



Uniform regularity for the compressible Navier-Stokes system with low Mach number in domains with boundaries



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ABSTRACT

We establish uniform with respect to the Mach number regularity estimates for the isentropic compressible Navier-Stokes system in smooth domains with Navier-slip condition on the boundary in the general case of ill-prepared initial data. To match the boundary layer effects due to the fast oscillations and the ill-prepared initial data assumption, we prove uniform estimates in an anisotropic functional framework with only one normal derivative close to the boundary. This allows to prove the local existence of a strong solution on a time interval independent of the Mach number and to justify the incompressible limit through a simple compactness argument.

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RÉSUMÉ

Nous établissons des estimations de régularité uniformes par rapport au nombre de Mach pour le système de Navier-Stokes compressible isentropique dans les domaines réguliers avec condition de Navier au bord dans le cas général de données initiales mal préparées. Pour être cohérent avec les effets de couche limite dus aux oscillations rapides et à l'hypothèse de données initiales mal préparées, nous prouvons des estimations uniformes dans un cadre fonctionnel anisotrope avec une seule dérivée normale proche du bord. Ceci permet de prouver l'existence locale d'une solution

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forte sur un intervalle de temps indépendant du nombre de Mach et de justifier la limite incompressible par un argument de compacité simple.

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1. Introduction

In this paper, we consider the following scaled isentropic compressible Navier-Stokes system $(CNS)_\varepsilon$

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \operatorname{div} \mathcal{L}u^\varepsilon + \frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \quad (t, x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\rho^\varepsilon(t, x)$ and $u^\varepsilon(t, x)$ are the density and the velocity of the fluid respectively, P is the pressure. The viscous stress tensor takes the form:

$$\mathcal{L}u^\varepsilon = 2\mu \mathbb{S}u^\varepsilon + \lambda \operatorname{div} u^\varepsilon \operatorname{Id}, \quad \mathbb{S}u^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + \nabla^t u^\varepsilon).$$

Here, μ, λ are viscosity parameters that are assumed to be constant and to satisfy the condition: $\mu > 0, 2\mu + 3\lambda > 0$. The parameter ε is the scaled Mach number which is assumed small, that is $\varepsilon \in (0, 1]$.

Since we are considering the system in a domain with boundaries, we shall supplement the system (1.1) with the Navier-slip boundary condition

$$u^\varepsilon \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^\varepsilon \mathbf{n}) + a\Pi u^\varepsilon = 0 \quad \text{on } \partial\Omega \quad (1.2)$$

where \mathbf{n} is the unit outward normal vector and a is a constant related to a slip length (our analysis can be easily extended to a smooth section of $T^*\partial\Omega \otimes T\partial\Omega$). We use the notation Πf for the tangential part of a vector f , $\Pi f^\varepsilon = f^\varepsilon - (f^\varepsilon \cdot \mathbf{n}) \cdot \mathbf{n}$. Let us remark that Navier-slip boundary conditions can be expressed as a non-homogeneous Dirichlet condition on $\operatorname{curl} u \times \mathbf{n}$,

$$\operatorname{curl} u \times \mathbf{n} = 2\Pi(-au + D\mathbf{n} \cdot u) \quad \text{on } \partial\Omega.$$

The special case $\operatorname{curl} u \times \mathbf{n} = 0$ corresponds to the choice $a = D\mathbf{n}$.

The aim of this paper is to study the uniform regularity (with respect to ε) and the low Mach number limit of system (1.1). Formally, due to the stiff term $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, the pressure (and hence the density ρ^ε) is expected to tend to a constant state. One thus expects to obtain in the limit a solution to the following incompressible Navier-Stokes system:

$$\begin{cases} \bar{\rho}(\partial_t u^0 + \operatorname{div}(u^0 \otimes u^0)) - \Delta u^0 + \nabla \pi = 0, \\ \operatorname{div} u^0 = 0, \quad (t, x) \in \mathbb{R}_+ \times \Omega \\ u^0|_{t=0} = u_0^0, \\ u^0 \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^0 \mathbf{n}) + a\Pi u^0 = 0 \quad (t, x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases} \quad (1.3)$$

This limit process is therefore frequently referred to as the incompressible limit.

The rigorous justification of this limit process has been studied extensively in different contexts depending on the generality of the system (isentropic or non-isentropic), the type of the system (Navier-Stokes or Euler), the type of solutions (strong solutions or weak solutions), the properties of the domain (whole space, torus or bounded domain with various boundary conditions), as well as the type of the initial data considered

(well-prepared or ill-prepared). Roughly speaking, in the case of the compressible Euler system, one proves first that the local strong solution exists on an interval of time independent of the Mach number, and then compactness arguments are developed to pass to the limit. In the case of the compressible Navier-Stokes system, one can either try to use the same approach as for the inviscid case (prove the existence of a strong solution on an interval of time independent of the Mach number and then try to pass to the limit) or try to pass to the limit directly from global weak solutions. Both approaches have been used in domains without boundaries (whole space or torus), nevertheless when a boundary is present the question of uniform regularity for general data is more subtle, as we shall see below, and has not been addressed.

More precisely, the mathematical justification of the low Mach number limit was initiated by Ebin [14], Klainerman-Majda [30,31] for *local strong solutions* of compressible fluids (Navier-Stokes or Euler), in the whole space with well-prepared data ($\operatorname{div} u_0^\varepsilon = O(\varepsilon)$, $\nabla P_0^\varepsilon = O(\varepsilon^2)$) and later, by Ukai [49] for ill-prepared data ($\operatorname{div} u_0^\varepsilon = O(1)$, $\nabla P_0^\varepsilon = O(\varepsilon)$). In the latter case, there are acoustic waves of amplitude 1 and frequency ε^{-1} in the system. These works were extended by several authors in different settings. For instance, one can refer to [2,5,39,40] for the non-isentropic system and ill-prepared initial data whenever the domain is the whole space or the torus, and also [29,44] for bounded domains with well-prepared initial data. Uniform (in Mach number) regularity estimates for the non-isentropic Euler equations in a bounded domain are established in [1]. The low Mach number limit of *weak solutions* for the viscous fluid system (1.1) was studied by Lions and the first author [32], [33] where the convergence of the global weak solutions of the isentropic Navier-Stokes system towards a solution of the incompressible system is established. The result holds for ill-prepared initial data and several different domains (whole space, torus and bounded domain with suitable boundary conditions). In general, for ill-prepared data, one can only obtain weak convergence in time, nevertheless, by using the dispersion of acoustic waves in the whole space, Desjardins and Grenier [11] could get local strong convergence. There are also many other related works, one can see for example [2,4,6,8,10,15,19,23,26,34]. For more exhaustive information, one can refer for example to the well-written survey papers by Alazard [3], Danchin [9], Feireisl [17], Gallagher [21], Jiang-Masmoudi [28], Schochet [45].

Let us focus now more specifically on the study of the low Mach limit of the isentropic compressible Navier-Stokes $(CNS)_\varepsilon$ system in domains with boundaries with *ill-prepared* initial data, which is more related to the interest of the current paper. As mentioned above, Lions and Masmoudi [32] studied the convergence of *weak solutions* to $(CNS)_\varepsilon$ in bounded domains with Navier-slip boundary condition. Later on, for low Mach limit in bounded domains with Dirichlet boundary condition, the authors in [12,27] noticed that, under some geometric assumption on the domain, the acoustic waves are damped in a boundary layer so that local in time strong convergence ($L^2_{t,x}$) holds. Recently, this result is extended by Feireisl et al. [18] and Xiong [51] to the case of Navier-slip boundary conditions with a of the order $\varepsilon^{-\frac{1}{2}}$. In this case, the boundary layer effect is comparable to the one in the Dirichlet case. One can also refer to [13,15,16] for the justification of convergence in unbounded domains by using the local energy decay for the acoustic system. Without one of the above properties of the domain, strong convergence does not hold for ill-prepared data.

In the current paper, our aim is to obtain uniform (with respect to ε) high order regularity estimates for $(CNS)_\varepsilon$ in bounded domains with ill-prepared initial data, in order to get the existence of a local strong solution on a time interval independent of ε . There are only a few papers addressing this issue. In [42], the authors establish uniform global (for small data) H^2 estimates under a (very) *well-prepared* initial data assumption, namely the second time derivative of the velocity needs to be uniformly bounded initially. For ill-prepared initial data, the situation is more subtle and a uniform H^2 estimate, even locally in time, cannot be expected. Indeed, at leading order, after linearization and symmetrization, the system (1.1) becomes:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon} L U^\varepsilon - \begin{pmatrix} 0 \\ \operatorname{div} \mathcal{L} u^\varepsilon \end{pmatrix} = 0, \quad L = \begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{pmatrix}, \quad U = (\sigma^\varepsilon, u^\varepsilon) \in \mathbb{R} \times \mathbb{R}_+^3. \quad (1.4)$$

Due to the presence of the diffusion term as well as the singular linear term, a boundary layer correction to the highly oscillating acoustic waves appear and create unbounded high order normal derivatives of the velocity. Note that here, we do not start from a small viscosity problem, nevertheless, at the scale $\tau = t/\varepsilon$ of the acoustic waves the system (1.4) behaves like a small viscosity perturbation of the acoustic system. For example, in the easiest case where the boundary is flat (for example $\Omega = \mathbb{R}_+^3$), we expect the following expansion of the solutions to (1.4) involving boundary layers

$$\begin{cases} \sigma^\varepsilon(t, x) = \sigma_0^I(\frac{t}{\varepsilon}, t, x) + \varepsilon^{\frac{3}{2}} \sigma^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \dots, \\ u^\varepsilon(t, x) = u_0^I(\frac{t}{\varepsilon}, t, x) + \sqrt{\varepsilon} \begin{pmatrix} u_{1,\tau}^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) \\ 0 \end{pmatrix} + \varepsilon u_2^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \dots \end{cases} \quad (1.5)$$

where $x = (y, z)$, $z > 0$, which suggests that $\|u_\tau\|_{L_t^2 H^1}$, $\|u_3^\varepsilon\|_{L_t^2 H^2}$, $\|\sigma^\varepsilon\|_{L_t^2 H^3}$ can be uniformly bounded whereas $\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L_{t,x}^2}$ and $\|\partial_z^2 u_\tau^\varepsilon\|_{L_{t,x}^2}$ will blow up as ε tends to 0.

In order to get uniform high order estimates, we shall thus need to use a functional framework based on conormal Sobolev spaces that minimize the use of normal derivatives close to the boundary in the spirit of [35,36] (conormal Sobolev spaces have been widely used to study initial boundary value problems for parabolic and hyperbolic equations, see for example [46], [25,48], [22,38,41]). Nevertheless, note that here we have to handle simultaneously the fast oscillations in time and a boundary layer effect so that the difficulties and the analysis will be different from the ones in [43,50] where compressible slightly viscous fluids are considered. Indeed, the energy estimates for conormal derivatives cannot be easily obtained since for example tangential vector fields do not commute with the singular part of the system, while in order to include ill-prepared data, it will be impossible to get uniform estimates for high order time derivatives as it is done in [43,50] in the study of the inviscid limit. We shall explain more these two difficulties below after the introduction of the various norms used in this paper.

1.1. Conormal Sobolev spaces and notations

To define the conormal Sobolev norms, we take a finite set of generators of vector fields that are tangent to the boundary of Ω : Z_j ($1 \leq j \leq M$). Due to the appearance in (1.5) of the ‘fast scale’ variable $\frac{t}{\varepsilon}$, it is also necessary to involve the scaled time derivative $Z_0 = \varepsilon \partial_t$. We set

$$Z^I = Z_0^{\alpha_0} \cdots Z_M^{\alpha_M}, \quad I = (\alpha_0, \alpha_1, \dots, \alpha_M) \in \mathbb{N}^{M+1}$$

Note that Z^I contains not only spatial derivatives but also the scaled time derivative $\varepsilon \partial_t$. We introduce the following Sobolev conormal spaces: for $p = 2$ or $+\infty$,

$$L_t^p H_{co}^m = \{f \in L^p([0, t], L^2(\Omega)), Z^I f \in L^p([0, t], L^2(\Omega)), |I| \leq m\},$$

equipped with the norm:

$$\|f\|_{L_t^p H_{co}^m} = \sum_{|I| \leq m} \|Z^I f\|_{L^p([0, t], L^2(\Omega))}, \quad (1.6)$$

where $|I| = \alpha_0 + \dots + \alpha_M$. For the space modeled on L^∞ , we shall use the following notation for the norm:

$$\|f\|_{m, \infty, t} = \sum_{|I| \leq m} \|Z^I f\|_{L^\infty([0, t] \times \Omega)}. \quad (1.7)$$

Since the number of time derivatives and spatial conormal derivatives need sometimes to be distinguished, we shall also use the notation:

$$\|f\|_{L_t^p \mathcal{H}^{j,l}} = \sum_{I=(k,\tilde{I}), k \leq j, |\tilde{I}| \leq l} \|Z^I f\|_{L^p([0,t], L^2(\Omega))} \quad (1.8)$$

and to simplify, we will use $\mathcal{H}^j = \mathcal{H}^{j,0}$. To measure pointwise regularity at a given time t (in particular also with $t = 0$), we shall use the semi-norms

$$\|f(t)\|_{H_{co}^m} = \sum_{|I| \leq m} \|(Z^I f)(t)\|_{L^2(\Omega)}, \quad \|f(t)\|_{\mathcal{H}^{j,l}} = \sum_{I=(k,\tilde{I}), k \leq j, |\tilde{I}| \leq l} \|Z^I f(t)\|_{L^2(\Omega)}. \quad (1.9)$$

Finally, to measure regularity along the boundary, we use

$$|f|_{L_t^p \tilde{H}^s(\partial\Omega)} = \sum_{j=0}^{[s]} |(\varepsilon \partial_t)^j f|_{L^p([0,t], H^{s-j}(\partial\Omega))}. \quad (1.10)$$

Let us recall, how the vector fields Z_j , $1 \leq j \leq M$ can be defined. We consider $\Omega \subset \mathbb{R}^3$ a smooth domain (the following construction and our results are actually valid as long as the boundary of Ω can be covered by a finite number of charts), therefore, there exists a covering such that:

$$\Omega \subset \Omega_0 \cup_{i=1}^N \Omega_i, \quad \Omega_0 \Subset \Omega, \quad \Omega_i \cap \partial\Omega \neq \emptyset, \quad (1.11)$$

and $\Omega_i \cap \Omega$ is the graph of a smooth function $z = \varphi_i(x_1, x_2)$.

In Ω_0 , we just take the vector fields ∂_k , $k = 1, 2, 3$. To define appropriate vector fields near the boundary, we use the local coordinates in each Ω_i :

$$\begin{aligned} \Phi_i : \quad & (-\delta_i, \delta_i) \times (0, \epsilon_i) \rightarrow \Omega_i \cap \Omega \\ & (y, z)^t \rightarrow \Phi_i(y, z) = (y, \varphi_i(y) + z)^t \end{aligned} \quad (1.12)$$

and we define the vector fields (up to some smooth cut-off functions compactly supported in Ω_i) as:

$$Z_k^i = \partial_{y^k} = \partial_k + \partial_k \varphi_i \partial_3, \quad k = 1, 2 \quad Z_3^i = \phi(z)(\partial_1 \varphi_1 \partial_1 + \partial_2 \varphi_1 \partial_2 - \partial_3), \quad (1.13)$$

where $\phi(z) = \frac{z}{1+z}$, and ∂_k , $k = 1, 2, 3$ are the derivations with respect to the original coordinates of \mathbb{R}^3 . We remark that if $\Omega = \mathbb{R}_+^3$, the conormal vector fields can be defined globally due to the flat boundary:

$$Z_1 = \partial_1, \quad Z_2 = \partial_2, \quad Z_3 = \frac{z}{1+z} \partial_z.$$

We shall denote by \mathbf{n} the unit outward normal to the boundary. In each Ω_i , we can extend it to Ω_i by setting

$$\mathbf{n}(\Phi_i(y, z)) = \frac{1}{|\mathbf{N}|} \mathbf{N}, \quad \mathbf{N}(\Phi_i(y, z)) = (\partial_1 \varphi_i(y), \partial_2 \varphi_i(y), -1)^t.$$

In the same way, the projection on vector fields tangent to the boundary,

$$\Pi = \text{Id} - \mathbf{n} \otimes \mathbf{n}$$

can be extended in Ω_i by using the extension of \mathbf{n} .

Let us observe that by identity

$$\Pi(\partial_{\mathbf{n}} u) = \Pi((\nabla u) \mathbf{n}) = 2\Pi(\mathbb{S} u) - \Pi((D u) \mathbf{n})$$

with $[(\nabla u)\mathbf{n}]_i = \sum_{j=1}^3 \mathbf{n}_j \partial_j u_i$, $[(Du)\mathbf{n}]_i = \sum_{j=1}^3 \partial_i u_j \mathbf{n}_j$, the boundary conditions (1.2) can be reformulated as:

$$u \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Pi(\partial_{\mathbf{n}} u) = \Pi[-2au + (D\mathbf{n})u] \quad (1.14)$$

where $[(D\mathbf{n})u]_i = \sum_{j=1}^3 \partial_i \mathbf{n}_j u_j$.

1.2. Main results

Let us introduce the new unknown

$$\sigma^{\varepsilon} = \frac{P(\rho^{\varepsilon}) - P(\bar{\rho})}{\varepsilon},$$

where $\bar{\rho}$ is a positive constant state, we can rewrite the system (1.1) into the following form which is more convenient to perform energy estimates:

$$\begin{cases} g_1(\varepsilon\sigma^{\varepsilon})(\partial_t\sigma^{\varepsilon} + u^{\varepsilon} \cdot \nabla\sigma^{\varepsilon}) + \frac{\operatorname{div} u^{\varepsilon}}{\varepsilon} = 0, \\ g_2(\varepsilon\sigma^{\varepsilon})(\partial_t u^{\varepsilon} + u^{\varepsilon} \cdot \nabla u^{\varepsilon}) - \operatorname{div} \mathcal{L}u^{\varepsilon} + \frac{\nabla\sigma^{\varepsilon}}{\varepsilon} = 0, \\ u^{\varepsilon}|_{t=0} = u_0^{\varepsilon}, \sigma^{\varepsilon}|_{t=0} = \sigma_0^{\varepsilon}, \end{cases} \quad (t, x) \in \mathbb{R}_+ \times \Omega \quad (1.15)$$

where the scalar functions g_1, g_2 are defined by

$$g_2(s) = \rho^{\varepsilon} = P^{-1}(\bar{P} + s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar{P} = -P(\bar{\rho}). \quad (1.16)$$

In order to establish uniform energy estimates, we shall use the following quantity

$$\mathcal{N}_{m,T}(\sigma^{\varepsilon}, u^{\varepsilon}) = \mathcal{E}_{m,T}(\sigma^{\varepsilon}, u^{\varepsilon}) + \mathcal{A}_{m,T}(\sigma^{\varepsilon}, u^{\varepsilon})$$

where $\mathcal{E}_{m,T}$ contains L^2 (in space) type quantities

$$\begin{aligned} \mathcal{E}_{m,T}(\sigma^{\varepsilon}, u^{\varepsilon}) &= \|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L_T^{\infty} \mathcal{H}^m} + \|\nabla(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L_T^{\infty} H_{co}^{m-2} \cap L_T^2 H_{co}^{m-1}} \\ &\quad + \varepsilon (\|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L_T^{\infty} H_{co}^m} + \|\nabla(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L_T^{\infty} H_{co}^{m-1}} + \|\nabla^2 u^{\varepsilon}\|_{L_T^{\infty} H_{co}^{m-2}}) + \varepsilon \|\nabla^2 \sigma\|_{L_T^{\infty} L^2}, \end{aligned} \quad (1.17)$$

and $\mathcal{A}_{m,T}$ involves L^{∞} (in space and time) type quantities

$$\begin{aligned} \mathcal{A}_{m,T}(\sigma^{\varepsilon}, u^{\varepsilon}) &= \|\nabla u^{\varepsilon}\|_{0, \infty, T} + \|(\nabla \sigma^{\varepsilon}, \operatorname{div} u^{\varepsilon}, \varepsilon^{\frac{1}{2}} \nabla u)\|_{[\frac{m-1}{2}], \infty, T} + \|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{[\frac{m+1}{2}], \infty, T} \\ &\quad + \varepsilon \|\nabla u^{\varepsilon}\|_{[\frac{m+1}{2}], \infty, T} + \varepsilon \|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{[\frac{m+3}{2}], \infty, T}. \end{aligned} \quad (1.18)$$

Note that the norms involved in the above definitions are defined in (1.6)-(1.8). See also Remarks 1.4, 1.5 and 2.5 for the comments on the norms appearing in $\mathcal{E}_{m,T}$ and $\mathcal{A}_{m,T}$.

Before stating our main result, we introduce the following definition.

Definition 1 (*Compatibility conditions*). We say that $(\sigma_0^{\varepsilon}, u_0^{\varepsilon})$ satisfy the compatibility conditions up to order m if:

$$(\varepsilon \partial_t)^j u^{\varepsilon}|_{t=0} \cdot \mathbf{n} = 0, \quad \Pi[\mathbb{S}((\varepsilon \partial_t)^j u^{\varepsilon}|_{t=0}) \mathbf{n}] = -a \Pi[(\varepsilon \partial_t)^j u|_{t=0}] \quad \text{on } \partial\Omega, j = 0, 1 \dots m-1.$$

Note that the restriction of the time derivatives of the solution at the initial time can be expressed inductively by using the equations. For example, we have

$$(\varepsilon \partial_t u^\varepsilon)(0) = \frac{1}{\rho_0^\varepsilon} (-\varepsilon u_0^\varepsilon \cdot \nabla u_0^\varepsilon + \varepsilon \operatorname{div} \mathcal{L} u_0^\varepsilon - \nabla \sigma_0^\varepsilon).$$

We thus define the admissible space for initial data as

$$Y_m = \left\{ (\sigma_0^\varepsilon, u_0^\varepsilon) \in H^2(\Omega)^4, Y_m^\varepsilon(\sigma_0^\varepsilon, u_0^\varepsilon) < +\infty, (\sigma_0^\varepsilon, u_0^\varepsilon) \text{ satisfy the compatibility conditions up to order } m \right\}$$

where

$$\begin{aligned} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) =: & \varepsilon \|(\sigma_0^\varepsilon, u_0^\varepsilon)\|_{H^2(\Omega)} + \|(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^{m-1}} \\ & + \sum_{|I| \leq [\frac{m-1}{2}]} \|Z^I(\nabla \sigma^\varepsilon, \nabla u^\varepsilon)(0)\|_{L^\infty(\Omega)} \end{aligned} \quad (1.19)$$

by using our notation (1.9). Note that as explained above, by using inductively the equations to express the time derivatives $Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$ can indeed be expressed in terms of the initial data only.

The following is our main uniform regularity result:

Theorem 1.1 (Uniform estimates). *Given an integer $m \geq 6$ and a C^{m+2} smooth bounded domain Ω . Consider a family of initial data such that $(\sigma_0^\varepsilon, u_0^\varepsilon) \in Y_m$, and*

$$\begin{aligned} \sup_{\varepsilon \in (0, 1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) & < +\infty, \\ -\bar{c}\bar{P} \leq \varepsilon \sigma_0^\varepsilon(x) & \leq \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0, 1], \end{aligned}$$

where $0 < \bar{c} < 1/4$ is a fixed constant, $\bar{P} = P(\bar{\rho})$. There exist $\varepsilon_0 \in (0, 1]$ and $T_0 > 0$, such that, for any $0 < \varepsilon \leq \varepsilon_0$, the system (1.15), (1.2) has a unique solution $(\sigma^\varepsilon, u^\varepsilon)$ which satisfies:

$$-2\bar{c}\bar{P} \leq \varepsilon \sigma^\varepsilon(t, x) \leq 2\bar{P}/\bar{c}, \quad \forall (t, x) \in [0, T_0] \times \Omega, \quad (1.20)$$

and

$$\sup_{\varepsilon \in (0, \varepsilon_0]} \mathcal{N}_{m, T_0}(\sigma^\varepsilon, u^\varepsilon) < +\infty. \quad (1.21)$$

Let us begin with a few comments about the above assumptions and our result.

Remark 1.2. In view of (1.20), there exists $c_0 \in (0, 1]$, such that:

$$c_0 \leq \rho^\varepsilon(t, x) = g_2(\varepsilon \sigma) \leq 1/c_0 \quad \forall (t, x) \in [0, T_0] \times \Omega$$

Moreover, as a consequence of (1.21), the following uniform estimates hold:

$$\sup_{\varepsilon \in (0, \varepsilon_0]} \left(\|(\sigma^\varepsilon, u^\varepsilon)\|_{L_{T_0}^\infty H_{co}^{m-1} \cap L_{T_0}^2 H_{co}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L_{T_0}^\infty H_{co}^{m-2} \cap L_{T_0}^2 H_{co}^{m-1}} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{0, \infty, t} \right) < +\infty,$$

in particular, we have a uniform estimate for $\|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty([0, T_0] \times \Omega)}$.

Remark 1.3. Because of the compatibility conditions, the assumption $\sup_{\varepsilon \in (0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) < +\infty$ imposes that the data are prepared (in the sense that it may depend on ε) on the boundary. Nevertheless, this is compatible with the fact that

$$(\operatorname{div} u^\varepsilon, \nabla \sigma^\varepsilon) = \mathcal{O}(1)$$

in the domain and thus ill-prepared data in the usual sense. Indeed, note that Y_m clearly contains smooth functions which vanish identically near the boundary. This kind of compatibility conditions also appears in the study of the incompressible limit of the Euler system in bounded domains [1].

Remark 1.4. The control of the weighted time derivatives $(\varepsilon \partial_t)^k$ up to highest order $k = m : \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty \mathcal{H}^m}$ is available since time derivation commutes with the space derivation. Moreover,

$$\|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^{m-1} \cap L_T^2 H_{co}^m} \lesssim \mathcal{E}_{m,T}(\sigma^\varepsilon, u^\varepsilon). \quad (1.22)$$

In other words, we can control the highest number of derivatives in the $L_t^2 L_x^2$ norm but lose the uniform control of the highest space conormal derivatives in $L_t^\infty L_x^2$. This is due to the bad commutation properties of the space conormal derivatives with the singular part of the system.

Remark 1.5. The solution constructed in Theorem 1.1 is a strong solution in the sense that for $\varepsilon > 0$ fixed $(\sigma^\varepsilon, u^\varepsilon) \in L^\infty([0, T_0], H^1 \times H^2)$, $u^\varepsilon \in L^2([0, T_0], H^3)$. Note that we further have a uniform control of the $L_t^\infty H^{m-1} \cap L_t^2 H^m$ norms in every compact set in the interior of the domain. Nevertheless, due to boundary layer effects (see (1.5)), we cannot expect uniform estimates for higher order normal derivatives near the boundary.

To prove Theorem 1.1, the crucial step is to show the following uniform a priori estimate which is the heart of this paper:

Proposition 1.6. *Let $c_0 \in (0, 1]$ be such that:*

$$\forall s \in [-3\bar{c}\bar{P}, 3\bar{P}/\bar{c}], c_0 \leq g_i(s) \leq 1/c_0, i = 1, 2, \quad |(g_1, g_2)|_{C^m([-3\bar{c}\bar{P}, 3\bar{P}/\bar{c}])} \leq 1/c_0 \quad (1.23)$$

where \bar{c} is such that for some $T \in (0, 1]$ the following assumption holds:

$$-3\bar{c}\bar{P} \leq \varepsilon \sigma^\varepsilon(t, x) \leq 3\bar{P}/\bar{c} \quad \forall (t, x) \in [0, T] \times \Omega, \forall \varepsilon \in [0, 1]. \quad (1.24)$$

Then, there exist $C(1/c_0) > 0$ and a polynomial Λ_0 (whose coefficients are independent of ε), such that, for any $\varepsilon \in (0, 1]$, we have for a smooth enough solution of (1.15) on $[0, T]$ the following estimate:

$$\mathcal{N}_{m,T}^2(\sigma^\varepsilon, u^\varepsilon) \leq C\left(\frac{1}{c_0}\right) Y_m^2(\sigma_0^\varepsilon, u_0^\varepsilon) + (T + \varepsilon)^{\frac{1}{2}} \Lambda_0\left(\frac{1}{c_0}, \mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon)\right), \quad (1.25)$$

where $Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$ is defined in (1.19).

This proposition is the consequence of Proposition 2.1 and 3.1, which will be established in Section 2 and Section 3 respectively.

By combining the uniform estimates (1.21) stated in Theorem 1.1 with a compactness argument, we get the following convergence result:

Theorem 1.7 (Convergence). *Under the assumptions of Theorem 1.1, let $(\sigma^\varepsilon, u^\varepsilon)$ the solution defined on $[0, T_0]$ given by Theorem 1.1 and assume that u_0^ε converges strongly in $L^2(\Omega)$ to some u_0^0 when ε tends to zero. Then, as ε tends to zero, ρ^ε (defined by (1.16)) converges to $\bar{\rho}$ in $L^\infty([0, T_0] \times \Omega)$ and u^ε converges in $L_w^2([0, T_0], L^2(\Omega))$ (weak convergence in time) to u^0 such that*

$$u^0 \in L_{T_0}^\infty \mathcal{H}^{0,m-1} \cap L_{T_0}^2 \mathcal{H}^{0,m}, \quad \nabla u^0 \in L_{T_0}^2 \mathcal{H}^{0,m-1} \cap L^\infty([0, T_0] \times \Omega). \quad (1.26)$$

Moreover, u^0 is the (unique in this class) weak solution to the incompressible Navier-Stokes system with Navier boundary condition (1.3).

Note that $L_{T_0}^2 \mathcal{H}^{0,m}$ is defined in (1.8) and involves only spatial conormal derivatives.

Remark 1.8. Due to the absence of uniform estimate for the second order normal derivatives and thus also for the strong trace of the normal derivative, u^0 has to be interpreted as the weak solution to (1.3) in the following usual sense: for any $\psi \in C^\infty([0, T_0] \times \bar{\Omega})$ with $\operatorname{div} \psi = 0, \psi \cdot \mathbf{n}|_{\partial\Omega} = 0$, the following identity holds: for every $0 < t \leq T_0$,

$$\begin{aligned} & \bar{\rho} \int_{\Omega} (u^0 \cdot \psi)(t, \cdot) \, dx + \mu \iint_{Q_t} \nabla u^0 \cdot \nabla \psi \, dx ds + \bar{\rho} \iint_{Q_t} (u^0 \cdot \nabla u^0) \cdot \psi \, dx ds \\ &= \bar{\rho} \int_{\Omega} (u_0^0 \cdot \psi)(0, \cdot) \, dx + \bar{\rho} \iint_{Q_t} u^0 \cdot \partial_t \psi \, dx ds + \mu \int_0^t \int_{\partial\Omega} \Pi(-2au^0 + (D\mathbf{n})u^0) \cdot \psi \, dS_y \, ds, \end{aligned} \quad (1.27)$$

where $Q_t = [0, t] \times \Omega$ and dS_y denotes the surface measure of $\partial\Omega$.

Remark 1.9. The convergence is weak in the time variable due to the lack of uniform estimate for $\partial_t(\sigma^\varepsilon, u^\varepsilon)$. This cannot be improved since in our bounded domain setting, there is no large time dispersion effect for the acoustic waves, and since because of our Navier boundary conditions with fixed slip length, there is no damping in the boundary layers of the acoustic waves. Nevertheless, if Ω is changed into an exterior domain, the strong convergence in time can be shown by using the RAGE Theorem [15,20], see Section 6 for this aspect.

Note that when ε tends to zero, we have convergence of the whole family u^ε and not only of subsequences due to the uniqueness for the limit system at this level of regularity.

1.3. Difficulties and strategies

We shall now explain the main difficulties and the main strategies in order to prove Proposition 1.6. As already mentioned the main feature of our problem is the presence of both fast time oscillations and a boundary layer in space. These two aspects are well-understood when they occur separately, but in order to handle them simultaneously some new ideas will be needed.

On the one hand, concerning the inviscid limit problem, one controls [35,43,50] the high order tangential derivatives by direct energy estimates, and then uses the vorticity to control the normal derivatives. Nevertheless, for the system with low Mach number, even the tangential derivative estimates are not easy to get, since the spatial tangential derivatives do not commute with $\nabla, \operatorname{div}$, defined with the standard derivations in \mathbb{R}^3 , and thus create singular commutators. Without this a priori knowledge on the tangential derivatives, the estimate of the vorticity cannot be performed as in [35] [36] because of the consequent lack of information on its trace on the boundary. On the other hand, for the compressible Euler system with low Mach number, uniform high regularity estimates are established for example in [1]. One can get uniform H^s ($s > 5/2$) estimates by using first $\varepsilon \partial_t$ derivatives and then recover space derivatives by using the equations to estimate

the divergence of the velocity and the gradient of the pressure and a direct energy estimates for the vorticity which solves a transport equation with a characteristic vector field. Here, in the case of viscous fluids, we face again the fact that the estimates of the vorticity are challenging due to the lack of information on its trace on the boundary at this stage.

In order to get the missing information, we shall first use the Leray projection (the precise definition (2.2) is in Section 3) to split the velocity into a compressible part and an incompressible part: $u^\varepsilon = \nabla \Psi^\varepsilon + v^\varepsilon$. On the one hand, the compressible part $\nabla \Psi^\varepsilon$ of the velocity can be controlled by $\operatorname{div} u^\varepsilon$ thanks to standard elliptic theory and hence by using the mass conservation equation and the energy estimates for $\varepsilon \partial_t$ derivatives. On the other hand, the incompressible part v^ε solves, up to the control of non-local commutators, a convection-diffusion equation without oscillations, and thus one can use direct energy estimates to get a control of $\|v^\varepsilon\|_{L_t^\infty H_{co}^{m-1}}$ and $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. Note that we cannot estimate the maximal number of derivatives m due to the lack of structure of the coupling terms involving the compressible part in the energy estimates. The key point here is that the diffusion (which on the other hand creates new difficulties in the control of the vorticity) allows to get the estimate of $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. This is still not enough to close an estimate since, because of the time oscillations, we cannot use Sobolev embedding in time to control $\|\nabla v^\varepsilon\|_{L_t^\infty H_{co}^{m-2}}$ as it is done in small viscosity problems for compressible fluids (see for example [43], [50]). Here, we only have estimates for powers of $\varepsilon \partial_t$ instead of ∂_t . Nevertheless, with the additional information obtained from v^ε , we can then reduce the matter to the study of $\|\omega^\varepsilon \times \mathbf{n}\|_{L_t^\infty H_{co}^{m-2}}$ where ω^ε is the vorticity, which solves the heat equation with a non-homogeneous Dirichlet boundary condition which can be controlled from the previous estimates. We shall get the estimate by using the Green's function of the heat equation.

Outline of the proof of Theorem 1.1. The uniform energy estimates will be more precisely achieved in the following steps: (we shall skip the ε dependence in the notations for the sake of simplicity).

Step 1: Uniform high-order $\varepsilon \partial_t$ derivatives and ε -dependent high-order conormal derivatives. In this step, we aim to prove two kinds of estimates. Namely, uniform estimates for high order $\varepsilon \partial_t$ derivatives, $\|(\sigma, u)\|_{L_t^\infty \mathcal{H}^m}$, and ε -dependent estimates: $\varepsilon \|(\sigma, u)\|_{L_t^\infty H_{co}^m}$, $\varepsilon \|(\nabla \sigma, \operatorname{div} u)\|_{L_t^\infty H_{co}^{m-1}}$. On the one hand, since the time derivative $\varepsilon \partial_t$ commutes with the spatial derivatives, we can get uniform estimates for high order time derivatives. Note that we use $\varepsilon \partial_t$ instead of ∂_t since we are dealing with ill-prepared data. On the other hand, as the spatial conormal vector fields do not commute with $\nabla, \operatorname{div}$, the singular part of the system, we need at this stage to add this additional ε weight to control the commutator.

Step 2: Uniform estimates for the incompressible part of the velocity. Let us denote by $v = \mathbb{P}u$, and $\nabla \Psi = \mathbb{Q}u$ the incompressible and compressible part of the velocity respectively, where \mathbb{P}, \mathbb{Q} are defined in (2.2). By applying the projection \mathbb{P} on the equation for the velocity and expanding the boundary conditions, we find that v solves:

$$\begin{cases} \bar{\rho} \partial_t v - \mu \Delta v + \nabla q + \frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u + g_2 u \cdot \nabla u = 0 & \text{in } \Omega \\ v \cdot \mathbf{n} = 0, \quad \Pi(\partial_n v) = \Pi(-2au + D\mathbf{n} \cdot \nabla \Psi + D\mathbf{n} \cdot u) & \text{on } \partial\Omega \end{cases} \quad (1.28)$$

where

$$\nabla q = -\mathbb{Q}\left(\frac{g_2 - 1}{\varepsilon} \varepsilon \partial_t u + g_2 u \cdot \nabla u - \mu \Delta v\right).$$

Note that the first boundary condition $v \cdot \mathbf{n} = 0$ is due to the definition of the projection \mathbb{P} while the second boundary condition is deduced from (1.14). The incompressible part v interacts with the compressible part $\nabla \Psi$ through the source term and the boundary condition. Due to the absence of singular terms, one can get the uniform estimates for v (namely $\|v\|_{L_t^\infty H_{co}^{m-1}}$ and $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$) by direct energy estimates. Nevertheless, for latter use in the proof, we need to track in the energy estimates the counts of time and spatial conormal derivatives.

Step 3: Uniform estimates for the compressible part of the system. In this step, we aim to get the control of $\|(\nabla\sigma, \operatorname{div} u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}$. This can be done by using the equations and induction arguments. Indeed, by rewriting the system (1.15),

$$\begin{aligned} -\operatorname{div} u &= g_1 \varepsilon \partial_t \sigma + \varepsilon g_1 u \cdot \nabla \sigma, \\ -\nabla \sigma &= g_2 \varepsilon \partial_t u + \varepsilon (g_2 u \cdot \nabla u - \operatorname{div} \mathcal{L}u). \end{aligned}$$

In view of the above two equations, one can ‘trade’ one spatial derivative by one (small scale) time derivative $\varepsilon \partial_t$. We can thus recover the high order spatial (conormal) derivatives by using iteratively this observation.

Step 4: Control of $L_t^\infty H_{co}^{m-2}$ norm of ∇u . In this step, we aim to get a uniform control of $\|\nabla u\|_{L_t^\infty H_{co}^{m-2}}$ which is quite useful to control $L_{t,x}^\infty$ type norms. The difficulty is the estimate close to the boundary. We can work in a local chart Ω_i . In light of the identities

$$\partial_{\mathbf{n}} u \cdot \mathbf{n} = \operatorname{div} u - (\Pi \partial_{y_1} u)^1 - (\Pi \partial_{y_2} u)^2, \quad \Pi(\partial_{\mathbf{n}} u) = \Pi(\omega \times \mathbf{n}) - \Pi[(D\mathbf{n})u],$$

where \mathbf{n} is an extension of the unit normal and Π projects on $(\mathbf{n})^\perp$, it suffices to control $\|\omega \times \mathbf{n}\|_{L_t^\infty H_{co}^{m-2}}$. We remark that the advantage of working on $\omega \times \mathbf{n}$ rather than ω is that the boundary condition for $\omega \times \mathbf{n}$ (see (2.33)) only involves lower order terms on the boundary. To estimate $\omega \times \mathbf{n}$, a natural attempt, used in [35], is to perform energy estimates on the equation for the ‘modified vorticity’ $w = \omega \times \mathbf{n} + 2\Pi(au - (D\mathbf{n})u)$ and to take advantage of the fact that w vanishes on the boundary. However, the equations for w still involve a stiff term $\frac{1}{\varepsilon} \nabla^\perp \sigma$, which is obviously an obstacle to obtain uniform energy estimates. We shall thus instead use a lifting of the boundary conditions by using Green’s function for the solution of the heat equation with non-homogenous boundary conditions and estimate the remainder by energy estimates.

Step 5: $L_{t,x}^\infty$ estimates. The control of the $L_{t,x}^\infty$ norms contained in $\mathcal{A}_{m,T}$ mainly stems from the Sobolev embedding and the maximum principle for the system solved by the vorticity. Note that at this stage, it is crucial to use the direct $L_t^\infty H_{co}^{m-1}$ for (σ, u) and $L_t^\infty H_{co}^{m-2}$ for $\nabla(\sigma, u)$ estimates obtained in the previous steps since because of the fast oscillations in time, uniform L^∞ estimates in time cannot be deduced from a Sobolev embedding in time.

The case $\Omega = \mathbb{R}_+^3$ where the boundary is flat can also be treated following the above steps and is indeed easier to analyze. Indeed, the spatial tangential derivatives can be controlled directly through energy estimates without weight in ε , since in this case the derivatives ∂_{y^i} commute with div or ∇ . The use of the step with the Helmholtz-Leray projection is thus not necessary. The details can be found in the PhD thesis [47] (see pages 39-40 and Section 6.8).

In a companion paper [37], we strengthen the strategies used in this paper to deal with the low Mach number limit problem for the free surface compressible Navier-Stokes system, where we are forced to deal with strong enough solutions in the absence of a suitable theory of weak solutions.

Organization of the paper. We will prove Proposition 1.6 by establishing uniform control of energy norms $\mathcal{E}_{m,T}$ and $L_{t,x}^\infty$ type norms $\mathcal{A}_{m,T}$ which is achieved in Section 2 and Section 3 respectively. Section 4 is then devoted to the proof of Theorem 1.1. In Section 5, we will justify the incompressible limit. Some remarks will be given in Section 6 regarding to the incompressible limit in exterior domain. In the appendix, we gather some useful product and commutator estimates as well as the proofs of some technical lemmas.

2. Uniform estimates-energy norm

In this section, we establish the a-priori estimates for the energy norm $\mathcal{E}_{m,T}$. Again, for notational convenience, we skip the ε -dependence of the solutions.

Proposition 2.1. *If the estimates (1.24), (1.23) are satisfied, then we can find a constant $C_1(1/c_0)$ that depends only on $1/c_0$ and a polynomial $\tilde{\Lambda}$ whose coefficients are independent of ε , such that for a smooth enough solution of (1.15), the following estimate holds on $[0, T]$ for $\varepsilon \in (0, 1]$:*

$$\mathcal{E}_{m,T}^2 \leq C_1 \left(\frac{1}{c_0} \right) Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \tilde{\Lambda} \left(\frac{1}{c_0}, \mathcal{N}_{m,T} \right). \quad (2.1)$$

As explained in the introduction, to overcome the difficulty due to the nontrivial commutators between the tangential spatial derivatives and the standard derivation $(\nabla, \operatorname{div})$, we need to split the velocity u into $u = \nabla \Psi + v$, where $\nabla \Psi, v$ are the compressible part and the incompressible part respectively (see (2.2) the precisely definition). On the one hand, the compressible part $\nabla \Psi$ satisfies the elliptic equation $\Delta \Psi = \operatorname{div} u$ with Neumann boundary condition, from which one can deduce the estimate of $\nabla^2 \Psi$ from that of $\operatorname{div} u$. On the other hand, since the incompressible part v is governed by a convection diffusion equation without oscillations, we can control its conormal derivatives by direct energy estimates. The estimates for $\partial_{\mathbf{n}} v$ will then be deduced from the ones for $\omega \times \mathbf{n}$.

2.1. Preliminaries: Leray projection

To define the compressible or acoustic part and the incompressible part of the velocity field, we shall use the Leray projection. One has the decomposition,

$$L_x^2(\Omega)^3 = H \oplus G$$

where

$$H = \{v \in L_x^2(\Omega)^3, \operatorname{div} v = 0, v \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \quad G = \{\nabla \Psi, \nabla \Psi \in L^2(\Omega)^3\}.$$

We denote \mathbb{P}, \mathbb{Q} the projectors that map $L_x^2(\Omega)^3$ to its subspaces H and G respectively, namely,

$$\begin{aligned} \mathbb{Q} : L^2(\Omega)^3 &\rightarrow G & \mathbb{P} : L^2(\Omega)^3 &\rightarrow H \\ f &\mapsto \mathbb{Q}f = \nabla \Psi & f &\mapsto f - \mathbb{Q}f \end{aligned} \quad (2.2)$$

where Ψ is defined as the unique solution of

$$\begin{cases} \Delta \Psi = \operatorname{div} f & \text{in } \Omega, \\ \partial_{\mathbf{n}} \Psi = f \cdot \mathbf{n} & \text{on } \partial\Omega, \\ \int_{\Omega} \Psi \, dx = 0. \end{cases} \quad (2.3)$$

Note that the solvability of the Neumann problem (2.3) in $H^1(\Omega)$ is well-known as an application of the Lax-Milgram theorem. Moreover, by Proposition 7.6, one has that for a C^{k+1} bounded domain,

$$\|\nabla \Psi(t)\|_{H_{co}^k} \lesssim \|f(t)\|_{H_{co}^k}, \quad \|\nabla^2 \Psi(t)\|_{H_{co}^{k-1}} \lesssim \|\operatorname{div} f(t)\|_{H_{co}^{k-1}} + \|f(t)\|_{H_{co}^{k-1}}. \quad (2.4)$$

Note that in these estimates, the time variable is just an external parameter.

Since $[\mathbb{P}, \partial_t] = 0$, (1.15) is equivalent to the following system:

$$\begin{cases} g_1(\partial_t \sigma + u \cdot \nabla \sigma) + \frac{\Delta \Psi}{\varepsilon} = 0, \\ \bar{\rho} \partial_t \nabla \Psi + \mathbb{Q} \left(\frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u + g_2 u \cdot \nabla u - \mu \Delta v - (2\mu + \lambda) \nabla \operatorname{div} u + \frac{\nabla \sigma}{\varepsilon} \right) = 0, \\ \bar{\rho} \partial_t v + \frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u + g_2 u \cdot \nabla u - \mu \Delta v + \nabla q = 0, \end{cases} \quad (2.5)$$

where

$$v = \mathbb{P}u, \quad \nabla\Psi = \mathbb{Q}u, \quad \nabla q = -\mathbb{Q}\left(\frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u - \mu\Delta v\right), \quad \bar{\rho} = g_2(0).$$

By taking the divergence of the third equations of (2.5) and noting that $\operatorname{div} v = 0, \varepsilon\partial_t u \cdot \mathbf{n}|_{\partial\Omega} = 0$, we see that ∇q is governed by the following elliptic equation:

$$\begin{cases} \Delta q = -\operatorname{div}\left(\frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u\right) & \text{in } \Omega, \\ \partial_{\mathbf{n}} q = -(g_2 u \cdot \nabla u) \cdot \mathbf{n} + \mu\Delta v \cdot \mathbf{n} & \text{on } \partial\Omega. \end{cases} \quad (2.6)$$

Proposition 2.1 can be shown by the first three steps outlined in the introduction, they will be handled in the following three subsections.

2.2. Step 1: Highest conormal estimates

For notational convenience, we denote Λ for a polynomial which may differ from line to line, and use the notation $\lesssim \cdot$ as $\leq C \cdot$ for some generic constant $C = C(1/c_0)$ that depends on $1/c_0$ but not on ε .

Let us state the main result of this subsection.

Lemma 2.2. *Suppose that (1.24) is satisfied, then for any $m \geq 0$, any $0 < T \leq 1$ and $\varepsilon \in (0, 1]$ we have:*

$$\begin{aligned} & \|(\sigma, u)\|_{L_T^\infty \mathcal{H}^m}^2 + \varepsilon^2 (\|(\sigma, u)\|_{L_T^\infty H_{co}^m}^2 + \|(\nabla\sigma, \operatorname{div} u)\|_{L_T^\infty H_{co}^{m-1}}^2) \\ & + \|\nabla u\|_{L_t^2 \mathcal{H}^m}^2 + \varepsilon^2 (\|\nabla u\|_{L_T^2 H_{co}^m}^2 + \|\nabla \operatorname{div} u\|_{L_T^2 H_{co}^{m-1}}^2) \\ & \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,T}^2. \end{aligned} \quad (2.7)$$

Proof. The estimate (2.7) can be derived from the following two lemmas. \square

Let us start with:

Lemma 2.3. *Under the same assumption as in Lemma 2.2, for any $0 < t \leq T$, the following estimates hold:*

$$\|(\sigma, u)\|_{L_t^\infty \mathcal{H}^m}^2 + \|\nabla u\|_{L_t^2 \mathcal{H}^m}^2 \lesssim \|(\sigma, u)(0)\|_{\mathcal{H}^m}^2 + \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) T^{\frac{1}{2}} \mathcal{E}_{m,T}^2, \quad (2.8)$$

$$\varepsilon^2 (\|(u, \sigma)(t)\|_{H_{co}^m}^2 + \|\nabla u\|_{L_t^2 H_{co}^m}^2) \lesssim \varepsilon^2 \|(\sigma, u)(0)\|_{H_{co}^m}^2 + \varepsilon^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,T}^2 + \varepsilon^2 \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2. \quad (2.9)$$

We recall that in our notations the norms at $t = 0$ involve the computation of powers of $\varepsilon\partial_t$ at $t = 0$.

Proof. Define $\sigma^I = Z^I \sigma, u^I = Z^I u$. Then (σ^I, u^I) satisfies:

$$\begin{cases} g_1(\partial_t \sigma^I + u \cdot \nabla \sigma^I) + \frac{\operatorname{div} u^I}{\varepsilon} = \mathcal{R}_{\sigma}^I, \\ g_2(\partial_t u^I + u \cdot \nabla u^I) - Z^I(\operatorname{div} \mathcal{L}u) + \frac{\nabla \sigma^I}{\varepsilon} = \mathcal{R}_u^I, \end{cases} \quad (2.10)$$

where

$$\begin{aligned} \mathcal{R}_{\sigma}^I &= -[Z^I, \frac{g_1}{\varepsilon}] \varepsilon \partial_t \sigma - [Z^I, g_1 u \cdot \nabla] \sigma - \frac{1}{\varepsilon} [Z^I, \operatorname{div}] u, \\ \mathcal{R}_u^I &= -[Z^I, \frac{g_2}{\varepsilon}] \varepsilon \partial_t u - [Z^I, g_2 u \cdot \nabla] u - \frac{1}{\varepsilon} [Z^I, \nabla] \sigma. \end{aligned}$$

We first show (2.8) which is easier. Assuming that $I = (j, 0, \dots, 0)$, $|j| \leq m$ which means that $Z^I = (\varepsilon \partial_t)^j$ involves only time derivatives. The advantage of this case is that the commutators do not include singular terms, that is the third terms in \mathcal{R}_σ^I and \mathcal{R}_u^I vanish.

For the sake of notational simplicity, we denote $(\sigma^j, u^j) = (\varepsilon \partial_t)^j(\sigma, u)$. Taking the scalar product of (2.10) by (σ^j, u^j) and taking benefits of the boundary conditions

$$u^j \cdot \mathbf{n} = 0, \quad \Pi(\partial_{\mathbf{n}} u^j) = \Pi(-2au^j + (D\mathbf{n})u^j) \quad \text{on } \partial\Omega, \quad (2.11)$$

as well as the relation $\partial_t g_2 + \operatorname{div}(g_2 u) = 0$, we get from standard integration by parts that:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (g_1 |\sigma^j|^2 + g_2 |u^j|^2)(t) \, dx + \iint_{Q_t} \mu |\nabla u^j|^2 + (\mu + \lambda) |\operatorname{div} u^j|^2 \, dx ds \\ & \leq \frac{1}{2} \int_{\Omega} (g_1 |\sigma^j|^2 + g_2 |u^j|^2)(0) \, dx + \left| \iint_{Q_t} (\partial_t g_1 + \operatorname{div}(g_1 u)) |\sigma^j|^2 \, dx ds \right| \\ & \quad + \mu \left| \int_0^t \int_{\partial\Omega} \Pi(\partial_{\mathbf{n}} u^j) \Pi u^j \, dS_y \, ds \right| + \|\mathcal{R}_\sigma^I\|_{L^2(Q_t)} \|\sigma^j\|_{L^2(Q_t)} + \|\mathcal{R}_u^I\|_{L^2(Q_t)} \|u^j\|_{L^2(Q_t)}, \end{aligned} \quad (2.12)$$

where we denote by dS_y the surface measure of $\partial\Omega$ and $Q_t = [0, t] \times \Omega$. The second term in the above right hand side can be controlled easily by $\Lambda_{1,\infty,t} \|\sigma^j\|_{L^2(Q_t)}$. Note that

$$\|\partial_t g_1\|_{0,\infty,t} \leq \sup_{[-3\bar{P}, 3\bar{P}/\bar{c}]} (|g'_1(s)|) \|\varepsilon \partial_t \sigma\|_{0,\infty,t} \leq \frac{1}{c_0} \|\varepsilon \partial_t \sigma\|_{0,\infty,t}.$$

The boundary term of the last line of (2.12) can be treated thanks to the boundary condition (2.11) and the trace inequality (7.10)

$$\mu \left| \int_0^t \int_{\partial\Omega} \Pi(\partial_{\mathbf{n}} u^j) \cdot \Pi u^j \, dS_y \, ds \right| \leq \frac{\mu}{4} \|\nabla u^j\|_{L^2(Q_t)}^2 + C_\mu \|u^j\|_{L^2(Q_t)}^2. \quad (2.13)$$

We now detail the estimate of $(\mathcal{R}_\sigma^I, \mathcal{R}_u^I)$ which vanish unless $j \neq 0$. For $1 \leq j \leq m$, by the commutator estimate (7.3) and the estimate (7.4) for g_1 ,

$$\begin{aligned} \|\mathcal{R}_\sigma^I\|_{L^2(Q_t)} & \lesssim \|\partial_t g_1\|_{L_t^2 \mathcal{H}^{m-1}} \|(\varepsilon \partial_t) \sigma\|_{[\frac{m}{2}] - 1, \infty, t} + \|\partial_t g_1\|_{[\frac{m-1}{2}], \infty, t} \|(\varepsilon \partial_t) \sigma\|_{L_t^2 \mathcal{H}^{m-1}} \\ & \quad + \|g_1 u\|_{L_t^2 \mathcal{H}^m} \|\nabla \sigma\|_{[\frac{m}{2}] - 1, \infty, t} + \|g_1 u\|_{[\frac{m+1}{2}], \infty, t} \|\nabla \sigma\|_{L_t^2 \mathcal{H}^{m-1}} \\ & \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) (\|\nabla \sigma\|_{L_t^2 \mathcal{H}^{m-1}} + \|(\sigma, u)\|_{L_t^2 \mathcal{H}^m}). \end{aligned} \quad (2.14)$$

In a similar way, we have:

$$\|\mathcal{R}_u^I\|_{L^2(Q_t)} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) (\|\nabla(\sigma, u)\|_{L_t^2 \mathcal{H}^{m-1}} + \|(\sigma, u)\|_{L_t^2 \mathcal{H}^m}). \quad (2.15)$$

Therefore, (2.8) is the consequence of (2.12)-(2.15). Note that we have used the fact that

$$\|(\sigma, u)\|_{L_t^2 \mathcal{H}^m} \lesssim T^{\frac{1}{2}} \|(\sigma, u)\|_{L_t^\infty \mathcal{H}^m} \lesssim T^{\frac{1}{2}} \mathcal{E}_{m,T}, \quad \|\nabla(\sigma, u)\|_{L_t^2 \mathcal{H}^{m-1}} \lesssim \mathcal{E}_{m,T}.$$

We are now ready to prove (2.9). Suppose now that Z^I involves at least one spatial derivative and $1 \leq |I| \leq m$. In this case, it seems unlikely to get a uniform estimate with respect to ε with this approach since $\mathcal{R}_\sigma^I, \mathcal{R}_u^I$ now contains singular terms. Taking the scalar product of system (2.10) by $\varepsilon^2(\sigma^I, u^I)$, and integrating by parts in space and time, we get in the same way as for (2.12) that:

$$\begin{aligned} & \varepsilon^2 \int_{\Omega} (g_1 |\sigma^I|^2 + g_2 |u^I|^2)(t) \, dx \\ & \leq \varepsilon^2 \int_{\Omega} (g_1 |\sigma^I|^2 + g_2 |u^I|^2)(0) \, dx + \iint_{Q_t} (\partial_t g_1 + \operatorname{div}(g_1 u)) |\sigma^I|^2 \, dx ds \\ & + 2\varepsilon^2 \iint_{Q_t} Z^I \operatorname{div} \mathcal{L}u \cdot u^I \, dx ds + \varepsilon^2 (\|\mathcal{R}_\sigma^I\|_{L^2(Q_t)} \|\sigma^I\|_{L^2(Q_t)} + \|\mathcal{R}_u^I\|_{L^2(Q_t)} \|u^I\|_{L^2(Q_t)}). \end{aligned} \quad (2.16)$$

Before going further, it will be convenient to introduce the notation:

$$\|f\|_{E_t^m} = \|f\|_{L_t^2 H_{co}^m} + \|\nabla f\|_{L_t^2 H_{co}^{m-1}}. \quad (2.17)$$

Note that from the definition of $\mathcal{E}_{m,t}$ in (1.17), one has indeed that: $\|u\|_{E_t^m} \lesssim \mathcal{E}_{m,t}$.

Let us now estimate the terms in the last line of (2.16). It follows from the commutator estimate (7.2) that:

$$\varepsilon \|(\mathcal{R}_\sigma^I, \mathcal{R}_u^I)\|_{L^2(Q_t)} \lesssim \|\nabla(\sigma, u)\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}} \|(\sigma, u)\|_{E_t^m} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right). \quad (2.18)$$

We remark that when controlling the extra term: $\frac{1}{\varepsilon} [Z^I, \nabla] \sigma$, we have used the following identity which can be shown by induction:

$$[Z^I, \partial_i] = \sum_{j=1}^3 \sum_{|J| \leq |I|-1} c_{I,J} Z^J \partial_j = \sum_{j=1}^3 \sum_{|J| \leq |I|-1} d_{I,J} \partial_j Z^J \quad (2.19)$$

where J is an $(M+1)$ multi-index and $c_{I,J}, d_{I,J}$ are smooth functions that depend on I, J, i and the derivatives (up to order $|I|$) of $\nabla \phi$, ∂_i is the derivation in the standard Euclidean coordinates.

It remains to estimate the third term in the right hand side of (2.16). Since, we have

$$\operatorname{div} \mathcal{L}u = \operatorname{div}(2\mu \mathbb{S}u + \lambda \operatorname{div} u \operatorname{Id}) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u,$$

one has by integrating by parts that:

$$\begin{aligned} & \iint_{Q_t} Z^I \mathcal{L}u \cdot u^I \, dx ds = - \iint_{Q_t} (\mu [Z^I, \nabla] u \cdot \nabla u^I + (\mu + \lambda) [Z^I, \operatorname{div}] u \operatorname{div} u^I) \, dx ds \\ & + \iint_{Q_t} (\mu [Z^I, \operatorname{div}] \nabla u + (\mu + \lambda) [Z^I, \nabla] \operatorname{div} u) u^I \, dx ds - \iint_{Q_t} \mu |\nabla u^I|^2 + (\mu + \lambda) |\operatorname{div} u^I|^2 \, dx ds \\ & + \int_0^t \int_{\partial\Omega} \mu u^I (Z^I \nabla u \cdot \mathbf{n}) + (\mu + \lambda) Z^I \operatorname{div} u (u^I \cdot \mathbf{n}) \, dS_y ds =: \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4. \end{aligned} \quad (2.20)$$

Let us begin with the \mathcal{K}_1 term. By (2.19) and the Young inequality, we get

$$\mathcal{K}_1 \leq \delta \mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta, \mu, \lambda} \|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2 \quad (2.21)$$

for $\delta > 0$ to be chosen sufficiently small independent of ε . Next, by (2.19) and integration by parts, \mathcal{K}_2 can be written as a combination of the following two types of terms (up to some smooth coefficients that depending on ϕ, \mathbf{n} and their derivatives up to order $m+1$):

$$\mathcal{K}_2^1 = \iint_{Q_t} Z^{\tilde{I}} \partial_i u \cdot \partial_j u^I \, dx ds, \quad \mathcal{K}_2^2 = \int_0^t \int_{\partial\Omega} Z^{\tilde{I}} \partial_i u \cdot u^I \mathbf{n}_j \, dx ds, \quad |\tilde{I}| \leq |I| - 1.$$

The term \mathcal{K}_2^1 can be estimated in the same way as \mathcal{K}_1 , we find again

$$\mathcal{K}_2^1 \leq \delta \mu \|\nabla u^I\|_{L^2(Q_t)}^2 + C_{\delta, \mu, \lambda} \|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2.$$

For \mathcal{K}_2^2 , we use the trace inequality (7.10) to get that:

$$\begin{aligned} \mathcal{K}_2^2 &\lesssim \int_0^t |Z^{\tilde{I}} \partial_i u|_{L^2(\partial\Omega)} |u^I \cdot \mathbf{n}_j|_{L^2(\partial\Omega)} ds \lesssim \int_0^t (|u|_{\tilde{H}^m(\partial\Omega)} + |\operatorname{div} u|_{\tilde{H}^{m-1}(\partial\Omega)}) |u^I \cdot \mathbf{n}_j|_{L^2(\partial\Omega)} ds \\ &\leq \delta \mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta, \mu, \lambda} (\|u\|_{E_t^m}^2 + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \end{aligned}$$

To get the second inequality, we have used that \tilde{I} does not contain conormal derivatives of the type Z_3^i since Z_3^i vanishes on the boundary and the identity:

$$\partial_{\mathbf{n}} u \cdot \mathbf{n} = \operatorname{div} u - (\Pi \partial_{y^1} u)^1 - (\Pi \partial_{y^2} u)^2, \quad (2.22)$$

as well as the boundary condition (1.14).

To summarize, we have thus proven that there exists an absolute constant $C > 0$ (independent of δ and of course ε) such that

$$\mathcal{K}_2 \leq C \delta \mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta, \mu, \lambda} (\|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2 + \|u\|_{E_t^m}^2). \quad (2.23)$$

Finally, we handle the term \mathcal{K}_4 in the right hand side of (2.20) which is nontrivial only if Z^I contains merely $\varepsilon \partial_t$ and tangential derivatives which read in local charts $\partial_{y^1}, \partial_{y^2}$. For the second term of \mathcal{K}_4 , since Z^I is assumed to contain at least one spatial derivative, it can be written as $Z^I = \partial_y Z^{\tilde{I}}$ (we denote $\partial_y = \partial_{y^1}$ or $\partial_y = \partial_{y^2}$). Moreover, since $u \cdot \mathbf{n}|_{\partial\Omega} = 0$, $u^I \cdot \mathbf{n} = [Z^I, \mathbf{n}]u$. Integrating by parts along the boundary, and then use the trace inequality (7.11), we find that

$$\begin{aligned} \int_0^t \int_{\partial\Omega} Z^I \operatorname{div} u (u^I \cdot \mathbf{n}) \, dS_y ds &\leq \int_0^t |Z^{\tilde{I}} \operatorname{div} u|_{H^{\frac{1}{2}}(\partial\Omega)} |\partial_y [Z^I, \mathbf{n}]u|_{H^{-\frac{1}{2}}(\partial\Omega)} \, ds \\ &\lesssim \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2 + \|u\|_{E_t^m}^2. \end{aligned} \quad (2.24)$$

For the first term of \mathcal{K}_4 , we can split it into two terms:

$$\begin{aligned} &\mu \int_0^t \int_{\partial\Omega} -u^I ([Z^I, \mathbf{n}] \nabla u) + [Z^I, \mathbf{n}] \partial_{\mathbf{n}} u (u^I \cdot \mathbf{n}) + [Z^I, \Pi] \partial_{\mathbf{n}} u \cdot \Pi u^I \, dS_y ds \\ &\quad - \mu \int_0^t \int_{\partial\Omega} Z^I (\partial_{\mathbf{n}} u \cdot \mathbf{n}) (u^I \cdot \mathbf{n}) + Z^I (\Pi \partial_{\mathbf{n}} u) \cdot \Pi u^I \, dS_y ds =: \mathcal{K}_{411} + \mathcal{K}_{412}. \end{aligned}$$

Thanks to the trace inequality and the Young's inequality, \mathcal{K}_{411} can be bounded as:

$$\mathcal{K}_{411} \leq \delta\mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu} (\|u\|_{E_t^m}^2 + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2).$$

Next, for \mathcal{K}_{412} , we use again the identity (2.22), as well as the boundary conditions (1.14). Integrating by parts along the boundary for the first term of \mathcal{K}_{412} , we get that by writing $Z^I = \partial_y Z^{\bar{I}}$

$$\begin{aligned} \mathcal{K}_{412} &= \mu \int_0^t |Z^{\bar{I}}(\partial_{\mathbf{n}} u \cdot \mathbf{n})|_{H^{\frac{1}{2}}(\partial\Omega)} |\partial_y [Z^I, \mathbf{n}] u|_{H^{-\frac{1}{2}}(\partial\Omega)} + |Z^I \Pi \partial_{\mathbf{n}} u|_{L^2(\partial\Omega)} |u^I|_{L^2(\partial\Omega)} \, ds \\ &\leq \delta\mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu} (\|u\|_{E_t^m}^2 + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \end{aligned}$$

To summarize, we get the following estimate for \mathcal{K}_4 :

$$\mathcal{K}_4 \leq 2\delta\mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu} (\|u\|_{E_t^m}^2 + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \quad (2.25)$$

Inserting (2.21), (2.23), (2.25) into (2.20), we get that:

$$\begin{aligned} \int_{Q_t} Z^I \mathcal{L} u \cdot u^I \, dx ds &\leq - \iint_{Q_t} \mu |\nabla u^I|^2 + (\mu + \lambda) |\operatorname{div} u^I|^2 \, dx ds \\ &\quad + (C + 3)\delta\mu \|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu} (\|u\|_{E_t^m}^2 + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \end{aligned} \quad (2.26)$$

Plugging (2.18) and (2.26) into (2.16) and summing up for $|I| \leq m$, we finally get (2.9) by choosing δ small enough (independent of ε). \square

Lemma 2.4. *Under the same assumption as in Lemma 2.2, for any $0 < t \leq T$, one has that:*

$$\varepsilon^2 \|(\nabla \sigma, \operatorname{div} u)(t)\|_{H_{co}^{m-1}(\Omega)}^2 + \varepsilon^2 \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}}^2 \lesssim \|(\nabla \sigma, \operatorname{div} u)(0)\|_{H_{co}^{m-1}}^2 + (T^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}}) \Lambda_{2,\infty,T} \mathcal{E}_{m,T}^2. \quad (2.27)$$

Proof. Applying the vector field Z^I with $0 \leq |I| \leq m-1$, we then find that $((\nabla \sigma)^I, u^I) = (Z^I \nabla \sigma, Z^I u)$ solves the system:

$$\begin{cases} g_1(\partial_t + u \cdot \nabla)(\nabla \sigma)^I + \frac{\nabla \operatorname{div} u^I}{\varepsilon} =: C_{\sigma}^I, \\ g_2 \partial_t u^I - \mu \operatorname{curl}(Z^I \omega) - (2\mu + \lambda) \nabla \operatorname{div} u^I + \frac{(\nabla \sigma)^I}{\varepsilon} =: C_u^I, \end{cases} \quad (2.28)$$

where $\omega = \operatorname{curl} u$ and

$$\begin{aligned} C_{\sigma}^I &= -[Z^I \nabla, g_1/\varepsilon] \varepsilon \partial_t \sigma - [Z^I \nabla, g_1 u \cdot \nabla] \sigma - [Z^I, \nabla \operatorname{div}] u / \varepsilon, \\ C_u^I &= -Z^I (g_2 u \cdot \nabla u) - [Z^I, g_2/\varepsilon] \varepsilon \partial_t u + \mu [Z^I, \operatorname{curl}] \omega + (2\mu + \lambda) [Z^I, \nabla \operatorname{div}] u. \end{aligned} \quad (2.29)$$

We take the scalar product of the equation (2.28)₁ by $(\nabla \sigma)^I$, and (2.28)₂ by $-\nabla \operatorname{div} u^I$, we then integrate in space and time and sum up the two equations to get that (note that the singular terms cancel):

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} (g_1 |(\nabla \sigma)^I|^2 + g_2 |\operatorname{div} u^I|^2)(t) \, dx + (2\mu + \lambda) \iint_{Q_t} |\nabla \operatorname{div} u^I|^2 \, dx ds \\
& \leq \frac{1}{2} \int_{\Omega} (g_1 |\nabla \sigma^I|^2 + g_2 |\operatorname{div} u^I|^2)(0) \, dx + \frac{1}{2} \left| \iint_{Q_t} (\partial_t g_1 + \operatorname{div}(g_1 u)) |\nabla \sigma^I|^2 \, dx ds \right| \\
& \quad + \left| \iint_{Q_t} (g'_2 \varepsilon \partial_t u^I \cdot \nabla \sigma) \operatorname{div} u^I \, dx ds \right| + \left| \int_0^t \int_{\partial \Omega} g_2 \partial_t u^I \cdot \mathbf{n} \operatorname{div} u^I \, dS_y ds \right| \\
& \quad + \mu \left| \iint_{Q_t} \operatorname{curl} Z^I \omega \nabla \operatorname{div} u^I \, dx ds \right| \\
& \quad + \|C_{\sigma}^I\|_{L^2(Q_t)} \|\nabla \sigma^I\|_{L^2(Q_t)} + \frac{1}{(2\mu + \lambda)} \|C_u^I\|_{L^2(Q_t)}^2 + \frac{2\mu + \lambda}{4} \|\nabla \operatorname{div} u^I\|_{L^2(Q_t)}^2.
\end{aligned} \tag{2.30}$$

Among the terms in the right hand side, the second and the third terms can be bounded by:

$$\Lambda \left(\frac{1}{c_0}, \|(\sigma, u)\|_{1, \infty, t} + \|(\nabla \sigma, \operatorname{div} u)\|_{0, \infty, t} \right) \|((\nabla \sigma)^I, \operatorname{div} u^I, \varepsilon \partial_t u^I)\|_{L^2(Q_t)}^2. \tag{2.31}$$

Next, we note that the fourth term vanishes if Z^I involves at least one conormal derivative Z_3^i which vanishes on the boundary. We thus suppose that $I = (l, I')$, $|I'| \geq 1$ and Z^I does not contain Z_3^i . Consequently, the trace inequality (7.10) leads to

$$\begin{aligned}
& \left| \int_0^t \int_{\partial \Omega} g_2 \partial_t u^I \cdot \mathbf{n} \operatorname{div} u^I \, dS_y ds \right| \lesssim \frac{1}{\varepsilon} \int_0^t |[Z^I, \mathbf{n}] \varepsilon \partial_t u(s)|_{L^2(\partial \Omega)} |\operatorname{div} u^I(s)|_{L^2(\partial \Omega)} \, ds \\
& \lesssim \frac{1}{\varepsilon} (\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + \|u\|_{L_t^2 H_{co}^{m-1}}) (\|\nabla \operatorname{div} u^I\|_{L^2(Q_t)}^{\frac{1}{2}} \|\operatorname{div} u^I\|_{L^2(Q_t)}^{\frac{1}{2}} + \|\operatorname{div} u^I\|_{L^2(Q_t)}) \\
& \leq \frac{2\mu + \lambda}{4} \|\nabla \operatorname{div} u\|_{L^2(Q_t)}^2 + C_{\mu, \lambda} (1 + \varepsilon^{-\frac{4}{3}}) \|(u, \nabla u)\|_{L_t^2 H_{co}^{m-1}}^2.
\end{aligned} \tag{2.32}$$

Note that since $\partial_t u \cdot \mathbf{n}|_{\partial \Omega} = 0$, one has $(Z^I \partial_t u \cdot \mathbf{n})|_{\partial \Omega} = ([Z^I, \mathbf{n}] \partial_t u)|_{\partial \Omega}$.

For the fifth term in the right hand side of (2.30) we first integrate by parts and then use the duality $\langle \cdot, \cdot \rangle_{H^{\frac{1}{2}}(\partial \Omega) \times H^{-\frac{1}{2}}(\partial \Omega)}$ to get that

$$\begin{aligned}
& \mu \left| \iint_{Q_t} \operatorname{curl} Z^I \omega \cdot \nabla \operatorname{div} u^I \, dx ds \right| = -\mu \int_0^t \int_{\partial \Omega} (Z^I \omega \times \mathbf{n}) \cdot \Pi \nabla \operatorname{div} u^I \, dS_y ds \\
& \leq \mu \int_0^t |Z^I \omega \times \mathbf{n}(s)|_{H^{\frac{1}{2}}(\partial \Omega)} |\operatorname{div} u^I(s)|_{H^{\frac{1}{2}}(\partial \Omega)} \, ds
\end{aligned}$$

We point out that for the derivation of the last line, the fact that $\Pi \nabla$ involves only tangential derivatives has been used. It remains to control $Z^I \omega \times \mathbf{n}$ on the boundary. One first deduces by (1.14) that on the boundary,

$$\omega \times \mathbf{n} = \Pi(\omega \times \mathbf{n}) = 2\Pi(\mathbf{S}u) - 2\Pi((\nabla u)^t \cdot \mathbf{n}) = 2\Pi(-au + D\mathbf{n} \cdot u) \text{ on } \partial \Omega, \tag{2.33}$$

which leads to:

$$\begin{aligned} |Z^I \omega \times \mathbf{n}(s)|_{H^{\frac{1}{2}}(\partial\Omega)} &\lesssim |Z^I(\omega(s) \times \mathbf{n})|_{H^{\frac{1}{2}}(\partial\Omega)} + |[Z^I, \mathbf{n}] \times \omega|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\lesssim |u(s)|_{\tilde{H}^{m-\frac{1}{2}}} + |\omega(s)|_{\tilde{H}^{m-\frac{3}{2}}} \lesssim |u(s)|_{\tilde{H}^{m-\frac{1}{2}}} + |\operatorname{div} u(s)|_{\tilde{H}^{m-\frac{3}{2}}} \end{aligned}$$

where we recall that we denote:

$$|f(t)|_{\tilde{H}^r} := \sum_{k \leq [r]} |(\varepsilon \partial_t)^k f(t)|_{H^{r-k}(\partial\Omega)}.$$

Note that by using the boundary condition (1.14) and the identity (2.22), we have that:

$$|\nabla u|_{\tilde{H}^s} \lesssim |u|_{\tilde{H}^{s+1}} + |\operatorname{div} u|_{\tilde{H}^s}.$$

Finally, owing to the trace inequality (7.11) and Young's inequality, one obtains that:

$$\begin{aligned} \mu \left| \iint_{Q_t} \operatorname{curl} Z^I \omega \cdot \nabla \operatorname{div} u^I \, dx ds \right| &\leq C \mu (\|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}} + \|\nabla u\|_{L_t^2 H_{co}^{m-1}} + \|u\|_{L_t^2 H_{co}^m}) (\|\operatorname{div} u^I\|_{L^2(Q_t)} + \|\nabla \operatorname{div} u^I\|_{L^2(Q_t)}) \\ &\leq \frac{2\mu + \lambda}{4} \|\nabla \operatorname{div} u^I\|_{L^2(Q_t)}^2 + C_{\mu, \lambda} (\|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}}^2 + \|u\|_{E_t^m}^2) \end{aligned} \quad (2.34)$$

where we use again the notation (2.17).

It remains to control the $L^2(Q_t)$ norm of C_σ^I, C_u^I in (2.30). Let us begin with the estimate C_σ^I . For the term:

$$[Z^I \nabla, \frac{g_1}{\varepsilon}] \varepsilon \partial_t \sigma = Z^I ((\nabla g_1 / \varepsilon) \varepsilon \partial_t \sigma) + [Z^I, g_1 / \varepsilon] (\varepsilon \partial_t) \nabla \sigma,$$

the product estimates (7.1) the commutator estimate (7.2) and the estimate (7.5) yield:

$$\begin{aligned} \| [Z^I \nabla, g_1 / \varepsilon] \varepsilon \partial_t \sigma \|_{L^2(Q_t)} &\lesssim \| (\varepsilon \partial_t \sigma, \nabla \sigma) \|_{L_t^2 H_{co}^{m-1}} \Lambda \left(\frac{1}{c_0}, \| \nabla \sigma \|_{[\frac{m}{2}-1, \infty, t]} + \| \sigma \|_{[\frac{m+1}{2}], \infty, t} \right) \\ &\lesssim \| \sigma \|_{E_t^m} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right). \end{aligned}$$

For the term

$$[Z^I \nabla, g_1 u \cdot \nabla] \sigma = Z^I (\nabla(g_1 u) \nabla \sigma) + [Z^I, g_1 u] \nabla \nabla \sigma,$$

since in the interior domain Ω_0 , the spatial conormal derivatives are equivalent to the derivations with respect to the standard coordinates in \mathbb{R}^3 . We thus have that:

$$\begin{aligned} \varepsilon \| \chi_0 [Z^I \nabla, g_1 u \cdot \nabla] \sigma \|_{L^2(Q_t)} &\lesssim (\| \tilde{\chi}_0(\sigma, u) \|_{L_t^2 H^m} + \| \tilde{\chi}_0 \nabla(\sigma, u) \|_{L_t^2 H^{m-1}}) \Lambda \left(\frac{1}{c_0}, \| \varepsilon(\sigma, u) \|_{[\frac{m}{2}]+1, \infty, t} \right) \\ &\lesssim \| (\sigma, u) \|_{E_t^m} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right), \end{aligned}$$

where $\operatorname{Supp}(\tilde{\chi}_0) \Subset \Omega$ and $\tilde{\chi}_0 \chi_0 = \chi_0$. It suffices to focus on the case near the boundary. Direct computations show that, in the local coordinates (1.12),

$$u \cdot \nabla f = u_1 \partial_{y^1} f + u_2 \partial_{y^2} f + u \cdot \mathbf{N} \partial_z f, \quad (2.35)$$

which leads to:

$$\begin{aligned} [Z^I \nabla, g_1 u \cdot \nabla] \sigma &= Z^I (\nabla(g_1 u) \nabla \sigma) + \sum_{j=1}^2 [Z^I, g_1 u_j] \partial_{y^j} \nabla \sigma \\ &\quad + [Z^I, (g_1 u \cdot \mathbf{N})/\phi] \phi \partial_z \nabla \sigma + ((g_1 u \cdot \mathbf{N})/\phi) [Z^I, \phi] \partial_z \nabla \sigma + (g_1 u \cdot \mathbf{N}) [Z^I, \partial_z] \nabla \sigma. \end{aligned} \quad (2.36)$$

With the help of the product and commutator estimates (7.1), (7.2) and the estimate (7.5) for g_1 , the first two terms in the right hand side of (2.36) can be bounded as:

$$\begin{aligned} &\varepsilon \|\chi_i Z^I (\nabla(g_1 u) \nabla \sigma)\|_{L^2(Q_t)} + \sum_{j=1}^2 \|\chi_i [Z^I, g_1 u_j] \partial_{y^j} \nabla \sigma\|_{L^2(Q_t)} \\ &\lesssim \|(\sigma, u)\|_{E_t^m} \Lambda\left(\frac{1}{c_0}, \varepsilon \|(\sigma, u)\|_{[\frac{m}{2}], \infty, t} + \|\nabla \sigma\|_{[\frac{m-1}{2}], \infty, t} + \varepsilon \|\nabla u\|_{[\frac{m}{2}], \infty, t}\right) \\ &\lesssim \|(\sigma, u)\|_{E_t^m} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right). \end{aligned} \quad (2.37)$$

To continue, we need to establish some estimates on $(g_1 u \cdot \mathbf{N})/\phi$. At first, since $(u \cdot \mathbf{n})|_{\partial\Omega} = 0$, one has by the fundamental theorem of calculus and the identity (2.22) that:

$$\begin{aligned} \|\chi_j(g_j u \cdot \mathbf{N})/\phi\|_{k, \infty, t} &\lesssim (\|\nabla(u \cdot \mathbf{N})\|_{k, \infty, t} + \|u\|_{k, \infty, t}) \|g\|_{k, \infty, t} \\ &\lesssim \Lambda\left(\frac{1}{c_0}, \|u\|_{k+1, \infty, t} + \|(\sigma, \operatorname{div} u)\|_{k, \infty, t}\right), \quad j = 1, 2. \end{aligned} \quad (2.38)$$

Next, thanks to Hardy inequality and product estimate (7.1), estimate (7.6) for g_j

$$\begin{aligned} \|\chi_i(g_j u \cdot \mathbf{N})/\phi\|_{L_t^2 H_{co}^{m-1}} &\lesssim \|\tilde{\chi}_i(u \cdot \mathbf{N})/\phi\|_{L_t^2 H_{co}^{m-1}} + \|(g_j - g_j(0))(u \cdot \mathbf{N})/\phi\|_{L_t^2 H_{co}^{m-1}} \\ &\lesssim (\|\tilde{\chi}_i(u, \nabla u)\|_{L_t^2 H_{co}^{m-1}} + \|g_j - g_j(0)\|_{L_t^2 H_{co}^{m-1}}) \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \\ &\lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|(\sigma, u)\|_{E_t^m}, \quad j = 1, 2, \end{aligned} \quad (2.39)$$

where $\tilde{\chi}_i$ is a cut-off function supported on the vicinity of Ω_i and $\tilde{\chi}_i \chi_i = \chi_i$. Therefore, since $\phi \partial_z$ can be spanned by Z_1^i, Z_2^i, Z_3^i , it follows from (2.38), (2.39), (7.2), (7.5) that:

$$\begin{aligned} &\varepsilon \|\chi_i [Z^I, (g_1 u \cdot \mathbf{N})/\phi] \phi \partial_z \nabla \sigma\|_{L^2(Q_t)} \\ &\lesssim \|(\nabla \sigma, (g_1 u \cdot \mathbf{N})/\phi)\|_{L_t^2 H_{co}^{m-1}} \Lambda\left(\frac{1}{c_0}, \|\nabla \sigma\|_{[\frac{m-1}{2}], \infty, t} + \varepsilon \|\tilde{\chi}_i(g_1 u \cdot \mathbf{N})/\phi\|_{[\frac{m}{2}], \infty, t}\right) \\ &\lesssim \|(\sigma, u)\|_{E_t^m} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right). \end{aligned} \quad (2.40)$$

Moreover, one gets by induction that (up to some coefficients that depend only on ϕ and its derivatives)

$$[Z^I, \phi](\partial_z f) = \sum_{|\tilde{I}| \leq |I|-1} *_{\tilde{I}} Z^{\tilde{I}}(\phi \partial_z f), \quad [Z^I, \partial_z] = \sum_{|\tilde{I}| \leq |I|-1} *_{\tilde{I}} \partial_z Z^{\tilde{I}} \quad (2.41)$$

Hence, by (2.38), the last two terms in (2.36) can be controlled by $\|\nabla \sigma\|_{L_t^2 H_{co}^{m-1}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right)$, which, together with (2.37), (2.40) leads to:

$$\varepsilon \|\chi_i [Z^I \nabla, g_1 u \cdot \nabla] \sigma\|_{L^2(Q_t)} \lesssim \|(\sigma, u)\|_{E_t^m} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right). \quad (2.42)$$

We switch to the estimate of the third term of C_σ^I defined in (2.29), which is nontrivial only if Z^I contains at least one spatial derivative, that is $|I'| \geq 1$. By induction, one has that (up to some coefficients which are regular enough)

$$[Z^I, \nabla \operatorname{div}] = \sum_{|\tilde{I}| \leq |I|-1, |\tilde{I}| \leq |I|-1} \sum_{j,k=1}^3 *_{jk\tilde{I}} \partial_{jk}^2 Z^{\tilde{I}} + *_{j\tilde{I}} \partial_j Z^{\tilde{I}},$$

which yields that:

$$\frac{1}{\varepsilon} \| [Z^I, \nabla \operatorname{div}] u \|_{L^2(Q_t)} \lesssim \frac{1}{\varepsilon} (\|\nabla^2 u\|_{L_t^2 H_{co}^{m-2}} + \|\nabla u\|_{L_t^2 H_{co}^{m-2}}).$$

To summarize, we have thus obtained from the above estimates that:

$$\varepsilon \|C_\sigma^I\|_{L^2(Q_t)} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|(\sigma, u)\|_{E_t^m} + \|(\nabla^2 u, \nabla u)\|_{L_t^2 H_{co}^{m-2}}. \quad (2.43)$$

By using the same argument, C_u^I (defined in (2.29)) can be controlled as follows:

$$\varepsilon \|C_u^I\|_{L^2(Q_t)} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|(\sigma, u)\|_{E_t^m} + \varepsilon \|\nabla^2 u\|_{L_t^2 H_{co}^{m-2}}. \quad (2.44)$$

Plugging (2.31), (2.32), (2.34), (2.43), (2.44) in (2.30), we arrive at

$$\begin{aligned} & \varepsilon^2 (\|((\nabla \sigma)^I, \operatorname{div} u^I)(t)\|_{L^2(\Omega)}^2 + \|\nabla \operatorname{div} u^I\|_{L^2(Q_t)}^2) \\ & \lesssim \varepsilon^2 \|((\nabla \sigma)^I, \operatorname{div} u^I)(0)\|_{L^2(\Omega)}^2 + \varepsilon^{\frac{3}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|(\sigma, u)\|_{E_t^m}^2 \\ & \quad + T^{\frac{1}{2}} \|\varepsilon \nabla^2 u\|_{L_t^\infty H_{co}^{m-2}} (\|\varepsilon \nabla^2 u\|_{L_t^2 H_{co}^{m-2}} + \|\nabla \sigma\|_{L_t^2 H_{co}^{m-1}}). \end{aligned} \quad (2.45)$$

We thus get (2.27) by summing up (2.45) for $0 \leq |I| \leq m-1$. \square

2.3. Step 2: Energy estimate for the incompressible part of velocity

In this subsection, we focus on the estimates of the incompressible part of the velocity $v = \mathbb{P}u$ which solves (2.5)₃.

In the following, we recall for convenience the definition of the $L_{t,x}^\infty$ norm:

$$\begin{aligned} \mathcal{A}_{m,t} &= \|\nabla u\|_{0,\infty,t} + \|(\sigma, u)\|_{[\frac{m+1}{2}],\infty,t} + \|(\nabla \sigma, \operatorname{div} u, \varepsilon^{\frac{1}{2}} \nabla u)\|_{[\frac{m-1}{2}],\infty,t} \\ & \quad + \|\varepsilon \nabla u\|_{[\frac{m+1}{2}],\infty,t} + \varepsilon \|(\sigma, u)\|_{[\frac{m+3}{2}],\infty,t}. \end{aligned} \quad (2.46)$$

Remark 2.5. In view of the first term in $\mathcal{A}_{m,t}$, we have only the uniform control of ∇u in $L_{t,x}^\infty$ space. Indeed, by some delicate analysis on the Green function for the vorticity in the local coordinates, it is possible to get the uniform control of the high order conormal derivatives of ∇u (say $\|\nabla u\|_{[\frac{m}{2}-2,\infty,t]}$). One can refer for instance to [37]. Nevertheless, involving only $\|\nabla u\|_{0,\infty,t}$ in $\mathcal{A}_{m,t}$ is enough for us to close our estimate. See Lemma 2.8 and Proposition 2.18.

We begin with some additional estimates on $\nabla \operatorname{div} u$:

Lemma 2.6. Suppose that (1.24) holds then for any $0 < t \leq T \leq 1$.

$$\|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}} \lesssim \|\nabla \sigma\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|(\sigma, u)\|_{E_t^m}, \quad (2.47)$$

$$\varepsilon \|\nabla \operatorname{div} u(t)\|_{H_{co}^{m-2}} \lesssim \varepsilon \|\nabla \sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}, \quad (2.48)$$

$$\|\nabla \operatorname{div} u(t)\|_{H_{co}^{m-3}} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \quad (2.49)$$

Proof. By the equation for σ , we have that:

$$\nabla \operatorname{div} u = g_1(0) \varepsilon \partial_t \nabla \sigma + \varepsilon \nabla \left(\frac{g_1(\varepsilon \sigma) - g_1(0)}{\varepsilon} \varepsilon \partial_t \sigma + g_1(\varepsilon \sigma) u \cdot \nabla \sigma \right). \quad (2.50)$$

We can control $\varepsilon \nabla \operatorname{div} u$ as follows, for $p = 2, +\infty$,

$$\|\nabla \operatorname{div} u\|_{L_t^p H_{co}^{m-2}} \lesssim \|\nabla \sigma\|_{L_t^p H_{co}^{m-1}} + \varepsilon \|\nabla((g_1 - g_1(0)) \partial_t \sigma, g_1 u \cdot \nabla \sigma)(t)\|_{L_t^p H_{co}^{m-2}} \quad (2.51)$$

Inequalities (2.47)-(2.48) can thus be derived from the following estimate:

$$\begin{aligned} & \varepsilon \|\nabla((g_1 - g_1(0)) \partial_t \sigma, g_1 u \cdot \nabla \sigma)(t)\|_{L_t^p H_{co}^{m-2}} \\ & \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) (\|\varepsilon \nabla(\sigma, u)\|_{L_t^p H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}} \|(\sigma, u, \nabla \sigma, \nabla u)\|_{L_t^p H_{co}^{m-2}}). \end{aligned}$$

Let us show the estimate of the term $g_1 u \cdot \nabla \nabla \sigma$, the other terms can be controlled in a similar way. Again, we focus only on the estimate near the boundary. Thanks to the identity (2.35), we have

$$\chi_i g_1 u \cdot \nabla \nabla \sigma = \chi_i g_1 u_y \cdot \partial_y \nabla \sigma + \chi_i g_1 \frac{u \cdot \mathbf{N}}{\phi} \phi \partial_z \nabla \sigma.$$

Therefore, by applying the product estimate (7.1) and inequality (2.38), we find

$$\begin{aligned} \varepsilon \|\chi_i(g_1 u \cdot \nabla \nabla \sigma)\|_{L_t^p H_{co}^{m-2}} & \lesssim \varepsilon \|(u_y, \chi_i u \cdot \mathbf{N}/\phi)\|_{L_t^p H_{co}^{m-2}} \|g_1 Z \nabla \sigma\|_{[\frac{m-1}{2}]_{-1, \infty}, t} \\ & \quad + \varepsilon \|g_1 Z \nabla \sigma(t)\|_{L_t^p H_{co}^{m-2}} \|(u_y, \chi_i u \cdot \mathbf{N}/\phi)\|_{[\frac{m}{2}]_{-1, \infty}, t} \\ & \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) (\|\varepsilon \nabla \sigma\|_{L_t^p H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}} \|(u, \nabla \sigma, \nabla u)\|_{L_t^p H_{co}^{m-2}}). \end{aligned} \quad (2.52)$$

Finally, one gets (2.49) by using similar arguments as in the derivation of (2.48), we skip the details. \square

Remark 2.7. By (2.7) and (2.48), we have that:

$$\varepsilon \|\nabla \operatorname{div} u\|_{L_t^\infty H_{co}^{m-2}} \lesssim Y_m(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{4}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \quad (2.53)$$

Lemma 2.8. Let

$$f = -\frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u - g_2 u \cdot \nabla u \quad (2.54)$$

and assume that (1.24) holds, then we have:

$$\|f\|_{L_t^2 H_{co}^{m-1}} + \|f\|_{L_t^\infty H_{co}^{m-2}} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \quad (2.55)$$

Proof. Since the higher order $L_{t,x}^\infty$ norm of $\partial_{\mathbf{n}} u$ is not included in the definition of $\mathcal{A}_{m,t}$, we need to use again the fact that $u \cdot \mathbf{n}$ vanishes on the boundary. More precisely, by using the product estimate (7.1), identity (2.35) and the estimate (2.39), we get for $(p, k) = (2, 1), (\infty, 2)$,

$$\|g_2 u \cdot \nabla u\|_{L_t^p H_{co}^{m-k}} \lesssim \|(\sigma, u, \nabla \sigma, \nabla u)\|_{L_t^p H_{co}^{m-k}} \Lambda\left(\frac{1}{c_0}, \|(\nabla \sigma, \operatorname{div} u)\|_{[\frac{m-1}{2}], \infty, t} + \|(\sigma, u)\|_{[\frac{m+1}{2}], \infty, t}\right).$$

The first term is a direct application of the product estimate (7.1), we omit the detail. \square

We split the estimate for v in the following three subsections.

2.3.1. Estimate of ∇q

We first give the estimate of ∇q that appears in (2.5)₃. Since q is governed by the elliptic equation (2.6) without singular terms, it can be easily estimated by standard elliptic regularity theory.

Lemma 2.9. *Under the assumptions (1.24), we have the following estimates: for $j + l \leq m - 1, l \geq 1$,*

$$\|\nabla q\|_{L_t^2 \mathcal{H}^{j,l}} + \varepsilon^{\frac{1}{2}} \|\nabla q\|_{L_t^2 \mathcal{H}^{m-1}} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \quad (2.56)$$

where E_t^m is defined in (2.17). Moreover,

$$\varepsilon \|\operatorname{curl} \omega(t)\|_{H_{co}^{m-2}} + \varepsilon \|\nabla q(t)\|_{H_{co}^{m-2}} \lesssim \|v(t)\|_{H_{co}^{m-1}} + Y_m(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{4}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \quad (2.57)$$

Proof. Recall that q is governed by (2.6), an elliptic equation with Neumann boundary conditions. We can apply (7.14) in the appendix by setting

$$f = -\frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u - g_2 u \cdot \nabla u, \quad g = \mu \Delta v \cdot \mathbf{n}$$

to get

$$\|\nabla q\|_{L_t^2 \mathcal{H}^{j,l}} \lesssim \|f\|_{L_t^2 H_{co}^{m-1}} + \sum_{|I| \leq m-1} |Z^I(\Delta v \cdot \mathbf{n})|_{L_t^2 H^{-\frac{1}{2}}(\partial\Omega)} \quad (2.58)$$

The first term in the right hand side has been controlled in (2.55), it remains to estimate the boundary term. By using the identity

$$(\nabla \times a) \cdot b = \nabla \cdot (a \times b) + a \cdot (\nabla \times b), \quad (2.59)$$

we have that:

$$-\Delta v \cdot \mathbf{n} = (\nabla \times \omega) \cdot \mathbf{n} = \operatorname{div}(\omega \times \mathbf{n}) + \omega \cdot \operatorname{curl} \mathbf{n}.$$

Near the boundary, it follows from (2.22) that:

$$\begin{aligned} \operatorname{div}(\omega \times \mathbf{n}) &= \partial_{\mathbf{n}}(\omega \times \mathbf{n}) \cdot \mathbf{n} + (\Pi \partial_{y^1}(\omega \times \mathbf{n}))^1 + (\Pi \partial_{y^2}(\omega \times \mathbf{n}))^2 \\ &= -(\omega \times \mathbf{n}) \cdot \partial_{\mathbf{n}} \mathbf{n} + (\Pi \partial_{y^1}(\omega \times \mathbf{n}))^1 + (\Pi \partial_{y^2}(\omega \times \mathbf{n}))^2. \end{aligned} \quad (2.60)$$

Therefore, by using the boundary condition (2.33), one has that for $|I| \leq m - 1$,

$$|Z^I(\operatorname{div}(\omega \times \mathbf{n}))|_{L_t^2 H^{-\frac{1}{2}}(\partial\Omega)} \lesssim |u|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}(\partial\Omega)} \quad (2.61)$$

where $L_t^2 \tilde{H}^s(\partial\Omega)$ is defined in (1.10). In view of the identity (2.22) and the boundary condition (1.14), we have for $l \geq 1$

$$\begin{aligned} |Z^I \omega|_{L_t^2 H^{-\frac{1}{2}}(\partial\Omega)} &\lesssim |u|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}} + |Z^I(\partial_{\mathbf{n}} u)|_{L_t^2 H^{-\frac{1}{2}}} \lesssim |u|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}} + |Z^I \operatorname{div} u|_{L_t^2 H^{-\frac{1}{2}}} \\ &\lesssim \|u\|_{E_t^m} + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}}. \end{aligned} \quad (2.62)$$

Moreover, if $Z^I = (\varepsilon \partial_t)^{m-1}$, we have by $L^2(\partial\Omega) \hookrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ and the trace inequality (7.10)

$$\varepsilon^{\frac{1}{2}} |Z^I \operatorname{div} u|_{L_t^2 H^{-\frac{1}{2}}} \lesssim \|(\operatorname{div} u, \varepsilon \nabla \operatorname{div} u)\|_{L_t^2 \mathcal{H}^{m-1}} \quad (2.63)$$

Collecting (2.58)-(2.63), and using (2.47), (2.55), one obtains that:

$$\begin{aligned} &\|\nabla q\|_{L_t^2 \mathcal{H}^{j,l}} + \varepsilon^{\frac{1}{2}} \|\nabla q\|_{L_t^2 \mathcal{H}^{m-1}} \\ &\lesssim \|f\|_{L_t^2 H_{co}^{m-1}} + \|u\|_{E_t^m} + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}} + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \end{aligned}$$

We are now ready to prove (2.57). By using the equation (2.5)₃, the elliptic estimate (7.14) and the product estimate (7.1), one finds:

$$\begin{aligned} &\varepsilon \|\Delta v(t)\|_{H_{co}^{m-2}} + \varepsilon \|\nabla q(t)\|_{H_{co}^{m-2}} \\ &\lesssim \|v(t)\|_{H_{co}^{m-1}} + \varepsilon \|f(t)\|_{H_{co}^{m-2}} + \varepsilon \sum_{|I| \leq m-2} |Z^I(\Delta v \cdot \mathbf{n})(t)|_{H^{-\frac{1}{2}}(\partial\Omega)} \end{aligned} \quad (2.64)$$

With the aid of the boundary condition (1.14), the identities (2.22), (2.60) and the estimates (2.7), (2.53), the boundary term can be treated as,

$$\begin{aligned} &\varepsilon \sum_{|I| \leq m-2} |Z^I(\Delta v \cdot \mathbf{n})|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\lesssim \varepsilon (\|\nabla u(t)\|_{H_{co}^{m-2}} + \|u(t)\|_{H_{co}^{m-1}}) + \varepsilon \|\nabla \operatorname{div} u(t)\|_{H_{co}^{m-2}} \\ &\lesssim Y_m(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{4}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \end{aligned} \quad (2.65)$$

Combined with (2.64) and the fact that $\Delta v = -\operatorname{curl} \omega$, this yields (2.57). \square

2.3.2. High order regularity estimates for v

This subsection is devoted to the high order estimates for v : $\|v\|_{L_t^\infty H_{co}^{m-1}}, \|\nabla v\|_{L_t^2 H_{co}^{m-1}}$.

Lemma 2.10. *Suppose that (1.24) is satisfied, then for any $j + l \leq m - 1, j, l \geq 0$ and for every $0 < t \leq T$, the following a-priori estimate holds:*

$$\begin{aligned} &\|v\|_{L_t^\infty \mathcal{H}^{j,l}}^2 + \varepsilon^2 \|\nabla v\|_{L_t^\infty \mathcal{H}^{j,l}}^2 + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 + \varepsilon^2 \|\operatorname{curl} \omega\|_{L_t^2 \mathcal{H}^{j,l}}^2 \\ &\lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda_{2,\infty,T} \mathcal{E}_{m,T}^2 + \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,l} \cap L_t^2 \mathcal{H}^{j+1,l-1}}^2 \end{aligned} \quad (2.66)$$

where we use the notation (1.8).

Remark 2.11. The estimate (2.66) will be used later (see Lemma 2.12) to get the high order spatial regularity for $\operatorname{div} u$, which in turn, together with (2.66), gives the control of v .

Proof. In view of (1.22), (2.7), it suffices to show that the left hand side of (2.66) can be controlled by:

$$C(1/c_0) \left(Y_m^2(\sigma_0, u_0) + \mathcal{W}_{m,T}^2 + \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,l} \cap L_t^2 \mathcal{H}^{j+1,l-1}}^2 \right)$$

where:

$$\mathcal{W}_{m,T}^2 = \|u\|_{L_t^\infty \mathcal{H}^{m-1}}^2 + \|\nabla u\|_{L_t^2 \mathcal{H}^{m-1}}^2 + \varepsilon^2 \|\nabla u\|_{L_t^2 H_{co}^m}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}. \quad (2.67)$$

This estimate will be obtained as the direct consequence of the following three inequalities:

$$\|v\|_{L_t^\infty \mathcal{H}^{m-1}}^2 + \|\nabla v\|_{L_t^2 \mathcal{H}^{m-1}}^2 \lesssim \|u\|_{L_t^\infty \mathcal{H}^{m-1}}^2 + \|\nabla u\|_{L_t^2 \mathcal{H}^{m-1}}^2, \quad (2.68)$$

$$\begin{aligned} \|v\|_{L_t^\infty \mathcal{H}^{j,l}}^2 + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 &\lesssim \|v(0)\|_{H_{co}^{m-1}}^2 + \|\nabla u\|_{L_t^2 \mathcal{H}^{m-1}}^2 \\ &\quad + \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,l}}^2 + T^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}^2, \quad l \geq 1, \end{aligned} \quad (2.69)$$

$$\begin{aligned} \varepsilon^2 \|\nabla v\|_{L_t^\infty \mathcal{H}^{j,l}}^2 + \varepsilon^2 \|\Delta v\|_{L_t^2 \mathcal{H}^{j,l}}^2 &\lesssim \varepsilon^2 \|(\nabla v, v)(0)\|_{H_{co}^{m-1}} + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l} \cap L_t^2 \mathcal{H}^{j+1,l-1}}^2 \\ &\quad + \varepsilon^2 \|\nabla u\|_{L_t^2 H_{co}^m}^2 + (T^{\frac{1}{2}} + \varepsilon) \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}^2. \end{aligned} \quad (2.70)$$

Note that since the Leray projector \mathbb{P} commutes with $\varepsilon \partial_t$, one has that: $\mathbb{P}((\varepsilon \partial_t)^j u) = (\varepsilon \partial_t)^j v$. Therefore, from the continuity of the projection, we have:

$$\|v(0)\|_{H_{co}^{m-1}} \lesssim \|u(0)\|_{H_{co}^{m-1}}.$$

The inequality (2.68) is a direct consequence of the definition of v and the elliptic estimates in Proposition 7.6. We thus focus on the other two inequalities. Let us first prove (2.69) and then sketch the proof of (2.70). By (1.28), v solves

$$\bar{\rho} \partial_t v - \mu \Delta v + \nabla q = - \left(\frac{g_2 - \bar{\rho}}{\varepsilon} \varepsilon \partial_t u + g_2 u \cdot \nabla u \right) =: f \quad (2.71)$$

supplemented with the boundary conditions:

$$v \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Pi(\partial_{\mathbf{n}} v) = \Pi(-2av + D\mathbf{n} \cdot v) + 2\Pi(-a\nabla\Psi + D\mathbf{n} \cdot \nabla\Psi). \quad (2.72)$$

We apply Z^I to the equation (2.71) with $I = (j, I')$, $0 \leq j + |I'| = j + l = k \leq m - 1$, $|I'| \geq 1$. Taking the scalar product by $Z^I v$, and then integrating in space and time, we get that:

$$\begin{aligned} \frac{1}{2} \bar{\rho} \int_{\Omega} |Z^I v(t)|^2 \, dx &\leq \frac{1}{2} \bar{\rho} \int_{\Omega} |(Z^I v)(0)|^2 \, dx + \mu \iint_{Q_t} Z^I(\Delta v) Z^I v \, dx ds \\ &\quad + \|Z^I v\|_{L^2(Q_t)} (\|\nabla q\|_{L_t^2 \mathcal{H}^{j,l}} + \|f\|_{L_t^2 H_{co}^{m-1}}). \end{aligned} \quad (2.73)$$

By (2.55) and (2.56), the second line in the above inequality can be bounded as:

$$\begin{aligned} \|Z^I v\|_{L^2(Q_t)} (\|\nabla q\|_{L_t^2 \mathcal{H}^{j,l}} + \|f\|_{L_t^2 H_{co}^{m-1}}) &\lesssim T^{\frac{1}{2}} \|u\|_{L_t^\infty H_{co}^{m-1}} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t} \\ &\lesssim T^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}^2. \end{aligned} \quad (2.74)$$

It remains to control the second term in the right hand side of (2.73), which is the following task. We split it into three terms:

$$\begin{aligned} \mu \iint_{Q_t} Z^I(\Delta v) \cdot Z^I v \, dx ds &= \mu \iint_{Q_t} [Z^I, \operatorname{div}] \nabla v \cdot Z^I v \, dx ds - \mu \iint_{Q_t} Z^I \nabla v \cdot \nabla Z^I v \, dx ds \\ &\quad + \mu \int_0^t \int_{\partial\Omega} Z^I \nabla v \cdot \mathbf{n} Z^I v \, dS_y \, ds =: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \end{aligned} \quad (2.75)$$

The estimate of $\mathcal{T}_1 - \mathcal{T}_3$ will be similar to that of $\mathcal{K}_1 - \mathcal{K}_4$ in (2.20).

We first estimate \mathcal{T}_2 . By integrating by parts, one has that:

$$\begin{aligned} \mathcal{T}_2 &= -\mu \iint_{Q_t} |Z^I \nabla v|^2 \, dx ds - \mu \iint_{Q_t} Z^I \nabla v [\nabla, Z^I] v \, dx ds \\ &\leq -\frac{\mu}{2} \|Z^I \nabla v\|_{L^2(Q_t)}^2 + \frac{\mu}{2} \|[\nabla, Z^I] v\|_{L^2(Q_t)}^2 \leq -\frac{\mu}{2} \|Z^I \nabla v\|_{L^2(Q_t)}^2 + C \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l-1}}^2. \end{aligned} \quad (2.76)$$

Note that in the last estimate, by (2.19), we know that $[\nabla, Z^I] v$ involves only lower order ($\leq k-1$) conormal derivatives of ∇v .

We now switch to the estimate of the boundary term \mathcal{T}_3 in (2.75), which vanishes if Z^I involves at least one weighted normal derivative Z_3^i . We thus assume that Z^I contains only time derivatives and spatial tangential derivatives.

$$\begin{aligned} \mathcal{T}_3 &= -\mu \int_0^t \int_{\partial\Omega} \left(-[Z^I, \mathbf{n}] \nabla v \cdot Z^I v + [Z^I, \mathbf{n}] \cdot \partial_{\mathbf{n}} v (Z^I v \cdot \mathbf{n}) + [Z^I, \Pi] \partial_{\mathbf{n}} v \cdot \Pi Z^I v \right) dS_y \, ds \\ &\quad + \mu \int_0^t \int_{\partial\Omega} (Z^I (\partial_{\mathbf{n}} v \cdot \mathbf{n}) (Z^I v \cdot \mathbf{n}) + Z^I (\Pi \partial_{\mathbf{n}} v) \cdot \Pi Z^I v) \, dS_y \, ds =: \mathcal{T}_{31} + \mathcal{T}_{32}. \end{aligned}$$

The first term \mathcal{T}_{31} can be dealt with thanks to Hölder inequality and the trace inequality (7.10)

$$\begin{aligned} \mathcal{T}_{31} &\lesssim \int_0^t |(\varepsilon \partial_t)^j \nabla v(s)|_{H^{l-1}(\partial\Omega)} |Z^I v(s)|_{L^2(\partial\Omega)} \, ds \\ &\lesssim \int_0^t (|(\varepsilon \partial_t)^j v|_{H^l(\partial\Omega)} + |(\varepsilon \partial_t)^j \nabla \Psi|_{H^l(\partial\Omega)}) |Z^I v|_{L^2(\partial\Omega)} \, ds \\ &\leq \delta \mu \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 + C(\delta, \mu) \|(u, \operatorname{div} u)\|_{L_t^2 H_{co}^k}^2. \end{aligned}$$

Note that in the second inequality, we have used the boundary condition (2.72) and the identity (since $\operatorname{div} v = 0$):

$$\partial_{\mathbf{n}} v \cdot \mathbf{n} = -(\Pi \partial_{y^1} v)^1 - (\Pi \partial_{y^2} v)^2, \quad (2.77)$$

to obtain that:

$$|(\varepsilon \partial_t)^j \nabla v(s)|_{H^{l-1}} \lesssim |(\varepsilon \partial_t)^j v(s)|_{H^l} + |(\varepsilon \partial_t)^j \nabla \Psi(s)|_{H^l}. \quad (2.78)$$

For the second term \mathcal{T}_{32} , since $l \geq 1$, we might as well assume that $Z^I = \partial_y Z^{\tilde{I}}$, where $\partial_y = \partial_{y^1}$ or ∂_{y^2} . In view of the boundary condition (2.72) and the identity (2.77), we have by integrating by parts along the boundary that:

$$\begin{aligned} \mathcal{T}_{32} &= \int_0^t \int_{\partial\Omega} Z^{\tilde{I}} (\partial_{\mathbf{n}} v \cdot \mathbf{n}) \partial_y \cdot ([Z^I, \mathbf{n} \cdot] v) + Z^I (\Pi \partial_{\mathbf{n}} v) \Pi Z^I v \, dS_y \, ds \\ &\lesssim \int_0^t |(\varepsilon \partial_t)^j v|_{H^l(\partial\Omega)}^2 + |(\varepsilon \partial_t)^j (v, \nabla \Psi)|_{H^l(\partial\Omega)} |(\varepsilon \partial_t)^j v|_{H^l(\partial\Omega)} \, ds \\ &\lesssim \delta \mu \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 + C(\delta, \mu) \|(u, \operatorname{div} u)\|_{L_t^2 \mathcal{H}^{j,l}}^2. \end{aligned} \quad (2.79)$$

It remains to control \mathcal{T}_1 . Owing to (2.19) and (2.78), one obtains again by integrating by parts that:

$$\begin{aligned} \mathcal{T}_1 &\lesssim \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l-1}} (\|v\|_{L_t^2 \mathcal{H}^{j,l}} + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}) + |(\varepsilon \partial_t)^j \nabla v(s)|_{H^{l-1}(\partial\Omega)} |v|_{H^l(\partial\Omega)} \\ &\lesssim \delta \mu \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 + C(\delta, \mu) (\|(u, \operatorname{div} u)\|_{L_t^2 \mathcal{H}^{j,l}}^2 + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l-1}}^2). \end{aligned} \quad (2.80)$$

Plugging (2.75)-(2.80) into (2.73) and summing up for all $I = (j, I')$, $|I'| = l$, one has by choosing δ small enough that

$$\begin{aligned} \|v(t)\|_{\mathcal{H}^{j,l}}^2 + \frac{\mu}{4} \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l}}^2 &\leq \|v(0)\|_{\mathcal{H}^{j,l}}^2 + C(\delta, \mu) \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l-1}}^2 + \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,l}}^2 \\ &\quad + T^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}^2. \end{aligned} \quad (2.81)$$

In view of inequalities (2.68) and (2.81), we obtain (2.69) by induction on l .

We are now in position to prove (2.70). As before, we apply Z^I to the equation (2.71) for v and we take the scalar product by $-\varepsilon^2 Z^I \Delta v$. One gets by integration by parts and by using Young's inequality that:

$$\begin{aligned} &\frac{1}{2} \bar{\rho} \varepsilon^2 \int_{\Omega} |\nabla Z^I v(t)|^2 \, dx + \frac{\mu}{2} \varepsilon^2 \iint_{Q_t} |Z^I(\Delta v)|^2 \, dx \, ds \\ &\leq \frac{1}{2} \bar{\rho} \varepsilon^2 \int_{\Omega} |\nabla Z^I v(0)|^2 \, dx + \varepsilon \iint_{Q_t} \varepsilon \partial_t Z^I v \cdot [Z^I, \Delta] v \, dx \, ds \\ &\quad + \varepsilon \int_0^t \int_{\partial\Omega} \varepsilon \partial_t Z^I v \cdot \partial_{\mathbf{n}} Z^I v \, dS_y \, ds + C_{\mu} \varepsilon^2 \|(\nabla q, f)\|_{L_t^2 H_{co}^{m-1}}^2. \end{aligned} \quad (2.82)$$

By induction, the following identity (up to some coefficients that depends on ϕ, φ and their derivatives up to order m) holds:

$$[Z^I, \Delta] = \sum_{\substack{|\tilde{I}| \leq |I|-1, |J| \leq |I|-1 \\ \tilde{I}_0 = j, J_0 = j}} \sum_{i,k=1}^3 (*Z^{\tilde{I}} \partial_{ik}^2 + *Z^J \partial_k).$$

This identity, combined with elliptic regularity theory yields:

$$\begin{aligned} \| [Z^I, \Delta] v \|_{L^2(Q_t)} &\lesssim \|\nabla^2 v\|_{L_t^2 \mathcal{H}^{j,l-1}} + \|\nabla v\|_{L_t^2 \mathcal{H}^{j,l-1}} \lesssim \|\Delta v\|_{L_t^2 \mathcal{H}^{j,l-1}} + |\partial_{\mathbf{n}} (\varepsilon \partial_t)^j v|_{H^{l-\frac{1}{2}}} \\ &\lesssim \|\Delta v\|_{L_t^2 \mathcal{H}^{j,l-1}} + \|(u, \nabla u)\|_{L_t^2 \mathcal{H}^{j,l}}. \end{aligned}$$

Note that in the last inequality, we have used (2.78) and the trace inequality (7.10). We thus control the second term in (2.82) as follows:

$$\varepsilon \iint_{Q_t} \varepsilon \partial_t Z^I v \cdot [Z^I, \Delta] v \, dx ds \lesssim \varepsilon^2 \|\Delta v\|_{L_t^2 \mathcal{H}^{j,l-1}}^2 + \|\varepsilon \partial_t v\|_{L_t^2 \mathcal{H}^{j,l}}^2 + \varepsilon \|u\|_{E_t^m}^2. \quad (2.83)$$

Moreover, the third term of (2.82) can be dealt with by arguments very similar to the ones for \mathcal{T}_3 :

$$\begin{aligned} & \varepsilon \int_0^t \int_{\partial\Omega} \varepsilon \partial_t Z^I v \cdot \partial_{\mathbf{n}} Z^I v \, dS_y \, ds \\ & \lesssim \varepsilon \int_0^t |Z^I \varepsilon \partial_t v|_{L^2} (|(\varepsilon \partial_t)^j v|_{H^{l+1}} + |(\varepsilon \partial_t)^j \nabla \Psi|_{H^l}) \, ds \\ & \lesssim \varepsilon (\|\nabla v\|_{L_t^2 \mathcal{H}^{j+1,l}}^{\frac{1}{2}} \|v\|_{L_t^2 \mathcal{H}^{j+1,l}}^{\frac{1}{2}} + \|v\|_{L_t^2 \mathcal{H}^{j+1,l}}) \cdot \\ & \quad (\|\nabla v\|_{L_t^2 \mathcal{H}^{j,l+1}}^{\frac{1}{2}} \|v\|_{L_t^2 \mathcal{H}^{j,l+1}}^{\frac{1}{2}} + \|v\|_{L_t^2 \mathcal{H}^{j,l+1}} + \|u, \operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,l}}) \\ & \lesssim \varepsilon^2 \|\nabla v\|_{L_t^2 H_{co}^m}^2 + \varepsilon \|(u, \operatorname{div} u)\|_{L_t^2 H_{co}^{m-1}}^2 + \|\nabla v\|_{L_t^2 \mathcal{H}^{j+1,l-1} \cap L_t^2 \mathcal{H}^{j,l}}^2 + \|v\|_{L_t^2 \mathcal{H}^m}^2 \end{aligned} \quad (2.84)$$

Inserting (2.83) and (2.84) into (2.82), and using (2.55), (2.56) to find

$$\varepsilon^2 \|(\nabla q, f)\|_{L_t^2 H_{co}^{m-1}}^2 \lesssim \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}^2,$$

we obtain (2.70) by induction. \square

2.4. Step 3: Uniform estimates for $(\nabla \sigma, \operatorname{div} u)$

In this subsection, we aim to get uniform control of higher spatial conormal derivatives of $(\nabla \sigma, \operatorname{div} u)$. More precisely, we prove uniform boundedness of $\|(\nabla \sigma, \operatorname{div} u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}$. This will be achieved by using the equation iteratively.

Lemma 2.12. *Assume that (1.24) holds, we then have that for every $0 < t \leq T$,*

$$\|(\nabla \sigma, \operatorname{div} u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}^2 \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \mathcal{E}_{m,T}^2 \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right). \quad (2.85)$$

Proof. We will prove the following two inequalities:

- $L_t^2 H_{co}^{m-1}$ estimate: for any $j, k \geq 0, j + k \leq m - 1$:

$$\begin{aligned} \|(\nabla \sigma, \operatorname{div} u)\|_{L_t^2 \mathcal{H}^{j,k}} & \lesssim Y_m(\sigma_0, u_0) + T^{\frac{1}{2}} \|(u, \sigma)\|_{L_t^\infty \mathcal{H}^m} \\ & \quad + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} + (T + \varepsilon)^{\frac{1}{4}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,T}. \end{aligned} \quad (2.86)$$

- $L_t^\infty H_{co}^{m-2}$ estimate: for any $j, l \geq 0$ and $j + l \leq m - 2$:

$$\begin{aligned} \|(\nabla \sigma, \operatorname{div} u)\|_{L_t^\infty \mathcal{H}^{j,l}} & \lesssim Y_m(\sigma_0, u_0) + \varepsilon \|(\nabla \operatorname{div} u, \operatorname{curl} \omega)\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}} \\ & \quad + \|(\sigma, u)\|_{L_t^\infty \mathcal{H}^{m-1}} + \varepsilon \|\nabla \sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,T}. \end{aligned} \quad (2.87)$$

These two inequalities, combined with the estimates (2.7), (2.57), (2.66) and the definition (2.46), yield (2.85).

The inequality (2.86) can be obtained by induction on the number of space conormal derivatives. Let us first prove (2.86) for $k = 0, j \leq m - 1$. By (2.50) and product estimate (7.1), we find that:

$$\|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{m-1}} \lesssim T^{\frac{1}{2}} \|\sigma\|_{L_t^\infty \mathcal{H}^m} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \quad (2.88)$$

Moreover, by the equations (1.15)₂ for u ,

$$\nabla \sigma = -\bar{\rho} \varepsilon \partial_t u + \varepsilon f - \varepsilon \mu \operatorname{curl} \omega + \varepsilon (2\mu + \lambda) \nabla \operatorname{div} u, \quad (2.89)$$

we thus have by (2.55), (2.66) that:

$$\begin{aligned} \|\nabla \sigma\|_{L_t^2 \mathcal{H}^{m-1}} &\lesssim \|u\|_{L_t^2 \mathcal{H}^m} + \varepsilon \|\operatorname{curl} \omega\|_{L_t^2 \mathcal{H}^{m-1}} + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \\ &\lesssim T^{\frac{1}{2}} \|u\|_{L_t^\infty \mathcal{H}^m} + \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{m-1}} + Y_m(\sigma_0, u_0) \\ &\quad + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} + (T + \varepsilon)^{\frac{1}{4}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}, \end{aligned} \quad (2.90)$$

which, together with (2.88), yields (2.86) for $k = 0, j \leq m - 1$.

Now suppose that (2.86) holds for $k = k_0 - 1$ with $k_0 \geq 1$, it suffices to prove that it is also true for $k = k_0$ and for every j such that $j + k_0 \leq m - 1$. We begin with the estimate of $\operatorname{div} u$, which again follows from the equation (2.50) and product estimate (7.1):

$$\begin{aligned} \|\operatorname{div} u\|_{L_t^2 \mathcal{H}^{j,k_0}} &\lesssim \|\varepsilon \partial_t \sigma\|_{L_t^2 \mathcal{H}^{j,k_0}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \\ &\lesssim \|(\sigma, \nabla \sigma)\|_{L_t^2 \mathcal{H}^{j+1,k_0-1}} + \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \lesssim \text{R.H.S. of (2.86)}. \end{aligned} \quad (2.91)$$

Next, one gets by equation (2.89), estimate (2.66) and the induction hypothesis that:

$$\begin{aligned} \|\nabla \sigma\|_{L_t^2 \mathcal{H}^{j,k_0}} &\lesssim \|u\|_{L_t^2 \mathcal{H}^{j+1,k_0}} + \varepsilon \|\operatorname{curl} \omega\|_{L_t^2 \mathcal{H}^{j,k_0}} + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,t} \\ &\lesssim \|(\operatorname{div} u, \nabla v)\|_{L_t^2 \mathcal{H}^{j+1,k_0-1}} + \varepsilon \|\operatorname{curl} \omega\|_{L_t^2 \mathcal{H}^{j,k_0}} + \varepsilon \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \mathcal{E}_{m,t} \\ &\lesssim \text{R.H.S. of (2.86)}. \end{aligned}$$

Let us switch to the proof of (2.87). By similar argument as in the derivation of (2.88), (2.90), one can find that:

$$\|(\nabla \sigma, \operatorname{div} u)\|_{L_t^\infty \mathcal{H}^{m-2}} \lesssim \|(\sigma, u)\|_{L_t^\infty \mathcal{H}^{m-1}} + \varepsilon \|(\nabla \operatorname{div} u, \operatorname{curl} \omega)\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}, \quad (2.92)$$

which proves (2.87) for $l = 0$. Suppose that it is true for $l = l_0 - 1 \leq m - 3$, we show that it also holds for $l = l_0$ and for any j , such that $j + l_0 \leq m - 2$. Let us start with the estimate of $\operatorname{div} u$. It follows from the equation (2.50), the product estimate (7.1) and the induction hypothesis that:

$$\begin{aligned} \|\operatorname{div} u\|_{L_t^\infty \mathcal{H}^{j,l_0}} &\lesssim \|\varepsilon \partial_t \sigma\|_{L_t^\infty \mathcal{H}^{j,l_0}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \\ &\lesssim \|(\sigma, \nabla \sigma)\|_{L_t^\infty \mathcal{H}^{j+1,l_0-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \end{aligned}$$

$$\begin{aligned} &\lesssim \|\sigma\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\nabla \sigma\|_{L_t^\infty \mathcal{H}^{j+1, l_0-1}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t} \\ &\lesssim \text{R.H.S. of (2.87)}. \end{aligned}$$

For the estimate of $\nabla \sigma$, we use the equation (2.89) and the product estimate (7.1) to obtain:

$$\begin{aligned} &\|\nabla \sigma\|_{L_t^\infty \mathcal{H}^{j, l_0}} \\ &\lesssim \|\varepsilon \partial_t u\|_{L_t^\infty \mathcal{H}^{j, l_0}} + \varepsilon \|(\nabla \operatorname{div} u, \operatorname{curl} \omega)\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}. \end{aligned} \quad (2.93)$$

It remains to bound $\|\varepsilon \partial_t u\|_{L_t^\infty \mathcal{H}^{j, l_0}}$. We use that for $j + l_0 \leq m - 2$,

$$\begin{aligned} \|\varepsilon \partial_t u\|_{L_t^\infty \mathcal{H}^{j, l_0}} &\lesssim \|v\|_{L_t^\infty H_{co}^{m-1}} + \|(\nabla \Psi, \nabla^2 \Psi)\|_{L_t^\infty \mathcal{H}^{j+1, l_0-1}} \\ &\lesssim \|v\|_{L_t^\infty H_{co}^{m-1}} + \|(u, \operatorname{div} u)\|_{L_t^\infty \mathcal{H}^{j+1, l_0-1}} \\ &\lesssim \|u\|_{L_t^\infty \mathcal{H}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}} + \sum_{k=1}^{l_0} \|\operatorname{div} u\|_{L_t^\infty \mathcal{H}^{j+k, l_0-k}}. \end{aligned} \quad (2.94)$$

Plugging (2.48) and (2.94) into (2.93) and using the induction hypothesis, we get that:

$$\|\nabla \sigma\|_{L_t^\infty \mathcal{H}^{j, l_0}} \lesssim \text{R.H.S. of (2.87)}.$$

We thus proved that (2.87) holds for $j + 1, l_0$ which ends the proof. \square

Remark 2.13. By Lemmas 2.10, 2.12, we get that:

$$\|(\sigma, u)\|_{E_t^m}^2 \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right). \quad (2.95)$$

2.5. Step 4: Uniform estimates for the gradient of the velocity

In this section, we will bound $\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}$, which, combined with (2.3), (2.87), gives the control of $\|\nabla u\|_{L_t^\infty H_{co}^{m-2}}$.

Lemma 2.14. *Suppose that (1.24) holds, then for any $0 < t \leq T$, we have the following estimate,*

$$\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}^2 \lesssim Y_m^2(\sigma_0, u_0) + \|v\|_{L_t^\infty H_{co}^{m-1}}^2 + T^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right). \quad (2.96)$$

Proof. Since in the interior domain, the conormal spatial derivatives are equivalent to the standard spatial derivatives, we only have to estimate ∇v near the boundary, say $\|\chi_i \nabla v\|_{L_t^\infty H_{co}^{m-2}}$ where $\chi_i, (i = 1 \cdots N)$ are smooth functions associated to the covering (1.11) and are compactly supported in Ω_i . Close to the boundary, it follows from the identity (2.77) and the following identity

$$\Pi(\partial_{\mathbf{n}} v) = \Pi((\nabla v - Dv)\mathbf{n}) + \Pi((Dv)\mathbf{n}) = \Pi(\omega \times \mathbf{n}) + \Pi \nabla(v \cdot \mathbf{n}) - \Pi((D\mathbf{n})v)$$

that:

$$\begin{aligned} \|\chi_i \nabla v\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \|\chi_i \Pi(\partial_{\mathbf{n}} v)\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}} \\ &\lesssim \|\chi_i(\omega \times \mathbf{n})\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}}. \end{aligned}$$

We thus reduce the problem to the estimate of $\chi_i(\omega \times \mathbf{n})$, which is the aim of the following lemma. \square

Lemma 2.15. *Under the assumption (1.24), the following estimate holds: for every $0 < t \leq T$,*

$$\|\chi_i(\omega \times \mathbf{n})\|_{L_t^\infty H_{co}^{m-2}(\Omega)}^2 \lesssim \|\chi_i(\omega \times \mathbf{n})(0)\|_{H_{co}^{m-2}}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right), \quad (2.97)$$

where χ_i is a smooth function compactly supported in Ω_i .

Proof. Note that the important feature of $\chi_i(\omega \times \mathbf{n})$ is that: it solves a transport-diffusion system **without** singular terms, with a non-homogeneous Dirichlet boundary condition. In order to perform the estimate, we split the system for $\chi_i(\omega \times \mathbf{n})$ into two parts, one of which just solves the heat equation with the nontrivial Dirichlet boundary condition and a remainder which is amenable to energy estimates since it satisfies a convection-diffusion equation with homogeneous Dirichlet boundary condition. To deal with the first system, the explicit formula for heat equation will play an important role. It is thus helpful to transform the problem to the half-space.

Let us set $\eta_i = \chi_i \omega \times \mathbf{n}$, $i \geq 1$. Direct computations show that ω solves the following system:

$$g_2 \partial_t \omega + g_2 u \cdot \nabla \omega - \mu \Delta \omega = g_2 \omega \cdot \nabla u - g_2 \omega \operatorname{div} u - \frac{\nabla g_2}{\varepsilon} \times (\varepsilon \partial_t u + \varepsilon u \cdot \nabla u) =: G^\omega \quad (2.98)$$

from which we obtain the equations satisfied by η_i (which is compactly supported in Ω_i)

$$\begin{cases} \bar{\rho} \partial_t \eta_i - \mu \Delta \eta_i = F_i^\omega & \text{in } \Omega_i \cap \Omega, \\ \eta_i = \chi_i \Pi(\omega \times \mathbf{n}) = 2\chi_i \Pi(-au + (D\mathbf{n})u) & \text{on } \Omega_i \cap \partial\bar{\Omega}, \end{cases} \quad (2.99)$$

where

$$F_i^\omega =: -\Delta(\chi_i \mathbf{n}) \times \omega - 2\nabla \omega \times \nabla(\chi_i \mathbf{n}) - (g_2 u \cdot \nabla \omega) \times (\chi_i \mathbf{n}) + \frac{\bar{\rho} - g_2}{\varepsilon} \varepsilon \partial_t \omega \times (\chi_i \mathbf{n}) + G^\omega \times (\chi_i \mathbf{n}).$$

Since we will use the local coordinate (1.12), it is useful to know the expressions of Laplacian in this new coordinates. By direct computation, we find that:

$$(\nabla f) \circ \Phi_i = P \nabla(f \circ \Phi_i), \quad (\operatorname{div} F) \circ \Phi_i = \operatorname{div}(P^*(F \circ \Phi_i)) \quad (\Delta f) \circ \Phi_i = \operatorname{div}(E \nabla(f \circ \Phi_i)) \quad (2.100)$$

where $\nabla = (\partial_{y^1}, \partial_{y^2}, \partial_z)^t$, $\operatorname{div} = (\nabla)^*$ represent the gradient and the divergence in the new coordinates and

$$\begin{pmatrix} 1 & 0 & -\partial_{y^1} \varphi_i \\ 0 & 1 & -\partial_{y^2} \varphi_i \\ 0 & 0 & 1 \end{pmatrix}, \quad E = P^* P = \begin{pmatrix} 1 & 0 & -\partial_{y^1} \varphi_i \\ 0 & 1 & -\partial_{y^2} \varphi_i \\ -\partial_{y^1} \varphi_i & -\partial_{y^2} \varphi_i & |\mathbf{N}|^2 \end{pmatrix}. \quad (2.101)$$

Let us set $\tilde{\eta}_i(t, y, z) = \eta_i(t, \Phi_i(y, z)) := (\eta_i \circ \Phi_i)(y, z)$, $(y, z) \in \Phi_i^{-1}(\Omega_i \cap \bar{\Omega})$. Denote also $\tilde{F}_i^\omega = F_i^\omega \circ \Phi_i$. Since $\operatorname{Supp} \chi_i|_{\bar{\Omega}} \Subset \Omega_i \cap \bar{\Omega}$, we can extend the definition of $\tilde{\eta}_i$ and \tilde{F}_i^ω from $\Phi_i^{-1}(\Omega_i \cap \bar{\Omega})$ to \mathbb{R}_+^3 by zero extension, which are still denoted by $\tilde{\eta}_i$, \tilde{F}_i^ω . Consequently, by (2.99) and (2.100), we find that $\tilde{\eta}_i$ satisfies:

$$\begin{cases} \bar{\rho} \partial_t \tilde{\eta}_i - \mu \operatorname{div}(E \nabla \tilde{\eta}_i) = F_{ni}^\omega & \text{in } \mathbb{R}_+^3, \\ \tilde{\eta}_i|_{z=0} = 2[\chi_i \Pi(-au + (D\mathbf{n})u)] \circ \Phi_i|_{z=0}. \end{cases} \quad (2.102)$$

Let us set $\mathcal{Z}_0 = \varepsilon \partial_t$, $\mathcal{Z}_j = \partial_{y^j}$, $j = 1, 2$, $\mathcal{Z}_3 = \phi(z) \partial_z$ and define

$$\|\tilde{\eta}_i\|_{m,t} = \sum_{|\alpha| \leq m} \|\mathcal{Z}^\alpha \tilde{\eta}_i\|_{L^2([0,t] \times \mathbb{R}_+^3)}, \quad \|\tilde{\eta}_i(t)\|_m = \sum_{|\alpha| \leq m} \|(\mathcal{Z}^\alpha \tilde{\eta}_i)(t)\|_{L^2(\mathbb{R}_+^3)}, \quad (2.103)$$

where $\mathcal{Z}^\alpha = \mathcal{Z}_0^{\alpha_0} \mathcal{Z}_1^{\alpha_1} \mathcal{Z}_2^{\alpha_2} \mathcal{Z}_3^{\alpha_3}$, $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, by the definition of the conormal spaces (1.6) and the vector fields (1.13) we find that:

$$\|\tilde{\eta}_i\|_{m,t} \approx \|\eta_i\|_{L_t^2 H_{co}^m(\Omega)}, \quad \|\tilde{\eta}_i(t)\|_m \approx \|\eta_i(t)\|_{H_{co}^m(\Omega)}. \quad (2.104)$$

Therefore, our following task is to establish an estimate for $\sup_{0 \leq t \leq T} \|\tilde{\eta}_i(t)\|_{m-2}$.

We shall write $\tilde{\eta}_i, \tilde{F}_i^\omega$ by $\tilde{\eta}, \tilde{F}^\omega$ for the sake of notational clarity. We write $\tilde{\eta} = \tilde{\eta}_h + \tilde{\eta}_{nh}$, where $\tilde{\eta}_h$ solves

$$\begin{cases} \bar{\rho} \partial_t \tilde{\eta}_h - \mu |\mathbf{N}|^2 \partial_z^2 \tilde{\eta}_h = 0 & \text{in } \mathbb{R}_+^3, \\ \tilde{\eta}_h|_{t=0} = 0, \tilde{\eta}_h|_{z=0} = \tilde{\eta}|_{z=0} \end{cases} \quad (2.105)$$

while $\tilde{\eta}_{nh}$ satisfies

$$\begin{cases} \bar{\rho} \partial_t \tilde{\eta}_{nh} - \mu \operatorname{div}(E \nabla \tilde{\eta}_{nh}) = H(\tilde{\eta}_h) + F^\omega & \text{in } \mathbb{R}_+^3, \\ \tilde{\eta}_{nh}|_{t=0} = \tilde{\eta}|_{t=0}, \tilde{\eta}_{nh}|_{z=0} = 0 \end{cases} \quad (2.106)$$

where

$$H(\tilde{\eta}_h) = \mu \sum_{i,j=1}^2 \partial_{y^i} (E_{ij} \partial_{y^j} \tilde{\eta}_h) + \mu \sum_{i=1}^2 \partial_{y^i} (E_{i3} \partial_z \tilde{\eta}_h) + \partial_z (E_{3i} \partial_{y^i} \tilde{\eta}_h).$$

Estimate (2.97) will be the consequence of the following two lemmas. \square

Lemma 2.16. *Adopting the notation introduced in (2.103), we have the following estimate: for any $0 < t \leq T$,*

$$\sup_{0 \leq t \leq T} \|\tilde{\eta}_h(t)\|_{m-2} + \|\tilde{\eta}_h\|_{m-1,T} \lesssim T^{\frac{1}{4}} \mathcal{E}_{m,T}. \quad (2.107)$$

Proof. Since $|\mathbf{N}|^2$ depends only on the tangential variable y^1, y^2 , the equation (2.105) can be seen as a heat equation on the half line with Dirichlet boundary condition, which can be solved explicitly:

$$\tilde{\eta}_h(t, y, z) = -2\tilde{\mu} \int_0^t \frac{|\mathbf{N}|^2}{(4\pi\tilde{\mu}|\mathbf{N}|^2(t-s))^{\frac{1}{2}}} \partial_z (e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2(t-s)}}) \tilde{\eta}|_{z=0}(s, y) ds$$

where $\tilde{\mu} = \mu/\bar{\rho}$. Taking a multi-index $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, since time derivation commutes with ∂_t, ∂_z^2 , we have that:

$$((\varepsilon \partial_t)^{\gamma_0} \tilde{\eta}_h)(t, y, z) = -2\tilde{\mu} \int_0^t \frac{|\mathbf{N}|^2}{(4\pi\tilde{\mu}|\mathbf{N}|^2(t-s))^{\frac{1}{2}}} \partial_z (e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2(t-s)}}) ((\varepsilon \partial_t)^{\gamma_0} \tilde{\eta})|_{z=0}(s, y) ds,$$

which, combined with (7.16) established in the appendix, yields that:

$$\|\mathcal{Z}^\gamma \tilde{\eta}_h(t)\|_{L_{y,z}^2(\mathbb{R}_+^3)} \lesssim \int_0^t (t-s)^{-\frac{3}{4}} |\tilde{\eta}|_{z=0}(s) |_{\tilde{H}^{|\gamma|}(\mathbb{R}_y^2)} ds. \quad (2.108)$$

The above inequality, combined with the boundary condition (2.102)₂ and the trace inequality (7.9), yields that:

$$\|\tilde{\eta}_h(t)\|_{m-2} \lesssim T^{\frac{1}{4}} \sup_{0 \leq s \leq t} |\tilde{\eta}(s)|_{\tilde{H}^{m-2}(\mathbb{R}_y^2)} \lesssim T^{\frac{1}{4}} \|(u, \nabla u)\|_{L_t^\infty H_{co}^{m-2}} \lesssim T^{\frac{1}{4}} \mathcal{E}_{m,T}.$$

Similarly, we apply a convolution inequality in the time variable (after extending $\tilde{\eta}(s)|_{z=0}$ to $s \in \mathbb{R}$ by zero extension) to (2.108), and use the boundary condition (2.102)₂ and the trace inequality (7.10) to obtain:

$$\|\tilde{\eta}_h(t)\|_{m-1,t} \lesssim T^{\frac{1}{4}} \|\tilde{\eta}\|_{L_t^2 \tilde{H}^{m-1}(\mathbb{R}_y^2)} \lesssim T^{\frac{1}{4}} \|(u, \nabla u)\|_{L_t^2 H_{co}^{m-1}} \lesssim T^{\frac{1}{4}} \mathcal{E}_{m,T}. \quad \square$$

Lemma 2.17. *Using the notation (2.103), the following energy inequality holds: for any $0 < t \leq T$,*

$$\|\tilde{\eta}_{nh}(t)\|_{m-2}^2 + \|\nabla \tilde{\eta}_{nh}\|_{m-2,t}^2 \lesssim \|\eta(0)\|_{H_{co}^{m-2}}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right). \quad (2.109)$$

Proof. Suppose that $0 \leq |\gamma| = k \leq m - 2$. Denote $\tilde{\eta}_{nh}^\gamma = \mathcal{Z}^\gamma \tilde{\eta}_{nh}$, then $\tilde{\eta}_{nh}^\gamma$ solves the system (note that $[\mathcal{Z}^\gamma, E] = 0$):

$$\begin{aligned} \bar{\rho} \partial_t \tilde{\eta}_{nh}^\gamma - \mu \operatorname{div}(E \nabla \tilde{\eta}_{nh}^\gamma) &= \mu [\mathcal{Z}^\gamma, \operatorname{div}](E \nabla \tilde{\eta}_{nh}^\gamma) + \mu \mathcal{Z}^\gamma H(\tilde{\eta}_h) + \mathcal{Z}^\gamma F^\omega \\ &=: \mathcal{R}_1^\gamma + \mathcal{R}_2^\gamma + \mathcal{Z}^\gamma F^\omega \end{aligned}$$

with the initial condition $\tilde{\eta}_{nh}^\gamma|_{t=0} = \mathcal{Z}^\gamma \tilde{\eta}|_{t=0}$ and the boundary condition $\tilde{\eta}_{nh}^\gamma|_{z=0} = 0$.

Standard energy estimates show that:

$$\begin{aligned} \bar{\rho} \|\tilde{\eta}_{nh}^\gamma(t)\|_{L^2(\mathbb{R}_+^3)}^2 + \int_0^t \int_{\mathbb{R}_+^3} E \nabla \tilde{\eta}_{nh}^\gamma \cdot \nabla \tilde{\eta}_{nh}^\gamma \, dx ds &= \bar{\rho} \|\tilde{\eta}_{nh}^\gamma(0)\|_{L^2(\mathbb{R}_+^3)}^2 + \int_0^t \int_{\mathbb{R}_+^3} (\mathcal{R}_1^\gamma + \mathcal{R}_2^\gamma + \mathcal{Z}^\gamma \widetilde{F^\omega}) \tilde{\eta}_{nh}^\gamma \, dx ds. \end{aligned} \quad (2.110)$$

At first, since we can find some $\kappa > 0$, such that $2|\mathbf{N}|^2 \leq 1/\kappa$, one has that $EX \cdot X = |PX|^2 \geq \frac{1}{2|\mathbf{N}|^2} |X|^2 \geq \kappa |X|^2$ and hence, we deduce that:

$$\int_0^t \int_{\mathbb{R}_+^3} E \nabla \tilde{\eta}_{nh}^\gamma \cdot \nabla \tilde{\eta}_{nh}^\gamma \, dx ds \geq \kappa \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2. \quad (2.111)$$

For the second term of the right hand side of (2.110), one needs to integrate by parts to avoid involving additional normal derivatives. Let us first study \mathcal{R}_1^γ which vanishes if $|\gamma| = 0$. By induction, one gets that for $k = |\gamma| \geq 1$,

$$[\mathcal{Z}^\gamma, \operatorname{div}] = [\mathcal{Z}^\gamma, \partial_z] = \sum_{\beta < \gamma} C_{\phi,\beta,\gamma} \partial_z \mathcal{Z}^\beta \quad (2.112)$$

where $C_{\phi,\beta,\gamma}$ are smooth functions that depend on ϕ and its derivatives. Consequently, by integration by parts and Young's inequality, we obtain that:

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{R}_1^\gamma \cdot \tilde{\eta}_{nh}^\gamma \, dx ds \leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta (\|\nabla \tilde{\eta}_{nh}\|_{k-1,t}^2 + \|\tilde{\eta}_{nh}\|_{k,t}^2). \quad (2.113)$$

Similarly, by taking benefits of the zero boundary condition of $\tilde{\eta}_{nh}^\gamma$, one integrates by parts to get:

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{R}_2^\gamma \tilde{\eta}_{nh}^\gamma \, dx ds \leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta (\|\tilde{\eta}_h\|_{k+1,t}^2 + \|\tilde{\eta}_{nh}\|_{k,t}^2). \quad (2.114)$$

We are now left to deal with the term:

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma \widetilde{F}^\omega \tilde{\eta}_{nh}^\gamma \, dx ds = \sum_{j=1}^5 \int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma \widetilde{F}_j^\omega \tilde{\eta}_{nh}^\gamma \, dx dt =: \sum_{j=1}^5 \mathcal{I}_j \quad (2.115)$$

where we denote that:

$$\begin{aligned} \widetilde{F}^\omega &= -\widetilde{\Delta}(\widetilde{\chi_i \mathbf{n}}) \times \widetilde{\omega} - 2\widetilde{\nabla \omega} \times \widetilde{\nabla(\chi_i \mathbf{n})} - (g_2 \widetilde{u} \cdot \widetilde{\nabla \omega}) \times \widetilde{(\chi_i \mathbf{n})} + \frac{\widetilde{(\rho - g_2)}}{\varepsilon} \widetilde{\varepsilon} \widetilde{\partial_t \omega} \times \widetilde{(\chi_i \mathbf{n})} + \widetilde{G^\omega} \times \widetilde{(\chi_i \mathbf{n})}. \\ &=: \widetilde{F}_1^\omega + \widetilde{F}_2^\omega + \widetilde{F}_3^\omega + \widetilde{F}_4^\omega + \widetilde{F}_5^\omega. \end{aligned}$$

Note that G^ω is defined in (2.98). Moreover, without much ambiguity, we denote \tilde{f} as $(\tilde{\chi}_i f) \circ \Phi_i$ where $\tilde{\chi}_i$ is a smooth function such that $\tilde{\chi}_i \chi_i = \chi_i$.

By the Cauchy-Schwarz inequality and the fact (2.104), \mathcal{I}_1 can be controlled by:

$$\mathcal{I}_1 \lesssim \|\tilde{\omega}\|_{k,t} \|\tilde{\eta}_{nh}\|_{k,t} \lesssim T^{\frac{1}{2}} \|\nabla u\|_{L_t^\infty H_{co}^{m-2}} \|\tilde{\eta}_{nh}\|_{k,t}. \quad (2.116)$$

Nevertheless, for \mathcal{I}_2 and \mathcal{I}_3 , as $\widetilde{F}_2^\omega, \widetilde{F}_3^\omega$ involve normal derivatives of ω , it is necessary to use integration by parts. By doing so, we can bound the term \mathcal{T}_2 as follows:

$$\mathcal{I}_2 \leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta (\|\tilde{\eta}_{nh}\|_{k,t}^2 + \|\widetilde{\nabla u}\|_{k,t}^2). \quad (2.117)$$

Next, for \mathcal{I}_3 , by noticing the expression

$$\begin{aligned} \widetilde{g_2 u} \cdot \widetilde{\nabla \omega} &= \partial_{y^1} (\widetilde{g_2 u}_1 \widetilde{\omega}) + \partial_{y^2} (\widetilde{g_2 u}_2 \widetilde{\omega}) + \partial_z ((\widetilde{g_2 u} \cdot \mathbf{N}) \widetilde{\omega}) \\ &\quad - (\partial_{y^1} \widetilde{g_2 u}_1 + \partial_{y^2} \widetilde{g_2 u}_2 + \partial_z (\widetilde{g_2 u} \cdot \mathbf{N})) \widetilde{\omega}, \end{aligned}$$

one performs an integration by parts again to get that:

$$\begin{aligned} \mathcal{I}_3 &\lesssim \|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t} + \|\tilde{\omega} (\partial_{y^j} (\widetilde{g_2 u}_j), \partial_z (\widetilde{g_2 u} \cdot \mathbf{N}))\|_{k,t} \|\tilde{\eta}_{nh}^\gamma\|_{0,t} \\ &\leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta \|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} + T^{\frac{1}{2}} (\sup_{s \in [0,t]} \|\tilde{\eta}_{nh}(s)\|_k) \|\tilde{\omega} (\partial_{y^j} (\widetilde{g_2 u}_j), \partial_z (\widetilde{g_2 u} \cdot \mathbf{N}))\|_{k,t} \end{aligned}$$

Here we used Einstein summation convention for $j = 1, 2$. By (2.104), (2.107) and the assumption $k \leq m-2$, one can have that:

$$\sup_{s \in [0,t]} \|\tilde{\eta}_{nh}\|_k \lesssim \sup_{s \in [0,t]} \|(\tilde{\eta}, \tilde{\eta}_h)(s)\|_k \lesssim \|\nabla u\|_{L_t^\infty H_{co}^{m-2}} + T^{\frac{1}{4}} \mathcal{E}_{m,t} \lesssim \mathcal{E}_{m,t}. \quad (2.118)$$

Moreover, since $k \leq m-2$, we have thanks to (2.104) that:

$$\begin{aligned} &\|\tilde{\omega} (\partial_{y^j} (\widetilde{g_2 u}_j), \partial_z (\widetilde{g_2 u} \cdot \mathbf{N}))\|_{m-2,t} \\ &\lesssim \|\omega\|_{0,\infty,t} \|Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})\|_{L_t^2 H_{co}^{m-2}} \\ &\quad + \|\omega\|_{L_t^\infty H_{co}^{m-2}} \left(\int_0^t \| [Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})] (s) \|_{m-3,\infty}^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

where Z_i stands for the tangential vector fields in Ω_i . By identity (2.22) and the Sobolev embedding (7.7) and estimate (2.47),

$$\begin{aligned} \left(\int_0^t \|Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})(s)\|_{m-3,\infty}^2 ds \right)^{\frac{1}{2}} &\lesssim \|u\|_{E_t^m} + \|\nabla \operatorname{div} u\|_{L_t^2 H_{co}^{m-2}} + \varepsilon \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right) \\ &\lesssim \|(\sigma, u)\|_{E_t^m} + \varepsilon^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right), \end{aligned}$$

which together with the previous inequality, yields:

$$\|\tilde{\omega}(\partial_{y^j}(\tilde{g_2} \tilde{u}_j), \partial_z(\tilde{g_2} u \cdot \mathbf{N}))\|_{m-2,t} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right).$$

Similarly, we have that:

$$\|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} \lesssim T^{\frac{1}{2}} \|\omega\|_{0,\infty,t} \|u\|_{L_t^\infty H_{co}^{m-2}} + \|u\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}.$$

Moreover, if $k \leq [\frac{m}{2}] - 2$,

$$\|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \|\nabla u\|_{L_t^2 H_{co}^{[\frac{m}{2}]-2}} \lesssim T^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}.$$

To summarize, we control \mathcal{T}_3 (defined in (2.115)) as follows:

$$\mathcal{T}_3 \leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right), \quad \text{if } k \leq [\frac{m}{2}] - 2, \quad (2.119)$$

and for $k \leq m - 2$,

$$\mathcal{T}_3 \leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right) + \|(\sigma, u)\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}}. \quad (2.120)$$

For \mathcal{I}_4 , the direct application of the Hölder inequality requires the control of the quantity $\|\frac{(\tilde{\rho} - \tilde{g}_2)}{\varepsilon} \tilde{\varepsilon} \tilde{\partial}_t \omega\|_{k,t}$, which further requires the estimate of $L_{t,x}^\infty$ type norm of $\partial_t \omega$. However, $\|\varepsilon \partial_t \omega\|_{\infty,t}$ (or $\|\nabla u\|_{1,\infty,t}$) does not appear in the $L_{t,x}^\infty$ type norms present in $\mathcal{A}_{m,T}$. To avoid this problem, since $\tilde{\varepsilon} \tilde{\partial}_t \omega = (P \nabla) \times \tilde{\varepsilon} \tilde{\partial}_t u$, we can integrate by parts in space before using product estimate. By doing so, we achieve that:

$$\begin{aligned} \mathcal{I}_4 &\leq \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta \|\tilde{\eta}_{nh}\|_{k,t}^2 + \|(\tilde{\nabla} \sigma, \tilde{\varepsilon} \tilde{\partial}_t u)\|_{k,t}^2 \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \\ &\lesssim \delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta \|\tilde{\eta}_{nh}\|_{k,t}^2 + T \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,t}\right) \mathcal{E}_{m,t}^2. \end{aligned} \quad (2.121)$$

Finally, regarding the term \mathcal{T}_5 (defined in (2.115)) we control it by Cauchy-Schwarz inequality as:

$$\mathcal{T}_5 \lesssim T^{\frac{1}{2}} \left(\sup_{s \in [0,t]} \|\tilde{\eta}_{nh}(s)\|_k \right) \|\tilde{G}^\omega\|_{k,t}.$$

By the estimate (2.118), the fact (2.104) and the Proposition 2.18, we get that:

$$\mathcal{T}_5 \lesssim (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right). \quad (2.122)$$

To summarize, we have found by collecting (2.116)-(2.121) that for $0 \leq k \leq m-2$,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma \widetilde{F}^\omega \tilde{\eta}_{nh}^\gamma \, dxdt \leq 3\delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 \\ & + C_\delta \left(\|(\tilde{\eta}, \tilde{\eta}_h)\|_{k,t}^2 + \|u\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} \right) + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right) \\ & \leq 3\delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta \|u\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right), \end{aligned} \quad (2.123)$$

and also for $0 \leq k \leq [\frac{m}{2}] - 2$,

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma \widetilde{F}^\omega \tilde{\eta}_{nh}^\gamma \, dxdt \leq 3\delta \|\nabla \tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right). \quad (2.124)$$

Inserting (2.113)-(2.114), (2.123)-(2.124) in (2.110), we obtain by choosing δ small enough that for any $0 \leq k \leq m-2$,

$$\begin{aligned} & \|\tilde{\eta}_{nh}(t)\|_k^2 + \|\nabla \tilde{\eta}_{nh}\|_{k,t}^2 \lesssim \|\eta(0)\|_{H_{co}^k}^2 + \|\nabla \tilde{\eta}_{nh}\|_{k-1,t}^2 \\ & + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right) + \|(\sigma, u)\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} \mathbb{I}_{\{k \geq [\frac{m}{2}]-1\}}, \end{aligned} \quad (2.125)$$

where the convention $\|\cdot\|_{l,t} = 0$ if $l < 0$ is used. We thus get by induction on $0 \leq k \leq [\frac{m}{2}] - 2$ that:

$$\|\tilde{\eta}_{nh}(t)\|_{[\frac{m}{2}]-2}^2 + \|\nabla \tilde{\eta}_{nh}\|_{[\frac{m}{2}]-2,t}^2 \lesssim \|\eta(0)\|_{H_{co}^{m-2}}^2 + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right), \quad (2.126)$$

which, together with (2.107) and (2.85) gives that:

$$\|\nabla u\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}}^2 \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right).$$

We then combine this estimate and (2.95) to obtain that:

$$\|u\|_{E_t^m} \|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right).$$

Therefore, we take benefits of the estimate (2.125) and the induction arguments to get (2.109). \square

Proposition 2.18. *Assume that (1.23) holds and let*

$$G^\omega = g_2 \omega \cdot \nabla u - g_2 \omega \operatorname{div} u - \frac{\nabla g_2}{\varepsilon} \times (\varepsilon \partial_t u + \varepsilon u \cdot \nabla u),$$

then we have:

$$\|\tilde{\chi}_i G^\omega\|_{L_t^2 H_{co}^{m-2}} \lesssim \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,t} \right).$$

Proof. Let us show the estimate of $\tilde{\chi}_i \omega \cdot \nabla u$, which is not direct since the higher order $L_{t,x}^\infty$ norm is not included in $\mathcal{A}_{m,T}$. Nevertheless, thanks to identity (2.35), one can write this term as:

$$\tilde{\chi}_i \omega \cdot \nabla u = \tilde{\chi}_i (\omega_1 \partial_{y^1} u + \omega_2 \partial_{y^2} u + (\omega \cdot \mathbf{N}) \partial_{\mathbf{n}} u).$$

Moreover, by identities (2.59) and (2.22),

$$\begin{aligned} \omega \cdot \mathbf{N} &= (\nabla \times u) \cdot \mathbf{N} \\ &= -(u \times \mathbf{N}) \partial_{\mathbf{n}} \mathbf{N} + (\Pi \partial_{y^1} (u \times \mathbf{N}))^1 + (\Pi \partial_{y^2} (u \times \mathbf{N}))^2 + u \cdot \operatorname{curl} \mathbf{N} \end{aligned}$$

which gives that for any $t \in [0, T]$, any $k \geq 0$,

$$\|(\omega \cdot \mathbf{N})(t)\|_{H_{co}^k} \lesssim \|u(t)\|_{H_{co}^{k+1}}, \quad \|(\omega \cdot \mathbf{N})(t)\|_{k,\infty} \lesssim \|u(t)\|_{k+1,\infty}$$

Therefore, by the Sobolev embedding (7.7), we have that:

$$\begin{aligned} &\|\tilde{\chi}_i \omega \cdot \nabla u\|_{L_t^2 H_{co}^{m-2}} \\ &\lesssim \|\nabla u\|_{0,\infty,t} \|(\partial_{y^i} u, \omega \cdot \mathbf{N})\|_{L_t^2 H_{co}^{m-2}} + \|\nabla u\|_{L_t^\infty H_{co}^{m-2}} \left(\int_0^t \|(\partial_{y^i} u, \omega \cdot \mathbf{N})(s)\|_{m-3,\infty}^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \|\nabla u\|_{0,\infty,t} \|u\|_{L_t^2 H_{co}^{m-1}} + \|\nabla u\|_{L_t^\infty H_{co}^{m-2}} \|u\|_{E_t^m} \lesssim \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,t}\right). \end{aligned}$$

The other two terms in the definition of G^ω are similar or easier to treat, we omit the details. \square

Remark 2.19. Collecting the results stated in Lemmas 2.4, 2.10, 2.14, 2.12, we find that:

$$\begin{aligned} &\|\varepsilon \nabla(\sigma, u)\|_{L_t^\infty H_{co}^{m-1}}^2 + \|\nabla(\sigma, u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}^2 + \|(\sigma, u)\|_{L_t^\infty H_{co}^{m-1}}^2 \\ &\lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,T}\right). \end{aligned} \quad (2.127)$$

2.6. ε -dependent estimate of $\nabla^2 u$

To finish the estimates for the energy norm, we are left to deal with $\|\varepsilon \nabla^2 u\|_{L_t^\infty H_{co}^{m-2}}, \varepsilon \|\nabla^2 \sigma\|_{L_t^\infty L^2}$.

Lemma 2.20. *Under the assumption (1.24), the following estimate holds:*

$$\|\varepsilon \nabla^2 u(t)\|_{H_{co}^{m-2}}^2 \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \Lambda\left(\frac{1}{c_0}, \mathcal{N}_{m,T}\right). \quad (2.128)$$

Proof. As u satisfies the equation:

$$\varepsilon \mu \Delta u = -(\mu + \lambda) \varepsilon \nabla \operatorname{div} u + g_2(\varepsilon \partial_t u + \varepsilon u \cdot \nabla u) + \nabla \sigma,$$

we have by elliptic regularity theory:

$$\begin{aligned} \|\varepsilon \nabla^2 u(t)\|_{H_{co}^{m-2}} &\lesssim \varepsilon \sum_{|I| \leq m-2} |Z^I \partial_{\mathbf{n}} u(t)|_{H^{\frac{1}{2}}} + \varepsilon \|\nabla \operatorname{div} u(t)\|_{H_{co}^{m-2}} \\ &\quad + \|u(t)\|_{H_{co}^{m-1}} + \|\nabla \sigma(t)\|_{H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}} \mathcal{E}_{m,T} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right). \end{aligned} \quad (2.129)$$

It follows from the boundary condition (1.14), the identity (2.22) and the trace inequality (7.9) that:

$$\varepsilon \sum_{|I| \leq m-2} |Z^I \partial_{\mathbf{n}} u(t)|_{H^{\frac{1}{2}}} \lesssim \varepsilon \|\nabla \operatorname{div} u(t)\|_{H_{co}^{m-2}} + \varepsilon \|(u, \nabla u)(t)\|_{H_{co}^{m-1}}. \quad (2.130)$$

Inserting (2.48) and (2.130) into (2.129), one arrives at:

$$\varepsilon \|\nabla^2 u(t)\|_{H_{co}^{m-2}} \lesssim \varepsilon \|\nabla(\sigma, u)(t)\|_{H_{co}^{m-1}} + \|\nabla \sigma(t)\|_{H_{co}^{m-2}} + \|u(t)\|_{H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}} \mathcal{E}_{m,T} \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,T} \right),$$

which, combined with (2.127) leads to (2.128). \square

Lemma 2.21. *Under the assumption (1.24), we have the following estimate for $\nabla^2 \sigma$:*

$$\|\varepsilon \nabla^2 \sigma\|_{L_t^\infty L^2}^2 + \|\nabla^2 \sigma\|_{L^2(Q_t)}^2 \lesssim Y_m^2(0) + (T + \varepsilon) \Lambda \left(\frac{1}{c_0}, \mathcal{N}_{m,T} \right). \quad (2.131)$$

Proof. By (2.50) and (2.89), one finds that $\nabla \sigma$ solves:

$$\varepsilon^2 g_1 (\partial_t + u \cdot \nabla) \nabla \sigma + \frac{1}{(2\mu + \lambda)} \nabla \sigma = G \quad (2.132)$$

where

$$G = -\varepsilon^2 (g'_1 S \varepsilon \partial_t \sigma + \nabla(g_1 u_k) \cdot \partial_k \sigma) - \varepsilon \frac{\mu}{(2\mu + \lambda)} \operatorname{curl} \omega - \frac{1}{(2\mu + \lambda)} g_2 (\varepsilon \partial_t u + \varepsilon u \cdot \nabla u).$$

By taking the divergence of the equation (2.132), one arrives at:

$$\varepsilon^2 g_1 (\partial_t + u \cdot \nabla) \Delta \sigma + \frac{1}{2\mu + \lambda} \Delta \sigma = \operatorname{div} G - \varepsilon^2 \left[g'_1 \nabla \sigma \cdot \varepsilon \partial_t \nabla \sigma + \sum_{i=1}^3 \partial_i (g_1 u) \cdot \nabla \partial_i \sigma \right] =: \tilde{G} \quad (2.133)$$

From an energy estimate, we find

$$\varepsilon^2 \|\Delta \sigma\|_{L_t^\infty L^2}^2 + \|\Delta \sigma\|_{L^2(Q_t)}^2 \lesssim T^{\frac{1}{2}} \|\Delta \sigma\|_{L^2(Q_t)} \|\tilde{G}\|_{L_t^\infty L^2} + T \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \|\varepsilon \Delta \sigma\|_{L_t^\infty L^2}^2. \quad (2.134)$$

We first observe that:

$$\|\tilde{G}\|_{L_t^\infty L^2} \lesssim \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,t} \right) \mathcal{E}_{m,t}^2.$$

Moreover, since in the local coordinate, we can find some coefficients a_{ij} that depends smoothly on \mathbf{n} , such that (we use the convention $\partial_{y^3} = \partial_{\mathbf{n}}$):

$$\Delta = \partial_{\mathbf{n}}^2 + \sum_{0 \leq i,j \leq 3, (i,j) \neq (3,3)} \partial_{y^i} (a_{ij} \partial_{y^j}) \quad (2.135)$$

which yields:

$$\|\partial_{\mathbf{n}} \nabla \sigma\|_{L_t^\infty L^2} \lesssim \|\Delta \sigma\|_{L_t^\infty L^2} + \|\nabla \sigma\|_{L_t^\infty H_{co}^1}.$$

We thus obtain (2.131) from (2.134). \square

2.7. Proof of Proposition 2.1

By collecting (2.7), (2.127), (2.128) and (2.131), we get (2.1).

Remark 2.22. In view of the formal expansion (1.5), one expects the first three normal derivatives of σ to be bounded in $L^2(Q_t)$. This can be achieved in the following way. By imposing additional assumption on σ_0 , namely $\varepsilon \nabla^2 \sigma_0 \in H_{co}^1(\Omega)$, $\nabla^3 \sigma_0 \in L^2(\Omega)$, one can show by following similar computations as in the proof of Lemma 2.21 that: $\varepsilon \nabla^2 \sigma \in L_t^\infty H_{co}^1$, $\nabla \sigma \in L_t^2 H_{co}^1$. These estimates at hand, one can carry out another energy estimate to control $\|\partial_n \Delta \sigma\|_{L^2(Q_t)}$, which further leads to the boundedness of $\|\nabla^3 \sigma\|_{L^2(Q_t)}$. We remark that in the latter energy estimate, the knowledge of $\|\varepsilon \nabla^3 u\|_{L^2(Q_t)}$ is needed. Nevertheless, this term can be bounded by all the controlled norms appearing in $\mathcal{N}_{m,T}$. More precisely, one has by equation for the velocity

$$\varepsilon \operatorname{div} \mathcal{L}u = g_2(\varepsilon \partial_t u + \varepsilon u \cdot \nabla u) + \nabla \sigma,$$

and thus by (2.135):

$$\begin{aligned} \|\varepsilon \nabla^3 u\|_{L^2(Q_t)} &\lesssim \|\varepsilon \nabla \operatorname{div} \mathcal{L}u\|_{L^2(Q_t)} + \|\varepsilon \nabla^2 u\|_{L_t^2 H_{co}^1} \\ &\leq \Lambda(1/c_0, \mathcal{A}_{m,T})(\|(\sigma, u)\|_{L_t^2 H_{co}^2} + \|\nabla(\sigma, u)\|_{L_t^2 H_{co}^1} + \|\nabla^2 \sigma\|_{L^2(Q_t)}). \end{aligned} \quad (2.136)$$

3. Uniform estimates – $L_{t,x}^\infty$ norms

In this section, we aim to control the $L_{t,x}^\infty$ norms appearing in $\mathcal{A}_{m,T}$. Part of them can be deduced directly from the Sobolev embedding in the conormal setting (see Proposition 7.4) and the norms controlled in the previous section. Moreover, we use the maximum principle for transport-diffusion equation (3.5) satisfied by ω and of the damped transport equation (2.132) for $\nabla \sigma$ to get the $L_{t,x}^\infty$ estimates of ∇u and $\nabla \sigma$ respectively.

We will prove the following proposition.

Proposition 3.1. *Assuming that (1.24), (1.23) hold, then there is a constant $C_2(1/c_0)$ depending only on $1/c_0$ and a polynomial $\bar{\Lambda}$ whose coefficients are independent of ε , such that:*

$$\mathcal{A}_{m,T} \leq C_2(1/c_0)(Y_m(\sigma_0, u_0) + \mathcal{E}_{m,T}) + (\varepsilon^{\frac{1}{2}} + T)\mathcal{A}_{m,T}\bar{\Lambda}(1/c_0, \mathcal{A}_{m,T}). \quad (3.1)$$

Proof. Let us recall that $\mathcal{A}_{m,T}$ is defined as:

$$\begin{aligned} \mathcal{A}_{m,T} &= \| \nabla u \|_{0,\infty,T} + \| (\nabla \sigma, \operatorname{div} u) \|_{[\frac{m-1}{2}],\infty,T} + \| (\sigma, u) \|_{[\frac{m+1}{2}],\infty,T} \\ &\quad + \| \varepsilon^{\frac{1}{2}} \nabla u \|_{[\frac{m-1}{2}],\infty,T} + \| \varepsilon \nabla u \|_{[\frac{m+1}{2}],\infty,T} + \varepsilon \| (\sigma, u) \|_{[\frac{m+3}{2}],\infty,T}. \end{aligned} \quad (3.2)$$

The last four terms of $\mathcal{A}_{m,T}$ can be controlled directly by the Sobolev embedding (7.7). For instance,

$$\begin{aligned} \| (\sigma, u) \|_{[\frac{m+1}{2}],\infty,T} &\lesssim \sup_{0 \leq s \leq T} (\| (\sigma, u)(s) \|_{H_{co}^{[\frac{m+5}{2}]}} + \| \nabla(\sigma, u)(s) \|_{H_{co}^{[\frac{m+3}{2}]}}) \lesssim \mathcal{E}_{m,T}, \\ \varepsilon^{\frac{1}{2}} \| \nabla u \|_{[\frac{m-1}{2}],\infty,T} &\lesssim \sup_{0 \leq s \leq T} (\| \nabla u(s) \|_{H_{co}^{[\frac{m+3}{2}]}} + \varepsilon \| \nabla^2 u(s) \|_{H_{co}^{[\frac{m+1}{2}]}}) \lesssim \mathcal{E}_{m,T}, \\ \varepsilon \| \nabla u \|_{[\frac{m+1}{2}],\infty,T} &\lesssim \varepsilon \sup_{0 \leq s \leq T} (\| \nabla u(s) \|_{H_{co}^{[\frac{m+5}{2}]}} + \| \nabla^2 u(s) \|_{H_{co}^{[\frac{m+3}{2}]}}) \lesssim \mathcal{E}_{m,T}. \end{aligned} \quad (3.3)$$

Note that we have $[\frac{m+3}{2}] + 1 \leq m - 2$, $[\frac{m+5}{2}] \leq m - 1$ if $m \geq 6$.

We remark also that $\| \operatorname{div} u \|_{[\frac{m-1}{2}],\infty,T}$ can be estimated by the other quantities in the definition of $\mathcal{A}_{m,T}$. Indeed, by using the equation satisfied by σ , we have that:

$$\| \operatorname{div} u \|_{[\frac{m-1}{2}],\infty,T} \lesssim \| \sigma \|_{[\frac{m+1}{2}],\infty,T} + \varepsilon \mathcal{A}_{m,T}^2. \quad (3.4)$$

It thus remains to control $\|\nabla u\|_{0,\infty,T}$, $\|\nabla \sigma\|_{[\frac{m-1}{2}],\infty,T}$. We note that away from the boundaries where the conormal Sobolev norm is equivalent to the usual Sobolev norm, these two terms can be bounded by the standard Sobolev embedding. Therefore, it suffices to control $\|\chi_i \partial_{\mathbf{n}} u\|_{0,\infty,T}$, $\|\chi_i \partial_{\mathbf{n}} \sigma\|_{[\frac{m-1}{2}],\infty,T}$, where χ_i , $(1 \leq i \leq N)$ are smooth functions compactly supported in Ω_i . Moreover, by identity (2.22) and

$$\Pi(\partial_{\mathbf{n}} u) = \omega \times \mathbf{n} + 2\Pi(-(D\mathbf{n})u),$$

we reduce our problem to the control of $\|\omega\|_{0,\infty,T}$, $\|\chi_i \partial_{\mathbf{n}} \sigma\|_{[\frac{m-1}{2}],\infty,T}$, which is the aim of the following two lemmas. \square

We begin with the estimate for $\|\omega\|_{0,\infty,T}$ which follows from the maximum principle of the transport-diffusion equation for the vorticity.

Lemma 3.2. *Under the assumption (1.24), the following estimate holds:*

$$\|\omega\|_{0,\infty,T} \lesssim \|\omega(0)\|_{L^\infty(\Omega)} + \mathcal{E}_{m,T} + (T + \varepsilon)\mathcal{A}_{m,T}^2. \quad (3.5)$$

Proof. Recall that ω solves (2.98) which is rewritten below for convenience:

$$g_2(\partial_t + u \cdot \nabla)\omega - \mu \Delta \omega = g_2(\omega \cdot \nabla u - \omega \operatorname{div} u) + \nabla g_2 \times [(\partial_t + u \cdot \nabla)u] = G^\omega \quad x \in \Omega.$$

Since $g_2(\varepsilon \sigma)$ satisfies the transport equation: $\partial_t g_2 + \operatorname{div}(g_2 u) = 0$, by the maximum principle, (one can refer to Proposition 13 of [43])

$$\|\omega(t)\|_{L^\infty(\Omega)} \leq \|\omega(0)\|_{L^\infty(\Omega)} + |\omega(t)|_{L^\infty(\partial\Omega)} + \frac{1}{\inf g_2} \int_0^t \|G^\omega(s)\|_{L^\infty(\Omega)} \, ds. \quad (3.6)$$

For the second term in the right hand side of (3.6), we use the boundary condition (1.14), the identity (2.22) and (3.3), (3.4) to get that:

$$|\omega(t)|_{L^\infty(\partial\Omega)} \lesssim |(u, \partial_y u, \operatorname{div} u)(t)|_{L^\infty(\partial\Omega)} \lesssim \mathcal{E}_{m,T} + \varepsilon \mathcal{A}_{m,T}^2.$$

For the last term, we have by the assumption (1.24) and the property (1.23) that there is some $C(1/c_0)$, such that:

$$\frac{1}{\inf g_2} \int_0^t \|G^\omega(s)\|_{L^\infty(\Omega)} \, ds \leq C(1/c_0)T\mathcal{A}_{m,T}^2,$$

which ends the proof. \square

In the following, we estimate $\|\chi_i \partial_{\mathbf{n}} \sigma\|_{[\frac{m-1}{2}],\infty,T}$:

Lemma 3.3. *Under the assumption (1.24), we have:*

$$\|\chi_i \partial_{\mathbf{n}} \sigma\|_{[\frac{m-1}{2}],\infty,T} \lesssim Y_m(\sigma_0, u_0) + \mathcal{E}_{m,T} + \varepsilon^{\frac{1}{2}} \mathcal{A}_{m,T} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) \quad (3.7)$$

where χ_i is a smooth function that is compactly supported in Ω_i .

Proof. We define $R = \chi_i \partial_{\mathbf{n}} \sigma = \chi_i \mathbf{n} \cdot \nabla \sigma$. By (2.132), R solves the following equation:

$$\varepsilon^2 g_1 (\partial_t R + u \cdot \nabla R) + \frac{1}{2\mu + \lambda} R = -\varepsilon^2 g_1 u \cdot \nabla (\chi_i \mathbf{n}_k) \partial_k \sigma + G \cdot \chi_i \mathbf{n} =: G_R \quad (3.8)$$

where

$$G = -\varepsilon^2 (g'_1 R \varepsilon \partial_t \sigma + \nabla(g_1 u_k) \cdot \partial_k \sigma) - \varepsilon \frac{\mu}{(2\mu + \lambda)} \operatorname{curl} w - \frac{1}{(2\mu + \lambda)} g_2 (\varepsilon \partial_t u + \varepsilon u \cdot \nabla u).$$

By applying Z^I ($|I| \leq [\frac{m-1}{2}]$) to the equation (3.8), we get by setting $R^I = Z^I R$ that

$$\varepsilon^2 g_1 (\partial_t R^I + u \cdot \nabla R^I) + \frac{1}{2\mu + \lambda} R^I = Z^I G_R + C_{R,1}^I + C_{R,2}^I =: \mathcal{H}^I$$

where $C_{R,1}^I = -\varepsilon^2 [Z^I, g_1/\varepsilon] \varepsilon \partial_t R$, $C_{R,2}^I = -\varepsilon^2 [Z^I, g_1 u \cdot \nabla] R$.

It is convenient to use the Lagrangian coordinates. Define the unique flow $X_t(x) = X(t, x)$ associated to u :

$$\begin{cases} \partial_t X(t, x) = u(t, X(t, x)) \\ X(0, x) = x \in \Omega. \end{cases} \quad (3.9)$$

Note that since $u \cdot \mathbf{n}|_{\partial\Omega} = 0$, and $u \in \operatorname{Lip}([0, T] \times \Omega)$, we have for each $t \in [0, T]$, $X_t : \Omega \rightarrow \Omega$ is a diffeomorphism. By using the characteristics method, $R^I(t, X_t(x))$ can then be expressed in the following way:

$$R^I(t, X_t(x)) = e^{-\Gamma(t, x)} R^I(0) + \int_0^t e^{-\Gamma(t-s, x)} \left(\frac{1}{\varepsilon^2 g_1} \mathcal{H}^I \right) (s, X_s(x)) \, ds \quad (3.10)$$

where $\Gamma(t, x) = \frac{1}{2\mu + \lambda} \int_0^t \frac{1}{\varepsilon^2 g_1(s, X_s(x))} \, ds \geq \frac{c_0 t}{(2\mu + \lambda) \varepsilon^2}$. Note that we have used assumption (1.24) and property (1.23). Taking the supremum in $(t, x) \in [0, T] \times \Omega$ on both sides of (3.10), and using that $X(t, \cdot)(0 \leq t \leq T)$ is a diffeomorphism of Ω , we arrive at:

$$\|R^I(t)\|_{L^\infty(\Omega)} \lesssim \|R^I(0)\|_{L^\infty(\Omega)} + \int_0^t e^{-\frac{t-s}{(2\mu + \lambda) c_1 \varepsilon^2}} \frac{1}{c_0 \varepsilon^2} \, ds \|\mathcal{H}^I\|_{\infty, T} \lesssim \|R^I(0)\|_{L^\infty(\Omega)} + \|\mathcal{H}^I\|_{\infty, T}. \quad (3.11)$$

We have thus reduced the problem to the estimate of $\|(C_{R,1}^I, C_{R,2}^I)\|_{\infty, T}$ and $\|G_R\|_{[\frac{m-1}{2}], \infty, T}$. By the identities (2.35), (2.41), and the definition of $\mathcal{A}_{m,T}$, we have:

$$\|(C_{R,1}^I, C_{R,2}^I)\|_{\infty, T} \leq \varepsilon \Lambda \left(\frac{1}{c_0}, \mathcal{A}_{m,T} \right) \mathcal{A}_{m,T}. \quad (3.12)$$

Moreover, G_R (defined in (3.8)) can be controlled as:

$$\|G_R\|_{[\frac{m}{2}] - 1, \infty, T} \lesssim \varepsilon^{\frac{1}{2}} \mathcal{A}_{m,T} \Lambda(1/c_0, \mathcal{A}_{m,T}) + \|(\sigma, u)\|_{[\frac{m+1}{2}], \infty, T} + \varepsilon \|\chi \operatorname{curl} \omega \cdot \mathbf{n}\|_{[\frac{m-1}{2}], \infty, T}.$$

Since $\operatorname{curl} \omega \cdot \mathbf{n} = \operatorname{div}(\omega \times \mathbf{n}) + \omega \cdot \operatorname{curl} \mathbf{n}$, the identity (2.60) yields

$$\varepsilon \|\chi \operatorname{curl} \omega \cdot \mathbf{n}\|_{[\frac{m-1}{2}], \infty, T} \lesssim \varepsilon \|\nabla u\|_{[\frac{m+1}{2}], \infty, T},$$

which further leads to:

$$\|G_R\|_{[\frac{m}{2}-1,\infty,T]} \lesssim \varepsilon^{\frac{1}{2}} \mathcal{A}_{m,T} \Lambda\left(\frac{1}{c_0}, \mathcal{A}_{m,T}\right) + \mathcal{E}_{m,T}. \quad (3.13)$$

Inserting (3.12)-(3.13) into (3.11), we get (3.7). \square

4. Proof of Theorem 1.1

Based on the uniform estimates established in previous sections, Theorem 1.1 can be shown by combining classical local existence results stated in below with a bootstrap argument:

Theorem 4.1. *Assume that $(\sigma_0^\varepsilon, u_0^\varepsilon) \in H^2(\Omega)$, and*

$$-\bar{c}\bar{P} \leq \varepsilon\sigma_0^\varepsilon(x) \leq \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0, 1].$$

There is some $T_\varepsilon > 0$ such that (1.15) has a unique strong solution which satisfies: $(\sigma^\varepsilon, u^\varepsilon) \in C([0, T^\varepsilon], H^2), u^\varepsilon \in L^2([0, T^\varepsilon], H^3)$. Moreover, the following property holds:

$$-3\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t, x) \leq 3\bar{P}/\bar{c} \quad \forall (t, x) \in [0, T^\varepsilon] \times \Omega. \quad (4.1)$$

This result can be obtained from Theorem 1.1 in [24]. Indeed, the statement in [24] requires higher regularity for the velocity (but not for the density), namely $u_0^\varepsilon \in H^3$, in order to include some time-space Hölder continuity and also to make the boundary condition (1.14) pointwisely satisfied. Nevertheless, we can use [24] to obtain Theorem 4.1. Indeed, to get the existence, we can approximate the initial velocity u_0 in H^2 satisfying the compatibility condition at order one by a velocity $u_0^N \in H^3$, still satisfying the compatibility condition, and such that u_0^N converges to u_0 in H^2 . This yields a local solution (ρ^N, u^N) with u^N in H^3 from Theorem 1.1 of [24]. Then, by using our a priori estimates, we can obtain that this local solution exists on an interval of time independent of N and use standard compactness argument to pass to the limit. A way to choose u_0^N is to use the approach of [24] based on the elliptic regularity for the Lamé operator, basically we take $u_0^N = P_N u_0$ where P_N is the L^2 projection on the N first eigenmodes.

By using Theorem 4.1, we can give the proof of Theorem 1.1.

4.1. Proof of Theorem 1.1

On the one hand, $(\sigma_0^\varepsilon, u_0^\varepsilon) \in H^2$, by Theorem 4.1, one can find some $T^\varepsilon > 0$ such that there is a unique solution of (1.15) satisfying: $(\sigma^\varepsilon, u^\varepsilon) \in C([0, T^\varepsilon], H^2), u^\varepsilon \in L^2([0, T^\varepsilon], H^3)$. Moreover, condition (4.1) holds. On the other hand, as $(\sigma_0^\varepsilon, u_0^\varepsilon) \in Y_m$, a higher regularity space, by standard propagation of regularity arguments (for example based on applying finite difference instead of derivatives) in the estimates of Section 3 and Section 4, we find that the estimates of Proposition 1.6 hold on $[0, T^\varepsilon]$. More specifically, we can find a constant $C(1/c_0)$ and an increasing polynomial Λ_0 that are independent of ε and T^ε , such that for any $0 \leq T \leq \min\{1, T^\varepsilon\}, 0 < \varepsilon \leq 1$,

$$\mathcal{N}_{m,T}^2(\sigma^\varepsilon, u^\varepsilon) \leq C\left(\frac{1}{c_0}\right) Y_m^2(\sigma_0^\varepsilon, u_0^\varepsilon) + (T + \varepsilon)^{\frac{1}{2}} \Lambda_0\left(\frac{1}{c_0}, \mathcal{N}_{m,T}\right). \quad (4.2)$$

Moreover, by using the characteristics method, we have that $\varepsilon\sigma$ can be expressed as,

$$\varepsilon\sigma^\varepsilon(t, x) = \varepsilon\sigma_0^\varepsilon(X^{-1}(t, x)) - \int_0^t (\operatorname{div} u^\varepsilon/g_1)(X(s, X^{-1}(t, x))) \, ds \quad (4.3)$$

where $X(t, \cdot)$ is the flow associated to u .

Let us define

$$\begin{aligned} T_*^\varepsilon &= \sup\{T \mid (\sigma^\varepsilon, u^\varepsilon) \in C([0, T], H^2), u^\varepsilon \in L^2([0, T], H^3)\}, \\ T_0^\varepsilon &= \sup\{T \leq \min\{T_*^\varepsilon, 1\} \mid \mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \leq 2\sqrt{C(1/c_0)}M, \\ &\quad -2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t, x) \leq 2\bar{P}/\bar{c} \quad \forall(t, x) \in [0, T] \times \Omega\} \end{aligned}$$

where $M > \sup_{\varepsilon \in (0, 1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$.

We now choose successively two constants $0 < \varepsilon_0 \leq 1$ and $0 < T_0 \leq 1$ (uniform in $\varepsilon \in (0, \varepsilon_0]$) which are small enough, such that:

$$(T_0 + \varepsilon_0)^{\frac{1}{2}}\Lambda_0(1/c_0, 2\sqrt{C(1/c_0)}M) < 1/2, \quad 2\sqrt{C(1/c_0)}MT_0/c_0 \leq \bar{c}\bar{P}.$$

In order to prove Theorem 1.1, it suffices to show that $T_0^\varepsilon \geq T_0$ for every $0 < \varepsilon \leq \varepsilon_0$. Suppose otherwise $T_0^\varepsilon < T_0$ for some $0 < \varepsilon \leq \varepsilon_0$, then in view of inequalities (4.2) and formula (4.3), we have by the definition of ε_0 and T_0 that:

$$\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \leq \sqrt{2C(1/c_0)}M, \quad \forall T \leq \tilde{T} = \min\{T_0, T_*^\varepsilon\}, \quad (4.4)$$

$$-2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t, x) \leq 2\bar{P}/\bar{c} \quad \forall(t, x) \in [0, \tilde{T}] \times \Omega. \quad (4.5)$$

We will prove that $\tilde{T} = T_0 \leq T_*^\varepsilon$. This fact, combined with the definition of T_0^ε and estimates (4.4), (4.5), yield $T_0^\varepsilon \geq T_0$, which is a contradiction with the assumption $T_0^\varepsilon < T_0$. To continue, we shall need the claim stated and proved below. Indeed, once the following claim holds, we have by (4.4) that $\|(\sigma^\varepsilon, u^\varepsilon)(T_0)\|_{H^2(\Omega)} < +\infty$. Combined with the local existence result stated in Theorem 4.1, this yields that $T_*^\varepsilon > T_0$.

Claim. For all $\varepsilon \in (0, 1]$, if $\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) < +\infty$, then $(\sigma^\varepsilon, u^\varepsilon) \in C([0, T], H^2)$, $u^\varepsilon \in L^2([0, T], H^3)$.

Proof of Claim. We see from the definition of $\mathcal{N}_{m,T}$ and the estimate (2.136) that:

$$\varepsilon u^\varepsilon \in L^2([0, T], H^3), \quad \varepsilon \partial_t u^\varepsilon \in L^2([0, T], H^1), \quad \varepsilon \sigma^\varepsilon \in L^\infty([0, T], H^2).$$

One deduces from interpolation that $\varepsilon u^\varepsilon \in C([0, T], H^2)$. Moreover, carrying out direct energy estimates for σ^ε in $H^2(\Omega)$, one gets that:

$$|\partial_t R^\varepsilon(t)| \leq K^\varepsilon (R^\varepsilon(t) + f^\varepsilon(t)) \quad (4.6)$$

where $K^\varepsilon = \Lambda(1/c_0, \|(\nabla \sigma^\varepsilon, \nabla u^\varepsilon, \varepsilon \nabla^2 u^\varepsilon)\|_{\infty, t})$ is uniformly bounded and

$$R^\varepsilon(t) = \|\varepsilon \sigma^\varepsilon(t)\|_{H^2}^2, \quad f^\varepsilon(t) = \|\varepsilon u^\varepsilon(t)\|_{H^3} \|\sigma^\varepsilon(t)\|_{H^2} \in L^1([0, T]).$$

Inequality (4.6) and the boundedness of $\|R^\varepsilon(\cdot)\|_{L^\infty([0, T])}$ leads to the fact that $R^\varepsilon(\cdot) \in C([0, T])$, which further yields that $\varepsilon \sigma^\varepsilon \in C([0, T], H^2)$. This ends the proof of the claim. Note that at this stage we do not require the norm $\|(\sigma^\varepsilon, u^\varepsilon)\|_{C([0, T], H^2)}$ to be bounded uniformly in ε . \square

5. Proof of Theorem 1.7

The convergence result follows from compactness arguments. At first, since $\sigma^\varepsilon = \frac{P(\rho^\varepsilon) - P(\bar{\rho})}{\varepsilon}$ is uniformly bounded in $L^\infty([0, T_0], W^{1,\infty}(\Omega)) \cap L^2([0, T_0], H^1(\Omega))$, we have that: $P(\rho^\varepsilon) \rightarrow P(\bar{\rho})$ in $L^\infty([0, T_0], W^{1,\infty}(\Omega)) \cap L^2([0, T_0], H^1(\Omega))$, which yields that $\rho^\varepsilon \rightarrow \bar{\rho}$ in $L^2([0, T_0], H^1(\Omega))$.

For the convergence of u^ε , let us split the velocity into compressible part and incompressible part: $u^\varepsilon = \nabla \Psi^\varepsilon + v^\varepsilon$ by using the Leray decomposition (2.2). We shall prove that the compressible part $\nabla \Psi^\varepsilon$ tends to 0 in $L_{t,w}^2 H^1(\Omega)$ whereas incompressible part of u^ε tends to u^0 in $L^2(Q_{T_0})$. Since $\nabla \Psi^\varepsilon$ is uniformly bounded in $L_t^2 H^2(\Omega)$, we have that, up to the extraction of a subsequence (that we do not mention explicitly) $\nabla \Psi^\varepsilon$ converges to $\mathbb{Q}u^0$ in $L_w^2([0, T_0], H^1(\Omega))$. Nevertheless, by the equation (2.50), $\operatorname{div} u^\varepsilon$ tends to 0 in the sense of distribution, which leads to $\mathbb{Q}u^0 = 0$. Because of this, one can indeed see that, without any extraction of the subsequences, $\nabla \Psi^\varepsilon \rightarrow 0$ in $L_w^2([0, T_0], H^1(\Omega))$.

We are now in position to prove the convergence of v^ε . By the equation of v^ε : (2.5)₃, $\partial_t v^\varepsilon$ is uniformly bounded in $L^2([0, T_0], H^{-1}(\Omega))$ whereas v^ε is uniformly bounded in $L^2([0, T_0], H^1(\Omega))$. Therefore, by Aubin-Lions lemma, $\{v^\varepsilon\}$ is compact in $L^2(Q_{T_0})$, which yields, up to extraction of subsequences, the convergence of v^ε (say to u^0) in $L^2(Q_{T_0})$.

In the following, we aim to justify that u^0 is the unique weak solution of the incompressible Navier-Stokes equation (1.3) satisfying (1.26). Let us rewrite the equations of v^ε as follows:

$$\bar{\rho} \partial_t v^\varepsilon - \mu \Delta v^\varepsilon + \nabla \pi^\varepsilon = F^\varepsilon = F_1^\varepsilon + F_2^\varepsilon, \quad (5.1)$$

where

$$F_1^\varepsilon = -(\rho^\varepsilon - \bar{\rho})(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon), \quad F_2^\varepsilon = -\bar{\rho}(v^\varepsilon \cdot \nabla u^\varepsilon + \nabla \Psi^\varepsilon \cdot \nabla v^\varepsilon).$$

Note that we put the gradient terms $\bar{\rho} \nabla(\partial_t \Psi^\varepsilon + \frac{1}{2} |\nabla \Psi^\varepsilon|^2)$ into the pressure $\nabla \pi^\varepsilon$. Let us write down the weak formulation for (5.1). Multiplying equation (5.1) by a test function $\psi \in (C^\infty([0, T_0] \times \bar{\Omega}))^3$ which satisfies $\operatorname{div} \psi = 0, \psi \cdot \mathbf{n}|_{\partial\Omega} = 0$, we obtain that for each $0 < t \leq T_0$,

$$\begin{aligned} & \bar{\rho} \int_{\Omega} (v^\varepsilon \cdot \psi)(t, \cdot) \, dx + \mu \iint_{Q_t} \nabla v^\varepsilon \cdot \nabla \psi \, dx ds + \iint_{Q_t} F^\varepsilon \cdot \psi \, dx ds \\ &= \bar{\rho} \int_{\Omega} (v^\varepsilon \cdot \psi)(0, \cdot) \, dx + \bar{\rho} \iint_{Q_t} v^\varepsilon \cdot \partial_t \psi \, dx ds + \mu \int_0^t \int_{\partial\Omega} \Pi \partial_{\mathbf{n}} v^\varepsilon \cdot \psi \, dS_y ds. \end{aligned} \quad (5.2)$$

It remains to pass to the limit to show that u^0 satisfies (1.27). We shall only focus on the last terms in both sides of (5.2), as the other terms are direct. Since $\rho^\varepsilon = g_2(\varepsilon \sigma^\varepsilon)$, we have that $(\rho^\varepsilon - \bar{\rho})/\varepsilon$ is uniformly bounded in $L^\infty(Q_{T_0})$, it then follows from the velocity equation in (1.15)₂ that

$$\iint_{Q_t} F_1^\varepsilon \cdot \psi \, dx ds = \iint_{Q_t} \frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon} (\operatorname{div} \mathcal{L} u^\varepsilon - \frac{\nabla \sigma^\varepsilon}{\varepsilon}) \psi \, dx ds.$$

We then observe that

$$\frac{1}{\varepsilon} \iint_{Q_t} \frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon} \nabla \sigma^\varepsilon \cdot \psi \, dx ds = \frac{1}{\varepsilon} \iint_{Q_t} \frac{g_2(\varepsilon \sigma^\varepsilon) - g_2(0)}{g_2(\varepsilon \sigma^\varepsilon)} \nabla \sigma^\varepsilon \cdot \psi \, dx ds = 0$$

by integrating by parts since

$$\frac{g_2(\varepsilon \sigma^\varepsilon) - g_2(0)}{g_2(\varepsilon \sigma^\varepsilon)} \nabla \sigma^\varepsilon = \frac{1}{\varepsilon} \nabla (G(\varepsilon \sigma^\varepsilon))$$

where $G(s)$ is such that

$$G'(s) = \frac{g_2(s) - g_2(0)}{g_2(s)}.$$

In a similar way, we have that

$$\begin{aligned} \iint_{Q_t} \frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon} \nabla \operatorname{div} u^\varepsilon \cdot \psi \, dx ds &= -\varepsilon \iint_{Q_t} \operatorname{div} u^\varepsilon G''(\varepsilon \sigma^\varepsilon) \nabla \sigma^\varepsilon \cdot \psi \, dx ds \\ \iint_{Q_t} \frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon} \Delta u^\varepsilon \cdot \psi \, dx ds &= -\varepsilon \iint_{Q_t} G''(\varepsilon \sigma^\varepsilon) ((\nabla \sigma^\varepsilon \cdot \nabla) u^\varepsilon) \cdot \psi \, dx ds + \int_0^t \int_{\partial\Omega} \frac{\rho^\varepsilon - \bar{\rho}}{\rho^\varepsilon} \Pi \partial_n u^\varepsilon \cdot \psi \, dx ds. \end{aligned}$$

These three above terms tend to zero, for the last one, we use that $\|\rho^\varepsilon - \bar{\rho}\|_{L^\infty(Q_t)} = O(\varepsilon)$ while $\Pi \partial_n u^\varepsilon$ is uniformly bounded in $L^2(\partial\Omega)$ by using the Navier-boundary condition and the trace inequality. This yields

$$\iint_{Q_t} F_1^\varepsilon \cdot \psi \, dx ds \rightarrow 0.$$

Next, since $\nabla \Psi^\varepsilon \rightarrow 0$, $\nabla u^\varepsilon \rightarrow \nabla u^0$, $v^\varepsilon \rightarrow u^0$ in $L^2(Q_t)$ and v^ε is uniformly bounded in $L^2([0, T_0], H^1(\Omega))$, we have that:

$$\iint_{Q_t} F_2^\varepsilon \cdot \psi \, dx ds \rightarrow \bar{\rho} \iint_{Q_t} (u^0 \cdot \nabla u^0) \cdot \psi \, dx ds.$$

Finally, for the boundary term in (5.2), we use the boundary condition for v^ε (see (2.72)):

$$\Pi(\partial_n v^\varepsilon) = \Pi(-2av^\varepsilon + (D\mathbf{n})v^\varepsilon) + 2\Pi(-a\nabla \Psi^\varepsilon + (D\mathbf{n})\nabla \Psi^\varepsilon).$$

As $v^\varepsilon \rightarrow u^0$ in $L^2(Q_t)$ and v^ε is uniformly bounded in $L^2([0, t], H^1(\Omega))$, $\nabla \Psi^\varepsilon \rightarrow 0$ in $L_w^2([0, t], H^1(\Omega))$, it follows from the trace inequality and the Hölder inequality that: $v^\varepsilon|_{\partial\Omega} \rightarrow u^0|_{\partial\Omega}$ in $L^2([0, t], L^2(\partial\Omega))$, $\nabla \Psi^\varepsilon \rightarrow 0$ in $L_w^2([0, t], L^2(\partial\Omega))$. This yields:

$$\mu \int_0^t \int_{\partial\Omega} \Pi \partial_n v^\varepsilon \cdot \psi \, dS_y \, ds \rightarrow \mu \int_0^t \int_{\partial\Omega} \Pi(-2au^0 + (D\mathbf{n})u^0) \cdot \psi \, dS_y \, ds.$$

Therefore, u^0 satisfies the formulation (1.27) and hence is a weak solution to (1.3). Next, due to the uniform boundedness of v^ε in $L_{T_0}^\infty H_{co}^{m-1}$ and ∇v^ε in $L_{T_0}^2 H_{co}^{m-1} \cap L^\infty(Q_{T_0})$, we get that u^0 has the additional regularity property (1.26). The uniqueness result is easy owing to the boundedness of the Lipschitz norm. Since any subsequence of u^ε will have an extracted subsequence that solves (1.27) and satisfies the additional regularity property (1.26), we finally get from the uniqueness that the whole family u^ε converges to u^0 . This ends the proof of Theorem 1.7.

6. Remarks on the exterior domains

In this short section, we make some remarks on the incompressible limit problem for (CNS) when the domain is an exterior domain, that is $\Omega = \mathbb{R}^3 \setminus K$, where $K \subset \mathbb{R}^3$ is a smooth compact domain. We first remark that the uniform regularity estimates can be obtained in the same way as in the bounded domain case (see Section 2-3). Indeed, to define conormal vector fields and conormal Sobolev spaces we only need a finite covering property of the boundary without requiring any boundedness of the domain. Moreover, the

Sobolev embeddings and the trace inequalities (Proposition 7.4, Lemma 7.5) widely used in Section 2-3 do not depend on the size of the domain.

Note that in the case of exterior domains, one can obtain the strong convergence in time of the velocities, namely $u^\varepsilon \rightarrow u^0$ in $L^2([0, T_0], L^2_{loc}(\Omega))$ by using some properties of the wave equation. Indeed, in the case of exterior domains, there is no point spectrum for the Neumann Laplacian and the RAGE Theorem (Theorem 5.8 of [7] for example) can be used. We refer to [15,20] where this approach was used for weak solutions of the Navier-Stokes-Fourier system, it can be easily used for solutions of the isentropic system enjoying our stronger uniform estimates.

7. Appendix

We state here the product and commutator estimates which are used throughout the paper:

Lemma 7.1. *For each $0 \leq t \leq T$, and for any integer $k \geq 2$, one has the (rough) product estimates*

$$\|(fg)(t)\|_{H_{co}^k} \lesssim \|f(t)\|_{H_{co}^k} \|g\|_{[\frac{k-1}{2}], \infty, t} + \|g(t)\|_{H_{co}^k} \|f\|_{[\frac{k}{2}], \infty, t}, \quad (7.1)$$

and commutator estimates:

$$\|[Z^I, f]g(t)\|_{L^2} \lesssim \|Zf(t)\|_{H_{co}^{k-1}} \|g\|_{[\frac{k}{2}]-1, \infty, t} + \|g(t)\|_{H_{co}^{k-1}} \|Zf\|_{[\frac{k-1}{2}], \infty, t}, \quad |I| = k, \quad (7.2)$$

$$\|[(\varepsilon \partial_t)^k, f]g(t)\|_{L^2} \lesssim \|(\varepsilon \partial_t f)(t)\|_{\mathcal{H}^{k-1}} \|g\|_{[\frac{k}{2}]-1, \infty, t} + \|g(t)\|_{\mathcal{H}^{k-1}} \|\varepsilon \partial_t f\|_{[\frac{k-1}{2}], \infty, t}. \quad (7.3)$$

Proof. This lemma follows from simply counting the derivatives hitting on f or g . For instance, to prove the product estimate (7.1) and the commutator estimate (7.2), one can use the following expansion:

$$\begin{aligned} Z^I(fg) &= \left(\sum_{|J| \leq [(k-1)/2]} + \sum_{|I-J| \leq [k/2]} \right) (C_{I,J} Z^J g Z^{I-J} f) \\ &= \left(\sum_{|J| \leq [k/2]-1} + \sum_{1 \leq |I-J| \leq [(k+1)/2]} \right) (C_{I,J} Z^J g Z^{I-J} f) + f Z^I g, \quad |I| = k. \quad \square \end{aligned}$$

As a corollary of Lemma 7.1 the following composition estimates hold:

Corollary 7.2. *Suppose that $h \in C^0(Q_t) \cap L_t^2 H_{co}^m$ with*

$$A_1 \leq h(t, x) \leq A_2, \quad \forall (t, x) \in Q_t.$$

Let $F(\cdot) : [A_1, A_2] \rightarrow \mathbb{R}$ be a smooth function satisfying

$$\sup_{s \in [A_1, A_2]} |F^{(m)}|(s) \leq B.$$

Then we have the composition estimate, for $p = 2, +\infty$

$$\|F(h(\cdot, \cdot)) - F(0)\|_{L_t^p H_{co}^m} \leq \Lambda(B, \|h\|_{[\frac{m}{2}], \infty, t}) \|h\|_{L_t^p H_{co}^m},$$

where $\Lambda(B, \|h\|_{[\frac{m}{2}], \infty, t})$ is a polynomial with respect to B and $\|h\|_{[\frac{m}{2}], \infty, t}$.

This Corollary, combined with Lemma 7.1, leads to the following estimates:

Corollary 7.3. *Let $g_1(\varepsilon\sigma), g_2(\varepsilon\sigma)$ defined in (1.16) and assume that (1.24), (1.23) hold. Then one has the following estimates: for $j = 1, 2, p = 2, +\infty$,*

$$\|Zg_j\|_{L_t^p \mathcal{H}^{m-1}} \leq \varepsilon \Lambda\left(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}], \infty, t}\right) \|(\sigma, Z\sigma)\|_{L_t^p \mathcal{H}^{m-1}}, \quad (7.4)$$

$$\|Zg_j\|_{L_t^p H_{co}^{m-1}} \leq \varepsilon \Lambda\left(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}], \infty, t}\right) \|\sigma\|_{L_t^p H_{co}^m}, \quad (7.5)$$

$$\|g_j(\varepsilon\sigma) - g_j(0)\|_{L_t^p H_{co}^m} \lesssim \varepsilon \Lambda\left(\frac{1}{c_0}, \|\sigma\|_{[\frac{m}{2}], \infty, t}\right) \|\sigma\|_{L_t^p H_{co}^m}. \quad (7.6)$$

We will use often the following Sobolev embedding inequality whose proof is similar to that of Proposition 12 and Proposition 20 of [35].

Proposition 7.4. *Let $\Omega = \mathbb{R}_+^3$ or a smooth bounded domain, we have the following Sobolev embedding inequality*

$$\|f(t)\|_{L^\infty(\Omega)} \lesssim \|\nabla f(t)\|_{H_{co}^{k+1}}^{\frac{1}{2}} \|f(t)\|_{H_{co}^{k+2}}^{\frac{1}{2}} + \|f(t)\|_{H_{co}^{k+2}}. \quad (7.7)$$

Proof. For the case of the half-space, this is a consequence of the inequality: for a function g defined on \mathbb{R}_+^3 ,

$$\|f(t)\|_{L^\infty(\mathbb{R}_+^3)} \lesssim \|\partial_z f(t)\|_{H_{co}^{s_1}(\mathbb{R}_+^3)}^{\frac{1}{2}} \|f(t)\|_{H_{co}^{s_2}(\mathbb{R}_+^3)}^{\frac{1}{2}} \quad (7.8)$$

where s_1, s_2 are positive and satisfy $s_1 + s_2 > 2$. One can refer to (Prop. 2.2) of [36] for the proof. The case of general smooth bounded domains follows by working in local coordinates. \square

The following trace inequalities are also used:

Lemma 7.5. *For multi-index $I = (I_0, \dots, I_M)$ with $|I| = k$, we have the following trace inequalities:*

$$|Z^I f(t)|_{L^2(\partial\Omega)}^2 \lesssim \|\nabla f(t)\|_{H_{co}^k} \|f(t)\|_{H_{co}^k} + \|f(t)\|_{H_{co}^k}^2. \quad (7.9)$$

$$\int_0^t |Z^I f(s)|_{L^2(\partial\Omega)}^2 ds \lesssim \|\nabla f\|_{L_t^2 H_{co}^k} \|f\|_{L_t^2 H_{co}^k} + \|f\|_{L_t^2 H_{co}^k}^2. \quad (7.10)$$

$$\int_0^t |Z^I f(s)|_{H^{\frac{1}{2}}(\partial\Omega)}^2 ds \lesssim \|\nabla f\|_{L_t^2 H_{co}^k}^2 + \|f\|_{L_t^2 H_{co}^k}^2. \quad (7.11)$$

In the next proposition, we state some elliptic estimates which are used frequently.

Proposition 7.6. *Given a bounded domain Ω with C^{k+1} boundary. Consider the following elliptic equation with Neumann boundary condition:*

$$\begin{cases} \Delta q = \operatorname{div} f & \text{in } \Omega \\ \partial_{\mathbf{n}} q = f \cdot \mathbf{n} + g & \text{on } \partial\Omega \\ \int_{\Omega} q dx = 0 \end{cases} \quad (7.12)$$

The system (7.12) has a unique solution in $H^1(\Omega)$ which satisfies the following gradient estimate:

$$\|\nabla q(t)\|_{L^2(\Omega)} \lesssim \|f(t)\|_{L^2(\Omega)} + |g(t)|_{H^{-\frac{1}{2}}(\partial\Omega)}. \quad (7.13)$$

Moreover, for $j + l = k$,

$$\|\nabla q(t)\|_{\mathcal{H}^{j,l}(\Omega)} \lesssim \|f(t)\|_{\mathcal{H}^{j,l}(\Omega)} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}(\partial\Omega)}. \quad (7.14)$$

$$\|\nabla^2 q(t)\|_{\mathcal{H}^{j,l}(\Omega)} \lesssim \|(f, \operatorname{div} f(t))\|_{\mathcal{H}^{j,l}(\Omega)} + |(f \cdot \mathbf{n}, g)(t)|_{\tilde{H}^{k+\frac{1}{2}}(\partial\Omega)}. \quad (7.15)$$

Proof. The existence of the weak solution in $H =: \{q \mid q \in H^1(\Omega), \int_{\Omega} q dx = 0\}$ as well as the gradient estimate (7.13) comes from Lax-Milgram Lemma. The estimates (7.14)-(7.15) are then standard regularity estimates for elliptic equations, that take into account the number of time derivatives (the time variable being only a parameter in this Lemma). \square

Finally, we state an elementary estimate of the heat kernel which is useful in the estimates of the vorticity.

Lemma 7.7. *Let*

$$K(s, y, z) = \tilde{\mu} |\mathbf{N}|^2 (4\pi \tilde{\mu} |\mathbf{N}|^2 s)^{-\frac{1}{2}} \partial_z (e^{-\frac{z^2}{4\tilde{\mu} |\mathbf{N}|^2 s}}), \quad \mathbf{N}(y) = (-\partial_1 \varphi(y), -\partial_2 \varphi(y), 1)^t$$

where $(y, z) \in \mathbb{R}_+^3$ and set $\mathcal{Z}^\beta = \partial_{y_1}^{\beta_1} \partial_{y_2}^{\beta_2} \mathcal{Z}_3^{\beta_3}$, $\mathcal{Z}_3 = \frac{z}{1+z} \partial_z$. We have the following estimate:

$$\|\mathcal{Z}^\beta K(s, y, \cdot)\|_{L_z^2(\mathbb{R}_+)} \leq C(\beta, \tilde{\mu}, |\varphi|_{C^{|\beta|+1}}) s^{-\frac{3}{4}}. \quad (7.16)$$

Proof. It suffices to prove that, for any $l \in \mathbb{N}$, there is a polynomial $P_{2|\beta|+1}$ with $2|\beta| + 1$ degree, such that:

$$|\mathcal{Z}^\beta K(s, y, z)| \leq C(\beta, \tilde{\mu}, |\varphi|_{C^{|\beta|+1}}) P_{2|\beta|+1} \left(\frac{z}{\sqrt{s}} \right) e^{-\frac{z^2}{4\tilde{\mu} |\mathbf{N}|^2 s}} s^{-1} \quad \forall s > 0, y \in \mathbb{R}^2. \quad (7.17)$$

By direct computation, one can see that, there exists a polynomial with degree $2(\beta_1 + \beta_2) + 1 : P_{2(\beta_1 + \beta_2) + 1}$, a smooth function depends on $\nabla_y \varphi$ and its derivatives up to order $\beta_1 + \beta_2 : F_{\beta_1 + \beta_2}(\nabla_y \varphi)$ such that

$$\partial_{y_1}^{\beta_1} \partial_{y_2}^{\beta_2} K(s, y, z) = P_{2(\beta_1 + \beta_2) + 1} \left(\frac{z}{\sqrt{s}} \right) F_{\beta_1 + \beta_2}(\nabla_y \varphi) e^{-\frac{z^2}{4\tilde{\mu} |\mathbf{N}|^2 s}} s^{-1}.$$

To prove (7.17), it suffices to show by induction arguments that, there exists a smooth function $F(|\mathbf{N}|^2)$, such that

$$\partial_z^{\beta_3} \left(P_{2(\beta_1 + \beta_2) + 1} \left(\frac{z}{\sqrt{s}} \right) e^{-\frac{z^2}{4\tilde{\mu} |\mathbf{N}|^2 s}} \right) = F(|\mathbf{N}|^2) e^{-\frac{z^2}{4\tilde{\mu} |\mathbf{N}|^2 s}} P_{2|\beta|+1} \left(\frac{z}{\sqrt{s}} \right) z^{-\beta_3}. \quad \square$$

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