(i, j) \in \mathcal{F}_{m+2}$, $j \leq m$: We use the same argument as in (9.14) to obtain the lower bound $\epsilon_0/2$.
- $(i, j) \in \mathcal{F}_{m+2}$, $j \in \{m+1, m+2\}$: (9.5) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right) \setminus \mathcal{B}_m^{0,-}(\bar{Z}_m)$, using part *(ii)* of Lemma 9.1 and similar arguments to the corresponding cases in the proof of Lemma 9.3. Let us note that the lower bound is in fact ϵ_0.
- $(i, j) = (m, m+1)$: Triangle inequality implies that for $t \geq \delta$ and $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d}$, such that $|v_{m+1} - \bar{v}_m| > \eta$, we have

$$
\begin{aligned}
|x_m(t) - x_{m+1}(t)| &= |\sqrt{2}\epsilon\omega_1 - t(\bar{v}_m - v_{m+1})| \geq |\bar{v}_m - v_{m+1}|t - \sqrt{2}\epsilon|\omega_1| \\
&\geq |\bar{v}_m - v_{m+1}|\delta - \sqrt{2}\epsilon > \eta\delta - \sqrt{2}\epsilon > \epsilon_0, \qquad (9.18)
\end{aligned}
$$

where to obtain (9.18) we use the fact that $\epsilon << \epsilon_0 << \eta\delta$. Let us note that the lower bound is in fact ϵ_0. Therefore, (9.5) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus V_{m,m+1}$, where

$$
V_{m,m+1} = \mathbb{S}_1^{2d-1} \times B_\eta^d(\bar{v}_m) \times \mathbb{R}^d. \qquad (9.19)
$$

- $(i, j) = (m, m+2)$: Same arguments as in the case $(i, j) = (m, m+1)$ yield that (9.5) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus V_{m,m+2}$, where

$$
V_{m,m+2} = \mathbb{S}_1^{2d-1} \times \mathbb{R}^d \times B_\eta^d(\bar{v}_m). \qquad (9.20)
$$

  The lower bound is in fact ϵ_0.
- $(i, j) = (m+1, m+2)$. Triangle inequality implies that for $t \geq \delta$ and $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d}$, such that $|v_{m+1} - v_{m+2}| > \eta$, we have

$$
\begin{aligned}
|x_{m+1}(t) - x_{m+2}(t)| &= |\sqrt{2}\epsilon(\omega_2 - \omega_1) - t(v_{m+1} - v_{m+2})| \\
&\geq |v_{m+1} - v_{m+2}|t - \sqrt{2}\epsilon|\omega_1 - \omega_2| \\
&\geq |v_{m+1} - v_{m+2}|\delta - \sqrt{2}\epsilon(|\omega_1| + |\omega_2|) \\
&\geq |v_{m+1} - v_{m+2}|\delta - 2\sqrt{2}\epsilon \\
&> \eta\delta - 2\sqrt{2}\epsilon > \epsilon_0, \qquad (9.21)
\end{aligned}
$$

where to obtain (9.21) we use the fact that $\epsilon << \epsilon_0 << \eta\delta$. Recalling from (8.7) the $2d$-strip

$$
W_{\eta,1,1}^{2d} = \{(w_1, w_2) \in \mathbb{R}^{2d} : |w_1 - w_2| \leq \eta\}, \qquad (9.22)
$$

we obtain that (9.5) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus U_{m+1,m+2}$, where

$$
U_{m+1,m+2} = \mathbb{S}_1^{2d-1} \times W_{\eta,1,1}^{2d}. \qquad (9.23)
$$

Notice that the lower bound is in fact ϵ_0 again.

Defining

$$\mathcal{B}_m^{\delta,-}(\bar{Z}_m) = \mathcal{B}_m^{0,-}(\bar{Z}_m) \cup V_{m,m+1} \cup V_{m,m+2} \cup U_{m+1,m+2}, \tag{9.24}$$

we conclude that (9.5) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{B}_m^{\delta,-}(\bar{Z}_m)$.

Let us note that the only case which prevents $Z_{m+2} \in G_{m+2}(\epsilon_0, \delta)$ is the case $1 \leq i < j \leq m$, where we obtain a lower bound of $\epsilon_0/2$. In all other cases we can obtain lower bound ϵ_0.

A similar argument shows that, for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{B}_m^{\delta,-}(\bar{Z}_m)$, (9.6) holds for all $1 \leq i < j \leq m+2$ except the case $1 \leq i < j \leq m$. However in this case, for any $1 \leq i < j \leq m$, we have $|\bar{x}_i(t) - \bar{x}_j(t)| = |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| > \epsilon_0$, since $\bar{Z}_m \in G_m(\epsilon_0, 0)$. This observation shows that (9.6) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{B}_m^{\delta,-}(\bar{Z}_m)$, as well.

We conclude that the set

$$\begin{aligned} \mathcal{B}_m^{-}(\bar{Z}_m) = (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap \Big[& V_{m,m+1} \cup V_{m,m+2} \\ & \cup U_{m+1,m+2} \cup \bigcup_{i=1}^{m-1} (U_{m+1}^i \cup U_{m+2}^i) \Big], \end{aligned} \tag{9.25}$$

is the set we need for the pre-collisional case.

Proof of *(ii)* Here we use the notation from (9.11). The proof follows the steps of the pre-collisional case, but we replace the velocities $(\bar{v}_m, v_{m+1}, v_{m+2})$ by the transformed velocities $(\bar{v}_m^*, v_{m+1}^*, v_{m+2}^*)$ and then pull-back. For details see [2]. It is worth mentioning that the m-particle needs special treatment since its velocity is transformed to $\bar{v}_m^*$. Following similar arguments to the precollisional case, we conclude that the appropriate set for the postcollisional case is given by

$$\begin{aligned} \mathcal{B}_m^{+}(\bar{Z}_m) = & (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap \Big[V_{m,m+1}^* \cup V_{m,m+2}^* \cup U_{m+1,m+2}^* \cup \\ & \bigcup_{i=1}^{m-1} (V_m^{i,*} \cup U_{m+1}^{i,*} \cup U_{m+2}^{i,*}) \Big], \end{aligned} \tag{9.26}$$

where

$$\begin{aligned} V_m^{i,*} &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : \bar{v}_m^* \in K_\eta^{d,i} \right\}, \\ U_{m+1}^{i,*} &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : v_{m+1}^* \in K_\eta^{d,i} \right\}, \\ U_{m+2}^{i,*} &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : v_{m+2}^* \in K_\eta^{d,i} \right\}, \\ V_{m,m+1}^* &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : (v_m^*, v_{m+1}^*) \in W_{\eta,1,1}^{2d} \right\}, \\ V_{m,m+2}^* &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : (v_m^*, v_{m+2}^*) \in W_{\eta,1,1}^{2d} \right\}, \\ U_{m+1,m+2}^* &= \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d} : (v_{m+1}^*, v_{m+2}^*) \in W_{\eta,1,1}^{2d} \right\}. \end{aligned} \tag{9.27}$$

Therefore, the set we need is

$$\mathcal{B}_m(\bar{Z}_m) = \mathcal{B}_m^{-}(\bar{Z}_m) \cup \mathcal{B}_m^{+}(\bar{Z}_m). \tag{9.28}$$

$\square$

We now use the results of Sect. 8 to estimate the measure of this set, up to the parameters chosen.

Proposition 9.4. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2) and $\epsilon << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \ldots, m\}$ and $\mathcal{B}_\ell(\bar{Z}_m)$ the set given in the statement of Proposition 9.2. Denoting by $|\cdot|$ the product measure on $\mathbb{S}_1^{2d-1} \times B_R^{2d}$, the following estimate holds:*

$$\left|\mathcal{B}_\ell(\bar{Z}_m)\right| \lesssim m R^{2d} \eta^{\frac{d-1}{4d+2}}.$$

Proof. Without loss of generality, we may assume that $\ell = m$.

Estimate of $\mathcal{B}_m^-(\bar{Z}_m)$. We recall (9.25).

- Estimate of the terms corresponding to $V_{m,m+1}, V_{m,m+2}, U_{m+1,m+2}$: Recalling (9.19) , we have $V_{m,m+1} = \mathbb{S}_1^{2d-1} \times B_\eta^d(\bar{v}_m) \times \mathbb{R}^d$. We have $(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap V_{m,m+1} \subseteq \mathbb{S}_1^{2d-1} \times \left(B_R^d \cap B_\eta^d(\bar{v}_m)\right) \times B_R^d$, so

$$|(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap V_{m,m+1}| \leq |\mathbb{S}_1^{2d-1}|_{\mathbb{S}_1^{2d-1}} |B_R^d \cap B_\eta^d(\bar{v}_m)|_d |B_R^d|_d \lesssim R^d \eta^d. \tag{9.29}$$

In a similar way, we obtain

$$|(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap V_{m,m+2}| \lesssim R^d \eta^d. \tag{9.30}$$

Recalling (9.23), we have $U_{m+1,m+2} = \mathbb{S}_1^{2d-1} \times W_{\eta,1,1}^{2d}$, thus $(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap U_{m+1,m+2} \subseteq \mathbb{S}_1^{2d-1} \times \left[\left(B_R^d \times B_R^d\right) \cap W_{\eta,1,1}^{2d}\right]$, hence

$$\begin{aligned}|(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap U_{m+1,m+2}| &\leq |\mathbb{S}_1^{2d-1}|_{\mathbb{S}_1^{2d-1}} |(B_R^d \times B_R^d) \cap W_{\eta,1,1}^{2d}|_{2d} \\ &\lesssim \int_{B_R^d} \int_{B_R^d} \mathbb{1}_{B_\eta^d(v_{m+1})}(v_{m+2})\, dv_{m+2}\, dv_{m+1} \\ &\lesssim R^d \eta^d.\end{aligned} \tag{9.31}$$

- Estimate of the terms corresponding to U_{m+1}^i, U_{m+2}^i , $i \in \{1, \ldots, m-1\}$: Fix $i \in \{1, \ldots, m-1\}$. Recalling the set $U_{m+1}^i = \mathbb{S}_1^{2d-1} \times K_\eta^{d,i} \times \mathbb{R}^d$, from (9.15), we have $(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap U_{m+1}^i \subseteq \mathbb{S}_1^{2d-1} \times \left[B_R^{2d} \cap \left(K_\eta^{d,i} \times \mathbb{R}^d\right)\right]$. Since $\eta << 1 << R$, Proposition 8.2 implies that

$$\begin{aligned}|(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+ \cap U_{m+1}^i| &\leq |\mathbb{S}_1^{2d-1}|_{\mathbb{S}_1^{2d-1}} |B_R^{2d} \cap \left(K_\eta^{d,i} \times \mathbb{R}^d\right)|_{2d} \\ &\lesssim |\left(B_R^d \cap K_\eta^{d,i}\right) \times B_R^d|_{2d} \lesssim R^{2d} \eta^{\frac{d-1}{2}}.\end{aligned} \tag{9.32}$$

In a similar way, we obtain

$$|(\mathbb{S}_1^{2d-1} \times B_R^{2d})^+ \cap U_{m+2}^i| \lesssim R^{2d} \eta^{\frac{d-1}{2}}. \tag{9.33}$$

Therefore, recalling (9.25), using estimates (9.29)–(9.33) and the facts that $s \geq 1$, $\eta << 1 << R$, sub-additivity implies

$$|\mathcal{B}_m^-(\bar{Z}_m)| \lesssim m R^{2d} \eta^{\frac{d-1}{2}} < m R^{2d} \eta^{\frac{d-1}{4d+2}}, \quad \text{since } \eta << 1. \tag{9.34}$$

Estimate of $\mathcal{B}_m^+(\bar{Z}_m)$: We recall (9.26). To estimate the measure of $\mathcal{B}_m^+(\bar{Z}_m)$, we will strongly rely on the properties of the transition map defined in Proposition 8.5.

Let us define $\Phi_{\bar{v}_m}:\mathbb{R}^{2d}\to\mathbb{R}$ by $\Phi_{\bar{v}_m}(v_{m+1},v_{m+2})=|v_{m+1}-\bar{v}_m|^2+|v_{m+2}-\bar{v}_m|^2+|v_{m+1}-v_{m+2}|^2$. We can easily see that given $r>0$ and $(v_{m+1},v_{m+2})\in\Phi_{\bar{v}_m}^{-1}(\{r^2\})$, we have

$$2r\leq|\nabla\Phi_{\bar{v}_m}(v_{m+1},v_{m+2})|\leq 4r. \tag{9.35}$$

Let also define the set $G_R^{2d}(\bar{v}_m):=[0\leq\Phi_{\bar{v}_m}\leq 16R^2]$. Notice that by triangle inequality and the fact that $\bar{v}_m\in B_R^d$, we have

$$B_R^{2d}\subseteq G_R^{2d}(\bar{v}_m). \tag{9.36}$$

Recall from (8.8) the set $\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}$. Then, Fubini's Theorem and the co-area formula yield

$$\begin{aligned}
|\mathcal{B}_m^+(\bar{Z}_m)|&=\int_{(\mathbb{S}_1^{2d-1}\times B_R^{2d})^+(\bar{v}_m)}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\\
&\leq\int_{G_R^{2d}(\bar{v}_m)}\int_{\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2} \qquad (9.37)\\
&=\int_0^{16R^2}\int_{\Phi_{\bar{v}_m}^{-1}(\{s\})}|\nabla\Phi_{\bar{v}_m}(v_{m+1},v_{m+2})|^{-1}\\
&\int_{\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\,ds\\
&=\int_0^{4R}2r\int_{\Phi_{\bar{v}_m}^{-1}(\{r^2\})}|\nabla\Phi_{\bar{v}_m}(v_{m+1},v_{m+2})|^{-1}\\
&\int_{\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\,dr\\
&\lesssim\int_0^{4R}\int_{\Phi_{\bar{v}_m}^{-1}(\{r^2\})}\int_{\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}(\omega_1,\omega_2)\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\,dr, \qquad (9.38)
\end{aligned}$$

where to obtain (9.37), we use (9.36), and to obtain (9.38) we use the lower bound of (9.35).

We estimate the integral $\int_{\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}}\mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}(\omega_1,\omega_2)\,d\omega_1\,d\omega_2$, for fixed $0<r\leq 4R$ and $(v_{m+1},v_{m+2})\in\Phi_{\bar{v}_m}^{-1}(\{r^2\})$. Let us introduce a parameter $0<\beta<1$, which will be chosen later in terms of η. Writing

$$\boldsymbol{\omega}=(\omega_1,\omega_2),\quad \boldsymbol{v}=(v_{m+1}-\bar{v}_m,v_{m+2}-\bar{v}_m), \tag{9.39}$$

we have $b(\boldsymbol{\omega},\boldsymbol{v})=\langle\boldsymbol{\omega},\boldsymbol{v}\rangle$. Inspired in part by [15] (Proposition 1), we decompose

$$\mathcal{S}^+_{\bar{v}_m,v_{m+1},v_{m+2}}=\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}\cup\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1},v_{m+2}},$$

where

$$\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}} = \left\{\boldsymbol{\omega} \in \mathcal{S}^{+}_{\bar{v}_m,v_{m+1},v_{m+2}} : \langle \boldsymbol{\omega}, \boldsymbol{v}\rangle > \beta|\boldsymbol{v}|\right\}, \tag{9.40}$$

$$\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1},v_{m+2}} = \left\{\boldsymbol{\omega} \in \mathcal{S}^{+}_{\bar{v}_m,v_{m+1},v_{m+2}} : 0 < \langle \boldsymbol{\omega}, \boldsymbol{v}\rangle \leq \beta|\boldsymbol{v}|\right\}. \tag{9.41}$$

Notice that $\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1},v_{m+2}}$ is the union of two unit $(2d-1)$-spherical caps of angle $\pi/2 - \arccos\beta$. Thus integrating in spherical coordinates, we have

$$\int_{\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1},v_{m+2}}} \mathbb{1}_{\mathcal{B}^+_m(\bar{Z}_m)}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 \lesssim \frac{\pi}{2} - \arccos\beta = \arcsin\beta. \tag{9.42}$$

Let us estimate the terms corresponding to $\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}$. Our purpose is to change variables under the transition map $\mathcal{J}_{\bar{v}_m,v_{m+1},v_{m+2}}$, and use part *(iv)* of Proposition 8.5.

Notice that for $\boldsymbol{\omega} \in \mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}$, the lower estimate of (8.14) and (9.40) imply

$$\mathrm{Jac}^{-1}(\mathcal{J}_{\bar{v}_m,v_{m+1},v_{m+2}})(\boldsymbol{\omega}) \lesssim r^{2d} b^{-2d}(\boldsymbol{\omega},\boldsymbol{v}) \leq r^{2d}\beta^{-2d}|\boldsymbol{v}|^{-2d} \lesssim \beta^{-2d}, \tag{9.43}$$

since by triangle inequality and Young's inequality, we have

$$\begin{aligned} r^2 &= |\bar{v}_m - v_{m+1}|^2 + |\bar{v}_m - v_{m+2}|^2 + |v_{m+1} - v_{m+2}|^2 \\ &\leq 3(|\bar{v}_m - v_{m+1}|^2 + |\bar{v}_m - v_{m+2}|^2) = 3|\boldsymbol{v}|^2. \end{aligned}$$

• Estimate of $V^*_{m,m+1}$, $V^*_{m,m+2}$, $U^*_{m+1,m+2}$ terms: By recalling (9.27)

$$V^*_{m,m+1} = \left\{(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \bar{v}^*_m - v^*_{m+1} \in B^d_\eta\right\},$$

and (8.11), given $\boldsymbol{\omega} = (\omega_1,\omega_2) \in \mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}$, we have

$$\bar{v}^*_m - v^*_{m+1} \in B^d_\eta \Leftrightarrow \boldsymbol{\nu} = (\nu_1,\nu_2) \in B^d_{\eta/r} \times \mathbb{R}^d. \tag{9.44}$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}} \mathbb{1}_{V^*_{m,m+1}}(\boldsymbol{\omega})\, d\boldsymbol{\omega} &= \int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}} (\mathbb{1}_{B^d_{\eta/r}\times\mathbb{R}^d} \circ \mathcal{J}_{\bar{v}_m,v_{m+1},v_{m+2}})(\boldsymbol{\omega})\, d\boldsymbol{\omega} \\ &\lesssim \beta^{-2d} \int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}} (\mathbb{1}_{B^d_{\eta/r}\times\mathbb{R}^d} \circ \mathcal{J}_{\bar{v}_m,v_{m+1},v_{m+2}})(\boldsymbol{\omega})\, \mathrm{Jac}\, \mathcal{J}_{\boldsymbol{v}_m,v_{m+1},v_{m+2}}(\boldsymbol{\omega})\, d\boldsymbol{\omega} \quad (9.45) \\ &\lesssim \beta^{-2d} \int_{\mathbb{E}_1^{2d-1}} \mathbb{1}_{B^d_{\eta/r}\times\mathbb{R}^d}(\boldsymbol{\nu})\, d\boldsymbol{\nu} \lesssim \beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}, \quad (9.46) \end{aligned}$$

where to obtain (9.45) we use (9.43), to obtain (9.46) we use part *(iv)* of Proposition 8.5 and part *(iii)* of Proposition 8.8. Thus

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1},v_{m+2}}} \mathbb{1}_{V^*_{m,m+1}}(\omega_1,\omega_2)\, d\omega_1\; d\omega_2 \lesssim \beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \tag{9.47}$$

In a similar manner, recalling from (9.27) the sets $V^*_{m,m+2}$, $U^*_{m+1,m+2}$ respectively, and parts *(iv)*, *(v)* of Proposition 8.8 respectively, we obtain the corresponding estimates.

• Estimate of $V_m^{i,*}, U_{m+1}^{i,*}, U_{m+2}^{i,*}, i \in \{1,\dots,m-1\}$ terms: Consider $i \in \{1,\dots,m-1\}$. By recalling (9.27), the set $V_m^{i,*}$ can be equivalently written as

$$V_m^{i,*} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : (\bar{v}_m^*, v_{m+1}^*) \in K_\eta^{d,i} \times \mathbb{R}^d\}.$$

Recalling also the operator $\boldsymbol{S_{12}}$ defined in (8.27), Lemma 8.6 implies

$$(\bar{v}_m^*, v_{m+1}^*) \in K_\eta^{d,i} \times \mathbb{R}^d \Leftrightarrow \left(\boldsymbol{S_{12}} \circ \mathcal{J}_{\bar{v}_m, v_{m+1}, v_{m+2}}\right)(\omega_1, \omega_2) \in \bar{K}_{\eta/r}^{d,i} \times \mathbb{R}^d, \quad (9.48)$$

where $K_\eta^{d,i}$ is a d-cylinder of radius η and $\bar{K}_{\eta/r}^{d,i}$ is a d-cylinder of radius η/r. Recalling $\mathcal{S} = \boldsymbol{S_{12}}(\mathbb{E}_1^{2d-1})$ from (8.28), and using the same reasoning to change variables under $\mathcal{J}_{\bar{v}_m, v_{m+1}, v_{m+2}}$ as in the estimate for $V_{m,m+1}^*$, we have

$$\begin{aligned}
&\int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}, v_{m+2}}} \mathbb{1}_{V_m^{i,*}}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \\
&\quad = \int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}, v_{m+2}}} \mathbb{1}_{(\bar{v}_m^*, v_{m+1}^*) \in K_\eta^{d,i} \times \mathbb{R}^d}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \\
&= \int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}, v_{m+2}}} (\mathbb{1}_{\bar{K}_{\eta/r}^{d,i} \times \mathbb{R}^d} \circ \boldsymbol{S_{12}} \circ \mathcal{J}_{\bar{v}_m, v_{m+1}, v_{m+2}})(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 && (9.49) \\
&\lesssim \beta^{-2d} \int_{\mathbb{E}_1^{2d-1}} (\mathbb{1}_{\bar{K}_{\eta/r}^{d,i} \times \mathbb{R}^d} \circ \boldsymbol{S_{12}})(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 && (9.50) \\
&\lesssim \beta^{-2d} \int_{\mathcal{S}} \mathbb{1}_{\bar{K}_{\eta/r}^{d,i} \times \mathbb{R}^d}(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 && (9.51) \\
&\lesssim \beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}, && (9.52)
\end{aligned}$$

where to obtain (9.49) we use (9.48), to obtain (9.50) we use estimate (9.43) and part *(iv)* of Proposition 8.5, to obtain (9.51) we make the linear transformation $(\theta_1, \theta_2) = \boldsymbol{S_{12}}(\nu_1, \nu_2)$ and use the fact that $\mathcal{S} = \boldsymbol{S_{12}}(\mathbb{E}_1^{2d-1})$, and to obtain (9.52) we use part *(i)* of Proposition 8.8.

Recalling $U_{m+1}^{i,*}, U_{m+2}^{i,*}$ from (9.27), and using respectively the map $\boldsymbol{S_{12}}$ from Lemma 8.6 and estimate *(ii)* from Proposition 8.8, the map $\boldsymbol{S_{13}}$ from Lemma 8.6 and estimate *(ii)* from Proposition 8.8, we obtain the corresponding estimates in a similar way.

We conclude that

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}, v_{m+2}}} \mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \lesssim m\beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\} \quad (9.53)$$

Therefore, recalling $\mathcal{S}^+_{\bar{v}_m, v_{m+1}, v_{m+2}} = \mathcal{S}^{1+}_{\bar{v}_m, v_{m+1}, v_{m+2}} \cup \mathcal{S}^{2+}_{\bar{v}_m, v_{m+1}, v_{m+2}}$, and using estimates (9.42), (9.53), we obtain the estimate:

$$\int_{\mathcal{S}^+_{\bar{v}_m, v_{m+1}, v_{m+2}}} \mathbb{1}_{\mathcal{B}_m^+(\bar{Z}_m)}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \lesssim \arcsin\beta + m\beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \quad (9.54)$$

Hence, (9.38) yields

$$
\begin{aligned}
|\mathcal{B}_m^+(\bar{Z}_m)| &\lesssim \int_0^{4R} \int_{\Phi_{\bar{v}_m}^{-1}(\{r^2\})} \left(\arcsin \beta + m\beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}\right) dv_{m+1}\, dv_{m+2}\, dr \\
&\lesssim \int_0^{4R} r^{2d-1} \left(\arcsin \beta + m\beta^{-2d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}\right) dr \\
&\lesssim mR^{2d}\left(\arcsin\beta + \beta^{-2d}\eta^{\frac{d-1}{2}}\right) \\
&\lesssim mR^{2d}\left(\beta + \beta^{-2d}\eta^{\frac{d-1}{2}}\right),
\end{aligned}
\tag{9.55}
$$

after using an estimate similar to (8.1) and the fact that $m \geq 1$, $\beta << 1$. Choosing $\beta = \eta^{\frac{d-1}{4d+2}} << 1$, since $d \geq 2$, we obtain

$$
|\mathcal{B}_m^+(\bar{Z}_m)| \lesssim mR^{2d}\eta^{\frac{d-1}{4d+2}}. \tag{9.56}
$$

Combining (9.28), (9.34), (9.56), we obtain the required estimate. □

10. Elimination of Recollisions

In this section we reduce the convergence proof to comparing truncated elementary observables. We first restrict to good configurations and then inductively reduce the convergence proof to truncated elementary observables, which will be comparable in the scaled limit.

10.1. Restriction to good configurations. Throughout this subsection, we consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ given in Theorems 5.5 and 5.8, the functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0, T] \to \mathbb{R}$ defined by (5.6), (N, ϵ) in the scaling (4.22) and initial data $F_{N,0} \in X_{N,\beta_0,\mu_0}$, $F_0 \in X_{\infty,\beta_0,\mu_0}$. Let $\boldsymbol{F_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$, $\boldsymbol{F} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ be the mild solutions of the corresponding BBGKY and Boltzmann hierarchies in $[0, T]$, given by Theorems 5.5 and 5.8 respectively.

For the convenience of a reader we recall the notation from Sect. 9. Specifically, given $m \in \mathbb{N}$, $\sigma > 0$ and $t_0 > 0$, we denote

$$
\begin{aligned}
\Delta_m^X(\sigma) &= \{\widetilde{X}_m \in \mathbb{R}^{dm} : |\widetilde{x}_i - \widetilde{x}_j| > \sigma, \quad \forall 1 \leq i < j \leq m\}, \quad m \geq 2, \quad \Delta_1(\sigma) = \mathbb{R}^d, \\
\Delta_m(\sigma) &= \Delta_m^X(\sigma) \times \mathbb{R}^{dm}, \\
G_m(\sigma, t_0) &= \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\sigma), \quad \forall t \geq t_0 \right\},
\end{aligned}
$$

where $Z_m(t)$ denotes the backwards free flow, given by: $Z_m(t) = (X_m - tV_m, V_m)$, for $t \geq 0$. Given $\epsilon, \epsilon_0 > 0$ with $\epsilon << \epsilon_0$ and $\delta > 0$, we define the new set

$$
G_m(\epsilon, \epsilon_0, \delta) := G_m(\epsilon, 0) \cap G_m(\epsilon_0, \delta). \tag{10.1}
$$

Inductively using Lemma 9.1 and Proposition 8.2, we obtain the following result. For more details on the proof see [2].

Proposition 10.1. *Let $s \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in* (9.2) *and $\epsilon << \alpha$. Then for any $X_s \in \Delta_s^X(\epsilon_0)$, there is a subset of velocities $\mathcal{M}_s(X_s) \subseteq B_R^{ds}$ of measure*

$$|\mathcal{M}_s(X_s)|_{ds} \leq C_{d,s} R^{ds} \eta^{\frac{d-1}{2}}, \tag{10.2}$$

such that $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta)$, for all $V_s \in B_R^{ds} \setminus \mathcal{M}_s(X_s)$.

Consider $s, n \in \mathbb{N}$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2), (N, ϵ) in the scaling (4.22) with $\epsilon << \alpha$, $0 \leq k \leq n$ and $t \in [0, T]$. Let us recall the observables $I_{s,k,R,\delta}^N(t)$, $I_{s,k,R,\delta}^\infty(t)$ defined in (7.11)–(7.12). We restrict the domain of integration to velocities giving good configurations. For convenience, given $X_s \in \Delta_s^X(\epsilon_0)$, we write $\mathcal{M}_s^c(X_s) = B_R^{ds} \setminus \mathcal{M}_s(X_s)$. We define

$$\widetilde{I}_{s,k,R,\delta}^N(t)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) f_{N,R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s, \tag{10.3}$$

$$\widetilde{I}_{s,k,R,\delta}^\infty(t)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) f_{R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s. \tag{10.4}$$

We now apply Proposition 10.1 and the a-priori estimates of Sect. 5 to restrict to initially good configurations.

Proposition 10.2. *Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in* (9.2)*, (N, ϵ) in the scaling* (4.22) *with $\epsilon << \alpha$, and $t \in [0, T]$. Then, the following estimates hold:*

$$\sum_{k=0}^n \|I_{s,k,R,\delta}^N(t) - \widetilde{I}_{s,k,R,\delta}^N(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C_{d,s,\mu_0,T} R^{ds} \eta^{\frac{d-1}{2}} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\sum_{k=0}^n \|I_{s,k,R,\delta}^\infty(t) - \widetilde{I}_{s,k,R,\delta}^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C_{d,s,\mu_0,T} R^{ds} \eta^{\frac{d-1}{2}} \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Remark 10.3. Under the assumptions of Proposition 10.2, given $X_s \in \Delta_s^X(\epsilon_0)$, the definition of $\mathcal{M}_s(X_s)$ implies that $\widetilde{I}_{s,0,R,\delta}^N(t)(X_s) = \widetilde{I}_{s,0,R,\delta}^\infty(t)(X_s)$ for all $t \in [0, T]$. Therefore, Proposition 10.2 reduces the convergence to controlling the differences $\widetilde{I}_{s,k,R,\delta}^N(t) - \widetilde{I}_{s,k,R,\delta}^\infty(t)$, for $k = 1, \ldots, n$, in the scaled limit.

10.2. Reduction to elementary observables. Here, given $s, n \in \mathbb{N}$, parameters α, $\epsilon_0, R, \eta, \delta$ as in (9.2) $1 \leq k \leq n$, (N, ϵ) in the scaling (4.22) with $\epsilon << \alpha$, and $t \in [0, T]$, inspired by notation used in [19,27], we expand $\widetilde{I}_{s,k,R,\delta}^N(t)$ and $\widetilde{I}_{s,k,R,\delta}^\infty(t)$, defined in (10.3)–(10.4), in terms of elementary observables.

For this purpose, given $\ell, N \in \mathbb{N}$ with $\ell < N$, $R > 1$, we decompose the truncated BBGKY hierarchy collisional operator (given in (4.17)–(4.20)) in the following way:

$$\mathcal{C}_{\ell,\ell+2}^{N,R} = \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+2}^{N,R,+,i} - \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+2}^{N,R,-,i},$$

$$\mathcal{C}_{\ell,\ell+2}^{N,R,+,i} g_{\ell+2}(Z_\ell) := A_{N,\epsilon,\ell} \int_{\mathbb{S}_1^{2d-1} \times B_R^{2d}} b_+(\omega_{\ell+1}, \omega_{\ell+2}, v_{\ell+1} - v_i, v_{\ell+2} - v_i)$$

$$\times g_{\ell+2}\mathbb{1}_{[E_{\ell+2}\le R^2]}(Z^{i*}_{\ell+2,\epsilon})\,d\omega_{\ell+1}\,d\omega_{\ell+2}\,dv_{\ell+1}\,dv_{\ell+2},$$

$$\mathcal{C}^{N,R,-,i}_{\ell,\ell+2}g_{\ell+2}(Z_\ell) := A_{N,\epsilon,\ell}\int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} b_+(\omega_{\ell+1},\omega_{\ell+2},v_{\ell+1}-v_i,v_{\ell+2}-v_i)$$

$$\times g_{\ell+2}\mathbb{1}_{[E_{\ell+2}\le R^2]}(Z^{i}_{\ell+2,\epsilon})\,d\omega_{\ell+1}\,d\omega_{\ell+2}\,dv_{\ell+1}\,dv_{\ell+2}.$$

For $s\in\mathbb{N}$ and $k\in\mathbb{N}$, let us denote $\mathcal{U}_{s,k}=\mathcal{A}_{s,k}\times\mathcal{B}_{s,k}$, where

$$\mathcal{A}_{s,k} := \left\{J=(j_1,\dots,j_k)\in\mathbb{N}^k : j_i\in\{-1,1\},\quad \forall i\in\{1,\dots,k\}\right\}, \tag{10.5}$$

$$\mathcal{B}_{s,k} := \left\{M=(m_1,\dots,m_k)\in\mathbb{N}^k : m_i\in\{1,\dots,s+2i-2\},\quad \forall i\in\{1,\dots,k\}\right\}. \tag{10.6}$$

Under this notation, given $s,n\in\mathbb{N}$, parameters $\alpha,\epsilon_0,R,\eta,\delta$ as in (9.2), $1\le k\le n$, (N,ϵ) in the scaling (4.22) with $\epsilon << \alpha$, and $t\in[0,T]$, the BBGKY hierarchy observable functional $\widetilde{I}^N_{s,k,R,\delta}(t)$ (given in (10.3)) can be expressed as a superposition of elementary observables

$$\widetilde{I}^N_{s,k,R,\delta}(t)(X_s) = \sum_{(J,M)\in\mathcal{U}_{s,k}}\left(\prod_{i=1}^k j_i\right)\widetilde{I}^N_{s,k,R,\delta}(t,J,M)(X_s), \tag{10.7}$$

$$\widetilde{I}^N_{s,k,R,\delta}(t,J,M)(X_s) := \int_{\mathcal{M}_s^c(X_s)}\phi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} T_s^{t-t_1}\mathcal{C}^{N,R,j_1,m_1}_{s,s+2}T_{s+2}^{t_1-t_2}\dots$$
$$\mathcal{C}^{N,R,j_k,m_k}_{s+2k-2,s+2k}T^{t_m}_{s+2k}f_0^{(s+2k)}(Z_s)\,dt_k\dots dt_1 dV_s. \tag{10.8}$$

Similarly, given $\ell,N\in\mathbb{N}$ with $\ell<N$, $R>1$, we decompose the truncated Boltzmann hierarchy collisional operator (given in (4.23)–(4.26)) as:

$$\mathcal{C}^{\infty,R}_{\ell,\ell+2} = \sum_{i=1}^{\ell}\mathcal{C}^{\infty,R,+,i}_{\ell,\ell+2}-\sum_{i=1}^{\ell}\mathcal{C}^{\infty,R,-,i}_{\ell,\ell+2},$$

$$\mathcal{C}^{\infty,R,+,i}_{\ell,\ell+2}g_{\ell+2}(Z_\ell) := \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} b_+(\omega_{\ell+1},\omega_{\ell+2},v_{\ell+1}-v_i,v_{\ell+2}-v_i)$$
$$g_{\ell+2}\mathbb{1}_{[E_{\ell+2}\le R^2]}(Z^{i*}_{\ell+2})\,d\omega_{\ell+1}\,d\omega_{\ell+2}\,dv_{\ell+1}\,dv_{\ell+2},$$

$$\mathcal{C}^{\infty,R,-,i}_{\ell,\ell+2}g_{\ell+2}(Z_\ell) := \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} b_+(\omega_{\ell+1},\omega_{\ell+2},v_{\ell+1}-v_i,v_{\ell+2}-v_i)$$
$$g_{\ell+2}\mathbb{1}_{[E_{\ell+2}\le R^2]}(Z^{i}_{\ell+2})\,d\omega_{\ell+1}\,d\omega_{\ell+2}\,dv_{\ell+1}\,dv_{\ell+2}.$$

Under this notation, given $s,n\in\mathbb{N}$, $t\in[0,T]$, parameters $\alpha,\epsilon_0,R,\eta,\delta$ as in (9.2), $1\le k\le n$, and $t\in[0,T]$, the Boltzmann hierarchy observable functional $\widetilde{I}^\infty_{s,k,R,\delta}(t)$ (given in (10.4)) can be expressed as a superposition of elementary observables

$$\widetilde{I}^\infty_{s,k,R,\delta}(t)(X_s) = \sum_{(J,M)\in\mathcal{U}_{s,k}}\left(\prod_{i=1}^k j_i\right)\widetilde{I}^\infty_{s,k,R,\delta}(t,J,M)(X_s), \tag{10.9}$$

$$\widetilde{I}^\infty_{s,k,R,\delta}(t,J,M)(X_s) := \int_{\mathcal{M}_s^c(X_s)}\phi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1}\mathcal{C}^{\infty,R,j_1,m_1}_{s,s+2}S_{s+2}^{t_1-t_2}\dots$$
$$\mathcal{C}^{\infty,R,j_k,m_k}_{s+2k-2,s+2k}S^{t_m}_{s+2k}f_0^{(s+2k)}(Z_s)\,dt_k\dots dt_1 dV_s. \tag{10.10}$$

10.3. Boltzmann pseudo-trajectories. In this subsection, we introduce an explicit discrete backwards in time construction of so called Boltzmann pseudo-trajectory, which lets us keep track of the collisions. Similar constructions, although continuous in time, can be found in [15,19,27]. Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $k \in \mathbb{N}$ and $t \in [0, T]$. Given $\delta > 0$, let us recall from (7.8) the set $\mathcal{T}_{k,\delta}(t)$.

Consider $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$, $J = (j_1, \ldots, j_k)$, $M = (m_1, \ldots, m_k)$, $(J, M) \in \mathcal{U}_{s,k}$, and for each $i = 1, \ldots, k$, we consider $(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}$. We inductively define the Boltzmann pseudo-trajectory of Z_s. Roughly speaking, the Boltzmann pseudo-trajectory is formulated as follows:

Assume we are given a configuration $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ at time $t_0 = t$. Z_s evolves under backwards free flow until the time t_1 when a pair of particles $(\omega_{s+1}, \omega_{s+2}, v_{s+1}, v_{s+2})$ is added to the m_1-particle, the adjunction being pre-collisional if $j_1 = -1$ and post-collisional if $j_1 = 1$. We then form an $(s+2)$-configuration and continue this process inductively until time $t_{k+1} = 0$. More precisely, given $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$:

Time $t_0 = t$: We initially define

$$Z_s^\infty(t_0^-) = \left(x_1^\infty(t_0^-), \ldots, x_s^\infty(t_0^-), v_1^\infty(t_0^-), \ldots, v_s^\infty(t_0^-)\right) := Z_s.$$

Time t_i, $i \in \{1, \ldots, k\}$: Consider $i \in \{1, \ldots, k\}$, and assume we know

$$Z_{s+2i-2}^\infty(t_{i-1}^-) = \left(x_1^\infty(t_{i-1}^-), \ldots, x_{s+2i-2}^\infty(t_{i-1}^-), v_1^\infty(t_{i-1}^-), \ldots, v_{s+2i-2}^\infty(t_{i-1}^-)\right).$$

We define $Z_{s+2i-2}^\infty(t_i^+) = \left(x_1^\infty(t_i^+), \ldots, x_{s+2i-2}^\infty(t_i^+), v_1^\infty(t_i^+), \ldots, v_{s+2i-2}^\infty(t_i^+)\right)$ as:

$$Z_{s+2i-2}^\infty(t_i^+) := \left(X_{s+2i-2}^\infty\left(t_{i-1}^-\right) - (t_{i-1} - t_i)\, V_{s+2i-2}^\infty\left(t_{i-1}^-\right), V_{s+2i-2}^\infty\left(t_{i-1}^-\right)\right).$$

We also define $Z_{s+2i}^\infty(t_i^-) = \left(x_1^\infty(t_i^-), \ldots, x_{s+2i}^\infty(t_i^-), v_1^\infty(t_i^-), \ldots, v_{s+2i}^\infty(t_i^-)\right)$ as:

$$\left(x_j^\infty(t_i^-), v_j^\infty(t_i^-)\right) := \left(x_j^\infty(t_i^+), v_j^\infty(t_i^+)\right) \quad \forall j \in \{1, \ldots, s+2i-2\} \setminus \{m_i\},$$

and if $j_i = -1$:

$$\begin{aligned}
\left(x_{m_i}^\infty(t_i^-), v_{m_i}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{m_i}^\infty(t_i^+)\right),\\
\left(x_{s+2i-1}^\infty(t_i^-), v_{s+2i-1}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{s+2i-1}\right),\\
\left(x_{s+2i}^\infty(t_i^-), v_{s+2i}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{s+2i}\right),
\end{aligned}$$

while if $j_i = 1$:

$$\begin{aligned}
\left(x_{m_i}^\infty(t_i^-), v_{m_i}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{m_i}^{\infty*}(t_i^+)\right),\\
\left(x_{s+2i-1}^\infty(t_i^-), v_{s+2i-1}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{s+2i-1}^*\right),\\
\left(x_{s+2i}^\infty(t_i^-), v_{s+2i}^\infty(t_i^-)\right) &:= \left(x_{m_i}^\infty(t_i^+), v_{s+2i}^*\right),\\
(v_{m_i}^{\infty*}(t_i^-), v_{s+2i-1}^*, v_{s+2i}^*) &= T_{\omega_{s+2i-1},\omega_{s+2i}}\left(v_{m_i}^\infty(t_i^+), v_{s+2i-1}, v_{s+2i}\right).
\end{aligned}$$

Time $t_{k+1} = 0$: We finally obtain

$$Z_{s+2k}^\infty(0^+) = Z_{s+2k}^\infty(t_{k+1}^+) = \left(X_{s+2k}^\infty\left(t_k^-\right) - t_k V_{s+2k}^\infty\left(t_k^-\right), V_{s+2k}^\infty\left(t_k^-\right)\right).$$

The sequence $Z_{s+2i}^\infty(t_i^+)$, $i = 0, \ldots, k+1$ is called Boltzmann pseudo-trajectory of Z_s.

The construction process is illustrated in Fig. 3 (to be read horizontally and from right to left):

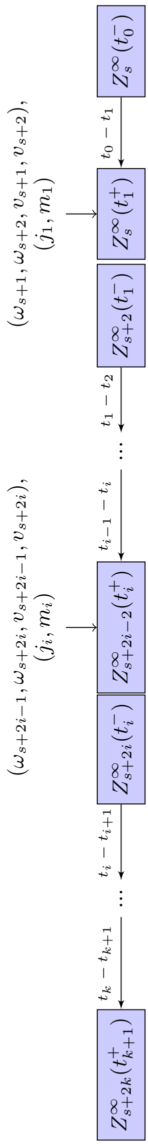

Fig. 3. Boltzmann pseudo-trajectory

10.4. Reduction to truncated elementary observables. We now use the Boltzmann pseudo-trajectory to define the truncated observables for the BBGKY hierarchy and Boltzmann hierarchy. The proof will then be reduced to the convergence of the corresponding truncated elementary observables. Given $\ell \in \mathbb{N}$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2) and $\epsilon << \alpha$, recall the set $G_\ell(\epsilon, \epsilon_0, \delta)$ from (10.1).

Let $s \in \mathbb{N}$, $X_s \in \Delta_s^X(\epsilon_0)$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{s,k}$ and $t \in [0, T]$ and $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$, where we recall from (7.8) the set $\mathcal{T}_{k,\delta}(t)$. By Proposition 10.1, for any $V_s \in \mathcal{M}_s^c(X_s)$, we have $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta)$. Since $t_0 - t_1 \geq \delta$, we obtain $Z_s^\infty(t_1^+) \in G_s(\epsilon_0, 0)$. Recalling notation from (9.3), Proposition 9.2 (see (9.6) for the pre-collisional case or (9.10) for the post-collisional case) yields there is a set $\mathcal{B}_{m_1}\left(Z_s^\infty\left(t_1^+\right)\right) \subseteq (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+\left(v_{m_1}^\infty\left(t_1^+\right)\right)$ such that

$$Z_{s+2}^\infty(t_2^+) \in G_{s+2}(\epsilon_0, 0), \quad \forall (\omega_{s+1}, \omega_{s+2}, v_{s+1}, v_{s+2}) \in \mathcal{B}_{m_1}^c\left(Z_s^\infty\left(t_1^+\right)\right).$$

Clearly this process can be iterated. In particular, given $i \in \{2, \ldots, k\}$, we have $Z_{s+2i-2}^\infty(t_i^+) \in G_{s+2i-2}(\epsilon_0, 0)$, so there exists a set $\mathcal{B}_{m_i}\left(Z_{s+2i-2}^\infty\left(t_i^+\right)\right) \subseteq (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+\left(v_{m_i}^\infty\left(t_1^+\right)\right)$ such that:

$$Z_{s+2i}^\infty(t_{i+1}^+) \in G_{s+2i}(\epsilon_0, 0), \quad \forall (\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathcal{B}_{m_i}^c\left(Z_{s+2i-2}^\infty\left(t_i^+\right)\right). \tag{10.11}$$

We finally obtain $Z_{s+2k}^\infty(0^+) \in G_{s+2k}(\epsilon_0, 0)$.

Let us now define the truncated elementary observables. Heuristically we will truncate the domains of adjusted particles in the definition of the observables $\widetilde{I}_{s,k,R,\delta}^N$, $\widetilde{I}_{s,k,R,\delta}^\infty$ (see (10.3)–(10.4)).

More precisely, let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.2), (N, ϵ) in the scaling (4.22) with $\epsilon << \alpha$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{s,k}$ and $t \in [0, T]$. For $X_s \in \Delta_s^X(\epsilon_0)$, Proposition 10.1 implies there is a set of velocities $\mathcal{M}_s(X_s) \subseteq B_R^{2d}$ such that $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta)$, for all $V_s \in \mathcal{M}_s^c(X_s)$. Following the reasoning above, we define the BBGKY hierarchy truncated observables as:

$$\begin{aligned} J_{s,k,R,\delta}^N(t, J, M)(X_s) := \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} & T_s^{t-t_1} \widetilde{\mathcal{C}}_{s,s+2}^{N,R,j_1,m_1} T_{s+2}^{t_1-t_2} \ldots \\ & \widetilde{\mathcal{C}}_{s+2k-2,s+2k}^{N,R,j_k,m_k} T_{s+2k}^{t_m} f_0^{(s+2k)}(Z_s)\, dt_k, \ldots dt_1 dV_s, \end{aligned} \tag{10.12}$$

where $\widetilde{\mathcal{C}}_{s+2i-2,s+2i}^{N,R,j_i,m_i} g_{N,s+2i} := \mathcal{C}_{s+2i-2,s+2i}^{N,R,j_i,m_i}$ $\left[g_{N,s+2i} \mathbb{1}_{(\omega_{s+2i-1},\omega_{s+2i},v_{s+2i-1},v_{s+2i}) \in \mathcal{B}_{m_i}^c\left(Z_{s+2i-2}^\infty(t_i^+)\right)}\right]$.

In the same spirit, for $X_s \in \Delta_s^X(\epsilon_0)$, we define the Boltzmann hierarchy truncated elementary observables as:

$$\begin{aligned} J_{s,k,R,\delta}^\infty(t, J, M)(X_s) := \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} & S_s^{t-t_1} \widetilde{\mathcal{C}}_{s,s+2}^{\infty,R,j_1,m_1} S_{s+2}^{t_1-t_2} \ldots \\ & \widetilde{\mathcal{C}}_{s+2k-2,s+2k}^{\infty,R,j_k,m_k} S_{s+2k}^{t_m} f_0^{(s+2k)}(Z_s)\, dt_k, \ldots dt_1 dV_s, \end{aligned} \tag{10.13}$$

where $\widetilde{\mathcal{C}}_{s+2i-2,s+2i}^{\infty,R,j_i,m_i} g_{s+2i} := \mathcal{C}_{s+2i-2,s+2i}^{\infty,R,j_i,m_i}\left[g_{s+2i} \mathbb{1}_{(\omega_{s+2i-1},\omega_{s+2i},v_{s+2i-1},v_{s+2i}) \in \mathcal{B}_{m_i}^c\left(Z_{s+2i-2}^\infty(t_i^+)\right)}\right]$.

Recalling the observables $\widetilde{I}_{s,k,R,\delta}^N$, $\widetilde{I}_{s,k,R,\delta}^\infty$ from (10.8), (10.10) and using Proposition 9.4 (since we integrate at least in one of the bad sets), we obtain:

Proposition 10.4. *Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.2), (N, ϵ) in the scaling (4.22) with $\epsilon << \alpha$ and $t \in [0, T]$. Then the following estimates hold uniformly in N:*

$$\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \|\widetilde{I}_{s,k,R,\delta}^{N}(t, J, M) - J_{s,k,R,\delta}^{N}(t, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))}$$
$$\leq C_{d,s,\mu_0,T}^{n} \|\phi_s\|_{L_{V_s}^\infty} R^{d(s+3n)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$
$$\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \|\widetilde{I}_{s,k,R,\delta}^{\infty}(t, J, M) - J_{s,k,R,\delta}^{\infty}(t, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))}$$
$$\leq C_{d,s,\mu_0,T}^{n} \|\phi_s\|_{L_{V_s}^\infty} R^{d(s+3n)} \eta^{\frac{d-1}{4d+2}} \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proof. As usual, it suffices to prove the estimate for the BBGKY hierarchy case and the Boltzmann hierarchy case follows similarly. Fix $k \in \{1, \dots, n\}$ and $(J, M) \in \mathcal{U}_{s,k}$. We first estimate the difference:

$$\widetilde{I}_{s,k,R,\delta}^{N}(t, J, M)(X_s) - J_{s,k,R,\delta}^{N}(t, J, M)(X_s). \tag{10.14}$$

Triangle and Cauchy-Scwhartz inequalities yield

$$\left|b(\omega_1, \omega_2, v_1 - v, v_2 - v)\right| \leq 4R, \quad \forall (\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}, \quad \forall v, v_1, v_2 \in B_R^d, \tag{10.15}$$

so

$$\int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} |b(\omega_1, \omega_2, v_1 - v, v_2 - v_2)|\, d\omega_1\, d\omega_2\, dv_1\, dv_2$$
$$\leq C_d R^{2d+1} \leq C_d R^{3d}, \quad \forall v \in B_R^d. \tag{10.16}$$

But in order to estimate the difference (10.14), we integrate at least once over $\mathcal{B}_{m_i}\left(Z_{s+2i-2}^{\infty}\left(t_i^+\right)\right)$ for some $i \in \{1, \dots, k\}$. Proposition 9.4 and the expression (10.15) yield the estimate:

$$\int_{\mathcal{B}_{m_i}(Z_{s+2i-2}^{\infty}(t_i^+))} |b(\omega_1, \omega_2, v_1-v, v_2-v)|\, d\omega_1\, d\omega_2\, dv_1\, dv_2 \leq C_d (s + 2i - 2) R^{2d+1} \eta^{\frac{d-1}{4d+2}}$$
$$\leq C_d (s + 2k) R^{3d} \eta^{\frac{d-1}{4d+2}}, \quad \forall v \in B_R^d. \tag{10.17}$$

Moreover, we have the elementary inequalities:

$$\|f_{N,0}^{(s+2k)}\|_{L^\infty} \leq e^{-(s+2k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0}, \tag{10.18}$$

$$\int_{T_{k,\delta}(t)} dt_1 \dots dt_k \leq \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} dt_1 \dots dt_k \leq \frac{T^k}{k!}. \tag{10.19}$$

Therefore, (10.16)–(10.19) imply

$$\left|\widetilde{I}_{s,k,R,\delta}^{N}(t, J, M)(X_s) - J_{s,k,R,\delta}^{N}(t, J, M)(X_s)\right| \leq$$
$$\leq \|\phi_s\|_{L_{V_s}^\infty} e^{-(s+2k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0} C_d R^{ds} C_d^{k-1} R^{3d(k-1)} (s + 2k) C_d R^{3d} \eta^{\frac{d-1}{4d+2}} \frac{T^k}{k!}$$
$$\leq C_{d,s,\mu_0,T}^{k} \|\phi_s\|_{L_{V_s}^\infty} \frac{(s + 2k)}{k!} R^{d(s+3k)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0}.$$

Adding for all $(J, M) \in \mathcal{U}_{s,k}$, we get $2^k s(s+2)\dots(s+2k-2)$ contributions, thus

$$\begin{aligned}
&\sum_{(J,M)\in\mathcal{U}_{s,k}} \|\widetilde{I}^N_{s,k,R,\delta}(t,J,M) - J^N_{s,k,R,\delta}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\leq C^k_{d,s,\mu_0,T}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)} \frac{(s+2k)^{k+1}}{k!} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0} \\
&\leq C^k_{d,s,\mu_0,T}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0},
\end{aligned} \tag{10.20}$$

since $\dfrac{(s+2k)^{k+1}}{k!} = \dfrac{(s+2k)(s+2k)^k}{k!} \leq C_s^k$, Summing over $k = 1, \dots, n$, we obtain the required estimate. $\square$

11. Convergence Proof

In Sect. 10.4, given $s, n \in \mathbb{N}$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2), (N, ϵ) in the scaling (4.22) with $\epsilon << \alpha$ and $t \in [0, T]$, we have reduced the convergence proof to controlling the differences $J^N_{s,k,R,\delta}(t,J,M) - J^\infty_{s,k,R,\delta}(t,J,M)$ for given $1 \leq k \leq n$ and $(J, M) \in \mathcal{U}_{s,k}$, where $J^N_{s,k,R,\delta}(t,J,M)$, $J^\infty_{s,k,R,\delta}(t,J,M)$ are given by (10.12)–(10.13), respectively. Throughout this section $s \in \mathbb{N}$ will be fixed. We also consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $F_0 \in X_{\infty,\beta_0,\mu_0}$ as in the statement of Theorem 6.9.

11.1. BBGKY pseudo-trajectories and proximity to the Boltzmann pseudo-trajectories. Consider $s \in \mathbb{N}$, (N, ϵ) in the scaling (4.22), $k \in \mathbb{N}$ and $t \in [0, T]$. Given $\delta > 0$ recall from (7.8) the set $\mathcal{T}_{k,\delta}(t)$. Let $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $J = (j_1, \dots, j_k)$, $M = (m_1, \dots, m_k)$, $(J, M) \in \mathcal{U}_{s,k}$, and for each $i = 1, \dots, k$, we consider $(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d}$.

In the same spirit as in Sect. 10.3 where we introduced the Boltzmann pseudo-trajectory, we define the BBGKY pseudo-trajectory, the main difference being that we take into account the interaction zone of the adjusted particles in each step. More precisely, given $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$:

Time $t_0 = t$: We initially define

$$Z_s^N(t_0^-) = \left(x_1^N(t_0^-), \dots, x_s^N(t_0^-), v_1^N(t_0^-), \dots, v_s^N(t_0^-)\right) := Z_s.$$

Time t_i, $i \in \{1, \dots, k\}$: Consider $i \in \{2, \dots, k\}$, and assume we know

$$Z^N_{s+2i-2}(t^-_{i-1}) = \left(x_1^N(t^-_{i-1}), \dots, x^N_{s+2i-2}(t^-_{i-1}), v_1^N(t^-_{i-1}), \dots, v^N_{s+2i-2}(t^-_{i-1})\right).$$

We define $Z^N_{s+2i-2}(t_i^+) = \left(x_1^N(t_i^+), \dots, x^N_{s+2i-2}(t_i^+), v_1^N(t_i^+), \dots, v^N_{s+2i-2}(t_i^+)\right)$ as:

$$Z^N_{s+2i-2}(t_i^+) := \left(X^N_{s+2i-2}\left(t^-_{i-1}\right) - (t_{i-1} - t_i)\, V^N_{s+2i-2}\left(t^-_{i-1}\right), V^N_{s+2i-2}\left(t^-_{i-1}\right)\right).$$

We also define $Z^N_{s+2i}(t_i^-) = \left(x_1^N(t_i^-), \dots, x^N_{s+2i}(t_i^-), v_1^N(t_i^-), \dots, v^N_{s+2i}(t_i^-)\right)$ as:

$$\left(x_j^N(t_i^-), v_j^N(t_i^-)\right) := \left(x_j^N(t_i^+), v_j^N(t_i^+)\right), \quad \forall j \in \{1, \dots, s+2i-2\} \setminus \{m_i\},$$

and if $j_i = -1$:

$$\left(x_{m_i}^N(t_i^-), v_{m_i}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+), v_{m_i}^N(t_i^+)\right),$$
$$\left(x_{s+2i-1}^N(t_i^-), v_{s+2i-1}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+) - \sqrt{2}\epsilon\omega_{s+2i-1}, v_{s+2i-1}\right),$$
$$\left(x_{s+2i}^N(t_i^-), v_{s+2i}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+) - \sqrt{2}\epsilon\omega_{s+2i}, v_{s+2i}\right),$$

while if $j_i = 1$:

$$\left(x_{m_i}^N(t_i^-), v_{m_i}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+), v_{m_i}^{N*}(t_i^+)\right),$$
$$\left(x_{s+2i-1}^N(t_i^-), v_{s+2i-1}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+) + \sqrt{2}\epsilon\omega_{s+2i-1}, v_{s+2i-1}^*\right),$$
$$\left(x_{s+2i}^N(t_i^-), v_{s+2i}^N(t_i^-)\right) := \left(x_{m_i}^N(t_i^+) + \sqrt{2}\epsilon\omega_{s+2i}, v_{s+2i}^*\right),$$
$$(v_{m_i}^{N*}(t_i^-), v_{s+2i-1}^*, v_{s+2i}^*) = T_{\omega_{s+2i-1},\omega_{s+2i}}\left(v_{m_i}^N(t_i^+), v_{s+2i-1}, v_{s+2i}\right).$$

Time $t_{k+1} = 0$: We finally obtain

$$Z_{s+2k}^N(0^+) = Z_{s+2k}^N(t_{k+1}^+) = \left(X_{s+2k}^N\left(t_k^-\right) - t_k V_{s+2k}^N\left(t_k^-\right), V_{s+2k}^N\left(t_k^-\right)\right).$$

The sequence $Z_{s+2i}^N(t_i^+)$, $i = 0, \ldots, k+1$ is called BBGKY pseudo-trajectory of Z_s. The construction can be illustrated by an analogous diagram to Figure 3.

We now state a proximity result for the corresponding BBGKY and Boltzmann pseudo-trajectories. The proof of this result follows inductively from the definition of the pseudo-trajectories, for more details see [2].

Lemma 11.1. *Let $s, n \in \mathbb{N}$, (N, ϵ) in the scaling (4.22), $1 \le k \le n$, $(J, M) \in \mathcal{U}_{s,k}$, $t \in [0, T]$ and $(t_1, \ldots, t_k) \in \mathcal{T}_k(t)$. Fix $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$. For each $i = 1, \ldots, k$, consider $(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}$. Then for all $i = 1, \ldots, k+1$ and $\ell = 1, \ldots, s+2i-2$, we have*

$$|x_\ell^N(t_i^+) - x_\ell^\infty(t_i^+)| \le \sqrt{2}\epsilon(i-1), \quad v_\ell^N(t_i^+) = v_\ell^\infty(t_i^+). \tag{11.1}$$

In particular, if $s < n$, there holds:

$$\left|X_{s+2i-2}^N(t_i^+) - X_{s+2i-2}^\infty(t_i^+)\right| \le \sqrt{6}n^{3/2}\epsilon, \quad \forall i = 1, \ldots, k+1. \tag{11.2}$$

11.2. Reformulation in terms of pseudo-trajectories. We will now re-write the Boltzmann hierarchy truncated elementary observables, defined in (10.13), and the BBGKY hierarchy truncated elementary observables, defined in (10.12), in terms of pseudo-trajectories.

Let $s, n \in \mathbb{N}$ with $s < n$, parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.2). For the Boltzmann hierarchy case, there is always free flow between the collision times. Therefore, for

$X_s \in \Delta_s^X(\epsilon_0)$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{s,k}$ and $t \in [0, T]$, the Boltzmann hierarchy truncated elementary observable can be written

$$J_{s,k,R,\delta}^{\infty}(t, J, M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}_{m_1}^c(Z_s^\infty(t_1^+))} \cdots \int_{\mathcal{B}_{m_k}^c(Z_{s+2k-2}^\infty(t_k^+))}$$
$$\prod_{i=1}^{k} b_+\left(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v_{m_i}^\infty\left(t_i^+\right), v_{s+2i} - v_{m_i}^\infty\left(t_i^+\right)\right) f_0^{(s+2k)}\left(Z_{s+2k}^\infty\left(0^+\right)\right)$$
$$\times \prod_{i=1}^{k} (d\omega_{s+2i-1}\, d\omega_{s+2i}\, dv_{s+2i-1}\, dv_{s+2i})\, dt_k \ldots dt_1\, dV_s. \tag{11.3}$$

It is not immediate to obtain a comparable expansion at the BBGKY level because of the recollisions. However, thanks to Proposition 9.2 and Lemma 11.1, this is possible for N large enough.

More precisely, fix $X_s \in \Delta_s^X(\epsilon_0)$, $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{s,k}$, $t \in [0, T]$ and $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$. Consider (N, ϵ) in the scaling (4.22) with N large enough such that $n^{3/2}\epsilon << \alpha$. By Proposition 10.1, given $V_s \in \mathcal{M}_s^c(X_s)$, we have $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta)$. By the definition of the set $G_s(\epsilon, \epsilon_0, \delta)$, see (10.1), we have $Z_s \in G_s(\epsilon, \epsilon_0, \delta) \Rightarrow Z_s(\tau) \in \mathring{\mathcal{D}}_{s,\epsilon}$, for all $\tau \geq 0$, thus

$$\Psi_s^{\tau - t_0} Z_s^N\left(t_0^-\right) = \Phi_s^{\tau - t_0} Z_s^N\left(t_0^-\right), \quad \forall \tau \in [t_1, t_0], \tag{11.4}$$

where Ψ_s, given in (3.29), denotes the ϵ-interaction zone flow of s-particles and Φ_s, given in (3.30), denotes the free flow of s-particles. We also have $Z_s = (X_s, V_s) \in G_s(\epsilon, \epsilon_0, \delta) \Rightarrow Z_s^\infty(t_1^+) \in G_s(\epsilon_0, 0)$. Moreover, for all $i \in \{1, \ldots, k\}$, we have seen that for all $(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1}, v_{s+2i}) \in \mathcal{B}_{m_i}^c(Z_{s+2i-2}^\infty(t_i^+))$

$$Z_{s+2i}^\infty(t_{i+1}^+) \in G_{s+2i}(\epsilon_0, 0). \tag{11.5}$$

Since $s < n$ and $n^{3/2}\epsilon << \alpha$, (11.2) from Lemma 11.1 implies

$$\left| X_{s+2i-2}^N(t_i^+) - X_{s+2i-2}^\infty(t_i^+) \right| \leq \frac{\alpha}{2}, \quad \forall i = 1, \ldots, k.$$

Then, Proposition 9.2 yields that for any $i = 1, \ldots, k$, we have

$$\Psi_{s+2i}^{\tau - t_i} Z_{s+2i}^N\left(t_i^-\right) = \Phi_{s+2i}^{\tau - t_i} Z_{s+2i}^N\left(t_i^-\right), \quad \forall \tau \in [t_{i+1}, t_i]. \tag{11.6}$$

Moreover, Lemma 11.1 also implies that $v_{m_i}^N(t_i^+) = v_{m_i}^\infty(t_i^+)$, for all $i = 1, \ldots, k$. Therefore, for N large enough such that $n^{3/2}\epsilon << \alpha$, (11.4), (11.6) yield the expansion

$$J_{s,k,R,\delta}^{N}(t, J, M)(X_s) = \boldsymbol{A_{N,\epsilon}^{s,k}} \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}_{m_1}^c(Z_s^\infty(t_1^+))} \cdots \int_{\mathcal{B}_{m_k}^c(Z_{s+2k-2}^\infty(t_k^+))}$$
$$\prod_{i=1}^{k} b_+\left(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v_{m_i}^\infty\left(t_i^+\right), v_{s+2i} - v_{m_i}^\infty\left(t_i^+\right)\right) f_{N,0}^{(s+2k)}\left(Z_{s+2k}^N\left(0^+\right)\right)$$
$$\times \prod_{i=1}^{k} (d\omega_{s+2i-1}\, d\omega_{s+2i}\, dv_{s+2i-1}\, dv_{s+2i})\, dt_k \ldots dt_1\, dV_s, \tag{11.7}$$

where, recalling (4.20), we denote

$$\boldsymbol{A_{N,\epsilon}^{s,k}} := \prod_{i=1}^{k} A_{N,\epsilon,s+2i-2} = 2^{k(d-2)}\epsilon^{k(2d-1)} \prod_{i=1}^{k}(N-s-2i+2)(N-s-2i+1). \tag{11.8}$$

Remark 11.2. Notice that for fixed $s, k \in \mathbb{N}$, (N, ϵ) in the scaling (4.22), there holds the estimate

$$0 < 1 - \boldsymbol{A_{N,\epsilon}^{s,k}} \leq 2^{\frac{d+1}{2}} \epsilon^{d-1/2} k(s+2k-1). \tag{11.9}$$

In particular $\boldsymbol{A_{N,\epsilon}^{s,k}} \nearrow 1$, as $N \to \infty$ and $\epsilon \to 0$ in the scaling (4.22).

Let us approximate the BBGKY hierarchy initial data by Boltzmann hierarchy initial data defining some auxiliary functionals. Let $s \in \mathbb{N}$ and $X_s \in \Delta_s^X(\epsilon_0)$. For $1 \leq k \leq n$, $(J, M) \in \mathcal{U}_{s,k}$ and $t \in [0, T]$, we define the auxiliary functional $\widehat{J}_{s,k,R,\delta}^{N}(t, J, M)$ which differs from $J_{s,k,R,\delta}^{N}(t, J, M)$ by the absence of the scaling factor $\boldsymbol{A_{N,\epsilon}^{s,k}}$ and the use of Boltzmann hierarchy initial data:

$$\begin{aligned}
\widehat{J}_{s,k,R,\delta}^{N}(t, J, M)(X_s) := &\int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}_{m_1}^c(Z_s^\infty(t_1^+))} \dots \int_{\mathcal{B}_{m_k}^c(Z_{s+2k-2}^\infty(t_k^+))} \\
&\prod_{i=1}^{k} b_+\left(\omega_{s+2i-1}, \omega_{s+2i}, v_{s+2i-1} - v_{m_i}^\infty\left(t_i^+\right), v_{s+2i} - v_{m_i}^\infty\left(t_i^+\right)\right) \\
&\quad f_0^{(s+2k)}\left(Z_{s+2k}^N\left(0^+\right)\right) \\
&\times \prod_{i=1}^{k} \left(d\omega_{s+2i-1}\, d\omega_{s+2i}\, dv_{s+2i-1}\, dv_{s+2i}\right) dt_k \dots dt_1\, dV_s.
\end{aligned} \tag{11.10}$$

Due to the scaling (4.22) and convergence of the initial data, we conclude that the auxiliary functionals approximate the BBGKY hierarchy truncated elementary observables $J_{s,k,R,\delta}^{N}$, defined in (11.7).

Proposition 11.3. *Let $s, n \in \mathbb{N}$, with $s < n$, α, ϵ_0, R, η, δ be parameters as in (9.2), and $t \in [0, T]$. Then for any $\zeta > 0$, there is $N^* = N^*(\zeta) \in \mathbb{N}$, such that for all (N, ϵ) in the scaling (4.22) with $N > N^*$, there holds:*

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \| J_{s,k,R,\delta}^{N}(t, J, M) - \widehat{J}_{s,k,R,\delta}^{N}(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\quad \leq C_{d,s,\mu_0,T}^{n} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \zeta^2.
\end{aligned} \tag{11.11}$$

In the case of tensorized initial data and approximation by conditioned BBGKY initial data (see Proposition 6.6), the estimate can be improved to

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \| J_{s,k,R,\delta}^{N}(t, J, M) - \widehat{J}_{s,k,R,\delta}^{N}(t, J, M) \|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\quad \leq C_{d,s,\beta_0,\mu_0,T}^{n} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \epsilon^{1/2},
\end{aligned} \tag{11.12}$$

for all (N, ϵ) in the scaling (4.22) with N large enough.

Proof. Fix $1 \leq k \leq n$ and $(J, M) \in \mathcal{U}_{s,k}$. Consider (N, ϵ) in the scaling (4.22) with N large enough such that $n^{3/2}\epsilon << \alpha$. Triangle inequality and the fact that $\Delta_s^X(\epsilon_0) \subseteq \Delta_s^X(\epsilon_0/2)$ yield

$$\begin{aligned}
&\|J_{s,k,R,\delta}^N(t,J,M) - \widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\quad \leq \|J_{s,k,R,\delta}^N(t,J,M) - \boldsymbol{A_{N,\epsilon}^{s,k}}\widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0/2))} \\
&\qquad + (1 - \boldsymbol{A_{N,\epsilon}^{s,k}})\|\widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))}.
\end{aligned} \tag{11.13}$$

We estimate each of the terms in (11.13). For the first term, let us fix $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$. Applying (11.5) for $i = k-1$, we obtain $Z_{s+2k-2}^\infty(t_k^+) \in G_{s+2k-2}(\epsilon_0, 0)$. Since $s < n$ and $n^{3/2}\epsilon << \alpha$, (11.2), applied for $i = k$, implies $|X_{s+2k-2}^N(t_k^+) - X_{s+2k-2}^\infty(t_k^+)| \leq \frac{\alpha}{2}$. Therefore, Proposition 9.2 (precisely expression (9.5) for the pre-collisional case, (9.9) for the post-collisional case) implies $Z_{s+2k}^N(0^+) \in G_{s+2k}(\epsilon_0/2, 0) \subseteq \Delta_{s+2k}(\epsilon_0/2)$. Thus (10.16), (10.18)–(10.19), (11.7)–(11.10) imply

$$\begin{aligned}
&\|J_{s,k,R,\delta}^N(t,J,M) - \boldsymbol{A_{N,\epsilon}^{s,k}}\widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0/2))} \\
&\quad \leq \frac{C_{d,s,T}^k}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|f_{N,0}^{(s+2k)} - f_0^{(s+2k)}\|_{L^\infty(\Delta_{s+2k}(\epsilon_0/2))} \\
&\quad \leq \frac{C_{d,s,T}^k}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|f_{N,0}^{(s+2k)} - f_0^{(s+2k)}\|_{L^\infty(\mathcal{D}_{s+2k,\epsilon})},
\end{aligned} \tag{11.14}$$

as long as $\epsilon < \epsilon_0/2\sqrt{2}$ (i.e. N large enough) so that $\Delta_{s+2k}(\epsilon_0/2) \subseteq \mathcal{D}_{s+2k,\epsilon}$. For the second term, using (10.16) we obtain

$$\|\widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq \frac{C_{d,s,\mu_0,T}^k}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|F_0\|_{\infty,\beta_0,\mu_0}. \tag{11.15}$$

Adding over all $(J, M) \in \mathcal{U}_{s,k}$, $k = 1, \ldots, n$, using (11.13)–(11.15), (11.9) and an argument similar to (10.20) to control the summation over $k = 1, \ldots, n$, for N large enough, we obtain the estimate

$$\begin{aligned}
&\sum_{k=1}^n \sum_{(J,M)\in\mathcal{U}_{s,k}} \|J_{s,k,R,\delta}^N(t,J,M) - \widehat{J}_{s,k,R,\delta}^N(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\quad \leq C_{d,s,\mu_0,T}^n \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \\
&\quad \times \left(\sup_{k\in\{1,\ldots,n\}} \|(f_{N,0}^{(s+2k)} - f_0^{(s+2k)})\|_{L^\infty(\mathcal{D}_{s+2k,\epsilon})} + \|F_0\|_{\infty,\beta_0,\mu_0}\epsilon^{d-1/2} \right).
\end{aligned}$$

Since n is fixed, the result follows from convergence in the level of initial data and the scaling estimate (11.9).

In the case of tensorized initial data and approximation by conditioned BBGKY initial data, the estimate can be improved to (11.12) using (6.3). ◻

Due to the proximity Lemma 11.1 and the uniform continuity assumption (6.14) on the Boltzmann hierarchy initial data, we also obtain the following

Proposition 11.4. *Let $s, n \in \mathbb{N}$ with $s < n$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.2) and $t \in [0, T]$. Then for any $\zeta > 0$, there is $N^* = N^*(\zeta) \in \mathbb{N}$, such that for all (N, ϵ) in the scaling (4.22) with $N > N^*$, there holds*

$$\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \|\widehat{J}^N_{s,k,R,\delta}(t, J, M) - J^\infty_{s,k,R,\delta}(t, J, M)\|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \zeta^2. \tag{11.16}$$

In the case of Hölder continuous $C^{0,\gamma}$, $\gamma \in (0, 1]$ tensorized initial data (see Remark 6.3), the estimate can be improved to

$$\sum_{k=1}^{n} \sum_{(J,M)\in\mathcal{U}_{s,k}} \|\widehat{J}^N_{s,k,R,\delta}(t, J, M) - J^\infty_{s,k,R,\delta}(t, J, M)\|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \epsilon^\gamma, \tag{11.17}$$

for all (N, ϵ) in the scaling (4.22).

Proof. Let $\zeta > 0$. Fix $1 \leq k \leq n$ and $(J, M) \in \mathcal{U}_{s,k}$. Since $s < n$, Lemma 11.1 yields

$$|Z^N_{s+2k}(0^+) - Z^\infty_{s+2k}(0^+)| \leq \sqrt{6} n^{3/2} \epsilon, \quad \forall Z_s \in \mathbb{R}^{2ds}. \tag{11.18}$$

Thus the continuity assumption (6.14) on F_0, (11.18) and the scaling (4.22) imply that there exists $N^* = N^*(\zeta) \in \mathbb{N}$, such that for all $N > N^*$, we have

$$|f_0^{(s+2k)}(Z^N_{s+2k}(0^+)) - f_0^{(s+2k)}(Z^\infty_{s+2k}(0^+))| \leq C^{s+2k-1} \zeta^2, \quad \forall Z_s \in \mathbb{R}^{2ds}. \tag{11.19}$$

In the same spirit as in the proof of Proposition 11.3, using (11.19), (10.16), (10.19), and summing over $(J, M) \in \mathcal{U}_{s,k}$, $k = 1, \ldots, n$, we obtain estimate (11.16).

In the case of tensorized $C^{0,\gamma}$ data, one can easily see by induction that for any $Z_{s+2k}, Z'_{s+2k} \in \mathbb{R}^{2d(s+2k)}$, we have

$$\begin{aligned} |f_0^{\otimes(s+2k)}(Z_{s+2k}) - f_0^{\otimes(s+2k)}(Z'_{s+2k})| &\leq \|f_0\|^{s+2k-1}_{L^\infty} [f_0]_{C^{0,\gamma}} \sqrt{2d(s+2k)} |Z_{s+2k} - Z'_{s+2k}|^\gamma \\ &\leq C^{s+2k-1} |Z_{s+2k} - Z'_{s+2k}|^\gamma. \end{aligned}$$

Thus by (11.18) we have

$$|f_0^{(s+2k)}(Z^N_{s+2k}(0^+)) - f_0^{(s+2k)}(Z^\infty_{s+2k}(0^+))| \leq C^{s+2k-1} \epsilon^\gamma,$$

and the estimate (11.17) follows in a similar manner as estimate (11.16). □

11.3. Proof of Theorem 6.9. We are now in the position to prove Theorem 6.9. Fix $\sigma > 0$, $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ and $t \in [0, T]$. Consider $n \in \mathbb{N}$ with $s < n$, and parameters $\alpha, \epsilon_0, R, \eta, \delta$ satisfying (9.2). Let $\zeta > 0$ small enough. Triangle inequality, Propositions 7.1, 10.2, 10.4, Remark 10.3, estimates (11.11), (11.16) and part *(i)* of Definition 6.1, yield that there is $N^*(\zeta) \in \mathbb{N}$ such that for all $N > N^*$, we have

$$\begin{aligned}\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C\left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C^n\right)\\ + C^n R^{4dn}\eta^{\frac{d-1}{4d+2}} + C^n R^{4dn}\zeta^2,\end{aligned} \tag{11.20}$$

where $C > 1$ is an appropriate constant.

We now choose parameters satisfying (9.2), depending only on ζ, such that the right hand side of (11.20) becomes less than ζ.

Choice of parameters: For ζ sufficiently small, we choose $n \in \mathbb{N}$ and the parameters $\delta, \eta, R, \epsilon_0, \alpha$ in the following order:

$$\begin{aligned}&\max\left\{s, \log_2(C\zeta^{-1})\right\} << n, \quad \delta << \zeta C^{-(n+1)},\\ &\max\left\{1, \sqrt{3}\beta_0^{-1/2}\ln^{1/2}(C\zeta^{-1})\right\} << R << \zeta^{-1/4dn}C^{-1/4d},\\ &\eta << \zeta^{\frac{8d+4}{d-1}}, \quad \epsilon_0 << \min\{\sigma, \eta\delta\}, \quad \alpha << \epsilon_0 \min\{1, R^{-1}\eta\}.\end{aligned} \tag{11.21}$$

Relations (11.21) imply the parameters chosen satisfy (9.2) and depend only on ζ. Then, (11.20)–(11.21) imply that we may find $N_0(\zeta) \in \mathbb{N}$, such that for all (N, ϵ) in the scaling (4.22) with $N > N_0$, there holds

$$\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\sigma))} \overset{\epsilon_0<\sigma}{\leq} \|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} < \zeta,$$

and Theorem 6.9 is proved.

Proof of Corollary 6.11. By Theorem 5.13 we have that $\boldsymbol{F} = (f^{\otimes s})_{s\in\mathbb{N}}$, where f is the mild solution of the ternary Boltzmann equation. Therefore, in the same spirit as before (using estimates (11.12), (11.17) instead of (11.11), (11.16)), for N large enough we have

$$\begin{aligned}&\|I_{\phi_s} f_N^{(s)}(t) - I_{\phi_s} f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))}\\ &\quad\leq C\left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C^n\right) + C^n R^{4dn}\eta^{\frac{d-1}{4d+2}} + C^n R^{4dn}\epsilon^{\gamma_*},\end{aligned} \tag{11.22}$$

where $\gamma_* = \min\{1/2, \gamma\} \in (0, \frac{1}{2}]$ and γ is the Hölder regularity of f_0. Consider $0 < r < \gamma_*$.

Choice of parameters: For N large enough (or equivalently for ϵ small enough), we choose $n \in \mathbb{N}$ and the parameters $\delta, \eta, R, \epsilon_0, \alpha$ in the following order:

$$\begin{aligned}&\max\left\{s, \log_2(C\epsilon^{\gamma_*})\right\} << n, \quad \delta << \epsilon^{\gamma_*} C^{-(n+1)},\\ &\max\left\{1, \sqrt{3}\beta_0^{-1/2}\ln^{1/2}(C\epsilon^{-\gamma_*})\right\} << R << \epsilon^{\frac{r-\gamma_*}{4dn}}C^{-1/4d},\\ &\eta << \epsilon^{\frac{4d+2)}{d-1}\gamma_*}, \quad \epsilon_0 << \min\{\sigma, \eta\delta\}, \quad \alpha << \epsilon_0 \min\{1, R^{-1}\eta\}.\end{aligned} \tag{11.23}$$

Then by (11.22), for N large enough, we take

$$\|I_{\phi_s} f_N^{(s)}(t) - I_{\phi_s} f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\sigma))} \overset{\epsilon_0<\sigma}{\leq} \|I_{\phi_s} f_N^{(s)}(t) - I_{\phi_s} f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} < \epsilon^r,$$

and Corollary 6.11 is proved. □

Acknowledgement. I.A. and N.P. acknowledge support from NSF grants DMS-1516228, DMS-1840314 and DMS-2009549. I.A. acknowledges support from the Simons Collaboration on Wave Turbulence. Authors are thankful to Thomas Chen, Irene M. Gamba, Philip Morrison, Maja Tasković and Joseph K. Miller for helpful discussions regarding physical and mathematical aspects of the problem. Authors would like to thank Ryan Denlinger for his constructive suggestions regarding geometric estimates in this paper. Finally, authors are grateful to the reviewers for their in depth comments and remarks which significantly improved the manuscript.

Appendix A. Auxiliary Results

In this appendix, we state two auxiliary results. For the proofs, see [2].

Lemma A.1. *Let $n \in \mathbb{N}$, $\lambda \neq 0$ and $w, u \in \mathbb{R}^n$. Denoting by I_n the $n \times n$ identity matrix, we have*

$$\det(\lambda I_n + wu^T) = \lambda^n (1 + \lambda^{-1} \langle w, u \rangle).$$

Lemma A.2. *Let $n \in \mathbb{N}$, $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and $\gamma \in \mathbb{R}$. Assume there is $\delta > 0$ with $\nabla \Psi(\boldsymbol{\omega}) \neq 0$ for $\boldsymbol{\omega} \in [\gamma - \delta < \Psi < \gamma + \delta]$. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and consider a C^1 map $F : \Omega \to \mathbb{R}^n$ of non-zero Jacobian in Ω. Then for any measurable $g : \mathbb{R}^n \to [0, +\infty]$ or $g : \mathbb{R}^n \to [-\infty, +\infty]$ integrable*

$$\begin{aligned} &\int_{[\Psi = \gamma]} g(\nu) \mathcal{N}_F(\nu, [\Psi \circ F = \gamma]) \, d\sigma(\nu) \\ &= \int_{[\Psi \circ F = \gamma]} (g \circ F)(\omega) |\operatorname{Jac} F(\omega)| \frac{|\nabla \Psi(F(\omega))|}{|\nabla(\Psi \circ F)(\omega)|} \, d\sigma(\omega), \end{aligned} \tag{A.1}$$

where given $\nu \in \mathbb{R}^n$ and $A \subseteq \Omega$, $\mathcal{N}_F(\nu, A) := \operatorname{card}(\{\omega \in A : F(\omega) = \nu\})$ is the Banach indicatrix of A.

References

1. Alexander, R.: The infinite hard sphere system. Ph.D. dissertation, Dept. Mathematics, Univ. California, Berkeley (1975)
2. Ampatzoglou, I.: Higher order extensions of the Boltzmann equation. Ph.D, dissertation, Dept. Mathematics, UT Austin (2020)
3. Ampatzoglou, I., Gamba, I.M., Pavlović, N., Tasković, M.: Global well-posedness for a binary–ternary Boltzmann equation, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire (2019). arXiv:1910.14476
4. Ampatzoglou, I., Pavlović, N.: Rigorous derivation of a binary–ternary Boltzmann equation for a dense gas of hard spheres (2020). arXiv:2007.00446
5. Bobylev, A.V., Gamba, I.M., Cercignani, C.: On the self-similar asymptotics for generalized non-linear kinetic Maxwell models. Commun. Math. Phys. **291**, 599–644 (2009)
6. Bobylev, A.V., Gamba, I.M., Cercignani, C.: Generalized kinetic Maxwell type models of granular gases. In: Capriz, G., Giovine, P., Mariano, P.M. (eds.) Mathematical Models of Granular Matter Series: Lecture Notes in Mathematics, vol. 1937. Springer, Berlin (2008) . (**ISBN: 978-3-540-78276-6**)
7. Bobylev, A.V., Pulvirenti, M., Saffirio, C.: From particle systems to the landau equation: a consistency result. Commun. Math. Phys. **319**, 683–702 (2013). https://doi.org/10.1007/s00220-012-1633-6
8. Boltzmann, L.: Weitere Studien uber das Warme gleichgenicht unfer Gasmolakular. Sitzungs-berichte der Akademie der Wissenschaften **66**, 275–370 (1872). Translation: Further studies on the thermal equilibrium of gas molecules, in Kinetic Theory 2, 88–174, Ed. S.G. Brush, Pergamon, Oxford (1966)
9. Boltzmann, L.: Lectures on Gas Theory, Translated by Stephen G. Brush. University of California Press, Berkeley-Los Angeles (1964)

10. Cercignani, C.: The Boltzmann Equation and Its Applications. Springer, Berlin (1988)
11. Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. Springer, New York (1994)
12. Choh, S.T., Uhlenbeck, G.E.: The kinetic theory of phenomena in dense gases. Ph.D. dissertation, University of Michigan (1958)
13. Cohen, E.G.D.: Fundamental Problems in Statistical Mechanics, vol. 1. E.G.D. Cohen, ed., p. 228. NHPC, Amsterdam (1968)
14. Cohen, E.G.D.: Fifty years of kinetic theory. Phys. A **194**, 229–257 (1993)
15. Denlinger, R.: The propagation of chaos for a rarefied gas of hard spheres in the whole space. Arch. Ration. Mech. Anal. **229**, 885–952 (2018)
16. Dobnikar, J., Bechinger, C., Brunner, M., von Gründberg, H.H.: Three-body interactions in colloidal systems. Phys. Rev. E **69**, 031402 (2004)
17. Dobnikar, J., Chen, Y., Rzehak, R., von Grünberg, H.. H.: Many-body interactions in colloidal suspensions. J. Phys. Condens. Matter **15**, S263–S268 (2003)
18. Dorfman, J., Cohen, E.: Difficulties in the kinetic theory of dense gases. J. Math. Phys. **8**, 282 (1967). https://doi.org/10.1063/1.1705194
19. Gallagher, I., Saint-Raymond, L., Texier, B.: From Newton to Boltzmann: hard spheres and short-range potentials. Zurich Adv. Lect. Math. Ser. **18**, 148 (2014)
20. Grad, H.: On the kinetic theory of rarefied gases. Commun. Pure Appl. Math. **2**, 331–407 (1949)
21. Grad, H.: Principles of the kinetic theory of gases. In: Flugge, S. (ed.) Handbuch der Physik, Thermodynamik der Gase, vol. 12, pp. 205–294. Springer, Berlin (1958) , Thermodynamik der Gase, vol. 12, pp. 205–294. Springer, Berlin (1958)
22. Green, M.: Boltzmann equation from the statistical mechanical point of view. J. Chem. Phys. **25**, 836 (1956). https://doi.org/10.1063/1.1743132
23. Hoegy, W., Sengers, J.: Three-particle collisions in a gas of hard spheres. Phys. Rev. A **2**(6), 2461 (1970)
24. Hollinger, B., Curtis, C.: Kinetic theory of dense gases. J. Chem. Phys. **33**, 1386 (1960). https://doi.org/10.1063/1.1731418
25. Hynninen, A.P., Dijkstra, M., van Roij, R.: Effect of three-body interactions on the phase behavior of charge-stabilized colloidal suspensions. Phys. Rev. E **69**, 061407 (2004)
26. King, F.: BBGKY hierarchy for positive potentials. Ph.D. dissertation, Dept. Mathematics, Univ. California, Berkeley (1975)
27. Lanford, O.E.: Time evolution of large classical systems. In: Moser, J. (ed.) Lect. Notes in Physics, vol. 38, pp. 1–111. Springer, Berlin (1975)
28. Pulvirenti, M., Saffirio, C., Simonella, S.: On the validity of the Boltzmann equation for short range potentials. Rev. Math. Phys. **26**(2), 1450001 (2014)
29. Russ, C., von Grünberg, H.H., Dijkstra, M., van Roij, R.: Three-body forces between charged colloidal particles. Phys. Rev. E **66**, 011402 (2002)
30. Sengers, J.: The three-particle collision term in the generalized Boltzmann equation. Acta Physica Austriaca Suppl X, 177–208 (1973)
31. Simonella, S.: Evolution of correlation functions in the hard sphere dynamics. J. Stat. Phys. **155**(6), 1191–1221 (2014)

Communicated by C. Mouhot