



Linear damping and depletion in flowing plasma with strong sheared magnetic fields



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ARTICLE INFO

Article history:

Received 29 March 2021

Available online 16 July 2021

MSC:

35Q35

76E05

76E25

76W05

Keywords:

MHD

Flowing plasma

Damping

Depletion

ABSTRACT

In this paper, we study the long-time behavior of the solution for the linearized ideal MHD around sheared velocity and magnetic field under Stern stability condition. We prove that the velocity and magnetic field will converge to sheared velocity and magnetic field as time approaches infinity. Moreover a new depletion phenomenon is proved: the horizontal velocity and magnetic field at the critical points will decay to 0 as time approaches infinity.

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RÉSUMÉ

Dans cet article on étudie le comportement en temps long de la solution des équations de la MHD idéale linéarisées autour d'un champ de vitesses et d'un champ magnétique cisailés avec la condition de stabilité de Stern. On démontre la convergence du champ des vitesses et celle du champ magnétique vers des états de cisaillement lorsque le temps tend vers l'infini. On découvre aussi un nouveau phénomène de déplétion : aux points critiques, les composantes horizontales du champ de vitesse et du champ magnétique décroissent vers 0, quand le temps tend vers l'infini.

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¹ The work of H.L. is supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.

² The work of N.M. is supported by NSF grant DMS-1716466 and by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.

³ The work of C.Z. is partially supported by the Fundamental Research Funds for the Central Universities under Grants No. 06108275.

1. Introduction

The appearance of large coherent structures is an important phenomenon in the magnetic fluid. The study of the long-time behavior of MHD waves is a very active field in physics and mathematics [5,11,24].

1.1. The linearized MHD system

In this paper, we consider the magnetic fluid described by the two-dimensional incompressible ideal MHD equations on the periodic domain $\mathbb{T}^2 = \{(x, y) | x \in \mathbb{T}, y \in \mathbb{T}\}$

$$\begin{cases} \partial_t \mathcal{U} + \mathcal{U} \cdot \nabla \mathcal{U} - \mathcal{H} \cdot \nabla \mathcal{H} + \nabla \mathcal{P} = 0, \\ \partial_t \mathcal{H} + \mathcal{U} \cdot \nabla \mathcal{H} - \mathcal{H} \cdot \nabla \mathcal{U} = 0, \\ \nabla \cdot \mathcal{U} = 0, \quad \nabla \cdot \mathcal{H} = 0, \end{cases} \quad (1)$$

with initial data $\mathcal{U}(0, x, y)$ and $\mathcal{H}(0, x, y)$. Here $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2)$, $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$ and \mathcal{P} denote the velocity field, magnetic field, and the total pressure of the magnetic fluid, respectively.

System (1) has an equilibrium $\mathcal{U}_s = (u(y), 0)$, $\mathcal{H}_s = (b(y), 0)$, $\mathcal{P}_s = \text{const.}$. We shall focus on the asymptotic behavior of the linearized 2D MHD equations around this equilibrium, which take the form

$$\begin{cases} \partial_t U_1 + u \partial_x U_1 + \partial_x p + u' U_2 - b \partial_x H_1 - b' H_2 = 0, \\ \partial_t U_2 + u \partial_x U_2 + \partial_y p - b \partial_x H_2 = 0, \\ \partial_t H_1 + u \partial_x H_1 + b' U_2 - b \partial_x U_1 - u' H_2 = 0, \\ \partial_t H_2 + u \partial_x H_2 - b \partial_x U_2 = 0, \\ \nabla \cdot U = 0, \quad \nabla \cdot H = 0. \end{cases} \quad (2)$$

We introduce the vorticity $\omega = \partial_x U_2 - \partial_y U_1$ and the current density $j = \partial_x H_2 - \partial_y H_1$ which satisfy the following equations:

$$\begin{cases} \partial_t \omega + u \partial_x \omega - b \partial_x j = u'' U_2 - b'' H_2, \\ \partial_t j + u \partial_x j - b \partial_x \omega = b'' U_2 - u'' H_2 + u' \partial_x H_1 - u' \partial_y H_2 + b' \partial_y U_2 - b' \partial_x U_1. \end{cases} \quad (3)$$

We further introduce the stream function ψ and the magnetic potential function ϕ , satisfying $U = (\partial_y \psi, -\partial_x \psi)$, $\omega = -\Delta \psi$ and $H = (\partial_y \phi, -\partial_x \phi)$, $j = -\Delta \phi$, which allow us to derive the following system, satisfied by (ψ, ϕ) :

$$\begin{cases} \partial_t (\Delta \psi) + u \partial_x (\Delta \psi) - b \partial_x (\Delta \phi) = u'' \partial_x \psi - b'' \partial_x \phi, \\ \partial_t (\Delta \phi) + u \partial_x (\Delta \phi) - b \partial_x (\Delta \psi) = b'' \partial_x \psi - u'' \partial_x \phi - 2u' \partial_x \partial_y \phi + 2b' \partial_x \partial_y \psi. \end{cases} \quad (4)$$

Taking the Fourier transform in x and inverting the operator $(\partial_y^2 - \alpha^2)$, we rewrite the system as

$$\begin{cases} \partial_t \begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix} (t, \alpha, y) = -i\alpha M_\alpha \begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix} (t, \alpha, y), \\ (\hat{U}_1, \hat{U}_2) = (\partial_y \hat{\psi}, -i\alpha \hat{\psi}), \quad (\hat{H}_1, \hat{H}_2) = (\partial_y \hat{\phi}, -i\alpha \hat{\phi}), \end{cases} \quad (5)$$

where $\alpha \neq 0$ and

$$M_\alpha = -\Delta_\alpha^{-1} \begin{bmatrix} u'' - u \Delta_\alpha & -b'' + b \Delta_\alpha \\ b \Delta_\alpha + b'' + 2b' \partial_y & -u \Delta_\alpha - u'' - 2u' \partial_y \end{bmatrix}. \quad (6)$$

For the homogeneous equilibrium $u = 0$, $b(y) = \text{const.}$, \widehat{U}_2 and \widehat{H}_2 satisfy a 1-D wave equation, which is stable but exhibits no decay. There are few rigorous mathematical results on the non-flowing plasma with inhomogeneous sheared magnetic field. In [18], Tataronis and Grossmann predicted that the vertical components of velocity and magnetic field may decay by phase mixing, to which a mathematically rigorous proof was given by Ren and Zhao in [15], under the condition that the magnetic field is positive and strictly monotone. If the positivity assumption on the magnetic field is removed, which allows the direction of the sheared magnetic field to change, then it turns out that magnetic reconnection occurs in infinite time, generating the magnetic island. This phenomenon was predicted by Hirota, Tatsuno and Yoshida [6] and later justified by Zhai, Zhang and Zhao [22]. For the flowing plasma $u \neq 0$, fewer mathematical rigorous results are available. We refer to [6,14,22] for the long time behaviors of the solutions to the MHD equations linearized around a flowing plasma.

Notations: Let us specify the notations to be used throughout the paper. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ and by $A \sim B$ an estimate of the form $C^{-1}B \leq A \leq CB$, where C is a constant. Given a function $f(x, y)$, we denote its Fourier transform in x -variable as $\widehat{f}(\alpha, y) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ix\alpha} f(x, y) dx$, where α is the wave number. We shall use the Japanese bracket notation $\langle x \rangle := \sqrt{|x|^2 + 1}$.

1.2. Vertical damping and horizontal depletion

In this paper, we focus on the long time behavior of the solution to the linearized MHD equation (5).

Our first main result states as follows:

Theorem 1.1 (Vertical damping). *Let $u, b \in C^3(\mathbb{T})$ be such that $b > |u| \geq 0$ and the critical points of $(u \pm b)$ are non-degenerate. Let $\alpha \neq 0$ be a fixed wave number and let $(\widehat{\psi}, \widehat{\phi})$ solve (5) with initial data $(\widehat{\psi}_0, \widehat{\phi}_0) \in (H^3 \times H^3)$. Then the following space-time estimate holds:*

$$\left\| (\widehat{\psi}, \widehat{\phi}) \right\|_{H_t^1 L_y^2} \leq C_\alpha \left\| (\widehat{\psi}_0, \widehat{\phi}_0) \right\|_{H_y^3}. \tag{7}$$

In particular, $\lim_{t \rightarrow \infty} \left\| (\widehat{U}_2, \widehat{H}_2) \right\|_{L_y^2} = 0$.

Remark 1.2. The condition $|u| < |b|$ is called Stern stability condition (see [16]).

Remark 1.3. Formally, the space-time estimate may indicate that $\left\| (\widehat{U}_2, \widehat{H}_2) \right\|_{L^2} \lesssim \frac{1}{\langle t \rangle^\beta}$ with $\beta > \frac{1}{2}$. The study of the precise decay rate shall be our forthcoming work.

Remark 1.4. For the case of flowing plasma with constant velocity or non-flowing plasma ($u = \text{const.}$), it holds that

$$\|U(t), H(t)\|_{H_x^k L_y^2} \sim \|U_0, H_0\|_{H_x^k L_y^2}, \quad k \geq 0, \tag{8}$$

which implies linear growth of vorticity and current density, i.e.,

$$\|\omega(t), j(t)\|_{L_{x,y}^2} \lesssim \langle t \rangle \|\omega_0, j_0\|_{L_{x,y}^2}.$$

The proof can be found in Appendix C. By (8), the total energy is almost conserved. Hence, the vertical damping in Theorem 1.1 shows that energy is transferred from the vertical direction to the horizontal direction. Whether similar energy conservation results are true for the flowing plasma with $u \neq \text{const.}$ remains an open question.

The vertical damping is induced by a certain mixing mechanism similar to the vorticity mixing that leads to inviscid damping for linearized Euler equations (see [9,10,19–21,23]). Readers may consult [2,4,7,8,12,13] for recent progress in nonlinear inviscid damping.

To better illustrate the mixing mechanism, let us recall the system in terms of (U_1, H_1) :

$$\begin{cases} \partial_t U_1 + u \partial_x U_1 - b \partial_x H_1 = L_1, \\ \partial_t H_1 + u \partial_x H_1 - b \partial_x U_1 = L_2, \end{cases} \quad (9)$$

where $(L_1, L_2) := (b'H_2 - u'U_2 - 2\partial_x \Delta^{-1}(b'\partial_x H_2 - u'\partial_x U_2), u'H_2 - b'U_2)$ can be seen as nonlocal forcing terms depending on U_2 and H_2 .

By the incompressibility condition, we can check that

$$\|(\partial_x U_1, \partial_x H_1)\|_{L_x^2 H_y^{-1}} \sim \|(U_2, H_2)\|_{L_{x,y}^2}.$$

Then the mixing of $(U_1 \pm H_1)$ would lead to the linear damping of (U_2, H_2) .

Let us consider a toy model, obtained by neglecting the nonlocal forcing terms (L_1, L_2) in the linearized system (9), i.e.,

$$\begin{cases} \partial_t U_1 + u \partial_x U_1 - b \partial_x H_1 = 0, \\ \partial_t H_1 + u \partial_x H_1 - b \partial_x U_1 = 0. \end{cases} \quad (10)$$

It is then easy to see that the Elsässer variables $Z_1^\pm := U_1 \pm H_1$ satisfy certain transport equations and then

$$\begin{aligned} (\widehat{U}_1 + \widehat{H}_1)(t, \alpha, y) &= \widehat{Z}_{1,in}^+(\alpha, y) e^{-i\alpha(u-b)t}, \\ (\widehat{U}_1 - \widehat{H}_1)(t, \alpha, y) &= \widehat{Z}_{1,in}^-(\alpha, y) e^{-i\alpha(u+b)t}, \end{aligned} \quad (11)$$

with $\widehat{Z}_{1,in}^\pm$ denoting the initial data.

Regarding the toy model (10), we have the following conclusions.

Lemma 1.5. *Let $u, b \in C^3(\mathbb{T})$ be such that $u \pm b$ have only non-degenerate critical points. Then the solution of (10) with initial data $(U_{1,in}, H_{1,in})$ satisfies*

$$\|(\partial_x U_1, \partial_x H_1)\|_{L_x^2 H_y^{-1}} \lesssim \frac{1}{\langle t \rangle^{\frac{1}{2}}} \|(U_{1,in}, H_{1,in})\|_{H_x^{\frac{1}{2}} H_y^1}. \quad (12)$$

Moreover, if the initial data $(U_{1,in}, H_{1,in})$ vanish at all the critical points of $(u \pm b)$, then it holds that

$$\|(\partial_x U_1, \partial_x H_1)\|_{L_x^2 H_y^{-1}} \lesssim \frac{1}{\langle t \rangle} \|(U_{1,in}, H_{1,in})\|_{H_x^{-1} H_y^2}. \quad (13)$$

The proof of (12) can be found in [20]; via the same dual method one can prove (13). In fact, the decay rate of $t^{-1/2}$ for the toy model (10) is optimal, as we know, via the classical stationary phase approximation (see Chapter VIII of [17]), that there exists a class of initial data such that the corresponding solutions satisfy

$$\|(\partial_x U_1, \partial_x H_1)\|_{L_x^2 H_y^{-1}} \sim \frac{1}{\langle t \rangle^{\frac{1}{2}}}.$$

The space-time estimate in Theorem (1.1), however, fails to hold for the toy model (10). Exploring the mechanisms behind the enhanced damping for the complete system (9), we found a new dynamical

phenomenon apart from velocity mixing: the depletion of horizontal velocity and magnetic field (U_1, H_1) at the critical points of $u \pm b$. This leads to our second main result, which states as follows:

Theorem 1.6 (Horizontal depletion). *Let u, b satisfy the same assumptions as in Theorem 1.1. Let y_0 be a critical point of $(u + b)$ or $(u - b)$, i.e., $u'(y_0) = b'(y_0)$ or $u'(y_0) = -b'(y_0)$. Let $(\widehat{U}_1, \widehat{H}_1)$ correspond to the solution to (5) with initial data $(\widehat{\psi}_0, \widehat{\phi}_0) \in (H^3 \times H^3)$. Then it holds that*

$$\lim_{t \rightarrow \infty} \left| (\widehat{U}_1, \widehat{H}_1)(t, \alpha, y_0) \right| = 0.$$

Remark 1.7. From (11), we can see that the horizontal depletion in Theorem 1.6 is not true for the toy model (10).

For a more precise description of the long time behavior of the horizontal components, we conjecture that there exist some final states $\widehat{Z}_{1,\infty}^\pm$ and some $\kappa > 0$ such that

$$\begin{aligned} (\widehat{U}_1 + \widehat{H}_1)(t, \alpha, y) &\sim (\widehat{Z}_{1,\infty}^+)(\alpha, y)e^{-i\alpha(u-b)t} + \mathcal{O}(t^{-\kappa}), \\ (\widehat{U}_1 - \widehat{H}_1)(t, \alpha, y) &\sim (\widehat{Z}_{1,\infty}^-)(\alpha, y)e^{-i\alpha(u+b)t} + \mathcal{O}(t^{-\kappa}). \end{aligned} \tag{14}$$

Theorem 1.6 reveals that $\widehat{Z}_{1,\infty}^\pm$ vanish at all the critical points of $(u \pm b)$. Our conjecture (14) shall imply the space-time estimate (7) and even a precise decay rate, provided that suitable regularity of the final states could be proven.

The parallel phenomenon exists in the realm of hydrodynamics. Vorticity depletion for the linearized 2D Euler equations around shear flows, first predicted by Bouchet and Morita in [3], was later mathematically proven in [20] by Wei, Zhang and Zhao. A similar vorticity depletion result for the 2D Euler equation linearized around a radially symmetric and strictly decreasing vorticity distribution is due to Bedrossian, Coti-Zelati and Vicol [1]. **As far as we know, this is the first paper studying the depletion of the horizontal velocity and magnetic field for the linearized MHD equations.**

Comparing the toy model (10) with the complete system (9), we observe at least three significant effects of the nonlocal terms L_1 and L_2 :

1. Altering the final state, as $\widehat{Z}_{1,in}^\pm$ on the right hand side of (11) are changed into some other final state $\widehat{Z}_{1,\infty}^\pm$ on the right hand side of (14);
2. Causing $\widehat{Z}_{1,\infty}^\pm$ to vanish at the critical points of $(u \pm b)$;
3. Enhancing the damping.

This demonstrates that the nonlocal terms L_1 and L_2 in (9) cannot simply be neglected or be regarded as mere perturbations on the toy model.

Remark 1.8. The significance of the nonlocal terms can also be seen from the resolvent estimate. Indeed, if we consider the toy model (10) instead, the Sturmian equation (16) in Section 3 would become $((u - c)^2 - b^2) \partial_y \Phi = F$, which is much simpler and yields an obvious estimate

$$|\partial_y \Phi| \lesssim |(u - c)^2 - b^2|^{-1}.$$

Yet, as we shall see in Section 3, at any critical point y_0 of $(u + b)$ or $(u - b)$, the solution Φ to the actual Sturmian equation (16) enjoys a non-trivial estimate

$$|\partial_y \Phi(y_0)| \lesssim |(u(y_0) - c)^2 - b(y_0)^2|^{-\frac{3}{4}}.$$

This $\frac{1}{4}$ -improvement, resulting from the effects of L_1 and L_2 , is the key to the depletion result. For more details, we refer to Section 3 and 4.

We have the following comments on the results in Theorem 1.1 and Theorem 1.6.

Remark 1.9. The vertical damping and horizontal depletion results also hold for the case of finite channel under slip boundary condition, provided that the critical points do not appear at the boundary.

Remark 1.10. To highlight the differences among the long time behaviors of the solutions to the linearized MHD equations in various cases, we show the following table:

Conditions				Results	References
Monotonicity	Uniform direction $b > 0$	Stern stability $ u \leq b $	Other conditions		
Yes	Yes	Yes	$u \equiv 0$	Damping	[15]
Yes	No	Yes	$u(0) = b(0) = 0$	Magnetic Island	[22]
Yes	No	No	$u = k_1 y, b = k_2 y$ $k_1 > k_2 \geq 0$	Damping	[14]
No	Yes	Yes $ u < b$	Non-degenerate critical points	Damping & Depletion	This paper

Comparing the results listed above, we have the following observations:

1. The first result of this paper, in comparison with those in [15] and [22], indicates that the unidirectionality of the sheared magnetic field, rather than the monotonicity, is responsible for the damping.
2. The stability condition $|u| < b$ in this paper seems to indicate magnetism-driven mechanisms, whilst the damping in [14] might be seen as fluid-driven.

1.3. Outline of the paper

The organization of the paper is as follows. In Section 2, we introduce the representation formula, which is a contour integral of the resolvent. The study of the resolvent is then reduced to that of the Sturmian equation. In Section 3, we study the Sturmian equation and establish a uniform estimate as well as the limiting absorption principle. This is the most technical part of the paper as our assumptions include rather general sheared velocity and magnetic field, which in turn results in a range of situations that need to be discussed separately from each other. A novelty here is the use of ODE techniques in dealing with the situation involving the critical points. In Section 4, we prove the main theorems by the resolvent estimate.

2. The Dunford integral and the Sturmian equation

The basic idea for the study of the long-time behavior of the solution to (5) is to acquire a precise formula of (ψ, ϕ) , which requires understanding the spectral properties of the linearized operator M_α . Indeed, it is easy to check that the spectrum $\sigma(M_\alpha) = \text{Ran}(u + b) \cup \text{Ran}(u - b)$. Then we have the Dunford integral

$$\begin{aligned} \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(t, \alpha, y) &= \frac{1}{2\pi i} \int_{\partial\Omega} e^{-i\alpha t c} (cI - M_\alpha)^{-1} \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(0, \alpha, y) dc \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\partial\Omega_\epsilon} e^{-i\alpha t c} (cI - M_\alpha)^{-1} \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(0, \alpha, y) dc, \end{aligned} \tag{15}$$

where Ω contains the spectrum $\sigma(M_\alpha)$ and Ω_ϵ is the ϵ -neighborhood of $\sigma(M_\alpha)$. Here the last equality holds due to the fact that for $c \notin \sigma(M_\alpha)$, the resolvent $(cI - M_\alpha)^{-1} \begin{pmatrix} \widehat{\psi} \\ \widehat{\phi} \end{pmatrix}(0, \alpha, y)$ is analytic in c . With the representation formula (15), we have reduced our problem to the study of the resolvent $(cI - M_\alpha)^{-1}$ and the limit of the contour integral.

Assume that

$$(cI - M_\alpha)^{-1} \begin{pmatrix} \widehat{\psi}_0 \\ \widehat{\phi}_0 \end{pmatrix}(\alpha, y) = \begin{pmatrix} \Psi_1 \\ \Phi_1 \end{pmatrix}(\alpha, y, c).$$

Then a direct calculation shows that (Ψ_1, Φ_1) solves the following system, for $c \notin \sigma(M_\alpha)$:

$$\begin{cases} (u - c)\Delta_\alpha \Psi_1 - u''\Psi_1 - b\Delta_\alpha \Phi_1 + b''\Phi_1 = \Delta_\alpha \widehat{\psi}_0, \\ (u - c)\Phi_1 - b\Psi_1 = -\widehat{\phi}_0. \end{cases}$$

Here $\Delta_\alpha = \partial_y^2 - \alpha^2$. Let $\Phi_1(\alpha, y, c) = b(y)\Phi(\alpha, y, c)$, and then

$$\Psi_1(\alpha, y, c) = (u(y) - c)\Phi(\alpha, y, c) + \widehat{\phi}_0(\alpha, y)/b(y).$$

Thus, we obtain

$$\begin{aligned} & \partial_y \left(\left((u(y) - c)^2 - b(y)^2 \right) \partial_y \Phi(\alpha, y, c) \right) - \alpha^2 \left((u(y) - c)^2 - b(y)^2 \right) \Phi(\alpha, y, c) \\ &= \Delta_\alpha \widehat{\psi}_0(\alpha, y) - (u(y) - c)\Delta_\alpha \left(\frac{\widehat{\phi}_0(\alpha, y)}{b(y)} \right) + u''(y) \frac{\widehat{\phi}_0(\alpha, y)}{b(y)} := F(\alpha, y, c). \end{aligned} \tag{16}$$

This is the so-called Sturmian type equation.

3. Uniform estimates and limiting absorption principle

Let Ω_{ϵ_0} denote the ϵ_0 -neighborhood of $\text{Ran}(u + b) \cup \text{Ran}(u - b)$ in \mathbb{C} . We shall study Equation (16) with $c \in \Omega_{\epsilon_0}$. We introduce the Elsässer variables $Z_\pm := u \pm b$. Hence, we can rewrite the equation of $\Phi(\alpha, y, c)$ as

$$\partial_y((Z_- - c)(Z_+ - c)\partial_y \Phi) - \alpha^2(Z_- - c)(Z_+ - c)\Phi = F. \tag{17}$$

This section is devoted to the proofs of the following propositions, which are crucial to our main results.

Proposition 3.1. *There exists $\epsilon_0 > 0$ such that for $c \in (\Omega_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-))$, the solution to (17) satisfies the following bound, uniform with respect to c*

$$\|\Phi(\alpha, \cdot, c)\|_{L^2} + \|(Z_- - c)(Z_+ - c)\partial_y \Phi(\alpha, \cdot, c)\|_{H^1} \leq C\|F(\alpha, \cdot, c)\|_{H^1}.$$

The estimate on Φ can be continued (in c) up to the boundary $\text{Ran } Z_+ \cup \text{Ran } Z_-$.

Proposition 3.2. *For $c \in (\text{Ran } Z_+ \cup \text{Ran } Z_-)$, there exist $\Phi^\pm(\alpha, \cdot, c) \in L^2$ such that as $\epsilon \rightarrow 0^+$, $\Phi(\alpha, \cdot, c \pm i\epsilon) \rightarrow \Phi^\pm(\alpha, \cdot, c)$ in L^r with $r \in (1, 2)$ and*

$$\|\Phi^\pm(\alpha, \cdot, c)\|_{L^2} \leq C\|F(\alpha, \cdot, c)\|_{H^1}.$$

We shall prove Proposition 3.1 by contradiction. Suppose that the proposition is false, then there exists a sequence $\{c_n, \Phi_n, F_n\}_{n=1}^\infty$ satisfying the equation

$$\partial_y((Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n) - \alpha^2(Z_- - c_n)(Z_+ - c_n)\Phi_n = F_n, \quad (18)$$

such that

- $c_n \in (\Omega_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-))$,
- $\|\Phi_n\|_{L^2} + \|(Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n\|_{H^1} = 1$,
- $F_n \rightarrow 0$ in H^1 as $n \rightarrow \infty$.

For convenience, we shall use the notation $q_n := (Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n$ from time to time in this section.

As $|c_n| \leq C$ and $\|\Phi_n\|_{L^2} + \|q_n\|_{H^1} = 1$, we know that

- $c_n \rightarrow c$ for some $c \in \overline{\Omega}_{\epsilon_0}$ up to a subsequence,
- $\Phi_n \rightarrow \Phi$ in L^2 up to a subsequence,
- $q_n \rightarrow q$ in H^1 up to a subsequence, for some $q \in H^1$.

(For convenience, we shall simply use the original c_n and Φ_n to denote the elements in the subsequence.) Moreover, $q = (Z_- - c)(Z_+ - c)\partial_y\Phi$, as seen from the identity

$$\begin{aligned} \langle q_n, f \rangle_{L^2} &= \langle (Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n, f \rangle_{L^2} \\ &= - \langle \Phi_n, Z'_-(Z_+ - c_n)f + (Z_- - c_n)Z'_+f + (Z_- - c_n)(Z_+ - c_n)f' \rangle_{L^2}, \quad \forall f \in C_0^\infty. \end{aligned}$$

We shall show in the following passages that

- the weak limit $\Phi \equiv 0$,
- in fact, strong convergences hold true, i.e., $\Phi_n \rightarrow 0$ in L^2 and $q_n \rightarrow 0$ in H^1 ,

which contradict the very assumption that $\|\Phi_n\|_{L^2} + \|(Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n\|_{H^1} = 1$.

Formally, by performing integration by parts on the limiting equation

$$\partial_y((Z_- - c)(Z_+ - c)\partial_y\Phi) - \alpha^2(Z_- - c)(Z_+ - c)\Phi = 0, \quad (19)$$

we can see that the weak limit $\Phi \equiv 0$. To prove this, we have to show that Φ belongs to a space for which the operation is allowed.

The limit c belongs to either $(\overline{\Omega}_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-))$ or $(\text{Ran } Z_+ \cup \text{Ran } Z_-)$.

Let us first consider the case when $c_n \rightarrow c = \text{Re } c + i\text{Im } c \notin (\text{Ran } Z_+ \cup \text{Ran } Z_-)$. In this case, it's straightforward to show that $\Phi_n \rightarrow \Phi$ in L^2 and $q_n \rightarrow q := (Z_- - c)(Z_+ - c)\partial_y\Phi$ in H^1 , where Φ is the classical solution to (19). Taking the inner product with $\overline{\Phi}$, integrating by parts and separating the real and imaginary parts, we obtain

$$\int_{\mathbb{T}} \left((u(y) - \text{Re } c)^2 - (b(y))^2 - (\text{Im } c)^2 \right) (|\partial_y\Phi(y, c)|^2 + \alpha^2|\Phi(y, c)|^2) dy = 0, \quad (20)$$

and

$$\int_{\mathbb{T}} (u(y) - \text{Re } c) (|\partial_y\Phi(y, c)|^2 + \alpha^2|\Phi(y, c)|^2) dy = 0. \quad (21)$$

Multiplying (21) by $2\operatorname{Re} c$ and adding it to (20) lead to

$$\int_{\mathbb{T}} (u(y)^2 - b(y)^2 - (\operatorname{Re} c)^2 - (\operatorname{Im} c)^2) (|\partial_y \Phi(y, c)|^2 + \alpha^2 |\Phi(y, c)|^2) dy = 0.$$

The condition $|u| < b$ then guarantees that $\Phi \equiv 0$.

The difficult situation is when $c_n \rightarrow c \in (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-)$ as $n \rightarrow \infty$, which renders Equation (18) degenerate at the sets $(Z_+)^{-1}(c) := \{y \in \mathbb{T} : Z_+(y) = c\}$ and $(Z_-)^{-1}(c) := \{y \in \mathbb{T} : Z_-(y) = c\}$. The condition $0 < |u| < b$ ensures that $\operatorname{Ran} Z_+ \cap \operatorname{Ran} Z_- = \emptyset$. Hence, if $c_n \rightarrow c \in \operatorname{Ran} Z_+$, then $c \notin \operatorname{Ran} Z_-$, and vice versa. In this section, we will always provide detailed proofs only for the case that $c_n \rightarrow c \in \operatorname{Ran} Z_+$, as those for the case $c_n \rightarrow c \in \operatorname{Ran} Z_-$ are essentially similar.

By our assumptions on $(u \pm b)$, the sets $(Z_s)^{-1}(c)$, $s = +$ or $-$, consist of two possible types of points:

1. points located in a monotone interval, i.e., $\{y \in \mathbb{T} : Z_s(y) = c \text{ and } |Z'_s(y)| > 0\}$,
2. critical points, where $Z'_s = 0$ and $|Z''_s| > 0$.

In our analysis, the two situations require separate treatments. To better distinguish between the two, we denote a critical point as y_0 .

3.1. Weak convergence of $\{\Phi_n\}_{n=1}^\infty$ to 0

In this subsection, we establish several lemmas implying that $\Phi_n \rightarrow \Phi \equiv 0$ as $c_n \rightarrow c$. As previously mentioned, to this end we need to prove that Φ is regular enough such that the desired integration by parts is justified. This is clear when $y \notin (Z_-)^{-1}(c) \cup (Z_+)^{-1}(c)$, as $\Phi \in H^3_{loc}(\mathbb{T} \setminus (Z_\pm)^{-1}(c))$ thanks to $|(Z_- - c)(Z_+ - c)| > C > 0$. Therefore, our consideration starts from the case when $y_c \in (Z_s)^{-1}(c)$ lies in a region where Z_s is strictly monotone, $s = +$ or $-$.

Lemma 3.3. *Let $\{c_n\}_{n=1}^\infty \subset (\Omega_{\epsilon_0} \setminus (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-))$ be such that $c_n \rightarrow c$ as $n \rightarrow \infty$ for a certain $c \in (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-)$. Let the triple $\{c_n, \Phi_n, F_n\}_{n=1}^\infty$ satisfy the equation (18) along with the following conditions*

- $\Phi_n \rightarrow \Phi$ in L^2 and $(Z_- - c_n)(Z_+ - c_n)\partial_y \Phi_n \rightarrow (Z_- - c)(Z_+ - c)\partial_y \Phi$ in H^1 in \mathcal{I} ,
- $F_n \rightarrow F$ in H^1 in \mathcal{I} ,

for some interval $\mathcal{I} := [y_1, y_2] \subset \mathbb{T}$ such that there exists $y_c \in (Z_s)^{-1}(c)$ in \mathcal{I} with $|Z'_s(y_c)| > 0$; $Z'_s(y_c)Z'_s(y) > 0, \forall y \in \mathcal{I}$, where $s = +$ or $-$.

Then $\Phi_n \rightarrow \Phi$ in $L^2(\mathcal{I})$ and $(Z_- - c_n)(Z_+ - c_n)\partial_y \Phi_n \rightarrow (Z_- - c)(Z_+ - c)\partial_y \Phi$ in $H^1(\mathcal{I})$.

In particular, if $F \equiv 0$, then $\Phi \in H^1(\mathcal{I})$ and

$$-\int_{\mathcal{I}} (Z_- - c)(Z_+ - c)\partial_y \Phi f' dy + \alpha^2 \int_{\mathcal{I}} (Z_- - c)(Z_+ - c)\Phi f dy = 0, \forall f \in H^1_{w,0}(\mathcal{I}), \tag{22}$$

where $H^1_{w,0} = \{f \in L^2 : ((y - y_c)\partial_y f) \in L^2 \text{ and } f|_{y=y_1} = f|_{y=y_2} = 0\}$.

Proof. To facilitate the proof, let us assume that $c \in \operatorname{Ran} Z_+$. Then for $\epsilon < \frac{\min_{y \in \mathbb{T}} b - \|u\|_{L^\infty}}{3}$, there exists $N > 0$ such that for any $n > N$, $|c_n - c| < \epsilon$. Thus there exists $C > 0$ such that $|Z_- - c_n| > C^{-1} > 0$ for $n > N$. Without loss of generality, let us also assume that $n > N$ and $\operatorname{Im} c_n > 0$. Thus, there exists $y_{c_n} \in \mathcal{I}$ such that $Z_+(y_{c_n}) = \operatorname{Re} c_n$ and $y_{c_n} \rightarrow y_c$ as $c_n \rightarrow c$.

We recall $q_n = (Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n$ with $c_n \in \Omega_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-)$. Dividing (18) by $(Z_- - c_n)(Z_+ - c_n)$ and differentiating, we see that q_n solves the following equation

$$\partial_y \left(\frac{\partial_y q_n}{(Z_- - c_n)(Z_+ - c_n)} \right) - \alpha^2 \left(\frac{q_n}{(Z_- - c_n)(Z_+ - c_n)} \right) = \partial_y \left(\frac{F_n}{(Z_- - c_n)(Z_+ - c_n)} \right). \tag{23}$$

By Equation (18), q_n also satisfies

$$q'_n = F_n + \alpha^2(Z_- - c_n)(Z_+ - c_n)\Phi_n, \tag{24}$$

$$q''_n - \alpha^2 q_n = F'_n + \alpha^2 Z'_-(Z_+ - c_n)\Phi_n + \alpha^2(Z_- - c_n)Z'_+\Phi_n. \tag{25}$$

From Equation (25) and the assumptions $\Phi_n \rightharpoonup \Phi$ in L^2 , $q_n \rightharpoonup q := (Z_- - c)(Z_+ - c)\partial_y\Phi$ in H^1 and $F_n \rightarrow F$ in H^1 , we can infer that

$$\|q''_n\|_{L^2} \leq \|F'_n\|_{L^2} + \alpha^2 (\|\Phi_n\|_{L^2} + \|q_n\|_{H^1}), \tag{26}$$

which, along with the assumption that $q_n \rightharpoonup (Z_- - c)(Z_+ - c)\partial_y\Phi$ in H^1 , implies strong convergence, i.e.,

$$(Z_- - c_n)(Z_+ - c_n)\partial_y\Phi_n \rightarrow (Z_- - c)(Z_+ - c)\partial_y\Phi \text{ in } H^1.$$

By our assumption and Sobolev embedding, $\|q_n\|_{L^\infty(\mathcal{I})} \leq \|q_n\|_{H^1(\mathcal{I})} < C$, it is true that

$$\|(y - y_{c_n})\partial_y\Phi_n\|_{L^p(\mathcal{I})} < C, \forall p < \infty.$$

Applying the compactness result from Appendix B, we have

$$\Phi_n \rightarrow \Phi \text{ in } L^2.$$

Now we consider the particular case $F_n \rightarrow 0$ in H^1 . As q_n satisfies, for $f \in C^2_0(\mathcal{I})$,

$$-\int_{-\pi}^{\pi} \frac{(q'_n f' + \alpha^2 q_n f)(y)}{(Z_- - c_n)(Z_+ - c_n)} dy = \int_{-\pi}^{\pi} \frac{(F_n f')(y)}{(Z_- - c_n)(Z_+ - c_n)} dy. \tag{27}$$

By the fact that

$$\int_{-\pi}^{\pi} \frac{\Gamma(y)}{(Z_- - c_n)(Z_+ - c_n)} dy = \int_{-\pi}^{\pi} \frac{1}{2b(y)} \frac{\Gamma(y)}{Z_- - c_n} dy - \int_{-\pi}^{\pi} \frac{1}{2b(y)} \frac{\Gamma(y)}{Z_+ - c_n} dy,$$

and

$$-\int_{-\pi}^{\pi} \text{Log}(Z_+(y) - c_n) \partial_y \left(\frac{\Gamma(y)}{2b(y)Z'_+(y)} \right) dy = \int_{-\pi}^{\pi} \frac{1}{2b(y)} \frac{\Gamma(y)}{Z_+ - c_n} dy,$$

we obtain that the right hand side of Equation (27) can be estimated as

$$\left| \int_{-\pi}^{\pi} \frac{(F_n f')(y)}{(Z_- - c_n)(Z_+ - c_n)} dy \right| \lesssim \left| \int_{-\pi}^{\pi} \frac{1}{2b(y)} \frac{(F_n f')(y)}{Z_- - c_n} dy \right|$$

$$+ \left| \int_{-\pi}^{\pi} \text{Log}(Z_+(y) - c_n) \partial_y \left(\frac{(F_n f')(y)}{2b(y)Z'_+(y)} \right) dy \right| \lesssim \|F_n\|_{H^1}.$$

Passing to the limit as $n \rightarrow \infty$, we can see that the right hand side of Equation (27) vanishes as $F_n \rightarrow 0$ in H^1 , and we obtain

$$0 = -\text{p.v.} \int_{-\pi}^{\pi} \frac{(q'f' + \alpha^2 qf)(y)}{(Z_- - c)(Z_+ - c)} dy + i\pi \frac{(q'f' + \alpha^2 qf)(y_c)}{2b(y_c)Z'_+(y_c)}, \quad \forall f \in C_0^1(\mathcal{I}). \tag{28}$$

Let χ be a smooth cut-off function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ near y_λ and $\text{supp } \chi \Subset \mathcal{I}$. Let us specify a test function $f(y) = \chi(y)q(y)b(y_c)Z'_+(y_c)$. The imaginary part of Equation (28) then reads $i\pi (|q'(y_c)|^2 + \alpha^2|q(y_c)|^2) = 0$, which in turn reveals that

$$q(y_c) = q'(y_c) = 0.$$

By the facts that $\partial_y \Phi(y, c) = \frac{q(y, c)}{(Z_- - c)(Z_+ - c)} \sim \frac{q}{y - y_c}$, $q(y_c) = 0$ and Hardy’s inequality, we have

$$\|\partial_y \Phi\|_{L^2(\mathcal{I})} \leq C\|q'\|_{L^2(\mathcal{I})}.$$

Thus, we conclude that $\Phi(\cdot, c) \in H^1(\mathcal{I})$.

Multiplying both sides of Equation (17) by $f \in H_{w,0}^1(\mathcal{I})$, passing to the limit as $n \rightarrow \infty$ and integrating by parts, we obtain the identity in (22). \square

We proceed to prove the analogue of Lemma 3.3 for $y_0 \in (Z_s)^{-1}(c)$ which is also a critical point of Z_s , $s = +$ or $-$. There are several scenarios depending on how c_n approaches c .

(1) In the case that $c = Z_s(y_0)$ and $Z_s''(y_0)(\text{Re } c_n - c) \geq 0$, by the assumptions on u and b , the intersection of $(Z_s)^{-1}(\text{Re } c_n)$ with a sufficiently small neighborhood of y_0 consists of two points $y_{c_n}^\ell$ and $y_{c_n}^r$ such that $y_{c_n}^\ell \leq y_0 \leq y_{c_n}^r$. (If $c = \text{Re } c_n$, equality holds and the two points shrink into one.) Furthermore, it’s true that modulo subsequences either $|\text{Re } c_n - c| \geq |\text{Im } c_n|$ or $|\text{Re } c_n - c| \leq |\text{Im } c_n|, \forall n \in \mathbb{N}$.

(2) In the case $Z_s''(y_0)(\text{Re } c_n - c) < 0$, any small neighborhood of y_0 no longer intersects $(Z_s)^{-1}(\text{Re } c_n)$. Instead, $\text{Re}(Z_s - c_n)$ is sign-definite as $\text{Re}(Z_s - c_n) = (Z_s - c) + (c - \text{Re } c_n)$.

We first consider the case (2) together with the case (1) when $|\text{Re } c_n - c| \leq |\text{Im } c_n|, \forall n \in \mathbb{N}$.

Lemma 3.4. *Let $\{c_n\}_{n=1}^\infty \subset (\Omega_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-))$ be such that $c_n \rightarrow c$ as $n \rightarrow \infty$ for a certain $c \in (\text{Ran } Z_+ \cup \text{Ran } Z_-)$. Let the triple $\{c_n, \Phi_n, F_n\}_{n=1}^\infty$ solve (18) and satisfy*

- $\Phi_n \rightarrow \Phi$ in L^2 and $(Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n \rightarrow (Z_+ - c)(Z_- - c)\partial_y \Phi$ in L^2 on \mathcal{I}_0 ,
- $F_n \rightarrow F$ in H^1 on \mathcal{I}_0 ,

for some interval $\mathcal{I}_0 := [y_1, y_2] \subset \mathbb{T}$ such that for $s = +$ or $-$, one of the following situations occurs:

1. There exists $y_0 \in (Z_s)^{-1}(c)$ in \mathcal{I}_0 at which $Z'_s(y_0) = 0$; $Z_s''(y_0)Z_s''(y) > 0, \forall y \in \mathcal{I}_0$. For n sufficiently large, $|\text{Re } c_n - c| \leq |\text{Im } c_n|$ and $Z_s''(y_0)(\text{Re } c_n - c) \geq 0$.
2. There exists $y_0 \in (Z_s)^{-1}(c)$ in \mathcal{I}_0 at which $Z'_s(y_0) = 0$; $Z_s''(y_0)Z_s''(y) > 0, \forall y \in \mathcal{I}_0$. For n sufficiently large, $Z_s''(y_0)(\text{Re } c_n - c) < 0$.

Then for n sufficiently large,

1. $(y - y_0)\partial_y \Phi_n \in L^2, (y - y_0)\partial_y \Phi \in L^2$ and

$$-\int_{\mathcal{I}_0} (Z_+ - c)(Z_- - c) (\partial_y \Phi f' + \alpha^2 \Phi f) \, dy = \int_{\mathcal{I}_0} F f \, dy, \quad \forall f \in H_{w,0}^1, \tag{29}$$

where $H_{w,0}^1 = \{f \in L^2 : ((y - y_0)\partial_y f) \in L^2 \text{ and } f|_{y=y_1} = f|_{y=y_2} = 0\}$;

2. there exists a constant C independent of c_n such that

$$|\Phi_n(y_0)| \leq C |(Z_+(y_0) - c_n)(Z_-(y_0) - c_n)|^{-\frac{1}{4}}; \tag{30}$$

3. there exists a constant C independent of c_n such that

$$|\partial_y \Phi_n(y_0)| \leq C |(Z_+(y_0) - c_n)(Z_-(y_0) - c_n)|^{-\frac{3}{4}}. \tag{31}$$

Proof. We recall that we shall only prove the part of the lemma for $Z_+(y_0) = c$, as the proof when $Z_-(y_0) = c$ is along the same lines. As $\text{Ran } Z_+ \cap \text{Ran } Z_- = \emptyset$, we can find some constant $C > 1$ such that $C^{-1} < (\text{Re } c_n - Z_-) < C$, provided that n is sufficiently large. Without loss of generality, we may assume that $Z_+''(y_0) > 0$, i.e., $Z_+(y_0)$ is a local minimum.

We can choose $h > 0$ small enough independent of n so that $\mathcal{I}_{2h} := [y_0 - 2h, y_0 + 2h] \subset \mathcal{I}_0$. We also denote $\mathcal{I}_h := [y_0 - h, y_0 + h]$. Let us introduce a cut-off function χ satisfying

1. $\chi \equiv 1$ on \mathcal{I}_h and $\chi \equiv 0$ outside \mathcal{I}_{2h} ,
2. $\chi \in C^1$ and $0 \leq \chi \leq 1$,

Integrating by parts, we have

$$\int_{\mathcal{I}_0} \Phi_n \overline{\Phi_n} \chi \, dy = - \int_{\mathcal{I}_0} (y - y_0) \Phi_n \partial_y (\overline{\Phi_n} \chi) \, dy - \int_{\mathcal{I}_0} (y - y_0) \partial_y \Phi_n \overline{\Phi_n} \chi \, dy,$$

which leads to

$$\|\sqrt{\chi} \Phi_n\|_{L^2(\mathcal{I}_{2h})} \leq 2 \|(y - y_0) \partial_y \Phi_n \sqrt{\chi}\|_{L^2(\mathcal{I}_{2h})} + \|\Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)}. \tag{32}$$

Multiplying Equation (18) by $\overline{\Phi_n} \chi$, integrating by parts and taking the real part and the imaginary part separately, we obtain the following identities

$$-\int_{\mathcal{I}_0} ((Z_+ - \text{Re } c_n)(Z_- - \text{Re } c_n) - (\text{Im } c_n)^2) (|\partial_y \Phi_n|^2 + \alpha^2 |\Phi_n|^2) \chi \, dy \tag{33}$$

$$= \text{Re} \langle F_n, \chi \overline{\Phi_n} \rangle_{L^2(\mathcal{I}_0)} + \text{Re} \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} (Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n \chi' \, dy,$$

$$-\int_{\mathcal{I}_0} \text{Im } c_n (Z_+ + Z_- - 2\text{Re } c_n) (|\partial_y \Phi_n|^2 + \alpha^2 |\Phi_n|^2) \chi \, dy$$

$$= \text{Im} \langle F_n, \chi \overline{\Phi_n} \rangle_{L^2(\mathcal{I}_0)} + \text{Im} \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} (Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n \chi' \, dy. \tag{34}$$

To estimate (34), we note that for h small enough and for n large enough, it holds that

$$|Z_+(y) - \operatorname{Re} c_n| \leq Ch + \operatorname{dist}(c_n, \operatorname{Ran} Z_+) \leq \epsilon_0 + Ch \leq Ch, \quad \forall y \in \mathcal{I}_{2h},$$

which leads to the following bound:

$$Z_+ + Z_- - 2\operatorname{Re} c_n \leq Z_-(y) - \operatorname{Re} c_n + |Z_+ - \operatorname{Re} c_n| \leq -C^{-1} + Ch \leq -\frac{1}{2C^{-1}}.$$

Hence, it follows from (34) that

$$\begin{aligned} & |\operatorname{Im} c_n| \left(\|\sqrt{\chi} \partial_y \Phi_n\|_{L^2(\mathcal{I}_{2h})}^2 + \alpha^2 \|\sqrt{\chi} \Phi_n\|_{L^2(\mathcal{I}_{2h})}^2 \right) \\ & \lesssim \int_{\mathcal{I}_{2h}} |F_n \chi \overline{\Phi_n}| \, dy + \|(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)}^2. \end{aligned} \tag{35}$$

In particular, (35) shows that $\left\| \sqrt{|\operatorname{Im} c_n|} \partial_y \Phi_n \right\|_{L^2(\mathcal{I}_h)} < C$.

As $(Z_+(y) - \operatorname{Re} c_n) = (Z_+(y) - Z_+(y_0) + c - \operatorname{Re} c_n)$, we can write (33) as

$$\begin{aligned} & \int_{\mathcal{I}_{2h}} (Z_+(y) - Z_+(y_0)) (\operatorname{Re} c_n - Z_-) (|\partial_y \Phi_n|^2 + \alpha^2 |\Phi_n|^2) \chi \, dy \\ & = \operatorname{Re} \langle F_n, \chi \overline{\Phi_n} \rangle_{L^2(\mathcal{I}_0)} + \operatorname{Re} \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} (Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n \chi' \, dy \\ & \quad + \int_{\mathcal{I}_{2h}} ((\operatorname{Re} c_n - c) (\operatorname{Re} c_n - Z_-) - |\operatorname{Im} c_n|^2) (|\partial_y \Phi_n|^2 + \alpha^2 |\Phi_n|^2) \chi \, dy. \end{aligned} \tag{36}$$

For the case $|\operatorname{Re} c_n - c| \leq |\operatorname{Im} c_n|$ and $Z_+''(y_0) (\operatorname{Re} c_n - c) \geq 0$, we notice that the term on right hand side of (36) containing $(\operatorname{Re} c_n - c)$ can be dominated by the estimate of (34). Hence, we have

$$\begin{aligned} & \int_{\mathcal{I}_{2h}} (Z_+(y) - Z_+(y_0)) (\operatorname{Re} c_n - Z_-) |\partial_y \Phi_n|^2 \chi \, dy \\ & \lesssim \left| \operatorname{Re} \int_{\mathcal{I}_{2h}} F_n \overline{\Phi_n} \, dy \right| + \left| \operatorname{Re} \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} (Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n \chi' \, dy \right| \\ & \quad + |\operatorname{Im} c_n| \int_{\mathcal{I}_{2h}} (\operatorname{Re} c_n - Z_-) (|\partial_y \Phi_n|^2 + \alpha^2 |\Phi_n|^2) \, dy \\ & \lesssim C \int_{\mathcal{I}_{2h}} |F_n \chi \overline{\Phi_n}| \, dy + C \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} |(Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n| \, dy, \end{aligned}$$

whereas in the case $Z_+''(y_0) (\operatorname{Re} c_n - c) < 0$, $(c - \operatorname{Re} c_n)$ has the same sign as $(Z_+(y) - c)$, from which we infer

$$\begin{aligned} & \int_{\mathcal{I}_{2h}} (Z_+(y) - Z_+(y_0)) (\operatorname{Re} c_n - Z_-) |\partial_y \Phi_n|^2 \chi \, dy \\ & \leq \int_{\tilde{\mathcal{I}}_{2h}} |\chi F_n \overline{\Phi_n}| \, dy + \left| \int_{\mathcal{I}_{2h} \setminus \mathcal{I}_h} (Z_+ - c_n)(Z_- - c_n) \overline{\Phi_n} \partial_y \Phi_n \chi' \, dy \right|. \end{aligned}$$

Using the fact that $(Z_+ - c) \sim (y - y_0)^2$ in \mathcal{I}_0 , we have, by (32) and (33),

$$\begin{aligned} \|(y - y_0) \partial_y \Phi_n \sqrt{\chi}\|_{L^2(\mathcal{I}_{2h})}^2 & \leq C \left(\|\Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)}^2 + \|(Z_+ - c_n) \partial_y \Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)}^2 \right) \\ & \quad + C \|F_n\|_{L^2(\mathcal{I}_0)}^2. \end{aligned} \tag{37}$$

Multiplying both sides of Equation (18) by $f \in H_0^1(\mathcal{I}_0)$ and passing to the limit as $n \rightarrow \infty$, we have

$$-\int_{\tilde{\mathcal{I}}_0} (Z_+ - c)(Z_- - c) (\partial_y \Phi f' + \alpha^2 \Phi f) \, dy = \int_{\tilde{\mathcal{I}}_0} F f \, dy, \quad \forall f \in H_0^1(\mathcal{I}_0).$$

Estimate (37) ensures that $(y - y_0) \partial_y \Phi \in L^2$ and

$$\|\Phi_n\|_{L^2(\mathcal{I}_0)} \lesssim \|(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)} + \|\Phi_n\|_{L^2(\mathcal{I}_{2h} \setminus \mathcal{I}_h)} + \|F_n\|_{L^2}. \tag{38}$$

Thus, by the density of $H_0^1(\mathcal{I}_0)$ in $H_{0,w}^1(\mathcal{I}_0)$, Equation (29) holds.

It follows from Gagliardo-Nirenberg inequality and the fact that $\|\sqrt{\operatorname{Im} c_n} \partial_y \Phi_n\|_{L^2} < C$ that

$$|\Phi_n(y_0)| \leq \|\Phi_n\|_{L^\infty} \lesssim \|\partial_y \Phi_n\|_{L^2}^{\frac{1}{2}} \|\Phi_n\|_{L^2}^{\frac{1}{2}} \lesssim |\operatorname{Im} c_n|^{-\frac{1}{4}} \lesssim |Z_+(y_0) - c_n|^{-\frac{1}{4}}.$$

Thus, Estimate (30) holds.

As for Estimate (31), we can infer from the fact that $(y - y_0) \partial_y \Phi_n \in L^2$ the existence of $\tilde{y} \in [y_0 - 2\delta_n, y_0 - \delta_n]$ such that $|\partial_y \Phi_n(\tilde{y})| < C \delta_n^{-\frac{3}{2}}$. We further set $\delta_n = \sqrt{|c - c_n|}$.

Since $q_n \in H^2$, as shown in (26), and $F_n \in H^1$, we can integrate (18) from \tilde{y} to y_0 , which results in

$$\begin{aligned} & (Z_+(y_0) - c_n)(Z_-(y_0) - c_n) \partial_y \Phi_n(y_0) \\ & = ((Z_+(\tilde{y}) - c) + (c - c_n)) (Z_-(\tilde{y}) - c_n) \partial_y \Phi_n(\tilde{y}) \\ & \quad + \int_{\tilde{y}}^{y_0} F_n(y) \, dy + \alpha^2 \int_{\tilde{y}}^{y_0} (Z_+(y) - c_n)(Z_-(y) - c_n) \Phi_n(y) \, dy. \end{aligned} \tag{39}$$

By Equation (39) and the fact that $|Z_+(\tilde{y}) - c| \leq C \delta_n^2 = C|c - c_n|$, we have

$$|\partial_y \Phi_n(y_0)| \leq C \frac{|c - c_n|}{|Z_+(y_0) - c_n| \delta_n^{\frac{3}{2}}} + \frac{\delta_n}{|Z_+(y_0) - c_n|} \leq \frac{C}{|Z_+(y_0) - c_n|^{\frac{3}{4}}}. \quad \square \tag{40}$$

We then address the scenario when $y_0 \in (Z_s)^{-1}(c)$ with $Z_s''(y_0) (\operatorname{Re} c_n - c) \geq 0$, $s = +$ or $-$, and $|\operatorname{Re} c_n - c| \geq |\operatorname{Im} c_n|$. We recall that $y_{c_n}^\ell$ and $y_{c_n}^r$ are the two points in a sufficiently small neighborhood of y_0 such that $y_{c_n}^\ell \leq y_0 \leq y_{c_n}^r$ and $\operatorname{Re} c_n = Z_s(y_{c_n}^\ell) = Z_s(y_{c_n}^r)$.

Lemma 3.5. *Let $\{c_n\}_{n=1}^\infty \subset (\Omega_{e_0} \setminus (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-))$ be such that $c_n \rightarrow c$ as $n \rightarrow \infty$ for a certain $c \in (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-)$. Let the triple $\{c_n, \Phi_n, F_n\}_{n=1}^\infty$ solve (18) and satisfy*

- $\Phi_n \rightarrow \Phi$ in L^2 and $(Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n \rightarrow (Z_+ - c)(Z_- - c)\partial_y \Phi$ in H^1 on \mathcal{I}_0 ,
- $F_n \rightarrow F$ in H^1 on \mathcal{I}_0 ,

for some interval $\mathcal{I}_0 := [y_1, y_2] \subset \mathbb{T}$ such that there exists $y_0 \in (Z_s)^{-1}(c)$ in \mathcal{I}_0 at which $Z'_s(y_0) = 0$; $Z''_s(y_0)Z''_s(y) > 0, \forall y \in \mathcal{I}_0$; $|\operatorname{Re} c_n - c| \geq |\operatorname{Im} c_n|$ for sufficiently large n and $Z''_s(y_0)(\operatorname{Re} c_n - c) \geq 0$, where $s = +$ or $-$.

Then $(y - y_0)\partial_y \Phi \in L^2$ and

$$-\int_{\mathcal{I}_0} (Z_+ - c)(Z_- - c) (\partial_y \Phi f' + \alpha^2 \Phi f) \, dy = \int_{\mathcal{I}_0} F f \, dy, \quad \forall f \in H^1_{w,0}, \tag{41}$$

where $H^1_{w,0} = \{f \in L^2 : ((y - y_0)\partial_y f) \in L^2 \text{ and } f|_{y=y_1} = f|_{y=y_2} = 0\}$.

Proof. As in Appendix A, we can construct a solution, which we denote by φ_n^ℓ (or φ_n^r), to the homogeneous equation

$$\partial_y((Z_+ - c_n)(Z_- - c_n)\partial_y \varphi) - \alpha^2(Z_+ - c_n)(Z_- - c_n)\varphi = 0, \tag{42}$$

$$\varphi(y_{c_n}^\ell) = 1, \quad \partial_y \varphi(y_{c_n}^\ell) = 0 \quad (\text{or } \varphi(y_{c_n}^r) = 1, \quad \partial_y \varphi(y_{c_n}^r) = 0), \tag{43}$$

on an interval $[y_1, y_0]$ (or $[y_0, y_2]$, respectively).

We note that $y_{c_n}^i \rightarrow y_0$ as $c_n \rightarrow c$, $i = \ell$ or r . In turn, we denote by φ^ℓ (or φ^r) the solution to

$$\partial_y((Z_+ - c)(Z_- - c)\partial_y \varphi) - \alpha^2(Z_+ - c)(Z_- - c)\varphi = 0,$$

$$\varphi(y_0) = 1, \quad \partial_y \varphi(y_0) = 0,$$

on $[y_1, y_0]$ (or $[y_0, y_2]$, respectively).

We shall use the following properties of φ_n^i and φ^i , $i = \ell$ or r , as proven in Appendix A:

1. $\|\partial_y \varphi_n^i\|_{L^\infty} + \|\varphi_n^i\|_{L^\infty} \leq C$ and $\|\partial_y \varphi^i\|_{L^\infty} + \|\varphi^i\|_{L^\infty} \leq C$,
2. $|\varphi_n^i| \geq \frac{1}{2}$ and $\varphi^i \geq 1$,
3. $|\varphi_n^i(y) - 1| \leq C|y - y_{c_n}^i|^2$, $y \in [y_1, y_{c_n}^\ell]$ (or $[y_{c_n}^r, y_2]$, respectively),
and $|\varphi^i(y) - 1| \leq C|y - y_0|^2$, $y \in [y_1, y_0]$ (or $[y_0, y_2]$, respectively),
4. $\lim_{n \rightarrow \infty} \varphi_n^i = \varphi^i$.

For $y \in [y_1, y_{c_n}^\ell]$ we can solve the inhomogeneous equation (18) explicitly by

$$\begin{aligned} \Phi_n(y) &= \frac{\varphi_n^\ell(y)}{\varphi_n^\ell(y_1)} \Phi_n(y_1) + \mu_n^\ell \varphi_n^\ell(y) \int_{y_1}^y \frac{1}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^\ell)^2)(y')} \, dy' \\ &\quad + \varphi_n^\ell(y) \int_{y_1}^y \frac{\int_{y_{c_n}^\ell}^{y'} (F_n \varphi_n^\ell)(y'') \, dy''}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^\ell)^2)(y')} \, dy', \end{aligned} \tag{44}$$

as (18) is equivalent to

$$\partial_y \left((Z_+ - c_n)(Z_- - c_n) (\varphi_n^i)^2 \partial_y \left(\frac{\Phi_n}{\varphi_n^i} \right) \right) = F_n \varphi_n^i, \quad i = \ell \text{ or } r. \tag{45}$$

Here the coefficient μ_n^ℓ is given by

$$\mu_n^\ell := (Z_+(y_1) - c_n)(Z_-(y_1) - c_n) \left((\varphi_n^\ell \Phi_n')(y_1) - ((\varphi_n^\ell)' \Phi_n)(y_1) \right) - \int_{y_{c_n}^\ell}^{y_1} (F_n \varphi_n^\ell)(y) dy.$$

We have for the last term in (44) that

$$\begin{aligned} \left| \varphi_n^\ell(y) \int_{y_1}^y \frac{\int_{y_{c_n}^\ell}^{y'} (F_n \varphi_n^\ell)(y'') dy''}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^\ell)^2)(y')} dy' \right| &\leq C \left| \int_{y_1}^y \frac{1}{(y' + y_{c_n}^\ell)} dy' \right| \\ &\leq C \ln \left(\frac{y + y_{c_n}^\ell}{y_1 + y_{c_n}^\ell} \right) \in L^2(y_1, y_{c_n}^\ell). \end{aligned} \tag{46}$$

For the second term on the right hand side of (44), we have the point-wise limit for $y \in [y_1, y_0)$,

$$\varphi_n^\ell(y) \int_{y_1}^y \frac{1}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^\ell)^2)(y')} dy' \rightarrow \varphi^\ell(y) \int_{y_1}^y \frac{1}{((Z_+ - c)(Z_- - c)(\varphi^\ell)^2)(y')} dy'.$$

The fact that $\operatorname{Re}(Z_+ - c) \sim (y - y_0)^2$ implies that

$$\varphi^\ell(y) \int_{y_1}^y \frac{1}{((Z_+ - c)(Z_- - c)(\varphi^\ell)^2)(y')} dy' \sim \frac{1}{(y - y_0)} \notin L^2(y_1, y_0). \tag{47}$$

On the other hand, from the facts that $\|\Phi_n\|_{L^2} \leq C$, $\|(Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n\|_{H^1} \leq C$ and $\|F_n\|_{L^\infty} < C$, we know that $\|(Z_+ - c_n)(Z_- - c_n)\Phi_n\|_{H^1} \leq C$ and $\{\mu_n^\ell\}_{n=1}^\infty$ is bounded. Thus, we further infer that up to a subsequence $\lim_{n \rightarrow \infty} \mu_n^\ell = \mu^\ell$ and that the second term on the right hand side of (44) converges pointwise. Similarly, we can confirm the pointwise convergence of the third term on the right hand side of (44). We can verify that the weak limit Φ can be written as

$$\begin{aligned} \Phi(y) &:= \frac{\varphi^\ell(y)}{\varphi^\ell(y_1)} \Phi(y_1) + \mu^\ell \varphi^\ell(y) \int_{y_1}^y \frac{1}{((Z_+ - c)(Z_- - c)(\varphi^\ell)^2)(y')} dy' \\ &+ \varphi^\ell(y) \int_{y_1}^y \frac{\int_{y_0}^{y'} (F \varphi^\ell)(y'') dy''}{((Z_+ - c)(Z_- - c)(\varphi^\ell)^2)(y')} dy', \quad y < y_0. \end{aligned} \tag{48}$$

Moreover, the assumption $\Phi_n \rightharpoonup \Phi$ in L^2 along with (46) indicates that the second term on the right hand side of (48) is uniformly bounded in L^2 . According to (47), it is therefore necessary that

$$\mu^\ell = 0, \text{ i.e., } \mu_n^\ell \rightarrow 0 \text{ up to a subsequence.}$$

Similarly, for $y \in [y_{c_n}^r, y_2]$ we have

$$\begin{aligned} \Phi_n(y) &= \frac{\varphi_n^r(y)}{\varphi_n^r(y_2)} \Phi_n(y_2) + \mu_n^r \varphi_n^r(y) \int_{y_2}^y \frac{1}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^r)^2)(y')} dy' \\ &\quad + \varphi_n^r(y) \int_{y_2}^y \frac{\int_{y_{c_n}^r}^{y'} (F_n \varphi_n^r)(y'') dy''}{((Z_+ - c_n)(Z_- - c_n)(\varphi_n^r)^2)(y')} dy', \end{aligned}$$

with

$$\mu_n^r := (Z_+(y_2) - c_n)(Z_-(y_2) - c_n) ((\varphi_n^r \Phi_n')(y_2) - ((\varphi_n^r)' \Phi_n)(y_2)) - \int_{y_{c_n}^r}^{y_2} (F_n \varphi_n^r)(y) dy,$$

and we can show that $\mu_n^r \rightarrow 0$ up to a subsequence.

Thus, from the formula (48) and its analogue on $[y_{c_n}^r, y_2]$, where $\mu^i \equiv 0$ for $i = \ell$ or r , we know that

$$\begin{aligned} |(y - y_0) \partial_y \Phi| &\lesssim |\Phi(y_1)| \|\partial_y \varphi^i(y)\|_{L^\infty} + \|F\|_{L^\infty} \|\partial_y \varphi^i(y)\|_{L^\infty} |y - y_0| (\ln |y - y_0| + 1) \\ &\quad + \|F\|_{L^\infty} \|\varphi^i(y)\|_{L^\infty}. \end{aligned}$$

As $\Phi_n \rightharpoonup \Phi$ in L^2 and $(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n \rightharpoonup (Z_+ - c)(Z_- - c) \partial_y \Phi$ in H^1 , multiplying Equation (18) by $f \in H_0^1(\mathcal{I}_0)$ and passing to the limit as $n \rightarrow \infty$, we have

$$- \int_{\mathcal{I}_0} (Z_+ - c)(Z_- - c) (\partial_y \Phi f' + \alpha^2 \Phi f) dy = \int_{\mathcal{I}_0} F f dy, \quad \forall f \in H_0^1(\mathcal{I}_0).$$

The desired result (41) then follows from a density argument. \square

In view of Lemma 3.3, Lemma 3.4 and Lemma 3.5, the use of $\text{sgn}(Z_+ - c)(Z_- - c)\Phi$ as a test function and integration by parts can be justified. As $F_n \rightarrow 0$ in H^1 as $n \rightarrow \infty$, it is not difficult to see that

$$\Phi_n \rightarrow 0 \text{ in } L^2(\mathbb{T}) \text{ and } (Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n \rightarrow 0 \text{ in } H^1(\mathbb{T}).$$

We shall rigorously prove this claim in Section 3.3. To complete the proof of Proposition 3.1, it remains to be shown that $\{\Phi_n\}_{n=1}^\infty$ and $\{(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n\}_{n=1}^\infty$ converges strongly in L^2 and H^1 , respectively, throughout \mathbb{T} , which amounts to proving the strong convergence of $\{\Phi_n\}_{n=1}^\infty$ and $\{(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n\}_{n=1}^\infty$ near the critical points in $(Z_+)^{-1}(c) \cup (Z_-)^{-1}(c)$. Indeed, away from $(Z_+)^{-1}(c) \cup (Z_-)^{-1}(c)$, it is clear that

$$\Phi_n \rightarrow 0 \text{ in } H_{loc}^1(\mathbb{T} \setminus ((Z_+)^{-1}(c) \cup (Z_-)^{-1}(c))), \tag{49}$$

$$(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n \rightarrow 0 \text{ in } H_{loc}^1(\mathbb{T} \setminus ((Z_+)^{-1}(c) \cup (Z_-)^{-1}(c))), \tag{50}$$

and the case of points in $(Z_\pm)^{-1}(c)$ at which $|Z'_\pm| > 0$ is already covered in Lemma 3.3.

3.2. Strong convergence of $\{\Phi_n\}_{n=1}^\infty$ near y_0

As for strong convergence of $\{\Phi_n\}_{n=1}^\infty$ and $\{(Z_+ - c_n)(Z_- - c_n) \partial_y \Phi_n\}_{n=1}^\infty$ when $c_n \rightarrow c = Z_s(y_0)$ with $Z'_s(y_0) = 0$, $s = +$ or $-$, we note that for the scenarios in Lemma 3.4, by (38) and (49), it is not difficult to

show that as $F_n \rightarrow 0$ in H^1 , $\Phi_n \rightarrow 0$ and $(Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n \rightarrow 0$ in $L^2(\mathcal{I}_0)$ and $H^1(\mathcal{I}_0)$, respectively via integration by parts, while a more delicate analysis is needed for the scenario in Lemma 3.5, i.e., $Z_s''(y_0) (\operatorname{Re} c_n - c) \geq 0$, $s = +$ or $-$, and $|\operatorname{Re} c_n - c| \geq |\operatorname{Im} c_n|$. We shall postpone the proof for the scenarios in Lemma 3.4 to Section 3.3 and focus primarily on that for the scenario in Lemma 3.5.

We shall prove the following lemma.

Lemma 3.6. *Let $y_0 \in ((Z_+)^{-1}(c) \cup (Z_-)^{-1}(c))$ be a critical point, i.e., $Z_s'(y_0) = 0$, and the interval $\mathcal{I}_0 := [y_1, y_2]$ be such that $y_0 \in \mathcal{I}_0$ and $Z_s''(y_0)Z_s''(y) > 0$, $s = +$ or $-$. Let $\{(c_n, \Phi_n, F_n)\}_{n=1}^\infty$ and Φ satisfy the conditions as in Lemma 3.5, then $\Phi_n \rightarrow \Phi$ in L^2 in \mathcal{I}_0 . In addition, the following estimates hold*

$$|\Phi_n(y_0)| \leq C|(Z_-(y_0) - c_n)(Z_+(y_0) - c_n)|^{-\frac{1}{4}}, \tag{51}$$

$$|\partial_y \Phi_n(y_0)| \leq C|(Z_-(y_0) - c_n)(Z_+(y_0) - c_n)|^{-\frac{3}{4}}. \tag{52}$$

To prove Lemma 3.6, we shall construct solutions to Equation (18) in \mathcal{I}_0 and obtain its explicit formula using ODE techniques.

Without loss of generality, we may assume that the critical point $y_0 = 0$. Let $\mathcal{I}_0 := [y_1, y_2]$ be such that $Z_+(y_1) = Z_+(y_2)$ (or $Z_-(y_1) = Z_-(y_2)$) with $y_1 < 0 < y_2$. We shall rewrite Equation (18) by introducing

$$\Phi_n^*(y) := \Phi_n(y) - \mathcal{L}_{\Phi_n}(y) := \Phi_n(y) - \frac{\Phi_n(y_2) - \Phi_n(y_1)}{(y_2 - y_1)}(y - y_1) - \Phi_n(y_1),$$

which leads to

$$\begin{cases} \partial_y ((Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n^*) - \alpha^2(Z_+ - c_n)(Z_- - c_n)\Phi_n^* = F_n^*, \\ \Phi_n^*(y_1) = \Phi_n^*(y_2) = 0, \end{cases} \tag{53}$$

with

$$\begin{aligned} F_n^*(y) = & F_n(y) - \frac{\Phi_n(y_2) - \Phi_n(y_1)}{(y_2 - y_1)}(Z_+'(y)(Z_-(y) - c_n) \\ & - \frac{\Phi_n(y_2) - \Phi_n(y_1)}{(y_2 - y_1)}(Z_-'(y)(Z_+(y) - c_n) + \alpha^2(Z_+ - c_n)(Z_- - c_n)\mathcal{L}_{\Phi_n}(y). \end{aligned}$$

We can see that to study Equation (53) is to study an equation of the following type

$$\begin{cases} \partial_y ((Z_+ - c_*)(Z_- - c_*)\partial_y \Phi_*) - \alpha^2(Z_+ - c_*)(Z_- - c_*)\Phi_* = F_*, \\ \Phi_*(y_1) = \Phi_*(y_2) = 0. \end{cases} \tag{54}$$

Here $c_* \in (\Omega_{e_0} \setminus (\operatorname{Ran} Z_+ \cup \operatorname{Ran} Z_-))$ satisfies the conditions $|\operatorname{Re} c_* - Z_+(0)| \geq |\operatorname{Im} c_*|$ and $Z_+'(0)(\operatorname{Re} c_* - Z_+(0)) \geq 0$ (or $|\operatorname{Re} c_* - Z_-(0)| \geq |\operatorname{Im} c_*|$ and $Z_-'(0)(\operatorname{Re} c_* - Z_-(0)) \geq 0$), while $F_* \in H^1$. For c^* close enough to $Z_+(0)$ (or $Z_-(0)$), we can find exactly two points y_*^i , $i = \ell$ or r , in a sufficiently small neighborhood of 0 such that $y_*^\ell \leq 0 \leq y_*^r$ and $Z_+(y_*^\ell) = Z_+(y_*^r) = \operatorname{Re} c^*$ (or $Z_-(y_*^\ell) = Z_-(y_*^r) = \operatorname{Re} c^*$). It turns out that the solution to Equation (54) enjoys the following estimate.

Lemma 3.7. *Let Φ_* be the solution to Equation (54). There exists some $C > 0$ independent of c_* such that*

$$\|\Phi_*\|_{L^2(y_1, y_2)} \leq C\|F_*\|_{L^\infty}.$$

Before we prove Lemma 3.7, we make some preparations. Recall that here we assume that 0 is a critical point of Z_+ , i.e. $Z'_+(0) = 0$ (or $Z'_-(0) = 0$) and that we restrict ourselves to the case $c_* \rightarrow Z_+(0)$ with $\text{Im } c_* > 0$, as the proofs for the other cases are along the same lines.

Without loss of generality, we may also assume that $Z''_+(0) > 0$, i.e., $Z_+(0)$ is a local minimum. We define $\sigma(c_*) \in \mathbb{C}$, which satisfies $(\sigma(c_*))^2 = c_* - Z_+(0)$ with $\text{Im } \sigma(c_*) > 0$.

To this end, we proceed to introduce the notations. We define the function $V(y)$ such that $(V(y))^2 = Z_+(y) - Z_+(0)$, i.e.,

$$V(y) = \begin{cases} -\sqrt{Z_+(y) - Z_+(0)} & \text{on } [y_1, 0], \\ \sqrt{Z_+(y) - Z_+(0)} & \text{on } [0, y_2]. \end{cases} \tag{55}$$

It can be verified via direct computations that $V \in C^2(\mathcal{I}_0)$ is monotone, and $V'(0) = \frac{\sqrt{2Z'_+(0)}}{2}$.

Denoting the solution to Equation (54) on $[y_1, 0]$ by Φ_*^ℓ and that on $[0, y_2]$ by Φ_*^r , we notice that Equation (54) is equivalent to

$$\partial_y \left((Z_- - c_*)(Z_+ - c_*) (\varphi_*^j)^2 \partial_y \left(\frac{\Phi_*^j}{\varphi_*^j} \right) \right) = F_* \varphi_*^j, \quad j = \ell \text{ or } r, \tag{56}$$

where φ_*^r and φ_*^ℓ are the solutions to the homogeneous equation

$$\partial_y ((Z_- - c_*)(Z_+ - c_*) \partial_y \varphi_*) - \alpha^2 (Z_- - c_*)(Z_+ - c_*) \varphi_* = 0, \tag{57}$$

$$\varphi_*(y_*^\ell) = 1, \quad \partial_y \varphi_*(y_*^\ell) = 0, \quad \text{for } y \in [y_1, 0], \tag{58}$$

$$\text{or } \varphi_*(y_*^r) = 1, \quad \partial_y \varphi_*(y_*^r) = 0, \quad \text{for } y \in [0, y_2], \tag{59}$$

with φ_*^r and φ_*^ℓ corresponding to condition (58) and corresponding to (59), respectively. The following properties of φ_*^j , $j = \ell$ or r , shall be useful –

1. $|\varphi_*^j| > \frac{1}{2}$,
2. $|\varphi_*^j(y) - 1| \leq C |y - y_*^j|^2$, $y \in [y_1, 0]$ (or $[0, y_2]$),
3. $|\partial_y \varphi_*^j(y)| \leq C |y - y_*^j|$, $y \in [y_1, 0]$ (or $[0, y_2]$).

We can integrate twice and obtain explicit solution formulae to Equation (54). On $[0, y_2]$, the solution to Equation (54) is given by

$$\begin{aligned} \Phi_*^r(y) &= \nu^r [F_*](c_*) \varphi_*^r(y) + \mu^r [F_*](c_*) \varphi_*^r(y) \int_0^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \\ &\quad + \varphi_*^r(y) \int_0^y \frac{\int_{y_*^r}^{y'} (F_* \varphi_*^r)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \\ &= \tilde{\mu}^r [F_*](c_*) \varphi_*^r(y) \int_{y_2}^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \\ &\quad + \varphi_*^r(y) \int_{y_2}^y \frac{\int_{y_*^r}^{y'} (F_* \varphi_*^r)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy', \end{aligned} \tag{60}$$

while the solution to Equation (54) on $[y_1, 0]$ is given by

$$\begin{aligned} \Phi_*^\ell(y) &= \varphi_*^\ell(y) \int_{y_1}^y \frac{\int_{y_*^\ell}^{y'} (F_* \varphi_*^\ell)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^\ell(y'))^2} dy' \\ &\quad + \tilde{\mu}^\ell[F_*](c_*) \varphi_*^\ell(y) \int_{y_1}^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^\ell(y'))^2} dy' \\ &= \mu^\ell[F_*](c_*) \varphi_*^\ell(y) \int_0^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^\ell(y'))^2} dy' + \nu^\ell[F_*](c_*) \varphi_*^\ell(y) \\ &\quad + \varphi_*^\ell(y) \int_0^y \frac{\int_{y_*^\ell}^{y'} (F_* \varphi_*^\ell)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^\ell(y'))^2} dy'. \end{aligned}$$

Here the coefficients $\mu^j[F_*](c_*)$, $\tilde{\mu}^j[F_*](c_*)$ and $\nu^j[F_*](c_*)$, $j = \ell$ or r , are determined by F_* and c_* in a way such that Φ_*^r and Φ_*^ℓ are well-defined and satisfy the conditions

$$\begin{cases} \Phi_*^\ell(y_1) = 0, & \Phi_*^r(y_2) = 0, \\ \Phi_*^\ell(0) = \Phi_*^r(0), & \partial_y \Phi_*^\ell(0) = \partial_y \Phi_*^r(0), \end{cases} \quad (61)$$

which gives us

$$\begin{cases} \mu^r[F_*](c_*) = \tilde{\mu}^r[F_*](c_*), & \mu^\ell[F_*](c_*) = \tilde{\mu}^\ell[F_*](c_*), \\ I^r(c_*) \mu^r[F_*](c_*) + \nu^r[F_*](c_*) = -T^r[F_*](c_*), \\ I^\ell(c_*) \mu^\ell[F_*](c_*) - \nu^\ell[F_*](c_*) = T^\ell[F_*](c_*), \\ \varphi_*^r(0) \nu^r[F_*](c_*) - \varphi_*^\ell(0) \nu^\ell[F_*](c_*) = 0, \\ \varphi_*^\ell(0) \mu^r[F_*](c_*) - \varphi_*^r(0) \mu^\ell[F_*](c_*) + (Z_-(0) - c_*)(Z_+(0) - c_*) (\varphi_*^\ell \varphi_*^r \partial_y \varphi_*^r)(0) \nu^r[F_*](c_*) \\ \quad - (Z_-(0) - c_*)(Z_+(0) - c_*) (\varphi_*^r \varphi_*^\ell \partial_y \varphi_*^\ell)(0) \nu^\ell[F_*](c_*) = L[F_*](c_*). \end{cases}$$

We rewrite the above set of equations in the form of matrix equation as

$$\begin{bmatrix} I^r & 0 & 1 & 0 \\ 0 & I^\ell & 0 & -1 \\ 0 & 0 & \varphi_*^r(0) & -\varphi_*^\ell(0) \\ \varphi_*^\ell(0) & -\varphi_*^r(0) & \mathcal{H}_0(c_*) (\varphi_*^\ell \varphi_*^r \partial_y \varphi_*^r)(0) & -\mathcal{H}_0(c_*) (\varphi_*^r \varphi_*^\ell \partial_y \varphi_*^\ell)(0) \end{bmatrix} \begin{bmatrix} \mu^r[F_*] \\ \mu^\ell[F_*] \\ \nu^r[F_*] \\ \nu^\ell[F_*] \end{bmatrix} = \begin{bmatrix} -T^r[F_*] \\ T^\ell[F_*] \\ 0 \\ L[F_*] \end{bmatrix},$$

where $\mathcal{H}_0(c_*) := (Z_-(0) - c_*)(Z_+(0) - c_*)$ and

$$I^\ell(c_*) := \int_{y_1}^0 \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*) (\varphi_*^\ell(y))^2} dy, \quad (62)$$

$$I^r(c_*) := \int_0^{y_2} \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*) (\varphi_*^r(y))^2} dy, \quad (63)$$

$$L[F_*](c_*) := \varphi_*^\ell(0) \int_0^{y_*^r} (F_* \varphi_*^r)(y) dy - \varphi_*^r(0) \int_0^{y_*^\ell} (F_* \varphi_*^\ell)(y) dy, \quad (64)$$

$$T^\ell[F_*](c_*) := \int_0^{y_1} \frac{\int_{y_*}^y (F_* \varphi_*^\ell)(z) dz}{(Z_-(y) - c_*)(Z_+(y) - c_*) (\varphi_*^\ell(y))^2} dy, \tag{65}$$

$$T^r[F_*](c_*) := \int_0^{y_2} \frac{\int_{y_*}^y (F_* \varphi_*^r)(z) dz}{(Z_-(y) - c_*)(Z_+(y) - c_*) (\varphi_*^r(y))^2} dy. \tag{66}$$

And hereafter, we denote

$$\begin{aligned} \mathcal{D}(c_*) := & (Z_-(0) - c_*)(Z_+(0) - c_*) \varphi_*^r(0) \varphi_*^\ell(0) (\varphi_*^\ell(0) \partial_y \varphi_*^r(0) - \varphi_*^r(0) \partial_y \varphi_*^\ell(0)) I^r(c_*) I^\ell(c_*) \\ & - (\varphi_*^r(0))^2 I^r(c_*) - (\varphi_*^\ell(0))^2 I^\ell(c_*), \end{aligned}$$

which is in fact the determinant of the matrix in the matrix equation above.

Continuing solving for the coefficients, we obtain

$$\begin{aligned} \mu^r[F_*](c_*) &= \tilde{\mu}^r[F_*](c_*) \\ &:= \frac{1}{\mathcal{D}(c_*)} (Z_-(0) - c_*)(Z_+(0) - c_*) \varphi_*^r(0) \varphi_*^\ell(0) (\partial_y \varphi_*^\ell(0) \varphi_*^r(0) - \partial_y \varphi_*^r(0) \varphi_*^\ell(0)) T^r[F_*](c_*) I^\ell(c_*) \\ &\quad - \frac{1}{\mathcal{D}(c_*)} (\varphi_*^\ell(0) L[F_*](c_*) I^\ell(c_*) + (\varphi_*^r(0))^2 T^r[F_*](c_*)) \\ &\quad - \frac{1}{\mathcal{D}(c_*)} (\varphi_*^r(0) \varphi_*^\ell(0) T^\ell[F_*](c_*)), \end{aligned} \tag{67}$$

$$\begin{aligned} \mu^\ell[F_*](c_*) &= \tilde{\mu}^\ell[F_*](c_*) \\ &:= \frac{1}{\mathcal{D}(c_*)} (Z_-(0) - c_*)(Z_+(0) - c_*) \varphi_*^r(0) \varphi_*^\ell(0) (\partial_y \varphi_*^\ell(0) \varphi_*^r(0) - \partial_y \varphi_*^r(0) \varphi_*^\ell(0)) T^\ell[F_*](c_*) I^r(c_*) \\ &\quad + \frac{1}{\mathcal{D}(c_*)} (\varphi_*^r(0) L[F_*](c_*) I^r(c_*) - (\varphi_*^\ell(0))^2 T^\ell[F_*](c_*)) \\ &\quad + \frac{1}{\mathcal{D}(c_*)} (\varphi_*^r(0) \varphi_*^\ell(0) T^r[F_*](c_*)), \end{aligned} \tag{68}$$

$$\begin{aligned} \nu^r[F_*](c_*) := & \frac{1}{\mathcal{D}(c_*)} (\varphi_*^\ell(0) L[F_*](c_*) I^r(c_*) I^\ell(c_*) - \varphi_*^r(0) \varphi_*^\ell(0) T^\ell[F_*](c_*) I^r(c_*)) \\ & + \frac{1}{\mathcal{D}(c_*)} ((\varphi_*^\ell(0))^2 T^r[F_*](c_*) I^\ell(c_*)), \end{aligned} \tag{69}$$

$$\begin{aligned} \nu^\ell[F_*](c_*) := & \frac{1}{\mathcal{D}(c_*)} (\varphi_*^r(0) L[F_*](c_*) I^\ell(c_*) I^r(c_*) + \varphi_*^r(0) \varphi_*^\ell(0) T^r[F_*](c_*) I^\ell(c_*)) \\ & + \frac{1}{\mathcal{D}(c_*)} ((\varphi_*^r(0))^2 T^\ell[F_*](c_*) I^r(c_*)). \end{aligned} \tag{70}$$

Thus, we can verify that

$$\Phi_*(y, c_*) = \begin{cases} \Phi_*^\ell(y, c_*) & \text{on } [y_1, 0], \\ \Phi_*^r(y, c_*) & \text{on } [0, y_2] \end{cases}$$

is well-defined and is the unique C^1 - solution to (54) on \mathcal{I}_0 .

To facilitate the estimation of $I^k(c_*)$ for $k = r$ or ℓ , we introduce also the following quantities

$$\begin{aligned}
 I_1^r(c_*) &:= \int_0^{y_2} \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*)} \left(\frac{1}{(\varphi_*^r(y))^2} - 1 \right) dy + \int_0^{y_2} \frac{1}{2b(y)(Z_-(y) - c_*)} dy, \\
 I_1^\ell(c_*) &:= \int_{y_1}^0 \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*)} \left(\frac{1}{(\varphi_*^\ell(y))^2} - 1 \right) dy + \int_{y_1}^0 \frac{1}{2b(y)(Z_-(y) - c_*)} dy, \\
 I_2^r(\sigma(c_*)) &:= -\frac{1}{2b(y_2)V'(y_2)} \ln \left(\frac{|V(y_2) - \sigma(c_*)|}{|V(y_2) + \sigma(c_*)|} \right) + \frac{i}{2b(y_2)V'(y_2)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_2) - \text{Re } \sigma(c_*)|} \right) \\
 &\quad + \int_0^{y_2} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 &\quad + \frac{i}{2b(y_2)V'(y_2)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_2) + \text{Re } \sigma(c_*)|} \right), \\
 I_2^\ell(\sigma(c_*)) &:= \frac{1}{2b(y_1)V'(y_1)} \ln \left(\frac{|V(y_1) - \sigma(c_*)|}{|V(y_1) + \sigma(c_*)|} \right) + \int_{y_1}^0 \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 &\quad + \frac{i}{2b(y_1)V'(y_1)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_1) + \text{Re } \sigma(c_*)|} \right) \\
 &\quad + \frac{i}{2b(y_1)V'(y_1)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_1) - \text{Re } \sigma(c_*)|} \right) - \frac{2i\pi}{2b(y_1)V'(y_1)} + \frac{2i\pi}{2b(0)V'(0)},
 \end{aligned}$$

where Log is the complex logarithm with the principal value of the argument in $(-\pi, \pi]$.

We shall prove the following auxiliary estimates on $I^k(\sigma(c_*))$, $k = \ell$ or r and $\mathcal{D}(c_*)$ which will help us characterize the behaviors of the coefficients $\mu^k[F_*](c_*)$, $\tilde{\mu}^k[F_*](c_*)$ and $\nu^k[F_*](c_*)$, $k = \ell$ or r , in the solution formulae to Equation (54).

Lemma 3.8. *Assume that $\text{Im } c_* > 0$ and $\text{Im } \sigma(c_*) > 0$. It holds that*

$$2\sigma(c_*)I^k(c_*) = -\frac{i\pi}{b(0)\sqrt{2Z_+''(0)}} + 2\sigma(c_*)I_1^k(c_*) + I_2^k(\sigma(c_*)), \quad k = \ell \text{ or } r. \tag{71}$$

Moreover, there exist some $\delta_0 > 0$ and a constant C depending only on α , such that if $|\sigma(c_*)| < \delta_0$, then the following estimates are true for $k = \ell$ or r

$$|I_1^k(c_*)| \leq C, \tag{72}$$

$$|I_2^k(\sigma(c_*))| \leq C|\sigma(c_*)|^{\frac{1}{4}}, \tag{73}$$

$$C^{-1} \leq |2\sigma(c_*)I^k(c_*)| \leq C. \tag{74}$$

In particular,

$$\lim_{c_* \rightarrow Z_+(0)} 2\sigma(c_*)I^k(c_*) = -\frac{i\pi}{b(0)\sqrt{2Z_+''(0)}}, \quad k = \ell \text{ or } r.$$

Proof. Splitting the integral $I^r(c_*)$ and utilizing the function V , we have

$$\begin{aligned}
 2\sigma(c_*)I^r(c_*) &= 2\sigma(c_*) \int_0^{y_2} \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*)(\varphi_*^r(y))^2} dy \\
 &= 2\sigma(c_*) \int_0^{y_2} \frac{1}{(Z_-(y) - c_*)(Z_+(y) - c_*)} \left(\frac{1}{(\varphi_*^r(y))^2} - 1 \right) dy \\
 &\quad + 2\sigma(c_*) \int_0^{y_2} \frac{1}{2b(y)(Z_-(y) - c_*)} dy - 2\sigma(c_*) \int_0^{y_2} \frac{1}{2b(y)(Z_+(y) - c_*)} dy \\
 &= 2\sigma(c_*)I_1^r(c_*) - \int_0^{y_2} \frac{1}{2b(y)(V(y) - \sigma(c_*))} dy + \int_0^{y_2} \frac{1}{2b(y)(V(y) + \sigma(c_*))} dy \\
 &= 2\sigma(c_*)I_1^r(c_*) - \int_0^{y_2} \frac{1}{2b(y)V'(y)} \partial_y (\text{Log}(V(y) - \sigma(c_*))) dy \\
 &\quad + \int_0^{y_2} \frac{1}{2b(y)V'(y)} \partial_y (\text{Log}(V(y) + \sigma(c_*))) dy.
 \end{aligned}$$

Integration by parts yields

$$\begin{aligned}
 2\sigma(c_*)I^r(c_*) &= 2\sigma(c_*)I_1^r(c_*) - \frac{\text{Log}(V(y) - \sigma(c_*))}{2b(y)V'(y)} \Big|_{y=0}^{y=y_2} + \frac{\text{Log}(V(y) + \sigma(c_*))}{2b(y)V'(y)} \Big|_{y=0}^{y=y_2} \\
 &\quad + \int_0^{y_2} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) (\text{Log}(V(y) - \sigma(c_*)) - \text{Log}(V(y) + \sigma(c_*))) dy \\
 &= 2\sigma(c_*)I_1^r(c_*) - \frac{1}{2b(y_2)V'(y_2)} \ln \left(\frac{|V(y_2) - \sigma(c_*)|}{|V(y_2) + \sigma(c_*)|} \right) \\
 &\quad + \int_0^{y_2} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 &\quad + \frac{i}{2b(y_2)V'(y_2)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_2) + \text{Re } \sigma(c_*)|} \right) \\
 &\quad + \frac{i}{2b(y_2)V'(y_2)} \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y_2) + \text{Re } \sigma(c_*)|} \right) - \frac{i\pi}{2b(0)V'(0)} \\
 &= 2\sigma(c_*)I_1^r(c_*) - \frac{i\pi}{2b(0)V'(0)} + I_2^r(\sigma(c_*)).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 2\sigma(c_*)I^\ell(c_*) &= 2\sigma(c_*)I_1^\ell(c_*) - \int_{y_1}^0 \frac{1}{2b(y)(V(y) - \sigma(c_*))} dy + \int_{y_1}^0 \frac{1}{2b(y)(V(y) + \sigma(c_*))} dy \\
 &= 2\sigma(c_*)I_1^\ell(c_*) - \frac{\text{Log}(V(y) - \sigma(c_*))}{2b(y)V'(y)} \Big|_{y=y_1}^{y=0}
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{y_1}^0 \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log}(V(y) - \sigma(c_*)) dy + \frac{\text{Log}(V(y) + \sigma(c_*))}{2b(y)V'(y)} \Big|_{y=y_1}^{y=0} \\
 & - \int_{y_1}^0 \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log}(V(y) + \sigma(c_*)) dy \\
 & = 2\sigma(c_*)I_1^\ell(c_*) + \frac{1}{2b(y_1)V'(y_1)} \ln \left(\frac{|V(y_1) - \sigma(c_*)|}{|V(y_1) + \sigma(c_*)|} \right) \\
 & + \int_{y_1}^0 \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 & - \frac{i}{2b(y_1)V'(y_1)} \left(2\pi - \arctan \frac{\text{Im} \sigma(c_*)}{|V(y_1) + \text{Re} \sigma(c_*)|} - \arctan \frac{\text{Im} \sigma(c_*)}{|V(y_1) - \text{Re} \sigma(c_*)|} \right) \\
 & - \frac{i}{2b(0)V'(0)} \left(-\pi + \arctan \frac{\text{Im} \sigma(c_*)}{|\text{Re} \sigma(c_*)|} - \arctan \frac{\text{Im} \sigma(c_*)}{|\text{Re} \sigma(c_*)|} \right) \\
 & = 2\sigma(c_*)I_1^\ell(c_*) - \frac{i\pi}{2b(0)V'(0)} + I_2^\ell(\sigma(c_*)).
 \end{aligned}$$

We have thus shown (71).

We have the bound on $I_1^k(c_*)$, $k = \ell$ or r , in (72) for $|\sigma(c_*)| < \delta_0 \ll 1$ by the fact that $|\varphi_*^k| \geq \frac{1}{2}$ and $|\varphi_*^k(y) - 1| \leq C|y - y_*^k|^2$, $k = \ell$ or r .

We proceed to prove (73). As $|V'(y_2)| > 0$, by taking $|\text{Im}(\sigma(c_*))| \ll |\text{Re}(\sigma(c_*))| < \delta_0$, we obtain the following

$$\left| \frac{i}{2b(y_2)V'(y_2)} \left(\arctan \frac{\text{Im} \sigma(c_*)}{|V(y_2) - \text{Re} \sigma(c_*)|} + \arctan \frac{\text{Im} \sigma(c_*)}{|V(y_2) + \text{Re} \sigma(c_*)|} \right) \right| \leq C|\sigma(c_*)|. \tag{75}$$

For sufficiently small δ_0 and $|\sigma(c_*)| < \delta_0$, we have

$$\frac{1}{2b(y_2)V'(y_2)} \ln \left(\frac{|V(y_2) - \sigma(c_*)|}{|V(y_2) + \sigma(c_*)|} \right) \leq C|\sigma(c_*)|, \tag{76}$$

as

$$\frac{1}{2b(y_2)V'(y_2)} \ln \left(\frac{|V(y_2) - \sigma(c_*)|}{|V(y_2) + \sigma(c_*)|} \right) = \frac{1}{2b(y_2)V'(y_2)} \ln \left(1 - \frac{2V(y_2)\text{Re} \sigma(c_*)}{|V(y_2) + \text{Re} \sigma(c_*)|^2 + |\text{Im}(\sigma(c_*))|^2} \right).$$

We split the integral as follows –

$$\begin{aligned}
 & \int_0^{y_2} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 & = \int_E \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 & \quad + \int_{E^c} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy \\
 & := K_1(\sigma(c_*)) + K_2(\sigma(c_*)),
 \end{aligned}$$

where the set E is defined as $E := \left\{ y \in [0, y_2] : V(y) < M\sqrt{|\sigma(c_*)|} \right\}$ for sufficiently large M independent of $\sigma(c_*)$, while $E^c := \left\{ y \in [0, y_2] : V(y) \geq M\sqrt{|\sigma(c_*)|} \right\}$.

For $|\sigma(c_*)| < \delta_0$ with sufficiently small δ_0 , we have the estimate

$$|K_1(\sigma(c_*))| \leq C\sqrt[4]{|\sigma(c_*)|} + C\sqrt{|\sigma(c_*)|} \leq C\sqrt[4]{|\sigma(c_*)|}, \tag{77}$$

as $\text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) \in L^2$ in the set E , in particular,

$$\text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) = \ln \left(\frac{|V(y) - \sigma(c_*)|}{|V(y) + \sigma(c_*)|} \right) + i \arg(V(y) - \sigma(c_*)) - i \arg(V(y) + \sigma(c_*)).$$

Meanwhile, in the set E^c , it holds that

$$\begin{aligned} \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) &= \ln \left(\frac{|V(y) - \sigma(c_*)|}{|V(y) + \sigma(c_*)|} \right) - i \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y) - \text{Re } \sigma(c_*)|} \right) \\ &\quad - i \arctan \left(\frac{\text{Im } \sigma(c_*)}{|V(y) + \text{Re } \sigma(c_*)|} \right), \end{aligned}$$

from which we infer that

$$|K_2(\sigma(c_*))| \leq C\sqrt{|\sigma(c_*)|}, \tag{78}$$

when $|\sigma(c_*)| < \delta_0$ for small enough δ_0 .

Combining (75), (76), (77) and (78), we have

$$|I_2^r(\sigma(c_*))| \leq C \left(\sqrt[4]{|\sigma(c_*)|} + |\sigma(c_*)| + |\sigma(c_*)| \right) \leq C\sqrt[4]{|\sigma(c_*)|}.$$

We note that the terms $\frac{1}{2b(y_1)V'(y_1)} \ln \left(\frac{|V(y_1) - \sigma(c_*)|}{|V(y_1) + \sigma(c_*)|} \right)$ and $\frac{i}{2b(y_1)V'(y_1)} \arctan \left(\frac{\text{Im } (\sigma(c_*))}{|V(y_1) \pm \text{Re } (\sigma(c_*))|} \right)$ enjoy estimates similar to (76) and (75), respectively.

Introducing the sets

$$\mathcal{E} := \left\{ y \in [y_1, 0] : V(y) < M\sqrt{|\sigma(c_*)|} \right\} \text{ and } \mathcal{E}^c := \left\{ y \in [y_1, 0] : V(y) \geq M\sqrt{|\sigma(c_*)|} \right\},$$

where M is supposed to be large and independent of $\sigma(c_*)$, we split the integral in $I_2^\ell(\sigma(c_*))$:

$$\int_{y_1}^0 \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy - \frac{2i\pi}{2b(y_1)V'(y_1)} + \frac{2i\pi}{2b(0)V'(0)} =: \mathcal{K}_1(\sigma(c_*)) + \mathcal{K}_2(\sigma(c_*)),$$

where

$$\begin{aligned} \mathcal{K}_1(\sigma(c_*)) &:= \int_{\mathcal{E}} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy, \\ \mathcal{K}_2(\sigma(c_*)) &:= \int_{\mathcal{E}^c} \partial_y \left(\frac{1}{2b(y)V'(y)} \right) \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) dy - \frac{2i\pi}{2b(y_1)V'(y_1)} + \frac{2i\pi}{2b(0)V'(0)}. \end{aligned}$$

On the set \mathcal{E} , we have $\text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) \in L^2$, as

$$\text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) = \ln \left(\frac{|V(y) - \sigma(c_*)|}{|V(y) + \sigma(c_*)|} \right) + i \arg(V(y) - \sigma(c_*)) - i \arg(V(y) + \sigma(c_*)).$$

For δ_0 small enough, it holds that

$$|\mathcal{K}_1(\sigma(c_*))| \leq C \sqrt[4]{|\sigma(c_*)|} + C \sqrt{|\sigma(c_*)|} \leq C \sqrt[4]{|\sigma(c_*)|}.$$

On the \mathcal{E}^c , we have

$$\begin{aligned} \text{Log} \left(\frac{V(y) - \sigma(c_*)}{V(y) + \sigma(c_*)} \right) &= \ln \left(\frac{|V(y) - \sigma(c_*)|}{|V(y) + \sigma(c_*)|} \right) - i \arctan \left(\frac{\text{Im}(\sigma(c_*))}{|V(y) - \text{Re}(\sigma(c_*))|} \right) \\ &\quad - i \arctan \left(\frac{\text{Im}(\sigma(c_*))}{|V(y) + \text{Re}(\sigma(c_*))|} \right) - 2i\pi, \end{aligned}$$

from which we can deduce that for δ_0 small enough

$$|\mathcal{K}_2(\sigma(c_*))| \leq C \sqrt{|\sigma(c_*)|}.$$

It is then evident that

$$|I_2^\ell(\sigma(c_*))| \leq C \sqrt[4]{|\sigma(c_*)|}.$$

By (72) and (73), we have (74). \square

From Lemma 3.8, we can deduce the following corollary.

Corollary 3.9. *It holds that for $|\sigma(c_*)| < \delta_0$ with $\delta_0 > 0$ small enough,*

$$\frac{C^{-1}}{|\sigma(c_*)|} \leq |\mathcal{D}(c_*)| \leq \frac{C}{|\sigma(c_*)|}. \tag{79}$$

In addition, if $\text{Im } c_* > 0$, then

$$\lim_{c_* \rightarrow Z_+(0)} \sigma(c_*) \mathcal{D}(c_*) = \frac{i\pi}{b(0) \sqrt{2Z_+''(0)}}.$$

Recalling (60) and its counterpart on $[y_1, 0]$, we are ready to prove Lemma 3.7

Proof of Lemma 3.7. It is easy to check that

$$|L[F_*](c_*)| \leq C (|y_*^r| + |y_*^l|) \|F_*\|_{L^\infty}.$$

We also have the following estimate for $T^\ell[F_*](c_*)$,

$$\begin{aligned} |T^\ell[F_*](c_*)| &\leq \left| \int_0^{y_1} \frac{\int_{y_*^\ell}^y F_*(z) \, dz}{(Z_-(y) - c_*)(Z_+(y) - c_*)} \, dy \right| \\ &\quad + \left| \int_0^{y_1} \int_{y_*^\ell}^y \left(\frac{F_*(z)}{(Z_-(y) - c_*)(Z_+(y) - c_*)} \right) \left(\frac{\varphi_*^\ell(z)}{\varphi_*^\ell(y)^2} - 1 \right) \, dz \, dy \right| \end{aligned}$$

$$\begin{aligned} &\leq C \|F_*\|_{L^\infty} \left(\int_0^{y_1} \frac{1}{|y| + |y_*^\ell|} dy + 1 \right) \\ &\leq C (1 + |\ln(|y_*^\ell|)|) \|F_*\|_{L^\infty}. \end{aligned}$$

Similarly,

$$|T^r[F_*](c_*)| \leq C (1 + |\ln(|y_*^r|)|) \|F_*\|_{L^\infty}.$$

By Lemma 3.8, Corollary 3.9 and the fact that $|y_*^r| \sim |y_*^\ell|$, we have the following bounds on the coefficients $\mu^j[F_*](c_*)$, $\tilde{\mu}^j[F_*](c_*)$ and $\nu^j[F_*](c_*)$, $j = \ell$ or r ,

$$|\mu^j[F_*](c_*)| \leq C |\sigma(c_*)| (1 + |\ln(|y_*^r|)|) \|F_*\|_{L^\infty}, \tag{80}$$

$$|\tilde{\mu}^j[F_*](c_*)| \leq C |\sigma(c_*)| (1 + |\ln(|y_*^r|)|) \|F_*\|_{L^\infty}, \tag{81}$$

$$|\nu^j[F_*](c_*)| \leq C (1 + |\ln(|y_*^r|)|) \|F_*\|_{L^\infty}. \tag{82}$$

We recall the explicit formula of Φ_*^r given by (60)

$$\begin{aligned} \Phi_*^r(y) &= \nu^r[F_*](c_*) \varphi_*^r(y) + \mu^r[F_*](c_*) \varphi_*^r(y) \int_0^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \\ &\quad + \varphi_*^r(y) \int_0^y \frac{\int_{y_*^r}^{y'} (F_* \varphi_*^r)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' =: J_1 + J_2 + J_3, \text{ for } y \in (0, y_*^r); \end{aligned}$$

and

$$\begin{aligned} \Phi_*^r(y) &= \tilde{\mu}^r[F_*](c_*) \varphi_*^r(y) \int_{y_2}^y \frac{1}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \\ &\quad + \varphi_*^r(y) \int_{y_2}^y \frac{\int_{y_*^r}^{y'} (F_* \varphi_*^r)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' =: \tilde{J}_1 + \tilde{J}_2, \text{ for } y \in (y_*^r, y_2). \end{aligned}$$

From (82), it is clear that for $y \in (0, y_*^r)$,

$$|J_1| \leq C (1 + |\ln(|y_*^r|)|) \|F_*\|_{L^\infty} \leq C (1 + |\ln(|y|)|) \|F_*\|_{L^\infty}.$$

From (80) and the fact that $(1 + |\ln(|y_*^r|)|) \lesssim |\sigma(c_*)|^{-\gamma}$, $0 < \gamma < \frac{1}{4}$, for c_* close to $Z_+(0)$, we infer that

$$\begin{aligned} |J_2| &\leq C \|F_*\|_{L^\infty} |\sigma(c_*)|^{1-\gamma} \left| \int_0^y \frac{1}{(|y'| + |\sigma(c_*)|)(y' - y_*^r)} dy' \right| \\ &\leq C \|F_*\|_{L^\infty} |\sigma(c_*)|^\gamma \left| \int_0^y \frac{1}{|y' - y_*^r|^{1+2\gamma}} dy' \right| \\ &\leq C |\sigma(c_*)|^\gamma \|F_*\|_{L^\infty} \left(\frac{1}{|y - y_*^r|^{2\gamma}} + 1 \right), \end{aligned} \tag{83}$$

which shows that $\|J_2\|_{L^2} \leq C\|F_*\|_{L^\infty}$. In a similar way, we can show that $\|\tilde{J}_1\|_{L^2} \leq C\|F_*\|_{L^\infty}$.

The term J_3 enjoys the following estimate

$$\left| \int_0^y \frac{\int_{y_*^r}^{y'} (F_* \varphi_*^r)(z) dz}{(Z_-(y') - c_*)(Z_+(y') - c_*) (\varphi_*^r(y'))^2} dy' \right| \leq \|F_*\|_{L^\infty} (1 + |\ln(|y_*^r| + |y|)|),$$

and the estimate for \tilde{J}_2 is similar.

Therefore, it holds that

$$\begin{aligned} \|\Phi_*^r\|_{L^2(0,y_2)} &\leq \|J_1\|_{L^2(0,y_*^r)} + \|J_2\|_{L^2(0,y_*^r)} + \|J_3\|_{L^2(0,y_*^r)} + \|\tilde{J}_1\|_{L^2(y_*^r,y_2)} + \|\tilde{J}_2\|_{L^2(y_*^r,y_2)} \\ &\leq C\|F_*\|_{L^\infty}. \end{aligned}$$

Similarly, we have $\|\Phi_*^\ell\|_{L^2(y_1,0)} \leq C\|F_*\|_{L^\infty}$, which concludes our proof. \square

Remark 3.10. In the case that $c_* \in \text{Ran } Z_-$ with $\text{Im } c_* > 0$, without loss of generality, we may assume that $Z'_-(0) = 0$ and $Z''_-(0) > 0$. We can instead let $V(y)^2 = Z_-(y) - Z_-(0)$ and $(\sigma(c_*))^2 = c_* - Z_-(0)$. Then by the same argument as in the proof of Lemma 3.8, we obtain

$$\lim_{c_* \rightarrow Z_-(0)} 2\sigma(c_*)I^k(c_*) = \frac{i\pi}{b(0)\sqrt{2Z''_-(0)}}, \quad k = \ell \text{ or } r$$

and

$$\lim_{c_* \rightarrow Z_-(0)} \sigma(c_*)\mathcal{D}(c_*) = -\frac{i\pi}{b(0)\sqrt{2Z''_-(0)}}.$$

Proof of Lemma 3.6. Recalling Lemma 3.3, which asserts the strong convergence

$$\Phi_n \rightarrow \Phi \text{ in } L^2 \text{ and } (Z_- - c_n)(Z_+ - c_n)\partial_y \Phi_n \rightarrow (Z_- - c)(Z_+ - c)\partial_y \Phi \text{ in } H^1,$$

away from the critical points in $Z_+^{-1}(c)$, along with Equation (53), we know that

$$\begin{aligned} F_n^* \rightarrow F_\infty^* := &F(y) - \frac{\Phi(y_2) - \Phi(y_1)}{(y_2 - y_1)} Z'_+(y)(Z_-(y) - c) \\ &- \frac{\Phi(y_2) - \Phi(y_1)}{(y_2 - y_1)} Z'_-(y)(Z_+(y) - c) + \alpha^2(Z_+ - c)(Z_- - c)\mathcal{L}_\Phi(y) \end{aligned} \tag{84}$$

in H^1 from our assumption that $F_n \rightarrow F$ in H^1 as $n \rightarrow \infty$.

From Lemma 3.8 and Corollary 3.9, we know that as $c_n \rightarrow c$ and $F_n^* \rightarrow F_\infty^*$ in H^1 ,

$$\tilde{\mu}^r[F_n^*](c_n)\varphi_*^r(y) \int_{y_2}^y \frac{1}{(Z_- - c_n)(Z_+ - c_n) (\varphi_*^r(y'))^2} dy' \rightarrow 0 \text{ in } L^2(0, y_2),$$

whose analogues also hold for the terms corresponding to $\tilde{\mu}^\ell[F_n^*](c_n)$, $\mu^\ell[F_n^*](c_n)$ and $\mu^r[F_n^*](c_n)$.

Therefore, we know from (60) that as $c_n \rightarrow c = Z_+(0)$ and $F_n^* \rightarrow F_\infty^*$ in H^1 ,

$$\Phi_n^* \rightarrow \Phi_\infty^* \text{ in } L^2(y_1, y_2), \text{ with } \Phi_\infty^*(y) := \begin{cases} \varphi^\ell(y) \int_{y_1}^y \frac{\int_0^{y'} (F_\infty^* \varphi^\ell)(z) dz}{(Z_- - c)(Z_+ - c)(\varphi^\ell(y'))^2} dy' & \text{on } [y_1, 0), \\ \varphi^r(y) \int_{y_2}^y \frac{\int_0^{y'} (F_\infty^* \varphi^r)(z) dz}{(Z_- - c)(Z_+ - c)(\varphi^r(y'))^2} dy' & \text{on } (0, y_2], \end{cases}$$

where φ^ℓ and φ^r are the solutions to the corresponding homogeneous equation, as constructed in Appendix A. It is then evident that $\Phi_n \rightarrow \Phi$ in $L^2(\mathcal{I}_0)$.

It follows from (60) and (82) that

$$\begin{aligned} |\Phi_n^*(0)| &\leq |\nu^r[F_n^*](c_n)| \leq C \left(1 + \left| \ln \left(\sqrt{|(c_n - Z_+(0))(c_n - Z_-(0))|} \right) \right| \right) \|F_n^*\|_{L^\infty} \\ &\leq C |(Z_+(0) - c_n)(Z_-(0) - c_n)|^{-\frac{1}{4}}. \end{aligned}$$

Differentiating (60) with $c_* = c_n$ at $y = 0$ yields

$$\begin{aligned} |\partial_y \Phi_n^*(0)| &= |\partial_y \Phi_*^r(0, c_n)| \\ &\leq |\nu^r[F_n^*](c_n) \partial_y \varphi_*^r(0)| + \left| \frac{1}{(Z_-(0) - c_n)(Z_+(0) - c_n) \varphi_*^r(0)} \left(\mu^r[F_n^*](c_n) + \int_{y_*}^0 (F_n^* \varphi_*^r)(z) dz \right) \right| \\ &\leq C (1 + |\ln(|y_*^r|)|) \left(\frac{1}{|(Z_-(0) - c_n)(Z_+(0) - c_n)|^{\frac{1}{2}}} + 1 \right) \\ &\leq C |(Z_-(0) - c_n)(Z_+(0) - c_n)|^{-\frac{3}{4}}. \end{aligned}$$

We note that $|(Z_- - c_n)(Z_+ - c_n)| \geq C^{-1} > 0$ at y_1 and y_2 as the two points are far away enough from $(Z_+)^{-1}(c)$. By the facts

$$(Z_- - c_n)(Z_+ - c_n) \partial_y \Phi_n \in L^\infty \text{ and } \Phi_n \in L^2,$$

we know that $|\Phi_n(y_1)| + |\Phi_n(y_2)| + |\partial_y \Phi_n(y_1)| + |\partial_y \Phi_n(y_2)| \leq C$. The estimates on $\Phi_n^*(y_0)$ and $\partial_y \Phi_n^*(y_0)$ then imply (51) and (52). \square

3.3. Proofs of Proposition 3.1 and Proposition 3.2

As in previous proofs, we restrict ourselves to the case $c \in \text{Ran } Z_+$. By our assumptions on u and b , given $c \in \text{Ran } Z_+$, we can assume that

$$(Z_+)^{-1}(c) = \{y_{c,1}, y_{c,2}, \dots, y_{c,k}; y_{0,1}, y_{0,2}, \dots, y_{0,m}\},$$

where $y_{c,i}$, $i = 1, 2, \dots, k$ are the points at which Z_+ is monotone, i.e., $|Z'_+(y_{c,i})| > 0$, whereas $y_{0,i}$, $i = 1, 2, \dots, m$ are the critical points, where $Z'_+(y_{0,i}) = 0$ (and $|Z''_+(y_{0,i})| \neq 0$).

For each $y_{c,i}$, there exists an interval \mathcal{I}_i such that $y_{c,i} \in \mathcal{I}_i$ and $Z'_+(y_{c,i})Z'_+(y) > 0, \forall y \in \mathcal{I}_i$, whereas for each critical point $y_{0,j}$ we may find an interval $\mathcal{I}_{0,j}$ containing $y_{0,j}$ such that $Z''_+(y_{0,j})Z''_+(y) > 0, \forall y \in \mathcal{I}_{0,j}$. The rest of \mathbb{T} , consisting of regions far away from the set $(Z_+)^{-1}(c)$, can also be covered by finitely many intervals, which we denote as $\{\mathcal{I}_{a,i}\}_{i=1}^{\tilde{n}}$. The intervals \mathcal{I}_i , $i = 1, 2, \dots, k$, $\mathcal{I}_{0,i}$, $i = 1, 2, \dots, m$, and $\mathcal{I}_{a,i}$, $i = 1, 2, \dots, \tilde{n}$, can be chosen in the way such that

$$[-\pi, \pi] = \left(\cup_{i=1}^k \mathcal{I}_i \right) \cup \left(\cup_{i=1}^m \mathcal{I}_{0,i} \right) \cup \left(\cup_{i=1}^{\tilde{n}} \mathcal{I}_{a,i} \right),$$

while each of the intervals overlaps only the ones next to it, with the size of the overlap not exceeding $\frac{1}{10} \min_{1 \leq i \leq k} |\mathcal{I}_i|$ and $\frac{1}{10} \min_{1 \leq i \leq m} |\mathcal{I}_{0,i}|$.

We can then construct a family of cut-off functions $\{\chi_j\}_{j=1}^{m+k+\tilde{n}}$ forming a smooth partition of unity of \mathbb{T} such that each χ_j is supported in $\tilde{\mathcal{I}}_j$, with $\tilde{\mathcal{I}}_j$ being one of the intervals from $\{\mathcal{I}_i\}_{i=1}^k \cup \{\mathcal{I}_{0,i}\}_{i=1}^m \cup \{\mathcal{I}_{a,i}\}_{i=1}^{\tilde{n}}$. The choice of the intervals ensures that $\chi_j \equiv 1$ near $(Z_+)^{-1}(c)$.

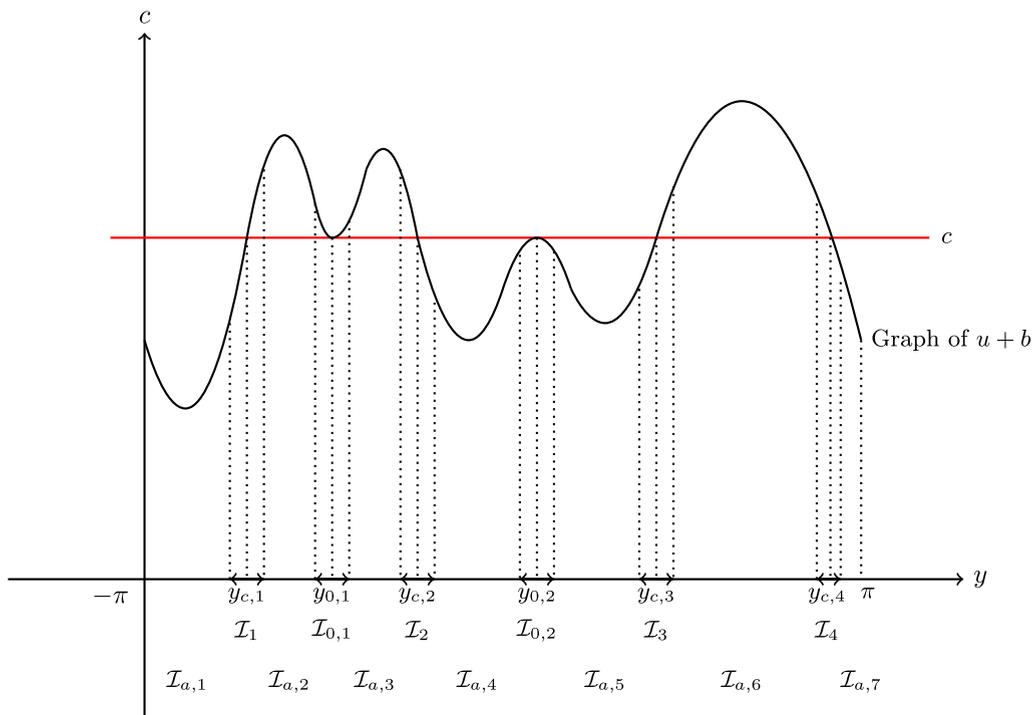


Fig. 1. Partition of $[-\pi, \pi]$.

See Fig. 1 for the partition of $[-\pi, \pi]$.

The proofs of the propositions are as follows.

Proof of Proposition 3.1 We note that on the intervals $\{\mathcal{I}_{a,i}\}_{i=1}^{\tilde{n}}$ we do not encounter any issue in integration by parts. As a consequence of Lemma 3.3, $q(y_c) = q'(y_c) = 0$ implies that $\partial_y \Phi(y_c) = 0$, allowing us to set the test function to be $f = \text{sgn}(Z_+ - c)\Phi\chi_j$ on $\tilde{\mathcal{I}}_j$ if $\tilde{\mathcal{I}}_j \in \{\mathcal{I}_i\}_{i=1}^k$, while Lemma 3.4 and Lemma 3.5 have enabled us also to set $\psi = \text{sgn}(Z_+ - c)\Phi\chi_j$ on $\tilde{\mathcal{I}}_j$ when $\tilde{\mathcal{I}}_j \in \{\mathcal{I}_{0,i}\}_{i=1}^m$.

Summing all of the integral identities and noticing that the terms containing χ'_j cancel each other, we have

$$\int_{-\pi}^{\pi} |Z_+ - c|(Z_- - c) \sum_{j=1}^{m+k} \chi_j (|\partial_y \Phi|^2 + \alpha^2 |\Phi|^2) dy = 0,$$

that is,

$$\int_{-\pi}^{\pi} |Z_+ - c|(Z_- - c) (|\partial_y \Phi|^2 + \alpha^2 |\Phi|^2) dy = 0.$$

Hence, the fact that $(Z_- - c)$ is sign-definite implies that $\Phi \equiv 0$, and $\Phi_n \rightarrow 0$ in L^2 .

It remains to be shown that $\{\Phi_n\}_{n=1}^{\infty}$ and $\{(Z_+ - c_n)(Z_- - c_n)\partial_y \Phi_n\}_{n=1}^{\infty}$ converge strongly in L^2 and H^1 , respectively, which is clear outside the set $(Z_+)^{-1}(c)$. Moreover, by Lemma 3.3 we already know that the desired strong convergence result holds outside the critical points $y_{0,1}, y_{0,2}, \dots, y_{0,k}$.

As for the critical points in $(Z_+)^{-1}(c)$, i.e., $y_{0,1}, y_{0,2}, \dots, y_{0,k}$, if $\{\Phi_n, c_n, F_n\}_{n=1}^{\infty}$ satisfies the conditions in Lemma 3.4, then by (37) and (49),

$$\Phi_n \rightarrow \Phi \equiv 0 \text{ in } L^2(\mathcal{I}_0).$$

Recalling $q_n = (Z_+ - c_n)(Z_- - c_n)\partial_y\Phi_n$, which satisfies Equation (25), i.e.,

$$q_n'' - \alpha^2 q_n = F_n' + \alpha^2 (Z_+'(Z_- - c_n) + Z_-'(Z_+ - c_n)) \Phi_n,$$

we have, from our assumptions $\|\Phi_n\|_{L^2} + \|q_n\|_{H^1} = 1$ and $F_n \rightarrow 0$ in H^1 , that

$$\|q_n''\|_{L^2} \lesssim \|\Phi_n\|_{L^2} + \|q_n\|_{L^2} + \|F_n\|_{H^1},$$

which together with the weak convergence $q_n \rightharpoonup q$ implies $q_n \rightarrow q$ in $H^1(\mathcal{I}_0)$, that is,

$$(Z_+ - c_n)(Z_- - c_n)\partial_y\Phi_n \rightarrow (Z_+ - c)(Z_- - c)\partial_y\Phi \equiv 0 \text{ in } H^1(\mathcal{I}_0).$$

Finally, if $\{\Phi_n, c_n, F_n\}_{n=1}^\infty$ satisfies the conditions in Lemma 3.5 instead, we recall Equation (53), in which both $y_1, y_2 \notin (Z_+)^{-1}(c) \cup (Z_-)^{-1}(c)$ are chosen uniform in n . Then by (49), we obtain that as $n \rightarrow \infty$, $|\Phi_n(y_1)| + |\Phi_n(y_2)| \rightarrow 0$, and thus $\|F_n^*\|_{L^\infty} \rightarrow 0$. Lemma 3.7 then ensures that $\Phi_n \rightarrow 0$ in $L^2(\mathcal{I}_0)$. By the same argument as above, we obtain that $q_n \rightarrow q$ in $H^1(\mathcal{I}_0)$.

Thus, we have shown that $\Phi_n \rightarrow 0$ in L^2 and $(Z_+ - c_n)(Z_- - c_n)\partial_y\Phi_n \rightarrow 0$ in H^1 , which contradicts the assumption that $\|\Phi_n\|_{L^2} + \|(Z_+ - c_n)(Z_- - c_n)\partial_y\Phi_n\|_{H^1} = 1$. As the same argument applies to the case $c \in \text{Ran}(Z_-)$, it must be that the uniform estimate

$$\|\Phi(\cdot, c)\|_{L^2} + \|(Z_+ - c)(Z_- - c)\partial_y\Phi(\cdot, c)\|_{H^1} \leq C\|F(\cdot, c)\|_{H^1}$$

holds true for $c \in (\Omega_{\epsilon_0} \setminus (\text{Ran } Z_+ \cup \text{Ran } Z_-))$. \square

Proof of Proposition 3.2 Suppose that there exists some $c \in \text{Ran } Z_+$ for which the limit $\Phi^+(y, c)$ do not exist, then we can find two sequences $\{c_{n,1}\}_{n=1}^\infty$ and $\{c_{n,2}\}_{n=1}^\infty$ such that $\text{Im } c_{n,j} > 0$ and $\text{Re } c_{n,j} = c$, $j = 1, 2$, $|c_{n,1} - c_{n,2}| \rightarrow 0$ as $n \rightarrow \infty$, while there exists some $\delta > 0$ such that

$$\|\Phi(\cdot, c_{n,1}) - \Phi(\cdot, c_{n,2})\|_{L^r} \geq \delta, \quad 1 < r < 2, \quad \forall n \in \mathbb{N}.$$

Up to a subsequence, $c_{n,j} \rightarrow c \in \text{Ran } Z_+$. By virtue of Proposition 3.1,

$$\|\Phi(\cdot, c_{n,j})\|_{L^2} + \|(Z_+ - c_{n,j})(Z_- - c_{n,j})\partial_y\Phi(\cdot, c_{n,j})\|_{H^1} \leq \|F(\cdot, c_{n,j})\|_{H^1} < C,$$

from which we know by (25) that

$$(Z_+ - c_{n,j})(Z_- - c_{n,j})\partial_y\Phi(\cdot, c_{n,j}) \in H^2,$$

and that there exist $\Phi_{c,j} \in L^2$ with $((Z_+ - c)(Z_- - c)\partial_y\Phi_{c,j}) \in H^2$, $j = 1, 2$, such that up to a subsequence,

$$\Phi(\cdot, c_{n,j}) \rightharpoonup \Phi_{c,j}(\cdot) \text{ in } L^2, \quad j = 1, 2.$$

As before, the situation at points in the monotone regions and that at the critical points have to be discussed separately.

As for $y_c \in (Z_+)^{-1}(c)$ at which Z_+ is strictly monotone, from the proof of Lemma 3.3 we have, for $\forall f \in C_0^1(\mathcal{I})$ and $j = 1, 2$, that

$$\begin{aligned} & -\text{p.v.} \int_{\mathcal{I}} \frac{(q'_{c,j}f' + \alpha^2 q_{c,j}f)(y)}{(Z_+(y) - c)(Z_-(y) - c)} dy + i\pi \frac{(q'_{c,j}f' + \alpha q_{c,j}f)(y_c)}{2b(y_c)Z_+'(y_c)} \\ & = -\text{p.v.} \int_{\mathcal{I}} \frac{F(y, c)f'(y)}{(Z_+(y) - c)(Z_-(y) - c)} dy + i\pi \frac{F(y_c, c)f'(y_c)}{2b(y_c)Z_+'(y_c)}, \end{aligned}$$

where $q_{c,j} := (Z_- - c)(Z_+ - c)\partial_y\Phi_{c,j}$, $j = 1, 2$. Subtracting, we have

$$-\text{p.v.} \int \frac{((q'_{c,1} - q'_{c,2})f' + \alpha^2(q_{c,1} - q_{c,2})f)(y)}{(Z_+(y) - c)(Z_-(y) - c)} dy + i\pi \frac{((q'_{c,1} - q'_{c,2})f' + \alpha^2(q_{c,1} - q_{c,2})f)(y_c)}{2b(y_c)Z'_+(y_c)} = 0,$$

whose imaginary part ensures that

$$(q_1 - q_2)(y_c) = (q'_1 - q'_2)(y_c) = 0.$$

By Hardy’s inequality, we know that $\partial_y(\Phi_{c,1} - \Phi_{c,2}) \in L^2(\mathcal{I})$.

In the case of a critical point $y_0 \in \mathcal{I}_0$, from Lemma 3.4 and Lemma 3.5 we know that $(y - y_0)\partial_y(\Phi_{c,1} - \Phi_{c,2}) \in L^2(\mathcal{I}_0)$ and

$$-\int_{\mathcal{I}_0} (Z_+ - c)(Z_- - c) (\partial_y\Phi_{c,j}f' + \alpha^2\Phi_{c,j}f) dy = \int_{\mathcal{I}_0} Ff dy, \quad \forall f \in H^1_{0,w}(\mathcal{I}_0), \quad j = 1, 2,$$

which implies that

$$-\int_{\mathcal{I}_0} (Z_+ - c)(Z_- - c) ((\partial_y\Phi_{c,1} - \partial_y\Phi_{c,2})f' + \alpha^2(\Phi_{c,1} - \Phi_{c,2})f) dy = 0, \quad \forall f \in H^1_{0,w}(\mathcal{I}_0).$$

By setting $f = \text{sgn}((Z_+ - c)(Z_- - c))(\Phi_{c,1} - \Phi_{c,2})$ and integrating by parts, similar to the procedure in the proof of Proposition 3.1, we can then conclude that $\Phi_{c,1} - \Phi_{c,2} \equiv 0$ and

$$(\Phi(\cdot, c_{n,1}) - \Phi(\cdot, c_{n,2})) \rightharpoonup 0 \quad \text{in } L^2.$$

We then aim to show that the weak convergence mentioned above is in fact strong. Away from the critical points in $(Z_+)^{-1}(c)$, this is ensured by Lemma 3.3.

Near y_0 , which can be any of the critical points $y_{0,1}, y_{0,2}, \dots, y_{0,k}$, the situation resembles those covered in Lemma 3.4, i.e., $|\text{Im } c_{n,j}| > |\text{Re } c_{n,j} - c|$ for $j = 1, 2$. (Note here $\text{Re } c_{n,j} = c$.) Therefore, by Lemma 3.4, we have

$$(y - y_0)\partial_y\Phi(y, c_{n,j}) \in L^2, \quad j = 1 \text{ or } 2,$$

which by the compactness result in Appendix B, yields the strong convergence locally near y_0 in this case, i.e.

$$\Phi(\cdot, c_{n,j}) \rightarrow \Phi_{c,j}(\cdot) \text{ in } L^r, \quad 1 < r < 2, \quad j = 1, 2.$$

Thus in both cases we have $\Phi(\cdot, c_{n,1}) - \Phi(\cdot, c_{n,2}) \rightarrow \Phi_{c,1}(\cdot) - \Phi_{c,2}(\cdot) = 0$ in L^r , which leads to a contradiction. \square

4. Linear damping and depletion

Recalling that $\Psi_1 = (u - c)\Phi + \frac{\hat{\phi}_0}{b}$ and $\Phi_1 = b\Phi$ we have, by the formula in (15) and Proposition 3.1, that

$$\begin{aligned} \widehat{\left(\frac{\psi}{\phi}\right)}(t, \alpha, y) &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\partial\Omega_\epsilon} e^{-i\alpha t c} \begin{pmatrix} \Psi_1 \\ \Phi_1 \end{pmatrix}(\alpha, y, c) \, dc \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \left(\int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} e^{-i\alpha t(c-i\epsilon)} \begin{pmatrix} u - (c-i\epsilon) \\ b \end{pmatrix} \Phi(\alpha, y, c-i\epsilon) \, dc \right. \\ &\quad \left. - \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} e^{-i\alpha t(c+i\epsilon)} \begin{pmatrix} u - (c+i\epsilon) \\ b \end{pmatrix} \Phi(\alpha, y, c+i\epsilon) \, dc \right). \end{aligned}$$

For $c \in (\text{Ran}(u+b) \cup \text{Ran}(u-b))$, we denote $c_\epsilon := c + i\epsilon$ with $-\epsilon_0 < \epsilon < \epsilon_0$. We recall that $\Phi(\alpha, y, c_\epsilon)$ solves

$$\partial_y \left((u - c_\epsilon)^2 - b^2 \right) \partial_y \Phi(\alpha, y, c_\epsilon) - \alpha^2 \left((u - c_\epsilon)^2 - b^2 \right) \Phi(\alpha, y, c_\epsilon) = F(\alpha, y, c_\epsilon), \tag{85}$$

which can also be written as

$$\partial_y \left((Z_+ - c_\epsilon)(Z_- - c_\epsilon) \right) \partial_y \Phi(\alpha, y, c_\epsilon) - \alpha^2 (Z_+ - c_\epsilon)(Z_- - c_\epsilon) \Phi(\alpha, y, c_\epsilon) = F(\alpha, y, c_\epsilon).$$

Let $\Phi^\pm(\alpha, y, c) = \lim_{\epsilon \rightarrow 0^+} \Phi(\alpha, y, c \pm i\epsilon)$, as defined in Proposition 3.2. For convenience, let us denote $\widetilde{\Phi}(\alpha, y, c) := \Phi^-(\alpha, y, c) - \Phi^+(\alpha, y, c)$. Then by Proposition 3.1 and Proposition 3.2, we have

$$\begin{aligned} \widehat{\left(\frac{\psi}{\phi}\right)}(t, \alpha, y) &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\partial\Omega_\epsilon} e^{-i\alpha t c} \begin{pmatrix} \Psi_1 \\ \Phi_1 \end{pmatrix}(\alpha, y, c) \, dc \\ &= \frac{1}{2\pi i} \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} e^{-i\alpha t c} \begin{pmatrix} (u-c)\widetilde{\Phi} \\ b\widetilde{\Phi} \end{pmatrix}(\alpha, y, c) \, dc. \end{aligned} \tag{86}$$

Proof of Theorem 1.1 Differentiating (86) in t yields

$$\partial_t \widehat{\left(\frac{\psi}{\phi}\right)}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} i\alpha c e^{-i\alpha t c} \begin{pmatrix} (c-u)\widetilde{\Phi} \\ -b\widetilde{\Phi} \end{pmatrix}(\alpha, y, c) \, dc. \tag{87}$$

By Plancherel’s theorem, we have the following estimates –

$$\begin{aligned} \left\| \widehat{\left(\frac{\psi}{\phi}\right)} \right\|_{L_t^2 L_y^2}^2 + \left\| \partial_t \widehat{\left(\frac{\psi}{\phi}\right)} \right\|_{L_t^2 L_y^2}^2 &= \int_{\mathbb{T}} \int_{-\infty}^{\infty} \left(\left| \widehat{\left(\frac{\psi}{\phi}\right)} \right|^2 + \left| \partial_t \widehat{\left(\frac{\psi}{\phi}\right)} \right|^2 \right) (t, \alpha, y) \, dt \, dy \\ &= \int_{\mathbb{T}} \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} (1 + (\alpha c)^2) \left| \begin{pmatrix} (u-c)\widetilde{\Phi} \\ b\widetilde{\Phi} \end{pmatrix}(\alpha, y, c) \right|^2 \, dc \, dy. \end{aligned}$$

Invoking Proposition 3.2 and the boundedness of b , we have

$$\begin{aligned} \left\| \widehat{\left(\frac{\psi}{\phi}\right)} \right\|_{L_t^2 L_y^2}^2 + \left\| \partial_t \widehat{\left(\frac{\psi}{\phi}\right)} \right\|_{L_t^2 L_y^2}^2 &\leq C_\alpha \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} \|\widetilde{\Phi}(\alpha, \cdot, c)\|_{L_y^2}^2 \, dc \\ &\leq C_\alpha \|F\|_{H_y^1}^2 \lesssim \left\| \begin{pmatrix} \widehat{\psi}_0 \\ \widehat{\phi}_0 \end{pmatrix} \right\|_{H_y^3}^2. \quad \square \end{aligned} \tag{88}$$

Proof of Theorem 1.6 Let y_0 be any critical point of $u + b$ or $u - b$. By the conclusion of Lemma 3.4, for $c \in \text{Ran}(u + b) \cup \text{Ran}(u - b)$, it holds that

$$|\Phi(\alpha, y_0, c \pm i\epsilon)| \leq C \left| ((u + b)(y_0) - (c \pm i\epsilon))((u - b)(y_0) - (c \pm i\epsilon)) \right|^{-\frac{1}{4}},$$

$$|\partial_y \Phi(\alpha, y_0, c \pm i\epsilon)| \leq C \left| ((u + b)(y_0) - (c \pm i\epsilon))((u - b)(y_0) - (c \pm i\epsilon)) \right|^{-\frac{3}{4}},$$

which implies uniform bounds on both $\Phi(\alpha, y_0, \cdot \pm i\epsilon_0)$ in L_c^ρ , $\rho \in [1, 4)$ and $\partial_y \Phi(\alpha, y_0, \cdot \pm i\epsilon)$ in L_c^p , $p \in [1, \frac{4}{3})$. Thus, there exists a subsequence $\epsilon_n \rightarrow 0^+$ as well as $\Lambda^\pm \in L_c^\rho$ and $\Theta^\pm \in L_c^p$ such that as $\epsilon_n \rightarrow 0^+$,

$$\Phi(\alpha, y_0, \cdot \pm i\epsilon_n) \rightharpoonup \Lambda^\pm(\alpha, y_0, \cdot),$$

$$\partial_y \Phi(\alpha, y_0, \cdot \pm i\epsilon_n) \rightharpoonup \Theta^\pm(\alpha, y_0, \cdot).$$

By (86), we have

$$\begin{aligned} \left(\begin{matrix} \widehat{U}_1 \\ \widehat{H}_1 \end{matrix} \right) (t, \alpha, y_0) &= \partial_y \left(\begin{matrix} \widehat{\psi} \\ \widehat{\phi} \end{matrix} \right) (t, \alpha, y_0) \\ &= \lim_{\epsilon_n \rightarrow 0^+} \frac{1}{2\pi i} \int_{\partial\Omega_{\epsilon_n}} e^{-i\alpha t c} \partial_y \begin{pmatrix} \Psi_1 \\ \Phi_1 \end{pmatrix} (\alpha, y_0, c) \, dc \\ &= \lim_{\epsilon_n \rightarrow 0^+} \frac{1}{2\pi i} \int_{\partial\Omega_{\epsilon_n}} e^{-i\alpha t c} \begin{pmatrix} (u - c)\partial_y \Phi + u' \Phi \\ b\partial_y \Phi + b' \Phi \end{pmatrix} (\alpha, y_0, c) \, dc \\ &= \frac{1}{2\pi i} \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} e^{-i\alpha t c} \begin{pmatrix} (u - c)(\Theta^- - \Theta^+) \\ b(\Theta^- - \Theta^+) \end{pmatrix} (\alpha, y_0, c) \, dc \\ &\quad + \frac{1}{2\pi i} \int_{\text{Ran}(u+b) \cup \text{Ran}(u-b)} e^{-i\alpha t c} \begin{pmatrix} u'(\Lambda^- - \Lambda^+) \\ b'(\Lambda^- - \Lambda^+) \end{pmatrix} (\alpha, y_0, c) \, dc. \end{aligned}$$

The desired conclusion follows from Riemann-Lebesgue lemma, as $(\Theta^- - \Theta^+)(\alpha, y_0, \cdot) \in L_c^1$ and $(\Lambda^- - \Lambda^+)(\alpha, y_0, \cdot) \in L_c^1$. \square

Appendix A. The homogeneous Sturmian equation

In this section, we shall first construct a regular solution to the homogeneous Sturmian equation on $[y_1, y_2]$ which contains only one critical point y_0 , i.e. $y_1 < y_0 < y_2$:

$$\partial_y((Z_- - c)(Z_+ - c)\partial_y \varphi) - \alpha^2(Z_- - c)(Z_+ - c)\varphi = 0, \tag{A.1}$$

for c in an ϵ_0 -strip \mathcal{S}_{ϵ_0} containing $Z_+(y_0)$:

$$\mathcal{S}_{\epsilon_0} = \left\{ c_r + i\epsilon : c_r \in [\min\{Z_+(y_0), Z_+(y_1)\}, \max\{Z_+(y_0), Z_+(y_1)\}], |\epsilon| < \epsilon_0 \right\}$$

with $Z'_+(y_0) = 0$ and $Z_+(y_1) = Z_+(y_2)$.

One may follow the same argument and construct a regular solution for the case c in an ϵ_0 -strip \mathcal{S}_{ϵ_0} containing $Z_-(y_0)$ with $Z'_-(y_0) = 0$. We omit the details here.

We note that $Z_+(y) \geq C^{-1} > 0 > -C^{-1} \geq Z_-(y)$, $Z_+(y_1) = Z_+(y_2)$, $Z'_+(y_0) = 0$ for the critical point $y_0 \in (y_1, y_2)$ and $|Z'_+(y)| > 0$ for $y \neq y_0$. Let

$$D_0 := \{c \in [\min\{Z_+(y_0), Z_+(y_1)\}, \max\{Z_+(y_0), Z_+(y_1)\}]\}$$

and

$$D_{\epsilon_0} := \{c = c_r + i\epsilon, \quad c_r \in [\min\{Z_+(y_0), Z_+(y_1)\}, \max\{Z_+(y_0), Z_+(y_1)\}], 0 < |\epsilon| < \epsilon_0\}.$$

Then $\mathcal{S}_{\epsilon_0} = D_0 \cup D_{\epsilon_0}$.

Given $c_r \in [\min\{Z_+(y_0), Z_+(y_1)\}, \max\{Z_+(y_0), Z_+(y_1)\}]$, when restricted to $[y_0, y_2]$, we can find $y^r \in [y_0, y_2]$ such that $Z_+(y^r) = c_r$. And when restricted to $[y_1, y_0]$, we can find $y^\ell \in [y_1, y_0]$ such that $Z_+(y^\ell) = c_r$.

Proposition Appendix A.1. *1. For $c \in \mathcal{S}_{\epsilon_0}$, there exists a solution $\varphi^r(y, c) \in C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$ of the Sturmian equation (A.5) and $\partial_y \varphi^r(y, c) \in C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$. Moreover, there exists $\epsilon_1 > 0$ such that for any $\epsilon_0 \in [0, \epsilon_1)$ and $(y, c) \in [y_0, y_2] \times \mathcal{S}_{\epsilon_0}$,*

$$|\varphi^r(y, c)| \geq \frac{1}{2}, \quad |\varphi^r(y, c) - 1| \leq C|y - y^r|^2,$$

where the constants ϵ_1, C may depend on α .

2. For $c \in D_0$, for any $y \in [y_0, y_2]$, there is a constant C (depends on α) such that,

$$\varphi^r(y, c) \geq \varphi^r(y', c) \geq 1, \quad \text{for } y_0 \leq y^r \leq y' \leq y \leq y_2 \quad \text{or } y_0 \leq y \leq y' \leq y^r \leq y_2.$$

Proposition Appendix A.2. *1. For $c \in \mathcal{S}_{\epsilon_0}$, there exists a solution $\varphi^\ell(y, c) \in C([y_1, y_0] \times \mathcal{S}_{\epsilon_0})$ of the Sturmian equation (A.5) and $\partial_y \varphi^\ell(y, c) \in C([y_1, y_0] \times \mathcal{S}_{\epsilon_0})$. Moreover, there exists $\epsilon_1 > 0$ such that for any $\epsilon_0 \in [0, \epsilon_1)$ and $(y, c) \in [y_1, y_0] \times \mathcal{S}_{\epsilon_0}$,*

$$|\varphi^\ell(y, c)| \geq \frac{1}{2}, \quad |\varphi^\ell(y, c) - 1| \leq C|y - y^\ell|^2,$$

where the constants ϵ_1, C may depend on α .

2. For $c \in D_0$, for any $y \in [y_1, y_0]$, there is a constant C (depends on α) such that,

$$\varphi^\ell(y, c) \geq \varphi^\ell(y', c) \geq 1, \quad \text{for } y_1 \leq y^\ell \leq y' \leq y \leq y_0 \quad \text{or } y_1 \leq y \leq y' \leq y^\ell \leq y_0.$$

In the following, we only give the proof of Proposition Appendix A.1 and Proposition Appendix A.2 can be similarly proved. To prove the existence result for Equation (A.1), we introduce the following adapted norms.

Definition Appendix A.3. For a function $f(y, c)$ on $[y_0, y_2] \times \mathcal{S}_{\epsilon_0}$, we define

$$\|f\|_{X_0} := \sup_{(y,c) \in [y_0, y_2] \times \mathcal{S}_{\epsilon_0}} \left| \frac{f(y, c)}{\cosh(A(y - y^r))} \right|, \quad \|f\|_{Y_0} := \|f\|_{X_0} + \frac{1}{A} \|\partial_y f\|_{X_0}.$$

In order to give the solution formula, we introduce the following integral operators.

Definition Appendix A.4. Let $y \in [y_0, y_2]$. The Sturmian integral operator S is defined by

$$Sf(y, c) := S_0 \circ S_1 f(y, c) = \int_{y^r}^y \frac{\int_{y^r}^{y'} (Z_-(y'') - c)(Z_+(y'') - c) f(y'', c) dy''}{(Z_-(y') - c)(Z_+(y') - c)} dy',$$

where

$$S_0f(y, c) := \int_{y^r}^y f(y', r)dy', \quad S_1f(y, c) := \frac{\int_{y^r}^y (Z_-(y'') - c)(Z_+(y'') - c)f(y'', c)dy''}{(Z_-(y) - c)(Z_+(y) - c)}.$$

Lemma Appendix A.5. *There exists a constant C independent of A such that*

$$\|S_0f\|_{X_0} \leq \frac{C}{A}\|f\|_{X_0}, \quad \|S_1f\|_{X_0} \leq \frac{C}{A}\|f\|_{X_0}, \quad \|Sf\|_{X_0} \leq \frac{C}{A^2}\|f\|_{X_0}.$$

Furthermore, there holds

$$\|Sf\|_{Y_0} \leq \frac{C}{A^2}\|f\|_{Y_0}.$$

Proof. For $c \in D_0$, we shall only prove the part of the lemma for $Z_+(y_0) = c$, as the proof when $Z_-(y_0) = c$ is along the same lines. As $\text{Ran } Z_+ \cap \text{Ran } Z_- = \emptyset$, we can find some positive C such that $C^{-1} < |Z_-(y) - c| < C$ for $y \in [y_0, y_2]$. Firstly, by definition, we have

$$\begin{aligned} \|S_0f\|_{X_0} &= \sup_{(y,c) \in [y_0, y_2] \times D_0} \left| \frac{1}{\cosh(A(y - y^r))} \int_{y^r}^y \cosh(A(y' - y^r)) \frac{f(y', c)}{\cosh(A(y - y^r))} dy' \right| \\ &\leq \sup_{(y,c) \in [y_0, y_2] \times D_0} \left| \frac{1}{\cosh(A(y - y^r))} \int_{y^r}^y \cosh(A(y' - y^r)) dy' \right| \|f\|_{X_0} \\ &\leq \frac{C}{A} \|f\|_{X_0}. \end{aligned} \tag{A.2}$$

And due to $|Z_+(y'') - c| \leq |Z_+(y) - c|$ for $y_0 \leq y'' \leq y \leq y_2$,

$$\begin{aligned} \|S_1f\|_{X_0} &\leq \sup_{(y,c) \in [y_0, y_2] \times D_0} \left| \frac{1}{\cosh(A(y - y^r))} \int_{y^r}^y \cosh(A(z - y^r)) \frac{f(y'', c)}{\cosh(A(z - y^r))} dy'' \right| \\ &\leq \frac{C}{A} \|f\|_{X_0}. \end{aligned} \tag{A.3}$$

Composing inequalities (A.2) and (A.3), we have

$$\|Sf\|_{X_0} \leq \frac{C}{A^2} \|f\|_{X_0}. \tag{A.4}$$

On the other hand, direct calculation shows $\partial_y Sf = S_1f$. By (A.3), we have

$$\|\partial_y Sf\|_{X_0} \leq \frac{C}{A} \|f\|_{X_0},$$

and then, combining with (A.4), it holds

$$\|Sf\|_{Y_0} \leq \frac{C}{A^2} \|f\|_{Y_0}.$$

In similar ways we can prove the inequalities for $c \in D_{\epsilon_0}$. \square

Proof of Proposition Appendix A.1. For $c \in \mathcal{S}_{\epsilon_0}$, we can solve the homogeneous Sturmian equation on $[y_0, y_2]$:

$$\begin{cases} \partial_y \left((Z_- - c)(Z_+ - c) \partial_y \varphi^r \right) = \alpha^2 (Z_- - c)(Z_+ - c) \varphi^r, \\ \varphi^r(y^r, c) = 1, \quad \partial_y \varphi^r(y^r, c) = 0, \end{cases} \tag{A.5}$$

where $Z_+(y^r) = c_r$, as previously defined. Integrating twice yields $\varphi^r = 1 + \alpha^2 S \varphi^r$. We choose A so that $\frac{C\alpha^2}{A^2} \leq \frac{1}{2} < 1$. Since $\|Sf\|_{Y_0} \leq \frac{C}{A^2} \|f\|_{Y_0}$, the operator $(I - \alpha^2 S)$ is invertible in the adapted space Y_0 and the solution to equation (A.5) is given by $\varphi^r = (I - \alpha^2 S)^{-1} 1$. As $\|\varphi^r\|_{Y_0} \leq \|1\|_{Y_0} + \alpha^2 \|S\varphi^r\|_{Y_0} \leq C + \frac{1}{2} \|\varphi^r\|_{Y_0}$, it holds that $\|\varphi^r\|_{Y_0} < C$.

We can rewrite S as

$$Sf(y, c) = |y - y^r|^2 \int_0^1 \int_0^1 f(y^r + (y - y^r)st) K_0(s, t, y, c) ds dt,$$

where

$$K_0(s, t, y, c) = t \frac{(Z_-(y^r + (y - y^r)st) - c)(Z_+(y^r + (y - y^r)st) - c)}{(Z_-(y^r + (y - y^r)t) - c)(Z_+(y^r + (y - y^r)t) - c)}.$$

Since $|K_0| \leq t$ and $K_0 \in C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$, S maps $C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$ to $C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$. Thus, we can deduce that $\varphi^r(y, c) \in C([y_0, y_2] \times \mathcal{S}_{\epsilon_0})$ from the formula $\varphi^r(y, c) = \sum_{k=0}^{\infty} \alpha^{2k} S^k 1$ for $c \in \mathcal{S}_{\epsilon_0}$ and the uniform convergence of the series.

Since S is a positive operator, $\varphi^r(y, c) \geq 1$ for $c \in \mathcal{S}_{\epsilon_0}$. By the continuity of $\varphi^r(y, c)$, there exists some $\epsilon \in (0, \epsilon_0]$ such that for c belonging to \mathcal{S}_{ϵ_0} , it holds that

$$|\varphi^r(y, c)| > \frac{1}{2}.$$

From the integral formula of S , we have

$$\begin{aligned} |\varphi^r(y, c) - 1| &\leq \alpha^2 \int_{y^r}^y \int_{y^r}^{y'} |\varphi^r(z, c)| \left| \frac{(Z_-(y'') - c)(Z_+(y'') - c)}{(Z_-(y') - c)(Z_+(y') - c)} \right| dy'' dy' \\ &\leq C \left(\alpha, \|\varphi^r\|_{L^\infty([y_0, y_2] \times \mathcal{S}_{\epsilon_0})} \right) |y - y^r|^2. \end{aligned}$$

Appendix B. A compactness lemma

We prove a useful compactness result.

Lemma Appendix B.1. Let $\{y_n\}_{n=1}^\infty \subset \mathcal{I} := [a, b]$ be such that $y_n \rightarrow \frac{a+b}{2}$. Let $\{f_n\}_{n=1}^\infty$ be a family of functions defined on \mathcal{I} satisfying the uniform bound

$$\|f_n\|_{L^r} + \|(y - y_n) \partial_y f_n\|_{L^p} \leq C, \quad 1 < r < p < \infty,$$

then there exist $f_\infty \in L^p$ and a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that as $j \rightarrow \infty$,

$$f_{n_j} \rightarrow f_\infty \text{ in } L^r, \quad 1 < r < p.$$

Proof. By virtue of Kolmogorov-Riesz theorem, the proof of the above compactness result amounts to showing that the family $\{f_n\}_{n=1}^\infty$ is equicontinuous in L^r . Let $\mathcal{I}_h := [y_n - h, y_n + h]$ and $\mathcal{I}_{2h} := [y_n - 2h, y_n + 2h]$ (Note that $\mathcal{I}_{2h} \subset \mathcal{I}$ for sufficiently small h .) We have

$$\begin{aligned} \|f_n(x+h) - f_n(x)\|_{L^r}^r &= \int_{\mathcal{I} \setminus \mathcal{I}_h} \left| \int_x^{x+h} f'_n(x') \, dx' \right|^r dx + \int_{\mathcal{I}_h} |f_n(x+h) - f_n(x)|^r dx \\ &:= I_* + I_{**}. \end{aligned}$$

Noticing that $|x - y_n| \geq h$ for $x \in \mathcal{I} \setminus \mathcal{I}_h$, we use Hardy-Littlewood maximal inequality to estimate I_* as follows –

$$\begin{aligned} I_* &= \int_{\mathcal{I} \setminus \mathcal{I}_h} h^r \left| \frac{1}{h} \int_x^{x+h} (x' - y_n) f'_n(x') \cdot \left(\frac{1}{x' - y_n} \right) dx' \right|^r dx \\ &\leq C_p \int_{\mathcal{I} \setminus \mathcal{I}_h} (h \| (x - y_n) f_n \|_{L^p})^r h^{-\frac{r}{p}} dx \leq C h^{r(1-\frac{1}{p})}. \end{aligned}$$

As for I_{**} , we let χ be a smooth cut-off function such that $\chi \equiv 1$ on \mathcal{I}_{2h} . Integration by parts yields the following identity –

$$\int_{\mathcal{I}} |f_n|^p \chi \, dy = - \int_{\mathcal{I}} (y - y_n) |f_n|^p \chi' \, dy - \frac{p}{2} \int_{\mathcal{I}} \chi (y - y_n) \left(f'_n |f_n|^p f_n^{-1} + \partial_y \overline{f_n} |f_n|^p \overline{f_n}^{-1} \right) dy,$$

which, along with Sobolev inequality $\|f_n\|_{L^\infty(\mathcal{I} \setminus \mathcal{I}_h)} \leq C \|f_n\|_{W^{1,q}(\mathcal{I} \setminus \mathcal{I}_h)}$, $q > 1$, leads to

$$\|f_n\|_{L^p(\mathcal{I}_h)}^p \leq C_p \|(y - y_n) \partial_y f_n\|_{L^p}^p + C_h \|f_n\|_{L^p(\mathcal{I} \setminus \mathcal{I}_h)}^p \leq C.$$

Hence, we can simply estimate I_{**} by Hölder’s inequality –

$$I_{**} \leq \int_{\mathcal{I}_h} (|f_n(x+h)|^r + |f_n(x)|^r) dx \leq 2 \|f_n\|_{L^p(\mathcal{I}_{2h})}^r h^{\frac{p-r}{p}} \leq C h^{1-\frac{r}{p}}.$$

Therefore, the desired compactness result is true. \square

Appendix C. Proof of Remark 1.4

For $u = 0$, taking the Fourier transform in x , we get for $\alpha \neq 0$,

$$\begin{cases} \partial_t \widehat{U}_1 + i\alpha \widehat{p} - i\alpha b \widehat{H}_1 - b' \widehat{H}_2 = 0, \\ \partial_t \widehat{U}_2 + \partial_y \widehat{p} - i\alpha b \widehat{H}_2 = 0, \\ \partial_t \widehat{H}_1 + b' \widehat{U}_2 - i\alpha b \widehat{U}_1 = 0, \\ \partial_t \widehat{H}_2 - i\alpha b \widehat{U}_2 = 0, \\ i\alpha \widehat{U}_1 + \partial_y \widehat{U}_2 = 0, \quad i\alpha \widehat{H}_1 + \partial_y \widehat{H}_2 = 0. \end{cases} \tag{C.1}$$

And we can diagonalize it to obtain

$$\begin{cases} \partial_{tt} \widehat{U}_2 + \alpha^2 \mathcal{A}_\alpha \widehat{U}_2 = 0, \\ \partial_{tt} (\widehat{H}_2/b) + \alpha^2 \mathcal{A}_\alpha (\widehat{H}_2/b) = 0, \end{cases} \tag{C.2}$$

where $\mathcal{A}_\alpha = (\partial_y^2 - \alpha^2)^{-1}(b^2(\partial_y^2 - \alpha^2) + 2bb'\partial_y)$, and the details can be seen in [15]. In order to prove the energy conservation law, we only need to prove the energy conservation on each frequency.

At first, we can show that

$$\begin{aligned} & \|\alpha^k \widehat{U}_1\|_{L_y^2}^2 + \|\alpha^k \widehat{U}_2\|_{L_y^2}^2 + \|\alpha^k \widehat{H}_2\|_{L_y^2}^2 + \|\alpha^k(\widehat{H}_1 - i(\alpha b)^{-1}b'\widehat{H}_2)\|_{L_y^2}^2 \\ &= \|\alpha^k \widehat{U}_{1,in}\|_{L_y^2}^2 + \|\alpha^k \widehat{U}_{2,in}\|_{L_y^2}^2 + \|\alpha^k \widehat{H}_{2,in}\|_{L_y^2}^2 + \|\alpha^k(\widehat{H}_{1,in} - i(\alpha b)^{-1}b'\widehat{H}_{2,in})\|_{L_y^2}^2. \end{aligned} \tag{C.3}$$

Indeed, taking L^2 inner product of (C.2)₂ with $\overline{\alpha^{2k}\partial_t(\partial_y^2 - \alpha^2)(\widehat{H}_2/b)}$, integrating by part, taking the real part, we obtain

$$\begin{aligned} & \operatorname{Re} \int_{\mathbb{T}} \alpha^k \partial_{tt}(\widehat{H}_2/b) \overline{\alpha^k \partial_t(\partial_y^2 - \alpha^2)(\widehat{H}_2/b)} dy \\ &= -\operatorname{Re} \left(\int_{\mathbb{T}} \alpha^k \partial_{tt} \partial_y(\widehat{H}_2/b) \overline{\alpha^k \partial_t \partial_y(\widehat{H}_2/b)} dy + \alpha^{2+2k} \int_{\mathbb{T}} \partial_{tt}(\widehat{H}_2/b) \overline{\partial_t(\widehat{H}_2/b)} dy \right) \\ &= -\frac{1}{2} \frac{d}{dt} \|\alpha^k \partial_t \partial_y(\widehat{H}_2/b)\|_{L_y^2}^2 - \frac{1}{2} \frac{d}{dt} \|\alpha^{1+k} \partial_t(\widehat{H}_2/b)\|_{L_y^2}^2 \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Re} \int_{\mathbb{T}} \mathcal{A}_\alpha(\widehat{H}_2/b) \overline{\alpha^{2k} \partial_t(\partial_y^2 - \alpha^2)(\widehat{H}_2/b)} dy \\ &= \operatorname{Re} \int_{\mathbb{T}} (b^2(\partial_y^2 - \alpha^2) + 2bb'\partial_y)(\widehat{H}_2/b) \overline{\alpha^{2k} \partial_t(\widehat{H}_2/b)} dy \\ &= -\operatorname{Re} \left(\int_{\mathbb{T}} \alpha^k b^2 \partial_y(\widehat{H}_2/b) \overline{\alpha^k \partial_t \partial_y(\widehat{H}_2/b)} dy + \alpha^{2k+2} \int_{\mathbb{T}} \widehat{H}_2 \partial_t \overline{\widehat{H}_2} dy \right) \\ &= -\frac{1}{2} \frac{d}{dt} \|\alpha^k \partial_y(\widehat{H}_2/b)\|_{L_y^2}^2 - \frac{1}{2} \frac{d}{dt} \|\alpha^{1+k} \widehat{H}_2\|_{L_y^2}^2, \end{aligned}$$

and then

$$\frac{d}{dt} \left(\|\alpha^k \partial_t \partial_y(\widehat{H}_2/b)\|_{L_y^2}^2 + \|\alpha^{1+k} \partial_t(\widehat{H}_2/b)\|_{L_y^2}^2 + \|\alpha^{1+k} \partial_y(\widehat{H}_2/b)\|_{L_y^2}^2 + \|\alpha^{1+k} \widehat{H}_2\|_{L_y^2}^2 \right) = 0.$$

By (C.1), we get (C.3) for $\alpha \neq 0$. Finally, we prove that for some constant $C > 0$ independent of t, α ,

$$C^{-1} \|\widehat{H}_1\|_{L_y^2} \leq \|\widehat{H}_1 - i(\alpha b)^{-1}b'\widehat{H}_2\|_{L_y^2} \leq C \|\widehat{H}_1\|_{L_y^2}.$$

By the condition that $\widehat{H}_2(t, \alpha, y_1) = \widehat{H}_2(t, \alpha, y_2) = 0$, we have $\|\widehat{H}_2\|_{L_y^2} \leq C \|\partial_y \widehat{H}_2\|_{L_y^2} \leq C \|\alpha \widehat{H}_1\|_{L_y^2}$. Thus,

$$\|\alpha^{-1+k} \partial_y(\widehat{H}_2/b)\|_{L_y^2} \leq C \|\alpha^{-1+k} \partial_y \widehat{H}_2\|_{L_y^2} + C \|\alpha^{-1+k} \widehat{H}_2\|_{L_y^2} \leq C \|\alpha^k \widehat{H}_1\|_{L_y^2}.$$

And on the other hand,

$$\begin{aligned} \|\alpha^k \widehat{H}_1\|_{L_y^2} &\leq \|\alpha^k(\widehat{H}_1 - i(\alpha b)^{-1}b'\widehat{H}_2)\|_{L_y^2}^2 + \|\alpha^k(\alpha b)^{-1}b'\widehat{H}_2\|_{L_y^2}^2 \\ &\leq C \|\alpha^{-1+k} \partial_y(\widehat{H}_2/b)\|_{L_y^2} \leq \|\alpha^k(\widehat{H}_1 - i(\alpha b)^{-1}b'\widehat{H}_2)\|_{L_y^2}^2, \end{aligned}$$

where we used the fact that $-i\alpha b\partial_y(\widehat{H}_2/b) = \widehat{H}_1 - i(\alpha b)^{-1}b'\widehat{H}_2$ in the last inequality. Since C is independent of t and α , we obtain the proof of Remark 1.4 by Plancherel identity.

For the vorticity and current density, we have that by taking Fourier transform in x ,

$$\begin{cases} \partial_t \widehat{\omega} = i\alpha b \widehat{j} - b'' \widehat{H}_2, \\ \partial_t \widehat{j} = i\alpha b \widehat{\omega} - 2i\alpha b' \widehat{U}_1 + b'' \widehat{U}_2. \end{cases} \tag{C.4}$$

By taking L^2 inner product of (C.4)₁ with $\overline{\widehat{\omega}}$ and (C.4)₂ with $\overline{\widehat{j}}$, and taking the real part, we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\widehat{\omega}\|_{L_y^2}^2 + \|\widehat{j}\|_{L_y^2}^2 \right) &= \operatorname{Re} \left(- \int_{\mathbb{T}} b'' \widehat{H}_2 \overline{\widehat{\omega}} dy - \int_{\mathbb{T}} 2i\alpha b' \widehat{U}_1 \overline{\widehat{j}} dy + \int_{\mathbb{T}} b'' \widehat{U}_2 \overline{\widehat{j}} dy \right) \\ &\leq \|b'' \widehat{H}_2\|_{L_y^2} \|\widehat{\omega}\|_{L_y^2} + \|2\alpha b' \widehat{U}_1\|_{L_y^2} \|\widehat{j}\|_{L_y^2} + \|b'' \widehat{U}_2\|_{L_y^2} \|\widehat{j}\|_{L_y^2}. \end{aligned}$$

And then, we get

$$\frac{d}{dt} \left(\|\widehat{\omega}\|_{L_y^2} + \|\widehat{j}\|_{L_y^2} \right) \leq C \left(\|\widehat{H}_2\|_{L_y^2} + \|\alpha \widehat{U}_1\|_{L_y^2} + \|\widehat{U}_2\|_{L_y^2} \right),$$

by integrating in time, it holds

$$\|\widehat{\omega}\|_{L_y^2} + \|\widehat{j}\|_{L_y^2} \leq \left(\|\widehat{\omega}_0\|_{L_y^2} + \|\widehat{j}_0\|_{L_y^2} \right) + C \int_0^t \left(\|\widehat{H}_2\|_{L_y^2} + \|\alpha \widehat{U}_1\|_{L_y^2} + \|\widehat{U}_2\|_{L_y^2} \right) d\tau.$$

Thus by using (C.3), we obtain the linear growth result of vorticity and current density.

Note that when considering the non-flowing plasma case, the system can be diagonalized to one with a self-adjoint operator \mathcal{A}_α .

For general case with constant background velocity ($u = \text{const.}$), we have

$$\begin{cases} \partial_t \widehat{U}_1 + i\alpha u \widehat{U}_1 + i\alpha \widehat{p} - i\alpha b \widehat{H}_1 - b' \widehat{H}_2 = 0, \\ \partial_t \widehat{U}_2 + i\alpha u \widehat{U}_2 + \partial_y \widehat{p} - i\alpha b \widehat{H}_2 = 0, \\ \partial_t \widehat{H}_1 + i\alpha u \widehat{H}_1 + b' \widehat{U}_2 - i\alpha b \widehat{U}_1 = 0, \\ \partial_t \widehat{H}_2 + i\alpha u \widehat{H}_2 - i\alpha b \widehat{U}_2 = 0, \\ i\alpha \widehat{U}_1 + \partial_y \widehat{U}_2 = 0, \quad i\alpha \widehat{H}_1 + \partial_y \widehat{H}_2 = 0. \end{cases}$$

Let us introduce $\widehat{U} = e^{i\alpha u t} \widehat{U}$, $\widehat{H} = e^{i\alpha u t} \widehat{H}$, $\widehat{p} = e^{i\alpha u t} \widehat{p}$ and $\widehat{\omega} = e^{i\alpha u t} \widehat{\omega}$, $\widehat{j} = e^{i\alpha u t} \widehat{j}$, then $(\widehat{U}, \widehat{H}, \widehat{p})$ solves (C.1) and $(\widehat{\omega}, \widehat{j})$ solves (C.2).

Thus Remark 1.4 follows from the fact that

$$\left\| (\widehat{U}, \widehat{H}, \widehat{\omega}, \widehat{j}) \right\|_{L_y^2} = \left\| (\widehat{U}, \widehat{H}, \widehat{\omega}, \widehat{j}) \right\|_{L_y^2}.$$

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