Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-022-10178-7

Curve Lengthening via Regularized Motion Against Curvature from the Strong FCH Flow

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Received: 6 September 2020 / Revised: 24 April 2022 / Accepted: 16 May 2022

Abstract
We present a rigorous analysis of the transient evolution of nearly circular bilayer interfaces evolving under the thin interface limit, $\varepsilon \ll 1$, of the mass preserving L^2-gradient flow of the strong scaling of the functionalized Cahn–Hilliard equation. For a domain $\Omega \subset \mathbb{R}^2$ we construct a bilayer manifold with boundary comprised of quasi-equilibria of the flow and a projection onto the manifold that associates functions u in an H^2 tubular neighborhood of the manifold with an interface Γ embedded in Ω. The linearization of the flow about the manifold does not present a clear spectral separation of modes normal and tangential to the manifold. The dimension of the parameterization of the interfaces and the bilayer manifold controls both the normal coercivity of the manifold and the coupling between normal and tangential modes, both of which increase with this dimension. The key step in the analysis is the identification of a range of dimensions in which coercivity dominates the coupling, permitting the closure of the nonlinear estimates that establish the asymptotic stability of the manifold. Orbits originating in a thin, forward invariant, tubular neighborhood ultimately converge to an equilibrium associated to a circular interface. Projections of these orbits yield interfacial evolution equivalent at leading order to the regularized curve-lengthening motion characterized by *normal motion against mean curvature*, regularized by a higher order Willmore expression. The curve lengthening is driven by absorption of excess mass from the regions of Ω away from the interface, leading to high dimensional dynamics that are ill-posed in the $\varepsilon \to 0^+$ limit.

Keywords Functionalized Cahn–Hilliard · Interfacial dynamics · Curve lengthening

The second author thanks the NSF DMS for their support through grant 1813203.

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Published online: 28 June 2022

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1 Introduction

Amphiphilic molecules are surfactants that form thin, bilayer interfaces comprised of two single-molecule layers. The functionalized Cahn–Hilliard (FCH) free energy, introduced in [11], models mixtures of amphiphilic molecules and solvent. It generalizes the energy proposed by Gompper and Goos [13], that was motivated by earlier studies of small-angle X-ray scattering data. The FCH free energy is given in terms of the volume fraction $u - b_-$ of the amphiphilic molecule over a domain Ω as

$$\mathcal{F}(u) := \int_\Omega \frac{\varepsilon}{2}\left(\Delta u - \frac{1}{\varepsilon^2}W'(u)\right)^2 - \varepsilon^{p-1}\left(\frac{\eta_1}{2}|\nabla u|^2 + \frac{\eta_2}{\varepsilon^2}W(u)\right)\mathrm{d}x, \tag{1.1}$$

where $W : \mathbb{R} \mapsto \mathbb{R}$ is a smooth tilted double well potential with local minima at $u = b_\pm$ with $b_- < b_+$, $W(b_-) = 0 > W(b_+)$, and $W''(b_-) > 0$. The state $u = b_-$ corresponds to pure solvent, while $u = b_+$ denotes a maximum packing of amphiphilic molecules. The system parameters $\eta_1 > 0$ and η_2 characterize key structural properties of the amphiphilic molecules. The small positive parameter $\varepsilon \ll 1$ characterizes the ratio of the length of the molecule to the domain size. The term ε^{p-1} is a distinguished limit of a second small parameter with the weak scaling $p = 2$ balancing the Willmore-type residual of the dominant squared term and the amphiphilic structure terms, while the strong scaling $p = 1$ places these latter terms in a position of dominance. We consider the strong scaling $p = 1$, and refer the interested reader to [12] for a detailed discussion of physical meaning of the parameters. The FCH is known to be bounded from below over subsets of $H^2(\Omega)$ that incorporate a wide range of boundary conditions, [20]. Work of Choksi and Ren [6] established the Ohta-Kawasaki free energy as a long-chain limit of a self-consistent mean field theory for diblock polymers. In particular their follow-up paper [7], considered diblocks immersed within a homopolymer, deriving a continuum model for the a diblock-homopolymer blend in the long-chain limit. This approach seems amenable to a short-chain limit, in which the homopolymer approximates a solvent and the Florey-Huggins parameters for each component of the diblock can be adjusted to mimic the hydrophilic-hydrophobic interactions of an amphiphilic diblock with a solvent (homopolymer) phase. Such a model is evocative of amphiphilic blends, deriving a continuum reduction would clarify the relation between the FCH and these important statistical physics models.

The goal of this work is to characterize the evolution of bilayer distributions under a mass-preserving gradient flow of the FCH energy. More specifically, to any smooth, embedded curve $\Gamma \subset \Omega$ we may associate a bilayer distribution $\Phi = \Phi_\Gamma \in H^2(\Omega)$ which is an approximate critical point of the FCH energy. In a neighborhood of Γ the bilayer distribution is expressed as

$$\Phi_\Gamma(x;\sigma) = \phi_0(z) + \frac{\varepsilon\sigma}{(W''(b_-))^2}, \tag{1.2}$$

where $z = z(x)$ is ε-scaled signed distance to Γ, ϕ_0 is the leading order bilayer profile defined as the unique non-trivial solution of the ODE

$$\partial_z^2\phi_0 = W'(\phi_0), \tag{1.3}$$

that is homoclinic to b_- as $z \to \pm\infty$. The solution is extended to be constant away from the front, with the constant σ determining the "bulk density" of surfactant. The system mass which is set by the initial data, u_0, and is scaled by ε,

$$\int_\Omega (u_0 - b_-)\,\mathrm{d}x = \varepsilon M_0. \tag{1.4}$$

For a bilayer distribution Φ_Γ with mass M_0 the bulk density σ and the length of Γ are slaved through the leading order relation

$$|\Gamma| = \frac{M_0}{m_0} - \frac{|\Omega|}{(W''(b_-))^2}\cdot\frac{\sigma}{m_0}, \tag{1.5}$$

where m_0 denotes the bilayer mass-per-unit-length,

$$m_0 := \int_{\mathbb{R}} (\phi_0 - b_-)\,\mathrm{d}z. \tag{1.6}$$

It is instructive to examine the leading order reduction of the FCH energy at $\Phi_\Gamma(\cdot;\sigma)$. Accounting for the mass-dependent slaving (1.5) the strong scaling of the FCH reduces to a Canham–Helfrich type energy [4, 15]

$$\mathcal{E}(\Gamma,\sigma) := \mathcal{F}(\Phi_\Gamma(\cdot;\sigma)) = \frac{m_1^2}{2}\int_\Gamma |\kappa|^2\,\mathrm{d}s + \frac{\nu_b}{2\varepsilon}\left(\sigma - \sigma_1^*\right)^2, \tag{1.7}$$

where the bulk coefficient $\nu_b > 0$ depends only upon the system parameters and the domain size $|\Omega|$, while σ_1^*, the leading order equilibrium value of the bulk parameter σ, and m_1 are given by

$$\sigma_1^* := -\frac{\eta_1+\eta_2}{2m_0}m_1^2, \quad m_1 := \|\phi_0'\|_{L^2(\mathbb{R})}. \tag{1.8}$$

The equilibrium σ_1^* represents the bulk density at which absorption of surfactant into a bilayer balances with ejection of surfactant out of the bilayer. The ε^{-1} scaling of the $(\sigma - \sigma_1^*)^2$ term in (1.7) shows the strong energetic preference for a bulk density σ close to equilibrium, and enforces an equilibrium length on Γ through the mass constraint (1.5).

The nature of the interfacial evolution arising from gradient flows of the reduced energy (1.7) is best understood through the normal velocity it induces on the interface Γ. Accounting for the slaving $\sigma = \sigma(|\Gamma|)$ from (1.5), formal arguments, [8], show that the full energy (1.1) and the reduced energy (1.7) both drive a geometric gradient flow that dissipates the energy (1.7), at leading order through the regularized curve lengthening normal velocity

$$V_{\mathrm{RCL}} = -\varepsilon^{-1} m_0(\sigma - \sigma_1^*)\kappa - m_1^2\left(\Delta_s + \frac{\kappa^2}{2}\right)\kappa. \tag{1.9}$$

If the bulk density is lower than the equilibrium value, $\sigma - \sigma_1^* < 0$, then the surface term induces a familiar mean curvature flow, which shortens the curve. However, if the bulk density exceeds the equilibrium value, $\sigma - \sigma_1^* > 0$, then the system dissipates total energy through a *curve-lengthening motion against curvature*, absorbing amphiphilic material from the bulk, see Fig. 1. We call this the regularized curve lengthening regime in which the higher-order Willmore term serves as a singular perturbation that regularizes the ill-posed motion against curvature.

We consider the mass-preserving $L^2(\Omega)$ gradient flow of the FCH energy (1.1), written in terms of the chemical potential $\mathrm{F} = \mathrm{F}(u)$, associated to $\mathcal{F}$ through the rescaled variational derivative

$$\mathrm{F}(u) := \varepsilon^3 \frac{\delta \mathcal{F}}{\delta u} = (\varepsilon^2 \Delta - W''(u))(\varepsilon^2 \Delta u - W'(u)) + \varepsilon^p(\eta_1 \varepsilon^2 \Delta u - \eta_2 W'(u)). \tag{1.10}$$

The mass-preserving FCH L^2-gradient flow takes the form

$$\partial_t u = -\Pi_0 \mathrm{F}(u), \tag{1.11}$$

subject to periodic boundary conditions on $\Omega \subset \mathbb{R}^2$. Here Π_0 is the zero-mass projection given by

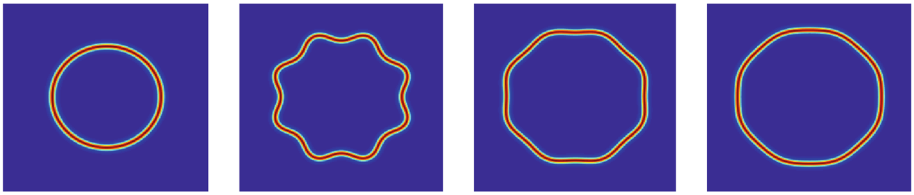
$$\Pi_0 f := f - \langle f \rangle_{L^2}, \tag{1.12}$$

in terms of the averaging operator

$$\langle f \rangle_{L^2} := \frac{1}{|\Omega|} \int_\Omega f \, \mathrm{d}x. \tag{1.13}$$

We provide a rigorous justification of the regularized curve lengthening flow via an asymptotically large dimensional center-stable manifold reduction in a vicinity of the equilibrium arising from the bilayer distribution with a circular interface Γ_0.

Previous work, [5], addressed this system and constructed a manifold with boundary contained in $H^2(\Omega)$ whose constituent points are refinements of the bilayer distributions that are quasi-equilibria of the system (1.11). More specifically, for ε and $\delta > 0$ independently small the work constructed a bilayer manifold, $\mathcal{M}_\delta$, and associated nonlinear projection that uniquely decomposes functions u from an open neighborhood of $\mathcal{M}_\delta$ into a point on the manifold (a bilayer distribution) and a perturbation that is orthogonal to the tangent plane of the manifold. The manifold is parameterized by an asymptotically large but finite set of meander parameters, grouped as a vector $\mathbf{p} = (\mathrm{p}_0, \cdots, \mathrm{p}_{N_1-1})$, residing in the bilayer domain $\mathcal{D}_\delta \subset \mathbb{R}^{N_1}$, defined in (2.8). A parameter vector also defines an interface $\Gamma_{\mathbf{p}}$, immersed in Ω, and the bilayer distribution $\Phi_{\mathbf{p}}$. The interfaces are constructed as perturbations of a fixed base interface Γ_0 and the construction of the projection requires that the base interface Γ_0



(a) $t = 0$ **(b)** $t = 367$ **(c)** $t = 930$ **(d)** $t = 1175$

Fig. 1 Numerical simulation of the strong FCH mass preserving L^2 gradient flow on $\Omega = [-2\pi, 2\pi]^2$ from initial data $u_0 = \Phi_\Gamma(x; \sigma)$ with Γ a circle and $\sigma = 2\sigma_1^*$, double the equilibrium value. Left to right, color coded contours of the evolving solution $u = u(t)$ at indicated times, show a meandering transient followed by relaxation to a circular equilibrium with larger radius. System parameters are $\varepsilon = 0.20$, $\eta_1 = 1.45$ and $\eta_2 = 2.0$ and well W as [17] (Color figure online)

and the scaled system mass M_0 are compatible in the sense that the background mass of surfactant is sufficiently close to the equilibrium value,

$$|\sigma_0(\Gamma_0, M_0) - \sigma_1^*| \leq \delta. \tag{1.14}$$

Here σ_0 is slaved to Γ_0 and M_0 through (1.5). We identify two slow spaces, the pearling and meander spaces, as small eigenvalue eigenspaces of the second variation of the FCH energy at $\Phi_{\mathbf{p}}$. The pearling eigenfunctions are associated to variations in the width of a bilayer interface and can be linearly unstable. To preclude this instability we impose an equilibrium pearling stability condition

$$(\mathbf{PSC}_*) \qquad \sigma_1^* S_1 + (\eta_1 - \eta_2)\lambda_0 > 0, \tag{1.15}$$

on the system parameters $\eta_1 > 0$ and $\eta_2 \in \mathbb{R}$, that renders the pearling modes linearly stable. Here $\lambda_0 < 0$ is the ground state eigenvalue of the linearization of (1.3) about ϕ_0, and $S_1 \in \mathbb{R}$ depends only upon the form of the double well, W. A detailed investigation of the onset of the pearling instability was conducted in [10, 17].

In this work we consider a bilayer manifold, $\mathcal{M}_\delta$ built around a *circular* base interface Γ_0 and in Theorem 4.2 identify conditions under which solutions to the flow (1.11) arising from initial data from an $O(\varepsilon^{5/2})$ tubular H^2-neighborhood of the bilayer manifold remain close for all time and ultimately converge to a unique, up to translation and system mass, equilibrium corresponding to the bilayer distribution with circular interface. Moreover Proposition 4.4 establishes that the normal velocity of the interface $\Gamma_{\mathbf{p}(t)}$ arising from the manifold projection of $u(t)$ agrees with the regularized-curve lengthening flow (1.9) to leading order. The most significant impact of the additional restrictions of the initial data is that the background density σ_0 associated to the manifold projection of the initial data u_0 must satisfy

$$|\sigma_0 - \sigma_1^*| \leq C\varepsilon^{1/2}\delta, \tag{1.16}$$

for some positive constant C. This restricts the length of the evolving interface to lie within $O(\varepsilon^{1/2}\delta)$ of the equilibrium interface length determined by the system mass.

There are two main issues that prevent the application of a standard center-stable manifold analysis. The first is that the pearling modes are associated to $O(\varepsilon)$ spectrum and overlap significantly with the spectrum of the meander modes that characterize the curvature flow. There is no spectral gap. This is remedied by inserting a third small parameter $\rho > 0$, a spectral cut-off which constrains the eigenvalues associated to the meander and pearling modes, see Definition 3.12. The value of ρ impacts the dimension, N_1, of the meander space and equivalently the dimension of the bilayer manifold. The choice is constrained by two competing requirements. The first is that ρ should be large enough so that the normal coercivity of the bilayer manifold, characterized in Lemma 3.14, is sufficiently strong to close the nonlinear estimates that establish the asymptotic stability. On the other hand, to prevent the motion of the interface from exciting the weakly-damped pearling modes, the spectral cut-off must be sufficiently small that the Laplace–Beltrami eigenmodes associated to the pearling and meander modes, have a large gap, see (3.36). This asymptotically large pearling-meander gap weakens the coupling between these spectral sets, see Lemma 3.13, and precludes the meander motion from driving a large pearling excursion. The second issue is that the tangent space of the bilayer manifold and the meander space are asymptotically large. To establish coercivity on the space perpendicular to the tangent space requires that it well-approximates the meander space. This is achieved in Sect. 2.1 through an *implicit* construction of the bilayer interfaces as perturbations of the base interface as a *Galerkin expansion in the Laplace Beltrami modes of the perturbed* interface. This implicit construction yields a tangent space that well-approximates the spectral meander space, allowing for a larger choice of spectral cut-off ρ. The nonlinear estimates of Theorem 4.2 culminate with Eq. (4.30) which combines the constraints on control parameters δ, ε, and ρ under which the argument closes. Specifically we find that the spectral cut-off must satisfy

$$\varepsilon^{\frac{1}{10}} \ll \rho \ll 1,$$

where the constants depend only upon system parameters. In turn, this condition on ρ sets a range of allowable dimensions $N_1 \sim \varepsilon^{-1}\rho^{1/4}$ for the bilayer manifold.

It is natural to compare the results for the bilayer interface dynamics of the FCH gradient flow with those derived for the front solutions of the Cahn–Hilliard equation. For the Cahn–Hilliard system, much of the initial work, notably [2, 18], focused on formal and rigorous derivations of the Mullins-Sekerka flow in the $\varepsilon \to 0$ limit. Quasi-stationary dynamics based upon a radial scaling and translation parameters were derived in [1, 3] in 2D and 3D.

The FCH gradient flows differ from Cahn–Hilliard flows in that its sharp interface limit, $\varepsilon \to 0$, is ill posed. The Γ-limits constructed in [21] consider the case $\eta_2 = \eta_1 < 0$, for which the Willmore and functionalization terms act in concert. The situation is fundamentally different when these two terms are in competition, as expressed in the the strong FCH with $\eta_1 > 0$. This competition leads to a wide variety of minimizing sequences from $H^2(\Omega)$ whose energies are bounded as $\varepsilon \to 0^+$ but are not readily associated to bilayers. These include the pearled interfaces constructed in [19], as well the cylindrical filaments studied in [8, 9]. Pearling can lower the free energy of a bilayer distribution by modulating the width of the level sets of u near an interface. Neither these higher codimension structures not has no analogy within the study of the front solutions in the Cahn–Hilliard model.

The remainder of this article is organized as follows. In Sect. 2, we present the local coordinates and implicit construction of the finite dimensional interface $\Gamma_{\mathbf{p}}$. In Sect. 3 we construct the bilayer distributions and define the map $\mathbf{p} \mapsto \Phi_{\mathbf{p}}$ which gives the bilayer manifold $\mathcal{M}_\delta$. We establish the coercivity of the linearization $\Pi_0\mathbb{L}_{\mathbf{p}}$ of the gradient flow about $\Phi_{\mathbf{p}}$ when the operator is restricted to act on the orthogonal complement to the slow space. Section 4 presents the main results, including the nonlinear estimates establish the asymptotic stability of the bilayer manifold and the estimation of the difference the normal velocity induced by the flow and RCL normal velocity, (1.9). For clarity of presentation, some estimates required in Sect. 4 are postponed to Sect. 5, in particular the impact of the evolution of the meander parameter vector $\mathbf{p}$ on the pearling and meander spaces are quantified there. Various technical estimates, including those that relate the smoothness of the interface $\Gamma_{\mathbf{p}}$ to $\mathbf{p}$ are presented in the “Appendix”, Sect. 6.

1.1 Notation

We present some general notation.

1. The symbol C generically denotes a positive constant whose value depends only on the system parameters η_1, η_2, mass parameter M_0, the domain Ω. In particular its value is independent of ε and ρ, so long as they are sufficiently small. The value of C may vary line to line without remark. In addition, $A \lesssim B$ indicates that quantity A is less than quantity B up to a multiplicative constant C as above, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The expression $f = O(a)$ indicates the existence of a constant C, as above, and a norm $|\cdot|$ for which
$$|f| \leq C|a|.$$
2. The quantity ν is a positive number, independent of ε, that denote an exponential decay rate. It may vary from line to line.
3. If a function space $X(\Omega)$ is comprised of functions defined on the whole spatial domain Ω, we will drop the symbol Ω.
4. We use $\mathbf{1}_E$ as the characteristic function of an index set $E \subset \mathbb{N}$, i.e. $\mathbf{1}_E(x) = 1$ if $x \in E$; $\mathbf{1}_E(x) = 0$ if $x \notin E$. We denote the usual Kronecker delta by
$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$
5. For a finite vector $\mathbf{q} = (\mathrm{q}_j)_j$, we denote the norms
$$\|\mathbf{q}\|_{l^k} = \left(\sum_j |\mathrm{q}_j|^k \right)^{1/k}, \qquad \text{for} k \in \mathbb{N}^+,$$
and $\|\mathbf{q}\|_{l^\infty} = \max_j |\mathrm{q}_j|$. For a matrix $\mathbb{Q} = (\mathbb{Q}_{ij})_{ij}$ as a map from l^2 to l^2 has operator norm l^2_* defined by
$$\|\mathbb{Q}\|_{l^2_*} = \sup_{\{\|\mathbf{q}\|_{l^2=1}\}} \|\mathbb{Q}\mathbf{q}\|_{l^2}.$$
We write
$$\mathrm{q}_j = O(a)e_j, \quad \mathbb{Q}_{ij} = O(a)\mathbb{E}_{ij},$$
where $\boldsymbol{e} = (e_j)_j$ is a vector with $\|\boldsymbol{e}\|_{l^2} = 1$ or $\mathbb{E}$ is a matrix with operator norm $\|\mathbb{E}\|_{l^2_*} = 1$ to imply that $\|\mathbf{q}\|_{l^2} = O(a)$ or $\|\mathbb{Q}\|_{l^2_*} = O(a)$ respectively. See (6.14)–(6.15) of Notation 6.5 for usage.
6. The matrix $e^{\theta\mathcal{R}}$ denotes rotation through the angle θ with the generator $\mathcal{R}$. More explicitly,
$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e^{\theta\mathcal{R}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

2 Bilayer Interfaces and Local Coordinates

The bilayer manifold is composed of a finite dimensional family of smooth closed interfaces immersed in Ω. We fix a base interface Γ_0 which is a circle of radius R_0. Without loss of generality we may rescale the domain so that $R_0 = 1$. We first define a family of closed interfaces parameterized by $\boldsymbol{\gamma} : \mathscr{I} \mapsto \Omega$, where $\mathscr{I}$ is the periodic interval of length 2π. We introduce the periodic distance $|\cdot|_{\mathscr{I}}$ defined as

$$|s|_{\mathscr{I}} = \min\left\{\left|s - 2\pi k\right| \ : k \in \mathbb{Z}\right\}.$$

The following class consists of smooth curves that do not self-intersect, quoted from [5].

Definition 2.1 Given $K, \ell > 0$ and an integer $k > 0$ the class $\mathcal{G}^k_{K,\ell}$ consists of closed curves Γ embedded in Ω whose parameterization $\boldsymbol{\gamma}$ has the properties (a) $\min_{s\in\mathscr{I}} |\boldsymbol{\gamma}'(s)| \geq \frac{1}{4}$ and $\|\boldsymbol{\gamma}\|_{W^{k,\infty}(\mathscr{I})} \leq K$ and (b) for any two points on $\mathscr{I}$ that satisfy $|s_1 - s_2|_{\mathscr{I}} > \frac{1}{8K}$ then $|\boldsymbol{\gamma}(s_1) - \boldsymbol{\gamma}(s_2)| > \ell$.

For each $\Gamma \in \mathcal{G}^2_{K,\ell}$ with $\ell \leq \frac{\pi}{16K}$ there exists a tubular neighborhood, Γ^ℓ of Γ with thickness ℓ, such that the change of coordinates $x \mapsto (s, r)$ through

$$x = \boldsymbol{\gamma}(s) + r\mathbf{n}(s), \tag{2.1}$$

is well defined, see [5]. Here $\mathbf{n} = e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}'/|\boldsymbol{\gamma}'|$ is the outer normal of Γ and $r = r(x)$ is the signed distance of x to the curve Γ. Introducing the scaled distance $z = r/\varepsilon \in [-\ell/\varepsilon, \ell/\varepsilon]$, we refer to (z, s) as the local coordinate near Γ.

2.1 Bilayer Interfaces

In the sequel we fix a base interface Γ_0 which is a circle with radius 1 and constant curvature $\kappa_0 = -1$. The local coordinate associated to Γ_0 is defined on all of Ω except for the center of the circular curve Γ_0. Let $\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0(s)$ be the arc-length parameterization of Γ_0, with $|\boldsymbol{\gamma}_0'(s)| = 1$ for all $s \in \mathscr{I}$. The associated Laplace–Beltrami operator $-\Delta_s : H^2(\mathscr{I}) \to L^2(\mathscr{I})$ has the scaled eigenvalues $\{\beta_k^2\}_{k=0}^\infty$ and normalized eigenfunctions $\{\theta_k\}_{k=0}^\infty$ which satisfy,

$$-\Delta_s\theta_k = \beta_k^2\theta_k. \tag{2.2}$$

In particular the ground state eigenmode is spatially constant,

$$\theta_0 = 1/\sqrt{2\pi}, \qquad \beta_0 = 0, \tag{2.3}$$

for $k = 1, 2, \ldots$, the higher modes are given by

$$\theta_{2k-1} = \frac{1}{\sqrt{2\pi}} \cos{(ks)}, \quad \theta_{2k} = \frac{1}{\sqrt{2\pi}} \sin{(ks)}; \quad \text{with } \ \beta_{2k-1} = \beta_{2k} = k. \tag{2.4}$$

In the following, we will introduce the set of bilayer interfaces whose components are perturbations of Γ_0 parameterized by $\mathbf{p} \in \mathbb{R}^{N_1}$. The dimension N_1 shall be defined in (3.30) and (3.35) in terms of the values of small parameters ε, and ρ introduced in (3.30). The parameter vector $\mathbf{p}$ is decomposed as

$$\mathbf{p} = (\mathrm{p}_0, \mathrm{p}_1, \mathrm{p}_2, \hat{\mathbf{p}}), \qquad \hat{\mathbf{p}} = (\mathrm{p}_3, \mathrm{p}_4, \ldots, \mathrm{p}_{N_1-1}), \tag{2.5}$$

in which p_0 scales the length of the bilayer interface, $\mathrm{p}_1, \mathrm{p}_2$ translate the interface, and $\hat{\mathbf{p}}$ controls the deviation of the bilayer interface from circularity. The following weighted spaces control $\hat{\mathbf{p}}$.

Definition 2.2 (*Weighted perturbation space*) Let $\mathbb{D}$ be the $(N_1-3)\times(N_1-3)$ diagonal matrix

$$\mathbb{D}=\mathrm{diag}\{\beta_3^2,\beta_4^2,\cdots\beta_{N_1-1}^2\}. \tag{2.6}$$

We say $\hat{\mathbf{p}}$ lies in $\mathbb{V}_r^k$ if $\|\hat{\mathbf{p}}\|_{\mathbb{V}_r^k}:=\|\mathbb{D}^{r/2}\hat{\mathbf{p}}\|_{l^k}<\infty$, or more precisely,

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_r^k}=\left(\sum_{j=0}^{N_1-1}\beta_j^{kr}|\mathrm{p}_j|^k\right)^{1/k}<\infty. \tag{2.7}$$

When $k=1$, we omit the superscript k and denote the space by $\mathbb{V}_r$.

The bilayer manifold is constructed as a graph over the domain introduced below.

Definition 2.3 (*Bilayer domain*) Fix $C\lesssim 1$ and let $\delta>0$ be a small parameter, independent of ε. The bilayer manifold domain is the set

$$\mathcal{D}_\delta:=\left\{\mathbf{p}\in\mathbb{R}^{N_1}\;\middle|\;\mathrm{p}_0>-1/2,\quad |\mathrm{p}_1|+|\mathrm{p}_2|+\|\hat{\mathbf{p}}\|_{\mathbb{V}_2}\le C,\quad \|\hat{\mathbf{p}}\|_{\mathbb{V}_1}\le C\delta\right\}. \tag{2.8}$$

We fix $K_0,\ell_0>0$ and a base interface $\Gamma_0\in\mathcal{G}^2_{K_0,\ell_0}$ and associate to each $\mathbf{p}\in\mathcal{D}_\delta$ a bilayer interface $\Gamma_{\mathbf{p}}$ with parameterization $\boldsymbol{\gamma}_{\mathbf{p}}$ associated to the $\mathbf{p}$-variation of Γ_0. This construction is implicit in $\mathbf{p}$ but is well-defined for $\mathbf{p}\in\mathcal{D}_\delta$, via Lemma 2.10 of [5]. Below we sketch the construction by two steps, see also Fig. 2.

(S1) The first step in the interface construction is to define the Laplace–Beltrami–Galerkin perturbation $\bar{p}$ associated to $\mathbf{p}$,

$$\bar{p}(\tilde{s}):=\sum_{i=3}^{N_1-1}\mathrm{p}_i\tilde{\theta}_i(\tilde{s}),\qquad \tilde{\theta}_i(\tilde{s}):=\theta_i\left(\frac{2\pi\tilde{s}}{|\Gamma_{\mathbf{p}}|}\right). \tag{2.9}$$

Here $\tilde{s}=\tilde{s}(s)$ is the *arc-length* parameterization of the perturbed curve $\Gamma_{\mathbf{p}}$, which takes values in $\mathscr{I}_{\mathbf{p}}:=[0,|\Gamma_{\mathbf{p}}|]$ and is defined implicitly as the solution of

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s}=|\boldsymbol{\gamma}'_{\mathbf{p}}|,\qquad \tilde{s}(0)=0. \tag{2.10}$$

(S2) The second step constructs the intermediate curve $\bar{\Gamma}_{\mathbf{p}}$, parameterized by

$$\bar{\boldsymbol{\gamma}}_{\mathbf{p}}(s):=\boldsymbol{\gamma}_0(s)+\bar{p}(\tilde{s})\mathbf{n}_0(s), \tag{2.11}$$

where $\mathbf{n}_0(s)$ is the outer normal vector of the circle Γ_0 parameterized in its arc-length parameter s. The scaled length of the intermediate curve,

$$A(\mathbf{p}):=|\Gamma_0|^{-1}\int_{\mathscr{I}}|\boldsymbol{\gamma}'_{\bar{p}}(s)|\,\mathrm{d}s, \tag{2.12}$$

is used to rescale the perturbed curve so that its length is controlled only by p_0,

$$\boldsymbol{\gamma}_{\mathbf{p}}(s):=\frac{1+\mathrm{p}_0}{A(\mathbf{p})}\boldsymbol{\gamma}_{\bar{p}}(s)+\mathrm{p}_1\theta_0\mathbf{E}_1+\mathrm{p}_2\theta_0\mathbf{E}_2,\quad \text{for}\quad s\in\mathscr{I}. \tag{2.13}$$

Then $\Gamma_{\mathbf{p}}$ is the perturbed interface defined by parameterization $\boldsymbol{\gamma}_{\mathbf{p}}$.

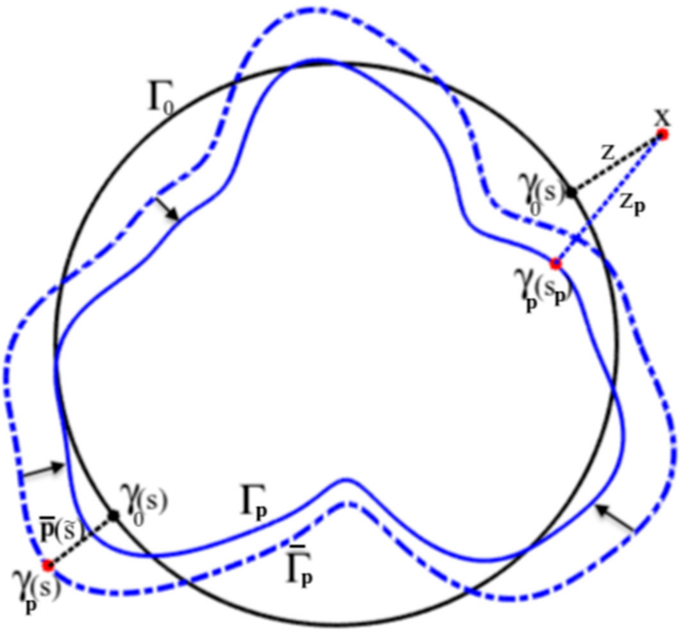


Fig. 2 The construction of the bilayer interface $\Gamma_\mathbf{p}$ in the translation-free case $\mathrm{p}_1 = \mathrm{p}_2 = 0$. First the base curve Γ_0 (solid black) is deformed along its normal vectors by $\bar{p}$, defined through the implicit local variable $\tilde{s} = \tilde{s}(s)$. Then the intermediate curve $\bar{\Gamma}_\mathbf{p}$ (dotted blue) is linearly rescaled, see black arrows, to $\Gamma_\mathbf{p}$ (solid blue) in such a way that its length is controlled only by p_0. Any point x in the reach, $\Gamma_\mathbf{p}^{2\ell}$ of $\Gamma_\mathbf{p}$ can be decomposed in the local coordinate of $\Gamma_\mathbf{p}$ as $x = \boldsymbol{\gamma}_\mathbf{p}(s_\mathbf{p}) + \mathbf{n}_\mathbf{p}(s_\mathbf{p})z_\mathbf{p}$, and equivalently in the local coordinate of Γ_0 as $x = \boldsymbol{\gamma}_0(s) + \mathbf{n}_0(s)z$. Here $(s_\mathbf{p}, z_\mathbf{p})$ and (s, z) are functions of x (Color figure online)

With the bilayer domain in Definition 2.3, we are at the point to introduce the set of bilayer interfaces,

$$\mathcal{I}_\delta = \mathcal{I}_\delta(\Gamma_0) := \{\Gamma_\mathbf{p} \big| \mathbf{p} \in \mathcal{D}_\delta\}. \tag{2.14}$$

We summarize the bilayer interface construction in the following Lemma.

Lemma 2.4 *Fix $K_0, \ell_0 > 0$ and base interface $\Gamma_0 \in \mathcal{G}^2_{K_0,\ell_0}$. There exist positive constants $K, \ell, \delta > 0$ independent of ε such that the set of bilayer interfaces, $\mathcal{I}_\delta$, resides in $\mathcal{G}^2_{K,2\ell}$.*

Proof This is an immediate consequence of Lemma 2.11 of [5]. We remark that the condition $\mathrm{p}_0 > -1/2$ in $\mathcal{D}_\delta$ prevents the shrinking of the base circle to a point, while the bound on $\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}$ prevents self intersection of the perturbed curve. The V_2-norm bound on $\hat{\mathbf{p}}$ controls the curvature of $\Gamma_\mathbf{p}$ guaranteeing the existence of K, ℓ for which $\Gamma_\mathbf{p} \in \mathcal{G}^2_{K,2\ell}$ for all $\mathbf{p} \in \mathcal{D}_\delta$. □

The implicit definition of $\Gamma_\mathbf{p}$ insures that the tangent plane is well conditioned with respect to the orthogonal basis $\{\tilde{\theta}_i\}_{i=0}^{N_1-1}$ of $L^2(\mathscr{I}_\mathbf{p})$, which satisfy

$$\int_{\mathscr{I}_\mathbf{p}} \tilde{\theta}_j \tilde{\theta}_k \, \mathrm{d}\tilde{s} = (1 + \mathrm{p}_0)\delta_{jk}, \quad j, k = 0, 1, \ldots, N_1 - 1. \tag{2.15}$$

In particular the rescaling by A removes the impact of the high-frequency terms, $\hat{\mathbf{p}}$, on the curve length. Indeed, the length of $\Gamma_{\mathbf{p}}$, given by $|\Gamma_{\mathbf{p}}| = 2\pi(1+\mathrm{p}_0)$, is controlled uniquely by p_0. The parameters p_1 and p_2 govern the rigid translation of the interface and are treated separately as rigid translations are not described as normal perturbations to the original interface. The projections of the rigid translations onto $\mathbf{n}_0$ satisfy

$$\theta_0\mathbf{E}_1 \cdot \mathbf{n}_0 = \frac{1}{\sqrt{2\pi}}\cos(s) = \theta_1, \quad \theta_0\mathbf{E}_2 \cdot \mathbf{n}_0 = \frac{1}{\sqrt{2\pi}}\sin(s) = \theta_2. \tag{2.16}$$

From (2.2) and (2.9) the scaled Laplace–Beltrami modes $\tilde{\theta}_j = \tilde{\theta}_j(\tilde{s})$ satisfy

$$-\tilde{\theta}_j''(\tilde{s}) = \beta_{\mathbf{p},j}^2\tilde{\theta}_j(\tilde{s}), \qquad \beta_{\mathbf{p},j} = \frac{\beta_j}{(1+\mathrm{p}_0)}. \tag{2.17}$$

Here and below, primes of $\tilde{\theta}_j$ denote their derivatives with respect to $\tilde{s}$. We remark that $\beta_{\mathbf{p},j}$ reduces to β_j when $\mathbf{p} = \mathbf{0}$. The orthogonality (2.15) implies

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_k^2} \sim \|\bar{p}\|_{H^k(\mathscr{I}_{\mathbf{p}})}, \quad \|\bar{p}^{(k)}\|_{L^\infty(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_k}, \tag{2.18}$$

for all $\mathbf{p} \in \mathcal{D}_\delta$.

When developing expansions of the interface $\Gamma_{\mathbf{p}}$ it is convenient to build in the uniform rescaling and translations associated to $(\mathrm{p}_0, \mathrm{p}_1, \mathrm{p}_2)$ so that the expansions are controlled by $\hat{\mathbf{p}}$. To this end we define the leading order perturbed interfacial map

$$\boldsymbol{\gamma}_{\mathbf{p},0} = (1+\mathrm{p}_0)\boldsymbol{\gamma}_0(s) + \mathrm{p}_1\theta_0\mathbf{E}_1 + \mathrm{p}_2\theta_0\mathbf{E}_2, \tag{2.19}$$

whose interface $\Gamma_{\mathbf{p},0}$ is a translated, scaled circle with constant curvature $\kappa_{\mathbf{p},0}$.

2.2 Local Coordinate Expansions

Since $\mathcal{I}_\delta(\Gamma_0) \subset \mathcal{G}^2_{K,2\ell}$, introduced in Definition 2.1, each $\Gamma_{\mathbf{p}}$ has a tubular neighborhood of width 2ℓ, denoted $\Gamma_{\mathbf{p}}^{2\ell}$, called the *reach* of $\Gamma_{\mathbf{p}}$, on which the pair $(s_{\mathbf{p}}, z_{\mathbf{p}})$ form a well-defined coordinate. In particular each $x \in \Gamma_{\mathbf{p}}^{2\ell}$ has a unique expression in the form

$$x = \boldsymbol{\gamma}_{\mathbf{p}}(s_{\mathbf{p}}) + \mathbf{n}_{\mathbf{p}}(s_{\mathbf{p}})z_{\mathbf{p}}, \tag{2.20}$$

as depicted in Fig. 2. We will also have occasion to use the coordinate system $(\tilde{s}_{\mathbf{p}}, z_{\mathbf{p}})$ posed in the arc-length scaling of $\Gamma_{\mathbf{p}}$.

For each $\mathbf{p} \in \mathcal{D}_\delta$, the local coordinates of $\Gamma_{\mathbf{p}}$ induce natural L^2-inner products and expressions for the Cartesian Laplacian in the local coordinates of $\Gamma_{\mathbf{p}}$. These results have been introduced in [5, 14], which we quote and adapt to our notation system in the following. For any $a > 0$ we define the interval

$$\mathbb{R}_a = [-a/\varepsilon, a/\varepsilon]. \tag{2.21}$$

For each $f, g \in L^2(\Omega)$ with support in $\Gamma_{\mathbf{p}}^{2\ell}$, their L^2-inner product can be written in local coordinates $(s_{\mathbf{p}}, z_{\mathbf{p}})$ as

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}} f(s_{\mathbf{p}}, z_{\mathbf{p}})g(s_{\mathbf{p}}, z_{\mathbf{p}})\varepsilon(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})|\boldsymbol{\gamma}_{\mathbf{p}}'|\,\mathrm{d}s_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}}, \tag{2.22}$$

or equivalently in the arc-length scaled variables $(\tilde{s}_{\mathbf{p}}, z_{\mathbf{p}})$ with $\mathrm{d}\tilde{s}_{\mathbf{p}} = |\boldsymbol{\gamma}_{\mathbf{p}}'|\,\mathrm{d}s_{\mathbf{p}}$ and $\tilde{s}_{\mathbf{p}} \in \mathscr{I}_{\mathbf{p}}$ as

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}_{2\ell}} \int_{\mathscr{I}_{\mathbf{p}}} f(s_{\mathbf{p}}, z_{\mathbf{p}}) g(s_{\mathbf{p}}, z_{\mathbf{p}}) \varepsilon(1 - \varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\, \mathrm{d}\tilde{s}_{\mathbf{p}} \mathrm{d}z_{\mathbf{p}}. \tag{2.23}$$

The ε-scaled Laplacian admits the local expansion

$$\varepsilon^2 \Delta_x = \partial_{z_{\mathbf{p}}}^2 + \varepsilon \mathrm{H}_{\mathbf{p}} \partial_{z_{\mathbf{p}}} + \varepsilon^2 \Delta_g, \qquad \Delta_g := \Delta_{s_{\mathbf{p}}} + \varepsilon z_{\mathbf{p}} D_{s_{\mathbf{p}},2}, \tag{2.24}$$

where $\mathrm{H}_{\mathbf{p}}$ is the extended curvature

$$\mathrm{H}_{\mathbf{p}}(s_{\mathbf{p}}, z_{\mathbf{p}}) := -\frac{\kappa(s_{\mathbf{p}})}{1 - \varepsilon z_{\mathbf{p}}\kappa(s_{\mathbf{p}})}, \tag{2.25}$$

$-\Delta_{s_{\mathbf{p}}} = -\partial_{\tilde{s}_{\mathbf{p}}}^2$ is the Laplace–Beltrami operator on the surface $\Gamma_{\mathbf{p}}$ and $D_{s_{\mathbf{p}},2}$ is a relatively bounded perturbation of $\Delta_{s_{\mathbf{p}}}$. In particular,

$$D_{s_{\mathbf{p}},2} = a(s_{\mathbf{p}}, z_{\mathbf{p}}) \Delta_{s_{\mathbf{p}}} + b(s_{\mathbf{p}}, z_{\mathbf{p}}) \partial_{s_{\mathbf{p}}}, \tag{2.26}$$

where the smooth coefficients a, b are given explicitly by

$$a(s_{\mathbf{p}}, z_{\mathbf{p}}) = (\varepsilon z_{\mathbf{p}})^{-1} \left(\frac{1}{|1 - \varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}}|^2} - 1 \right), \quad b(s_{\mathbf{p}}, z_{\mathbf{p}}) = \frac{(\varepsilon z_{\mathbf{p}})^{-1}}{2|\boldsymbol{\gamma}_{\mathbf{p}}'|^2} \partial_{s_{\mathbf{p}}} a(s_{\mathbf{p}}, z_{\mathbf{p}}). \tag{2.27}$$

We say a function $f = f(s_{\mathbf{p}})$ lies in $L^2(\mathscr{I}_{\mathbf{p}})$ if

$$\|f\|_{L^2(\mathscr{I}_{\mathbf{p}})}^2 := \int_{\mathscr{I}} f^2(s_{\mathbf{p}}) |\boldsymbol{\gamma}_{\mathbf{p}}'|\, \mathrm{d}s_{\mathbf{p}} < \infty \quad \text{or equivalently} \quad \int_{\mathscr{I}_{\mathbf{p}}} f^2(s_{\mathbf{p}})\, \mathrm{d}\tilde{s}_{\mathbf{p}} < \infty.$$

3 Bilayer Manifold and Linear Stability

The bilayer manifold $\mathcal{M}_\delta$ is introduced as the graph of the bilayer distributions $\{\Phi_{\mathbf{p}}\}$, defined in Lemma 3.7, over the domain $\mathcal{D}_\delta$, from Definition 2.3. In addition the residual of the vector field evaluated at $\Phi_{\mathbf{p}}$ is characterized and the coercivity of the linearized operator on the space perpendicular to the tangent plane to $\mathcal{M}_\delta$ is established.

3.1 Bilayer Distributions and Bilayer Manifold

In this subsection we develop the bilayer distributions , $\{\Phi_{\mathbf{p}}\}$, that include the equilibrium $\Phi_{\mathbf{p}^*}$ of the FCH flow. We deduce Lipschitz estimates on $\Phi_{\mathbf{p}}$ for $\mathbf{p}$ near $\mathbf{p}^*$. A bilayer distribution is constructed through a matching of an inner description in the reach $\Gamma_{\mathbf{p}}^{2\ell}$ to an outer distribution on the remainder of Ω. The inner construction begins with ϕ_0 defined on $L^2(\mathbb{R})$ as the nontrivial homoclinic solution of

$$\partial_z^2 \phi_0 - W'(\phi_0) = 0, \qquad \lim_{|z|\to\infty} \phi_0(z) = b_-. \tag{3.1}$$

The orbit ϕ_0 is unique up to translation, even about $z = 0$, and converges to the smaller minima, b_-, of W as z tends to $\pm\infty$ at the exponential rate $\sqrt{W''(b_-)} > 0$. The linearization L_0 of (3.1) about ϕ_0,

$$\mathrm{L}_0 := -\partial_z^2 + W''(\phi_0), \tag{3.2}$$

is a Sturm–Liouville operator on the real line whose coefficients decay exponentially fast to constants at $z = \infty$. The following Lemma follows from classic results, see for example Chapter 2.3.2 of [16].

Lemma 3.1 *The spectrum of* L_0 *is real, and uniformly positive except for two point spectra:* $\lambda_0 < 0$ *and* $\lambda_1 = 0$*. The ground state eigenfunction* ψ_0 *of* L_0 *is even and positive, with ground state eigenvalue* $\lambda_0 < 0$*. The operator* L_0 *has an inverse that is well defined on the* L^2 *perp of its kernel, span*$\{\phi_0'\}$*, and both* L_0 *and its inverse preserve even and odd parity.*

The base profile ϕ_0 is a function of one variable. For an interface $\Gamma_{\mathbf{p}}$ the first step in the construction of $\Phi_{\mathbf{p}}$ is introduction of the dressing of base profile with respect to $\Gamma_{\mathbf{p}}$, as defined below.

Definition 3.2 (*Dressing*) Let $\mathbf{p} \in \mathcal{D}_\delta$. Given a function $f(z) : \mathbb{R} \to \mathbb{R}$ which tends to a constant f^∞ and whose derivatives of all orders are continuous and tend to zero at an ε-independent exponential rate as $z \to \pm\infty$, we define the dressed function, $f^d \in L^2(\Omega)$, of f with respect to the interface $\Gamma_{\mathbf{p}}$ via the relation

$$f^d(x) := f(z_{\mathbf{p}}(x))\chi(\varepsilon|z_{\mathbf{p}}(x)|/\ell) + f^\infty(1 - \chi(\varepsilon|z_{\mathbf{p}}(x)|/\ell)), \qquad \forall x \in \Omega.$$

Here $\chi : \mathbb{R} \to \mathbb{R}$ is a fixed smooth cut-off function satisfying: $\chi(r) = 1$ if $r \leq 1$ and $\chi(r) = 0$ if $r \geq 2$. Where there is no ambiguity we abuse notation and use $f(z_{\mathbf{p}})$ to denote the dressing of f with respect to $\Gamma_{\mathbf{p}}$.

We introduce the dressed operator as follows.

Definition 3.3 (*Dressed operator*) Let $\mathrm{L} : \mathrm{D} \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ be a self-adjoint differential operator with smooth coefficients whose derivatives of all order decay to zero at an exponential rate at ∞. We define the space $\mathcal{S}$ to consist of the functions f as in Definition 3.2. Then to each $\mathbf{p} \in \mathcal{D}_\delta$ the dressed operator $\mathrm{L}_{\mathbf{p}} : \mathrm{D} \cap \mathcal{S} \mapsto L^2(\Omega)$ and its r'th power, $r \in \mathbb{N}$ are given by

$$\mathrm{L}_{\mathbf{p}}^r f := (\mathrm{L}^r f)^d. \tag{3.3}$$

If $r < 0$ then we assume that $f \in \mathcal{R}(\mathrm{L})$ and the inverse $\mathrm{L}_{\mathbf{p}}^{-1} f$ decays exponentially to a constant at $\pm\infty$.

Since the function 1 is orthogonal to ϕ_0' in $L^2(\mathbb{R})$ we may define $\mathrm{L}_0^{-k}1$ on the real line $\mathbb{R}$. Its dressing, subject to $\Gamma_{\mathbf{p}}$, is denoted $B_{\mathbf{p},k} \in L^2(\Omega)$ and called the background function,

$$B_{\mathbf{p},k}(x) := \mathrm{L}_{\mathbf{p},0}^{-k} 1 = (\mathrm{L}_0^{-k}1)^d, \qquad x \in \Omega. \tag{3.4}$$

It provides the leading-order far-field variation in the bilayer distributions and satisfies $B_{\mathbf{p},k} \to B_k^\infty$ as $|z| \to \infty$, where the far-field value $B_k^\infty = (W''(b_-))^{-k}$. When $\mathbf{p} = \mathbf{0}$, we drop the subscript $\mathbf{p}$ and denote $B_{\mathbf{p},k}$ as B_k. An important role is played by the mass of the background function,

$$\overline{B}_{\mathbf{p},k} := \int_\Omega B_{\mathbf{p},k} \, \mathrm{d}x. \tag{3.5}$$

With this notation the first correction ϕ_1 to the pulse profile is defined as

$$\phi_1^d(\sigma) = \phi_1(z_{\mathbf{p}}; \sigma) := \sigma B_{\mathbf{p},2} + \frac{\eta_d}{2} \mathrm{L}_{\mathbf{p},0}^{-1} \left(z_{\mathbf{p}} \phi_0'(z_{\mathbf{p}})\right). \tag{3.6}$$

It depends upon the bulk density and meander parameters, σ and $\mathbf{p}$, respectively. As a function of $z_{\mathbf{p}}$, ϕ_1 is smooth and is even about $z_{\mathbf{p}} = 0$.

The existence of equilibrium bilayer distributions associated to circular interfaces, up to exponentially small terms, was established in [10]. The system mass M_0 determines the equilibrium radius p_0^* and bulk density σ^*, and hence, up to the choice of rigid-body translations p_1 and p_2, the system mass fully characterizes the circular equilibrium state.

Lemma 3.4 *Let $\varepsilon > 0$ and let Γ_* be a circular interface with curvature κ_*, centered at the origin, and strictly contained within the periodic domain Ω. Let z_* denote the ε-scaled distance to Γ_*. Then for each ε sufficiently small there exists a unique constant $\sigma^* = \sigma_1^* + \varepsilon\sigma_{\geq 2}^*(\kappa_*, \varepsilon)$, a uniformly (in ε) bounded function $\phi_{\geq 3}^* = \phi_{\geq 3}^*(z_*; \kappa_*, \varepsilon)$ which decays exponentially fast to a constant as $z \to \infty$, and a uniformly (in ε) smooth function $\phi_e = \phi_e(x; \kappa_*, \varepsilon, \Omega)$ and a $\nu > 0$ such that,*

$$\begin{aligned}\Phi_*(x) :=& \phi_0(z_*(x)) + \varepsilon\phi_1(z_*(x); \sigma^*) + \varepsilon^2\phi_2(z_*(x); \sigma^*, \kappa_*) + \varepsilon^3\phi_{\geq 3}^*(z_*(x); \sigma^*, \kappa_*)\\ &+ e^{-\ell\nu/\varepsilon}\phi_e(x; \sigma^*, \kappa_*),\end{aligned}$$

is an equilibrium of (1.11) subject to periodic boundary conditions on Ω. Translations of periodic extensions of Φ_* are also exact equilibrium.

The exponential correction ϕ_e is nontrivial outside the reach $\Gamma_*^{2\ell}$, and it arises from the interaction of the radial equilibrium inside the reach with the period box $\Omega = [-L, L]^2$.

The dressing process introduces functions that decay rapidly outside of $\Gamma_{\mathbf{p}}^{2\ell}$, and depend upon $\Gamma_{\mathbf{p}}$ only through $z_{\mathbf{p}}$. Frequently functions will arise that decay to a constant outside of the reach, but that depend explicitly upon the interfacial map $\gamma_{\mathbf{p}}$ and its derivatives up to a certain order. These functions enjoy certain classes of estimates, and the following notation allows them to be grouped.

Definition 3.5 Let $\Gamma_{\mathbf{p}} \in \mathcal{G}_{K,2\ell}^2$, we say a scalar function $h \in L^2(\Omega)$ lies in $\mathcal{H}_k(\boldsymbol{\gamma}_{\mathbf{p}})$ if it takes the form

$$h = h_0(z_{\mathbf{p}}; \boldsymbol{\gamma}_{\mathbf{p}}) + h^\infty,$$

where h^∞ is a constant and h_0 has its support inside $\Gamma_{\mathbf{p}}^{2\ell}$ and depends upon $s_{\mathbf{p}}$ only through a smooth function of the first k derivatives of $\boldsymbol{\gamma}_{\mathbf{p}}$. In addition, we assume that there exists a constant $\nu > 0$ such that $h_0(z_{\mathbf{p}}; \boldsymbol{\gamma}_{\mathbf{p}})e^{\nu|z_{\mathbf{p}}|}$ is uniformly bounded independent of ε on $\Gamma_{\mathbf{p}}^{2\ell}$.

The space $\bar{\mathcal{H}}_k(\boldsymbol{\gamma}_{\mathbf{p}})$ consists of scalar functions $h \in L^2(\mathscr{I}_{\mathbf{p}})$ arising as smooth functions of the first k derivatives of $\boldsymbol{\gamma}_{\mathbf{p}}$.

Remark 3.6 These comments clarify the usage of the $\mathcal{H}_k$ and $\bar{\mathcal{H}}_k$ notation.

(a) The geometric quantities $|\boldsymbol{\gamma}_{\mathbf{p}}'|$ and $\mathbf{n}_{\mathbf{p}} \cdot \mathbf{n}_0$ lie in $\bar{\mathcal{H}}_1(\boldsymbol{\gamma}_{\mathbf{p}})$ while $\kappa_{\mathbf{p}}$ lies in $\bar{\mathcal{H}}_2(\boldsymbol{\gamma}_{\mathbf{p}})$. For the dressing $\phi_0(z_{\mathbf{p}})$ of ϕ_0 associated to $\Gamma_{\mathbf{p}}$, then $\phi_0(z_{\mathbf{p}}; \boldsymbol{\gamma}_{\mathbf{p}}) \in \mathcal{H}_0(\boldsymbol{\gamma}_{\mathbf{p}})$ with constant value b_- while $|\boldsymbol{\gamma}_{\mathbf{p}}'|^2\phi_0'(z_{\mathbf{p}}) \in \mathcal{H}_1(\boldsymbol{\gamma}_{\mathbf{p}})$ and $\phi_0'(z_{\mathbf{p}})\kappa_{\mathbf{p}} \in \mathcal{H}_2(\boldsymbol{\gamma}_{\mathbf{p}})$, both with constant value 0.
(b) For any natural number m, the action of the operator $\varepsilon^m\nabla_{s_{\mathbf{p}}}^m$ on $\boldsymbol{\gamma}_{\mathbf{p}}$ is bounded independent of ε for N_1 restricted as in (3.35). Thus $h \in \mathcal{H}_k$ implies that $\varepsilon^m\nabla_{s_{\mathbf{p}}}^m h$ enjoys the same estimates as h, see Lemma 6.4 for details.
(c) In usage, functions in $\bar{\mathcal{H}}_k(\boldsymbol{\gamma}_{\mathbf{p}})$ arise as integrals in $z_{\mathbf{p}}$ over $\mathbb{R}_{2\ell}$ of functions from $\mathcal{H}_k(\boldsymbol{\gamma}_{\mathbf{p}})$.

The following Lemma presents the bilayer distributions and their residuals. They include the equilibrium bilayer distribution Φ_* associated to the circular interface Γ_*.

Lemma 3.7 *Let ϕ_0 and ϕ_1 be as defined in (3.1), (3.6) respectively. Let $\phi_{\geq 3}(z_{\mathbf{p}})$ denote the dressing of $\phi^*_{\geq 3}$ with respect to $\Gamma_{\mathbf{p}}$ with $\mathbf{p} \in \mathcal{D}_\delta$. Let $\phi_{\mathbf{p},e}$ be the translation of ϕ_e defined in Lemma 3.4. Specifically we take*

$$\phi_{\geq 3}(z_{\mathbf{p}}) := \phi^*_{\geq 3}(z_{\mathbf{p}}; \sigma^*, \kappa_*), \qquad \phi_{\mathbf{p},e} = \phi_e(x - \mathrm{p}_1\theta_0\mathbf{E}_1 - \mathrm{p}_2\theta_0\mathbf{E}_2; \kappa_*, \sigma^*).$$

Then there exists $\nu > 0$ and a function $\phi_2 \in \mathcal{H}_2$ (see Definition 3.5) such that the bilayer distribution

$$\Phi_{\mathbf{p}}(x;\sigma) := \phi_0(z_{\mathbf{p}}) + \varepsilon\phi_1(z_{\mathbf{p}};\sigma) + \varepsilon^2\phi_{\geq 2}(z_{\mathbf{p}};\boldsymbol{\gamma},\sigma) + e^{-\ell\nu/\varepsilon}\phi_{\mathbf{p},e}(x), \tag{3.7}$$

with $\phi_{\geq 2} := \phi_2 + \varepsilon\phi_{\geq 3}(z_{\mathbf{p}})$ has the residual

$$\mathrm{F}(\Phi_{\mathbf{p}}) = \mathrm{F}_m(s_{\mathbf{p}}, z_{\mathbf{p}}) + e^{-\ell\nu/\varepsilon}\mathrm{F}_e(x), \tag{3.8}$$

with $\mathrm{F}_m = \varepsilon\sigma + \varepsilon^2\mathrm{F}_2 + \varepsilon^3\mathrm{F}_3 + \varepsilon^4\mathrm{F}_{\geq 4}$, whose expansion terms take the form

$$\begin{aligned}\mathrm{F}_2 &= \kappa_{\mathbf{p}}(\sigma - \sigma_1^*) f_2(z_{\mathbf{p}}),\\ \mathrm{F}_3 &= -\phi_0'\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} + f_3(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}),\\ \mathrm{F}_{\geq 4} &= \Delta_g f_{4,1}(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}) + f_{4,2}(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}).\end{aligned} \tag{3.9}$$

Here f_2 has far field value zero and has odd parity in $z_{\mathbf{p}}$, while $f_2, f_3, f_{4,1}, f_{4,2} \in \mathcal{H}_2$. In addition, the projections of $\mathrm{F}_2, \mathrm{F}_3$ satisfy

$$\begin{aligned}\int_{\mathbb{R}_{2\ell}} \mathrm{F}_2\,\phi_0'\,\mathrm{d}z_{\mathbf{p}} &= m_0(\sigma_1^* - \sigma)\kappa_{\mathbf{p}} + O(e^{-\ell\nu/\varepsilon});\\ \int_{\mathbb{R}_{2\ell}} \mathrm{F}_3\,\phi_0'\,\mathrm{d}z_{\mathbf{p}} &= m_1^2\left(-\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} - \frac{\kappa_{\mathbf{p}}^3}{2} + \alpha\kappa_{\mathbf{p}}\right) + O(e^{-\ell\nu/\varepsilon}).\end{aligned} \tag{3.10}$$

Here $\alpha = \alpha(\sigma;\eta_1,\eta_2)$ depends smoothly on σ.

Proof This is adapted from Lemma 3.2 of [5], subject to the incorporation of lower order terms terms in $\Phi_{\mathbf{p}}$ that do not affect the form of $\mathrm{F}_{2,3,4}$. Explicit formulations of ϕ_2 and α are given in [5]. They are omitted here as they do not impact the results. □

As constructed the bilayer distribution converges to an equilibrium of the FCH system if the meander and bulk density parameters converge. In light of Lemma 3.4, we assume a priori that there exist some $\mathbf{p}^*$ and σ^* in the form of

$$\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_1^*, \mathrm{p}_2^*, 0, \cdots, 0), \quad \sigma^* = \sigma_1^* + O(\varepsilon). \tag{3.11}$$

such that $\mathbf{p}(t) \to \mathbf{p}^*$ and $\sigma(t) \to \sigma^*$ as $t \to \infty$. The FCH gradient flow preserves system mass, (1.4), which is set by the initial data. We constrain the bulk density parameter σ so that the mass of $\Phi_{\mathbf{p}}$ equals the system mass. From the form (3.7) of $\Phi_{\mathbf{p}}$ with $\phi_1 = \phi_1(\sigma)$ given by (3.6) we deduce that the mass constraint

$$\langle \Phi_{\mathbf{p}}(x) - b_- \rangle_{L^2} = \varepsilon\frac{M_0}{|\Omega|}, \tag{3.12}$$

is satisfied precisely if

$$\sigma(\mathbf{p}) = \frac{1}{\overline{B}_{\mathbf{p},2}}\left\{M_0 - \int_\Omega \left[\frac{1}{\varepsilon}\left(\phi_0(z_{\mathbf{p}}) - b_- + \varepsilon^2\phi_{\geq 2} + e^{-\ell\nu/\varepsilon}\phi_{\mathbf{p},e}\right) + \frac{\eta_d}{2}\left(\mathrm{L}_{\mathbf{p},0}^{-1}(z_{\mathbf{p}}\phi_0')\right)\right]\mathrm{d}x\right\}. \tag{3.13}$$

The following result shows that at leading order σ depends upon $(\mathrm{p}_0, \hat{\mathbf{p}})$ only through p_0.

Lemma 3.8 *Let $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough and system mass M_0 independent of ε. Suppose the bilayer distribution $\Phi_{\mathbf{p}}$ with $\mathbf{p} \in \mathcal{D}_\delta$ satisfies the mass constraint (3.12), then*

$$\sigma(\mathbf{p}) = \sigma_0 - \frac{c_0 m_1^2}{m_0}\mathrm{p}_0 + \varepsilon \mathcal{C}(\mathrm{p}_0)\mathrm{p}_0 + O\left(\varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\right). \tag{3.14}$$

Here $\mathcal{C}$ is a smooth function of p_0 that is bounded uniformly independent of ε. The base bulk density $\sigma_0 = \sigma_0(\Gamma_0, M_0)$ is independent of $\mathbf{p}$, and c_0 is a fixed positive constant. Indeed,

$$\sigma_0(\Gamma_0, M_0) = \frac{M_0 - m_0|\Gamma_0|}{B_2^\infty |\Omega|} + O(\varepsilon^2), \qquad c_0 := \frac{2\pi m_0^2}{\overline{B}_2 m_1^2} > 0. \tag{3.15}$$

Proof We address the terms on the right-hand side of (3.13) one-by-one. First, using the local coordinate in $\Gamma_{\mathbf{p}}^{2\ell}$ we rewrite

$$\int_\Omega (\phi_0(z_{\mathbf{p}}) - b_-)\,\mathrm{d}x = \varepsilon \int_{\mathscr{I}} \int_{\mathbb{R}_{2\ell}} (\phi_0(z_{\mathbf{p}}) - b_-)|\boldsymbol{\gamma}_{\mathbf{p}}'|(1 - \varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\,\mathrm{d}s_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}.$$

Since $\phi_0(z_{\mathbf{p}}) - b_- = \phi_0^d - b_-$ has far field value zero and has even parity with respect to $z_{\mathbf{p}}$, we have

$$\int_\Omega (\phi_0(z_{\mathbf{p}}) - b_-)\,\mathrm{d}x = \varepsilon|\Gamma_{\mathbf{p}}| \int_{\mathbb{R}_{2\ell}} (\phi_0(z) - b_-)\chi(\varepsilon z/\ell)\,\mathrm{d}z.$$

With m_0 defined as in (1.6), there exists a constant C_1, independent of $\mathbf{p}$, such that

$$\int_\Omega (\phi_0(z_{\mathbf{p}}) - b_-)\,\mathrm{d}x = \varepsilon|\Gamma_{\mathbf{p}}| \left(m_0 + C_1 e^{-\ell\nu/\varepsilon}\right). \tag{3.16}$$

The remaining leading order terms depend only on $z_{\mathbf{p}}$ have far-field value zero. We deduce that

$$\int_\Omega \mathrm{L}_{\mathbf{p},0}^{-1}(z_{\mathbf{p}}\phi_0')\,\mathrm{d}x = C_2\varepsilon|\Gamma_{\mathbf{p}}|, \qquad C_2 := \int_{\mathbb{R}_{2\ell}} \mathrm{L}_0^{-1}(z\phi_0')\chi(\varepsilon z/\ell)\,\mathrm{d}z. \tag{3.17}$$

The constant $\overline{B}_{\mathbf{p},2}$ defined in (3.13) is the mass of the dressed function $B_{\mathbf{p},2}$ introduced in (3.4)–(3.5). Since $B_{\mathbf{p},2}$ approaches B_2^∞ as $|z| \to \infty$, we may rewrite

$$\overline{B}_{\mathbf{p},2} = \int_\Omega \left(B_{\mathbf{p},2} - B_2^\infty\right)\,\mathrm{d}x + B_2^\infty|\Omega|.$$

The integrand of the first term above has far-field value zero and is even with respect to $z_{\mathbf{p}}$. From this we deduce

$$\overline{B}_{\mathbf{p},2} = \varepsilon C_3|\Gamma_{\mathbf{p}}| + B_2^\infty|\Omega|, \qquad C_3 := \int_{\mathbb{R}_{2\ell}} (B_2 - B_2^\infty)\chi(\varepsilon z/\ell)\,\mathrm{d}z. \tag{3.18}$$

Finally, the term $\phi_{\geq 2}$ lies in the function family $\mathcal{H}_2$ (see Definition 3.5). Subtracting the far field value $\phi_{\geq 2}^\infty$, integrating out $z_{\mathbf{p}}$ under the local coordinate and applying Lemma 6.9 for $j = 0, k = 0$ we find

$$\int_\Omega \phi_{\geq 2}\,\mathrm{d}x = \phi_{\geq 2}^\infty|\Omega| + \varepsilon f(\mathrm{p}_0) + O(\varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}), \tag{3.19}$$

for some smooth function $f = f(\mathrm{p}_0)$. Combining (3.16)–(3.17) and (3.18)–(3.19) with (3.13) yields

$$\sigma(\mathbf{p}) = \frac{1}{B_2^\infty|\Omega| + \varepsilon C_3|\Gamma_\mathbf{p}|} \left(M_0 - \frac{e^{-\ell\nu/\varepsilon}}{\varepsilon} M_e - \varepsilon^2 \phi_{\geq 2}^\infty |\Omega| - (m_0 + C\varepsilon)|\Gamma_\mathbf{p}| + \varepsilon^2 f(\mathrm{p}_0) + O(\varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}) \right)$$

where $C = C_1\varepsilon^{-1}e^{-\ell\nu/\varepsilon} + C_2 \lesssim 1$ and

$$M_e := \int_\Omega \phi_{\mathbf{p},e}(x)\,\mathrm{d}x = \int_\Omega \phi_e(x)\,\mathrm{d}x,$$

is a mass correction arising from the exponential small correction $\phi_{\mathbf{p},e}$ introduced in Lemma 3.4. The result follows from (6.2) by extracting the leading order terms, introducing the $\mathbf{p}$ independent constants c_0 and

$$\sigma_0 := \frac{1}{B_2^\infty|\Omega|} \left\{ M_0 - \frac{e^{-\ell\nu/\varepsilon}}{\varepsilon} M_e - \varepsilon^2 \phi_{\geq 2}^\infty |\Omega| - m_0|\Gamma_0| \right\} \lesssim 1,$$

and taking ε_0 small enough. □

The bilayer manifold is defined as the graph of bilayer distributions $\{\Phi_\mathbf{p}(\sigma) : \mathbf{p} \in \mathcal{D}_\delta\}$ where each $\Phi_\mathbf{p}$ is associated to the interface $\Gamma_\mathbf{p}$ with $\sigma = \sigma(\mathbf{p})$ satisfying the mass relation. The scaled system mass M_0 introduced in (1.4) and the length of Γ_0 form an *admissible pair* if they balance in the sense that

$$|M_0 - m_0|\Gamma_0|| \lesssim 1, \tag{3.20}$$

for all $\varepsilon \in (0, \varepsilon_0)$. In light of (3.15) we see that the pair (Γ_0, M_0) is admissible if and only if σ_0 is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0)$.

Definition 3.9 (*Bilayer manifold*) Fix $N_1 > 0$. Given a circular base interface Γ_0 with radius 1 and system mass M_0 which form an admissible pair, (3.20), we define the N_1-dimensional bilayer manifold $\mathcal{M}_\delta(\Gamma_0, M_0)$ to be the graph of the map $\mathbf{p} \mapsto \Phi_\mathbf{p}(\sigma(\mathbf{p}))$ over the domain $\mathcal{D}_\delta$, where the bilayer distribution $\Phi_\mathbf{p}(\sigma)$ is introduced in (3.7) with bulk density $\sigma = \sigma(\mathbf{p})$ satisfying the mass relation (3.13).

The particular choice of N_1 is controlled through the spectral parameter ρ, see Definition 3.12 and (3.35). In the sequel we assume that the bulk density parameter satisfies the mass relation (3.13). Recalling that we also assume $\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_1^*, \mathrm{p}_2^*, \mathbf{0})$ is the equilibrium meander parameter vector and σ^* is the equilibrium bulk density, hence the mass relation implies $\sigma(\mathbf{p}^*) = \sigma^*$. With this relation we have the following result.

Corollary 3.10 *For* $\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_1^*, \mathrm{p}_2^*, \mathbf{0}) \in \mathcal{D}_\delta$ *the bulk density parameter* $\sigma = \sigma(\mathbf{p})$ *depends at leading-order only upon the interface parameter* p_0*, satisfying*

$$\sigma^* - \sigma(\mathbf{p}) = \frac{c_0 m_1^2}{m_0}(\mathrm{p}_0 - \mathrm{p}_0^*) + O\left(\varepsilon|\mathrm{p}_0 - \mathrm{p}_0^*|, \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\right). \tag{3.21}$$

Our analysis requires Lipschitz estimates on the residual $\mathrm{F}(\Phi_\mathbf{p})$ for $\mathbf{p}$ near $\mathbf{p}^*$.

Lemma 3.11 *For* $\mathbf{p} \in \mathcal{D}_\delta$ *the components of the residual* $\mathrm{F}(\Phi_\mathbf{p})$ *given in* (3.8) *satisfy*

$$\|\mathrm{F}_2\|_{L^2} \lesssim \varepsilon^{1/2}|\sigma - \sigma^*| + \varepsilon^{3/2},$$
$$\|\mathrm{F}_3 - \mathrm{F}_3^\infty\|_{L^2} + \|\mathrm{F}_{\geq 4} - \mathrm{F}_{\geq 4}^\infty\|_{L^2} \lesssim \varepsilon^{1/2}\left(\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + 1\right),$$

and the exponential residual satisfies $\|\mathrm{F}_e\|_{L^2} \lesssim 1$. *The overall residual satisfies the Lipschitz estimate*

$$\|\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}})\|_{L^2} \lesssim \varepsilon^{5/2}|\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^{5/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} + \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}.$$

Proof The L^2-bounds of the difference between F_k and its bulk value for $k = 2, 3, 4$ follow from the expressions for $\mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4$ given in Lemma 3.7, see [5] for details. The L^2-estimate of $\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})$ follows from comparing it to the zero residual of Φ_*. Indeed we write

$$\|\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})\|_{L^2} = \|\Pi_0\mathrm{F}(\Phi_{\mathbf{p}}) - \Pi_0\mathrm{F}(\Phi_*)\|_{L^2}, \tag{3.22}$$

where Φ_* is the equilibrium solution associated with bulk density state σ^* and interface Γ_* with parameterization

$$\boldsymbol{\gamma}_* := \boldsymbol{\gamma}_{\mathbf{p}^*} + \theta_0\Big((\mathrm{p}_1 - \mathrm{p}_1^*)\mathbf{E}_1 + (\mathrm{p}_2 - \mathrm{p}_2^*)\mathbf{E}_2\Big). \tag{3.23}$$

obtained by translating $\Gamma_{\mathbf{p}^*}$ to place its center at $(\mathrm{p}_1, \mathrm{p}_2)$. The triangle inequality and the expansion of $\mathrm{F} = \mathrm{F}(\Phi_{\mathbf{p}})$ from Lemma 3.7, yield the estimate

$$\begin{aligned}\|\Pi_0(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}(\Phi_*))\|_{L^2} \le{}& \varepsilon^2\|\Pi_0(\mathrm{F}_2 - \mathrm{F}_2(\Phi_*))\|_{L^2} + \varepsilon^3\|\Pi_0(\mathrm{F}_3 - \mathrm{F}_3(\Phi_*))\|_{L^2}\\ &+\varepsilon^4\|\Pi_0(\mathrm{F}_{\ge 4} - \mathrm{F}_{\ge 4}(\Phi_*))\|_{L^2} + e^{-\ell\nu/\varepsilon}\|\mathrm{F}_e - \mathrm{F}_e(\Phi_*)\|_{L^2}.\end{aligned} \tag{3.24}$$

We use the form of the $\mathrm{F}_k(\Phi_*)$ residuals to establish that they are Lipschitz in $(\mathrm{p}_0 - \mathrm{p}_0^*, \hat{\mathbf{p}})$. We observe from Lemma 3.7 that F_2 admits the general form $\mathrm{F}_2 = \kappa_{\mathbf{p}} f_2(z_{\mathbf{p}})(\sigma - \sigma_1^*)$, while $\mathrm{F}_2(\Phi_*) = \kappa_* f_2(z_*)(\sigma^* - \sigma_1^*)$. We deduce that

$$\|\mathrm{F}_2 - \mathrm{F}_2(\Phi_*)\|_{L^2}^2 \le |\sigma - \sigma^*|^2 \int_\Omega \kappa_{\mathbf{p}}^2 f_2^2(z_{\mathbf{p}})\,\mathrm{d}x + |\sigma^* - \sigma_1^*|^2 \int_\Omega |\kappa_{\mathbf{p}} f_2(z_{\mathbf{p}}) - \kappa_* f_2(z_*)|^2\,\mathrm{d}x.$$

Note that the function f_2 has far field zero. The integrals contribute a factor of ε since the integrands are bounded and they support near the interfaces $\Gamma_{\mathbf{p}}$ and Γ_*, respectively. We decompose the second integrand as $\kappa_{\mathbf{p}}\left(f_2(z_{\mathbf{p}}) - f_2(z_*)\right) + \left(\kappa_{\mathbf{p}} - \kappa_*\right) f_2(z_*)$ which we bound by $|z_{\mathbf{p}} - z_*| + |\kappa_{\mathbf{p}} - \kappa_*|$ in its support set. Using the estimates of Lemmas 6.6, 6.2 and 6.1, recalling $|\sigma^* - \sigma_1^*| \lesssim \varepsilon$ we arrive at the bound

$$\|\Pi_0\mathrm{F}_2 - \Pi_0\mathrm{F}_2(\Phi_*)\|_{L^2}^2 \lesssim \varepsilon|\sigma - \sigma^*|^2 + \varepsilon\left(|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2\right). \tag{3.25}$$

The L^2 bounds of $\mathrm{F}_3 - \mathrm{F}_3(\Phi_*)$ and $\mathrm{F}_{\ge 4} - \mathrm{F}_{\ge 4}(\Phi_*)$ involve higher derivatives of the perturbed curve $\boldsymbol{\gamma}_{\mathbf{p}}$ from (2.13) which are controlled with through (2.15), specifically

$$\|\varepsilon^2\Delta_{s_{\mathbf{p}}}\boldsymbol{\gamma}_{\mathbf{p}}^{(k)}\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \|\boldsymbol{\gamma}_{\mathbf{p}}^{(k)} - \boldsymbol{\gamma}_*^{(k)}\|_{L^2(\mathscr{I})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_k^2} + |\mathrm{p}_0 - \mathrm{p}_0^*|.$$

Using the bound above and the form of $\mathrm{F}_3, \mathrm{F}_4$ in Lemma 3.7, we establish that

$$\|\Pi_0\mathrm{F}_3 - \Pi_0\mathrm{F}_3(\Phi_*)\|_{L^2}^2 + \|\Pi_0\mathrm{F}_4 - \Pi_0\mathrm{F}_4(\Phi_*)\|_{L^2}^2 \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2 + \varepsilon^{-1}|\mathrm{p}_0 - \mathrm{p}_0^*|^2. \tag{3.26}$$

The term F_e incorporates residual from ϕ_e and from the dressing process. However ϕ_e in Lemma 3.7 cancels with the corresponding term in Φ_*, and this component of the residual is due solely to the dressing process, which makes a contribution

$$\|\mathrm{F}_e - \mathrm{F}_e(\Phi_*)\|_{L^2} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}. \tag{3.27}$$

Combining these bounds in (3.25)–(3.27) with (3.24), applying Corollary 3.10 completes the proof. □

3.2 Linearized Operator and Slow Spaces

The nonlinear stability analysis hinges upon the properties of the linearization of the flow (1.11) about a bilayer distribution $\Phi_{\mathbf{p}}$ introduced in Lemma 3.7. The linearization takes the form $\Pi_0\mathbb{L}_{\mathbf{p}}$ where

$$\begin{aligned}\mathbb{L}_{\mathbf{p}} := \frac{\delta^2\mathcal{F}}{\delta u^2}\Big|_{u=\Phi_{\mathbf{p}}} =& (\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}}) + \varepsilon\eta_1)(\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}})) \\ &- (\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}}))W'''(\Phi_{\mathbf{p}}) + \varepsilon\eta_d W''(\Phi_{\mathbf{p}}),\end{aligned} \tag{3.28}$$

denotes the second variational derivative of the free energy $\mathcal{F}$ at $\Phi_{\mathbf{p}}$ and $\eta_d := \eta_1 - \eta_2$. When restricted to functions with support within the $\Gamma_{\mathbf{p}}^{2\ell}$, the Cartesian Laplacian admits the expansion (2.24) and the leading order operator takes the form

$$\mathbb{L}_{\mathbf{p},0} := \mathcal{L}_{\mathbf{p}}^2, \qquad \text{where} \quad \mathcal{L}_{\mathbf{p}} := \mathrm{L}_{\mathbf{p},0} - \varepsilon^2\Delta_{s_{\mathbf{p}}}. \tag{3.29}$$

An analysis of the spectrum of the leading order operator $\mathbb{L}_{\mathbf{p},0}$ led to the definition of slow spaces $\mathcal{Z}_{\mathbf{p}}^0$ and $\mathcal{Z}_{\mathbf{p}}^1$, called the pearling and meander spaces respectively. For each $\mathbf{p} \in \mathcal{D}_\delta$ these slow spaces are spanned by the products of a collection of Laplace–Beltrami eigenmodes of $\Gamma_{\mathbf{p}}$ and the associated dressings of the normalized ground-state and first excited state eigenvectors, ψ_0 and ψ_1 respectively, of the operator L_0 defined in Lemma 3.1. These spaces are sufficiently accurate approximations of the small-eigenvalue eigenspaces of $\mathbb{L}_0$ to generate coercivity estimates of this operator on the orthogonal compliment of the combined slow space $\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1$, see [10, 14, 17]. However these spaces are only invariant under the action of the full operator $\mathbb{L}_{\mathbf{p}}$ up to order of ε. This is not sufficient to close the nonlinear energy estimates required to establish stability and accurately recover the normal velocity. Consequently the modified space slow spaces were introduced in [5] and are summarized below. This definition uses the dressed and scaled version $\tilde{\psi}_k(z_{\mathbf{p}})$ of $\tilde{\psi}_k$ defined as

$$\tilde{\psi}_k(z_{\mathbf{p}}) := \varepsilon^{-1/2}\psi_k(z_{\mathbf{p}}).$$

Definition 3.12 (*Slow spaces*) For $k = 0, 1$, fixed $\rho > 0$, and $\mathbf{p} \in \mathcal{D}_\delta$ we introduce the disjoint index sets:

$$\Sigma_k = \Sigma_k(\rho) := \left\{ j \,\Big|\, \Lambda_{ki}^2 := \left(\lambda_k + \varepsilon^2\beta_i^2\right)^2 \le \rho \right\}, \quad \text{and} \quad \Sigma := \Sigma_0 \cup \Sigma_1, \tag{3.30}$$

and the slow space $\mathcal{Z}_* = \mathcal{Z}_*(\mathbf{p}, \rho) \subset L^2$, as the union of the pearling and meander spaces, $\mathcal{Z}_*^0$ and $\mathcal{Z}_*^1$,

$$\mathcal{Z}_* := \mathcal{Z}_*^0 \cup \mathcal{Z}_*^1 \qquad \text{with} \qquad \mathcal{Z}_*^k := \text{span}\left\{ Z_{\mathbf{p},*}^{ki}, i \in \Sigma_k \right\}, k = 0, 1. \tag{3.31}$$

The modified basis functions take the form

$$Z_{\mathbf{p},*}^{ki} := \left(\tilde{\psi}_k + \varepsilon\tilde{\varphi}_{1,i}\right)\tilde{\theta}_i + \varepsilon^2\tilde{\varphi}_{2,i}\tilde{\theta}_i', \tag{3.32}$$

for $k = 0, 1$ and $i \in \Sigma_k$. For $l = 1, 2$ the correction functions $\tilde{\varphi}_{l,i} = \varepsilon^{-1/2}\varphi_{l,i}(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}) \in \mathcal{H}_2$ (see Definition 3.5) have far-field value zero and are defined in Lemma 4.6 of [5] so as to satisfy

$$\int_{\mathbb{R}} \varphi_{l,i}(z, \boldsymbol{\gamma}_{\mathbf{p}})\psi_k(z)\,\mathrm{d}z = 0, \quad i \in \Sigma_k, k = 0, 1. \tag{3.33}$$

For $k = 0, 1$ and $i \in \Sigma_k$ the leading order term in $Z^{ki}_{\mathbf{p},*}$, obtained by setting $\varphi_{l,i} = 0$, is denoted

$$Z^{ki}_{\mathbf{p}} := \tilde{\psi}_{I(i)}\tilde{\theta}_i. \tag{3.34}$$

The Laplace–Beltrami eigenvalues satisfy the classical Weyl's Law asymptotics $\beta_j \sim j$, [22] or (2.4), from which we deduce that

$$N_0 := |\Sigma_0| \sim \varepsilon^{-1}\rho^{1/2}, \qquad N_1 := |\Sigma_1| \sim \varepsilon^{-1}\rho^{1/4}. \tag{3.35}$$

A key point in the analysis is that for ρ sufficiently small, independent of ε, the Laplace–Beltrami eigenvalues associated to the pearling index set Σ_0 are asymptotically well-separated from those of the meander index set Σ_1. Specifically with this restriction on ρ it is straightforward to determine $C > 0$, independent of ε, such that the pearling-meander gap

$$|\beta_i - \beta_j| \geq C\varepsilon^{-1}, \qquad i \in \Sigma_0,\ j \in \Sigma_1, \tag{3.36}$$

holds. This gap, together with (2.15) and (3.33), affords the basis functions of $\mathcal{Z}_*$ an enhanced orthogonality that is essential to establishing Lemma 3.13. As outlined in Section 4.2 of [5], for $i, j \in \Sigma$, they satisfy

$$\left\langle Z^{I(i)i}_{\mathbf{p},*}, Z^{I(j)j}_{\mathbf{p},*}\right\rangle_{L^2} = \begin{cases} (1+\mathrm{p}_0)\,\delta_{ij} + O\left(\varepsilon^2, \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\right)\mathbb{E}_{ij}, & I(i) = I(j); \\ O\left(\varepsilon^2, \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right)\mathbb{E}_{ij}, & I(i) \neq I(j). \end{cases} \tag{3.37}$$

Here $\mathbb{E}$ is a norm-one $(N_0 + N_1) \times (N_0 + N_1)$ matrix, and here and below the indicator function I takes values $I(i) = 0$ if $i \in \Sigma_0$ and $I(i) = 1$ if $i \in \Sigma_1$.

We denote the L^2 linear projection on the subspace $\mathcal{Z}^k_*$ by $\Pi_{\mathcal{Z}^k_*}$ for $k = 0, 1$. In particular, for any $u \in L^2$ there exists a unique vector $(u_i)_{i\in\Sigma_k} \in l^2(\mathbb{R}^{N_k})$ such that $\Pi_{\mathcal{Z}^k_*}u = \sum_{i\in\Sigma_k} u_i Z^{ki}_{\mathbf{p},*}$. When restricted to $\mathcal{Z}_*$ the bilinear form of the full linearized operator $\Pi_0\mathbb{L}_{\mathbf{p}}\big|_{\mathcal{Z}_*}$, induces an $(N_0 + N_1) \times (N_0 + N_1)$ matrix $\mathbb{M}^*$ with entries

$$\mathbb{M}^*_{ij} = \langle \Pi_0\mathbb{L}_{\mathbf{p}} Z^{I(i)i}_{\mathbf{p},*}, Z^{I(j)j}_{\mathbf{p},*}\rangle_{L^2}. \tag{3.38}$$

We decompose $\mathbb{M}^*$ into a block structure corresponding to the pearling and meander spaces,

$$\mathbb{M}^* = \begin{pmatrix} \mathbb{M}^*(0,0) & \mathbb{M}^*(0,1) \\ \mathbb{M}^*(1,0) & \mathbb{M}^*(1,1) \end{pmatrix}, \tag{3.39}$$

where the blocks has entries $\mathbb{M}^*_{ij}(k,l) = \mathbb{M}^*_{ij}$ for $i \in \Sigma_k$, $j \in \Sigma_l$. A detailed analysis of $\mathbb{M}^*$ is given in [5]. In particular, the *dynamic pearling stability condition*

$$(\mathbf{PSC}) \qquad \sigma S_1 + \eta_d\lambda_0 > 0, \tag{3.40}$$

compares the bulk density $\sigma = \sigma(\mathbf{p})$ to the ground state eigenvalue λ_0 of L_0 in terms of the shape-factor S_1 which depends only upon the form of the well W, see [17]. When the pearling-stability condition holds the pearling sub-block $\mathbb{M}^*(0,0)$ is positive definite, that is,

$$\mathbf{q}^T\mathbb{M}^*(0,0)\mathbf{q} \geq \frac{\varepsilon}{2}(1+\mathrm{p}_0)(\sigma S_1 + \eta_d\lambda_0)\|\mathbf{q}\|^2_{l^2}, \quad \text{for all } \mathbf{q} \in \mathbb{R}^{N_0}. \tag{3.41}$$

The modified slow spaces $\mathcal{Z}^0_*$ and $\mathcal{Z}^1_*$, together with the pearling–meander gap, reduces the strength of couplings between the meander space and the pearling space, (3.37), as well as between the meander space and the fast decay space $\mathcal{Z}^\perp_*$. The later coupling becomes $O(\varepsilon^2)$ as recorded in the following Lemma.

Lemma 3.13 *For ε_0, ρ, and δ sufficiently small in terms of $|\Omega|$ and given system parameters. Then for all $\varepsilon \in (0, \varepsilon_0)$ we have the meander-fast decay space coupling bound*

$$\|\Pi_{\mathcal{Z}_*^1} \mathbb{L}_{\mathbf{p}} v^\perp\|_{L^2} \lesssim \left(\varepsilon^2 + \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right) \|v^\perp\|_{L^2}$$

for any $v^\perp \in (\mathcal{Z}_*^1)^\perp$ and $\mathbf{p} \in \mathcal{D}_\delta$.

Proof This is a direct adaptation of Theorem 4.11 of [5]. □

The bilinear form induced by the linearized operator $\Pi_0 \mathbb{L}_{\mathbf{p}}$ is uniformly coercive on the set perpendicular to the slow space $\mathcal{Z}_*$.

Lemma 3.14 *Fix $\rho, \varepsilon_0, \delta > 0$ sufficiently small with smallness depending only on the domain and given system parameters. Then there exists $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{p} \in \mathcal{D}_\delta$, and $w \in \mathcal{Z}_*^\perp$ the following coercivity estimates hold*

$$\begin{aligned} \langle \mathbb{L}_{\mathbf{p}} w, w\rangle >_{L^2} \geq C\rho^2 \left(\varepsilon^4 \|w\|_{H^2}^2 + \|w\|_{L^2}^2\right) \quad \textit{and} \quad \|\mathbb{L}_{\mathbf{p}} w\|_{L^2}^2 \\ \geq C\rho^2 \langle < \mathbb{L}_{\mathbf{p}} w, w\rangle >_{L^2} . \end{aligned} \tag{3.42}$$

Proof This is a direct adaptation of Theorem 4.13 of [5]. □

4 Nonlinear Stability and the Main Results

In this section the nonlinear estimates are developed which establish the main result: stability of the bilayer manifold defined around the circular base interface. Moreover the normal velocity of the interfaces $\Gamma_{\mathbf{p}}$ is captured through the projection of the flow onto the modified meander space. Technical details involving the projected flow are postponed to Sect. 5.

4.1 Decomposition of the Flow

To define the nonlinear manifold projection we restrict the perturbation parameters $\mathbf{p}$ to lie in a smaller space $\mathcal{O}_{m,\delta} \subset \mathcal{D}_\delta$, defined in (2.8), given by

$$\mathcal{O}_{m,\delta} := \left\{\mathbf{p} \in \mathbb{R}^{N_1} \;\middle|\; |\mathrm{p}_0| + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} \leq m\delta\right\}. \tag{4.1}$$

We denote by $\mathcal{V}_R(\mathcal{M}_\delta, \mathcal{O}_{m,\delta})$ the tubular neighborhood of thickness R in the H^2 inner norm,

$$\|u\|_{H^2_{\mathrm{in}}} := \sqrt{\|u\|_{L^2(\Omega)}^2 + \varepsilon^4 \|u\|_{H^2(\Omega)}^2}, \tag{4.2}$$

that surrounds the bilayer manifold $\mathcal{M}_\delta$ restricted to $\mathcal{O}_{m,\delta}$,

$$\mathcal{V}_R(\mathcal{M}_\delta, \mathcal{O}_{m,\delta}) := \left\{u \in H^2 \;\middle|\; \min_{\mathbf{p}\in\mathcal{O}_{m,\delta}} \|u - \Phi_{\mathbf{p}}\|_{H^2_{\mathrm{in}}} < R, \quad \langle u - b_-\rangle_{L^2} = \frac{\varepsilon M_0}{|\Omega|}\right\}. \tag{4.3}$$

Definition 4.1 (*Manifold projection*) Let $\mathcal{U}$ be a neighborhood of $\mathcal{M}_\delta$. We say $\Pi_{\mathcal{M}_\delta} u := \Phi_{\mathbf{p}}(\sigma)$ is a projection of $\mathcal{U}$ onto $\mathcal{M}_\delta$ and $\Pi^\perp_{\mathcal{M}_\delta} u := v^\perp$ is its complement if for each $u \in \mathcal{U}$ there exist a unique $\mathbf{p} \in \mathcal{D}_\delta$ and mass-free meander-orthogonal perturbation $v^\perp \in (\mathcal{Z}_*^1(\mathbf{p}))^\perp$ such that

$$u = \Phi_{\mathbf{p}} + v^\perp. \tag{4.4}$$

We call $\mathbf{p}$ and $\Gamma_{\mathbf{p}}$ the meander parameter vector and interface associated to u, respectively.

The following, from Lemma 5.2 of [5], establishes the existence of the manifold projection.

Lemma 4.2 *Let $\mathcal{M}_\delta = \mathcal{M}_\delta(\Gamma_0, M_0)$ be the bilayer manifold of Definition 3.9. Let $\delta, \varepsilon_0 > 0$ be sufficiently small, then for all $\varepsilon \in (0, \varepsilon_0)$ there exists a manifold projection $\Pi_{\mathcal{M}_\delta}$ defined on the tubular neigborhood $\mathcal{U} = \mathcal{V}_{\delta\varepsilon}(\mathcal{M}_\delta, \mathcal{O}_{2,\delta})$. Moreover, for $u \in \mathcal{U}$ of the form $u = \Phi_{\mathbf{p}_0} + v$ with $\mathbf{p}_0 \in \mathcal{D}_\delta$ and massless perturbation $v \in H^2$ satisfying $\|v\|_{H^2_{\rm in}} \leq \delta\varepsilon$, then u's meander parameter vector $\mathbf{p}$ and meander-orthogonal perturbation, $v^\perp$ satisfy*

$$\|\mathbf{p} - \mathbf{p}_0\|_{l^2} \lesssim \varepsilon^{1/2}\|v\|_{L^2}; \qquad \|v^\perp\|_{H^2_{\rm in}} \lesssim \|v\|_{H^2_{\rm in}}.$$

Assume a priori that a solution $u = u(t)$ of the FCH gradient flow satisfies $u \in \mathcal{V}_{C\varepsilon^{5/2}}(\mathcal{M}_\delta, \mathcal{O}_{2,\delta})$ on the interval $[0, T]$. Then for ε_0 sufficiently small, depending on δ, we have $u \in \mathcal{U}$ and may decompose u as

$$u(x,t) = \Phi_{\mathbf{p}}(x;\sigma) + v^\perp(x,t), \quad v^\perp \in (\mathcal{Z}^1_*)^\perp, \quad \int_\Omega v^\perp \,\mathrm{d}x = 0. \tag{4.5}$$

The FCH gradient flow (1.11) can be written in terms of the pair $(\mathbf{p}, v^\perp) = (\mathbf{p}(t), v^\perp(t))$ with the bulk density parameter $\sigma = \sigma(\mathbf{p}(t))$ given through (3.13). Substituting the decomposition (4.5) into the FCH gradient flow leads to an equation for $\Phi_{\mathbf{p}}$ and $v^\perp$:

$$\partial_t \Phi_{\mathbf{p}} = -\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}) + \mathscr{R}[v^\perp], \tag{4.6}$$

where $\mathscr{R}[v^\perp]$ is the meander-orthogonal remainder contributed by $v^\perp$. Specifically

$$\mathscr{R}[v^\perp] := -\partial_t v^\perp - \Pi_0 \mathbb{L}_{\mathbf{p}} v^\perp - \Pi_0 \mathrm{N}(v^\perp), \tag{4.7}$$

where $\mathrm{N}(v^\perp)$ is the genuinely nonlinear term defined by

$$\mathrm{N}(v^\perp) := \mathrm{F}(\Phi_{\mathbf{p}} + v^\perp) - \mathrm{F}(\Phi_{\mathbf{p}}) - \mathbb{L}_{\mathbf{p}} v^\perp. \tag{4.8}$$

A key to the nonlinear stability analysis is that the operator $\Pi_0\mathbb{L}_{\mathbf{p}}$ is uniformly coercive on the space L^2-orthogonal to the modified slow space $\mathcal{Z}_*(\mathbf{p})$. However the modified slow space includes the modified pearling space $\mathcal{Z}^0_*(\mathbf{p})$ on which the operator is only weakly coercive, see (3.41). This dichotomy motivates a further decomposition of the meander-orthogonal perturbation $v^\perp$ in pearling and "fast modes" as

$$v^\perp = Q(x,t) + w(x,t), \quad w \in \mathcal{Z}^\perp_*(\mathbf{p}, \rho), \tag{4.9}$$

where $Q = \Pi_{\mathcal{Z}^0_*} v^\perp \in \mathcal{Z}^0_*$ admits the Galerkin expansion

$$Q = \sum_{j \in \Sigma_0} \mathrm{q}_j Z^{0j}_{\mathbf{p},*},$$

for some $\mathbf{q} = \mathbf{q}(t) = (\mathrm{q}_i)_{i\in\Sigma_0}$. The decomposition is well defined by Lemma 4.10 of [5] which follows from the enhanced orthogonality estimate (3.37) and establishes the bounds

$$\|Q\|_{H^2_{\rm in}} + \|w\|_{H^2_{\rm in}} \lesssim \|v^\perp\|_{H^2_{\rm in}}; \qquad \|Q\|_{H^2_{\rm in}} \sim \|\mathbf{q}\|_{l^2}.$$

The decomposition (4.9) of $v^\perp$, allows the evolution Eq. (4.6) to be rewritten in terms of Q and w:

$$\partial_t Q + \Pi_0\mathbb{L}_{\mathbf{p}} Q + \partial_t w + \Pi_0 \mathbb{L}_{\mathbf{p}} w = -\partial_t \Phi_{\mathbf{p}} - \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}) - \Pi_0 \mathrm{N}(v^\perp). \tag{4.10}$$

The orthogonality of (4.9) and the coercivity of $\mathbb{L}_{\mathbf{p}}$ on pearling and fast spaces induces L^2 estimates on $\mathbf{q}$ (Lemma 6.15) and H^2-estimates on w (Lemma 6.16).

The following result, Theorem 5.13 of [5], establishes that the bilayer manifold for an admissible pair (Γ_0, M_0), see (3.20), is stable up to its boundary. Specifically, orbits that start within a thin tubular neighborhood of $\mathcal{M}_\delta$ stay within a comparable tubular neighborhood until such as time, T, that the meander parameter vector $\mathbf{p}$ reaches the boundary of the domain $\mathcal{O}_\delta$. The result requires a strengthening of the admissible pair condition that correlates the system mass M_0 and the base interface Γ_0, here expressed in terms of the difference between the bulk parameter σ_0 and its leading order equilibrium value σ_1^*.

Theorem 4.1 (Theorem 5.13 [5]) *Consider the mass preserving flow* (1.10)–(1.11) *subject to periodic boundary condition and initial data $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_\delta(\Gamma_0, M_0), \mathcal{O}_\delta)$. Let (Γ_0, M_0) be a admissible pair that satisfies*

$$|\sigma_0(|\Gamma_0|, M_0) - \sigma_1^*| \lesssim \delta,$$

where $\sigma_1^* = \sigma_1^*(\eta_1, \eta_2)$, introduced in (1.8)*, satisfies the equilbrium pearling stability condition* ($\mathbf{PSC}_*$)*,* (1.15)*. Then for δ small enough depending only on domain and system parameters, there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the solution u lies in the projection valid domain $\mathcal{U}(\mathcal{M}_b)$ so long as $\mathbf{p} \in \mathcal{O}_{2,\delta}$.*

Moreover the following statements hold on the residence interval $[0, T]$.

1. *The solution of the mass preserving flow* (1.11) *can be decomposed as*
$$u(x) = \Phi_{\mathbf{p}}(\boldsymbol{x}; \sigma) + v^\perp(x), \qquad v^\perp(x) = Q(x) + w(x) \tag{4.11}$$
where $\Phi_{\mathbf{p}} \in \mathcal{M}_\delta(\Gamma_0, M_0)$ and $Q = \Pi_{\mathcal{Z}_*^0} v^\perp \in \mathcal{Z}_*^0$ is the projection of $v^\perp$ to the pearling slow space.
2. *The meander-orthogonal perturbation $v^\perp$ remains in $\mathcal{V}_{C\varepsilon^{5/2}\rho^{-2}}$ for some positive constant C, that is*
$$\|v^\perp\|_{H^2_{\rm in}} \lesssim \|w\|_{H^2_{\rm in}}(t) + \|\mathbf{q}\|_{l^2}(t) \le C\varepsilon^{5/2}\rho^{-2}, \qquad \forall t \in [0, T]. \tag{4.12}$$

4.2 Nonlinear Stability of the Bilayer Manifold

The heart of the analysis lies in the bounds on the dynamic meander parameter vector $\mathbf{p}$. These are recovered via energy and continuity arguments which hinge on an appropriate choice of the spectral parameter ρ which controls the dimension of the meander space. The initial energy in the system is restricted to control the extend of the transient excursion. This energy is measured principally by the difference between the initial and equilibrium curve length, $|\mathrm{p}_0 - \mathrm{p}_0^*|$, and secondarily by the weighted $\mathbb{V}_2^2$- and $\mathbb{V}_3^2$-norms of $\hat{\mathbf{p}}$ that control the deviation of the initial curve from circularity. Correspondingly we introduce the parameter set

$$\mathcal{O}_{m,\delta}^\circ := \left\{ \mathbf{p} \in \mathbb{R}^{N_1} \;\middle|\; \varepsilon^{-1/2}|\mathrm{p}_0 - \mathrm{p}_0^*| + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} < m\delta \right\}. \tag{4.13}$$

As usual we omit the subscript m if $m = 1$. Our analysis requires that $\mathcal{O}_{m,\delta}^\circ \subset \mathcal{O}_{m,\delta}$, defined in (4.1). This containment is established in the following Lemma.

Lemma 4.3 *Fix $\delta > 0$, and let $\varepsilon_0 > 0$ be sufficiently small. Then for any admissible pair (Γ_0, M_0) whose base bulk density σ_0, (see* (3.15)*), is sufficiently close to the equilibrium bulk density σ_1^*, (see* (1.8)*), so as to satisfy*

$$|\sigma_0(\Gamma_0, M_0) - \sigma_1^*| \le \frac{c_0 m_1^2}{2m_0}\delta, \tag{4.14}$$

then $\mathcal{O}^\circ_{m,\delta} \subset \mathcal{O}_{m,\delta}$ *for* $m \geq 1$.

Proof Pick any $\mathbf{p} \in \mathcal{O}^\circ_{m,\delta}$, we show $\mathbf{p} \in \mathcal{O}_{m,\delta}$ for $m \geq 1$. By the definition of $\mathcal{O}_{m,\delta}$ in (4.1), it suffices to show $|\mathrm{p}_0| \leq \delta$. From Lemma 3.8 and the triangle inequality we have

$$|\mathrm{p}_0| \leq \frac{m_0}{c_0 m_1^2}|\sigma(\mathbf{p}) - \sigma_1^*| + \frac{m_0}{c_0 m_1^2}|\sigma_1^* - \sigma_0| + C\varepsilon.$$

Recalling $\sigma^* = \sigma_1^* + O(\varepsilon)$, the bound above, together with Corollary 3.10 and $|\mathrm{p}_0 - \mathrm{p}_0^*| \leq m\varepsilon^{1/2}$ for $\mathbf{p} \in \mathcal{O}_{m,\delta}$ yields

$$|\mathrm{p}_0| \leq |\mathrm{p}_0 - \mathrm{p}_0^*| + \frac{\delta}{2} + C\varepsilon < \delta,$$

for $\varepsilon \in (0, \varepsilon_0)$ and ε_0 sufficiently small with respect to δ. $\square$

Our main result establishes the asymptotic stability of the bilayer manifold with circular base interface Γ_0. More specifically we assume that (Γ_0, M_0) forms an admissible pair, that the base interface Γ_0 is circle, and the pair satisfies (4.14) for which $\sigma_1^* = \sigma_1^*(\eta_1, \eta_2)$, introduced in (1.8), satisfies the equilibrium pearling stability condition (**PSC**$_*$), (1.15).

Theorem 4.2 *Consider the mass preserving gradient flow* (1.10)–(1.11) *subject to periodic boundary conditions. Let* (Γ_0, M_0) *be an admissible pair that satisfies* (4.14)*, and* $\sigma_1^* = \sigma_1^*(\eta_1, \eta_2)$ *given in* (1.8) *satisfies the equilibrium pearling stability condition* (**PSC**$_*$)*,* (1.15)*. For* δ*,* ρ*, and* ε_0 *sufficient small, then for all* $\varepsilon \in (0, \varepsilon_0)$*, the solution* u *of the mass preserving gradient flow arising from initial data* $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_\delta(\Gamma_0, M_0), \mathcal{O}^\circ_\delta)$ *defined in* (4.3)*, remains in a slightly bigger set for all* $t \in [0, \infty)$*. Indeed the solution admits the decomposition* (4.11) *and there are constants* $C, c > 0$ *independent of* $\delta, \rho, \varepsilon \in (0, \varepsilon_0)$*, and choice of initial data for which the orthogonal perturbation* $v^\perp$ *satisfies*

$$\|v^\perp\|_{H^2_{\mathrm{in}}} \leq C\varepsilon^{5/2}\rho^2, \qquad \|v^\perp\|_{H^2_{\mathrm{in}}} \leq Ce^{-c\varepsilon^4 t}, \tag{4.15}$$

and the projected meander parameter vector relaxes to an equilibrium value $\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_{1,2}^*, \mathbf{0})$ *according to*

$$|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} \leq 4\varepsilon\delta^2 e^{-c\varepsilon^4 t}. \tag{4.16}$$

The equilibrium curve length parameter p_0^* *admits the approximation*

$$\mathrm{p}_0^* = -\frac{m_0}{c_0 m_1^2}(\sigma_1^* - \sigma_0) + O(\varepsilon), \quad \sigma_1^* = -\frac{\eta_1 + \eta_2}{2m_0} m_1^2, \quad \sigma_0 = \frac{M_0 - 2\pi m_0}{B_2^\infty |\Omega|} + O(\varepsilon^2), \tag{4.17}$$

where the positive constants c_0, m_0, m_1 *are defined in* (3.15) *and* (1.6) *while* η_1, η_2 *are system parameters. For all* $k \leq 4$ *we have the temporal* L^2 *bound*

$$\varepsilon^4 \int_0^\infty e^{c\varepsilon^4 t}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_k^2}\,\mathrm{d}t \leq 4\delta^2. \tag{4.18}$$

The translation parameters $\mathrm{p}_1, \mathrm{p}_2$ *remain within* $O(\delta)$ *of their initial values* $\mathrm{p}_1(0), \mathrm{p}_2(0)$ *and converge to* $\mathrm{p}_1^*, \mathrm{p}_2^*$ *as* $t \to \infty$.

Proof Since $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_\delta, \mathcal{O}_\delta^\circ)$, there exist $\mathbf{p}_0 \in \mathcal{O}_\delta^\circ$ and $v_0 \in H^2_{\text{in}}$ satisfying $\|v_0\|_{H^2_{\text{in}}} \leq \varepsilon^{5/2}$ such that $u_0 = \Phi_{\mathbf{p}_0} + v_0$. Lemma 4.2 affords the decomposition $u_0 = \Phi_{\mathbf{p}(0)} + v_0^\perp$ where $\Phi_{\mathbf{p}(0)} = \Pi_{\mathcal{M}_\delta} u_0$ and $v_0^\perp$ is the orthogonal perturbation. The distance from $\mathbf{p}(0)$ to $\mathbf{p}_0 \in \mathcal{O}_\delta^\circ$ can be bounded by

$$\|\mathbf{p}(0) - \mathbf{p}_0\|_{l^2} \lesssim \varepsilon^3, \qquad \|\hat{\mathbf{p}}(0) - \hat{\mathbf{p}}_0\|_{\mathbb{V}_3^2} \lesssim N_1^3 \|\hat{\mathbf{p}}(0) - \hat{\mathbf{p}}_0\|_{l^2} \lesssim \rho^{3/4},$$

where we also applied Lemma 6.1 and bounded N_1 from (3.35). Note that $\mathbf{p}_0 \in \mathcal{O}_\delta^\circ$. Hence for ε_0, ρ small enough depending on δ, the triangle inequality implies that the initial meander parameter components satisfy

$$\varepsilon^{-1/2}|\mathrm{p}_0(0) - \mathrm{p}_0^*| + \|\hat{\mathbf{p}}(0)\|_{\mathbb{V}_3^2} \leq \frac{3\delta}{2}, \tag{4.19}$$

and there exists $T > 0$ such that

$$(\mathbf{A}) \qquad \varepsilon^{-1/2}|\mathrm{p}_0(t) - \mathrm{p}_0^*| + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}(t) < 2\delta, \quad \|\dot{\mathbf{p}}\|_{l^2}(t) < 2\varepsilon^3 \qquad \forall t \in [0, T). \tag{4.20}$$

We show $T = \infty$ in the following.

The equilibrium pearling stability condition (**PSC**$_*$) holds by assumption. Under the a priori assumption (**A**) the dynamic pearling stability condition (**PSC**) (3.40) holds uniformly. Indeed, since $\sigma^* = \sigma_1^* + O(\varepsilon)$, (**PSC**) holds if σ stays sufficiently close to σ^*, which follows from Corollary 3.10 by

$$|\sigma - \sigma^*| \lesssim |\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \lesssim \varepsilon^{1/2}\delta. \tag{4.21}$$

Choosing ε_0 small enough the dynamic pearling stability condition (**PSC**) holds uniformly on $[0, T]$ and Theorem 4.1 applies. From (4.12) this in turn affords the following uniform bounds on w, $\mathbf{q}$

$$\langle \mathbb{L}_{\mathbf{p}} w, w \rangle_{L^2} \lesssim \|w\|^2_{H^2_{\text{in}}} \lesssim \varepsilon^5 \rho^{-2}, \qquad \|\mathbf{q}\|^2_{l^2} \lesssim \varepsilon^5 \rho^{-4} \qquad \forall t \in [0, T). \tag{4.22}$$

Since $v^\perp = Q + w$ is an orthogonal decomposition and $\|Q\|_{H^2_{\text{in}}} \sim \|\mathbf{q}\|_{l^2}$, we may estimate $v^\perp$ with the aid of the coercivity Lemma 3.14,and the nonlinear term $\mathrm{N}(v^\perp)$ from Lemma 6.17,

$$\|v^\perp\|^2_{H^2_{\text{in}}} \lesssim \rho^{-2} \langle \mathbb{L}_{\mathbf{p}} w, w \rangle_{L^2} + \|\mathbf{q}\|^2_{l^2}, \qquad \|\mathrm{N}(v^\perp)\|^2_{L^2} \lesssim \varepsilon^{-2} \left(\rho^{-2} \langle \mathbb{L}_{\mathbf{p}} w, w \rangle_{L^2} + \|\mathbf{q}\|_{l^2} \right)^2. \tag{4.23}$$

The first H^2_{in} bound of $v^\perp$ on $[0, T)$ in (4.15) follows from (4.23), particularly,

$$\|v^\perp\|_{H^2_{\text{in}}}(t) \lesssim \varepsilon^{5/2} \rho^{-2}, \qquad \forall t \in [0, T). \tag{4.24}$$

The estimate above and assumptions in (4.20), supersede assumptions in (5.1) of Sect. 5 so long as $\varepsilon_0 \lesssim \delta^2 \rho^4$. Hence the results of Sect. 5 apply on $[0, T)$. Particularly Lemma 5.14 applies. Combining these estimates (4.22)–(4.23) with Lemma 5.14 recovers the a priori estimate of $\|\dot{\mathbf{p}}\|_{l^2}$ in (4.20). In fact, for ε_0 small enough we have

$$\|\dot{\mathbf{p}}\|_{l^2} \leq \varepsilon^3 \qquad \forall t \in [0, T). \tag{4.25}$$

For future use, we may rewrite the bounds in (4.23) as

$$\|v^\perp\|^2_{H^2_{\text{in}}} \lesssim \rho^{-4} \|\mathbb{L}_{\mathbf{p}} w\|^2_{L^2} + \|\mathbf{q}\|^2_{l^2}, \qquad \|\mathrm{N}(v^\perp)\|^2_{L^2} \lesssim \varepsilon^3 \rho^{-8} \|\mathbb{L}_{\mathbf{p}} w\|^2_{L^2} + \varepsilon^3 \rho^{-4} \|\mathbf{q}\|^2_{l^2}. \tag{4.26}$$

To recover the a priori assumptions (**A**) it remains to bound the interface length residual $|\mathrm{p}_0-\mathrm{p}_0^*|$ and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}$. From Lemma 5.14 and (4.26) we obtain the dynamic bound on $\|\dot{\mathbf{p}}\|_{l^2}$

$$\|\dot{\mathbf{p}}\|_{l^2}^2 \lesssim \varepsilon^6|\mathrm{p}_0-\mathrm{p}_0^*|^2+\varepsilon^8\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2+\varepsilon^4\rho^{-8}\|\mathbb{L}_{\mathbf{p}}w\|_{L^2}^2+\varepsilon^4\rho^{-4}\|\mathbf{q}\|_{l^2}^2. \tag{4.27}$$

We break the remainder of the proof into three steps.

Step 1: Uniform estimates of $|\mathrm{p}_0-\mathrm{p}_0^*|$ *and* $\hat{\mathbf{p}}$ *in* $\mathbb{V}_3^2$ We introduce a mixed energy:

$$\mathcal{E}_1(t):=\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}^2+\varepsilon^{-1}|\mathrm{p}_0-\mathrm{p}_0^*|^2+\varepsilon^{-2}\rho^{-10}\left\langle\mathbb{L}_{\mathbf{p}}w,w\right\rangle_{L^2}+\varepsilon^{-3}\rho^{-5}\|\mathbf{q}\|_{l^2}^2,$$

and a positive time-dependent function

$$A_1(t):=\varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}^2+\varepsilon^2|\mathrm{p}_0-\mathrm{p}_0^*|^2+\varepsilon^{-2}\rho^{-10}\|\mathbb{L}_{\mathbf{p}}w\|_{L^2}^2+\varepsilon^{-2}\rho^{-5}\|\mathbf{q}\|_{l^2}^2.$$

Combining the first estimate on w from Lemma 6.16, $\mathbf{q}$-estimate from Lemma 6.15, and the $\mathbb{V}_3^2$-estimate of $\hat{\mathbf{p}}$ from Lemma 5.15, and the $\|\dot{\mathbf{p}}\|_{l^2}$ bound (4.27) we find a revised positive constant c_* independent of ε,ρ,δ for which the $\mathcal{E}_1$-dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+c_*A_1(t)\lesssim B_1(t)+\varepsilon^5\rho^{-10}+\varepsilon^{-2}\|v^\perp\|_{L^2}^2+\varepsilon^{-5}\|\mathrm{N}(v^\perp)\|_{L^2}^2 \tag{4.28}$$

holds. Here we have introduced

$$\begin{aligned}B_1(t):=&\varepsilon^4(\delta^2+\varepsilon\rho^{-10}+\varepsilon^2\rho^{-2})\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2+\varepsilon^2(\delta^2+\varepsilon\rho^{-10})|\mathrm{p}_0-\mathrm{p}_0^*|^2\\&+\varepsilon^{-2}\rho^{-10}(\rho+\varepsilon\rho^{-8})\|\mathbb{L}_{\mathbf{p}}w\|_{L^2}^2+\varepsilon^{-2}\rho^{-5}(\varepsilon^2\rho^{-8}+\varepsilon^3\rho^{-9})\|\mathbf{q}\|_{l^2}^2.\end{aligned}$$

For $\rho\le 1$ there exists a constant C, depending only on system parameters, such that

$$B_1(t)\le C(\delta^2+\varepsilon\rho^{-10}+\rho)A_1(t). \tag{4.29}$$

This implies the existence of $\epsilon_1>0$, independent of ε,δ,ρ, such that for any combination of ε,δ,ρ that satisfies

$$\delta^2+\varepsilon\rho^{-10}+\rho<\epsilon_1, \tag{4.30}$$

the function $B_1(t)$ can be absorbed into the positive term $A_1(t)$ on the left-hand side of the $\mathcal{E}_1$-dissipation inequality. Particularly this can be achieved by choosing positive parameters δ,ρ small enough independent of ε, and $\varepsilon\in(0,\varepsilon_0)$ with ε_0 small enough, depending on ρ,δ. The bounds on the orthogonal perturbation $v^\perp$ and nonlinear terms $\mathrm{N}(v^\perp)$ from (4.26) and estimates on w, $\mathbf{q}$ from (4.22) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\frac{c_*}{2}A_1(t)\lesssim\varepsilon^5\rho^{-10}.$$

Since $\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}\le\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}$, we deduce that $A_1(t)\ge\varepsilon^4\mathcal{E}_1(t)$ and $\mathcal{E}_1$-dissipation inequality above reduces to the simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\frac{c_*}{2}\varepsilon^4\mathcal{E}_1(t)\lesssim\varepsilon^5\rho^{-10}.$$

Multiplying by $e^{c_*\varepsilon^4t/2}$ and integrating, we derive the uniform bound

$$\mathcal{E}_1(t)\le e^{-c_*\varepsilon^4t/2}\mathcal{E}_1(0)+C\varepsilon\rho^{-10}.$$

In view of the definition of $\mathcal{E}_1(0)$ and the initial bound, (4.19) and (4.22) with $t=0$, the right hand side is strictly less than $4\delta^2$ for ε_0 small enough depending on ρ,δ. Again from the definition of $\mathcal{E}_1(t)$, the first assumption in (**A**) holds uniformly on $[0, T)$, which combined with the l^2-bound of $\dot{\mathbf{p}}$ in (4.25) yields $T = \infty$.

Step 2. Decay estimates To obtain a decay estimate, we introduce a second mixed energy

$$\mathcal{E}_2(t) := \|\hat{\mathbf{p}}\|^2_{\mathbb{V}^2_2} + \varepsilon^{-1}|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \rho^{-10}\langle \mathbb{L}_{\mathbf{p}} w, w\rangle_{L^2} + \varepsilon^{-1}\rho^{-5}\|\mathbf{q}\|^2_{l^2} \tag{4.31}$$

and a time-dependent function

$$A_2(t) := \varepsilon^4\|\hat{\mathbf{p}}\|^2_{\mathbb{V}^2_4} + \varepsilon^2|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \rho^{-10}\|\mathbb{L}_{\mathbf{p}} w\|^2_{L^2} + \rho^{-5}\|\mathbf{q}\|^2_{l^2}. \tag{4.32}$$

From the definition of $\mathcal{E}_2$, the estimates (4.19) and (4.22) yield the initial bound

$$\mathcal{E}_2(0) < 4\delta^2. \tag{4.33}$$

Combining the second estimate on w from Lemma 6.16, the $\mathbf{q}$-estimate from Lemma 6.15, and the $\mathbb{V}^2_2$-estimate on $\hat{\mathbf{p}}$ from Lemma 5.14, yields a revised constant $c_* > 0$ independent of $\varepsilon, \rho, \delta$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t) + c_* A_2(t) \lesssim \varepsilon^{-1}\rho^{-10}\|\dot{\mathbf{p}}\|^2_{l^2} + \varepsilon\|v^{\perp}\|^2_{L^2} + \varepsilon^{-3}\|\mathrm{N}(v^{\perp})\|^2_{L^2}. \tag{4.34}$$

The remainder of *Step 2* follows the approach of *Step 1*. We employ the upper bound on $\dot{\mathbf{p}}$ from (4.27) and the estimates on $v^{\perp}$, $\mathrm{N}(v^{\perp})$ from (4.26) to eliminate these terms from the right-hand side so long as δ, ρ and ε are small enough satisfying (4.30). We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t) + \frac{c_*}{2}A_2(t) \le 0,$$

Since $\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2} \le \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}$, the coercivity of $\mathbb{L}_{\mathbf{p}}$ from (3.42) allows us to bound the positive term on the left-hand side from below, $A_2(t) \ge \varepsilon^4\mathcal{E}_2(t)$. This yields the $\mathcal{E}_2$-dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t) + \frac{c_*\varepsilon^4}{4}\mathcal{E}_2(t) + \frac{c_*}{4}A_2(t) \le 0.$$

Multiplying both sides by $e^{c_*\varepsilon^4 t/4}$ and integrating with respect to time from 0 to t yields

$$e^{c_*\varepsilon^4 t/4}\mathcal{E}_2(t) + \int_0^t e^{c_*\varepsilon^4\tau/4}A_2(t)\,\mathrm{d}\tau \le \mathcal{E}_2(0),$$

from which we deduce the asymptotic decay of the $\mathcal{E}_2$ on the ε^{-4} time-scale,

$$\mathcal{E}_2(t) \le e^{-c_*\varepsilon^4 t/2}\mathcal{E}_2(0) \qquad \forall t \in [0, \infty). \tag{4.35}$$

The decay estimates in (4.15)–(4.16) follow with a use of the first inequality in (4.23). Moreover the relaxation of the weighted norms is controlled by the initial energy,

$$\int_0^\infty e^{c_*\varepsilon^4\tau/4}A_2(\tau)\,\mathrm{d}\tau \le \mathcal{E}_2(0). \tag{4.36}$$

The bound (4.33) on $\mathcal{E}_2(0)$, and definition of A_2 in (4.34) yield the temporal estimate (4.18) for $k = 4$, which implies them for $k < 4$.

Step 3. Relaxation of the translation parameters The decay from *step 2* shows that the translation parameters converge to some equilibrium points close to their initial values. In fact for $k = 1, 2$ and any $t_1 \le t_2$ on $[0, \infty)$

$$|\mathrm{p}_k(t_2) - \mathrm{p}_k(t_1)| \le \int_{t_1}^{t_2} |\dot{\mathrm{p}}_k(\tau)|\,\mathrm{d}\tau \le \int_{t_1}^{t_2} \|\dot{\mathbf{p}}\|_{l^2}\,\mathrm{d}\tau,$$

which combined with (4.27) implies

$$|\mathrm{p}_k(t_2) - \mathrm{p}_k(t_1)| \lesssim \int_{t_1}^{t_2} \left(\varepsilon^3 |\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^4 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + \varepsilon^2 \rho^{-4} \|\mathbb{L}_{\mathbf{p}} w\|_{L^2} + \varepsilon^2 \rho^{-2} \|\mathbf{q}\|_{l^2} \right) \mathrm{d}\tau.$$

We use the weighted norm relaxation estimate (4.36) and Hölder's inequality to bound the right-hand side. The integral of the $|\mathrm{p}_0 - \mathrm{p}_0^*|$-term satisfies

$$\begin{aligned} \int_{t_1}^{t_2} \varepsilon^3 |\mathrm{p}_0 - \mathrm{p}_0^*| \,\mathrm{d}\tau &\le \varepsilon^3 \left(\int_{t_1}^{t_2} e^{c_* \varepsilon^4 \tau/4} |\mathrm{p}_0 - \mathrm{p}_0^*|^2 \,\mathrm{d}\tau \right)^{1/2} \left(\int_{t_1}^{t_2} e^{-c_* \varepsilon^4 \tau/4} \,\mathrm{d}\tau \right)^{1/2}, \\ &\lesssim \left(\int_0^\infty e^{c_* \varepsilon^4 \tau/4} \varepsilon^2 |\mathrm{p}_0 - \mathrm{p}_0^*|^2 \,\mathrm{d}\tau \right)^{1/2} e^{-c_* \varepsilon^4 t_1/8} \lesssim e^{-c_* \varepsilon^4 t_1/8} \mathcal{E}_2^{1/2}(0). \end{aligned}$$

The other terms have similar or better bounds and from (4.33) we obtain

$$|\mathrm{p}_k(t_2) - \mathrm{p}_k(t_1)| \lesssim e^{-c_* \varepsilon^4 t_1/8} \delta.$$

We deduce that $\mathrm{p}_{1,2}(t)$ converges to some unique equilibrium value $\mathrm{p}_{1,2}^*$ as time tends to ∞. Moreover, taking $t_1 = 0$, $t_2 = t$ yields

$$|\mathrm{p}_k(t) - \mathrm{p}_k(0)| \lesssim \delta, \qquad \forall t \in [0, \infty).$$

We conclude that $\mathrm{p}_{1,2}(t)$ stays in a $C\delta$-neighborhood of its initial datum for some positive constant independent of ε, ρ, δ. The proof is complete. □

4.3 Recovery of the Normal Velocity

The projection $\Pi_{\mathcal{M}_\delta}$ of an orbit $u = u(t)$ of the system (4.6) onto the bilayer manifold $\mathcal{M}_\delta$, defines the meander parameters and induces a normal velocity on the associated interface $\Gamma_{\mathbf{p}}(t)$. Some elements of this analysis are postponed to Sect. 5 to streamline the presentation. As indicated in Remark 5.4 and Eq. (5.28) of Corollary 5.8, the flow induced by the manifold projection, $\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}}$, is equivalent at leading order to a finite dimensional Galerkin projection of a geometric flow. More specifically, at leading order the flow satisfies

$$\Pi_{G_1} \left(\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}} \right) = 0, \tag{4.37}$$

with the velocity given by a rescaled, **p**-parmeterized version of V_{RCL}, (1.9),

$$V_{\mathbf{p}} := \varepsilon^3 \frac{m_0}{m_1^2} (\sigma_1^* - \sigma) \kappa_{\mathbf{p}} - \varepsilon^4 \left(\Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} + \frac{\kappa_{\mathbf{p}}^3}{2} + \alpha \kappa_{\mathbf{p}} \right), \tag{4.38}$$

and $\Pi_{G_1} : L^2(\mathscr{I}_{\mathbf{p}}) \mapsto L^2(\mathscr{I}_{\mathbf{p}})$ is the projection onto the Galerkin space $G_1 \subset L^2(\mathscr{I}_{\mathbf{p}})$ spanned by the first N_1 Laplace–Beltrami modes of $\Gamma_{\mathbf{p}}$,

$$\Pi_{G_1} f := \frac{1}{1 + \mathrm{p}_0} \sum_{j=0}^{N_1 - 1} \tilde{\theta}_j \int_{\mathscr{I}_{\mathbf{p}}} f(\tilde{s}_{\mathbf{p}}) \tilde{\theta}_j(\tilde{s}_{\mathbf{p}}) \,\mathrm{d}\tilde{s}_{\mathbf{p}}. \tag{4.39}$$

There are two sources of error that differentiate the flow induced by the manifold projection and that defined by $V_{\mathbf{p}}$. The first are the lower order terms in (4.37). The second is the fact that the conditions on ρ, in particular those imposed in (4.30) require that $\varepsilon^{-1/10} \ll \rho \ll 1$ and hence from the approximation (3.35) the dimension N_1 of the Galerkin expansion must reside in the tight range

$$\varepsilon^{-\frac{39}{40}} \ll N_1 \ll \varepsilon^{-1}. \tag{4.40}$$

Thus there is an ε-dependent Galerkin truncation error. The following result quantifies these errors.

Proposition 4.4 *There exists a $C > 0$, independent of ρ, δ and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\mathbf{p} \in \mathcal{O}^{\circ}_{2,\delta}$ the normal velocity satisfies the error bound*

$$\|\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \le C\left(\varepsilon^{9/2}\rho^{-4} + \varepsilon^4\delta\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right). \tag{4.41}$$

Proof From the triangle inequality we have the relation

$$\begin{aligned}\|\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \le \left\|\Pi_{G_1}\left(\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\right)\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \left\|\Pi^{\perp}_{G_1}\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}}\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} \\ + \left\|\Pi^{\perp}_{G_1} V_{\mathbf{p}}\right\|_{L^2(\mathscr{I}_{\mathbf{p}})},\end{aligned} \tag{4.42}$$

where $\Pi^{\perp}_{G_1} = \mathrm{I} - \Pi_{G_1}$ is the complement to the Galerkin projection onto $G_1 \subset L^2(\mathscr{I}_{\mathbf{p}})$, the space spanned by the first N_1 Laplace–Beltrami eigenmodes. For the first term, from (5.28) of Corollary 5.8 and Lemma 5.11 we find that

$$\begin{aligned}\left\|\Pi_{G_1}\left(\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\right)\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon^{-1}\|v^{\perp}\|_{L^2}\|\dot{\mathbf{p}}\|_{l^2} + (\varepsilon^{5/2} + \varepsilon^{5/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4})\|v^{\perp}\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^{\perp})\|_{L^2} \\ + \varepsilon^4|\sigma^*_1 - \sigma| + \varepsilon^5 + \varepsilon^5\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} + \varepsilon\|\dot{\mathbf{p}}\|_{l^2}.\end{aligned}$$

From the estimates (4.22)–(4.23) and the relation $\|v^{\perp}\|_{L^2} \le \|v^{\perp}\|_{H^2_{\mathrm{in}}}$ arising from the definition of H^2 inner norm, see (4.2), we bound the L^2-norm of $v^{\perp}$ and $\mathrm{N}(v^{\perp})$ as

$$\|v^{\perp}\|_{L^2} \lesssim \varepsilon^{5/2}\rho^{-2}, \qquad \|\mathrm{N}(v^{\perp})\|_{L^2} \lesssim \varepsilon^4\rho^{-4}.$$

With these estimates, (4.21) combined with (4.16), and $|\sigma^* - \sigma^*_1| \le \varepsilon$, then for ε_0 small enough depending on ρ, the first term on the right-hand side of (4.42) can be estimated as

$$\left\|\Pi_{G_1}\left(\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\right)\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon^{9/2}\rho^{-4} + \varepsilon^5\rho^{-2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} + \varepsilon\|\dot{\mathbf{p}}\|_{l^2}. \tag{4.43}$$

Bounding the last two terms in (4.42) requires a standard Sobolev estimate of the L^2 projection to the high frequency space $G^{\perp}_1$ in terms of H^1-norm of a function. In fact for any function $f = f(s_{\mathbf{p}}) \in H^1(\mathscr{I}_{\mathbf{p}})$,

$$\|\Pi^{\perp}_{G_1} f\|_{L^2(\mathscr{I}_{\mathbf{p}})} \le \beta^{-1}_{N_1}\|f\|_{H^1(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon\rho^{-1/4}\|f\|_{H^1(\mathscr{I}_{\mathbf{p}})}. \tag{4.44}$$

Here we used $\beta_{N_1} \sim N_1$ with N_1 bounded from below by (3.35). Applying the Sobolev estimate (4.44) twice to $\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}}$ and utilizing the identity (5.4) yields the bound

$$\|\Pi^{\perp}_{G_1}\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon^2\rho^{-1/2}\left\|\sum_{j\in\Sigma_1}\dot{\mathrm{p}}_j\xi_j\right\|_{H^2(\mathscr{I}_{\mathbf{p}})},$$

where (ξ_j), with its components given in Lemma 6.6, is bounded in $H^2(\mathscr{I}_{\mathbf{p}})$-norm, independent of ε for all $\mathbf{p} \in \mathcal{O}_{2,\delta}$. Hence for $\varepsilon \in (0, \varepsilon_0)$ and $\varepsilon_0 \ll \rho$ we deduce that

$$\|\Pi^{\perp}_{G_1}\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon\|\dot{\mathbf{p}}\|_{l^2}. \tag{4.45}$$

To bound the last term in (4.42), we deduce from the definition of $V_{\mathbf{p}}$ in (4.38) and triangle inequality that

$$\|\Pi^{\perp}_{G_1} V_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon^3|\sigma - \sigma^*_1|\|\Pi^{\perp}_{G_1}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \varepsilon^4\|\Pi^{\perp}_{G_1}\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \varepsilon^4\|\Pi^{\perp}_{G_1}\kappa^3_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})}.$$

From the Corollary 6.3 and the estimate (4.21), since $\mathbf{p} \in \mathcal{O}^\circ_{2,\delta}$ we have

$$\begin{aligned}\|\Pi^\perp_{G_1} V_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} &\left(\varepsilon^3|\sigma - \sigma_1^*| + \varepsilon^4\right) \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3} + \varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} \\ &\lesssim \varepsilon^{9/2}\rho^{-1/4} + \varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}.\end{aligned} \tag{4.46}$$

Returning the estimates (4.43) and (4.45)–(4.46) to (4.42) we obtain

$$\|\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \varepsilon^{9/2}\rho^{-4} + \varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} + \varepsilon\|\dot{\mathbf{p}}\|_{l^2}.$$

The l^2-bound of $\dot{\mathbf{p}}$ with $\mathbf{p} \in \mathcal{O}^\circ_{2,\delta}$ from (4.27) completes the estimation. □

Remark 4.5 The dominant source of error in comparing the exact and formal normal velocity arises from the truncation error in the Galerkin projection of the surface diffusion term $\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}$. For a general $\mathbf{p} \in \mathcal{O}^\circ_{2,\delta}$, the largest term in $V_{\mathbf{p}}$ is generically the surface diffusion term which scales like $\varepsilon^4\|\Delta_s\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \sim \delta\varepsilon^3$, while its Galerkin residual $\varepsilon^4\delta\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} \lesssim \varepsilon^3\delta\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3} \lesssim \delta^2\varepsilon^3$ is smaller. The $L^2(\mathscr{I}_{\mathbf{p}})$ norms of the other terms in $V_{\mathbf{p}}$ typically scale like $\varepsilon^{7/2}$ or ε^4.

5 Dynamics of the Meander Parameters

The dynamics of the meander parameter vector $\mathbf{p}$ and hence of the interface $\Gamma_{\mathbf{p}}$ is determined by the projection of the mass preserving gradient flow (1.10)–(1.11), equivalently (4.6), onto the slow meander space $\mathcal{Z}^1_*$, which approximates the tangent plane of the bilayer manifold. The dependence of the bilayer distributions, the slow space $\mathcal{Z}_*$, and the local coordinate $(z_{\mathbf{p}}, s_{\mathbf{p}})$ on $\mathbf{p}$ makes the analysis somewhat technical. We break the projection of the system into three subsections, characterizing the projection of $\partial_t\Phi_{\mathbf{p}}$, of the residual $\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})$, and of the remainder $\mathscr{R}[v^\perp]$ in turn. In the final two subsections the projection estimates are used to recover the evolution of $\mathbf{p}$ and deduce energy estimates for its relaxation back to equilbrium. The analysis is conducted on the time interval $t \in [0, T)$ for which $u(\cdot, t) \in \mathcal{V}_{\varepsilon^2\delta}(\mathcal{M}_\delta(\Gamma_0, M_0), \mathcal{O}_{2,\delta})$ and

$$\|\dot{\mathbf{p}}(t)\|_{l^2} \le 2\varepsilon^3, \qquad \mathbf{p}(t) \in \mathcal{O}_{2,\delta} \subset \mathcal{D}_\delta \qquad \text{and} \qquad \varepsilon^{-2}\|v^\perp\|_{H^2_{\mathrm{in}}}(t) \le \delta. \tag{5.1}$$

These assumptions are strong enough to validate the manifold projection of Lemma 4.2 and to resolve the leading-order dynamics of $\mathbf{p}$. However they are weaker than (4.20) and (4.24) enforced in Sect. 4, and hence the results of this section hold under the assumptions of Theorem 4.2.

5.1 Projection of $\partial_t\Phi_{\mathbf{p}}$

The projection of $\partial_t\Phi_{\mathbf{p}}$ onto the meander slow space $\mathcal{Z}^1_*$ involves the $N_1 \times N_1$ matrix $\mathbb{T}$ whose (k, j)th component is defined by

$$\mathbb{T}_{kj} := \left\langle \frac{\partial\Phi_{\mathbf{p}}}{\partial \mathrm{p}_j}, Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2} \qquad \text{for} k, j \in \Sigma_1. \tag{5.2}$$

This matrix can be viewed as an approximation of the first fundamental form of the bilayer manifold induced by graph of $\Phi_{\mathbf{p}}$ over the approximate tangent space $\mathcal{Z}^1_*$. The asymptotic form of $\partial_{\mathrm{p}_j}\Phi_{\mathbf{p}}$ follows from Lemmas 6.6 and 3.7. This is presented below.

Lemma 5.1 *For $j \in \Sigma_1$, the bilayer distribution $\Phi_{\mathbf{p}}$ given in Lemma 3.7 satisfies*

$$\frac{\partial \Phi_{\mathbf{p}}}{\partial \mathrm{p}_j} = \frac{1}{\varepsilon}(\phi_0' + \varepsilon\phi_1')\xi_j(s_{\mathbf{p}}) + \varepsilon \mathrm{R}_j.$$

Here $\xi_j(s_{\mathbf{p}}) = \varepsilon \frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j}$ depends on $\mathbf{p}$ and is given in Lemma 6.6. The remainder $\mathrm{R} = (\mathrm{R}_j)_{j=1}^{N_1}$ lies in $L^2(\mathbb{R}^{N_1})$ and it's projection to the meander slow space satisfies the estimate

$$\|\Pi_{\mathcal{Z}_*^1}\mathrm{R}\|_{L^2} \lesssim \varepsilon^{1/2}.$$

Proof This is a consequence of Lemma 6.4 of [5]. □

Introducing the canonical unit basis $\{\mathbf{B}_k\}_{k\in\Sigma_1}$ of $\mathbb{R}^{N_1}$, the chain rule and (5.2) lead to the expression

$$\langle \mathbb{T}\dot{\mathbf{p}}, \mathbf{B}_k\rangle = \left\langle \partial_t \Phi_{\mathbf{p}}, Z_{\mathbf{p},*}^{1k}\right\rangle_{L^2} \qquad \text{for } k \in \Sigma_1. \tag{5.3}$$

Up to a multiplicative constant, the leading order term of the the inner product on the right-hand side above has the leading-order expression

$$\int_{\mathscr{I}_{\mathbf{p}}} \partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} \tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}}.$$

From Lemma 6.6 and the chain rule we have the relation

$$\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} = -\varepsilon \frac{\partial z_{\mathbf{p}}}{\partial t} = -\sum_{j\in\Sigma_1} \dot{\mathrm{p}}_j \xi_j(s_{\mathbf{p}}). \tag{5.4}$$

The projection of $\xi_j(s_{\mathbf{p}})$ to the Galerkin space G_1, (4.39), involves an $(N_1 - 3) \times N_1$ matrix $\mathbb{U}$ with (j, k)th component given by

$$\mathbb{U}_{jk} := \frac{1}{1+\mathrm{p}_0}\int_{\mathscr{I}_{\mathbf{p}}} \tilde{s}_{\mathbf{p}} \tilde{\theta}_j' \tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}}, \qquad j = 3, 4, \cdots, N_1 - 1, \quad k \in \Sigma_1. \tag{5.5}$$

With this notation, the projection of ξ_j to G_1 has the following approximation, expressed component-wise in relation to a unit-norm matrix $\mathbb{E}$.

Lemma 5.2 *With $\xi_j(s_{\mathbf{p}})$ defined in Lemma 6.6, it holds that for $j, k \in \Sigma_1$*

$$-\frac{1}{(1+\mathrm{p}_0)}\int_{\mathscr{I}_{\mathbf{p}}} \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} = \begin{cases} 1/\theta_0 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\mathbb{E}_{00}, & j = 0, k = 0; \\ \mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} - \hat{\mathbf{p}}^T \mathbb{U} B_k + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{k0} & j = 0, k \geq 1; \\ \delta_{jk} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\mathbb{E}_{kj} & j = 1, 2, k \in \Sigma_1; \\ \delta_{jk} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{kj}, & j \geq 3, k \in \Sigma_1. \end{cases}$$

Moreover when $j \in \Sigma_1, k \geq 3$, we have the following approximation for the weighted projection

$$-\frac{1}{(1+\mathrm{p}_0)}\int_{\mathscr{I}_{\mathbf{p}}} \xi_j(s_{\mathbf{p}})\beta_k\tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} = \begin{cases} \beta_k \mathrm{p}_k - \hat{\mathbf{p}}^T \mathbb{U} B_k + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{k0} & j = 0; \\ \beta_k \delta_{jk} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\mathbb{E}_{kj} & j = 1, 2; \\ \beta_k \delta_{jk} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})\mathbb{E}_{kj} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\beta_j, & j \geq 3. \end{cases}$$

The proof of the Lemma is postponed to the "Appendix".

To complete the reduction of $\mathbb{T}$ we introduce the translation vector

$$\mathrm{p}_{1,2} = \mathrm{p}_1 \mathbf{B}_1 + \mathrm{p}_2 \mathbf{B}_2, \tag{5.6}$$

the constants m_2, m_3

$$m_2 = \frac{1}{2} \int_{\mathbb{R}} \mathrm{L}_0^{-1}(z\phi_0')\,\mathrm{d}z; \qquad m_3 = \frac{1}{2} \int_{\mathbb{R}} |z\phi_0'|^2\,\mathrm{d}z, \tag{5.7}$$

and the scalar function $\mu_0 = \mu_0(\sigma, \mathrm{p}_0)$,

$$\mu_0(\sigma, \mathrm{p}_0) := \frac{m_1^2}{m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2)} \frac{1}{1 + \mathrm{p}_0} = 1 + O(\mathrm{p}_0, \varepsilon). \tag{5.8}$$

Lemma 5.3 *Suppose the assumptions in (5.1) hold, there exists a unit vector $\boldsymbol{e} = (e_k)_{k \in \Sigma_1}$ for which (5.3) admits the expansion:*

1. if $k = 0$,

$$-\frac{\varepsilon^{1/2}\mu_0}{m_1} \langle \mathbb{T}\dot{\mathbf{p}}, \mathbf{B}_0 \rangle = \frac{1}{\theta_0}\dot{\mathrm{p}}_0 + O\left((\varepsilon^2 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}) \|\dot{\mathbf{p}}\|_{l^2} \right). \tag{5.9}$$

2. if $k \geq 1$,

$$-\frac{\varepsilon^{1/2}\mu_0}{m_1} \langle \mathbb{T}\dot{\mathbf{p}}, \mathbf{B}_k \rangle = \dot{\mathrm{p}}_k + \dot{\mathrm{p}}_0 \left(\mathrm{p}_k \mathbf{1}_{\{k \geq 3\}} - \hat{\mathbf{p}}^T \mathbb{U} \mathbf{B}_k \right) + O\left(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} |\dot{\mathrm{p}}_{1,2}| + (\varepsilon^2 + \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2}) \|\dot{\mathbf{p}}\|_{l^2} \right) e_k, \tag{5.10}$$

where $\mathbb{U}$ is given in (5.5) and the indicator function $\mathbf{1}_{\{k \geq j\}}(k)$ has value 1 for $k \geq j$ and 0 for $k < j$.

Proof Starting with the algebraic relation

$$\langle \mathbb{T}\dot{\mathbf{p}}, \mathbf{B}_k \rangle_{l^2} = \sum_{j \in \Sigma_1} \mathbb{T}_{kj} \dot{\mathrm{p}}_j, \tag{5.11}$$

we calculate $\mathbb{T}_{kj}$ for $k, j \in \Sigma_1$ and then sum over j. From the definition of ϕ_0, ϕ_1 in (3.1), (3.6), we have the following identity

$$\int_{\mathbb{R}} (\phi_0' + \varepsilon\phi_1')\phi_0'\,\mathrm{d}z = m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2), \tag{5.12}$$

where m_1, m_2, m_3 are defined in (1.8) and (5.7). Using the expressions for $\partial_{\mathrm{p}_j} \Phi_{\mathbf{p}}$ from Lemma 5.1 and for $Z^{1k}_{\mathbf{p},*}$ with $k \in \Sigma_1$ from (3.32), the equality $\psi_1 = \phi_0'/m_1$, the local expressions for the dressed eigenfunctions of L_0, and the orthogonality (3.33) we find

$$\mathbb{T}_{kj} = \frac{1}{\varepsilon^{1/2}} \int_{\mathbb{R}_{2\ell}} \int_{\mathscr{I}_{\mathbf{p}}} (\phi_0'(z_{\mathbf{p}}) + \varepsilon\phi_1'(z_{\mathbf{p}})) \frac{\phi_0'}{m_1} \xi_j(s_{\mathbf{p}}) \tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}} + O(\varepsilon^{3/2}\mathbb{E}_{kj}).$$

Here from Lemma 6.5 the remainder matrix $\mathbb{E}$ may be taken with unit l^2_* norm. From (5.12) we simplify this expression

$$\mathbb{T}_{kj} = \frac{m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2)}{\varepsilon^{1/2}} \int_{\mathscr{I}_{\mathbf{p}}} \xi_j(s_{\mathbf{p}}) \tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} + O(\varepsilon^{3/2}\mathbb{E}_{kj}). \tag{5.13}$$

To complete the proof for $k \geq 1$ we employ the three cases on j in Lemma 5.2 and return to the summation (5.11). The second identity of the Lemma then follows with the reduction of μ_0 from (5.8). The first identity, for $k = 0$, follows from similar arguments with the simplification that the terms involving $\mathbb{U}$ are small and placed in the remainder. □

Remark 5.4 By the definition of $\mathbb{T}_{kj}$, the identities (5.13) and (5.11) in the proof of Lemma 5.3 imply

$$\left\langle \partial_t \Phi_{\mathbf{p}}, Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} = \frac{m_1}{\varepsilon^{1/2}} \sum_{j\in\Sigma_1} \dot{\mathrm{p}}_j \int_{\mathscr{I}_{\mathbf{p}}} \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + O(\varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2})e_k.$$

When combined with identity (5.4) the result above implies

$$\left\langle \partial_t \Phi_{\mathbf{p}}, Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} = -\frac{m_1}{\varepsilon^{1/2}} \int_{\mathscr{I}_{\mathbf{p}}} \partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + O(\varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2})e_k. \tag{5.14}$$

Corollary 5.5 *With the same assumptions as Lemma 5.3, the β-weighted projection satisfies*

$$\begin{aligned} -\frac{\varepsilon^{1/2}\mu_0}{m_1} \langle \mathbb{T}\dot{\mathbf{p}}, \beta_k \mathbf{B}_k\rangle =& \beta_k \dot{\mathrm{p}}_k + \dot{\mathrm{p}}_0 \left(\beta_k \mathrm{p}_k - \hat{\mathbf{p}^T} \mathbb{U} \beta_k \mathbf{B}_k\right) + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2} \|\dot{\mathrm{p}}_{1,2}\|_{l^2})e_k \\ & O((\varepsilon^2\beta_k + \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3})\|\dot{\mathbf{p}}\|_{l^2})e_k + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}^2_2}\|\dot{\mathbf{p}}\|_{\mathbb{V}^2_1})e_k. \end{aligned}$$

for $k \geq 3$ and $\boldsymbol{e}$ a unit vector in $l^2(\mathbb{R}^{N_1-3})$.

Proof Following the proof of Lemma 5.3, we estimate $\beta_k \mathbb{T}_{kj}$. From (5.13), we obtain

$$\beta_k \mathbb{T}_{kj} = -\frac{m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2)}{\varepsilon^{1/2}} \int_{\mathscr{I}_{\mathbf{p}}} \beta_k \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + O(\varepsilon^{3/2}\beta_k \mathbb{E}_{kj}).$$

The corollary follows from Lemma 5.2. □

Lemma 5.6 *As defined in* (5.5)*, the matrix $\mathbb{U} = (\mathbb{U}_{lk})$ for $l, k = 3, \cdots N_1 - 1$ satisfies the norm bound*

$$\|\mathbb{U}^T\|_{l^*_2} \lesssim \|\mathbb{D}^{1/2}\|_{l^*_2}. \tag{5.15}$$

Proof Taking the inner product of $\mathbb{U}$ with $\hat{\mathbf{p}} = (\mathrm{p}_3, \cdots, \mathrm{p}_{N_1-1}) \in l^2$, and the canonical vector $\mathbf{B}_k$

$$\left\langle \mathbb{U}^T \hat{\mathbf{p}}, \mathbf{B}_k \right\rangle_{l^2} = \sum_{k=3}^{N_1-1} \mathbb{U}_{lk}\mathrm{p}_l = \frac{1}{1+\mathrm{p}_0} \int_{\mathscr{I}_{\mathbf{p}}} \bar{p}'(\tilde{s}_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}.$$

Here we recall $\bar{p} = \sum_{l=3}^{N_1-1} \mathrm{p}_l \tilde{\theta}(\tilde{s}_{\mathbf{p}})$, from (2.18) satisfies $\|\bar{p}'\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_1} = \|\mathbb{D}^{1/2}\hat{\mathbf{p}}\|_{l^2}$. The result then follows from Lemma 6.5. □

5.2 Projection of the Residual

The projection of the bilayer distribution residual $\mathrm{F}(\Phi_{\mathbf{p}})$ from Lemma 3.7 to the meander slow space $\mathcal{Z}^1_*$ drives the dynamics of meander parameters $\mathbf{p}$. This is characterized in the following Lemma. At leading order the projection arises from the normal velocity $V_{\mathbf{p}}$ defined in (4.38), to emphasize this we introduce the term-wise projections of the curvature and Willmore terms of $V_{\mathbf{p}}$,

$$
\begin{aligned}
V_k^M(\mathbf{p}) &:= \int_{\mathscr{I}_{\mathbf{p}}} \kappa_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}, \\
V_k^W(\mathbf{p}) &:= \int_{\mathscr{I}_{\mathbf{p}}} \left(-\Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} - \frac{\kappa_{\mathbf{p}}^3}{2} + \alpha \kappa_{\mathbf{p}} \right) \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}.
\end{aligned}
\tag{5.16}
$$

Lemma 5.7 *Under the assumptions in (5.1), there exists a unit vector $\boldsymbol{e} = (e_k)_{k\in\Sigma_1}$ such that:*

1. *For $k = 0$,*

$$
\begin{aligned}
\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} =& \varepsilon^{5/2} m_1 \left(\mathscr{A}(\mathrm{p}_0) - \frac{c_0}{\theta_0} \mathrm{p}_0 \right) + O\left(\varepsilon^{5/2} |\mathrm{p}_0 - \mathrm{p}_0^*| \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \right) \\
&+ O\left(\varepsilon^{7/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{9/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} \right),
\end{aligned}
$$

where $\mathscr{A}(\mathrm{p}_0)$ is a smooth function of p_0 and given by

$$
\begin{aligned}
\mathscr{A}(\mathrm{p}_0) :=& -\frac{2\pi\theta_0 m_0}{m_1^2} \left(\sigma_1^* - \sigma_0 \right) + 2\pi\theta_0 \varepsilon \left[\frac{1}{2(1+\mathrm{p}_0)^2} - \alpha(\sigma_1^*) \right] \\
&+ \varepsilon \mathcal{C}_1(\mathrm{p}_0)\mathrm{p}_0 + \varepsilon \mathcal{C}_2(\mathrm{p}_0)(\mathrm{p}_0 - \mathrm{p}_0^*) + \varepsilon^2 \mathcal{C}_3(\mathrm{p}_0).
\end{aligned}
$$

Here $\mathcal{C}_k(\mathrm{p}_0)$ for $k = 1, 2, 3$ are smooth functions of p_0 with uniform bounds independent of $\varepsilon \in (0, \varepsilon_0)$.

2. *For $k \geq 1$ and $k \in \Sigma_1$,*

$$
\begin{aligned}
\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} =& -\varepsilon^{5/2} m_1 c_k(\mathrm{p}_0) \mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} \\
&+ O(\varepsilon^{7/2} |\mathrm{p}_0 - \mathrm{p}_0^*| \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^{5/2} |\mathrm{p}_0 - \mathrm{p}_0^*| \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2}) e_k \\
&+ O(\varepsilon^{7/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon^{9/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) e_k.
\end{aligned}
$$

Here c_k depends only on p_0 and admits the form

$$
c_k(\mathrm{p}_0) := (\beta_k^2 - 1) \left[c_0(\mathrm{p}_0 - \mathrm{p}_0^*) + O(\varepsilon |\mathrm{p}_0 - \mathrm{p}_0^*|^2) + \varepsilon \left(-\frac{m_0}{m_1^2} \sigma_2^* + \frac{2\beta_k^2 - 3}{2(1+\mathrm{p}_0)^2} + \alpha(\sigma_1^*) \right) \right].
$$

Proof Adding and subtracting the far-field value of the residual and using the definition of the mass projection Π_0 and the decomposition of $Z^{1k}_{\mathbf{p},*}$, we break the projection in dominant and remainder terms

$$
\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} = \mathcal{I}_k + \mathscr{R}_k. \tag{5.17}
$$

The dominant term $\mathcal{I}_k$ and remainder $\mathscr{R}_k := \mathscr{R}_{k,1} + \mathscr{R}_{k,2}$ are defined as

$$
\begin{aligned}
\mathcal{I}_k &:= \int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty \right) Z^{1k}_{\mathbf{p}} \, \mathrm{d}x, \\
\mathscr{R}_{k,1} &:= \varepsilon \int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty \right) (\tilde{\varphi}_{1,k} \tilde{\theta}_k + \tilde{\varphi}_{2,k} \varepsilon \tilde{\theta}_k') \, \mathrm{d}x, \\
\mathscr{R}_{k,2} &:= \frac{1}{|\Omega|} \int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty \right) \mathrm{d}x \int_\Omega Z^{1k}_{\mathbf{p},*} \, \mathrm{d}x.
\end{aligned}
\tag{5.18}
$$

The approximations of the remainder terms $\mathscr{R}_{k,0}, \mathscr{R}_{k,1}$ are given in section 6.3. To approximate the dominant term $\mathcal{I}_k$ we replace $Z_{\mathbf{p}}^{1k}$ by its definition (3.34), and replace $\psi_1(z_{\mathbf{p}})$ with $\phi_0'(z_{\mathbf{p}})/m_1$. Rewriting $\mathcal{I}_k$ in the local coordinates we find

$$\mathcal{I}_k = \frac{\varepsilon^{1/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty)\phi_0'\tilde{\theta}_k(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}.$$

Using the expansion of $\mathrm{F}(\Phi_{\mathbf{p}})$ from Lemma 3.7, we decompose $\mathcal{I}_k = \sum_{j=1}^3 \mathcal{I}_{k,j}$, where

$$\begin{aligned}
\mathcal{I}_{k,1} &= \frac{\varepsilon^{5/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \left(\mathrm{F}_2 - \mathrm{F}_2^\infty\right)\phi_0'\tilde{\theta}_k(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},\\
\mathcal{I}_{k,2} &= \frac{\varepsilon^{7/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \left(\mathrm{F}_3 - \mathrm{F}_3^\infty\right)\phi_0'\tilde{\theta}_k(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},\\
\mathcal{I}_{k,3} &= \frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \left(\mathrm{F}_{\geq 4} - \mathrm{F}_{\geq 4}^\infty + e^{-\ell\nu/\varepsilon}\varepsilon^{-4}\mathrm{F}_e\right)\phi_0'\tilde{\theta}_k(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},
\end{aligned} \tag{5.19}$$

and address these terms one by one. First, from Lemma 3.7, F_2 has far-field value $\mathrm{F}_2^\infty = 0$, while both ϕ_0' and F_2 have odd parity in $z_{\mathbf{p}}$, we deduce

$$\mathcal{I}_{k,1} = \frac{\varepsilon^{5/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \mathrm{F}_2\phi_0'\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}.$$

Changing the integrating order and using (3.10) yields the reduction

$$\mathcal{I}_{k,1} = \frac{\varepsilon^{5/2}m_0}{m_1}(\sigma_1^* - \sigma)V_k^M(\mathbf{p}), \tag{5.20}$$

where the curvature projection V_k^M is defined in (5.16). For $\mathcal{I}_{k,2}$, F_3 has the projection (3.10) so that

$$\begin{aligned}
\mathcal{I}_{k,2} &= \varepsilon^{7/2}m_1V_k^W + \frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (\mathrm{F}_3 - \mathrm{F}_3^\infty)\phi_0'\tilde{\theta}_k z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},\\
&= \varepsilon^{7/2}m_1V_k^W + \frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (\phi_0'\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} + f_3 - f_3^\infty)\phi_0'\tilde{\theta}_k z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}.
\end{aligned}$$

Here $f_3 = f_3(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}) \in \mathcal{H}_2$ since it depends only on second and lower derivatives of $\boldsymbol{\gamma}_{\mathbf{p}}$. The function $|\phi_0'|^2 z_{\mathbf{p}}$ has odd parity and does not contribute to the integral. We deduce

$$\mathcal{I}_{k,2} = \varepsilon^{7/2}m_1V_k^W + \frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (f_3 - f_3^\infty)\phi_0'\tilde{\theta}_k z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}. \tag{5.21}$$

Addressing the integral, we change the order of integration order and integrate in $z_{\mathbf{p}}$, leaving a function $h = h(\boldsymbol{\gamma}_{\mathbf{p}}) \in \bar{\mathcal{H}}_2$ (see Definition 3.5). With this notation we deduce

$$\frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (f_3 - f_3^\infty)\phi_0'\tilde{\theta}_k z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}} = \frac{\varepsilon^{9/2}}{m_1}\int_{\mathscr{I}_{\mathbf{p}}} h(\boldsymbol{\gamma}_{\mathbf{p}})\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}}.$$

Applying Lemma 6.9 to bound the right-hand side of this identity and returning the estimate to $\mathcal{I}_{k,2}$ in (5.21) yields the asymptotic form

$$\mathcal{I}_{k,2} = \varepsilon^{7/2}m_1V_k^W + \varepsilon^{9/2}C(\mathrm{p}_0)\delta_{k0} + O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})e_k. \tag{5.22}$$

For $\mathcal{I}_{k,3}$ similar arguments yield the expansion

$$\mathcal{I}_{k,3} = \varepsilon^{9/2}C(\mathrm{p}_0)\delta_{k0} + O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_k. \tag{5.23}$$

Combining the estimates (5.20), (5.22), and (5.23) yields the asymptotic expansion

$$\mathcal{I}_k = \frac{\varepsilon^{5/2}m_0}{m_1}(\sigma_1^* - \sigma)V_k^M(\mathbf{p}) + \varepsilon^{7/2}m_1V_k^W(\mathbf{p}) + \varepsilon^{9/2}C(\mathrm{p}_0)\delta_{k0} + O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_k. \tag{5.24}$$

Rearranging and including the error estimates on $\mathscr{R}_{k,1}, \mathscr{R}_{k,2}$ from Lemma 6.11, we reduce (5.17) to

$$\begin{aligned}\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Z_{\mathbf{p},*}^{1k}\right\rangle_{L^2} =& \frac{\varepsilon^{5/2}m_0}{m_1}(\sigma_1^* - \sigma)\left(V_k^M(\mathbf{p}) + \varepsilon C_1(\mathrm{p}_0)\delta_{k0} + O(\varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\right)\\ &+ \varepsilon^{7/2}m_1V_k^W(\mathbf{p}) + \varepsilon^{9/2}C_2(\mathrm{p}_0)\delta_{k0} + O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_k.\end{aligned} \tag{5.25}$$

where C_1, C_2 are smooth functions of p_0. To reduce these expressions to their final forms we consider the cases $k = 0$ and $k \neq 0$ separately. For $k = 0$, using the form of V_0^M, V_0^W presented in Corollary 6.8, we rewrite the expansion above as

$$\begin{aligned}\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Z_{\mathbf{p},*}^{10}\right\rangle_{L^2} =& \frac{\varepsilon^{5/2}m_0}{m_1}(2\pi\theta_0 + \varepsilon C_1(\mathrm{p}_0))(\sigma - \sigma_1^*)\\ &+ \varepsilon^{7/2}m_1\pi\theta_0\left(\frac{1}{(1+\mathrm{p}_0)^2} - \alpha\right) + \varepsilon^{9/2}C(\mathrm{p}_0)\\ &+ O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{5/2}|\sigma - \sigma^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}).\end{aligned}$$

Here we used $\sigma_1^* = \sigma^* + O(\varepsilon)$ to simplify the error term. The coefficient $\alpha = \alpha(\sigma)$ is smooth and affords the expansion

$$\alpha(\sigma) = \alpha(\sigma_1^*) + O(\varepsilon, |\sigma - \sigma^*|). \tag{5.26}$$

Using the first expansion of Corollary 3.10 and deducing that $|\sigma - \sigma^*| \lesssim |\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$ from the second expansion, we arrive at the expression

$$\begin{aligned}\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Z_{\mathbf{p},*}^{10}\right\rangle_{L^2} =& \varepsilon^{5/2}m_1\left(\mathscr{A}(\mathrm{p}_0) - \frac{c_0}{\theta_0}\mathrm{p}_0\right)\\ &+ O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{5/2}|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}).\end{aligned} \tag{5.27}$$

Here $\mathscr{A} = \mathscr{A}(\mathrm{p}_0)$ takes the form presented in part 1 of Lemma 5.7. For the case $k \neq 0$, we replace σ_1^* with $(\sigma^* - \varepsilon\sigma_{\geq 2}^*)$, and reduce (5.25) to

$$\begin{aligned}\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Z_{\mathbf{p},*}^{1k}\right\rangle_{L^2} =& \frac{\varepsilon^{5/2}m_0}{m_1}(\sigma^* - \sigma)V_k^M + \varepsilon^{7/2}\left(V_k^W - \frac{m_0}{m_1^2}\sigma_{\geq 2}^*V_k^M\right)\\ &+ O(\varepsilon^{7/2}|\sigma^* - \sigma|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_k\end{aligned}$$

Using this expansion, together with the expansions of V_k^M, V_k^W from Corollary 6.8, and the bound $|\sigma - \sigma^*| \lesssim |\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$ and the expansion of α from (5.26) yields part 2 of Lemma 5.7 with $c_k = c_k(\mathrm{p}_0)$ as defined therein. □

Collecting the results on the projection of $\partial_t\Phi_{\mathbf{p}}$ and of the residual $\mathrm{F}(\Phi_{\mathbf{p}})$ we can bound the Galerkin meander projection of the difference between the normal velocity of $\Gamma_{\mathbf{p}}$ and the curvature induced velocity defined in (4.38).

Corollary 5.8 *There exists a unit vector* $\boldsymbol{e} \in l^2(\mathbb{R}^{N_1})$ *such that the Galerkin projections of the normal and curvature induced velocity,*(4.38)*, satisfy*

$$\int_{\mathscr{I}_{\mathbf{p}}} \left(\partial_t \boldsymbol{\gamma}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} - V_{\mathbf{p}}\right) \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = -\frac{\varepsilon^{1/2}}{m_1} \langle \mathscr{R}[v^\perp], Z^{1k}_{\mathbf{p},*} \rangle >_{L^2} \\ + O(\varepsilon^4 |\sigma_1^* - \sigma|, \varepsilon^5, \varepsilon^5 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon \|\dot{\mathbf{p}}\|_{l^2}) e_k, \quad (5.28)$$

for $k \in \Sigma_1$.

Proof From (5.25) we have

$$\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} = \frac{\varepsilon^{5/2} m_0}{m_1} (\sigma_1^* - \sigma) V_k^M(\mathbf{p}) + \varepsilon^{7/2} m_1 V_k^W(\mathbf{p}) + O(\varepsilon^{7/2}(\sigma_1^* - \sigma), \varepsilon^{9/2}) \delta_{k0} \\ + O(\varepsilon^{7/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^{9/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) e_k$$

with V_k^M, V_k^W defined in (5.16). The result follows by combining this with (5.14), multiplying by $-\varepsilon^{1/2}/m_1$, and using the formulation (4.6) of the gradient flow to replace $\partial_t \Phi_{\mathbf{p}} + \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}})$ with $\mathscr{R}[v^\perp]$. □ The meander parameter $\mathrm{p}_0 = \mathrm{p}_0(t)$ controls the length of the interface $\Gamma_{\mathbf{p}}$. Its equilibrium value p_0^* satisfies

$$\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*} \right\rangle_{L^2} \Big|_{\mathbf{p}=\mathbf{p}^*} = 0 \quad (5.29)$$

for $\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_1^*, \mathrm{p}_2^*, \mathbf{0})$.

Lemma 5.9 *Suppose* $|\sigma_0(\Gamma_0, M_0) - \sigma_1^*| \lesssim 1$, *then there exists* ε_0 *small enough such that for each* $\varepsilon \in (0, \varepsilon_0)$ p_0^* *is well defined through* (5.29) *and admits the approximation* $\mathrm{p}_0^* = \mathrm{p}_{0,0}^* + \varepsilon \mathrm{p}_{0,1}^* + O(\varepsilon^2)$ *with*

$$\begin{aligned} \mathrm{p}_{0,0}^* &= -\frac{m_0}{c_0 m_1^2} \left(\sigma_1^* - \sigma_0(\Gamma_0, M_0)\right); \\ \mathrm{p}_{0,1}^* &= \frac{1}{c_0} \left(\frac{1}{2(1+\mathrm{p}_{0,0}^*)^2} - \alpha(\sigma_1^*) \right) + \frac{\theta_0}{c_0} \mathcal{C}_1(\mathrm{p}_{0,0}^*) \mathrm{p}_{0,0}^*. \end{aligned} \quad (5.30)$$

Proof From part 1 of Lemma 5.7 and (5.29), p_0^* solves

$$c_0 \mathrm{p}_0 / \theta_0 = \mathscr{A}(\mathrm{p}_0). \quad (5.31)$$

Here $\mathscr{A}$ can be decomposed as $\mathscr{A}(\mathrm{p}_0) = \mathscr{A}_0 + \varepsilon \mathscr{A}_1(\mathrm{p}_0)$ with

$$\begin{aligned} \mathscr{A}_0 &:= -\frac{m_0}{m_1^2} 2\pi \theta_0 \left(\sigma_1^* - \sigma_0(\Gamma_0, M_0)\right), \\ \mathscr{A}_1(\mathrm{p}_0) &:= 2\pi\theta_0 \left(\frac{1}{2(1+\mathrm{p}_0)^2} - \alpha(\sigma_1^*) \right) + \mathcal{C}_1(\mathrm{p}_0)\mathrm{p}_0 + \mathcal{C}_2(\mathrm{p}_0)(\mathrm{p}_0 - \mathrm{p}_0^*) + \varepsilon \mathcal{C}_3(\mathrm{p}_0), \end{aligned} \quad (5.32)$$

where the base bulk density σ_0 has the form (3.15). The system (5.31) is linear in p_0 for $\varepsilon = 0$, and hence has a unique solution. The smooth continuation of this unique solution, for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough, is a simple application of the implicit function theorem since $\mathscr{A}_1$ is smooth in p_0. Recalling that $\theta_0 = 1/\sqrt{2\pi}$, a regular perturbation expansion $\mathrm{p}_0^* = \mathrm{p}_{0,0}^* + \varepsilon \mathrm{p}_{0,1}^* + O(\varepsilon^2)$, yields (5.30). □

From Lemma 3.8, the equilibrium bulk density $\sigma^* = \sigma(\mathbf{p}^*)$ with $\mathbf{p}^* = (\mathrm{p}_0^*, \mathrm{p}_1^*, \mathrm{p}_2^*, \mathbf{0})$ depends only on the meander length parameter p_0^* through

$$\sigma^* = \sigma_0 - \frac{c_0 m_1^2}{m_0}\mathrm{p}_0^* + \varepsilon \mathrm{p}_0^* \mathcal{C}(\mathrm{p}_0^*). \tag{5.33}$$

From the approximation of p_0^* from Lemma 5.9 , we find the expansion $\sigma^* = \sigma_1^* + \varepsilon\sigma_2^* + O(\varepsilon^2)$ with

$$\sigma_2^* = -\frac{c_0\theta_0 m_1^2}{m_0}\mathrm{p}_{0,1}^* = \frac{m_1^2}{m_0}\left(\frac{-1}{2(1+\mathrm{p}_{0,0}^*)^2} + \alpha(\sigma_1^*) + \mathrm{p}_{0,0}^*\mathcal{C}(\mathrm{p}_{0,0}^*)\right). \tag{5.34}$$

These relations give the map from the system mass to a unique, up to translation, equilibrium profile with parameters $(\sigma, \mathrm{p}_0, \hat{\mathbf{p}}) = (\sigma^*, \mathrm{p}_0^*, \mathbf{0})$. Returning to (5.31) we may expand $\mathscr{A}(\mathrm{p}_0)$ around p_0^* to write the residual in the form

$$\mathscr{A}(\mathrm{p}_0) - \frac{c_0}{\theta_0}\mathrm{p}_0 = \frac{c_0}{\theta_0}(\mathrm{p}_0^* - \mathrm{p}_0) + O(\varepsilon)(\mathrm{p}_0^* - \mathrm{p}_0). \tag{5.35}$$

This allows the reformulation of the projection of the residual $\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})$ onto $Z_{\mathbf{p},*}^{10}$, given in part 1 of Lemma 5.7, in terms of the small quantities $\mathrm{p}_0 - \mathrm{p}_0^*$ and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$,

$$\begin{aligned}\int_\Omega \Pi_0\mathrm{F}(\Phi_{\mathbf{p}})Z_{\mathbf{p},*}^{10}\,\mathrm{d}x =& \varepsilon^{5/2}\frac{m_1c_0}{\theta_0}(\mathrm{p}_0^* - \mathrm{p}_0) + O(\varepsilon^{7/2}|\mathrm{p}_0 - \mathrm{p}_0^*|)\\ &+ O\left(\varepsilon^{5/2}|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right).\end{aligned} \tag{5.36}$$

Using (5.34) to eliminate σ_2^* and the a priori assumption $|\mathrm{p}_0(t)| \ll 1$ for all $t \in [0, T)$, we rewrite c_k as,

$$c_k(\mathrm{p}_0) = c_0(\mathbb{D}_{kk} - 1)(\mathrm{p}_0 - \mathrm{p}_0^*) + \varepsilon(\mathbb{D}_{kk} - 1)^2 + O(\varepsilon|\mathrm{p}_0|\mathbb{D}_{kk}), \tag{5.37}$$

where $\mathbb{D}$ is the diagonal matrix defined in (2.6) that induces the norm $\mathbb{V}_2^2$.

We estimate weighted projections by absorbing the factors of β_k into higher $\mathbb{V}_j^2$ norms of $\mathbf{p}$. This requires modifications of these error terms.

Corollary 5.10 *It holds that the weighted projections*

$$\begin{aligned}\int_\Omega \Pi_0\mathrm{F}(\Phi_{\mathbf{p}})\beta_k Z_{\mathbf{p},*}^{1k}\,\mathrm{d}x =& -\varepsilon^{5/2}m_1c_k\beta_k\mathrm{p}_k + O(\varepsilon^{5/2}|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})e_k\\ &+ O(\varepsilon^{7/2}|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon^{7/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}, \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2})e_k\end{aligned}$$

for $k \geq 3$.

Proof Multiplying (5.17) by β_k yields

$$\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Z_{\mathbf{p},*}^{1k}\right\rangle_{L^2} = \beta_k\mathcal{I}_k + \beta_k\mathscr{R}_k,$$

with $\beta_k\mathscr{R}_k = \beta_k\mathscr{R}_{k,1} + \beta_k\mathscr{R}_{k,2}$ given in Lemma 6.14. We focus on $\beta_k\mathcal{I}_k$ and which we expand as $\beta_k\mathcal{I}_{k,1} + \beta_k\mathcal{I}_{k,2} + \beta_k\mathcal{I}_{k,3}$, given in (5.19). Utilizing (5.20) and (5.21), we have

$$\begin{aligned}\beta_k\mathcal{I}_k =& \frac{\varepsilon^{5/2}m_0}{m_1}(\sigma_1^* - \sigma)\beta_k V_k^M(\mathbf{p}) + \varepsilon^{7/2}m_1\beta_k V_k^W(\mathbf{p})\\ &+ \frac{\varepsilon^{9/2}}{m_1}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}}(f_3 - f_3^\infty)\phi_0'\beta_k\tilde{\theta}_k z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}} + \beta_k\mathcal{I}_{k,3}.\end{aligned}$$

Integrating in $z_{\mathbf{p}}$ and recalling the form of $f_3 = f_3(z_{\mathbf{p}}; \boldsymbol{\gamma}_{\mathbf{p}})$, then Lemma 6.12 and Eq. (6.12) with $l = 1$ allow us to rewrite and estimate the integral on the second line of the equality above as

$$\varepsilon^{9/2} \int_{\mathscr{I}_{\mathbf{p}}} h(\boldsymbol{\gamma}_{\mathbf{p}}) \beta_k \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\varepsilon^{9/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}) e_k$$

The bounds on $\beta_k \mathcal{I}_{k,3}$ can be achieved by similar arguments that exploit the form of $\mathrm{F}_{\geq 4}$ and F_e. These terms collectively contribute an error of order $O(\varepsilon^{9/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2})$. The remainder of the reduction follows the lines of the proof of Lemma 5.7, using Lemmas 6.13 and 6.14. The details are omitted. □

5.3 Projection of the Orthogonal Remainder $\mathscr{R}[v^\perp]$

The orthogonal remainder $\mathscr{R}[v^\perp]$ appearing in (4.6) is induced by the meander-orthogonal perturbation $v^\perp$ and lifts the solution u off of the bilayer manifold $\mathcal{M}_\delta$. In this section we estimate its projection, which requires control of the impact of the flow on the meander basis functions $Z_{\mathbf{p},*}^{kj}$ defined in (3.32) for $k = 0, 1$ and $j \in \Sigma_k$. Under the assumptions that hold here, Lemma 6.5 of [5] establishes the bounds

$$\|\partial_t Z_{\mathbf{p},*}^{I(j)j}\|_{L^2} \lesssim \varepsilon^{-1} \|\dot{\mathbf{p}}\|_{l^2}, \qquad \forall j \in \Sigma, \tag{5.38}$$

where the indicator function I is from (3.37). This bound helps estimate the residual projections.

Lemma 5.11 *Under the assumptions in (5.1), there exists a unit vector $\boldsymbol{e} = (e_k)_{k=0}^{N_1 - 1}$ such that the projection of the orthogonal remainder $\mathscr{R}[v^\perp]$, defined in (4.7), to the meander space $\mathcal{Z}_*^1$ satisfies the bound*

$$\left\langle \mathscr{R}[v^\perp], Z_{\mathbf{p},*}^{1k} \right\rangle = O\left(\varepsilon^{-3/2} \|v^\perp\|_{L^2} \|\dot{\mathbf{p}}\|_{l^2}, (\varepsilon^2 + \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|v^\perp\|_{L^2}, \|\mathrm{N}(v^\perp)\|_{L^2}\right) e_k,$$

and the weighted estimate

$$\left\langle \mathscr{R}[v^\perp], \beta_k Z_{\mathbf{p},*}^{1k} \right\rangle = O\left(\varepsilon^{-5/2} \|v^\perp\|_{L^2} \|\dot{\mathbf{p}}\|, (\varepsilon^{3/2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2} + \varepsilon^{1/2}) \|v^\perp\|_{L^2}, \varepsilon^{-1} \|\mathrm{N}(v^\perp)\|_{L^2}\right) e_k$$

for $k \geq 3$ and $k \in \Sigma_1$.

Proof We break the orthogonal remainder into its three components as presented in (4.7), and rewrite the projection of $\partial_t v^\perp$ as

$$\left\langle \partial_t v^\perp, Z_{\mathbf{p},*}^{1k} \right\rangle_{L^2} = \partial_t \left\langle v^\perp, Z_{\mathbf{p},*}^{1k} \right\rangle_{L^2} + \left\langle v^\perp, \partial_t Z_{\mathbf{p},*}^{1k} \right\rangle_{L^2}.$$

The first item on the right-hand side is zero since $v^\perp$ is perpendicular to the meander slow space $\mathcal{Z}_*^1$; and the second term is bounded through Hölder's inequality and estimate (5.38). Combining these, we deduce

$$\left\langle \partial_t v^\perp, Z_{\mathbf{p},*}^{1k} \right\rangle_{L^2} = O(\varepsilon^{-1} \|\dot{\mathbf{p}}\|_{l^2} \|v^\perp\|_{L^2}), \qquad k \in \Sigma_1, \tag{5.39}$$

which together with the upper bound on the meander dimension $N_1 \leq \varepsilon^{-1}$ and approximate orthogonality of (3.37) we deduce

$$\|\Pi_{\mathcal{Z}_*^1} \partial_t v^\perp\|_{L^2} \lesssim \varepsilon^{-1} N_1^{1/2} \|\dot{\mathbf{p}}\|_{l^2} \|v^\perp\|_{L^2} \lesssim \varepsilon^{-3/2} \|\dot{\mathbf{p}}\|_{l^2} \|v^\perp\|_{L^2}.$$

To bound the projection of the second term, $\Pi_0\mathbb{L}_{\mathbf{p}}v^{\perp}$, of the orthogonal remainder, we turn to Lemma 3.13. This yields the estimate

$$\left\|\Pi_{\mathcal{Z}^1_*}\Pi_0\mathbb{L}_{\mathbf{p}}v^{\perp}\right\|_{L^2} \lesssim (\varepsilon^2+\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4})\|v^{\perp}\|_{L^2}. \tag{5.40}$$

The third term in the orthogonal residual expansion is the nonlinearity, $\mathrm{N}(v^{\perp})$, which enjoys the simple bound

$$\|\Pi_{Z^1_*}\Pi_0\mathrm{N}(v^{\perp})\|_{L^2} \lesssim \|\mathrm{N}(v^{\perp})\|_{L^2}. \tag{5.41}$$

The unweighted estimate of Lemma 5.11 follows. To derive the weighted estimate we observe that the terms β_k are uniformly bounded by ε^{-1} for $k\in\Sigma_1$, so that the unweighted estimate immediately implies

$$\left\langle \mathscr{R}[v^{\perp}], \beta_k Z^{1k}_{\mathbf{p},*}\right\rangle = O\left(\varepsilon^{-2}\|v^{\perp}\|_{L^2}\|\dot{\mathbf{p}}\|, (\varepsilon+\varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4})\|v^{\perp}\|_{L^2}, \varepsilon^{-1}\|\mathrm{N}(v^{\perp})\|_{L^2}\right) e_k.$$

From the definition of $\mathbb{V}^2_k$, and Young's inequality we have the embedding estimate

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} \le \|\hat{\mathbf{p}}\|^{1/2}_{\mathbb{V}^2_3}\|\hat{\mathbf{p}}\|^{1/2}_{\mathbb{V}^2_5} \lesssim \varepsilon^{1/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_5}+\varepsilon^{-1/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}.$$

The weighted estimate follows by the a priori bound on $\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}$ for $\mathbf{p}\in\mathcal{O}_{2,\delta}$. □

5.4 Dynamics of the Meander Parameter Vector p

The results of Lemmas 5.3, 5.7 and 5.11 provide a detailed description of the dynamics of the meander parameter vector $\mathbf{p}$ as induced by the gradient flow. For simplicity of presentation, we introduce two time-dependent functions

$$E(t) := \varepsilon+\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}+|\mathrm{p}_0|+\varepsilon^{-3/2}\|v^{\perp}\|_{L^2};\quad E_w(t) := \varepsilon+\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}+|\mathrm{p}_0|+\varepsilon^{-2}\|v^{\perp}\|_{L^2}. \tag{5.42}$$

It is immediate that $E(t)\le E_w(t)$ for any $t\in\mathbb{R}^+$. The assumptions in (5.1) make these two quantities small in $L^\infty(\mathbb{R}^+)$, explicitly they imply

$$E(t)\lesssim\delta,\qquad E_w(t)\lesssim\delta, \tag{5.43}$$

which allow the extraction of the main flow of $\mathbf{p}$, as presented below.

Theorem 5.1 *Suppose that the assumptions in (5.1) hold, then there exists a positive constant ϵ_1 independent of ε, ρ with the following property. If $\delta\le\epsilon_1$, then the meander parameter vector $\mathbf{p}=(\mathrm{p}_0,\mathrm{p}_{1,2},\hat{\mathbf{p}})$ evolves according to*

$$\begin{aligned}
\dot{\mathrm{p}}_0 &= -\varepsilon^3 c_0(\mathrm{p}_0-\mathrm{p}_0^*)+d_0,\\
\dot{\mathrm{p}}_k &= O\left(\varepsilon^3|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}\right)+d_k \quad \text{for} \quad k=1,2,\\
\dot{\hat{\mathbf{p}}} &= -\varepsilon^3\left[c_0(\mathbb{D}+\mathbb{U}^T)(\mathrm{p}_0-\mathrm{p}_0^*)+\varepsilon(\mathbb{D}-\mathbb{I})^2\right]\hat{\mathbf{p}}+\hat{d}.
\end{aligned} \tag{5.44}$$

Here $\mathbb{I}$ is the $(N_1-3)\times(N_1-3)$ identity matrix, and $\mathbb{U}$ is defined in (5.5). The vector $d=(d_0,d')$ with $d'=(d_1,d_2,\hat{d})$ has components which are error terms that satisfy

$$\begin{aligned}
|d_0| &\lesssim \varepsilon^3E(t)|\mathrm{p}_0-\mathrm{p}_0^*|+\varepsilon^4E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}+\varepsilon^{5/2}\|v^{\perp}\|_{L^2}+\varepsilon^{1/2}\|\mathrm{N}(v^{\perp})\|_{L^2};\\
\|d'\|_{l^2} &\lesssim \varepsilon^3E(t)|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}+\varepsilon^4(\varepsilon+\varepsilon^{-2}\|v^{\perp}\|_{L^2})|\mathrm{p}_0-\mathrm{p}_0^*|+\varepsilon^4E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\\
&\quad+\varepsilon^{5/2}\|v^{\perp}\|_{L^2}+\varepsilon^{1/2}\|\mathrm{N}(v^{\perp})\|_{L^2}.
\end{aligned}$$

Moreover the rate of change $\dot{\mathbf{p}} = (\dot{\mathrm{p}}_0, \dot{\mathrm{p}}_{1,2}, \dot{\hat{\mathbf{p}}})$ *admits the upper bound*

$$\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^3 |\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^4 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + |d_0| + \|d'\|_{l^2}. \tag{5.45}$$

Proof Projecting Eq. (4.6) onto $Z^{1k}_{\mathbf{p},*}$ in L^2, using identity (5.3) yields

$$\langle \mathbb{T}\dot{\mathbf{p}}, \mathbf{B}_k\rangle_{l^2} = -\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2} + \left\langle \mathscr{R}[v^\perp], Z^{1k}_{\mathbf{p},*}, \right\rangle_{L^2}, \tag{5.46}$$

where $\mathscr{R}[v^\perp]$ defined in (4.7) is the orthogonal remainder contributed by $v^\perp$. We first address the case $k = 0$. Multiplying the identity (5.46) with $k = 0$ by $-\varepsilon^{1/2}\frac{\mu_0}{m_1}\theta_0$ and applying Lemmas 5.3 and identity (5.36) yields the ODE

$$\dot{\mathrm{p}}_0 = \varepsilon^3 c_0(\mathrm{p}_0 - \mathrm{p}_0^*) + d_0, \tag{5.47}$$

where the remainder d_0 satisfies the bound

$$\begin{aligned}|d_0| \lesssim & \varepsilon^3 \left(\varepsilon + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} + |\mu_0 - 1|\right) |\mathrm{p}_0 - \mathrm{p}_0^*| + (\varepsilon^2 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\|\dot{\mathbf{p}}\|_{l^2} + \varepsilon^4 \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} + \varepsilon^5\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\\ & + \varepsilon^{1/2}\left|\left\langle \mathscr{R}[v^\perp], Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2}\right|.\end{aligned}$$

Recalling the quantity $\mu_0 = 1 + O(|\mathrm{p}_0|, \varepsilon)$, introduced in (5.8), and the first estimate of the projection of the orthogonal remainder from Lemma 5.11, we introduce $E(t)$ in (5.42) and obtain the simplified bound

$$|d_0| \lesssim \varepsilon^3 E(t)|\mathrm{p}_0 - \mathrm{p}_0^*| + E(t)\|\dot{\mathbf{p}}\|_{l^2} + \varepsilon^4 E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + \varepsilon^{5/2}\|v^\perp\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2}. \tag{5.48}$$

For $k \geq 1$, we multiply identity (5.46) by $-\varepsilon^{1/2}\mu_0/m_1$ and apply Lemmas 5.3 and 5.7 to deduce

$$\dot{\mathrm{p}}_k = -\dot{\mathrm{p}}_0\left(\mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} - \hat{\mathbf{p}}^T \mathbb{U}\mathbf{B}_k\right) - \varepsilon^3 \mu_0 c_k \mathrm{p}_k + d_k. \tag{5.49}$$

The remainder $d' = (d_k)_{k=1}^{N_1-1}$ can be bounded in l^2 by collecting the remainder estimates from Lemmas 5.3 and 5.7 and applying Lemma 5.11,

$$\begin{aligned}\|d'\|_{l^2} \lesssim & \varepsilon^3 E(t)|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} + \varepsilon^4 E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + \left(\varepsilon^2 + \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} + \varepsilon^{-1}\|v^\perp\|_{L^2}\right)\|\dot{\mathbf{p}}\|_{l^2}\\ & + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\dot{\mathrm{p}}_{1,2}\|_{l^2} + \varepsilon^{5/2}\|v^\perp\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2}.\end{aligned} \tag{5.50}$$

Replacing $\dot{\mathrm{p}}_0$ on the right-hand side of (5.49) with the right-hand side of (5.47), again using the expansion $\mu_0 = 1 + O(|\mathrm{p}_0|, \varepsilon)$, and replacing c_k with (5.37) yields

$$\begin{aligned}\dot{\mathrm{p}}_k = & - c_0(\mathrm{p}_0 - \mathrm{p}_0^*)\left(\mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} - \hat{\mathbf{p}}^T \mathbb{U}\mathbf{B}_k\right) - \varepsilon^3 c_k \mathrm{p}_k + \tilde{d}_k,\\ = & - \varepsilon^3\left[c_0(\mathrm{p}_0 - \mathrm{p}_0^*)\mathbb{D}_{kk} + \varepsilon(\mathbb{D}_{kk} - 1)^2\right]\mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} + c_0(\mathrm{p}_0 - \mathrm{p}_0^*)\hat{\mathbf{p}}\mathbb{U}\mathbf{B}_k + \tilde{d}_k.\end{aligned} \tag{5.51}$$

Here we have introduced the revised error term

$$\tilde{d}_k = d_k + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} d_0, \varepsilon^3(|\mathrm{p}_0| + \varepsilon)|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^4(|\mathrm{p}_0| + \varepsilon)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) e_k.$$

In the remainder of the proof we omit the tilde on d_k. From (5.50) the revised version of $d' = (d_k)_{k=1}^{N_1-1}$ enjoys the l^2-bound

$$\begin{aligned}\|d'\|_{l^2} \lesssim & \varepsilon^3 E(t)|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}|d_0| + \varepsilon^4 E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + \left(\varepsilon^2 + \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2}\right.\\ & \left. + \varepsilon^{-1}\|v^\perp\|_{L^2}\right)\|\dot{\mathbf{p}}\|_{l^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\dot{\mathrm{p}}_{1,2}\|_{l^2} + \varepsilon^{5/2}\|v^\perp\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2}.\end{aligned} \tag{5.52}$$

The bounds (5.48) and (5.52) are not yet closed since they depend on $\dot{\mathbf{p}}$, however from (5.44) we derive

$$\begin{aligned} |\dot{\mathrm{p}}_0| &\lesssim \varepsilon^3|\mathrm{p}_0-\mathrm{p}_0^*|+|d_0|,\\ \|\dot{\hat{\mathbf{p}}}\|_{l^2} &\lesssim \varepsilon^3|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}+\varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}+\|d'\|_{l^2},\\ |\dot{\mathrm{p}}_{1,2}| &\lesssim \varepsilon^3|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}+\|d'\|_{l^2}. \end{aligned} \tag{5.53}$$

The estimate (5.45) follows directly. Using this bound (5.45) and the bound on $\dot{\mathrm{p}}_{1,2}$ from (5.53) on the right-hand side of (5.48) and (5.50), and remarking that $\varepsilon^2+\varepsilon^{-1}\|v^\perp\|_{L^2}+\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\ll 1$, then algebraic rearrangements lead to the bounds

$$\begin{aligned} |d_0| &\lesssim \varepsilon^3E(t)|\mathrm{p}_0-\mathrm{p}_0^*|+E(t)\|d'\|_{l^2}+\varepsilon^4E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}+\varepsilon^{5/2}\|v^\perp\|_{L^2}+\varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2};\\ \|d'\|_{l^2} &\lesssim \varepsilon^3E(t)|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}+(\varepsilon^5+\varepsilon^2\|v^\perp\|_{L^2})|\mathrm{p}_0-\mathrm{p}_0^*|+\varepsilon^4E(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}+E(t)d_0\\ &\quad+\varepsilon^{5/2}\|v^\perp\|_{L^2}+\varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2}. \end{aligned}$$

Using the estimate on $\|d'\|_{l^2}$ to eliminate it from the right-hand side of the estimate on $|d_0|$ yields the final upper bound for d_0. The final l^2-estimate for d' follows from the estimate above and the a priori assumptions (5.1). □

Corollary 5.12 *Suppose the assumptions of* Theorem 5.1 *hold. Then the meander parameter vector evolution takes the form*

$$\mathbb{D}^{1/2}\dot{\hat{\mathbf{p}}}=-\varepsilon^3\left[c_0(\mathbb{D}+\mathbb{U}^T)(\mathrm{p}_0-\mathrm{p}_0^*)+\varepsilon(\mathbb{D}-\mathbb{I})^2\right]\mathbb{D}^{1/2}\hat{\mathbf{p}}+\hat{d}_w, \tag{5.54}$$

where the weighted remainder $\hat{d}_w=(d_{w,k})_{k=3}^{N_1-1}$ *satisfies*

$$\|\hat{d}_w\|_{l^2}\lesssim\varepsilon^3E_w(t)|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}+\varepsilon^4|\mathrm{p}_0-\mathrm{p}_0^*|+E_w(t)|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}+\varepsilon\|v^\perp\|_{L^2}+\varepsilon^{-1/2}\|\mathrm{N}(v^\perp)\|_{L^2}.$$

Proof Multiplying the Eq. (5.46) by the weight β_k we have

$$\langle\mathbb{T}\dot{\mathbf{p}},\beta_k\mathbf{B}_k\rangle_{l^2}=-\left\langle\Pi_0\mathrm{F}(\Phi_{\mathbf{p}}),\beta_kZ^{1k}_{\mathbf{p},*}\right\rangle_{L^2}-\left\langle\mathscr{R}[v^\perp],\beta_kZ^{1k}_{\mathbf{p},*}\right\rangle_{L^2}.$$

Multiply this result by $-\varepsilon^{-1/2}\mu_0/m_1$ and apply Corollaries 5.5 and 5.10, this yields

$$\beta_k\dot{\mathrm{p}}_k=-\dot{\mathrm{p}}_0\left(\beta_k\mathrm{p}_k-\beta_k\hat{\mathbf{p}^T}\mathbb{U}\mathbf{B}_k\right)-\varepsilon^3\mu_0c_k\beta_k\mathrm{p}_k+d_{w,k}. \tag{5.55}$$

Here from Lemma 5.11 the weighted remainder $\hat{d}_w=(d_{w,k})$ can be bounded as

$$\begin{aligned} \|\hat{d}_w\|_{l^2}\lesssim&\varepsilon^3\left(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}+\varepsilon\right)|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}+\varepsilon^4E_w(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}+\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\dot{\mathrm{p}}_{1,2}\|_{l^2}\\ &+(\varepsilon+\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}+\varepsilon^{-2}\|v^\perp\|_{L^2})\|\dot{\mathbf{p}}\|_{l^2}+\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2}\|\dot{\mathbf{p}}\|_{\mathbb{V}_1^2}\\ &+\varepsilon\|v^\perp\|_{L^2}+\varepsilon^{-1/2}\|\mathrm{N}(v^\perp)\|_{L^2}. \end{aligned}$$

As in the unweighted case, we use the first equation of (5.44) to substitute for $\dot{\mathrm{p}}_0$ on the right-hand side of (5.55), replace c_k with its definition (5.37), and recall that $\mu_0=1+O(\varepsilon,|\mathrm{p}_0|)$. These manipulations yield the equation

$$\beta_k\dot{\mathrm{p}}_k=-\varepsilon^3\left[c_0(\mathrm{p}_0-\mathrm{p}_0^*)\mathbb{D}_{kk}+\varepsilon(\mathbb{D}_{kk}-1)^2\right]\beta_k\mathrm{p}_k\mathbf{1}_{\{k\geq3\}}+c_0(\mathrm{p}_0-\mathrm{p}_0^*)\beta_k\hat{\mathbf{p}}\mathbb{U}\mathbf{B}_k+\tilde{d}_{w,k},$$

where from Lemma 5.6 the revised remainder takes the form

$$\tilde{d}_{w,k}=d_{w,k}+O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}d_0,\varepsilon^3E_w|\mathrm{p}_0-\mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2},\varepsilon^4E_w\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}).$$

We drop the tilde, and the weighted evolution (5.54) presented in the Lemma follows. The revised from of the remainder $\hat{d}_w$ satisfies the bound

$$\|\hat{d}_w\|_{l^2} \lesssim \varepsilon^3 E_w(t)|\mathrm{p}_0 - \mathrm{p}_0^*|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}|d_0| + \varepsilon^4 E_w(t)\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\dot{\mathrm{p}}_{1,2}\|_{l^2} \\ + (\varepsilon + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} + \varepsilon^{-2}\|v^\perp\|_{L^2})\|\dot{\mathbf{p}}\|_{l^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2\|\dot{\mathbf{p}}\|_{\mathbb{V}_1^2} + \varepsilon\|v^\perp\|_{L^2} + \varepsilon^{-1/2}\|\mathrm{N}(v^\perp)\|_{L^2}. \tag{5.56}$$

As the unweighted case, this system is not closed as it contains $\dot{\mathbf{p}} = (\dot{\mathrm{p}}_0, \dot{\mathrm{p}}_{1,2}, \dot{\hat{\mathbf{p}}})$. From Eq. (5.54) which we have established, the definition of $\mathbb{V}_1^2$, and the bounds on $\dot{\mathrm{p}}_0, \dot{\mathrm{p}}_{1,2}$ from Theorem 5.1 we find

$$\begin{aligned}\|\dot{\mathbf{p}}\|_{\mathbb{V}_1^2} &\lesssim |\dot{\mathrm{p}}_0| + |\dot{\mathrm{p}}_{1,2}| + \|\mathbb{D}^{1/2}\dot{\hat{\mathbf{p}}}\|_{l^2} \\ &\lesssim \varepsilon^3|\mathrm{p}_0 - \mathrm{p}_0^*| + |d_0| + \|d'\|_{l^2} + \varepsilon^4\|\mathbf{p}\|_{\mathbb{V}_5^2} + \|\hat{d}_w\|_{l^2}.\end{aligned}$$

Using this estimate and the bounds on d_0, $\|d'\|_{l^2}$, $\|\dot{\mathbf{p}}\|_{l^2}$ from Theorem 5.1 on the right-hand side of (5.56), we obtain the desired bound on $\hat{d}_w$. □

Lemma 5.13 *The diagonal matrices $\mathbb{D}$ and $\mathbb{D} - \mathbb{I}$ are uniformly comparable as maps from l^2 to l^2, in particular,*

$$\frac{1}{2}\|\mathbb{D}\|_{l_2^*} \le \|\mathbb{D} - \mathbb{I}\|_{l_2^*} \le \|\mathbb{D}\|_{l_2^*}.$$

Proof Since $\mathbb{D}$ and $\mathbb{D} - \mathbb{I}$ are both diagonal, we only need compare their diagonal elements. Indeed, their kth diagonal term are β_k^2 and $\beta_k^2 - 1$, for $\mathbb{D}$ and $\mathbb{D} - \mathbb{I}$ respectively. The result follows directly from the relationship

$$\frac{1}{2}\beta_k^2 \le \beta_k^2 - 1 \le \beta_k^2$$

since $\beta_k^2 > 2$ for $k \ge 3$, see (2.4). □

5.5 Energy Estimates on p

We derive energy estimates on $\mathbf{p}$ from its dynamics established in Theorem 5.1 and Corollary 5.12. These estimates are used in Theorem 4.2 to show that the residence time T of orbits $u = u(t)$ that start in the thin tubular neighborhood is infinite, and that the orbits converge to a translation of the circular equilibrium.

Lemma 5.14 *Under the a priori assumptions in (5.1) with δ small enough independent of ε and ρ, the projected meander parameters satisfy the energy estimates*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon^3 c_0|\mathrm{p}_0 - \mathrm{p}_0^*|^2 \lesssim \varepsilon^5\delta^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2 + \varepsilon^2\|v^\perp\|_{L^2}^2 + \varepsilon^{-2}\|\mathrm{N}(v^\perp)\|_{L^2}^2;$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2 + \frac{\varepsilon^4}{32}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2 \lesssim \varepsilon^2|\mathrm{p}_0 - \mathrm{p}_0^*|^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2 + \varepsilon\|v^\perp\|_{L^2}^2 + \varepsilon^{-3}\|\mathrm{N}(v^\perp)\|_{L^2}^2.$$

Moreover, the time derivative of $\mathbf{p}$ has the following l^2 bound

$$\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^3|\mathrm{p}_0 - \mathrm{p}_0^*| + \varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + \varepsilon^{5/2}\|v^\perp\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^\perp)\|_{L^2}.$$

Proof Multiplying the p_0 evolution from Theorem 5.1 by $2(\mathrm{p}_0 - \mathrm{p}_0^*)$ and applying Young's inequality yields the bound

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon^3 c_0 |\mathrm{p}_0 - \mathrm{p}_0^*|^2 \lesssim \varepsilon^{-3} d_0^2.$$

Using the bound on the remainder d_0 from Theorem 5.1 with $E(t)$ bounded by (5.43), the inequality of p_0 follows provided that δ is chosen small enough.

For the first estimate on $\hat{\mathbf{p}}$, we note that $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} = \|\mathbb{D}\hat{\mathbf{p}}\|_{l^2}$ and take the inner product of the evolution equation for $\hat{\mathbf{p}}$ in Theorem 5.1 with $2\mathbb{D}^2\hat{\mathbf{p}}$. This yields the equality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} = &-\varepsilon^3 2c_0(\mathrm{p}_0 - \mathrm{p}_0^*)\left\langle(\mathbb{D} + \mathbb{U}^T)\hat{\mathbf{p}}, \mathbb{D}^2\hat{\mathbf{p}}\right\rangle_{l^2} \\ &-2\varepsilon^4\left\langle(\mathbb{D} - \mathbb{I})^2\hat{\mathbf{p}}, \mathbb{D}^2\hat{\mathbf{p}}\right\rangle_{l^2} + 2\left\langle\hat{d}, \mathbb{D}^2\hat{\mathbf{p}}\right\rangle_{l^2}.\end{aligned} \tag{5.57}$$

By Hölder's inequality and the bound $\|\mathbb{U}^T\|_{l_2^*} \lesssim \|\mathbb{D}\|_{l_2^*}$ from Corollary 5.6, the first term on the right-hand side of (5.57) can be bounded from above by

$$\begin{aligned}-\varepsilon^3 2c_0(\mathrm{p}_0 - \mathrm{p}_0^*)\left\langle(\mathbb{D} + \mathbb{U}^T)\hat{\mathbf{p}}, \mathbb{D}^2\hat{\mathbf{p}}\right\rangle_{l^2} &\lesssim \varepsilon^3|\mathrm{p}_0 - \mathrm{p}_0^*|\|\mathbb{D}\hat{\mathbf{p}}\|_{l^2}\|\mathbb{D}^2\hat{\mathbf{p}}\|_{l^2} \\ &\le C\varepsilon^2\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_3^2}|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \frac{\varepsilon^4}{32}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_5^2}\end{aligned}$$

for a constant C independent of $\varepsilon \in (0, \varepsilon_0)$. From Lemma 5.13, the second term on the right-hand side of (5.57) is negative and can be bounded from above by

$$-2\varepsilon^4\left\langle(\mathbb{D} - \mathbb{I})^2\hat{\mathbf{p}}, \mathbb{D}^2\hat{\mathbf{p}}\right\rangle_{l^2} \le -2\varepsilon^4\|(\mathbb{D} - \mathbb{I})^2\hat{\mathbf{p}}\|^2_{l^2} \le -\frac{\varepsilon^4}{8}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2}.$$

Employing Hölder's and Young's inequalities to bound the third item on the right-hand side of (5.57) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} + \frac{\varepsilon^4}{16}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2} \lesssim \varepsilon^2|\mathrm{p}_0 - \mathrm{p}_0^*|^2\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} + \varepsilon^{-4}\|\hat{d}\|^2_{l^2}.$$

It remains to obtain an bound on $\|\hat{d}\|_{l^2}$. In fact, $\|\hat{d}\|_{l^2} \le \|d'\|_{l^2}$ and the latter is bounded in Theorem 5.1. We note that $E(t) \lesssim \delta$ and $\varepsilon^{-2}\|v^\perp\|_{L^2} \le \delta$. Using the bound on $\|d'\|_{l^2}$ from Theorem 5.1, we have

$$\varepsilon^{-4}\|\hat{d}\|^2_{l^2} \lesssim \varepsilon^2\delta^2|\mathrm{p}_0 - \mathrm{p}_0^*|^2\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2} + \varepsilon^4|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon^4\delta^2\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2} + \varepsilon\|v^\perp\|_{L^2} + \varepsilon^{-3}\|\mathrm{N}(v^\perp)\|^2_{L^2}.$$

Absorbing these terms involving $\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}$ and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$ for δ suitably small (independent of ε), we obtain the first estimate on $\hat{\mathbf{p}}$. Finally, the l^2-bound on $\dot{\mathbf{p}}$ follows from the bound on $\|\dot{\mathbf{p}}\|_{l^2}$ and these estimates on $d = (d_0, d')$ in Theorem 5.1. □

We require bounds on the evolution of weighted norms of $\mathbf{p}$.

Lemma 5.15 *Under the a priori assumptions (5.1) with δ small enough independent of ε, there exists a strictly positive constant c_* independent of $\varepsilon \in (0, \varepsilon_0)$, ρ, δ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_3^2} + c_*\varepsilon^4\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_5^2} \lesssim \varepsilon^2|\mathrm{p}_0 - \mathrm{p}_0^*|^2\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_3^2} + \varepsilon^4|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon^{-2}\|v^\perp\|^2_{L^2} + \varepsilon^{-5}\|\mathrm{N}(v^\perp)\|^2_{L^2}.$$

Proof Since $\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} = \|\mathbb{D}^{3/2}\hat{\mathbf{p}}\|_{l^2}$ we take the l^2-inner product of the weighted evolution equation in Corollary 5.12 with $2\mathbb{D}^{5/2}\hat{\mathbf{p}}$. This yields the equality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}^2 &= -\varepsilon^3 2c_0(\mathrm{p}_0-\mathrm{p}_0^*)\left\langle(\mathbb{D}+\mathbb{U}^T)\mathbb{D}^{1/2}\hat{\mathbf{p}},\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2}\\ &\quad -2\varepsilon^4\left\langle(\mathbb{D}-\mathbb{I})^2\mathbb{D}^{1/2}\hat{\mathbf{p}},\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2}+2\left\langle\hat{d}_w,\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2}.\end{aligned}\tag{5.58}$$

By Hölder's inequality and the bound $\|\mathbb{U}^T\|_{l_2^*}\lesssim\|\mathbb{D}\|_{l_2^*}$ from Corollary 5.6, the first term on the right-hand side of (5.58) can be bounded from above by

$$\begin{aligned}-\varepsilon^3 c_0(\mathrm{p}_0-\mathrm{p}_0^*)\left\langle(\mathbb{D}+\mathbb{U}^T)\mathbb{D}^{1/2}\hat{\mathbf{p}},\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2} &\lesssim \varepsilon^3|\mathrm{p}_0-\mathrm{p}_0^*|\|\mathbb{D}^{3/2}\hat{\mathbf{p}}\|_{l^2}\|\mathbb{D}^{5/2}\hat{\mathbf{p}}\|_{l^2}\\ &\leq C\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}^2|\mathrm{p}_0-\mathrm{p}_0^*|^2+\frac{\varepsilon^4}{64}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}^2.\end{aligned}$$

The second inequality above follows from an application of Young's inequality and the definition of $\mathbb{V}_3^2$, $\mathbb{V}_5^2$ in terms of $\mathbb{D}$. To address the second term on the right-hand side of (5.58) we recall from Lemma 5.13 that the diagonal matrices satisfy $\frac{1}{2}\mathbb{D}\leq\mathbb{D}-\mathbb{I}\leq\mathbb{D}$ so that

$$-2\varepsilon^4\left\langle(\mathbb{D}-\mathbb{I})^2\mathbb{D}^{1/2}\hat{\mathbf{p}},\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2}\leq-2\varepsilon^4\|(\mathbb{D}-\mathbb{I})^{5/2}\hat{\mathbf{p}}\|_{l^2}^2\leq-\frac{\varepsilon^4}{16}\|\mathbb{D}^{5/2}\hat{\mathbf{p}}\|_{l^2}^2\leq-\frac{\varepsilon^4}{16}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}^2.$$

To bound the third term on the right-hand side of (5.58) we apply Hölder's and Young's inequalities, and deduce that

$$\left\langle\hat{d}_w,\mathbb{D}^{5/2}\hat{\mathbf{p}}\right\rangle_{l^2}\leq\frac{\varepsilon^4}{64}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}^2+C\varepsilon^{-4}\|\hat{d}_w\|_{L^2}^2.$$

Using the l^2-bound of $\hat{d}_w$ from Corollary 5.12, returning these three estimates above to (5.58), and taking $c_*=\frac{1}{32}$ completes the proof provided that ε_0 and δ are small enough. □

Acknowledgements Both authors thank Gurgen Hayrapetyan for sharing preliminary results on this problem for the weak FCH that arose out of his thesis.

6 Appendix

This section contains technical results whose proof was deferred from the main presentation.

6.1 Elementary Embeddings in the Weighted Space

The following embeddings are direct results of Hölder's inequality and the asymptotic form of β_j introduced in (2.4), details are omitted.

Lemma 6.1 *Suppose that* $\hat{\mathbf{p}}\in l^\infty(\mathbb{R}^{N_1})$, *then* $\|\hat{\mathbf{p}}\|_{\mathbb{V}_0^2}=\|\hat{\mathbf{p}}\|_{l^2}$ *and*

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_k}\lesssim\|\hat{\mathbf{p}}\|_{\mathbb{V}_{k+1}^2},\quad\|\hat{\mathbf{p}}\|_{\mathbb{V}_{k+1}^r}\lesssim N_1\|\hat{\mathbf{p}}\|_{\mathbb{V}_k^r},\quad\|\hat{\mathbf{p}}\|_{\mathbb{V}_k}\lesssim N_1^{k+1/2}\|\hat{\mathbf{p}}\|_{l^2}.$$

In addition, for any vector $\mathbf{a}\in l^2(\mathbb{R}^m)$ *we have the dimension dependent bound*

$$\|\mathbf{a}\|_{l^1}\leq m^{1/2}\|\mathbf{a}\|_{l^2}.\tag{6.1}$$

6.2 Geometric Quantities and Their Bounds

Lemma 6.2 (Geometric quantities of $\Gamma_{\mathbf{p}}$) *Let $\mathbf{p} \in \mathcal{D}_\delta$ with $\mathcal{D}_\delta$ given by (2.8) in Definition 2.3. The length normalization $A(\mathbf{p})$ depends quadradically upon $\hat{\mathbf{p}}$, and the length of $\Gamma_{\mathbf{p}}$ depends only on p_0,*

$$A(\mathbf{p}) = 1 + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_1}); \qquad |\Gamma_{\mathbf{p}}| = (1 + \mathrm{p}_0)|\Gamma_0|. \tag{6.2}$$

The gradient of A with respect to $\mathbf{p}$ satisfies

$$\|\nabla_{\mathbf{p}} A\|_{l^2} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}.$$

The curvature of $\Gamma_{\mathbf{p}}$, defined by

$$\kappa_{\mathbf{p}} := \boldsymbol{\gamma}''_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} / |\boldsymbol{\gamma}'_{\mathbf{p}}|^2, \quad \textit{with} \quad \mathbf{n}_{\mathbf{p}} = e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}'_{\mathbf{p}} \big/ |\boldsymbol{\gamma}'_{\mathbf{p}}| \tag{6.3}$$

admits the expansion

$$\kappa_{\mathbf{p}}(s) = \kappa_{\mathbf{p},0} + \mathcal{Q}_1 + \mathcal{Q}_2, \qquad \kappa_{\mathbf{p},0} = -\frac{1}{1 + \mathrm{p}_0}, \tag{6.4}$$

where the linear approximation is given by

$$\mathcal{Q}_1 = \frac{1}{1 + \mathrm{p}_0} \sum_{j=3}^{N_1 - 1} (1 - \beta_j^2)\mathrm{p}_j \tilde{\theta}_j,$$

and the quadratic remainder $\mathcal{Q}_2$ satisfies

$$\|\mathcal{Q}_2\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2}, \qquad \|\mathcal{Q}_2\|_{H^1(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2},$$
$$\|\mathcal{Q}_2\|_{H^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}; \qquad \|\mathcal{Q}_2\|_{H^3(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}.$$

The curvature $\kappa_{\mathbf{p}}$ and normal $\mathbf{n}_{\mathbf{p}}$ depend only on p_0 and $\hat{\mathbf{p}}$, and satisfy the following bounds

$$\|\kappa_{\mathbf{p}}\|_{L^\infty} + \|\varepsilon^2 \Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}\|_{L^\infty} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2}; \quad |\mathbf{n}_{\mathbf{p}} - \mathbf{n}_0| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_1}, \tag{6.5}$$

Moreover, the perturbed and original normal satisfy the relation

$$\mathbf{n}_{\mathbf{p}} \cdot \mathbf{n}_0 = 1 + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_1}). \tag{6.6}$$

Proof The length of $\Gamma_{\mathbf{p}}$ follows from its definition, and the approximation of $A(\mathbf{p})$ and its gradient estimate are given in Lemma 2.11 of [5]. Taking the derivative of $\boldsymbol{\gamma}_{\mathbf{p}}$ in (2.13) and using $\mathbf{n}'_0 = -\kappa_0 \boldsymbol{\gamma}'_0$ we find

$$\boldsymbol{\gamma}'_{\mathbf{p}} = \frac{1 + \mathrm{p}_0}{A(\mathbf{p})}\Big[\big(1 - \kappa_0 \bar{p}(\tilde{s})\big)\boldsymbol{\gamma}'_0 + \bar{p}'(\tilde{s})|\boldsymbol{\gamma}'_{\mathbf{p}}|\mathbf{n}_0(s)\Big], \tag{6.7}$$

and hence for $\hat{\mathbf{p}} \in \mathbb{V}_2$ we have the approximations

$$|\boldsymbol{\gamma}'_{\mathbf{p}}| = \frac{1 + \mathrm{p}_0}{A(\mathbf{p})}(1 - \kappa_0 \bar{p}) + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_1}), \qquad |\boldsymbol{\gamma}'_{\mathbf{p}}|' = O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}). \tag{6.8}$$

To obtain an approximation of the curvature $\kappa_{\mathbf{p}}$ we take an additional s derivative of (6.7). Using the relation between the tangent and normal we find the equality

$$\boldsymbol{\gamma}''_{\mathbf{p}} = \frac{1 + \mathrm{p}_0}{A(\mathbf{p})}\left[\left(\kappa_0 + \frac{(1 + \mathrm{p}_0)^2}{A^2(\mathbf{p})}\bar{p}''(\tilde{s}) - \kappa_0^2 \bar{p} + \mathcal{Q}_{2,0}\right)\mathbf{n}_0 - 2\kappa_0 \bar{p}'(\tilde{s})|\boldsymbol{\gamma}'_{\mathbf{p}}|\boldsymbol{\gamma}'_0\right].$$

Here $\mathcal{Q}_{2,0}(\boldsymbol{\gamma}_{\mathbf{p}})$ takes the explicit form

$$\mathcal{Q}_{2,0} = \left(|\boldsymbol{\gamma}_{\mathbf{p}}'|^2 - \frac{(1+\mathrm{p}_0)^2}{A(\mathbf{p})}\right)\bar{p}'' + \bar{p}'|\boldsymbol{\gamma}_{\mathbf{p}}'|'$$

and is an intermediate quadratic remainder from (6.8). From the tangent and normal relation: $\mathbf{n} = e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}'$, we deduce from (6.7) that

$$e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}' = \frac{1+\mathrm{p}_0}{A(\mathbf{p})}\Big[(1-\kappa_0\bar{p}(\tilde{s}))\mathbf{n}_0 - \bar{p}'(\tilde{s})|\boldsymbol{\gamma}_{\mathbf{p}}'|\boldsymbol{\gamma}_0'\Big], \tag{6.9}$$

which when dotted with the approximation for $\boldsymbol{\gamma}_{\mathbf{p}}''$ implies

$$e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'\cdot\boldsymbol{\gamma}_{\mathbf{p}}'' = \left(\frac{1+\mathrm{p}_0}{A(\mathbf{p})}\right)^2\left[\kappa_0 + \frac{(1+\mathrm{p}_0)^2}{A^2(\mathbf{p})}\bar{p}'' - 2\kappa_0^2\bar{p} + \mathcal{Q}_{2,1}\right],$$

in which $\mathcal{Q}_{2,1}$ is a quadratic term given by

$$\mathcal{Q}_{2,1} = \mathcal{Q}_{2,0}(1-\kappa_0\bar{p}) + \left[-\kappa_0\bar{p}\left(\frac{(1+\mathrm{p}_0)^2}{A^2(\mathbf{p})}\bar{p}'' - \kappa_0\bar{p}\right) + \left(2\kappa_0\bar{p}'|\boldsymbol{\gamma}_{\mathbf{p}}'| + \kappa_0'\bar{p}\right)\bar{p}'|\boldsymbol{\gamma}_{\mathbf{p}}'|\right].$$

Finally, in light of (6.8) we rewrite

$$\frac{1}{|\boldsymbol{\gamma}_{\mathbf{p}}'|^3} = \left(\frac{A(\mathbf{p})}{1+\mathrm{p}_0}\right)^3\left(1 + 3\kappa_0\bar{p} + \mathcal{Q}_{2,2}\right), \qquad \mathcal{Q}_{2,2} := \frac{1}{|\boldsymbol{\gamma}_{\mathbf{p}}'|^3} - 1 - 3\kappa_0\bar{p},$$

and substituting these expressions in (6.3) we obtain the curvature expansion,

$$\kappa_{\mathbf{p}} = \frac{A(\mathbf{p})}{1+\mathrm{p}_0}\left[\kappa_0 + \kappa_0^2\bar{p} + \frac{(1+\mathrm{p}_0)^2}{A^2(\mathbf{p})}\bar{p}''(\tilde{s}) + \mathcal{Q}_{2,3}\right]$$

where $A(\mathbf{p}) = 1 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}^2)$ and the final quadratic remainder takes the form

$$\mathcal{Q}_{2,3} := \frac{A(\mathbf{p})}{(1+\mathrm{p}_0)^2}e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'\cdot\boldsymbol{\gamma}_{\mathbf{p}}''\,\mathcal{Q}_{2,2} + \frac{(1+\mathrm{p}_0)^3}{A^3(\mathbf{p})}\frac{1}{|\boldsymbol{\gamma}_{\mathbf{p}}'|^3}\,\mathcal{Q}_{2,1} + 3\kappa_0\bar{p}\left[\frac{(1+\mathrm{p}_0)^2}{A^2(\mathbf{p})}\bar{p}'' - 2\kappa_0\bar{p}\right].$$

The form of the expansion (6.4) follows from the definition of $\bar{p}$, (6.2), and $(A(\mathbf{p})-1)$ is quadratic. The H^k estimates of $\mathcal{Q}_2$ follows directly from the formulas for the quadratic terms, the independence of $A(\mathbf{p})$ from $\tilde{s}_{\mathbf{p}}$, and the embedding estimates in Lemma 6.1.

The curvature bounds in (6.5) follow directly from these expansions and the embedding estimate of Lemma 6.1. To establish that the normals are nearly parallel, from the definition (6.3) of $\mathbf{n}_{\mathbf{p}}$ we have

$$\mathbf{n}_{\mathbf{p}}\cdot\mathbf{n}_0 = \frac{e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'}{|\boldsymbol{\gamma}_{\mathbf{p}}'|}\cdot\mathbf{n}_0.$$

The estimate (6.6) follows directly by (6.9) and (6.8). This completes the proof. □ Recall that $\Pi_{G_1}^{\perp} = I - \Pi_{G_1}$ is the complement to the Garlerkin projection onto $G_1 \subset L^2(\mathscr{I}_{\mathbf{p}})$. The curvature expansion and remainder estimates in the Lemma 6.2 above imply the following estimates.

Corollary 6.3 *With the same assumptions as in Lemma 6.2, it holds that*

$$\|\Pi_{G_1}^{\perp}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \|\Pi_{G_1}^{\perp}\kappa_{\mathbf{p}}^3\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}^4);$$
$$\|\Pi_{G_1}^{\perp}\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}.$$

Proof The curvature admits the expansion (6.4), for which the first two terms $\kappa_{\mathbf{p},0}$, $\mathcal{Q}_1 \in G_1$, and hence $\Pi_{G_1}^{\perp}\kappa_{\mathbf{p}} = \Pi_{G_1}^{\perp}\mathcal{Q}_2$. Applying the $H^1(\mathscr{I}_{\mathbf{p}})$-bound of $\mathcal{Q}_2$ from Lemma 6.2 we find

$$\|\Pi_{G_1}^{\perp}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\mathcal{Q}_2\|_{H^1(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}.$$

From (6.4) that we expand

$$\begin{aligned}\kappa_{\mathbf{p}}^3 &= \kappa_{\mathbf{p},0}^3 + 3\kappa_{\mathbf{p},0}^2\mathcal{Q}_1 + 3\kappa_{\mathbf{p},0}\mathcal{Q}_1^2 + 3\kappa_{\mathbf{p},0}^2\mathcal{Q}_2 + 3\kappa_{\mathbf{p},0}\mathcal{Q}_2^2 + 3\mathcal{Q}_1^2\mathcal{Q}_2 + 3\mathcal{Q}_1\mathcal{Q}_2^2\\ &\quad + 6\kappa_{\mathbf{p},0}\mathcal{Q}_1\mathcal{Q}_2 + \mathcal{Q}_1^3 + \mathcal{Q}_2^3.\end{aligned}$$

Note the first two terms lies in G_1 and $|\kappa_{\mathbf{p},0}| \lesssim 1$ for $\mathbf{p} \in \mathcal{D}_\delta$. Then from the Sobolev embedding $L^\infty(\mathscr{I}_{\mathbf{p}}) \nearrow H^1(\mathscr{I}_{\mathbf{p}})$, and definition of $\mathcal{Q}_1$ and the $L^2(\mathscr{I}_{\mathbf{p}})$, $H^1(\mathscr{I}_{\mathbf{p}})$-bounds of $\mathcal{Q}_2$ from Lemma 6.2 we derive

$$\|\Pi_{G_1}^{\perp}\kappa_{\mathbf{p}}^3\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}^4).$$

Similarly, since G_1 is invariant under $\Delta_{s_{\mathbf{p}}}$ and $\Pi_{G_1}^{\perp}\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} = \Pi_{G_1}^{\perp}\Delta_{s_{\mathbf{p}}}\mathcal{Q}_2$, applying the $H^2(\mathscr{I}_{\mathbf{p}})$-bound of $\mathcal{Q}_2$ from Lemma 6.2 implies

$$\|\Pi_{G_1}^{\perp}\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\mathcal{Q}_2\|_{H^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}.$$

The corollary follows. □

For a function $h \in \mathcal{H}_2(\boldsymbol{\gamma}_{\mathbf{p}})$, see (3.5), its value at the perturbed interface $\gamma_{\mathbf{p}}$ will frequently be compared to its value at leading order circular interface $\gamma_{\mathbf{p},0}$ defined in (2.19). This leads to the decomposition

$$h(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}) = h(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p},0}) + \left(h(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}}) - h(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p},0})\right). \tag{6.10}$$

The following Lemma provides Lipschitz estimates on the second term of the decomposition.

Lemma 6.4 *Suppose $h = h(\boldsymbol{\gamma}_{\mathbf{p}})$ lies in the function family $\bar{\mathcal{H}}_2(\boldsymbol{\gamma}_{\mathbf{p}})$ as introduced in Definition 3.5, and is decomposed as in (6.10). Then the leading order term $h(\boldsymbol{\gamma}_{\mathbf{p},0})$ is independent of $\tilde{s}_{\mathbf{p}}$ and if $\hat{\mathbf{p}} \in \mathbb{V}_2$, then*

$$\|h(\boldsymbol{\gamma}_{\mathbf{p}}) - h(\boldsymbol{\gamma}_{\mathbf{p},0})\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}. \tag{6.11}$$

If moreover $\hat{\mathbf{p}} \in \mathbb{V}_3^2$, then for $l \geq 1$,

$$\left\|\varepsilon^{l-1}\partial_{s_{\mathbf{p}}}^l\left(h(\boldsymbol{\gamma}_{\mathbf{p}}) - h(\boldsymbol{\gamma}_{\mathbf{p},0})\right)\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}. \tag{6.12}$$

Proof These estimates in (6.11)–(6.12), can be derived directly by the approximations of $|\boldsymbol{\gamma}_{\mathbf{p}}'|$, $\kappa_{\mathbf{p}}$ and $\mathbf{n}_{\mathbf{p}}$ in (6.8), (6.4). We only need to verify that $h(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p},0})$ is independent of $s_{\mathbf{p}}$. This holds true since $|\boldsymbol{\gamma}_{\mathbf{p},0}'| = 1 + \mathrm{p}_0$ by (6.8), $\kappa_{\mathbf{p},0}$ admits form in (6.4) and

$$\mathbf{n}_0 \cdot \mathbf{n}_{\mathbf{p},0} = 1, \quad \text{for} \quad \mathbf{n}_{\mathbf{p},0} = \frac{e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p},0}'}{|\boldsymbol{\gamma}_{\mathbf{p},0}'|}. \tag{6.13}$$

Here we used (6.9) with $\hat{\mathbf{p}} = 0$. □

Lemma 6.5 *Recalling the notation of Sect. 1.1, if $f \in L^2(\mathscr{I}_{\mathbf{p}})$, then there exists a unit vector $\boldsymbol{e} = (e_i)$ such that*

$$\int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\theta}_i \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|f\|_{L^2(\mathscr{I}_{\mathbf{p}})})e_i. \tag{6.14}$$

If in addition $f \in L^\infty$ on $\mathscr{I}_{\mathbf{p}}$, then for any vector $\mathbf{a} = (\mathrm{a}_j) \in l^2$, we have

$$\left| \sum_j \int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\theta}_i \mathrm{a}_j \tilde{\theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \right| \lesssim \|\mathbf{a}\|_{l^2} \|f\|_{L^\infty} e_i, \tag{6.15}$$

and there exists a matrix $\mathbb{E} = (\mathbb{E}_{ij})$ with l^2_* norm one, such that

$$\int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\theta}_i\tilde{\theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|f\|_{L^\infty})\mathbb{E}_{ij}. \tag{6.16}$$

Proof The estimates follow from Plancherel and classic applications of Fourier theory. □

The following Lemma estimates the $\mathbf{p}$-variation of the local coordinate associated to $\Gamma_{\mathbf{p}}$. In particular it provides estimates on the difference between $(s_{\mathbf{p}}, z_{\mathbf{p}})$ and (s, z) in terms of $\mathbf{p}$. It is equivalent to Lemma 6.2 of [5] and the proof is omitted.

Lemma 6.6 *Let $(s_{\mathbf{p}}, z_{\mathbf{p}})$ be the local coordinate subject to $\Gamma_{\mathbf{p}}$ on $\Gamma_{\mathbf{p}}^{2\ell}$. Assuming (2.8) the tangent coordinate $s_{\mathbf{p}}$ satisfies*

$$\|\nabla_{\mathbf{p}} s_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim 1;$$

while $z_{\mathbf{p}}$ have the $\mathbf{p}$-gradient

$$\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} = \varepsilon^{-1}\xi_j(s_{\mathbf{p}}),$$

where ξ_j is a function of $s_{\mathbf{p}}$ given explicitly by

$$\xi_j(s_{\mathbf{p}}) = \begin{cases} -\left(\dfrac{1+\bar{p}}{A}\left(1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0}\ln A\right) - \dfrac{\tilde{s}_{\mathbf{p}}\bar{p}'}{A}\right)\mathbf{n}_0\cdot\mathbf{n}_{\mathbf{p}}, & j=0 \\ -\theta_0\mathbf{E}_j\cdot\mathbf{n}_{\mathbf{p}}, & j=1,2; \\ -\left(\tilde{\theta}_j - \dfrac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}\ln A}{A}(1+\bar{p})\right)\mathbf{n}_0\cdot\mathbf{n}_{\mathbf{p}} & j\geq 3. \end{cases}$$

Moreover, we have the estimate

$$|s_{\mathbf{p}} - s| \lesssim \|\mathbf{p}\|_{l^1}, \quad |z_{\mathbf{p}} - z| \leq \varepsilon^{-1}\|\mathbf{p}\|_{l^1}. \tag{6.17}$$

6.3 Results on the Projection of the Normal Velocity

Lemma 6.7 *Under the assumption (2.8), the curvature of $\Gamma_{\mathbf{p}}$ admits the following projection identities:*

$$\int_{\mathscr{I}_{\mathbf{p}}} \kappa_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = -2\pi\theta_0\delta_{k0} - (\beta_k^2 - 1)\mathrm{p}_k \mathbf{1}_{\{k\ge 3\}} + O\left(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}^2_2}\right) e_k,$$

$$\int_{\mathscr{I}_{\mathbf{p}}} \kappa^3_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = -\frac{2\pi\theta_0}{(1+\mathrm{p}_0)^2}\delta_{k0} - \frac{3(\beta_k^2-1)}{(1+\mathrm{p}_0)^2}\mathrm{p}_k \mathbf{1}_{\{k\ge 3\}} + O\left(\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}\right) e_k,$$

$$\int_{\mathscr{I}_{\mathbf{p}}} \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \frac{(\beta_k^2-1)\beta_k^2}{(1+\mathrm{p}_0)^2}\mathrm{p}_k \mathbf{1}_{\{k\ge 3\}} + O\left(\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right) e_k \mathbf{1}_{\{k\ge 1\}},$$

for $k \in \Sigma_1$. Here $\boldsymbol{e} = (e_j)_{j=0}^{N_1-1}$ denotes a possibly different unit vector in each line.

Proof The curvature admits the expansion as in Lemma 6.2, and the quadratic term $\mathcal{Q}_2$ contributes

$$\int_{\mathscr{I}_{\mathbf{p}}} \mathcal{Q}_2 \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O\left(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}^2_2}\right) e_k, \tag{6.18}$$

while from the orthogonality (2.15) the projection of the linear term takes the form

$$\int_{\mathscr{I}_{\mathbf{p}}} \mathcal{Q}_1 \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = (1-\beta_k^2)\mathrm{p}_k \mathbf{1}_{\{k\ge 3\}}. \tag{6.19}$$

The leading order term $\kappa_{\mathbf{p},0}$ contributes

$$\int_{\mathscr{I}_{\mathbf{p}}} \kappa_{\mathbf{p},0} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = -2\pi\theta_0\delta_{k0}. \tag{6.20}$$

Combining (6.4) with the identities (6.18)–(6.20) yields the first result of the Lemma. For the $\kappa^3_{\mathbf{p}}$ projection, we expand

$$\kappa^3_{\mathbf{p}} = \kappa^3_{\mathbf{p},0} + 3\kappa^2_{\mathbf{p},0}\mathcal{Q}_1 + \tilde{\mathcal{Q}}_2, \tag{6.21}$$

where $\tilde{\mathcal{Q}}_2$ denotes quadratic terms in $\mathbf{p}$ and satisfies

$$|\tilde{\mathcal{Q}}_2| \lesssim |\mathcal{Q}_1|^3 + |\mathcal{Q}_2|^3 + |\mathcal{Q}_1|^2|\mathcal{Q}_2| + |\mathcal{Q}_1||\mathcal{Q}_2|^2$$

Since $|\mathcal{Q}_1| + |\mathcal{Q}_2| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \lesssim 1$, the assumption (2.8) and the estimates above imply

$$\|\tilde{\mathcal{Q}}_2\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\mathcal{Q}_1\|_{L^\infty}\|\mathcal{Q}_1\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \|\mathcal{Q}_2\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_2}.$$

Here we also used the embedding estimate in the perturbation space $\mathbb{V}$, see Lemma 6.1. Since $\kappa_{\mathbf{p},0}$ is independent of $s_{\mathbf{p}}$, the second identity of Lemma follows from (6.19), (6.20), the definition of $\kappa_{\mathbf{p},0}$ in (6.4), and Hölder's inequality. For the Laplace–Beltrami curvature projection, we integrate by parts and use (2.17), to find

$$\int_{\mathscr{I}_{\mathbf{p}}} \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = -\beta^2_{\mathbf{p},k} \int_{\mathscr{I}_{\mathbf{p}}} \mathcal{Q}_1 \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + \int_{\mathscr{I}_{\mathbf{p}}} \Delta_{s_{\mathbf{p}}} \mathcal{Q}_2 \, \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}. \tag{6.22}$$

The first term on the right-hand side is dominant, and can be estimated by (6.19). The second term on the right-hand side is higher order and can be bounded by

$$\left|\int_{\mathscr{I}_{\mathbf{p}}} \Delta_{s_{\mathbf{p}}} Q_2\, \tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}}\right| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} e_k. \tag{6.23}$$

The result follows. □

Combining the identities in Lemma 6.7, yields the following result.

Corollary 6.8 *For $V_k^M = V_k^M(\mathbf{p})$, $V_k^W = V_k^W(\mathbf{p})$ defined in (5.16), there exists a unit vector $\boldsymbol{e} = (e_k)$ such that*

$$\begin{aligned}
V_k^M(\mathbf{p}) =& -2\pi\theta_0\delta_{k0} - (\beta_k^2 - 1)\mathrm{p}_k \mathbf{1}_{\{k\geq 3\}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2) e_k,\\
V_k^W(\mathbf{p}) =& 2\pi\theta_0 \left(\frac{1}{2(1+\mathrm{p}_0)^2} - \alpha\right)\delta_{k0} - \frac{\beta_k^2 - 1}{(1+\mathrm{p}_0)^2}\left[\frac{2\beta_k^2 - 3}{2} - \alpha(1+\mathrm{p}_0)^2\right]\mathrm{p}_k \mathbf{1}_{k\geq 3}\\
&+ O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) e_k \mathbf{1}_{\{k\geq 1\}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}) e_k,
\end{aligned}$$

for $k = 0, \ldots, N_1 - 1$.

Lemma 6.9 *Let $h = h(\boldsymbol{\gamma}_{\mathbf{p}})$ lie in $\bar{\mathcal{H}}_2$ (see Definition 3.5). Then for j=0,1,2*

$$\begin{aligned}
\int \nabla_{s_{\mathbf{p}}}^j h(\boldsymbol{\gamma}_{\mathbf{p}})\tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} &= \mathcal{C}(\mathrm{p}_0)\delta_{k0}\delta_{j0} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_{2+j}^2}),\\
\int \nabla_{s_{\mathbf{p}}}^j h(\boldsymbol{\gamma}_{\mathbf{p}})\varepsilon\tilde{\theta}_k' \,\mathrm{d}\tilde{s}_{\mathbf{p}} &= O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_{2+j}^2}).
\end{aligned}$$

Proof From decomposition (6.10) the function h can be rewritten as $h(\boldsymbol{\gamma}_{\mathbf{p}}) = h(\boldsymbol{\gamma}_{\mathbf{p},0}) + (h(\boldsymbol{\gamma}_{\mathbf{p}}) - h(\boldsymbol{\gamma}_{\mathbf{p},0}))$, and the integral of the leading order term $h(\boldsymbol{\gamma}_{\mathbf{p},0})$ reduces to

$$\int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}}^j h(\boldsymbol{\gamma}_{\mathbf{p},0})\tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} = \delta_{j0} h(\boldsymbol{\gamma}_{\mathbf{p},0}) \int_{\mathscr{I}_{\mathbf{p}}} \theta_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} = \mathcal{C}(\mathrm{p}_0)\delta_{k0}\delta_{j0}, \tag{6.24}$$

where the constant $\mathcal{C}(\mathrm{p}_0)$ depends only on p_0. Here we note $h(\boldsymbol{\gamma}_{\mathbf{p},0})$ is independent of $s_{\mathbf{p}}$ by Lemma 6.4. Moreover, we have the bound

$$\left|\int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}}^j \left(h(\boldsymbol{\gamma}_{\mathbf{p}}) - h(\boldsymbol{\gamma}_{\mathbf{p},0})\right) \tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}}\right| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} e_k. \tag{6.25}$$

The proof is complete. □

Lemma 6.10 *Imposing assumptions (2.8), then there exist smooth functions $C_k = C_k(\mathrm{p}_0)$ for $k = 1, 2$ such that*

$$\int_{\Omega} \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty\right) \mathrm{d}x = C_1(\mathrm{p}_0)\varepsilon^4 + C_2(\mathrm{p}_0)\varepsilon^4(\sigma - \sigma_1^*) + O\left(\varepsilon^4\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}, \varepsilon^5\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right).$$

Proof This is a direct result of the form of F given in Lemma 3.7 and the Lemma 6.9, details are omitted. □

Lemma 6.11 *There exists a unit vector* $\boldsymbol{e} = (e_k)_{k=0}^{N_1-1}$ *such that the remainders defined in* (5.18) *satisfy*

$$\mathscr{R}_{k,1}(\mathbf{p}) = \varepsilon^{7/2}(\sigma_1^* - \sigma)\left(C_1(\mathrm{p}_0)\delta_{k0} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})e_k\right) + \varepsilon^{9/2}C_2(\mathrm{p}_0) + O(\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_k$$

$$\mathscr{R}_{k,2}(\mathbf{p}) = \varepsilon^{11/2}\Big(C_1(\mathrm{p}_0) + C_2(\mathrm{p}_0)(\sigma - \sigma_1^*)\Big)\delta_{k0} + O\left(\varepsilon^{11/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\right)e_k$$

where C_1 *and* C_2 *are smooth functions of* p_0.

Proof For $\mathscr{R}_{k,1}$ in (5.18), we expand F as in Lemma 3.7 that is, $\mathrm{F} - \mathrm{F}_m^\infty = \varepsilon^2\mathrm{F}_2 + \varepsilon^3(\mathrm{F}_3 - \mathrm{F}_3^\infty) + \varepsilon^4(\mathrm{F}_4 - \mathrm{F}_4^\infty) + e^{-\ell\varepsilon/\nu}\mathrm{F}_e$ by noting $\mathrm{F}_2^\infty = 0$. Integrating out $z_{\mathbf{p}}$ and using that the functions $\tilde{\varphi}_{1,k} = \varepsilon^{-1/2}\varphi_{1,k}(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}})$, $\tilde{\varphi}_{2,k} = \varepsilon^{-1/2}\varphi_{2,k}(z_{\mathbf{p}};\boldsymbol{\gamma}_{\mathbf{p}})$ from (3.32) belong to the function family $\mathcal{H}_2$ as introduced in Definition 3.5, the leading order contribution from F_2 takes the form

$$\varepsilon^{7/2}(\sigma_1^* - \sigma)\int_{\mathscr{I}_{\mathbf{p}}}\left(h_1(\boldsymbol{\gamma}_{\mathbf{p}})\tilde{\theta}_k + h_2(\boldsymbol{\gamma}_{\mathbf{p}})\varepsilon\tilde{\theta}_k'\right)\mathrm{d}\tilde{s}_{\mathbf{p}}.$$

The dependence of $\varphi_{1,k}, \varphi_{2,k}$ on $s_{\mathbf{p}}$ is uniform in k so that Lemma 6.5 applies. Applying Lemma 6.9 with $j = 0$ we see that this term provides the leading order contribution to $\mathscr{R}_{k,1}$. From the form of $\mathrm{F}_3, \mathrm{F}_4$ in Lemma 3.7 and Lemma 6.9 we find that the remaining terms can be bounded by $\varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}$.

To estimate $\mathscr{R}_{k,2}$ we turn to the definition of $Z_{\mathbf{p},*}^{1k}$ with $k \in \Sigma_1$ and use that $\psi_1 = \phi_0'/m_1$ has odd parity to derive

$$\mathscr{R}_{k,2} = C\int_{\Omega}\left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty\right)\mathrm{d}x\left(\varepsilon^{3/2}\int_{\mathscr{I}_{\mathbf{p}}}\left(h_1(\boldsymbol{\gamma}_{\mathbf{p}})\tilde{\theta}_k + h_2(\boldsymbol{\gamma}_{\mathbf{p}})\varepsilon\tilde{\theta}_k'\right)\mathrm{d}\tilde{s}_{\mathbf{p}}\right). \tag{6.26}$$

Applying Lemma 6.9 to h_1, h_2, the identity (6.26) reduces to

$$\mathscr{R}_{k,2} = \varepsilon^{3/2}\left(C(\mathrm{p}_0)\delta_{k0} + O\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})e_k\right)\int_{\Omega}\left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty\right)\mathrm{d}x, \tag{6.27}$$

which, combined with Lemma 6.10, yields the revised functions $C_1(\mathrm{p}_0)$, $C_2(\mathrm{p}_0)$ which appear in the statement of the Lemma. □

6.4 Weighted Estimates

The proof of the weighted estimates in Corollaries 5.5 and 5.10 are based on the following Lemma which primarily follows from integration by parts.

Lemma 6.12 *Let* $f = f(s_{\mathbf{p}})$ *be a function of* $s_{\mathbf{p}}$. *Then if* $f \in H^1(\mathscr{I}_{\mathbf{p}})$ *there exists a unit vector* $(e_k) \in l^2$ *such that*

$$\int_{\mathscr{I}_{\mathbf{p}}} f(s_{\mathbf{p}})\beta_k\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|\nabla_{s_{\mathbf{p}}}f\|_{L^2(\mathscr{I}_{\mathbf{p}})})e_k,$$

for $k \geq 3$. *Moreover if* $f \in W^{1,\infty}(\mathscr{I}_{\mathbf{p}})$, *then there exists a matrix* $\mathbb{E}$ *bounded in the* l_*^2 *norm such that*

$$\int_{\mathscr{I}_{\mathbf{p}}} f(s_{\mathbf{p}})\tilde{\theta}_j\beta_k\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|\nabla_{s_{\mathbf{p}}}f\|_{L^\infty})\mathbb{E}_{kj} + O(\|f\|_{L^\infty})\beta_j\mathbb{E}_{kj}.$$

Proof We observe from (2.17) that if we denote

$$\theta_k(\tilde{s}_{\mathbf{p}}) := -(1+\mathrm{p}_0)\beta_{\mathbf{p},k}^{-1}\tilde{\theta}_k'(\tilde{s}_{\mathbf{p}}), \tag{6.28}$$

then

$$\theta_k'(\tilde{s}_{\mathbf{p}}) = \beta_k\tilde{\theta}_k(\tilde{s}_{\mathbf{p}}).$$

Hence through integration by parts

$$\int_{\mathscr{I}_{\mathbf{p}}} f(s_{\mathbf{p}})\beta_k\tilde{\theta}_k \,\mathrm{d}\tilde{s}_{\mathbf{p}} = -\int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} f(s_{\mathbf{p}})\theta_k(\tilde{s}_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}.$$

In light of Lemma 6.5, we only need to show that $\{\theta_k, k \ge 3\}$ are orthogonal to each other. Since

$$\tilde{\theta}_k' = \begin{cases} -\beta_{\mathbf{p},k}\tilde{\theta}_{k+1} & \text{if } k \text{ is odd;} \\ \beta_{\mathbf{p},k}\tilde{\theta}_{k-1} & \text{if } k \text{ is even.} \end{cases}$$

The orthogonality of θ_k follows from its definition and orthogonality of $\tilde{\theta}_k$ in (2.15). The first estimate follows from the identity $\beta_{\mathbf{p},j} = \beta_j/(1+\mathrm{p}_0)$. The second estimate is derived from similar arguments using Lemma 6.5. □

Lemma 6.13 *For $k \ge 3$ and $k \in \Sigma_1$, the quantities V_k^M, V_k^W defined in (5.16) satisfy the weighted approximations,*

$$\beta_k V_k^M(\mathbf{p}) = -(\beta_k^2-1)\beta_k \mathrm{p}_k + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})e_k,$$

$$\beta_k V_k^W(\mathbf{p}) = -\frac{\beta_k^2-1}{(1+\mathrm{p}_0)^2}\left[\frac{2\beta_k^2-3}{2} - \alpha(1+\mathrm{p}_0)^2\right]\beta_k\mathrm{p}_k + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2})e_k.$$

Proof For the first approximation, since $\kappa_{\mathbf{p}}$ admits expansion (6.4) we can rewrite the definition (5.16) of V_k^M as

$$\beta_k V_k^M(\mathbf{p}) = -\kappa_{\mathbf{p},0}\int_{\mathscr{I}_{\mathbf{p}}}\beta_k\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} + \beta_k\int_{\mathscr{I}_{\mathbf{p}}}\mathcal{Q}_1\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} + \int_{\mathscr{I}_{\mathbf{p}}}\mathcal{Q}_2\beta_k\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}}.$$

The first term is zero since $\tilde{\theta}_k$ has no mass for any $k \ge 3$; by (6.18) the second term equals

$$\beta_k\int_{\mathscr{I}_{\mathbf{p}}}\mathcal{Q}_1\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} = (1-\beta_k^2)\beta_k\mathrm{p}_k,$$

and with the aid of Lemma 6.12 and the $H^1(\mathscr{I}_{\mathbf{p}})$ estimate of $\mathcal{Q}_2$ from Lemma 6.2 the third term is bounded as

$$\int_{\mathscr{I}_{\mathbf{p}}}\mathcal{Q}_2\beta_k\tilde{\theta}_k\,\mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})e_k.$$

The first estimate follows. The approximation of $V_k^W(\mathbf{p})$ is derived from similar arguments through the use of the higher-order estimates on quadratic term $\mathcal{Q}_2$ afforded by Lemma 6.2. □

Lemma 6.14 *For $k \ge 3$, the reminders defined in (5.18) satisfy the weighted estimates*

$$\beta_k\mathscr{R}_{k,1} = O(\varepsilon^{7/2}|\sigma_1^* - \sigma|\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_5^2})e_k;$$

$$\beta_k\mathscr{R}_{k,2} = O(\varepsilon^{11/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})e_k.$$

Proof This follows from arguments similar to those for Lemma 6.11, using the weighted estimates from Lemma 6.12. The details are omitted. □

Finally, we give a proof a Lemma 5.2 deferred from Sect. 5.

Proof of Lemma 5.2 We first address the unweighted approximation for the three cases $j = 0$, $j = 1, 2$ and $j \geq 3$.

Case 1: $j = 0$. We use Lemma 6.6 to replace $\xi_0(s_{\mathbf{p}})$ in the integral to obtain

$$-\int_{\mathscr{I}_{\mathbf{p}}} \xi_0(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \int_{\mathscr{I}_{\mathbf{p}}} \left[\frac{1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0} \ln A}{A}(1+\bar{p}) - \frac{\tilde{s}_{\mathbf{p}}\bar{p}'}{A}\right] \mathbf{n}_0 \cdot \mathbf{n}_{\mathbf{p}}\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}.$$

From (6.6) and (6.2) both the normal projection $\mathbf{n}_{\mathbf{p}} \cdot \mathbf{n}_0$ and length normalization A take the value one up to a quadratic correction. This yields the leading order approximation

$$-\int_{\mathscr{I}_{\mathbf{p}}} \xi_0(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \int_{\mathscr{I}_{\mathbf{p}}} \left[\Big(1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0} \ln A\Big)(1+\bar{p}) - \tilde{s}_{\mathbf{p}}\bar{p}'\right] \tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{k0}.$$

For $k \geq 1$ the eigenmode $\tilde{\theta}_k$ has zero mass in $L^2(\mathscr{I}_{\mathbf{p}})$. Using the definition (2.9) of $\bar{p}$, the orthogonality (2.15), and introducing $\mathbb{U}$ from (5.5) we arrive at the expansion

$$\begin{aligned}-\int_{\mathscr{I}_{\mathbf{p}}} \xi_0(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} &= (1+\mathrm{p}_0)\Big(1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0} \ln A\Big)2\pi\tilde{\theta}_0\delta_{k0} - (1+\mathrm{p}_0)\hat{\mathbf{p}}^T\mathbb{U}\mathbf{B}_k \\ &\quad + \Big(1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0} \ln A\Big)(1+\mathrm{p}_0)\mathrm{p}_k\mathbf{1}_{\{k\geq 3\}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{k0}.\end{aligned}$$

Since $|\partial_{\mathrm{p}_0} \ln A| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$ from Lemma 6.2, and $\tilde{\theta}_0 = \theta_0 = 1/\sqrt{2\pi}$, we rewrite this expansion as

$$\begin{aligned}-\int_{\mathscr{I}_{\mathbf{p}}} \xi_0(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} &= (1+\mathrm{p}_0)\Big(1+O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\Big)\frac{1}{\theta_0}\delta_{k0} - (1+\mathrm{p}_0)\hat{\mathbf{p}}^T\mathbb{U}\mathbf{B}_k \\ &\quad + (1+\mathrm{p}_0)\mathrm{p}_k\mathbf{1}_{\{k\geq 3\}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2)\mathbb{E}_{k0}.\end{aligned} \tag{6.29}$$

The second approximation for $k \geq 1$ follows directly. The first estimate for $k = 0$ follows by placing terms involving $\mathbb{U}$ into the error.

Case 2: $j = 1, 2$. Using Lemma 6.6 to replace $\xi_j(s_{\mathbf{p}})$ in (5.13), we have

$$-\int \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \int_{\mathscr{I}_{\mathbf{p}}} \theta_0 \mathbf{E}_j \cdot \mathbf{n}_{\mathbf{p}}\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}$$

Since $\mathbf{n}_{\mathbf{p}} = \mathbf{n}_0 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1})$ by Lemma 6.2 and applying identity (2.16) we find

$$\begin{aligned}-\int \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} &= \int_{\mathscr{I}_{\mathbf{p}}} \theta_j\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\mathbb{E}_{kj}, \\ &= (1+\mathrm{p}_0)\delta_{jk} + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2})\mathbb{E}_{kj}.\end{aligned}$$

Here we used $\|\tilde{\theta}_j - \theta_j\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}$ by its definition in (2.9).

Case 3: $j \geq 3$. We follow the approach for the case $j = 1$. Using Lemmas 6.6 and 5.1, we write

$$-\int_{\mathscr{I}_{\mathbf{p}}} \xi_j(s_{\mathbf{p}})\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \int_{\mathscr{I}_{\mathbf{p}}} \left[\tilde{\theta}_j - \frac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_j} \ln A}{A}(1+\bar{p})\right] \mathbf{n}_0 \cdot \mathbf{n}_{\mathbf{p}}\tilde{\theta}_k \, \mathrm{d}\tilde{s}_{\mathbf{p}}.$$

Approximating the normal projection $\mathbf{n}_0 \cdot \mathbf{n}_\mathbf{p}$ with (6.6) and using the orthogonality of $\{\tilde\theta_k\}$ in (2.15) we derive

$$-\int_{\mathscr{I}_\mathbf{p}} \xi_j(s_\mathbf{p})\tilde\theta_k \, \mathrm{d}\tilde s_\mathbf{p} = \int_{\mathscr{I}_\mathbf{p}} \left[\tilde\theta_j - \frac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_j} \ln A}{A}(1+\bar p)\right]\tilde\theta_k \, \mathrm{d}\tilde s_\mathbf{p} + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2})\mathbb{E}_{kj}$$
$$= (1+\mathrm{p}_0)\delta_{jk} + O(\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_2^2})\mathbb{E}_{kj}.$$

Here we also used the l^2 upper bound of $\nabla_\mathbf{p} A$ in Lemma 6.2. The last identity for the case $j \geq 3$ follows.

To deal with the weighted case for $k \geq 3$ we use a similar derivation which leads to the estimate

$$\int_{\mathscr{I}_\mathbf{p}} \xi_j(s_\mathbf{p})\beta_k\tilde\theta_k \, \mathrm{d}\tilde s_\mathbf{p} = (1+\mathrm{p}_0)\beta_k\delta_{kj} - (1+\mathrm{p}_0)\hat{\mathbf{p}^T}\mathbb{U}\mathbf{B}_k\delta_{j0} + \int_{\mathscr{I}_\mathbf{p}} \mathrm{R}[\xi_j]\beta_k\tilde\theta_k \, \mathrm{d}\tilde s_\mathbf{p}.$$

Here the remainder term $\mathrm{R}[\xi_j]$ is given by

$$\mathrm{R}[\xi_j] = \begin{cases} -\dfrac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_0}\ln A}{A}(1+\bar p)\mathbf{n}_0\cdot\mathbf{n}_\mathbf{p} - 1 - \tilde s_\mathbf{p}\bar p'\left(\dfrac{\mathbf{n}_0\cdot\mathbf{n}_\mathbf{p}}{A} - 1\right) & j = 0; \\ \theta_0\left(\mathbf{E}_j\cdot\mathbf{n}_\mathbf{p} - \tilde\theta_j\right) & j = 1, 2; \\ \tilde\theta_j\mathbf{n}_0\cdot(\mathbf{n}_\mathbf{p} - \mathbf{n}_0) - \dfrac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}\ln A}{A}(1+\bar p)\mathbf{n}_0\cdot\mathbf{n}_\mathbf{p}, & j \geq 3. \end{cases}$$

We observe $\mathrm{R}[\xi_j]$ involves only the zero and first derivatives: $\bar p$, $\bar p'$. Again we note that by Lemma 6.2 both $\mathbf{n}_\mathbf{p}\cdot\mathbf{n}_0$ and A are equal to one up to some quadratic errors. The contribution from the remainder is estimated through Lemma 6.12. □

6.5 Control of q, *w*

Taking the L^2 projection of the evolution Eq. (4.10) with Q and $\partial_t Q$ we obtain l^2 estimates on the evolution of the pearling parameter vector $\mathbf{q}$.

Lemma 6.15 *Under the assumptions of Theorem 4.1, then there exists $C > 0$ independent of ε such that the pearling parameter vector $\mathbf{q} = (\mathrm{q}_k(t))_{k\in\Sigma_0}$ obeys*

$$\partial_t\|\mathbf{q}\|^2_{l^2} + C\varepsilon\|\mathbf{q}\|^2_{l^2} \lesssim \varepsilon\rho^{-4}\|\mathbb{L}_\mathbf{p} w\|^2_{L^2} + \varepsilon^{-1}\|\mathrm{N}(v^\perp)\|^2_{L^2} + \varepsilon^8\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2}.$$

Moreover, the l^2-norm of the time derivative $\dot{\mathbf{q}}$ can be bounded by

$$\|\dot{\mathbf{q}}\|^2_{l^2} \lesssim \|\mathbf{q}\|^2_{L^2} + \varepsilon^2\|w\|^2_{L^2} + \varepsilon\|\dot{\mathbf{p}}\|^2_{l^2} + \|\mathrm{N}(v^\perp)\|^2_{L^2} + \varepsilon^9\|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2}.$$

Proof The proof is a simplification of that of Lemma 5.4 of [5] since the base interface Γ_0 is a circle, so the contribution from the geometric quantities of the base interface Γ_0 are zero. □

Taking L^2-inner product of (4.10) with $\mathbb{L}_\mathbf{p} w$ we develop two H^2 estimates of w by dealing with the residual differently. These estimates have utility on different time scales.

Lemma 6.16 *Under the same assumptions of Theorem 4.1, the function $w \in \mathcal{Z}_*^\perp$, obeys*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbb{L}_\mathbf{p} w, w\right\rangle_{L^2} + \frac{1}{2}\|\mathbb{L}_\mathbf{p} w\|^2_{L^2} \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|^2_{l^2} + \varepsilon^2\rho^{-4}\|\mathbf{q}\|^2_{l^2}$$
$$+\varepsilon^5|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \varepsilon^7(1 + \|\hat{\mathbf{p}}\|^2_{\mathbb{V}_4^2}) + \|\mathrm{N}(v^\perp)\|^2_{L^2};$$

The fast mode also obeys the second estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathbb{L}_{\mathbf{p}} w, w\right\rangle_{L^2} + \frac{1}{2}\|\mathbb{L}_{\mathbf{p}} w\|_{L^2}^2 \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^2\rho^{-4}\|\mathbf{q}\|_{l^2}^2 + \varepsilon^5(|\mathrm{p}_0 - \mathrm{p}_0^*|^2 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}^2) + \varepsilon^7\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2 + \|\mathrm{N}(v^\perp)\|_{L^2}^2. \tag{6.30}$$

Proof The first estimate is derived in Lemma 5.3 of [5], we address the second estimate. Taking the L^2-inner product of (4.10) with $\mathbb{L}_{\mathbf{p}} w$ we estimate each term as in the first case, except for the residual $\mathrm{F}(\Phi_{\mathbf{p}})$. This yields the bound

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathbb{L}_{\mathbf{p}} w, w\right\rangle_{L^2} + \|\mathbb{L}_{\mathbf{p}} w\|_{L^2}^2 \lesssim \left(\varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2} + \varepsilon\rho^{-2}(\|\mathbf{q}\|_{l^2} + \|\dot{\mathbf{q}}\|_{l^2}) + \|\mathrm{N}\|_{L^2}\right)\|\mathbb{L}_{\mathbf{p}} w\|_{L^2} - 2\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), \mathbb{L}_{\mathbf{p}} w\right\rangle_{L^2}.$$

Applying Hölder's and Young's inequalities this reduces to the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathbb{L}_{\mathbf{p}} w, w\right\rangle_{L^2} + \|\mathbb{L}_{\mathbf{p}} w\|_{L^2}^2 \lesssim \varepsilon\|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^2\rho^{-4}(\|\mathbf{q}\|_{l^2}^2 + \|\dot{\mathbf{q}}\|_{l^2}^2) + \|\mathrm{N}\|_{L^2}^2 + \|\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})\|_{L^2}^2.$$

The second estimate on w follows from the L^2-bound on the residual $\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})$ given in Lemma 3.11, and the l^2-bound on $\dot{\mathbf{q}}$ in Lemma 6.15. □

The Lemmas 6.16, 6.15 and Theorem 5.1, incorporate L^2-bounds of the nonlinear term $\mathrm{N}(v^\perp)$. This quantity, and the L^∞ norm of the orthogonal perturbation are bounded in terms of the fast and the pearling modes in Lemma 5.9 of [5], which we quote below for completeness.

Lemma 6.17 *If $\|v^\perp\|_{L^\infty(\Omega)}$ is bounded independent of ε, then*

$$\|\mathrm{N}(v^\perp)\|_{L^2} \lesssim \varepsilon^{-1}\left(\rho^{-2}\left\langle \mathbb{L}_{\mathbf{p}} w, w\right\rangle_{L^2} + \|\mathbf{q}(t)\|_{l^2}^2\right), \tag{6.31}$$

Moreover, if $v^\perp = w + Q$ as in (4.9) then it admits the upper bound

$$\|v^\perp\|_{L^\infty} \lesssim \varepsilon^{-1}\left(\rho^{-1}\left\langle \mathbb{L}_{\mathbf{p}} w, w\right\rangle_{L^2}^{1/2} + \|\mathbf{q}(t)\|_{l^2}\right).$$

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