Constructing Linear Systems with Particular Kinds of Solution Sets

Jessica L. Smith
Florida State University

Inyoung Lee
Arizona State University

Michelle Zandieh
Arizona State University

Christine Andrews-Larson
Florida State University

Systems of equations are a core topic in linear algebra courses. Solving systems with no or infinitely many solutions tends to be less intuitive for students. In this study, we examined two students' reasoning about the relationship between the structure of a system of linear equations and its solution set, particularly when creating systems with a certain number of equations and unknowns. Using data from a paired teaching experiment, we found that both students favored the notion of parallel planes, geometrically and numerically, in the case of a system having no solution or infinitely many solutions. We also found that algebraic or numerical approaches were used as the main way of developing systems with a unique solution, especially in systems with more than two equations and two unknowns. Throughout the tasks, one student generally used geometric approaches and the other toward algebraic and numerical approaches.

Keywords: solution sets, linear systems of equations, Realistic Mathematics Education, paired teaching experiment, linear algebra

Systems of linear (SLE)equations are a fundamental topic of introductory linear algebra courses. Systems are one way to model relationships among multiple quantities (Smith & Thompson, 2007). Students from applied science, technology, engineering, and mathematics make up a dominant portion of those enrolled in linear algebra in the U.S. and Canada, and applications related to linear systems are an important component of their learning in this course (Andrews-Larson et al., in press). In applied contexts, SLE often do not have a unique solution, so in this work we focus on students' reasoning about SLE with non-unique solutions.

Students at a variety of levels have exhibited greater levels of success solving systems of linear equations with unique solutions as compared to solving inconsistent systems or systems with an infinite number of solutions (Harel, 2017; Huntley et al., 2007; Oktaç, 2018), and this has been linked to procedural and rule-based approaches to students' solving processes. Oktaç (2018), for example, found that interpreting the result "x=x" was not intuitive for students. Such approaches tend to be unhelpful to students when linking algebraic and geometric representations of systems and their solution sets. It has more recently been documented that, in the context of introductory linear algebra, though students are relatively successful at rewriting systems as augmented matrices and row reducing with technological assistance, many students experience a disconnect in reinterpreting solutions in relation to the original system (Zandieh & Andrews-Larson, 2019). In this paper, we identify resources in students' reasoning that may help address this disconnect, drawing on data from a teaching experiment. Our research question is: How did students reason about the relationship between the structure of a system of linear equations and its solution set, including their reasoning about the graphical representation of the solution set?

Theoretical Framing

In this paper we focus on how students symbolize systems of linear equations that have one, infinitely many or no solutions, and how the students relate their symbolizations to various graphical representations. In doing this we follow Larson and Zandieh (2013) who describe a system of equations as one way to symbolize a relationship between variables and scalars. They compare and contrast this symbolization to others such as a vector equation (a linear combination of vectors equaling another vector) or a linear transformation stated in function notation. Collectively they refer to these as interpretations of the matrix equation $A\vec{x}=\vec{b}$. Paired with these symbolic expressions are different graphical representations. For example, a system of two equations with two unknowns is depicted graphically as two lines with the intersection of the lines denoting the solution. In contrast a linear combination graphically is indicated by scalars stretching each of two vectors with the vectors being added in a tip to tail manner.

In a later paper, Zandieh and Andrews-Larson (2019) extend their work to symbolizations of augmented matrices and discuss how the solution(s) to a system of equations may be expressed using both implicit and explicit notation. For example, the statement of a system of equations to be solved is already an implicit description of the solution space. In this paper we focus on two students' growing recognition of how the components of the symbolic expression that implicitly describes the solution set (i.e., the system of linear equations) relates to the graphical inscriptions that illustrate the solution space.

Methods

This proposal reports findings from a paired teaching experiment (PTE) conducted as part of a larger NSF-funded project aimed at extending inquiry-oriented curricula in linear algebra. The task sequence was designed to support students' reinvention of ways to reason about solution sets to systems of linear equations (SLE). We conducted a task-based PTE with two students for four consecutive days on Zoom to see how students reason about solutions to SLE related to the task and how their reasoning evolved throughout the sequence of tasks (Steffe & Thompson, 2000). The participants we call Student R and Student L were undergraduate math majors also studying to be secondary teachers; one was a white woman and one a white man, respectively. They had no experience with linear algebra but had taken Calculus I at the time of the PTE. Two authors conducted the PTE with one leading the interview and the other taking notes and asking clarifying questions. We present findings from the last day of the PTE, in which students started working individually on the given tasks and shared their initial thoughts with each other. The leading interviewer prompted with questions about their thinking in the moment and leveraged their ideas to help them advance their mathematical thinking. While this proposal emphasizes the last day of the PTE, relevant results from the first three days are included in the task description to give readers the basic context of the instructional sequence.

The Day 1 and 2 tasks were designed to support students in reinventing the notion of large or infinite solution sets corresponding to a SLE. On Day 1, the students worked with a constraint of meal plans that included three variables, B, L, D (# of breakfasts, lunches, and dinners), where B+L+D=210. They used ordered *n-tuples* to organize a large set of solutions satisfying the constraint. On Day 2, the students made predictions about how the set of solutions to B+L+D=210 and 5B+7L+10D=1500 (an added meal plan constraint) would look geometrically. The students agreed the graphs of the two equations would intersect more than once because they found several solutions to both equations (for more details, see Smith et al., 2021b). Day 3 focused on graphing the two meal plan constraints in GeoGebra. This was the first time the students used GeoGebra in the PTE. Based on this work they concluded that each constraint constitutes a plane and the solutions found on Day 2 are on the same line where the two planes intersect. They were guided to the next task called "Intersections of Three Planes" (Wawro et al., 2013). Given the new SLE where the third equation is the sum of the first two equations, the students were asked to find the closest graphic from options (a) through (f), focusing on the intersection of the three planes. At first, they were not allowed to do any calculations nor use GeoGebra. The options consisted of graphics such as three planes having no intersection and intersecting at a point, line, or plane, as shown in Figure 1. Results from these tasks will be discussed in the next section.

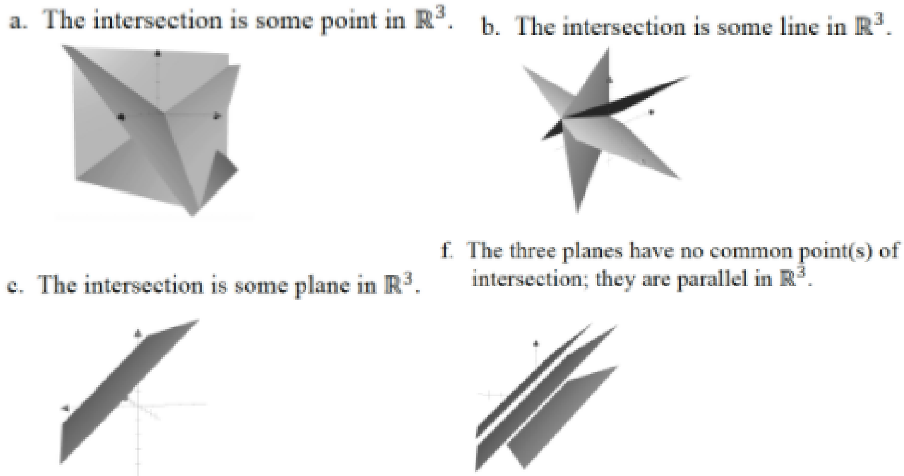

Figure 1. Examples of options in the Intersections of Three Planes task (Wawro et al., 2013).

On Day 4, the students continued to work on the Intersection of Three Planes task with a different approach. Using GeoGebra as an aid, they engaged in the activity to construct a SLE that looks like one of the options' graphics. The students started manipulating the given SLE (where the third equation is the sum of the first two) to satisfy the graphics. The students then worked on a task involving creating SLE that met a specified number of equations, unknowns, and solutions described in a table, called the Example Generation task. In this paper, we use abbreviations to report these more succinctly. For example, 2E2U refers to 2 equations and 2 unknowns and 3E2U refers to 3 equations and 2 unknowns. This task also included a prompt to make generalizations about SLE regarding the number of solutions, equations, and unknowns.

Our data sources include Zoom video recordings, students' written work, and field notes. The Day 3 and 4 video recordings were transcribed in spreadsheets. Two of the authors watched and reviewed the videos and transcripts and created reflective notes focusing on the students' reasoning about creating SLE with specific solution sets using an emergent coding method (Glaser & Strauss, 2017). We paid special attention to what students were referring to when reasoning about solutions, including students' attention to specific traits of the SLE as written and traits of its graph. We discussed the students' quotes and gestures that offered insight into their mathematical reasoning about SLE. Based on the findings, we were able to describe the differences and similarities between students' reasoning about SLE in relation to specified solution sets.

Findings

We found that Student R and Student L developed geometric and numeric strategies, respectively, for reasoning about parallel planes, and leveraged these strategies heavily in constructing systems of linear equations with specific types of solution sets. In this section, we illustrate how this strategy emerged from students' efforts to construct a system with particular visual characteristics in GeoGebra and provide evidence regarding how students leveraged these strategies to construct systems with no solution, unique solution, and infinite solutions -- even as the numbers of equations and unknowns in these systems varied.

Intersections of Three Planes Task

When working on the modified version of the Intersections of Three Planes task, students looked at scenario (a) in which three planes intersect at a point and initially noted that they did not think it was possible. Student R drew on her knowledge of systems of line equations by drawing three lines as a way to represent three planes to see if they could intersect in a point.

R: I didn't even think (a) was possible at first, but I drew out a picture of what I thought it could be. And I was like, maybe, I guess it is possible for planes to intersect in a point. But I thought only lines could intersect in a point.

Int (interviewer): Can you show what you drew? [Student R shows Figure 2] ... Tell me how you were thinking about what you drew.

R: I thought, these are lines technically, but I was like, even if they were extending, if they were all going different directions, then I just drew three random lines that all intersected only in the middle.

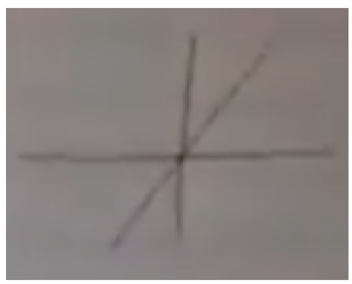

Figure 2. Student R's two-dimensional reasoning about three planes intersecting at a point.

Student (R) drew on her prior knowledge to make sense of a new type of intersection. She conjectured, through the lens of intersecting lines as shown in Figure 2, that it could be possible for three planes to intersect at a point. Both students revisited this idea in the Example Generation task.

The rest of this section will highlight students' initial reasoning with parallel planes that they used later in the Example Generation task. When starting the modified Intersections of Three Planes task, Student R stated that she initially thought the parallel option *"would be easier because… if you just double the values, then it'll be the same plane. But maybe if you manipulate the number that it's set equal to, to not be double, if you doubled the coefficients of the variables… [then] I think we can make the planes parallel."* She pointed out that every value cannot be multiplied by the same number, but parallel planes might be achievable by changing the constant by a different factor from the coefficients. While working on making three parallel planes, Student R referred to the "slopes" of the equations in terms of the coefficients in each. She reasoned that to make parallel lines they needed to create equivalent slopes in the equations, then expanded that reasoning to planes.

Int: So why do you think you have to make them the same for all of them?

R: If we want the slope to be the same, they all have to be multiplied by the same. I can't think of the words… We want the slopes to be there's a word that I can't think of, related…

L: Parallel. Yeah, related.

R: So like, if you change the coefficient of one number, it's no longer gonna be parallel because then the slopes of the equations aren't the same anymore.

Int: So like, what if your original equation was x plus three y plus z and you're trying to make one parallel by changing the coefficients? [Referring to L's eq7: x+3y+z=6]

L: ... like 2x+6y+2z is equal to like, whatever. I believe that would make it as well… Then, so then 2x+6y+2z=6.

Int: So you doubled one side but not the other. So it kept the slope the same.

L: Right. Exactly.

Both students reasoned that all the coefficients in an equation must be multiplied by the same number to have the same "slope" as the original equation. They said that if one coefficient is not multiplied, then the plane will no longer be parallel to the original. Once again, it was mentioned that the 'right side' (or constant) cannot be changed by the same amount as the coefficients.

The students used the parallel planes they had just developed to recreate a system consisting of two parallel planes with a third plane intersecting both. When asked about the number of solutions to the SLE, Student R relied on the fact that two of the planes are parallel and said "*Zero. Because two of the planes are parallel. And those are never going to have a solution that makes them both true.*" Student L came to the same conclusion but by reasoning numerically, stating, "*And also because you have x+y+z is two, and x+y+z is eight. So that doesn't really make sense either, because you can't have three numbers that equal two and also equal eight, when you add them together.*"

Overall, we found that students' previous knowledge regarding systems was essential to students' development of SLE in relation to graphical representations. Students wondered about the possibility of a unique solution for a system with three planes. Because they had never seen it previously, they explored this possibility by drawing on their knowledge of a system of equations in a 2-dimensional setting. The students developed initial reasoning for parallel planes, seemingly relying on what they knew about parallel lines. This reasoning about parallel lines and planes became a way for students to reason about systems with infinitely many and no solutions.

Example Generation Task

Constructing systems with no solutions. In this section, we will describe our findings regarding students' reasoning while creating SLE with no solutions. The students' work on the example generation task began in the context of 2E2U (two equations and two unknowns). In this case, the students started with one equation, x+y=3, and created another equation by multiplying each coefficient by two to get, 2x+2y=3. As they explained with planes in the previous task, the constant cannot be multiplied by the same number as the coefficients, or else the second equation is actually the same line as the first. When students subsequently worked to construct an example of a system with 3E2U with no solution, they added another parallel line to their previous example. The interviewer probed on whether this was the only way a SLE with no solution could be constructed.

Int: So would it have to be parallel?

R: I guess not. We could do like we just did with the last one we did with three variables. Maybe. That also has no solutions.

L: If it wasn't parallel, wouldn't they eventually come to a point or something?

R: Well, equation one and equation two are parallel. But if I change the third equation, instead of it being parallel as well, if I make it not parallel, then it's kind of like what we just did. How there's no solution because these two lines are parallel.

Student R mentioned their previous example in which there were two parallel planes and one plane intersecting the two, except with lines this time. The students used similar thinking to create a system of 3E3U with no solution with parallel planes.

Constructing systems with infinitely many solutions. When constructing systems of equations with infinitely many solutions, the pair began by working on the case with 2E2U. Student R created two copies of the same line by multiplying x+y=3 (equation one) by 2 and explaining, "*If we wanted infinitely many solutions, we can just do like 2x+2y=6 with equation one. And then they're the exact same line."* The students used a similar process to construct a system of 3E2U and voiced that multiplying the second and third equations by some value is the only way they understand creating a system of lines with infinitely many solutions. Student L wondered, "*Is there? I'm not sure if there's another way to have infinitely many solutions*." When shifting to the cases with 2E3U and 3E3U, the pair leveraged their previous work, and created equations representing the same planes.

Constructing systems with unique solutions. In working to construct a system of 2E2U with a unique solution, Student L used a heavily numeric approach and started with one equation, found a viable solution to that equation, then created a second equation in which the previously viable solution satisfies. In other words, Student L started with the equation x+y=3 and a solution of x=2, y=1 and developed 2x+y=5 as a second equation. He explained, *"So I just used x+y is equal to three. And then I said, okay, x is equal to two y is equal to one. So, then I was just like, well, two times two is four plus that one is five. So that should give us the one solution. It'll be two, one."* The students only referenced the geometry of this SLE by describing (2, 1) as the point of intersection. In moving to 3E2U, Student L did something similar by building another equation based on a selected solution that satisfies the two originally developed equations: *"Just like how we did it with the, the two equations and two unknowns. I was just thinking, well, we have x+y is equal to three. And then if x is two, y is one. So then, in 2x+y, it would make it five. And then if you were to add like another one to just like another coefficient to that x, it would make it 3x+y. And now you just go to seven."* Once again, neither student explicitly referenced geometry as they built this SLE.

In shifting to the case with 2E3U, student L stuck with his numerical approach, presumably because it had been successful previously. Student R thought more geometrically, arguing that it was impossible for the system to have a single solution because two planes cannot intersect at a single point. She explained, "*When we were doing the algebra part of it, I was more like maybe there is only one solution. But because when you look at it, it's hard to just tell from the numbers. There's no way just to look at it and be like there's a million solutions. I know what they all are. But when you look at the graph, it becomes clear that there's no way that these two planes are going to ever intersect in just a point because they're planes. And there's only two of them."* Student L tried a guess-and-check method, but he later accepted Student R's justification because he could not find a system that had only one solution. In the case of 3E3U, the students questioned whether it was possible to have a single solution, thus revisiting their previous query. Their intuition told them that it was possible. They constructed the SLE in GeoGebra by creating two planes that intersect in a line and then added a plane that intersected that line at one point.

Making Generalizations

At the end of the interview, the interviewer asked students for generalizations about systems and their solutions. We found that both students attended to ideas they developed when reasoning about parallel planes, such as looking at coefficients and constants.

R: I would say pay attention to the **coefficients** and maybe look for like **how the equations could be maybe parallel… or how they're intersecting**… Parallel will tell you that there would be no solutions for the system.

L: I would also say, make sure to pay attention to the **constant**. So like whatever you're setting it equal to, x+y is equal to three. So for instance, it had no solutions, we know that the constant is just going to stay the same, because the lines are going to be parallel… The constant is going to change based on how the other equations change.

Student R eventually came to reason about the number of equations and unknowns, something that had not been deeply discussed previously. She stated, *"In the beginning, we were trying to find a solution to that system of equations that had only two equations and three unknowns. Now I know in the future, don't waste your time looking, doing substitution. It's not gonna work... Maybe because there's* ***more unknowns than equations****, you don't have enough information to use substitution or anything because there's not going to be a single solution. I don't know if that's true."* It could be that the design of the Example Generation task led Student R to connect that the reason two planes could not have one solution was because there were fewer equations than unknowns (along with her geometric reasoning), leading her to conjecture that there can never be one solution to a system with that trait.

Discussion

On Day 4 of the PTE, Student R generally oriented her reasoning around a geometric approach to make sense of no, unique, and infinitely many solutions. She predicted how the graphs of solutions would look as she modified the coefficients in the SLE and used the same strategy when constructing SLE in the Example Generation Task. On the other hand, Student L leveraged a numerical approach to reason about solutions to various SLE. He began by choosing specific tuples satisfying a first equation or concluded no solution by examining the coefficients and constants in equations. Then, the parallel graphics became sensible to him. The two students' different ways of reasoning seemed to reflect their interpretation of solution to SLE: It is the intersection of graphs [R] and the point(s) in the intersections of graphs that satisfies all equations in the SLE [L]. In the case of 2E3U, Student R's geometric approach allowed her to conclude that this type of system can never have a unique solution. She subsequently conjectured that this might be true in all cases where the number of equations is less than the number of variables. Throughout the task sequence, Student R's geometric reasoning was more useful in some cases, and Student L's numerical reasoning in other cases, but both students relied on their understanding of parallel lines and planes in many of the infinitely many and no solutions cases. This study demonstrates ways students can reason about solutions to SLE without having learned row reduction, pointing out potential areas for connections between students' prior knowledge and what is to be learned.

Acknowledgements

The work presented here was supported by the National Science Foundation (NSF) under Grant Numbers 1914793, 1914841, and 1915156. Any opinions, findings, and conclusions or recommendations in this article do not necessarily reflect the views of the NSF.

References

Andrews-Larson, C. (2015). Roots of linear algebra: An historical exploration of linear systems. *Primus*, *25*(6), 507-528.

Andrews-Larson, C. J., Siefken, J., & Simha, R. (in press). Report on a US-Canadian Faculty Survey on Undergraduate Linear Algebra: Could Linear Algebra Be an Alternate First Collegiate Math Course? *Notices of the American Mathematical Society*.

Glaser, B. G., & Strauss, A. L. (2017). *Discovery of grounded theory: Strategies for qualitative research*. Routledge.

Harel, G. (2017). The learning and teaching of linear algebra: Observations and generalizations. *Journal of Mathematical Behavior*, 46, 69-95.

Huntley, M. A., Marcus, R., Kahan, J., & Miller, J. L. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *Journal of Mathematical Behavior*, 26, 115-139.

Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax= b. *For the Learning of Mathematics*, *33*(2), 11-17.

Oktaç, A. (2018). Conceptions about system of linear equations and solution. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and strategies in teaching linear algebra* (pp. 71–101). Springer.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. *Algebra in the Early Grades*, 95-132.

Smith, J., Andrews-Larson, C., Reinholz, D. L., Stone-Johnstone, A., & Mullins, B. (2019, March). Examined inquiry-oriented instructional moves with an eye toward gender equity. In *Proceedings of the 22nd annual conference on Research in Undergraduate Mathematics Education* (pp. 1035-1041).

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Zandieh, M. & Andrews-Larson, C. (2019). Symbolizing while solving linear systems. *ZDM* Mathematics Education, 51, 1183-1197.