What is a Vector to Students?

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This study presents linear algebra students' vector conception found in the least-squares solution context through an IOLA (Inquiry-Oriented Linear Algebra) CTE (Classroom Teaching Experiment). Students' reflection writings after the CTE are the data source. Using a previously found student conception of vector in another study as a basic framing, the CTE data have been analyzed to investigate how students used the word 'vector' and what they referred to. This study offers a framework, a tool to be useful in a wide range of describing student conception of a vector emphasizing their natural way of thinking of a vector and their use of the vector.

Keywords: Vector, Linear Combination, Span, Linear Algebra, Student Thinking

Vectors are widely used in mathematical sciences. In Calculus, a vector is represented by an arrow or a directed line segment which has both magnitude and direction. Students work with functions of two or more variables incorporating vectors in that course. In linear algebra, vectors are extensively used as students learn the key concepts such as linear (in)dependence, span, basis, and vector spaces. While a significant body of studies have explored student conception of domain-specific contents, less study on student thinking of vectors has been done in linear algebra. This proposal foregrounds vectors and investigates students' notion of vectors to answer the research questions: How do linear algebra students think of a vector? And how do they use the vector and what do they refer to?

Literature

There is little literature exploring students' thinking of a vector in mathematics. Hillel (2000) identified three modes of vector representation: directed line segments (geometric mode), n-tuples (algebraic mode), and elements of vector spaces (abstract mode). In the geometric mode, students interpret a vector as an arrow having the magnitude and the direction with initial and terminal points. In the algebraic mode, students perform vector operations component-wise. In the abstract mode, a vector is described as an element of a vector space, and its vector operation is performed according to the axioms of that vector space. Along the similar strand, Sierpinska (2000) defined the terms, "synthetic-geometric thinking" that mathematical objects are given to students' mind directly being seen as a shape lying in space, and "analytic-arithmetic thinking" and "analytic-structural thinking" that mathematical objects are being component-coordinatized in a given dimension. Watson et al. (2003) found that students have the imagery of "a journey" when adding vectors. For example, students think that the equality in $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ holds because it is moving from A to B and finally to C. But this may become problematic when students interpret the commutative property for addition such as $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ since the right side does not give a satisfying meaning to them with the journey reasoning. Kwon (2011, 2013) focused on the progress students make from embodied vector to abstract vector and provided a vector framework attending to the ontological and epistemological aspects. It incorporated Hillel (2000) and Siepinska (2000)'s modes of description and reasoning into the framework. Appova & Berezovski (2013) identified students' misconceptions and error patterns about vector operations. The students in their study did not distinguish between a vector and a scalar in operations, for example, a vector minus a scalar equaling another vector or a scalar. Mikula and Heckler (2017) developed a framework to design essential skills on vector math and implemented an online training for introductory physics students. They categorized the skills required for vector operations such as addition, subtraction, dot products, and cross products using both **i, j, k** and arrow representations.

Theoretical Framework

This study is grounded in the theoretical perspective which makes a distinction between students' mathematics and mathematics of students (Steffe & Thompson, 2000). Students' mathematics refers to the students' mathematical reality which we cannot access directly, whereas our interpretation of the students' mathematical reality is referred to as mathematics of students. Even though models constructed in this study are not representing their mathematics perfectly, it is still worth constructing the models because students' mathematics is indicated by observable behaviors such as the students' gestures, written work, and discourse occurring in their mathematical activities. In this paper, the hypothesis that various ways a student thinks about a vector are closely related to their written description of the vector and to their use of the vector in task setting, is the theoretical foundation in developing the conceptual framework and analyzing data.

The conceptual framework used to analyze data in this study was developed as a tool to be useful in a wide range of describing student conception of a vector. It was initially the result of the author's unpublished class project in which task-based clinical interviews were conducted with students who had no or little experience with linear algebra at that time. The initial framework consists of five different vector conceptions: Vector as a point, Vector as a displacement, Vector as a direction, Vector as an equivalence class, and Vector as a linear combination of other vectors. Its geometric description of the categories is shown in Figure 1. Even though a vector can be in any dimensional space, the descriptions of the categories are illustrated in R^2because the interview tasks mainly included 2D vectors. Vector as an arrow is set as a default because it was the initial description of vector most students made in the study.

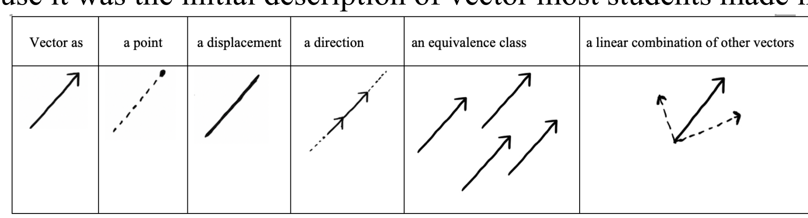

Figure 1. Geometric description of vector conception

Using the initial framework as a basic lens, the author began coding students' data from a classroom teaching experiment (CTE) implementing a recently designed unit about least squares method in linear algebra. The framework was further developed by the author as the vectors in the task of the CTE include various aspects categorized under "student conception of a vector" and "how they use the vector" in the problem context.

Vector conception categories

Vector as an arrow: Students describe the tip of a vector as referring to a line segment.

Vector as a point: Students use the word 'point' to describe a vector or they attend to the tip of a vector considering the tail as if it were at origin.

Vector as a segment: Students describe a vector as a connected segment between two locations.

Vector as a direction: Students use the word vector referring to the span of the vector or the direction of the vector.

Use of vector categories

Over and up alternations: Students use vectors in over and up process alternately.

Scaling: Students use vectors in scaling process.

Constructing a space: Students use vectors in generating a space such as a line or a plane.

Methods

This study reports students' notion of vector and their use of vector in a classroom teaching experiment (CTE) conducted as a part of a broader NSF-funded linear algebra project aimed at developing instructional sequences. A team of mathematics education researchers designed a new unit about least squares methods in linear algebra adopting the instructional design heuristics of Realistic Mathematics Education (RME) informed by Freudenthal (1991). The classroom teaching experiment (CTE) was used as a method to test the instructional sequence of the domain-specific mathematical activities and to see how students in the classroom reason and how the reasoning evolves with the task sequence (Cobb, 2000).

The Task "Meeting Gauss"

The designed task referred to as "Meeting Gauss" begins with an experientially real situation: you want to meet Gauss using three modes of transportation- carpet ($\boldsymbol{v_1}$), hoverboard ($\boldsymbol{v_2}$), and jetski ($\boldsymbol{v_3}$). However, there is no way for you to reach Gauss because Gauss ($\boldsymbol{g}$) is located off the plane spanned by the transportation vectors. Now, Gauss also needs to move to meet somewhere you can reach, but he wants his trip to be the shortest as possible (Figure 2). The big guiding questions students were asked in the task are (a) Where should Gauss meet you? (b) Along what vector would Gauss travel to get to your meeting point? (c) What distance would Gauss's trip be? (d) How would you get to your meeting point using your modes of transportation? To answer these questions, students need to think of vectors and use them in various ways. The instructor in the CTE notated the Gauss travel vector as $\boldsymbol{e}$ and the vector from the origin (Home) to Meetup as $\boldsymbol{p}$.

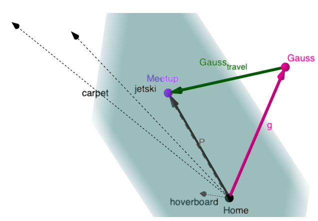

Figure 2. Illustration of Meeting Gauss task in GeoGebra

Data Source and Analytic Method

The CTE was conducted with students in two introductory linear algebra classes (33 students, 38 students) separately at a large public university of the Southeastern United States. The students were STEM majors who had taken Calculus 1 or 2 as a prerequisite. The CTE lasted for four consecutive class days on Zoom at the end of semester. Students' reflection writings on the first two days of CTE is the data source. Students wrote reflections each day after the CTE. The reflections were transcribed into a spreadsheet, and the author highlighted the word vector whenever students mentioned it in their writing. Their reflections were coded line by line using the initial framework, a new set of codes emerged, the author used the updated framework to complete the coding. The author specially focused on students' explanation *before and after* the word vector when coding. When students used the symbolic notations from previous math classes to refer to a vector such as $< a, b, c >$, (a, b, c) or $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, their meaning is also analyzed. All the names used in Results are pseudonyms.

Results

As shown in Figure 2, the ***e*** vector that Gauss travels and the ***p*** vector from Home to Meetup are important pieces in answering the questions of the task sequence of least squares. While Gauss can travel around freely to meet you, your movement is restricted by the transportation vectors given in the problem context. The analysis on students' reflection writings revealed that their conception of a vector and their use of a vector are multifaceted. In this section, their notion of vector and their use of vectors in the Meeting Gauss task will be briefly presented focusing on the two vectors ***e*** and ***p***.

Vector Conception- Describing the e vector that Gauss travels.

Vector as an arrow: In the initial framework, this category was defined as a default, however, the author noticed that vector as an arrow conception is a way of thinking characterized when students describe the tip of a vector as referring to a line segment. For example, *"Gauss should meet you at the shortest point **from the tip of his vector line** to you"*(Anton). Anton describes the path Gauss travels referring to it as an arrow.

Vector as a point: Students use the word 'point' to describe a vector or they attend to the tip of a vector considering the tail as if it were at origin. Damien wrote, *"Gauss should meet us at the point (1.35,0.06, 2.21) …the vector (Gauss travels) should be [1.35; 0.06;2.21] as that is **the point he needs to reach anyways**"*. Damien found the coordinates of Meetup using the GeoGebra applet and used them as the coordinates for the Gauss travel vector ***e***. The tail of the vector ***e*** is at the Gauss location, not at the origin. Vectors and points have the same coordinates only if the vector has a tail at the origin. Another student, Annalisa noted "*I understand everything up until we had to actually find **the point/vector** to meet at*". Annalisa interprets the Meetup in two ways, a point and a vector. She did not mention anything other than this quote in her writing, but still this is an indication that a vector could be identified as a point or vice versa.

Vector as a segment: Students conceive a vector as a connected segment between two locations. Specifically, they use the word 'displacement' or 'magnitude' to describe a vector attending to the length of the vector. For example, Owen noted, *"Gauss will travel along (-1-a,1-b,4-c) which is **the displacement vector between Gauss's old and new location**."* Owen seemed to write the Meetup as (a,b,c) and subtract it from the ***g*** vector to describe the Gauss travel vector ***e***. His description of a vector includes two locations and a line segment connecting them. Another student, Luka wrote *"We described the entire situation as a right triangle, where **the hypotenuse is the vector between the origin and Gauss**, the vertical component is **the vector between the plane and Gauss**, and the horizontal component is **the net distance travelled via hoverboard and magic carpet**"*. Luka's description illustrates that each side on the right triangle is a vector which connects the locations among Home, Gauss, and Meetup.

Vector as a direction: Students use the word 'vector' referring to the span of the vector or the direction of the vector. Span of a vector refers to a line scaled either extended or shrunk along the vector. Consider the following examples, *"Gauss would need to move to a point that is **along the line of the vectors** given"(Bianca) "The direction he'd travel can be found with the cross product: [[i j k] [1 1 1] [6 3 8]] = 5i - 2j - 3k =* ***[5 -2 -3] <- so either that direction or***

reversed. *"* (Vaki). Their description for the ***e*** vector indicates that they attend to the scaled line or the direction Gauss travels rather than the length of the vector or the vector itself.

Use of Vector- Traveling from Home (origin) to Meetup

Over and up alternations: Students use vectors to get to Meetup stair-wise. Hanora noted,*"Sometimes it might be shorter if I moved a certain distance so he can reach me without using many* ***modes of transportation to make zig-zags****."* and Dilon wrote "*Gauss should meet us along one of our vectors. If he wants the shortest trip he should meet. He would use V1 to get there and his distance would be very short,* ***only 1 unit in each direction***.." Hanora and Dilon seemed to understand the task context differently than intended, but Hanora's description on the use of transportation vectors indicates that vectors move in over-and-up manner to get to the Meetup. Also, Dilon's explanation reveals that he recognizes $\boldsymbol{v_1} = [1,1,1]$ consists of stair-wise movement 1 unit in each direction.

Scaling: Students use vectors to get to Meetup by scaling. Vaki noted, *"We would get to the meeting place by* ***using the hoverboard for 1.5 hours and the carpet for 0.5 hours****."* The hours are the amount spent on each transportation, which indicates that Vaki's vector moves in scaling manner. Iliana answered, "*I would get to the meeting point by using a combination of my transportation methods, or by using a* ***linear combination of the given vectors***." Also, Lurenya mentioned, "*Since Gauss is getting to the plane that is the* ***span of the three modes of transportation****,* ***a linear combination of the three vectors*** *will get us there as well*." In Iliana and Lureyna's description, they would use three vectors to reach the Meetup using the span or linear combination of the transportation vectors, which indicates that they find vectors varying in length.

Constructing a space: Students use vectors in generating a space such as a line or a plane. Ani noted, "*The way I am thinking about this is to locate a spot that is within* ***the area that the 3 vectors span*** *and that Gauss could go to. To do that, I would* ***find the plane that the 3 vectors together span****...*" and Cecil "*What I'm imagining is that if we* ***created a "basis plane" with these two vectors*** *then we can map that plane as well as the point Gauss is at".* There were many students who wrote the plane to be generated by the transportation vectors when they use the word linear combination and/or span.

Discussion

This study, to investigate student conception of vectors and their use of vectors, began as students' data were analyzed with a different focus. The author noticed that the Meeting Gauss task scenario entails various ways of thinking with vectors as exploring ways to answer the guiding questions. By foregrounding vectors, student conception and its use were briefly presented using the conceptual framework developed as analyzing their data. The author finds this study important in developing instructional sequence of least squares and even linear algebra because vector is often backgrounded in studies even though student thinking of vector is multifaceted.

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References

Appova, A., & Berezovski, T. (2013). Commonly identified students' misconceptions about vectors and vector operations. 11.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–333). Mahwah: Lawrence Erlbaum Associates Inc.

Freudenthal, H. (1991). Revisiting mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *on the teaching of linear algebra* (pp. 191-207). Kluwer Academic Publishers.

Kwon, O. H. (2011). Student approaches and difficulties in understanding and use of vectors. In Brown, S., Larsen, S., Marrongelle, K., and Oertman, M., editors, *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education*, volume 1, pages 187 – 202. SIGMMA RUME.

Mikula, B. D., & Heckler, A. F. (2017). Framework and implementation for improving physics essential skills via computer-based practice: Vector math. *Physical Review Physics Education Research*, 13(1), 010122. https://doi.org/10.1103/PhysRevPhysEducRes.13.010122

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), on the teaching of linear algebra (pp. 209-246). Kluwer Academic Publishers.

Steffe, L. P., & Thompson, P. W. (2000a). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* . Dordrecht, The Netherlands: Kluwer.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, 22(8), 577-599.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.