



Stability of the Couette Flow for a 2D Boussinesq System Without Thermal Diffusivity

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Abstract

In this paper, we prove the stability of the Couette flow for a 2D Navier–Stokes Boussinesq system without thermal diffusivity for the initial perturbation in Gevrey- $\frac{1}{s}$, $(1/3 < s \leq 1)$. The synergism of density mixing, vorticity mixing and velocity diffusion leads to the stability.

1. Introduction

The stability of shear flow in a stratified medium is of interest in many fields of research, such as fluid dynamics, geophysics, astrophysics, mathematics, etc.. Density stratification can strongly affect the dynamic of fluids like air in the atmosphere or water in the ocean and the stability question of stratified flows dates back to Taylor 1914 [76] and Goldstein 1931 [38] and since then there has been an active search towards the understanding of the stability of density-stratified flows. The question that many researchers want to answer is for a given steady state is it (asymptotically) stable relative to small disturbances?

This is the problem of the hydrodynamic stability, which is one of the most classical problems in the study of fluid dynamics and its investigation dates back to Rayleigh, Orr, Summerfeld, Bénard among others; see for instance the book of Drazin and Reid [35] and reference therein.

In this paper, we consider the 2D Navier–Stokes Boussinesq system without thermal diffusivity in $\mathbb{T} \times \mathbb{R}$:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = -\bar{\rho} g e_2 + \nu \Delta v, \\ \partial_t \bar{\rho} + v \cdot \nabla \bar{\rho} = 0, \\ \nabla \cdot v = 0. \end{cases} \quad (1.1)$$

Here $(x, y) \in \mathbb{T} \times \mathbb{R}$, $v = (v^x, v^y)$ is the velocity field, P is the pressure and $\bar{\rho}$ is the density and $g = 1$ being the normalized gravitational constant and $e_2 = (0, 1)$

is the unit vector in the vertical direction and ν is the kinematic viscosity. The first equation is the Navier–Stokes equation with the buoyancy forcing term $-\bar{\rho}ge_2$ in the vertical direction. The second equation is the transport equation of the density and the third equation is the incompressibility condition which represents the mass conservation.

The Boussinesq system (1.1) attracted the attention of many mathematicians first due to its wide range of applications, (see for example [23,62,71,78]) and second, due to the fact that the 2D Boussinesq model retain some key features of the 3D Euler and Navier–Stokes equations. For instance it has been known that the inviscid 2D Boussinesq equations are identical to the incompressible axi-symmetric swirling 3D Euler equations, as pointed out in [61]. During the last decades many interesting results were obtained in different directions.

One of the important research directions is to find the minimal dissipation in the Boussinesq system that yields a global existence under the lowest possible regularity. When both the viscosity and diffusivity are present in the Boussinesq system, then the system is known to be globally well-posed for smooth and arbitrary large initial data; see for instance [16,37] and also [77]. In the absence of the diffusivity, the global existence was proved in [43] (see also [18]). An extension of the results in [18,43] to a rough initial data in some Besov type spaces has been obtained in [41]. More importantly, it has been proved in [26], that the L^2 regularity of the initial data is enough to prove the global existence of the solution; see also [27] where a similar result, under some extra assumptions, has been proved in the presence of the diffusivity only. If the diffusion or the viscosity acts on the horizontal direction on one of the equations only, the authors in [1,2,28,33] showed a global existence result for initial data with different regularities. Under the same regularity assumption as in [28], the uniqueness of the solution was shown in [52]. Recently, and by considering only partial dissipation on the vertical direction in both equations, a global existence result was obtained in [55] under very low regularity assumptions. We recall that in the absence of viscosity and diffusion, the global well-posedness of the inviscid Boussinesq system is still largely open; see [18] and [19] for investigation in this direction.

Despite the large literature on the Boussinesq system, the asymptotic stability of solutions has not been well studied. For the non-flowing steady states $v_s = 0$, $\rho_s = y$, in [24,75], the authors studied the stability problem of 2D Boussinesq with different settings.

For the flowing steady states as in the case of Couette flow,

$$v_s = (y, 0), \quad \rho_s = -r_0 y + 1, \quad p_s = \int_0^y \rho_s(y_1) dy_1 = y - \frac{r_0}{2} y^2, \quad (1.2)$$

the asymptotic stability problem is very challenging.

The goal of this paper is to study the stability of the Couette flow described by the steady state (1.2). Before stating our main results, let us first recall previous works about the stability problem of flowing steady states.

The linear inviscid 2D Boussinesq system with shear flows has been extensively studied starting from the work of Taylor [76], Goldstein [38] and Synge [74]. We also refer to the book of Lin [57]. It has been proved that the stability of the solutions

of the linearized inviscid 2D Boussinesq system with a shear flow is determined by the competition between the stabilizing forces and the vertical shear flow $U(y)$.

In general if ρ_s is the steady state for the density, we define the local Richardson number $\gamma(y)$ to be such that

$$\gamma(y)^2 = -\frac{g\partial_y\rho_s(y)}{(\partial_y U(y))^2}.$$

This number measures the ratio of the stabilizing effect of the gravity to the destabilizing effect of the shear. We assume that $\partial_y\rho_s \leq 0$ (stable stratification) so that $\gamma^2 \geq 0$.

The Richardson number is one of the control parameters of the stability of stratified shear flows. The Miles–Howard theorem [44,67] guarantees that any flow in the inviscid non-diffusive limit is linearly stable if the local Richardson number everywhere exceeds the value $1/4$, however unstable modes can arise for Richardson number smaller than $1/4$ [34].

The introduction of viscosity or diffusivity in stratified shear flows may seem to lead to stability, however this is not always correct. As shown in Miller and Lindzen [68] for some particular geometry, the addition of viscosity may allow over-reflection and subsequent instability even if the Richardson number is everywhere greater than $1/4$. In fact, they proved a normal mode instability for a Richardson number as large as 0.349. This is in contrast to the Miles–Howard theorem in the inviscid case, where it is shown that unstable modes cannot exist for any flow with a Richardson number greater than $1/4$. Hence, it seems an interesting problem to investigate the stability of the stratified shear flows when viscosity is added.

In the physics literature, there have been a lot of work devoted to the stability of the Couette flow in the linearized stratified inviscid flow; see for example [15, 17, 22, 31, 36, 40, 42, 50], though these are less mathematically rigorous results.

In [85] Yang and Lin studied the linear asymptotic stability of the steady state (1.2) for the 2D Euler Boussinesq system ($\nu = 0$):

$$\begin{cases} \partial_t \omega + y \partial_x \omega = -\gamma^2 \partial_x \theta, \\ \partial_t \theta + y \partial_x \theta = u^y, \\ u = (u_1, u_2) = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega; \end{cases} \quad (1.3)$$

see also [13] for more linear results of general shear flows. They showed that the decay rates depend crucially on the Richardson number. More precisely, they obtained the following decay rates for the velocity components (which confirms the decay rate stated in [36]):

$$\begin{aligned} (u_{1\neq}, u_{2\neq}, \theta_{\neq}, \omega_{\neq}) &\lesssim \left(\langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{-\frac{3}{2}}, \langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{\frac{1}{2}} \right) \quad \text{if } \gamma^2 > \frac{1}{4}, \\ (u_{1\neq}, u_{2\neq}, \theta_{\neq}, \omega_{\neq}) &\lesssim \ln(e + |t|) \times \left(\langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{-\frac{3}{2}}, \langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{\frac{1}{2}} \right) \quad \text{if } \gamma^2 = \frac{1}{4}, \\ (u_{1\neq}, u_{2\neq}, \theta_{\neq}, \omega_{\neq}) &\lesssim \langle t \rangle^{\sqrt{\frac{1}{4}-\gamma^2}} \times \left(\langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{-\frac{3}{2}}, \langle t \rangle^{-\frac{1}{2}}, \langle t \rangle^{\frac{1}{2}} \right) \quad \text{if } 0 < \gamma^2 < \frac{1}{4}, \end{aligned}$$

where $f_{\neq} = f - \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx$ denotes the non-zero mode. Let us also point out that the linearized Euler Boussinesq ($\nu = 0$) system around (1.2) with $r_0 = 0$

$$\begin{cases} \partial_t \omega + y \partial_x \omega = -\partial_x \theta, \\ \partial_t \theta + y \partial_x \theta = 0, \\ u = (u_1, u_2) = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega, \end{cases} \quad (1.4)$$

is a couple system with two transport equations (transport diffusion equation if $\nu \neq 0$). The behavior of the solutions are easy to obtain. Indeed, we have

$$(u_{1\neq}, u_{2\neq}, \theta_{\neq}, \omega_{\neq}) \lesssim (1, \langle t \rangle^{-1}, 1, t),$$

which is same as the limit behavior of the solutions of (1.3) as $\gamma^2 = r_0 \rightarrow 0$. However, since the limit process $\gamma^2 = r_0 \rightarrow 0$ is a singular limit, the system (1.3) does not converge to (1.4).

Let us also point out here that if $\theta = 0$ then (1.4) is the linearized Euler (Navier–Stokes) equation whose solutions behave as follows:

$$(u_{1\neq}, u_{2\neq}, \omega_{\neq}) \lesssim (\langle t \rangle^{-1}, \langle t \rangle^{-2}, 1).$$

The vorticity does not grow and behaves better. The buoyancy forcing term $\partial_x \theta$ leads to a growth of the vorticity even at the linear level, which destabilizes the system.

In recent papers [29, 63], the authors investigated the stability of the Couette flow for the 2D Navier–Stokes Boussinesq system with both dissipation and thermal diffusion in an infinite channel and a finite channel. They also considered the problem with a weaker stabilization mechanism and studied the partial dissipation case. The mechanism leading to stability is the so-called inviscid damping and enhanced dissipation which we will introduce later.

In this paper, we study the system without thermal diffusivity which is the natural and the physical setting. Also mathematically it is much more interesting and challenging. The stability problem of systems with various diffusion terms is always a difficult problem. We refer to [73, 79] for similar challenges appearing in MHD.

In order to state our main result, we introduce the perturbation: $v = u + (y, 0)$, $P = p + p_s$ and $\bar{\rho} = \varrho + \rho_s$, then (u, p, ϱ) satisfies

$$\begin{cases} \partial_t u + y \partial_x u + \begin{pmatrix} u^y \\ 0 \end{pmatrix} + u \cdot \nabla u + \nabla p = -\varrho e_2 + \Delta u, \\ \partial_t \varrho + y \partial_x \varrho - r_0 u^y + u \cdot \nabla \varrho = 0, \\ \nabla \cdot u = 0. \end{cases}$$

For $r_0 > 0$, we introduce $\theta = \frac{1}{r_0} \varrho$, the Richardson number $\gamma = \sqrt{r_0}$ and the vorticity

$$\omega = \nabla \times u = \partial_x u^y - \partial_y u^x$$

which satisfies

$$\begin{cases} \partial_t \omega + y \partial_x \omega + u \cdot \nabla \omega = -\gamma^2 \partial_x \theta + \Delta \omega, \\ \partial_t \theta + y \partial_x \theta + u \cdot \nabla \theta = u^y, \\ u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega. \end{cases} \quad (1.5)$$

We also study the case $r_0 = 0$, and let $\theta = \varrho$, see the perturbation (θ, ω) satisfies

$$\begin{cases} \partial_t \omega + y \partial_x \omega + u \cdot \nabla \omega = -\partial_x \theta + \Delta \omega, \\ \partial_t \theta + y \partial_x \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega. \end{cases} \quad (1.6)$$

The dissipation term $\Delta \omega$ can hopefully stabilize the system. However comparing to the full diffusion case, in this system, the perturbed density θ does not decay, which leads to a linear growth of the vorticity due to the presence of the buoyancy force term $\partial_x \theta$.

Let us point out that due to the dissipation term, the behavior of the solutions changes a lot. Our main result reads as follows:

Theorem 1.1. *Let (ω, θ) solve (1.5) with $\gamma \neq 0$. Let (u, ψ) be the corresponding velocity field and stream function. For all $\frac{1}{3} < s \leq 1$ and $\lambda_0 > \lambda' > 0$, there exists an $\epsilon_0 = \epsilon_0(\lambda_0, \lambda', s, \gamma) \leq \frac{1}{2}$ such that for all $\epsilon \leq \epsilon_0$ if $(\omega_{in}, \theta_{in})$ and (u_{in}, ψ_{in}) satisfy*

$$\int u_{in} dx dy = \int \omega_{in} dx dy = \int \theta_{in} dx dy = 0,$$

$\int |y \omega_{in}(x, y)| dx dy + \int |y \theta_{in}(x, y)| dx dy < \epsilon$ and

$$\left\| \frac{1}{2\pi} \int_{\mathbb{T}} \psi_{in}(x, \cdot) dx \right\|_{L^1} \leq \epsilon \quad (1.7)$$

and

$$\|\omega_{in}\|_{\mathcal{G}^{\lambda_0}}^2 + \|\theta_{in}\|_{\mathcal{G}^{\lambda_0}}^2 = \sum_k \int (|\hat{\omega}_{in}(k, \eta)|^2 + |\hat{\theta}_{in}(k, \eta)|^2) e^{2\lambda_0 |k, \eta|^s} d\eta \leq \epsilon^2,$$

then there exists θ_∞ with $\int \theta_\infty dx dy = 0$ and $\|\theta_\infty\|_{\mathcal{G}^{\lambda'}} \lesssim \epsilon$ such that

$$\|\theta(t, x + ty + \Phi(t, y), y) - \theta_\infty(x, y)\|_{\mathcal{G}^{\lambda'}} \lesssim \frac{\epsilon^2}{\langle t \rangle} \ln(e + t) + \frac{\epsilon}{\langle t \rangle^3} \quad (1.8)$$

where $\Phi(t, y)$ is given explicitly by

$$\Phi(t, y) = \frac{1}{2\pi} \int_0^t \int_{\mathbb{T}} U^x(\tau, x, y) dx d\tau$$

Moreover, it holds that

$$\begin{aligned}\left\|\omega(t, x, y) - \frac{1}{2\pi} \int \omega(t, x, \cdot) dx\right\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^2}, \\ \left\|u^x(t, x, y) - \frac{1}{2\pi} \int u^x(t, x, \cdot) dx\right\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^3}, \\ \|u^y(t, x, y)\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^4}.\end{aligned}$$

Let us remark that the Gevrey regularity of the initial perturbations does not change for different Richardson numbers. The size of perturbations ϵ may vary for different Richardson numbers. We prove Theorem 1.1 in this paper. One can easily follow the same proof and obtain the following stability result for (1.6):

Theorem 1.2. *Let (ω, θ) solve (1.6). Let (u, ψ) be the corresponding velocity field and stream function. For all $\frac{1}{3} < s \leq 1$ and $\lambda_0 > \lambda' > 0$, there exists an $\epsilon_0 = \epsilon_0(\lambda_0, \lambda', s) \leq \frac{1}{2}$ such that for all $\epsilon \leq \epsilon_0$ if $(\omega_{in}, \theta_{in})$ and (u_{in}, ψ_{in}) satisfy*

$$\int u_{in} dx dy = \int \omega_{in} dx dy = \int \theta_{in} dx dy = 0,$$

$$\int |y\omega_{in}(x, y)| dx dy + \int |y\theta_{in}(x, y)| dx dy < \epsilon \text{ and}$$

$$\left\|\frac{1}{2\pi} \int_{\mathbb{T}} \psi_{in}(x, \cdot) dx\right\|_{L^1} \leq \epsilon$$

and

$$\|\omega_{in}\|_{\mathcal{G}^{\lambda_0}}^2 + \|\theta_{in}\|_{\mathcal{G}^{\lambda_0}}^2 = \sum_k \int (|\hat{\omega}_{in}(k, \eta)|^2 + |\hat{\theta}_{in}(k, \eta)|^2) e^{2\lambda_0|k, \eta|^s} d\eta \leq \epsilon^2,$$

then there exists θ_∞ with $\int \theta_\infty dx dy = 0$ and $\|\theta_\infty\|_{\mathcal{G}^{\lambda'}} \lesssim \epsilon$ such that

$$\|\theta(t, x + ty + \Phi(t, y), y) - \theta_\infty(x, y)\|_{\mathcal{G}^{\lambda'}} \lesssim \frac{\epsilon^2}{\langle t \rangle} \ln(e + t)$$

where $\Phi(t, y)$ is given explicitly by

$$\Phi(t, y) = \frac{1}{2\pi} \int_0^t \int_{\mathbb{T}} U^x(\tau, x, y) dx d\tau$$

Moreover, it holds that

$$\begin{aligned}\left\|\omega(t, x, y) - \frac{1}{2\pi} \int \omega(t, x, \cdot) dx\right\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^2}, \\ \left\|u^x(t, x, y) - \frac{1}{2\pi} \int u^x(t, x, \cdot) dx\right\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^3}, \\ \|u^y(t, x, y)\|_{L^2} &\lesssim \frac{\epsilon}{\langle t \rangle^4}.\end{aligned}$$

The results are surprising at first glance. Normally the dissipation term may have smoothing effect on the system. Then the infinite regularity assumptions on initial perturbations are not necessary. However, the buoyancy force term $\partial_x \theta$ brings the trouble. One may find some evidences about the necessity of Gevrey-3₋ regularity assumptions on the initial perturbation in Section 3. As mentioned, the buoyancy force term $\partial_x \theta$ leads to a time growth of the vorticity in the inviscid model. The presence of the viscosity term $\Delta \omega$ is a physical setting which also stabilizes the equation of vorticity but destabilizes the equation of density. It causes a significant change of the behavior even at the linear level comparing to the inviscid case. Although the vorticity ω decays as $\frac{1}{t^2}$, the density θ becomes worse and does not decay any more. To characterize the mixing effects of the buoyancy force term $\partial_x \theta$ and dissipation term $\Delta \omega$, we introduce an important good unknown K in this paper, see Section 2 for more details.

The mechanism leading to stability is the synergism of density mixing, vorticity mixing and velocity diffusion. It is similar to the inviscid damping caused by vorticity mixing. In [70], Orr observed an important phenomenon that the velocity tends to 0 as $t \rightarrow \infty$. This phenomenon is called inviscid damping. In [8], Bedrossian and Masmoudi proved nonlinear inviscid damping around the Couette flow in Gevrey class 2₋ (see also [46]). Nonlinear asymptotic stability and inviscid damping are sensitive to the topology of the perturbation. There are also some negative results. In [59], Lin and Zeng constructed periodic solutions near Couette flow. Recently, Deng and Masmoudi [30] proved some instability for initial perturbations in Gevrey class 2₊. For general shear flows, due to the presence of the nonlocal term the inviscid damping is a challenging problem even at the linear level. For the linear inviscid damping we refer to [17, 39, 48, 49, 81, 86] for the results for general monotone flows. For non-monotone flows such as the Poiseuille flow and the Kolmogorov flow, another dynamic phenomenon should be taken into consideration, which is the so-called vorticity depletion phenomenon, predicted by Bouchet and Morita [14] and later proved by Wei, Zhang and Zhao [82, 83]. Very recently, Ionescu and Jia [47], Masmoudi and Zhao [66] proved the nonlinear inviscid damping for stable monotone shear flow independently. The inviscid damping is the analogue in hydrodynamics of the Landau damping found by Landau [51] and later proved by Mouhot and Villani [69] (see also [3, 9]), which shows the rapid decay of the electric field of the Vlasov equation around homogeneous equilibrium. See [12, 45, 60, 72, 84, 87] for similar phenomena in various system.

It remains a very interesting problem to study the nonlinear asymptotic stability/instability of shear flow for the Euler Boussinesq system.

We also remark that when $\theta = 0$, the system (1.6) reduces to the 2D Navier Stokes. The stability problem of 2D Couette flow has previously been investigated. One may refer to [10, 11, 54, 64, 65] for infinite channel case, and to [7, 21] for finite channel case and to [4–6, 20, 80] for stability results of 3D Couette flow. We also refer to references [25, 32, 56, 58] for the stability results of other shear flows.

In a forthcoming paper, the small viscosity case will be studied, where the Richardson number will play an important role. Of course, under the assumption that the initial perturbations are sufficiently small (depending on the viscosity), and

by following the proof in this paper, one can prove the stability results. The main problem in the small viscosity case should be the optimality of the size.

1.1. Notation and conventions

See [8, Appendix A] for the Fourier analysis conventions we are taking. A convention we generally use is to denote the discrete x (or z) frequencies as subscripts. By convention we always use Greek letters such as η and ξ to denote frequencies in the v direction and lowercase Latin characters commonly used as indices such as k and l to denote frequencies in the x or z direction (which are discrete). Another convention we use is to denote M, N, K as dyadic integers. That is $M, N, K \in \mathbb{D}$ where

$$\mathbb{D} = \left\{ \frac{1}{2}, 1, 2, 4, 8, \dots, 2^j, \dots \right\}.$$

When a sum is written with indices K, M, M', N or N' it will always be over a subset of $\mathbb{D}$. We will mix use same A for $Af = (A_k(t, \eta) \hat{f}_k(t, \eta))^\vee$ or $A\hat{f} = A_k(t, \eta) \hat{f}_k(t, \eta)$, where A is a Fourier multiplier.

We use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \leq Cg$ (we analogously define $g \gtrsim f$). Similarly, we use the notation $f \approx g$ when there exists $C > 0$ such that $C^{-1}g \leq f \leq Cg$.

We will denote the l^1 vector norm $|k, \eta| = |k| + |\eta|$, which by convention is the norm taken in our work. Similarly, given a scalar or vector in $\mathbb{R}^n$ we denote

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}.$$

We use a similar notation to denote the x or z average of a function: $\langle f \rangle = \frac{1}{2\pi} \int f(x, y) dx = f_0$. We also frequently use the notation $f_\# = P_\# f = f - f_0$. We denote the standard L^p norms by $\|\cdot\|_p$ for $1 \leq p \leq \infty$.

For any f defined on $\mathbb{R}$, we make common use of the Gevery- $\frac{1}{s}$ norm with Sobolev correction defined by

$$\|f\|_{\mathcal{G}^{\lambda, \sigma; s}} = \sum_k \int \left| \hat{f}_k(\eta) \right|^2 e^{2\lambda|k, \eta|^s} \langle k, \eta \rangle^{2\sigma} d\eta.$$

For $\eta \geq 0$, we define $E(\eta) \in \mathbb{Z}$ to be the integer part. We define for $\eta \in \mathbb{R}$ and $1 \leq |k| \leq E(|\eta|^{\frac{1}{3}})$ with $\eta k \geq 0$, $t_{k, \eta}^- = \frac{\eta}{k} - \frac{|\eta|}{2|k|^3}$, $t_{k, \eta}^+ = \frac{\eta}{k} + \frac{|\eta|}{2|k|^3}$ and the critical intervals

$$I_{k, \eta} = \begin{cases} [t_{k, \eta}^-, t_{k, \eta}^+] & \text{if } \eta k \geq 0 \text{ and } 1 \leq |k| \leq E(|\eta|^{\frac{1}{3}}), \\ \emptyset & \text{otherwise.} \end{cases}$$

We also introduce $I_{k, \eta} \stackrel{\text{def}}{=} [t_{k, \eta}^-, t_{k, \eta}^+] \subset [\frac{2\eta}{2k+1}, \frac{2\eta}{2k-1}] \stackrel{\text{def}}{=} \bar{I}_{k, \eta}$.

2. Main Difficulties, Ideas and Sketch of the Proof

We next give the proof of Theorem 1.1, starting the primary steps as propositions which are proved in subsequent sections. The stability or instability of the steady state (1.2) for the 2D nonlinear Euler Boussinesq system (i.e., (1.1) with $v = 0$) is unknown due to the growth of vorticity, and to authors knowledge, even no partial result is available. The complexity of the problem becomes clear from the linear behavior of the vorticity of the linearized problem. In fact as shown in [85], the vorticity $w(t)$ grows roughly like $\sqrt{t}$. This seems far away from the situation of the Euler equation discussed in [8] where the vorticity stays bounded, and even without any time growth, a Gevrey-2 regularity in [8] was necessary to close the estimates and prove stability. See also [30] for a negative result if the regularity is below Gevrey-2. Hence, it seems that a stability result for the 2D nonlinear Euler Boussinesq system may not be possible even for analytic regularity, since a small perturbation of the steady state (1.2) may amplify by a very large factor. Therefore, from the stability point of view, the presence of the viscosity term Δv in (1.1) is completely justified both physically and mathematically. However the presence of viscosity will lead to other complications, since it acts as a stabilizing factor only for the vorticity equation, but due to the presence of the buoyancy forcing term $-\bar{\rho}ge_2$, the viscosity has a destabilizing effect on the equation of density, since in the absence of viscosity, ρ decays roughly like $\frac{1}{\sqrt{t}}$, but in the presence of viscosity, ρ does not decay at all. This leads to a major difficulty in the analysis and due to this fact and from the growth of the toy model in Section 3, it seems that a Gevrey-3 regularity is needed.

Hence, in order to use the damping of the vorticity equation for the density equation, we introduce a new unknown K that connect the vorticity and the density (see the definition of K in (2.2) for the linear problem and the adapted one (2.3c) for the nonlinear problem). The unknown K creates somehow a balance between the buoyancy term and the viscosity term. However, in terms of analysis it leads to some extra terms that we should control carefully. (See the definition of $\mathbf{H}$ in (2.4)).

Another issue in the proof is that even in the presence of viscosity, it seems not possible to use the nonlinear coordinate systems introduced in [10], since this leads to a shear flow term in the equation of density, which cannot be controlled. So, due to this we rely on an inviscid change of coordinates as in [8]. However, due to the presence of the viscosity, the control of the coordinate systems is different from the one in [8] and the extra L^1 -control (1.7) is needed to get enough decay of the coordinates.

Also, compared to [8] the norm introduced here (see (2.8)) contains the two extra components $\mathcal{M}_k(t, \eta)$ and $B_k(t, \eta)$. The multiplier $\mathcal{M}_k(t, \eta)$ is used to control the growth in appropriate time regime and together with $B_k(t, \eta)$ they have been also used as “ghost” weight in phase place to control the growth coming from some linear terms.

One of the key ideas in the proof of Theorem (1.1) is the construction of time-dependent norm which contains several components, each component is introduced to control the growth predicted by the toy model in different time regimes. (See Section 3 for more details). Armed with such a norm, and by applying energy

estimates, we were able to allow the loss of regularity at specific frequency and time and it enables us also to pay regularity for time decay in order to close the energy estimates. Another complicated issue in the proof is the absence of any damping in the density equation, for this reason we need to find a nice combination that connects the density to the velocity (see the definition of K in (2.3c)). This combination allows us to transfer damping from the velocity equation to the density equation. Another important remark, which is well known in this Gevrey-type estimates is that by allowing λ to shrink (see (2.9)) we were able to introduce the CK_λ terms that will play a role of an extra damping term that will help to control many terms in some specific time regime.

2.1. Linearized behavior and an important good unknown

Before beginning the proof of Theorem 1.1, we discuss the linearized behavior in more detail and mention some of the main challenges that must be overcome for a nonlinear result. The linearized equation of (1.5) or (1.6) can be written as:

$$\begin{cases} \partial_t \omega + y \partial_x \omega = -\gamma^2 \partial_x \theta + \Delta \omega, \\ \partial_t \theta + y \partial_x \theta = \gamma_1 u^y, \\ u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega, \end{cases} \quad (2.1)$$

where the parameters γ, γ_1 varies in different cases. Let us consider the simple case: $\gamma = 1$ and $\gamma_1 = 0$, which are the parameters of the linearized equation of (1.6). Now we introduce the linear change of coordinates:

$$z = x - ty, \quad f(t, z, y) = \omega(t, x, y), \quad \rho(t, z, y) = \theta(t, x, y).$$

From (2.1) with $\gamma = 1$ and $\gamma_1 = 0$, we have

$$\begin{cases} \partial_t f = -\partial_z \rho + (\partial_v - t \partial_z)^2 f + \partial_{zz} f, \\ \partial_t \rho = 0, \end{cases}$$

which gives us that $\hat{\rho}(t, k, \eta) = \hat{\rho}_{in}(k, \eta)$, and, for $k \neq 0$,

$$\hat{f}(t, k, \eta) = e^{-\int_0^t (ks-\eta)^2 + k^2 ds} \left(\hat{f}_{in}(k, \eta) - ik \hat{\rho}_{in}(k, \eta) \int_0^t e^{\int_0^\tau (ks-\eta)^2 + k^2 ds} d\tau \right).$$

By the fact that

$$e^{-\int_0^t (ks-\eta)^2 + k^2 ds} \int_0^t e^{\int_0^\tau (ks-\eta)^2 + k^2 ds} d\tau \lesssim \frac{1}{(kt - \eta)^2 + k^2},$$

we get that

$$|\hat{f}(t, k, \eta)| \lesssim e^{-\int_0^t (ks-\eta)^2 + k^2 ds} |\hat{f}_{in}(k, \eta)| + \frac{k}{(kt - \eta)^2 + k^2} |\hat{\rho}_{in}(k, \eta)|.$$

However for the nonlinear system and the case $\gamma_1 \neq 0$ even at the linear level, we can not write the precise formula for the solutions. It is not just a technical difficulty.

Indeed, to obtain the behavior of the solution, we should balance the effect of the buoyancy force term $\partial_x \theta$ and dissipation term $\Delta \omega$. We define the following “good” unknown:

$$K = -\gamma^2 \partial_z \rho + \Delta_L f \quad (2.2)$$

which has good properties. Indeed we have

$$\partial_t \hat{K} + (k^2 + (\eta - kt)^2) \hat{K} = -\gamma^2 \frac{k^2}{k^2 + (\eta - kt)^2} \hat{f} + 2k(\eta - kt) \hat{f},$$

which gives us that

$$\begin{aligned} & \frac{1}{2} \partial_t |\hat{K}|^2 + (k^2 + (\eta - kt)^2) |\hat{K}|^2 \\ & \leq \gamma^2 \frac{k^2}{k^2 + (\eta - kt)^2} |f| |\hat{K}| + 2k|\eta - kt| |f| |\hat{K}| \\ & \leq \gamma^2 \frac{k^2}{k^2 + (\eta - kt)^2} |\hat{f}| |\hat{K}| + 10k^2 |\hat{f}|^2 + \frac{1}{10} (\eta - kt)^2 |\hat{K}|^2. \end{aligned}$$

Note that K satisfies a diffusion equation (or transport diffusion equation in the (t, x, y) coordinate) with forcing terms that decay fast enough.

Therefore we get $|\hat{K}(t, k, \eta)| \lesssim |\hat{K}_{in}(k, \eta)|$ and

$$|\hat{f}(t, k, \eta)| \lesssim \frac{1}{(kt - \eta)^2 + k^2} (|k| |\hat{\rho}_{in}| + |\hat{K}_{in}|).$$

The good unknown K characterizes the mixing effects of the buoyancy force term $\partial_x \theta$ and dissipation term $\Delta \omega$, which is one of the key structures we found in this system. For the nonlinear system, K will change slightly due to the nonlinear change of coordinates, but we will use, without ambiguity, the same notation, see (2.3c) below.

2.2. Coordinate transform

In order to tackle the nonlinear problem (1.5), we make suitable nonlinear change of variables. The basic idea of this change of variables is to get a rid of the zero mode. There are two different types of change of coordinates: one is the inviscid one for the Euler equation, see [8], the other one is the viscous one for the Navier–Stokes equation, see [10]. The coordinate systems were chosen in a very natural way in both cases. However, in this paper, the diffusion term only appears in the vorticity equation, the density equation is a transport equation. It is always a challenge problem when the diffusion terms are different in one system, see [79]. Here we use the inviscid change of coordinates.

Let

$$u_0^x(t, y) = -\frac{1}{2\pi} \int_{\mathbb{T}} \partial_y \psi(t, x, y) dx = \frac{1}{2\pi} \int_{\mathbb{T}} u^x(t, x, y) dx,$$

then system (2.1) can be rewritten as

$$\begin{cases} \partial_t \omega + (y + u_0^x(t, y)) \partial_x \omega + (\nabla^\perp \psi_\# \cdot \nabla) \omega = -\gamma^2 \partial_x \theta - \Delta \omega, \\ \partial_t \theta + (y + u_0^x(t, y)) \partial_x \theta + (\nabla^\perp \psi_\# \cdot \nabla) \theta = \gamma_1 \partial_x \psi. \end{cases}$$

To remove the zero mode from the above system, we introduce the change of variables $(t, x, y) \rightarrow (t, z, v)$ with

$$\begin{cases} z = x - tv, \\ v = y + \frac{1}{t} \int_0^t \frac{1}{2\pi} \int_{\mathbb{T}} u^x(\tau, x, y) dx d\tau. \end{cases}$$

We then have

$$\frac{d}{dt} (t(\partial_y v(t, y) - 1)) = -\omega_0(t, y).$$

Let

$$\begin{aligned} f(t, z, v) &= \omega(t, x, y), \quad \rho(t, z, v) = \theta(t, x, y), \quad \phi(t, z, v) = \psi(t, x, y), \\ v'(t, v) &= \partial_y v(t, y), \quad v''(t, v) = \partial_{yy} v(t, y), \quad g(t, v) = \partial_t v(t, y), \\ \tilde{u}_0(t, v) &= u_0^x(t, y), \quad h(t, v) = v'(t, v) - 1. \end{aligned}$$

Then $v''(t, v) = \frac{1}{2} \partial_v ((v')^2 - 1)$ and ϕ satisfies the equation

$$\Delta_t \phi \stackrel{\text{def}}{=} \left[\partial_{zz} + (v')^2 (\partial_v - t \partial_z)^2 + v'' (\partial_v - t \partial_z) \right] \phi = f,$$

where Δ_t can be regarded as a perturbation of Δ_L .

Hence, we obtain the following equations for f :

$$\partial_t f + \mathbf{u} \cdot \nabla_{z,v} f = -\gamma^2 \partial_z \rho + \Delta_t f, \quad \Delta_t \phi = f,$$

with

$$\mathbf{u}(t, z, v) = \begin{pmatrix} 0 \\ g \end{pmatrix} + v' \nabla_{z,v}^\perp P_\# \phi.$$

We also obtain the following equation for ρ :

$$\begin{aligned} \partial_t \rho + \mathbf{u} \cdot \nabla_{z,v} \rho &= \gamma_1 \partial_z \phi, \\ \Delta_t \phi &= f, \end{aligned} \tag{2.3a}$$

Following the argument of the linearized system, let us introduce

$$K = -\gamma^2 \partial_z \rho + \Delta_t f. \tag{2.3b}$$

Hence, K satisfies the equation

$$\partial_t K + \mathbf{u} \cdot \nabla_{z,v} K - \Delta_t K = \mathbf{H} - \gamma^2 \gamma_1 \partial_{zz} \phi - 2(\partial_v - t \partial_z) \partial_z f, \tag{2.3c}$$

where

$$\begin{aligned} \mathbf{H} &= \gamma^2 v' \nabla_{z,v}^\perp \partial_z P_\# \phi \cdot \nabla_{z,v} \rho - 2h(\partial_v - t \partial_z) \partial_z f + 2f_0 v' \partial_z (\partial_v - t \partial_z) f \\ &\quad - 2(v')^3 (\partial_v - t \partial_z) \nabla_L^\perp \phi_\# \cdot \nabla_L (\partial_v - t \partial_z) f - 2v' \nabla_L^\perp \partial_z \phi_\# \cdot \nabla_L \partial_z f \\ &\quad - \partial_z (v'' v' (\partial_v - t \partial_z) \phi_\# (\partial_v - t \partial_z) f). \end{aligned} \tag{2.4}$$

We also have

$$\partial_t f_0 + g \partial_v f_0 - (v')^2 \partial_{vv} f_0 - v'' \partial_v f_0 + v' < \nabla^\perp \phi_\# \cdot \nabla_{z,v} f_\# > = 0. \quad (2.5a)$$

Therefore,

$$\partial_t h + g \partial_v h = -\frac{1}{t} (f_0 + h) \quad (2.5b)$$

and

$$\partial_t g + \frac{2}{t} g + g \partial_v g = -\frac{1}{t} v' < \nabla^\perp \phi_\# \cdot \nabla \tilde{u}_\# > + \frac{1}{t} v' \partial_v f_0,$$

where $\tilde{u}(t, z, v) = u(t, x, y)$. Let

$$\bar{h} = -\frac{1}{t} (f_0 + h) = v' \partial_v g. \quad (2.5c)$$

Then $\bar{h}$ satisfies

$$\partial_t \bar{h} + \frac{2}{t} \bar{h} + g \partial_v \bar{h} + \frac{1}{t} K_0 = \frac{1}{t} v' < \nabla_{z,v}^\perp \phi_\# \cdot \nabla_{z,v} f > .$$

In the sequel, we will perform the estimates using system (2.3) together with (2.5).

2.2.1. Discussion of the nonlinear change of coordinates In this paper, we use the inviscid nonlinear change of coordinates (see [10] for the viscous nonlinear change of coordinates).

By the definition of g , an easy calculation shows that

$$g(t, v) = \partial_t v(t, y) = \frac{1}{t^2} \int_0^t s \partial_t u_0^x(s, y) ds,$$

where the zero mode of velocity satisfies

$$\partial_t u_0^x - \partial_{yy} u_0^x = -\partial_y < \partial_x \psi u^x > . \quad (2.6)$$

In order to get enough decay of g in lower regularity, we introduce the equation of the stream function. By the fact that $\int u_{in}(x, y) dx dy = \int \theta_{in}(x, y) dx dy = 0$, we have for any $t \geq 0$, $\int u(t, x, y) dx dy = 0$. Thus the average of the stream function $\psi_0(t, y) = \frac{1}{2\pi} \int \psi(t, x, y) dx$ satisfies the 1D nonlinear heat equation

$$\partial_t \psi_0 - \partial_{yy} \psi_0 = < \partial_x \psi u^x > \quad (2.7)$$

and $u_0^x(t, y) = -\partial_y \psi_0(t, y)$.

2.3. Main energy estimate

We will use a carefully designed time-dependent norms written as

$$\|A(t, \nabla)K\|_2^2 = \sum_k \int_{\eta} |A_k(t, \eta) \widehat{K}_k(t, \eta)|^2 d\eta,$$

and

$$\|A(t, \nabla)\rho\|_2^2 = \sum_k \int_{\eta} |A_k(t, \eta) \widehat{\rho}_k(t, \eta)|^2 d\eta.$$

The multiplier A has several components

$$A_k(t, \eta) = e^{\lambda(t)|k, \eta|^s} \langle k, \eta \rangle^\sigma J_k(t, \eta) \mathcal{M}_k(t, \eta) B_k(t, \eta). \quad (2.8)$$

The index $\lambda(t)$ is the bulk Gevrey- $\frac{1}{s}$ regularity and will be chosen to satisfy (see [8])

$$\lambda(t) = \frac{3}{4}\lambda_0 + \frac{1}{4}\lambda', \quad t \leq 1$$

and

$$\frac{d}{dt}\lambda(t) = -\frac{\delta_\lambda}{\langle t \rangle^{2\tilde{q}}}(1 + \lambda(t)), \quad t > 1, \quad (2.9)$$

where $\delta_\lambda \approx \lambda_0 - \lambda'$ is a small parameter that ensures $\lambda(t) > \lambda_0/2 + \lambda'/2$ and $\tilde{q}$ is a parameter that will be determined by the proof. Let us also remark here that to study analytic data, $s = 1$, we would need to add an additional Gevrey- $\frac{1}{s'}$ correction to A with $s' \in (\frac{1}{3}, 1)$ as an intermediate regularity so that we may take advantage of certain beneficial properties of Gevrey spaces.

The main multipliers for dealing with the nonlinear interaction are

$$J_k(t, \eta) = \frac{e^{\mu|\eta|^{\frac{1}{3}}}}{\Theta_k(t, \eta)} + e^{\mu|k|^{\frac{1}{3}}},$$

and

$$\mathcal{M}_k(t, \eta) = \frac{e^{4\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}}{g(t, \eta)} + e^{4\pi\delta_L^{-1}|k|^{\frac{1}{3}}},$$

with $\delta_L > 0$ being a small enough constant that will be determined by the linear nonlocal term. The weights $\Theta_k(t, \eta)$ and $g(t, \eta)$ are constructed in Section 3.

The multiplier B is defined as follows:

$$B_k(t, \eta) = \exp\left(\delta_B^{-1} \int_0^t \frac{b(s, k, \eta)}{1 + (s - \frac{\eta}{k})^2} ds\right).$$

Here

$$b(t, k, \eta) = \chi\left(\frac{100}{t}\right) \chi\left(\frac{\eta}{t^3}\right) \chi\left(\frac{\eta}{k^3}\right) \chi_1\left(\frac{\eta}{kt}\right)$$

where $0 \leq \chi(x) \leq 1$ is a smooth cut-off function satisfying $\chi(x) \equiv 1$ for $|x| \leq 8$ and $\text{supp } \chi \subset [-10, 10]$ and $0 \leq \chi_1(x) \leq 1$ is a smooth cut-off function satisfying $\chi_1(x) \equiv 1$ when $\frac{1}{2} \leq |x| \leq \frac{3}{2}$ and $\text{supp } \chi \subset [\frac{1}{3}, \frac{5}{2}]$.

Thus we get that $B_0(t, \eta) \equiv 1$ and

$$B_k(t, \eta) \approx_{\delta_B} 1.$$

Note that

$$\sup_{t,k} |\partial_\eta b(t, k, \eta)| \lesssim \frac{1}{\langle \eta \rangle}.$$

With this special norm, we can define our main energy:

$$E(t) = \frac{1}{2} \|A(t, \nabla) K\|_2^2 + \frac{1}{2} \|A(t, \nabla) \rho\|_2^2, \quad E_d(t) = \frac{1}{2} \langle t \rangle \|A \langle \partial_v \rangle^2 h\|_2^2$$

and

$$E_{lo, f_0}(t) = \sum_{k=0}^3 \frac{t^k}{4^k} \|\partial_v^k f_0\|_{\mathcal{G}^{\lambda(t), \beta; s}}^2,$$

$$E_{lo, g}(t) = \langle t \rangle^4 \|\partial_v^3 g\|_{\mathcal{G}^{\lambda(t), \beta; s}}^2, \quad E_{lo, h}(t) = \langle t \rangle^2 \|\partial_v^2 h\|_{\mathcal{G}^{\lambda(t), \beta; s}}^2.$$

The assistant energy is

$$E_{as, \psi_0}(t) = \max \left(\langle t \rangle \|\psi_0\|_{L^\infty}^2, \langle t \rangle^4 \|\partial_{yyy} \psi_0\|_{L^\infty}^2, \langle t \rangle^{\frac{5}{2}} \|\psi_0\|_{\dot{H}^2}^2, \langle t \rangle^{\frac{9}{2}} \|\psi_0\|_{\dot{H}^4}^2 \right),$$

$$E_{as, g}(t) = \max \left(\frac{\langle t \rangle^4 \|g\|_{L^\infty}^2}{(1 + \ln \langle t \rangle)^2}, \langle t \rangle^{\frac{7}{2}} \|g\|_{L^2}^2, \epsilon_1^2 \langle t \rangle^{4-\epsilon_1} \|g\|_{\dot{H}^{\frac{1}{2}-\epsilon_1}}^2, \epsilon_2^2 \langle t \rangle^4 \|g\|_{\dot{H}^{\frac{1}{2}+\epsilon_2}}^2 \right),$$

where $0 < \sqrt{\epsilon} \leq \epsilon_1^2$ with $0 < \epsilon_1 \leq 3s + 1 - 4\tilde{q}$ and $0 < \epsilon_2 < \frac{1}{2}$.

Let us make a technical remark about the assistant energy here: We use the inviscid change of coordinate which brings us new challenge since the equation are not both inviscid. It leads to the appearance of the term $\frac{1}{t} v' \partial_v f_0$ in the equation of g . Then it holds that $\|g\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}$, which decays not fast enough to control the main energy. However, the assistant energy together with Proposition 2.3 shows that the homogeneous norms $\|g\|_{\dot{H}^{\frac{1}{2}-\epsilon_1}}, \|g\|_{\dot{H}^{\frac{1}{2}+\epsilon_2}}, \|\partial_v g\|_{\mathcal{G}^{\lambda, \beta; s}}$, the Fourier L^1 norms $\|\hat{g}\|_{L^1}, \|g\|_{\mathcal{G}_1^{\lambda, \beta; s}}$ and $\|g\|_{L^\infty}$ have better decay rate. Luckily we can use these norms of g to close our energy estimate.

It is natural to compute the time evolution of $\|A(t, \nabla) K\|_2^2$ and $\|A(t, \nabla) \rho\|_2^2$. To lighten notations, we define, for $\varphi \in \{K, \rho\}$, the following:

$$CK_{\lambda, \varphi} = -\dot{\lambda}(t) \|\nabla|^{\frac{s}{2}} A \varphi\|_{L^2}^2,$$

$$CK_{\Theta, \varphi} = \sum_k \int_\eta \frac{\partial_t \Theta_k(t, \eta)}{\Theta_k(t, \eta)} e^{\lambda(t)|k, \eta|^s} \langle k, \eta \rangle^\sigma \frac{e^{\mu|\eta|^{\frac{1}{3}}}}{\Theta_k(t, \eta)}$$

$$\mathcal{M}_k(t, \eta) B_k(t, \eta) A_k(t, \eta) |\widehat{\varphi}_k(t, \eta)|^2 d\eta,$$

$$\begin{aligned}
\mathrm{CK}_{M,\varphi} &= \sum_k \int_{\eta} \frac{\partial_t \mathbf{g}(t, \eta)}{\mathbf{g}(t, \eta)} e^{\lambda(t)|k, \eta|^s} \langle k, \eta \rangle^{\sigma} \frac{e^{4\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}}{\mathbf{g}(t, \eta)} \\
&\quad \mathbf{J}_k(t, \eta) \mathbf{B}_k(t, \eta) \mathbf{A}_k(t, \eta) |\widehat{\varphi}_k(t, \eta)|^2 d\eta, \\
\mathrm{CK}_{B,\varphi} &= \delta_B^{-1} \sum_k \int_{\eta} \frac{b(t, k, \eta)}{1 + (t - \frac{\eta}{k})^2} |\mathbf{A}_k(t, \eta) \widehat{\varphi}_k(t, \eta)|^2 d\eta.
\end{aligned}$$

Here CK stands for ‘Cauchy–Kovalevskaya’. In what follows, we define

$$\tilde{\mathbf{A}}_k(t, \eta) = e^{\lambda(t)|k, \eta|^s} \langle k, \eta \rangle^{\sigma} \frac{e^{\mu|\eta|^{\frac{1}{3}}}}{\Theta_k(t, \eta)} \mathcal{M}_k(t, \eta) \mathbf{B}_k(t, \eta)$$

and

$$\tilde{\tilde{\mathbf{A}}}_k(t, \eta) = e^{\lambda(t)|k, \eta|^s} \langle k, \eta \rangle^{\sigma} \frac{e^{4\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}}{\mathbf{g}(t, \eta)} \mathbf{J}_k(t, \eta) \mathbf{B}_k(t, \eta)$$

which satisfy $\tilde{\mathbf{A}} \leq \mathbf{A}$, $\tilde{\tilde{\mathbf{A}}} \leq \mathbf{A}$. In particular if $|k| \leq |\eta|$ then $\mathbf{A} \lesssim \tilde{\mathbf{A}}$ and $\mathbf{A} \lesssim \tilde{\tilde{\mathbf{A}}}$.

First, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int |\mathbf{A}\rho|^2 dz dv &= -\mathrm{CK}_{\lambda, \rho} - \mathrm{CK}_{\Theta, \rho} - \mathrm{CK}_{M, \rho} - \mathrm{CK}_{B, \rho} \\
&\quad - \int \mathbf{A}\rho \mathbf{A}(\mathbf{u} \cdot \nabla \rho) dz dv + \int \mathbf{A}\rho \mathbf{A}(\partial_z \phi) dz dv \\
&= -\mathrm{CK}_{\lambda, \rho} - \mathrm{CK}_{\Theta, \rho} - \mathrm{CK}_{M, \rho} - \mathrm{CK}_{B, \rho} - \mathrm{NL}_{\rho} + \Pi_{\rho}.
\end{aligned} \tag{2.10}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int |\mathbf{A}K|^2 dz dv &= -\mathrm{CK}_{\lambda, K} - \mathrm{CK}_{\Theta, K} - \mathrm{CK}_{M, K} - \mathrm{CK}_{B, K} \\
&\quad + \int \mathbf{A}K \mathbf{A}(\Delta_t K) dz dv - \int \mathbf{A}K \mathbf{A}(\mathbf{u} \cdot \nabla K) dz dv \\
&\quad + \int \mathbf{A}K \mathbf{A}(\mathbf{H}) dz dv + \gamma^2 \gamma_1 \int \mathbf{A}K \mathbf{A}(\partial_{zz} \phi) dz dv \\
&\quad - 2 \int \mathbf{A}K \mathbf{A}((\partial_v - t \partial_z) \partial_z f) dz dv \\
&= -\mathrm{CK}_{\lambda, K} - \mathrm{CK}_{\Theta, K} - \mathrm{CK}_{M, K} - \mathrm{CK}_{B, K} \\
&\quad + \mathbf{E} - \mathrm{NL}_K^1 - \mathrm{NL}_K^2 + \Pi_K^1 + \Pi_K^2.
\end{aligned} \tag{2.11}$$

2.4. Bootstrap argument and main propositions

We prove the theorem by a bootstrap argument. In order to avoid discussing the fake singularity in the equations as $t \rightarrow 0_+$, let us first give the following local-wellposedness theory.

Lemma 2.1. *Under the same assumptions of Theorem 1.1 or 1.2, we have*

$$\sup_{t \in [0, 1]} E(t) + E_d(t) \leq \epsilon^2.$$

The lemma is easy to obtain by using the energy method to the transport equation and we omit the proof here.

The goal is next to prove by a continuity argument that this energy $E(t) + E_d(t)$ together with $E_{l o, g}(t)$, $E_{l o, h}(t)$, $E_{a s, \psi_0}(t)$ and $E_{a s, f_0}(t)$ are uniformly bounded for all time if ϵ is small enough. We define the following controls referred to in the sequel as the bootstrap hypotheses for $t \geq 1$ and some constants C_0 , $K_d \geq 1$ independent of ϵ and determined in the proof,

$$\begin{aligned} E(t) + K_d^{-1} E_d(t) &\leq 10C_0 \epsilon^2, \\ \int_1^t &\left(CK_{\lambda, K} + CK_{\Theta, K} + CK_{M, K} + CK_{B, K} + \|\nabla_L A K\|_2^2 \right. \\ &\quad \left. + CK_{\lambda, \rho} + CK_{\Theta, \rho} + CK_{M, \rho} + CK_{B, \rho} \right)(s) ds \leq 10C_0 \epsilon^2, \end{aligned} \quad (2.12)$$

$$\int_1^t \left(CK_{\lambda, h} + CK_{\Theta, h} + CK_{M, h} + \|A \langle \partial_v \rangle^2 h\|_2^2 \right)(s) ds \leq 10K_d C_0 \epsilon^2 \quad (2.13)$$

where

$$\begin{aligned} CK_{\lambda, h}(t) &= -\dot{\lambda}(t) \langle t \rangle \left\| |\partial_v|^{\frac{5}{2}} A \langle \partial_v \rangle^2 h \right\|_2^2, \\ CK_{\Theta, h}(t) &= \langle t \rangle \left\| A \langle \partial_v \rangle^2 \sqrt{\frac{\partial_t \Theta}{\Theta}} h \right\|_2^2, \quad CK_{M, h}(t) = \langle t \rangle \left\| A \langle \partial_v \rangle^2 \sqrt{\frac{\partial_t g}{g}} h \right\|_2^2. \end{aligned}$$

The main proposition of this paper is as follows:

Proposition 2.2. (Bootstrap) *There exists an $\epsilon_0 \in (0, \frac{1}{2})$ depending only on λ , λ' , s and σ such that if $\epsilon < \epsilon_0$, and on $[1, T^*]$ the bootstrap hypotheses (2.12)–(2.13) hold, then for $\forall t \in [1, T^*]$,*

$$\begin{aligned} E(t) + K_d^{-1} E_d(t) &\leq 8C_0 \epsilon^2 \\ \int_1^t &\left(CK_{\lambda, K} + CK_{\Theta, K} + CK_{M, K} + CK_{B, K} + \|\nabla_L A K\|_2^2 \right. \\ &\quad \left. + CK_{\lambda, \rho} + CK_{\Theta, \rho} + CK_{M, \rho} + CK_{B, \rho} \right)(s) ds \leq 8C_0 \epsilon^2, \\ \int_1^t &\left(CK_{\lambda, h} + CK_{\Theta, h} + CK_{M, h} + \|A \langle \partial_v \rangle^2 h\|_2^2 \right)(s) ds \leq 8K_d C_0 \epsilon^2, \end{aligned}$$

from which it follows that $T^* = +\infty$.

The main purpose of this paper is to prove Proposition 2.2, which follows from the following propositions by taking ϵ , δ_L , δ_B sufficiently small and M_0 sufficiently large. (See Section 9.4 for the determination of M_0 .)

First we control the lower energy and the assistant energy and obtain the following proposition. The proof is given in Section 4.

Proposition 2.3. *Under the bootstrap hypotheses, for any $0 < \sqrt{\epsilon} \leq \epsilon_1^2$ with $0 < \epsilon_1 \leq 3s + 1 - 4\tilde{q}$ and $0 < \epsilon_2 < \frac{1}{2}$, it holds that*

$$\langle t \rangle^2 \left\| A \partial_v^3 g \right\|_2^2 + \int_1^t \langle s \rangle^2 \left\| A \partial_v^3 g \right\|_2^2(s) ds \lesssim \epsilon^2.$$

and

$$E_{as, \psi_0}(t) \lesssim \epsilon^2, \quad E_{as, g}(t) \lesssim \epsilon^2,$$

As a corollary, it holds that

$$\|h\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle}, \quad \|f_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{5}{4}}, \quad \|\partial_v f_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}.$$

We also get that

$$E_{lo, f_0}(t) + E_{lo, g}(t) + E_{lo, h}(t) \lesssim \epsilon^2.$$

As a corollary, it holds that

$$\|g\|_{\mathcal{G}_1^{\lambda, \beta; s}} \stackrel{\text{def}}{=} \int_{\mathbf{R}} \langle \xi \rangle^\beta e^{\lambda(t)|\xi|^s} |\hat{g}(t, \xi)| d\xi \lesssim \frac{\epsilon}{\epsilon_1^{3/2}} \langle t \rangle^{-2+\frac{\epsilon_1}{2}} \lesssim \sqrt{\epsilon} \langle t \rangle^{-2+\frac{\epsilon_1}{2}}.$$

Proposition 2.4. *Under the bootstrap hypotheses, it holds that*

$$\begin{aligned} |\text{NL}_\rho| + |\text{NL}_K^1| &\lesssim \frac{\epsilon^3}{\langle t \rangle^2} + \epsilon^2 \langle t \rangle^2 \|A \partial_v^3 g\|_2^2 + \sqrt{\epsilon} \text{CK}_{\lambda, \rho} + \epsilon \text{CK}_{\Theta, \rho} + \epsilon \text{CK}_{M, \rho} \\ &\quad + \sqrt{\epsilon} \text{CK}_{\lambda, K} + \epsilon \text{CK}_{\Theta, K} + \epsilon \text{CK}_{M, K} \\ &\quad + \epsilon \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t g}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2. \end{aligned}$$

The proof of Proposition 2.4 is given in Section 6 and it is one of the main parts of the proof.

Proposition 2.5. *Under the bootstrap hypotheses, it holds that*

$$\begin{aligned} |\text{NL}_K^2| &\lesssim \frac{\epsilon^3}{\langle t \rangle^2} + \epsilon \|\nabla_L A K\|_2^2 + \frac{\epsilon}{\langle t \rangle^2} \left\| A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ &\quad + \epsilon \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t g}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2. \end{aligned}$$

The proof of Proposition 2.5 is the subject of Section 8.

In Propositions 2.6, 2.7 and 2.8, we state the estimates of the linear terms Π_ρ , E , Π_K^1 and Π_K^2 appearing in (2.10) and (2.11). The proofs of such estimates are given in Section 9.

Proposition 2.6. *Under the bootstrap hypotheses, it holds that*

$$|\Pi_\rho| \leq \frac{C\epsilon}{\langle t \rangle^2} \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq f} \phi \right\|_2 + C_1 \delta_L \text{CK}_{M,\rho} \\ + \delta_L \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq f} \phi \right\|_2^2,$$

where C_1 is a constant independent of δ_L .

Proposition 2.7. *Under the bootstrap hypotheses, it holds that*

$$E \leq -\frac{7}{8} \|\nabla_L A K\|_2^2 + \frac{C\epsilon^3}{\langle t \rangle^2}.$$

Proposition 2.8. *Under the bootstrap hypotheses, it holds that*

$$|\Pi_K^1| + |\Pi_K^2| \lesssim \frac{C}{\langle t \rangle^4} \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq f} \phi \right\|_2^2 + \frac{C}{\langle t \rangle^4} \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2 \\ + \frac{1}{8} \|\nabla_L A K\|_2^2 + C_1 \delta_L \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq f} \phi \right\|_2^2 \\ + C_1 \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{5}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2 \\ + C_1 \delta_B \left\| \sqrt{\frac{b(t, \nabla) \partial_{zz}}{\Delta_L}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2,$$

where C_1 is a constant independent of δ_L , δ_B and M_0 .

Proposition 2.9. *Under the bootstrap hypotheses, it holds that*

$$\left\| A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2 \lesssim \epsilon. \quad (2.14)$$

The proof of Proposition 2.9 is the subject of Section 5.1.

Proposition 2.10. *Under the bootstrap hypotheses, for some $C_1 \geq 1$ independent of M_0 , it holds that*

$$\left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{5}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2 \leq C_1 M_0^{2\tilde{q}-3s} (\text{CK}_{\lambda,K} + \text{CK}_{\lambda,\rho}) + C\epsilon^2 \text{CK}_{\lambda,h} \quad (2.15)$$

and for some $C_2 \geq 1$ independent of δ_L , it holds that

$$\left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \tilde{A} \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2 \\ \leq C_2 (\text{CK}_{M,K} + \text{CK}_{\lambda,K} + \text{CK}_{M,\rho} + \text{CK}_{\lambda,\rho}) + C\epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2, \quad (2.16)$$

and for some $C_3 \geq 1$ independent of δ_B , it holds that

$$\left\| \sqrt{\frac{b(t, \nabla) \partial_{zz}}{\Delta_L}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq f} \right\|_2^2 \leq C_3 (\text{CK}_{B,K} + \text{CK}_{B,\rho}) + C \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2.$$

The proof of Proposition 2.10 is given in Section 5.3.

Proposition 2.11. *Under the bootstrap hypotheses, it holds that*

$$\left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2 \lesssim \epsilon.$$

The proof of Proposition 2.11 is detailed in Section 5.2

Proposition 2.12. *Under the bootstrap hypotheses, it holds that*

$$\begin{aligned} & \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t \tilde{g}}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2^2 \\ & \leq C_1 (\text{CK}_{\lambda,K} + \text{CK}_{\Theta,K} + \text{CK}_{M,K} + \text{CK}_{\lambda,\rho} + \text{CK}_{\Theta,\rho} + \text{CK}_{M,\rho}) \\ & \quad + C_2 (\epsilon^2 \text{CK}_{\lambda,h} + \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2), \end{aligned} \quad (2.17)$$

where the constant C_1 is independent of δ_L and the constant C_2 may depend on δ_L .

The proof of Proposition 2.12 is the subject of Section 5.4.

2.5. Conclusion of the proof

Now, to prove the estimates of the main Theorems 1.1 and 1.2, we need to recover the estimates on the original systems (1.5) and (1.6). So, we need to undo the coordinate transform and transfer the uniform energy bound on $E(t) + E_d(t)$ in the (z, v) variables into the original (x, y) variables. This requires the use of an inverse function theorem in Gevrey spaces. We refer to [8, Remark 7 and Section 2.4] for more details. Here we only focus on the scattering results.

Our goal now is to prove (1.8). We have by using the second equation in (2.3a), and for some $\lambda' < \lambda_1 < \lambda_0/2 + \lambda'/2 < \lambda$

$$\begin{aligned} \|\rho(t) - \rho_\infty\|_{\mathcal{G}^{\lambda_1,0;s}} & \lesssim \left\| \int_t^\infty g \partial_v \rho \, ds \right\|_{\mathcal{G}^{\lambda_1,0;s}} + \left\| \int_t^\infty v' \nabla_{z,v}^\perp P_{\neq \phi} \cdot \nabla_{z,v} \rho(s) \, ds \right\|_{\mathcal{G}^{\lambda_1,0;s}} \\ & \quad + \left\| \int_t^\infty \partial_z \phi(s) \, ds \right\|_{\mathcal{G}^{\lambda_1,0;s}}. \end{aligned} \quad (2.18)$$

We estimate first $\|g \partial_v \rho\|_{\mathcal{G}^{\lambda_1,0;s}}$. We have by using the physical definition of the Gevrey spaces (see (B.2)) together with Lemma B.2

$$\begin{aligned} \|g \partial_v \rho\|_{\mathcal{G}^{\lambda_1;s}} & = \|g \partial_v \rho\|_{\ell^2 L^2; \lambda_1} \lesssim \|g \partial_v \rho\|_{\ell^1 L^2; \lambda_1} \\ & \lesssim \|\partial_v \rho\|_{\ell^1 L^2; \lambda_1} \|g\|_{\ell^1 L^\infty; \lambda_1}. \end{aligned}$$

We have that

$$\begin{aligned} \|g\|_{\ell^1 L^\infty; \lambda_1} &\lesssim \|g\|_{L^\infty} + \|\partial_v^3 g\|_{\ell^1 L^\infty; \lambda_1} \\ &\lesssim \|g\|_{L^\infty} + \|\partial_v^3 g\|_{\ell^2 L^\infty; \lambda_2} \lesssim \|g\|_{L^\infty} + \|\partial_v^3 g\|_{\mathcal{G}^{\lambda_2, 3; s}} \end{aligned} \quad (2.19)$$

where we have used (B.3) for some $\lambda_0/2 + \lambda'/2 > \lambda_3 > \lambda_2 > \lambda_1$.

We also have, by (B.3), that

$$\|\partial_v \rho\|_{\ell^1 L^2; \lambda_1} \lesssim_{\lambda_2 - \lambda_1} \|\partial_v \rho\|_{\ell^2 L^2; \lambda_2}. \quad (2.20)$$

Hence, (2.19) and (2.20) together with the bootstrap assumption, yield

$$\|g \partial_v \rho\|_{\mathcal{G}^{\lambda_1, 0; s}} \lesssim \|g\|_{L^\infty} \|\rho\|_{\mathcal{G}^{\lambda_2, 0; s}} + \|\partial_v^3 g\|_{\mathcal{G}^{\lambda_3, 0; s}} \|\rho\|_{\mathcal{G}^{\lambda_2, 0; s}} \lesssim \epsilon^2 \frac{\ln(e+t)}{\langle t \rangle^2}. \quad (2.21)$$

The last two terms can be easily estimated using the lossy elliptic estimates (see Lemma 5.1). Indeed, we have

$$\left\| \int_t^\infty v' \nabla_{z, v}^\perp P_\neq \phi \cdot \nabla_{z, v} \rho(s) ds \right\|_{\mathcal{G}^{\lambda_1, 0; s}} \lesssim \epsilon^2 \int_t^\infty \frac{1}{\langle s \rangle^4} ds \lesssim \frac{\epsilon^2}{\langle t \rangle^3}. \quad (2.22)$$

and

$$\left\| \int_t^\infty \partial_z \phi(s) ds \right\|_{\mathcal{G}^{\lambda_1, 0; s}} \lesssim \frac{\epsilon}{\langle t \rangle^3}. \quad (2.23)$$

Hence, keeping in mind (2.18) and collecting (2.21), (2.22) and (2.23), we obtain

$$\|\rho(t) - \rho_\infty\|_{\mathcal{G}^{\lambda_1, 0; s}} \lesssim \frac{\epsilon^2}{\langle t \rangle} \ln(e+t) + \frac{\epsilon}{\langle t \rangle^3}.$$

Again by the same argument in [8, Section 2.3], we have the estimate in (x, y) coordinate and get (1.8).

3. Growth Mechanism and Construction of Weights

In this section, we will construct the key weights Θ and g which come from a toy model that capture the growth of the solutions (the worst case scenario) in the nonlinear interaction.

From the argument of the linearized equation, due to the dissipation term, the good unknown K decays. We may focus on the nonlinear interaction in the equation of ρ . Since we must pay regularity to deduce decay on the velocity u , it is natural to consider the frequency interactions in the product $u \cdot \nabla \rho$ with the frequencies of u much larger than ρ , which is called the reaction. This leads us to study a simpler model

$$\partial_t \rho - \partial_v \phi_\neq \partial_z \rho_{lo} = 0.$$

This contribution was chosen as Δ_t loses ellipticity in v , not in z . Suppose that instead of $\Delta_t \phi = f$ and $\Delta_t f - \gamma^2 \partial_z \rho = K$, we had $\Delta_L \phi = f$ and $\Delta_L f - \gamma^2 \partial_z \rho = 0$, then on the Fourier side:

$$\partial_t \hat{\rho}(t, k, \eta) = \frac{i\gamma^2}{2\pi} \sum_{l \neq 0} \int_{\xi} \frac{\xi l(k-l) \hat{\rho}(t, l, \xi)}{(l^2 + |\xi - lt|^2)^2} \hat{\rho}_{lo}(t, k-l, \eta - \xi) d\xi.$$

Since ρ_{lo} weakens interactions between well-separated frequencies, let us consider a discrete model with η as a fixed parameter:

$$\partial_t \hat{\rho}(t, k, \eta) = \frac{i\gamma^2}{2\pi} \sum_{l \neq 0} \frac{\eta l(k-l) \hat{\rho}(t, l, \eta)}{(l^2 + |\eta - lt|^2)^2} \hat{\rho}_{lo}(t, k-l, 0). \quad (3.1)$$

This is an infinite system of ODEs. The coefficient $\frac{\eta l}{(l^2 + |\eta - lt|^2)^2}$ of $\hat{\rho}(t, l, \eta)$ in the summation on the right hand side of (3.1) becomes large when $|t - \frac{\eta}{l}| \leq \frac{\eta}{l^3}$. Meanwhile, other coefficients $\frac{\eta \ell}{(\ell^2 + |\eta - \ell t|^2)^2}$, $\ell \neq l$ is small $|t - \frac{\eta}{\ell}| \leq \frac{\eta}{\ell^3}$. Thus for any l and $|t - \frac{\eta}{l}| \leq \frac{\eta}{l^3}$, we divide $\{\hat{\rho}(t, \ell, \eta)\}_{\ell \neq 0}$ into two parts $\rho_R = \hat{\rho}(t, l, \eta)$ and $\rho_{NR} = \hat{\rho}(t, \ell, \eta)$, $\ell \neq l$ where R and NR stand for resonance and non resonance. Due to the existence of $(k-l)$ in the summation of (3.1), the resonance part will never act on the resonance part. By choosing the largest coefficient of all non resonance, we have for $k = 1, 2, \dots, E(|\eta|^{\frac{1}{3}})$ and $|t - \frac{\eta}{k}| \lesssim \frac{\eta}{k^3}$, the growth of the solutions to (3.1) is captured by the following ODE system:

$$\begin{cases} \partial_t \rho_R = \kappa \frac{k^5}{\eta^3} \rho_{NR}, \\ \partial_t \rho_{NR} = \kappa \frac{\eta k}{(k^2 + (\eta - kt)^2)^2} \rho_R, \end{cases}$$

This leads to the weight Θ which is constructed in the next section.

We will also take into account the growth for $|t - \frac{\eta}{k}| \gtrsim \frac{\eta}{k^3}$ with $k = 1, 2, \dots, E(|\eta|^{\frac{2}{3}})$, which leads to the weight g .

3.1. Construction of Θ

The construction is done backward in time, starting with $k = 1$. For $t \in I_{k,\eta} = [t_{k,\eta}^-, t_{k,\eta}^+]$ with $|k| \leq E(|\eta|^{\frac{1}{3}})$. Let Θ_{NR} be a non-decreasing function of time with $\Theta_{NR}(t, \eta) = 1$ for $t \geq 2|\eta|$. For definiteness, we remark here that for $|\eta| < 1$, $\Theta_{NR}(t, \eta) = 1$, which will be consequence of the ensuing the definition. Hence we may safely assume $|\eta| > 1$ for the duration of the section. For $k \geq 1$, we assume that $\Theta_{NR}(\frac{2\eta}{2k-1}, \eta)$ was computed.

To compute Θ_{NR} on the interval $\bar{I}_{k,\eta}$, we define for $k = 1, 2, 3, \dots, E(|\eta|^{\frac{1}{3}})$ and $t \in [t_{k,\eta}^+, \frac{2\eta}{2k-1}]$,

$$\Theta_R(t, \eta) = \Theta_{NR}(t, \eta) = \Theta_{NR}\left(\frac{2\eta}{2k-1}, \eta\right).$$

For $t \in [t_{k,\eta}^-, t_{k,\eta}^+]$, we define

$$\begin{aligned}\Theta_{\text{NR}}(t, \eta) &= \left(\frac{k^3}{2\eta} \left[1 + b_{k,\eta} \left| t - \frac{\eta}{k} \right| \right] \right)^{C\kappa} \Theta_{\text{NR}}(t_{k,\eta}^+, \eta), \\ \forall t &\in \mathbf{I}_{k,\eta}^{\text{R}} = \left[\frac{\eta}{k}, t_{k,\eta}^+ \right], \\ \Theta_{\text{NR}}(t, \eta) &= \left(1 + a_{k,\eta} \left| t - \frac{\eta}{k} \right| \right)^{-1-C\kappa} \Theta_{\text{NR}}\left(\frac{\eta}{k}, \eta\right), \\ \forall t &\in \mathbf{I}_{k,\eta}^{\text{L}} = \left[t_{k,\eta}^-, \frac{\eta}{k} \right].\end{aligned}$$

The constant $b_{k,\eta}$ is chosen to ensure that $2 \frac{k^3}{\eta} \left[1 + b_{k,\eta} \frac{\eta}{2k^3} \right] = 1$, hence for $k \geq 1$, we have

$$b_{k,\eta} = 4 \left(1 - \frac{1}{2} \frac{k^3}{\eta} \right).$$

The constant $a_{k,\eta}$ is chosen to ensure that $\Theta_{\text{NR}}(t_{k,\eta}^+, \eta) = \Theta_{\text{NR}}(t_{k,\eta}^-, \eta) \left(\frac{2\eta}{k^3} \right)^{1+2C\kappa}$, hence for $k \geq 1$, we have

$$a_{k,\eta} = 4 \left(1 - \frac{1}{2} \frac{k^3}{\eta} \right).$$

On each interval $\mathbf{I}_{k,\eta}$, we define $\Theta_{\text{R}}(t, \eta)$ by

$$\begin{aligned}\Theta_{\text{R}}(t, \eta) &= \frac{k^3}{2\eta} \left(1 + b_{k,\eta} \left| t - \frac{\eta}{k} \right| \right) \Theta_{\text{NR}}(t, \eta), \quad \forall t \in \mathbf{I}_{k,\eta}^{\text{R}} = \left[\frac{\eta}{k}, t_{k,\eta}^+ \right], \\ \Theta_{\text{R}}(t, \eta) &= \frac{k^3}{2\eta} \left(1 + a_{k,\eta} \left| t - \frac{\eta}{k} \right| \right) \Theta_{\text{NR}}(t, \eta), \quad \forall t \in \mathbf{I}_{k,\eta}^{\text{L}} = \left[t_{k,\eta}^-, \frac{\eta}{k} \right].\end{aligned} \quad (3.2)$$

For $t \in [\frac{2\eta}{2k+1}, t_{k,\eta}^-]$, we define

$$\Theta_{\text{R}}(t, \eta) = \Theta_{\text{NR}}(t, \eta) = \Theta_{\text{NR}}(t_{k,\eta}^-, \eta).$$

Due to the choice of $a_{k,\eta}$ and $b_{k,\eta}$, we get that $\Theta_{\text{R}}(t_{k,\eta}^\pm, \eta) = \Theta_{\text{NR}}(t_{k,\eta}^\pm, \eta)$, $\Theta_{\text{R}}(\frac{\eta}{k}, \eta) = \frac{k^3}{\eta} \Theta_{\text{NR}}(\frac{\eta}{k}, \eta)$ and, for $t \in \mathbf{I}_{k,\eta}$,

$$\begin{aligned}\partial_t \Theta_{\text{R}} &\approx \frac{k^3}{\eta} \Theta_{\text{NR}}, \\ \partial_t \Theta_{\text{NR}} &\approx \frac{\kappa \eta}{k^3(1 + |t - \frac{\eta}{k}|^2)} \Theta_{\text{R}}.\end{aligned}$$

Then we define

$$\Theta_k(t, \eta) = \begin{cases} \Theta_k(t_{\text{E}(|\eta|^{\frac{1}{3}}), \eta}, \eta), & t < t_{\text{E}(|\eta|^{\frac{1}{3}}), \eta} \\ \Theta_{\text{NR}}(t, \eta), & t \in \left[t_{\text{E}(|\eta|^{\frac{1}{3}}), \eta}, 2\eta \right] \setminus \mathbf{I}_{k,\eta} \\ \Theta_{\text{R}}(t, \eta), & t \in \mathbf{I}_{k,\eta} \\ 1, & t \geq 2\eta \end{cases} \quad (3.3)$$

3.2. Construction of $g(t, \eta)$

Again the construction is done backward in time, starting with $k = 1$. For $t \in \bar{I}_{k,\eta} = [\frac{2\eta}{2k+1}, \frac{2\eta}{2k-1}]$ for $1 \leq |k| \leq E(|\eta|^{\frac{2}{3}})$. Let g be a non-decreasing function of time with $g(t, \eta) = 1$ for $t \geq 2|\eta|$. For definiteness, we remark here for $|\eta| < 1$, $g(t, \eta) = 1$, which will be consequence of the ensuing the definition. Hence we may safely assume $|\eta| > 1$ for the duration of the section. For $k \geq 1$, we assume that $g(\frac{2\eta}{2k-1}, \eta)$ was computed. To compute g on the interval $\bar{I}_{k,\eta}$, we define for $k = 1, 2, 3, \dots, E(|\eta|^{\frac{1}{3}})$ and $t \in [\frac{2\eta}{2k+1}, \frac{2\eta}{2k-1}]$,

$$\partial_t g = \frac{\delta_L^{-1}}{1 + |t - \frac{\eta}{k}|^2} g, \quad g|_{t=\frac{2\eta}{2k-1}} = g\left(\frac{2\eta}{2k-1}, \eta\right),$$

and for $k = E(|\eta|^{\frac{1}{3}}) + 1, \dots, E(|\eta|^{\frac{2}{3}})$ and $t \in [\frac{2\eta}{2k+1}, \frac{2\eta}{2k-1}]$

$$\partial_t g = \frac{\delta_L^{-1} \frac{\eta}{k^3}}{1 + |t - \frac{\eta}{k}|^2} g, \quad g|_{t=\frac{2\eta}{2k-1}} = g\left(\frac{2\eta}{2k-1}, \eta\right).$$

3.3. The total growth predicted by the toy model

Lemma 3.1. *For $\eta > 1$, there exists $\mu = 6(1 + 2C\kappa)$ such that*

$$\frac{\Theta_k(2\eta, \eta)}{\Theta_k(0, \eta)} = \frac{1}{\Theta_k(0, \eta)} = \frac{1}{\Theta_k\left(t_{E(|\eta|^{\frac{1}{3}}), \eta}, \eta\right)} \approx \frac{1}{|\eta|^{\frac{\mu}{12}}} e^{\frac{\mu}{2} |\eta|^{\frac{1}{3}}}.$$

Proof. The proof of Lemma 3.1 can be done as in [8]. Indeed, the total growth over $\bigcup_{k=N}^2 I_{k,\eta}$ is given by the product $(N = E(|\eta|^{\frac{1}{3}}))$

$$\left(\frac{\eta}{N^3}\right)^c \left(\frac{\eta}{(N-1)^3}\right)^c \cdots \left(\frac{\eta}{1^3}\right)^c = \left[\frac{\eta^N}{(N!)^3}\right]^c \quad (3.4)$$

with $c = 2C\kappa + 1$. Using the Stirling formula: $N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$, we get

$$\frac{\eta^N}{(N!)^3} \sim \frac{1}{(2\pi)^{3/2} \sqrt{\eta}} e^{3\eta^{1/3}} \left[\frac{\eta^{1/2}}{N^{3/2}} \left(\frac{\eta}{N^3}\right)^N e^{3N-3\eta^{1/3}} \right].$$

Since $|N - \eta^{1/3}| \leq 1$, then it holds that

$$\frac{\eta^N}{(N!)^3} \sim \frac{1}{\sqrt{\eta}} e^{3\eta^{1/3}}.$$

Hence, this together with (3.4), yields (3.6). $\square$

Lemma 3.2. For $\eta > 1$, it holds that

$$\begin{aligned}
 & \frac{1}{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta)} \\
 &= \frac{g(2\eta, \eta)}{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta)} = \frac{g(t_{E(|\eta|^{\frac{1}{3}}), \eta}, \eta)}{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta)} \frac{g(2\eta, \eta)}{g(t_{E(|\eta|^{\frac{1}{3}}), \eta}, \eta)} \\
 &= \exp \left(\delta_L^{-1} \sum_{E(|\eta|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{2}{3}})} \frac{\eta}{k^3} \left(\arctan \left(\frac{|\eta|}{|k|(2|k|+1)} \right) + \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right) \\
 & \quad \times \exp \left(\delta_L^{-1} \sum_{k=1}^{E(|\eta|^{\frac{1}{3}})} \left(\arctan \left(\frac{|\eta|}{|k|(2|k|+1)} \right) + \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right).
 \end{aligned}$$

This gives us that for any $t \geq 0$,

$$1 \leq \frac{1}{g(t, \eta)} \leq e^{3\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}. \quad (3.5)$$

Proof. We have for $t \in \bar{I}_{k, \eta}$, if $|k| \leq E(|\eta|^{\frac{1}{3}})$,

$$g(t, \eta) = \exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\eta|}{|k|} \right) - \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right) g\left(\frac{2\eta}{2k-1}, \eta\right),$$

and if $E(|\eta|^{\frac{2}{3}}) \geq |k| \geq E(|\eta|^{\frac{1}{3}}) + 1$,

$$g(t, \eta) = \exp \left(\delta_L^{-1} \frac{\eta}{k^3} \left(\arctan \left(t - \frac{|\eta|}{|k|} \right) - \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right) g\left(\frac{2\eta}{2k-1}, \eta\right),$$

This yields for $|k| \leq E(|\eta|^{\frac{1}{3}})$,

$$\begin{aligned}
 \frac{g(t_{k-1, \eta}, \eta)}{g(t_{k, \eta}, \eta)} &= \exp \left(\delta_L^{-1} \left(\arctan \left(t_{k-1, \eta} - \frac{|\eta|}{|k|} \right) - \arctan \left(t_{k, \eta} - \frac{|\eta|}{|k|} \right) \right) \right) \\
 &= \exp \left(\delta_L^{-1} \left(\arctan \left(\frac{|\eta|}{|k|(2|k|+1)} \right) + \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right)
 \end{aligned} \quad (3.6)$$

and $E(|\eta|^{\frac{2}{3}}) \geq |k| \geq E(|\eta|^{\frac{1}{3}}) + 1$

$$\frac{g(t_{k-1, \eta}, \eta)}{g(t_{k, \eta}, \eta)} = \exp \left(\delta_L^{-1} \frac{\eta}{k^3} \left(\arctan \left(\frac{|\eta|}{|k|(2|k|+1)} \right) + \arctan \left(\frac{|\eta|}{|k|(2|k|-1)} \right) \right) \right). \quad (3.7)$$

Hence, we obtain (recall that $t_{0,\eta} = 2\eta$)

$$\begin{aligned} \frac{g(2\eta, \eta)}{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta)} &= \frac{g(t_0, \eta, \eta)}{g(t_{t_1, \eta}, \eta)} \frac{g(t_{t_1, \eta}, \eta)}{g(t_{t_2, \eta}, \eta)} \cdots \frac{g(t_{t_{E(|\eta|^{\frac{2}{3}})-1, \eta}, \eta}}{g(t_{t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta}} \\ &= \prod_{k=1}^{E(|\eta|^{\frac{1}{3}})} \exp \left(\delta_L^{-1} \left(\arctan \left(t_{k-1, \eta} - \frac{|\eta|}{|k|} \right) - \arctan \left(t_{k, \eta} - \frac{|\eta|}{|k|} \right) \right) \right) \\ &\quad \times \prod_{\substack{E(|\eta|^{\frac{2}{3}}) \\ E(|\eta|^{\frac{1}{3}})+1}}^{E(|\eta|^{\frac{2}{3}})} \exp \left(\delta_L^{-1} \frac{\eta}{k^3} \left(\arctan \left(t_{k-1, \eta} - \frac{|\eta|}{|k|} \right) - \arctan \left(t_{k, \eta} - \frac{|\eta|}{|k|} \right) \right) \right), \end{aligned}$$

which gives the lemma. $\square$

Lemma 3.3. *Let ξ, η be such that there exists some $\alpha \geq 1$ with $\frac{1}{\alpha}|\xi| \leq |\eta| \leq \alpha|\xi|$ and let k, n be such that $t \in \bar{I}_{k, \eta}$ and $t \in \bar{I}_{n, \xi}$ (note that $k \approx n$). Then at least one of the following holds:*

- (a) $k = n$ and $t \in I_{k, \xi} \cap I_{k, \eta}$;
- (b) $k = n$ and $|t - \frac{\eta}{k}| \geq \frac{1}{10\alpha} \frac{|\eta|}{k^3}$ and $|t - \frac{\xi}{k}| \geq \frac{1}{10\alpha} \frac{|\xi|}{k^3}$;
- (c) $k = n$ and $|\xi - \eta| \gtrsim_{\alpha} \frac{|\eta|}{k^2}$;
- (d) $|t - \frac{\eta}{k}| \geq \frac{1}{10\alpha} \frac{|\eta|}{k^2}$ and $|t - \frac{\xi}{n}| \geq \frac{1}{10\alpha} \frac{|\xi|}{n^2}$;
- (e) $|\xi - \eta| \gtrsim_{\alpha} \frac{|\eta|}{|n|}$.

Moreover if $t \in I_{k, \eta} \cap I_{n, \xi}$, then at least one of the following holds:

- (a') $k = n$;
- (b') $|t - \frac{\eta}{k}| \gtrsim_{\alpha} \frac{|\eta|}{k^3}$ and $|t - \frac{\xi}{n}| \gtrsim_{\alpha} \frac{|\xi|}{n^3}$;
- (c') $|\xi - \eta| \gtrsim_{\alpha} \frac{|\eta|}{|n|^2}$.

Proof. To see that $k \approx n$ note that

$$\frac{|k|}{|n|} = \frac{|\eta|}{|\xi|} \frac{|tk|}{|\eta|} \frac{|\xi|}{|tn|} \approx_{\alpha} 1.$$

If $n = k$ and $t \in I_{k, \xi} \cap I_{k, \eta}$, then there is nothing to prove. Suppose $n \neq k$ but (a) and (b) are false, which means one of the two inequalities in (b) fails. Without loss of generality suppose $|t - \frac{\xi}{k}| < \frac{1}{10\alpha} \frac{|\xi|}{k^3}$. Then, $t \notin I_{k, \eta}$ gives us that

$$|\xi - \eta| \geq k \left| \frac{\xi}{k} - \frac{\eta}{k} \right| \geq k \left| \frac{\eta}{k} - t \right| - k \left| t - \frac{\xi}{k} \right| \geq \frac{|\eta|}{2k^2} - \frac{1}{10\alpha} \frac{|\xi|}{k^2} \gtrsim \frac{|\eta|}{k^2}.$$

This proves (c).

Suppose $n \neq k$ and (d) is false. Without loss of generality suppose that $|t - \frac{\xi}{n}| < \frac{1}{10\alpha} \frac{|\xi|}{n^2}$. Then, $t \in I_{k, \eta}$ gives us $t \notin I_{n, \eta}$, which gives us that

$$\left| \frac{\xi}{n} - \frac{\eta}{n} \right| \geq \left| \frac{\eta}{n} - t \right| - \left| t - \frac{\xi}{n} \right| \geq \frac{|\eta|}{2n(n+1)} - \frac{1}{10\alpha} \frac{|\xi|}{n^2} \gtrsim \frac{|\eta|}{n^2}.$$

This proves (e). The proof of (a'), (b') and (c') is similar. $\square$

By the construction of Θ , it is easy to check that the following lemma holds.

Lemma 3.4. For $t \in I_{k,\eta}$ and $t > E(|\eta|^{\frac{1}{3}})$, we have the following with $\tau = t - \frac{\eta}{k}$:

$$\frac{\partial_t \Theta_{NR}(t, \eta)}{\Theta_{NR}(t, \eta)} \approx \frac{1}{(1 + |\tau|)} \approx \frac{\partial_t \Theta_R(t, \eta)}{\Theta_R(t, \eta)}.$$

Lemma 3.5. For all t, ξ, η , it holds that

$$\frac{g(t, \xi)}{g(t, \eta)} + \frac{g(t, \eta)}{g(t, \xi)} \lesssim e^{C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}}.$$

Proof. It is enough to prove

$$e^{-C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}} \lesssim \frac{g(t, \xi)}{g(t, \eta)} \lesssim e^{C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}}.$$

Without loss of generality, we assume that $|\xi| \leq |\eta|$. If $|\eta| \leq 1$, we have $g(t, \xi) = g(t, \eta) = 1$, so there is nothing to prove. If $|\xi| < 1 \leq |\eta|$, then we have $|\eta| \leq |\eta - \xi| + 1$ and hence we have from (3.5)

$$e^{-C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}} \lesssim \frac{g(t, \xi)}{g(t, \eta)} = \frac{1}{g(t, \eta)} \lesssim e^{C\delta_L^{-1}|\eta|^{\frac{1}{3}}} \lesssim e^{C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}}.$$

So from now on, we assume that $\min(|\xi|, |\eta|) \geq 1$.

If $|\xi| < \frac{|\eta|}{2}$, then it holds that $|\eta| \leq 2|\eta - \xi|$ and hence $|\xi| \leq |\eta - \xi|$. Therefore, we have by using (3.5),

$$e^{-C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}} \lesssim e^{-C\delta_L^{-1}|\xi|^{\frac{1}{3}}} \lesssim \frac{g(t, \xi)}{g(t, \eta)} \lesssim e^{C\delta_L^{-1}|\eta|^{\frac{1}{3}}} \lesssim e^{C\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}}.$$

Now, we may focus on the case $|\eta|/2 \leq |\xi| \leq |\eta|$. We need to discuss the following time regimes:

Case 1 $t \geq 2|\eta|$;

Case 2 $t \leq \min(t_{E(|\xi|^{\frac{2}{3}}), |\xi|}, t_{E(|\eta|^{\frac{2}{3}}), |\eta|})$;

Case 3 $\min(t_{E(|\xi|^{\frac{2}{3}}), |\xi|}, t_{E(|\eta|^{\frac{2}{3}}), |\eta|}) \leq t \leq \max(t_{E(|\xi|^{\frac{2}{3}}), |\xi|}, t_{E(|\eta|^{\frac{2}{3}}), |\eta|})$;

Case 4 $\max(t_{E(|\xi|^{\frac{2}{3}}), |\xi|}, t_{E(|\eta|^{\frac{2}{3}}), |\eta|}) \leq t \leq \min(t_{E(|\xi|^{\frac{1}{3}}), |\xi|}, t_{E(|\eta|^{\frac{1}{3}}), |\eta|})$;

Case 5 $\min(t_{E(|\xi|^{\frac{1}{3}}), |\xi|}, t_{E(|\eta|^{\frac{1}{3}}), |\eta|}) \leq t \leq \max(t_{E(|\xi|^{\frac{1}{3}}), |\xi|}, t_{E(|\eta|^{\frac{1}{3}}), |\eta|})$;

Case 6 $\max(t_{E(|\xi|^{\frac{1}{3}}), |\xi|}, t_{E(|\eta|^{\frac{1}{3}}), |\eta|}) \leq t \leq 2|\xi|$;

Case 7 $2|\xi| \leq t \leq 2|\eta|$.

Now, we discuss each of the above cases separately. Throughout the proof, we will use the following notations:

$$F(k, \eta) = \arctan\left(\frac{|\eta|}{|k|(2|k| + 1)}\right) + \arctan\left(\frac{|\eta|}{|k|(2|k| - 1)}\right)$$

and

$$\tilde{F}(k, \eta) = \frac{|\eta|}{|k|^3} F(k, \eta). \quad (3.8)$$

Case 1. For $t \geq 2|\eta|$, then $g(t, \xi) = g(t, \eta) = 1$, so there is nothing to prove.

Case 2. For $t \leq \min(t_{E(|\xi|^{\frac{2}{3}}), |\xi|}, t_{E(|\eta|^{\frac{2}{3}}), |\eta|})$, then we have

$$\frac{g(t, \xi)}{g(t, \eta)} = \frac{g(0, \xi)}{g(0, \eta)} = \exp(G_1(k, \eta) - G_1(k, \xi)) \exp(G_2(k, \eta) - G_2(k, \xi)) \quad (3.9)$$

with

$$G_1(k, \eta) = \delta_L^{-1} \sum_{k=1}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) \quad \text{and} \quad G_2(k, \eta) = \delta_L^{-1} \sum_{E(|\eta|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{2}{3}})} \tilde{F}(k, \eta).$$

We have

$$\begin{aligned} & \exp(G_1(k, \eta) - G_1(k, \xi)) \\ & \leq \exp \left(\delta_L^{-1} \sum_{k=\min\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}+1}^{\max\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}} \left(\arctan \left(\frac{\max\{|\xi|, |\eta|\}}{|k|(2|k|+1)} \right) \right. \right. \\ & \quad \left. \left. + \arctan \left(\frac{\max\{|\xi|, |\eta|\}}{|k|(2|k|-1)} \right) \right) \right) \times \exp \left(\delta_L^{-1} \sum_{k=1}^{\min\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}} |F(k, \xi) - F(k, \eta)| \right). \end{aligned}$$

By the fact that

$$|F(k, \xi) - F(k, \eta)| \lesssim \frac{|\xi - \eta|^{\frac{1}{3}} |k^2|}{|\xi|^{\frac{4}{3}}},$$

we obtain that

$$\begin{aligned} \exp(G_1(k, \eta) - G_1(k, \xi)) & \leq \exp \left(\pi \delta_L^{-1} \left| E(|\eta|^{\frac{1}{3}}) - E(|\xi|^{\frac{1}{3}}) \right| \right) \\ & \exp \left(C \delta_L^{-1} \sum_{k=1}^{\min\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}} \frac{|\xi - \eta|^{\frac{1}{3}} |k^2|}{|\xi|} \right) \quad (3.10) \\ & \lesssim e^{C \delta_L^{-1} |\eta - \xi|^{\frac{1}{3}}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \exp(G_2(k, \eta) - G_2(k, \xi)) & \leq \exp \left(\delta_L^{-1} \sum_{k=\min\{E(|\xi|^{\frac{2}{3}}), E(|\eta|^{\frac{2}{3}})\}+1}^{\max\{E(|\xi|^{\frac{2}{3}}), E(|\eta|^{\frac{2}{3}})\}} \frac{|\eta|}{|k|^3} \right. \\ & \quad \left. \left(\arctan \left(\frac{\max\{|\xi|, |\eta|\}}{|k|(2|k|+1)} \right) + \arctan \left(\frac{\max\{|\xi|, |\eta|\}}{|k|(2|k|-1)} \right) \right) \right) \\ & \times \exp \left(\delta_L^{-1} \sum_{k=\max\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}}^{\min\{E(|\xi|^{\frac{2}{3}}), E(|\eta|^{\frac{2}{3}})\}} |\tilde{F}(k, \xi) - \tilde{F}(k, \eta)| \right) \end{aligned}$$

We have

$$|\tilde{F}(k, \xi) - \tilde{F}(k, \eta)| \leq \frac{|\eta - \xi|}{|k|^3}. \quad (3.11)$$

Consequently, we obtain

$$\begin{aligned} \exp(G_2(k, \eta) - G_2(k, \xi)) &\leq \exp\left(\pi \delta_L^{-1} \left|E(|\eta|^{\frac{2}{3}}) - E(|\xi|^{\frac{2}{3}})\right| |\eta|^{-1}\right) \\ &\quad \exp\left(C \delta_L^{-1} \sum_{k=\max\{E(|\xi|^{\frac{1}{3}}), E(|\eta|^{\frac{1}{3}})\}}^{\min\{E(|\xi|^{\frac{2}{3}}), E(|\eta|^{\frac{2}{3}})\}} \frac{|\eta - \xi|}{|k|^3}\right) \quad (3.12) \\ &\lesssim e^{C \delta_L^{-1} |\eta - \xi|^{\frac{1}{3}}}. \end{aligned}$$

Plugging (3.10) and (3.12) into (3.9), we obtain the desired result.

Case 3. If $t_{E(|\xi|^{\frac{2}{3}}), \xi} \leq t \leq t_{E(|\eta|^{\frac{2}{3}}), \eta}$, we have

$$\frac{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \xi)}{g(t_{E(|\eta|^{\frac{2}{3}}), \eta}, \eta)} \geq \frac{g(t, \xi)}{g(t, \eta)} \geq \frac{g(0, \xi)}{g(0, \eta)},$$

and if $t_{E(|\eta|^{\frac{2}{3}}), \eta} \leq t \leq t_{E(|\xi|^{\frac{2}{3}}), \xi}$, we have

$$\frac{g(t_{E(|\xi|^{\frac{2}{3}}), \xi}, \xi)}{g(t_{E(|\xi|^{\frac{2}{3}}), \xi}, \eta)} \leq \frac{g(t, \xi)}{g(t, \eta)} \leq \frac{g(0, \xi)}{g(0, \eta)}.$$

Thus we can deduce the estimates in Case 3 by the estimate of Case 4 and Case 2. Similar argument can also apply to Case 7. Indeed, we have:

Case 7. If $2|\xi| \leq t \leq 2|\eta|$, we have $1 \geq g(t, \eta) \geq g(2|\xi|, \eta)$ and $g(t, \xi) = g(2|\xi|, \xi) = 1$, which implies

$$1 \geq \frac{g(t, \eta)}{g(t, \xi)} \geq \frac{g(2|\xi|, \eta)}{g(2|\xi|, \xi)}.$$

Thus we can deduce the estimates in Case 7 by the estimate of Case 6.

For the cases 4, 5 and 6, we have $\max(t_{E(|\eta|^{\frac{2}{3}}), \eta}, t_{E(|\xi|^{\frac{2}{3}}), \xi}) \leq t \leq 2|\xi|$. Let j and n be such that $t \in \bar{I}_{n, \eta} \cap \bar{I}_{j, \xi}$. Then we have $j \approx n \geq j$.

Case 6. Let t be such that $\max(t_{E(|\xi|^{\frac{1}{3}}), |\xi|}, t_{E(|\eta|^{\frac{1}{3}}), |\eta|}) \leq t \leq 2|\xi|$. We have in this case

$$\begin{aligned} g(t, \eta) &= \exp\left(-\delta_L^{-1} \sum_{k=1}^{n-1} F(k, \eta)\right) \\ &\quad \times \exp\left(\delta_L^{-1} \left(\arctan\left(t - \frac{|\eta|}{|n|}\right) - \arctan\left(\frac{|\eta|}{|n|(2|n| - 1)}\right)\right)\right) \end{aligned}$$

and

$$g(t, \xi) = \exp \left(-\delta_L^{-1} \sum_{k=1}^{j-1} F(k, \xi) \right) \\ \times \exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j|-1)} \right) \right) \right).$$

By the fact that $n \geq j$ we have

$$\frac{g(t, \xi)}{g(t, \eta)} = \frac{\exp \left(\delta_L^{-1} \sum_{k=1}^{j-1} F(k, \eta) \right)}{\exp \left(\delta_L^{-1} \sum_{k=1}^{j-1} F(k, \xi) \right)} \underbrace{\exp \left(\delta_L^{-1} \sum_{k=j}^{n-1} F(k, \eta) \right)}_{\text{when } j \leq n-1} \\ \times \frac{\exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j|-1)} \right) \right) \right)}{\exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{|\eta|}{|n|(2|n|-1)} \right) \right) \right)}$$

We get

$$\left| \arctan \left(\frac{|\eta|}{|k|(2|k| \pm 1)} \right) - \arctan \left(\frac{|\xi|}{|k|(2|k| \pm 1)} \right) \right| \lesssim \frac{|\eta - \xi|}{\langle \xi \rangle}. \quad (3.13)$$

Since $t \in \bar{I}_{n,\eta} \cap \bar{I}_{j,\xi}$, then it holds that $|jt - \xi| = |j||t - \frac{\xi}{j}| \leq \frac{|\xi|}{|j|} \approx t$ and also $|\eta - nt| \leq \frac{|\eta|}{|n|} \approx t$, hence, we obtain the inequality

$$|j - n| \leq \frac{|jt - \xi| + |\eta - \xi| + |\eta - nt|}{t} \lesssim 1 + \frac{|\eta - \xi||n|}{|\eta|} \lesssim 1 + |\eta - \xi|^{\frac{1}{3}}.$$

Hence, we obtain

$$\frac{g(t, \xi)}{g(t, \eta)} \lesssim \exp \left(C \delta_L^{-1} \frac{|\eta - \xi|}{\langle \xi \rangle} n \right) \exp(|n - j|),$$

and similarly we have $\frac{g(t, \eta)}{g(t, \xi)} \lesssim \exp \left(C \delta_L^{-1} \frac{|\eta - \xi|}{\langle \xi \rangle} n \right) \exp(|n - j|)$.

By the fact that $n \lesssim |\xi|^{\frac{1}{3}}$ and $|\eta - \xi| \lesssim |\xi|$, we get

$$\frac{g(t, \eta)}{g(t, \xi)} + \frac{g(t, \xi)}{g(t, \eta)} \lesssim \exp \left(C \delta_L^{-1} |\eta - \xi|^{\frac{1}{3}} \right).$$

Case 4. In the proof of Lemma 3.10, you may need more precise formula: (similar for the Case 5.) Let j and n be such that $t \in \bar{I}_{n,\eta} \cap \bar{I}_{j,\xi}$. Using the fact

that $|\eta| \geq |\xi|$ which gives that $E(|\eta|^{\frac{1}{3}}) \geq E(|\xi|^{\frac{1}{3}})$, then $|n| \geq E(|\eta|^{\frac{1}{3}})$ and $|j| \geq E(|\xi|^{\frac{1}{3}})$. By the definition of $g(t, \eta)$, we get that

$$\begin{aligned} g(t, \eta) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \eta) \right) \underbrace{\exp \left(-\delta_L^{-1} \sum_{E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) \right)}_{\text{when } E(|\eta|^{\frac{1}{3}}) \geq E(|\xi|^{\frac{1}{3}})+1} \\ &\quad \times \exp \left(-\delta_L^{-1} \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{n-1} \tilde{F}(k, \eta) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \frac{|\eta|}{|n|^3} \left(\arctan \left(t - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{|\eta|}{|n|(2|n|-1)} \right) \right) \right) \end{aligned} \quad (3.14)$$

and if $E(|\eta|^{\frac{1}{3}}) \geq |j| \geq E(|\xi|^{\frac{1}{3}})$,

$$\begin{aligned} g(t, \xi) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \xi) \right) \exp \left(-\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{j-1} \tilde{F}(k, \xi) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \frac{|\xi|}{|j|^3} \left(\arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j|-1)} \right) \right) \right) \end{aligned}$$

and if $E(|\eta|^{\frac{1}{3}}) + 1 \leq |j|$

$$\begin{aligned} g(t, \xi) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \xi) \right) \exp \left(-\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \tilde{F}(k, \xi) \right) \\ &\quad \times \exp \left(-\delta_L^{-1} \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{j-1} \tilde{F}(k, \xi) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \frac{|\xi|}{|j|^3} \left(\arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j|-1)} \right) \right) \right) \end{aligned} \quad (3.15)$$

Thus we have that

$$\begin{aligned} &\frac{g(t, \eta)}{g(t, \xi)} + \frac{g(t, \xi)}{g(t, \eta)} \\ &\lesssim \exp \left(\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} |F(k, \eta) - F(k, \xi)| \right) \exp \left(E(|\eta|^{\frac{1}{3}}) - E(|\xi|^{\frac{1}{3}}) \right) \end{aligned}$$

$$\begin{aligned}
& \times \exp \left(\delta_L^{-1} \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{j-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \right) \exp \left(\delta_L^{-1} \sum_{k=j}^{n-1} \tilde{F}(k, \eta) \right) \\
& \lesssim \exp \left(C \delta_L^{-1} |\xi|^{\frac{1}{3}} \frac{|\xi - \eta|}{\xi} \right) \\
& \exp \left(E(|\eta|^{\frac{1}{3}}) - E(|\xi|^{\frac{1}{3}}) \right) \exp \left(\delta_L^{-1} \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{j-1} \frac{|\xi - \eta|}{k^3} \right) \exp \left(\frac{\delta_L^{-1} |n - j| |\eta|}{n^3} \right).
\end{aligned}$$

By the fact that $|\eta - \xi| \lesssim |\xi| \approx |\eta|$, $|n| \gtrsim |\eta|^{\frac{1}{3}}$ and

$$|j - n| \leq \frac{|jt - \xi| + |\eta - \xi| + |\eta - nt|}{t} \lesssim 1 + \frac{|\eta - \xi||n|}{|\eta|},$$

we get

$$\begin{aligned}
\frac{g(t, \eta)}{g(t, \xi)} + \frac{g(t, \xi)}{g(t, \eta)} & \lesssim \exp \left(C \delta_L^{-1} |\xi - \eta|^{\frac{1}{3}} \right) \exp \left(\frac{\delta_L^{-1} |\eta - \xi|}{n^2} \right) \\
& \lesssim \exp \left(C \delta_L^{-1} |\xi - \eta|^{\frac{1}{3}} \right).
\end{aligned}$$

Case 5. Let j and n be such that $t \in \bar{\mathbf{I}}_{n, \eta} \cap \bar{\mathbf{I}}_{j, \xi}$. If $t_{E(|\xi|^{\frac{1}{3}}), |\xi|} \leq t \leq t_{E(|\eta|^{\frac{1}{3}}), |\eta|}$, we have $|j| \leq E(|\xi|^{\frac{1}{3}}) \leq E(|\eta|^{\frac{1}{3}}) \leq |n| \approx |j| \approx |\xi|^{\frac{1}{3}} \approx |\eta|^{\frac{1}{3}}$, we write

$$\begin{aligned}
g(t, \eta) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{j-1} F(k, \eta) \right) \exp \left(-\delta_L^{-1} \sum_{k=j}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) \right) \\
& \exp \left(-\delta_L^{-1} \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{n-1} \tilde{F}(k, \eta) \right) \\
& \times \exp \left(\delta_L^{-1} \frac{|\eta|}{|n|^3} \left(\arctan \left(t - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{|\eta|}{|n|(2|n| - 1)} \right) \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
g(t, \xi) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{j-1} F(k, \xi) \right) \exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\eta|}{|j|} \right) \right. \right. \\
& \quad \left. \left. - \arctan \left(\frac{|\eta|}{|j|(2|j| - 1)} \right) \right) \right)
\end{aligned}$$

By the fact that

$$|j - n| \leq \frac{|jt - \xi| + |\eta - \xi| + |\eta - nt|}{t} \lesssim 1 + \frac{|\eta - \xi||n|}{|\eta|},$$

we have

$$\begin{aligned} \frac{g(t, \xi)}{g(t, \eta)} + \frac{g(t, \eta)}{g(t, \xi)} &\lesssim \exp \left(\delta_L^{-1} \sum_{k=1}^{j-1} |F(k, \xi) - F(k, \eta)| \right) \exp(|j - n|) \\ &\lesssim \exp(C\delta_L^{-1} |\xi - \eta|^{\frac{1}{3}}). \end{aligned}$$

If $t_{E(|\xi|^{\frac{1}{3}}), |\xi|} \geq t \geq t_{E(|\eta|^{\frac{1}{3}}), |\eta|}$, then $E(|\xi|^{\frac{1}{3}}) \leq |j| \leq |n| \leq E(|\eta|^{\frac{1}{3}})$, we write

$$\begin{aligned} g(t, \eta) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \eta) \right) \exp \left(-\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{n-1} F(k, \eta) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \left(\arctan \left(t - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{|\eta|}{|n|(2|n| - 1)} \right) \right) \right) \end{aligned}$$

and

$$\begin{aligned} g(t, \xi) &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \eta) \right) \exp \left(-\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{j-1} \tilde{F}(k, \eta) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \frac{|\xi|}{|j|^3} \left(\arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j| - 1)} \right) \right) \right). \end{aligned}$$

By the fact that

$$|j - n| \leq \frac{|jt - \xi| + |\eta - \xi| + |\eta - nt|}{t} \lesssim 1 + \frac{|\eta - \xi||n|}{|\eta|},$$

we have

$$\begin{aligned} \frac{g(t, \xi)}{g(t, \eta)} + \frac{g(t, \eta)}{g(t, \xi)} &\lesssim \exp \left(\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} |F(k, \xi) - F(k, \eta)| \right) \exp(|j - n|) \\ &\lesssim \exp(C\delta_L^{-1} |\xi - \eta|^{\frac{1}{3}}). \end{aligned}$$

Thus we have proved the lemma. $\square$

Lemma 3.6. For all t, η, ξ , we have

$$\frac{\Theta_{NR}(t, \eta)}{\Theta_{NR}(t, \xi)} \lesssim e^{\mu|\eta - \xi|^{\frac{1}{3}}}.$$

Proof. One may follow the proof of Lemma 3.5 and deduce to prove for $\eta, \xi \geq 0$, $\eta/2 \leq \xi \leq \eta$ and $t \in \max(t_{E(|\eta|^{\frac{1}{3}}), \eta}, t_{E(|\xi|^{\frac{1}{3}}), \xi}) \leq t \leq 2|\xi|$, it holds that

$$e^{-\mu|\eta - \xi|^{\frac{1}{3}}} \lesssim \frac{\Theta_{NR}(t, \eta)}{\Theta_{NR}(t, \xi)} \lesssim e^{\mu|\eta - \xi|^{\frac{1}{3}}}. \quad (3.16)$$

Let j and n be such that $t \in \bar{I}_{n,\eta} \cap \bar{I}_{j,\xi}$. Then we have $n \approx j \leq n$. We consider the following cases:

Case $j = n$ and $t \in \bar{I}_{n,\eta}^R \cap \bar{I}_{j,\xi}^R$ or $t \in \bar{I}_{n,\eta}^L \cap \bar{I}_{j,\xi}^L$ or $t \in \bar{I}_{n,\eta}^L \cap \bar{I}_{j,\xi}^R$: We can use the same argument in the proof of Lemma 3.5 to prove (3.16) and we omit the details here.

Case $j = n$, $|t - \frac{\eta}{n}| \gtrsim \frac{|\eta|}{n^3}$ and $|t - \frac{\xi}{j}| \gtrsim \frac{|\xi|}{j^3}$: It holds that

$$\text{for } t \geq \frac{\eta}{n}, \Theta_{NR}(t, \eta) \approx \Theta_{NR}(t_{n,\eta}^+, \eta);$$

$$\text{for } t \geq \frac{\xi}{n}, \Theta_{NR}(t, \xi) \approx \Theta_{NR}(t_{n,\xi}^+, \xi);$$

$$\text{for } t \leq \frac{\eta}{n}, \Theta_{NR}(t, \eta) \approx \left(\frac{\eta}{n^3}\right)^{-1-2C\kappa} \Theta_{NR}(t_{n,\eta}^+, \eta) \approx \Theta_{NR}(t_{n,\eta}^-, \eta);$$

$$\text{for } t \leq \frac{\xi}{n}, \Theta_{NR}(t, \xi) \approx \left(\frac{\xi}{n^3}\right)^{-1-2C\kappa} \Theta_{NR}(t_{n,\xi}^+, \xi) \approx \Theta_{NR}(t_{n,\xi}^-, \xi).$$

Thus if $t \geq \frac{\eta}{n}$ and $t \leq \frac{\xi}{n}$ or $t \leq \frac{\eta}{n}$ and $t \geq \frac{\xi}{n}$, then we have $|\eta - \xi| \geq \frac{|\eta|}{n^2}$ and

$$\begin{aligned} \frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} &\lesssim \left(\frac{\eta}{\xi}\right)^{c(n-1)+C\kappa} \left(\frac{\eta}{n^3}\right)^{1+2C\kappa} \lesssim \left(1 + \frac{\eta - \xi}{\xi}\right)^{c|\eta|^{\frac{1}{3}}} \left(\frac{1 + |\eta - \xi|}{n}\right)^{1+2C\kappa} \\ &\lesssim e^{c|\eta - \xi|^{\frac{1}{3}}} \end{aligned}$$

and if $t \geq \frac{\eta}{n}$ and $t \geq \frac{\xi}{n}$ or $t \leq \frac{\eta}{n}$ and $t \leq \frac{\xi}{n}$, then we have

$$\frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} \lesssim \left(\frac{\eta}{\xi}\right)^{c(n-1)+C\kappa} \lesssim e^{c|\eta - \xi|^{\frac{1}{3}}}.$$

Case $j = n$, $|\eta - \xi| \gtrsim \frac{|\eta|}{n^2}$, similarly we have

$$\frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} \lesssim \left(\frac{\eta}{\xi}\right)^{c(n-1)+C\kappa} \left(\frac{\eta}{n^3}\right)^{1+2C\kappa} \lesssim e^{c|\eta - \xi|^{\frac{1}{3}}}.$$

Case $j = n - 1$: If $t \in \bar{I}_{n,\eta}^L$ then $t_{n-1,\xi} \leq \frac{\eta}{n}$. If $t \in \bar{I}_{n-1,\xi}^R$, then $\frac{\xi}{n-1} < t_{n-1,\eta}$. In either one of these case, we deduce that $\frac{|\xi|}{n^2} \lesssim \frac{\eta - \xi}{n}$ and thus

$$\frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} \lesssim \left(\frac{\eta}{\xi}\right)^{c(n-1)+C\kappa} \left(\frac{\eta}{n^3}\right)^{1+2C\kappa} \lesssim e^{c|\eta - \xi|^{\frac{1}{3}}}. \quad (3.17)$$

Next assume that $t \in \bar{I}_{n,\eta}^R \cap \bar{I}_{n-1,\xi}^L$ there are only two possibilities: One is $\frac{|\xi|}{n^2} \lesssim \frac{\eta - \xi}{n}$ which we can conclude in a similar way to the above (3.17); the other one is $|t - \frac{\eta}{n}| \gtrsim \frac{\eta}{n^2} \geq \frac{\eta}{n^3}$ and $|t - \frac{\xi}{j}| \gtrsim \frac{\xi}{j^2} \geq \frac{\xi}{j^3}$, which gives us that $\Theta_{NR}(t, \eta) \approx \Theta_{NR}(t_{n,\eta}^+, \eta)$ and $\Theta_R(t, \xi) \approx \Theta_{NR}(t_{n-1,\xi}^-, \xi) = \Theta_{NR}(t_{n,\xi}^+, \xi)$. Thus we have

$$\frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} \lesssim \left(\frac{\eta}{\xi}\right)^{c(n-1)+C\kappa} \lesssim e^{c|\eta - \xi|^{\frac{1}{3}}}.$$

Case $j < n-1$: In this case, it is easy to see that $\frac{\xi}{n^2} \lesssim \frac{\eta-\xi}{n}$ and we can conclude in a similar way to (3.17). $\square$

Lemma 3.7. *Let $t \leq \frac{1}{2} \min\{|\xi|^{\frac{1}{3}}, |\eta|^{\frac{1}{3}}\}$. Then*

$$\left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_l(t, \xi)} - 1 \right| \lesssim \frac{\langle k-l, \xi-\eta \rangle}{(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}}} e^{C\delta_L^{-1}|k-l, \xi-\eta|^{\frac{1}{3}}}$$

Proof. By Lemma 3.5, we get that

$$\frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_l(t, \xi)} \lesssim \frac{\mathbf{g}(t, \xi)}{\mathbf{g}(t, \eta)} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}} + e^{C\delta_L^{-1}|k-l|^{\frac{1}{3}}} \lesssim e^{C\delta_L^{-1}|k-l, \xi-\eta|^{\frac{1}{3}}}. \quad (3.18)$$

If $(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}} \lesssim \max\{1, |k-l| + |\xi-\eta|\}$, we get the lemma.

From now on, we assume that

$$|k-l| + |\xi-\eta| \leq \frac{1}{100} \left(|k|^{\frac{2}{3}} + |l|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}} + |\xi|^{\frac{2}{3}} \right) \leq \frac{1}{100} (|k| + |l| + |\eta| + |\xi|).$$

Case 1: $\frac{1}{10}(|k| + |l|) \leq |\eta| + |\xi| \leq 10(|k| + |l|)$: In this case, we have

$$|k-l, \xi-\eta| \lesssim |k| \approx |l| \approx |\xi| \approx |\eta|,$$

and

$$\begin{aligned} \left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_l(t, \xi)} - 1 \right| &\leq \frac{\left| \frac{e^{4\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}}{\mathbf{g}(t, \eta)} - \frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{\mathbf{g}(t, \xi)} \right|}{\frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{\mathbf{g}(t, \xi)} + e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}}} + \frac{\left| e^{4\pi\delta_L^{-1}|k|^{\frac{1}{3}}} - e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}} \right|}{\frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{\mathbf{g}(t, \xi)} + e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}}} \\ &\leq \frac{\mathbf{g}(t, \xi)}{\mathbf{g}(t, \eta)} \left| |\xi|^{\frac{1}{3}} - |\eta|^{\frac{1}{3}} \right| e^{4\pi\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}} + \left| \frac{\mathbf{g}(t, \xi)}{\mathbf{g}(t, \eta)} - 1 \right| \\ &\quad + \left| |k|^{\frac{1}{3}} - |l|^{\frac{1}{3}} \right| e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}} \\ &\lesssim \frac{|\xi-\eta|}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}} + \left| \frac{\mathbf{g}(0, \xi)}{\mathbf{g}(0, \eta)} - 1 \right| \\ &\quad + \frac{|l-k|}{|l|^{\frac{2}{3}} + |k|^{\frac{2}{3}}} e^{4\pi\delta_L^{-1}|l-k|^{\frac{1}{3}}}. \end{aligned}$$

Recall that (3.6) and (3.7), we have

$$\frac{1}{\mathbf{g}(0, \eta)} = \exp \left(\delta_L^{-1} \sum_{\mathbf{E}(|\eta|^{\frac{2}{3}})} \tilde{\mathbf{F}}(k, \eta) \right) \exp \left(\delta_L^{-1} \sum_{k=1}^{\mathbf{E}(|\eta|^{\frac{1}{3}})} \mathbf{F}(k, \eta) \right).$$

and

$$\frac{1}{g(0, \xi)} = \exp \left(\delta_L^{-1} \sum_{E(|\xi|^{\frac{2}{3}})+1}^{E(|\xi|^{\frac{2}{3}})} \tilde{F}(k, \xi) \right) \exp \left(\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} F(k, \xi) \right).$$

This gives (if $|\xi| \leq |\eta|$)

$$\begin{aligned} \frac{g(0, \xi)}{g(0, \eta)} &= \exp \left(\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} (F(k, \eta) - F(k, \xi)) \right) \exp \left(\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \sum_{E(|\eta|^{\frac{1}{3}})+1}^{E(|\xi|^{\frac{2}{3}})} (\tilde{F}(k, \eta) - \tilde{F}(k, \xi)) \right) \exp \left(-\delta_L^{-1} \sum_{E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \tilde{F}(k, \xi) \right) \\ &\quad \times \exp \left(\delta_L^{-1} \sum_{E(|\xi|^{\frac{2}{3}})+1}^{E(|\eta|^{\frac{2}{3}})} \tilde{F}(k, \eta) \right) \\ \frac{g(0, \eta)}{g(0, \xi)} &= \exp \left(-\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} (F(k, \eta) - F(k, \xi)) \right) \exp \left(-\delta_L^{-1} \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) \right) \\ &\quad \times \exp \left(-\delta_L^{-1} \sum_{E(|\eta|^{\frac{1}{3}})+1}^{E(|\xi|^{\frac{2}{3}})} (\tilde{F}(k, \eta) - \tilde{F}(k, \xi)) \right) \exp \left(\delta_L^{-1} \sum_{E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \tilde{F}(k, \xi) \right) \\ &\quad \times \exp \left(-\delta_L^{-1} \sum_{E(|\xi|^{\frac{2}{3}})+1}^{E(|\eta|^{\frac{2}{3}})} \tilde{F}(k, \eta) \right) \end{aligned}$$

By using the fact that $|e^a e^b - 1| \lesssim (|a| + |b|)e^{a+b}$, we have

$$\begin{aligned} &\left| \frac{g(0, \xi)}{g(0, \eta)} - 1 \right| + \left| \frac{g(0, \eta)}{g(0, \xi)} - 1 \right| \\ &\lesssim \left(\delta_L^{-1} \sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} |F(k, \eta) - F(k, \xi)| + \delta_L^{-1} \sum_{E(|\eta|^{\frac{1}{3}})+1}^{E(|\xi|^{\frac{2}{3}})} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \right. \\ &\quad \left. + \sum_{\min\{E(|\eta|^{\frac{1}{3}}), E(|\xi|^{\frac{1}{3}})\}}^{\max\{E(|\eta|^{\frac{1}{3}}), E(|\xi|^{\frac{1}{3}})\}} 1 + \sum_{\min\{E(|\eta|^{\frac{2}{3}}), E(|\xi|^{\frac{2}{3}})\}}^{\max\{E(|\eta|^{\frac{2}{3}}), E(|\xi|^{\frac{2}{3}})\}} \frac{\max\{|\xi|, |\eta|\}}{|k|^3} \right) \\ &\max \left\{ \frac{g(0, \xi)}{g(0, \eta)}, \frac{g(0, \eta)}{g(0, \xi)} \right\}. \end{aligned}$$

Now applying Lemma 3.5, we deduce that

$$\begin{aligned} & \left| \frac{g(0, \xi)}{g(0, \eta)} - 1 \right| + \left| \frac{g(0, \eta)}{g(0, \xi)} - 1 \right| \\ & \lesssim \left(\sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} |F(k, \eta) - F(k, \xi)| + \sum_{\substack{E(|\xi|^{\frac{2}{3}}) \\ E(|\eta|^{\frac{1}{3}})+1}} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \right. \\ & \quad \left. + |E(|\eta|^{\frac{1}{3}}) - E(|\xi|^{\frac{1}{3}})| + \frac{|E(|\eta|^{\frac{2}{3}}) - E(|\xi|^{\frac{2}{3}})|}{|\xi|} \right) e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}. \end{aligned}$$

We have for $|\xi| \approx |\eta|$ and $1 \leq |k| \leq \min\{E(|\eta|^{\frac{1}{3}}), E(|\xi|^{\frac{1}{3}})\}$ that

$$|F(k, \eta) - F(k, \xi)| \lesssim \frac{|\xi - \eta|}{k^2 + |\xi|^2}, \quad (3.19)$$

and for $\max\{E(|\eta|^{\frac{1}{3}}), E(|\xi|^{\frac{1}{3}})\} \leq |k| \leq \min\{E(|\eta|^{\frac{2}{3}}), E(|\xi|^{\frac{2}{3}})\}$ that

$$|\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \lesssim \frac{\eta}{k^3} \frac{|\xi - \eta|}{k^2 + |\xi|^2} + \frac{|\eta - \xi|}{k^3} \lesssim \frac{|\eta - \xi|}{k^3}, \quad (3.20)$$

which gives us that

$$\left| \frac{g(0, \eta)}{g(0, \xi)} - 1 \right| + \left| \frac{g(0, \xi)}{g(0, \eta)} - 1 \right| \lesssim \frac{|\xi - \eta|}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}$$

Case 2: $|\eta| + |\xi| \geq 10(|k| + |l|)$: We have $|\xi| \geq 4(|k| + |l|)$ and

$$\begin{aligned} \left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_l(t, \xi)} - 1 \right| & \leq \frac{\left| \frac{e^{4\pi\delta_L^{-1}|\eta|^{\frac{1}{3}}}}{g(t, \eta)} - \frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{g(t, \xi)} \right|}{\frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{g(t, \xi)} + e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}}} + \frac{\left| e^{4\pi\delta_L^{-1}|k|^{\frac{1}{3}}} - e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}} \right|}{\frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{g(t, \xi)} + e^{4\pi\delta_L^{-1}|l|^{\frac{1}{3}}}} \\ & \leq \frac{g(t, \xi)}{g(t, \eta)} \left| |\xi|^{\frac{1}{3}} - |\eta|^{\frac{1}{3}} \right| e^{4\pi\delta_L|\eta-\xi|^{\frac{1}{3}}} + \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \\ & \quad + \frac{\left| |k|^{\frac{1}{3}} - |l|^{\frac{1}{3}} \right| e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}}}{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}} - 4\pi\delta_L^{-1}|l|^{\frac{1}{3}}} \\ & \lesssim \frac{|\xi - \eta|}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}} + \frac{||k| - |l|| e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}}}{|\xi|} \\ & \lesssim \frac{|k - l, \xi - \eta|}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}. \end{aligned}$$

Case 3: $|\eta| + |\xi| \leq \frac{1}{10}(|k| + |l|)$: We have

$$|k - l| + |\xi - \eta| \leq \frac{11}{1000}(|k| + |l|), \quad \frac{1011}{989}|l| \geq |k| \geq \frac{989}{1011}|l|,$$

thus $2|\xi| \leq |\xi - \eta| + |\eta| + |\xi| \leq \frac{111}{1000}(|k| + |l|) \leq \frac{222}{989}|l|$, which implies $|l| \geq 8|\xi|$. Thus

$$\frac{e^{4\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}}}{g(t, \xi)} \leq Ce^{7\pi\delta_L^{-1}|\xi|^{\frac{1}{3}}} \leq Ce^{3.5\pi\delta_L^{-1}|l|^{\frac{1}{3}}},$$

and then

$$\begin{aligned} \left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_l(t, \xi)} - 1 \right| &\leq \left(\frac{g(t, \xi)}{g(t, \eta)} \left| |\xi|^{\frac{1}{3}} - |\eta|^{\frac{1}{3}} \right| e^{4\pi\delta_L|\eta-\xi|^{\frac{1}{3}}} + \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \right) e^{-0.5\pi\delta_L^{-1}|l|^{\frac{1}{3}}} \\ &\quad + \left| |k|^{\frac{1}{3}} - |l|^{\frac{1}{3}} \right| e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}} \\ &\lesssim \frac{|\xi - \eta|}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}} e^{-0.5\pi\delta_L^{-1}|l|^{\frac{1}{3}}} + \frac{||k| - |l|| e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}}}{|l|^{\frac{2}{3}} + |k|^{\frac{2}{3}}} \\ &\lesssim \frac{|k - l, \xi - \eta|}{|l|^{\frac{2}{3}} + |k|^{\frac{2}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}. \end{aligned}$$

Thus we proved the lemma. $\square$

Lemma 3.8. *Let $t \leq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}$. Then*

$$\left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_k(t, \xi)} - 1 \right| \lesssim \frac{\langle \xi - \eta \rangle}{(|k| + |\eta| + |\xi|)^{\frac{1}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}.$$

Proof. By the argument in the proof of Lemma 3.7 (see Case 3) and Lemma 3.5, we have for $|\eta| + |\xi| \leq \frac{1}{5}|k|$,

$$\begin{aligned} \left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_k(t, \xi)} - 1 \right| &\leq \left(\frac{g(t, \xi)}{g(t, \eta)} \left| |\xi|^{\frac{1}{3}} - |\eta|^{\frac{1}{3}} \right| e^{4\pi\delta_L|\eta-\xi|^{\frac{1}{3}}} + \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \right) e^{-0.5\pi\delta_L^{-1}|k|^{\frac{1}{3}}} \\ &\lesssim \frac{1}{|k|^{\frac{1}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}} \lesssim \frac{\langle \xi - \eta \rangle}{(|k| + |\eta| + |\xi|)^{\frac{1}{3}}} e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}. \end{aligned}$$

Now we focus on the case $|\eta| + |\xi| > \frac{1}{5}|k|$. We have from Lemma 3.5 that

$$\left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_k(t, \xi)} - 1 \right| \lesssim 1 + e^{C\delta_L^{-1}|\xi-\eta|^{\frac{1}{3}}}$$

So, if $(|\xi| + |\eta|)^{\frac{1}{3}} \lesssim \max\{1, |\eta - \xi|\}$, then we obtain the lemma. So from now on, we assume that $|\eta|^{\frac{1}{3}} + |\xi|^{\frac{1}{3}} \gtrsim 1$ and

$$|\xi - \eta| \leq \frac{1}{100}(|\eta|^{\frac{1}{3}} + |\xi|^{\frac{1}{3}}) \leq \frac{1}{100}(|\eta| + |\xi|). \quad (3.21)$$

We have

$$\left| \frac{\mathcal{M}_k(t, \eta)}{\mathcal{M}_k(t, \xi)} - 1 \right| \leq \left| \frac{g(t, \xi)}{g(t, \eta)} \left[e^{4\pi\delta_L^{-1}(|\xi|^{\frac{1}{3}} - |\eta|^{\frac{1}{3}})} - 1 \right] \right| + \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right|. \quad (3.22)$$

Using Lemma 3.5, we estimate the first term as

$$\left| \frac{g(t, \xi)}{g(t, \eta)} \left[e^{4\pi\delta_L^{-1}(|\xi|^{1/3} - |\eta|^{1/3})} - 1 \right] \right| \lesssim \frac{|\eta - \xi|}{|\eta|^{2/3} + |\xi|^{2/3}} e^{C\delta_L^{-1}|\eta - \xi|^{1/3}}.$$

Now, we estimate the second term in (3.22). We first consider the time regime $t \leq \min(t_{E(|\xi|^{2/3}), |\xi|}, t_{E(|\eta|^{2/3}), |\eta|})$. In this case, we have (see (3.9))

$$\left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| = \left| \frac{g(0, \xi)}{g(0, \eta)} - 1 \right| \lesssim \frac{|\xi - \eta|}{|\xi|^{2/3} + |\eta|^{2/3}} e^{C\delta_L^{-1}|\xi - \eta|^{1/3}}.$$

Now, we consider the time regime: $\min(t_{E(|\xi|^{2/3}), |\xi|}, t_{E(|\eta|^{2/3}), |\eta|}) \leq t \leq \max(t_{E(|\xi|^{2/3}), |\xi|}, t_{E(|\eta|^{2/3}), |\eta|})$ which corresponds to Case 3 in the proof of Lemma 3.5. Let $t \in \bar{I}_{n, \eta} \cap \bar{I}_{j, \xi}$

If $t_{E(|\xi|^{2/3}), \xi} \leq t \leq t_{E(|\eta|^{2/3}), \eta}$, we have

$$\frac{g(t_{E(|\eta|^{2/3}), \eta}, \xi)}{g(t_{E(|\eta|^{2/3}), \eta}, \eta)} \geq \frac{g(t, \xi)}{g(t, \eta)} \geq \frac{g(0, \xi)}{g(0, \eta)},$$

This gives that if $\frac{g(t, \xi)}{g(t, \eta)} \leq 1$, then we have

$$\left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \leq \left| \frac{g(0, \xi)}{g(0, \eta)} - 1 \right| \lesssim \frac{|\xi - \eta|}{|\xi|^{2/3} + |\eta|^{2/3}} e^{C\delta_L^{-1}|\xi - \eta|^{1/3}}$$

and if $\frac{g(t, \xi)}{g(t, \eta)} \geq 1$, then we have

$$\left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \leq \left| \frac{g(t_{E(|\eta|^{2/3}), \eta}, \xi)}{g(t_{E(|\eta|^{2/3}), \eta}, \eta)} - 1 \right|.$$

Similarly, we have for $t_{E(|\eta|^{2/3}), \eta} \leq t \leq t_{E(|\xi|^{2/3}), \xi}$,

$$\left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| \leq \left| \frac{g(t_{E(|\xi|^{2/3}), \xi}, \xi)}{g(t_{E(|\xi|^{2/3}), \xi}, \eta)} - 1 \right| + \frac{|\xi - \eta|}{|\xi|^{2/3} + |\eta|^{2/3}} e^{C\delta_L^{-1}|\xi - \eta|^{1/3}}.$$

Thus we deduce the problem to consider the time regime $\max\{t_{E(|\eta|^{2/3}), \eta}, t_{E(|\xi|^{2/3}), \xi}\} \leq$

$t \leq \frac{1}{2} \min\{|\xi|^{2/3}, |\eta|^{2/3}\}$, which corresponds to Case 4 in proof of Lemma 3.5.

Without loss of generality, let us assume $|\eta| \geq |\xi|$ and prove

$$\left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| + \left| \frac{g(t, \eta)}{g(t, \xi)} - 1 \right| \lesssim \frac{|\eta - \xi|}{|\eta|^{1/3} + |\xi|^{1/3}} e^{C\delta_L^{-1}|\eta - \xi|^{1/3}}$$

So, let n and j be such that $t \in \bar{I}_{n, \eta} \cap \bar{I}_{j, \xi}$. Then we have $|n| \geq |j|$,

Claim: It holds that $\frac{2|\eta|}{2|n|-1} < \frac{1.5|\xi|}{1.5n-1} < \frac{|\xi|}{n-1}$ which implies $||n| - |j|| \leq 1$.

Indeed if not, then $\frac{2|\eta|}{2|n|-1} \geq \frac{1.5|\xi|}{1.5|n|-1}$ which implies $||\eta| - |\xi|| \geq \frac{2|\eta| - 1.5|\xi|}{3|n|}$. It holds from (3.21) that

$$\frac{99}{101}|\xi| \leq |\eta| \leq \frac{101}{99}|\xi|, \quad |\eta - \xi| \leq \frac{3}{100}|\xi|^{\frac{1}{3}}.$$

Thus we get that

$$\frac{3}{100}|\xi|^{\frac{1}{3}} \geq |\eta - \xi| \geq \frac{2|\eta| - 1.5|\xi|}{3|n|} \geq \frac{|\xi|}{7E(|\eta|^{\frac{1}{3}})} \geq \frac{|\xi|}{12|\xi|^{\frac{2}{3}}} = \frac{1}{12}|\xi|^{\frac{1}{3}},$$

which leads a contradiction.

Therefore, it holds that $n \geq E(|\eta|^{\frac{1}{3}})$, $j \geq E(|\xi|^{\frac{1}{3}})$ and $E(|\eta|^{\frac{1}{3}}) \geq E(|\xi|^{\frac{1}{3}})$. For $j = n$ (hence in this case $|j| \geq E(|\eta|^{\frac{1}{3}}) + 1$), thus recalling (3.14) and (3.15) and using the inequality $|e^a e^b - 1| \lesssim (|a| + |b|)e^{a+b}$, we get

$$\begin{aligned} & \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| + \left| \frac{g(t, \eta)}{g(t, \xi)} - 1 \right| \\ & \lesssim \left(\sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} \delta_L^{-1} |F(k, \eta) - F(k, \xi)| + \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \delta_L^{-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \right. \\ & \quad + \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{j-1} \delta_L^{-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| + \delta_L^{-1} \left| \frac{|\xi|}{|j|^3} \arctan \left(t - \frac{|\xi|}{|j|} \right) \right. \\ & \quad \left. - \frac{|\eta|}{|j|^3} \arctan \left(t - \frac{|\eta|}{|j|} \right) \right| + \delta_L^{-1} \left| \frac{|\eta|}{|j|^3} \arctan \left(\frac{|\eta|}{|j|(2|j|-1)} \right) \right. \\ & \quad \left. - \frac{|\xi|}{|j|^3} \arctan \left(\frac{|\xi|}{|j|(2|j|-1)} \right) \right| \Big) \max \left\{ \frac{g(t, \xi)}{g(t, \eta)}, \frac{g(t, \eta)}{g(t, \xi)} \right\} \end{aligned}$$

First applying (3.19), we have

$$\sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} \delta_L^{-1} |F(k, \eta) - F(k, \xi)| \lesssim |\xi|^{\frac{1}{3}} \frac{|\xi - \eta|}{k^2 + |\xi|^2} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{5}{3}} + |\eta|^{\frac{5}{3}}}. \quad (3.23)$$

Similarly, using (3.20), we have

$$\sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \delta_L^{-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \lesssim |E(|\eta|^{\frac{1}{3}}) - E(|\xi|^{\frac{1}{3}})| \frac{|\eta - \xi|}{|\xi|} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}} \quad (3.24)$$

and

$$\sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{j-1} \delta_L^{-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| \lesssim \frac{|\eta - \xi|}{|\eta|^{\frac{2}{3}}} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}}}. \quad (3.25)$$

Also, we have

$$\begin{aligned}
 & \left| \frac{|\xi|}{|j|^3} \arctan \left(t - \frac{|\xi|}{|j|} \right) - \frac{|\eta|}{|j|^3} \arctan \left(t - \frac{|\eta|}{|j|} \right) \right| \\
 & \lesssim \frac{|\xi|}{|j|^3} \left| \arctan \left(t - \frac{|\xi|}{|j|} \right) - \arctan \left(t - \frac{|\eta|}{|j|} \right) \right| \\
 & \quad + \left| \arctan \left(t - \frac{|\eta|}{|j|} \right) \right| \left| \frac{|\xi|}{|j|^3} - \frac{|\eta|}{|j|^3} \right| \\
 & \lesssim \frac{|\xi|}{|j|^3} \frac{|\eta - \xi|}{|j|} + \frac{|\eta - \xi|}{|j|^3} \\
 & \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{1}{3}} + |\eta|^{\frac{1}{3}}}.
 \end{aligned}$$

Similarly, for the last term, we have by applying (3.13)

$$\begin{aligned}
 & \left| \frac{|\eta|}{|j|^3} \arctan \left(\frac{|\eta|}{|j|(2|j| - 1)} \right) - \frac{|\xi|}{|j|^3} \arctan \left(\frac{|\xi|}{|j|(2|j| - 1)} \right) \right| \\
 & \lesssim \frac{|\eta|}{|j|^3} \left| \arctan \left(\frac{|\eta|}{|j|(2|j| - 1)} \right) - \arctan \left(\frac{|\xi|}{|j|(2|j| - 1)} \right) \right| \\
 & \quad + \left| \arctan \left(\frac{|\xi|}{|j|(2|j| - 1)} \right) \right| \left| \frac{|\xi|}{|j|^3} - \frac{|\eta|}{|j|^3} \right| \\
 & \lesssim \frac{|\eta|}{|j|^3} \frac{|\eta - \xi|}{\langle \xi \rangle} + \frac{|\eta - \xi|}{|j|^3} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi| + |\eta|}.
 \end{aligned} \tag{3.26}$$

Collecting (3.23)–(3.26) together with Lemma 3.5 yield Lemma 3.8 in the above time regime with $j = n$.

Now, we discuss the case $n \neq j$. We then have

$$\frac{|\eta|}{|n|} \leq \frac{2|\xi|}{2|n| - 1} \leq t \leq \frac{2|\eta|}{2|n| - 1} < \frac{1.5|\xi|}{1.5n - 1} < \frac{|\xi|}{n - 1} \tag{3.27}$$

It is clear that since $t \approx \frac{|\eta|}{|n|} \leq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}$, then it holds that $j = n - 1 \geq 2|\eta|^{\frac{2}{3}} - 1 \geq |\eta|^{\frac{2}{3}} + 1$. Hence, using (3.14) and (3.15), we have

$$\begin{aligned}
 & \left| \frac{g(t, \xi)}{g(t, \eta)} - 1 \right| + \left| \frac{g(t, \eta)}{g(t, \xi)} - 1 \right| \\
 & \lesssim \delta_L^{-1} \left(\sum_{k=1}^{E(|\xi|^{\frac{1}{3}})} |F(k, \eta) - F(k, \xi)| + \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} F(k, \eta) + \sum_{k=E(|\xi|^{\frac{1}{3}})+1}^{E(|\eta|^{\frac{1}{3}})} \tilde{F}(k, \xi) \right. \\
 & \quad \left. + \sum_{k=E(|\eta|^{\frac{1}{3}})+1}^{n-1} |\tilde{F}(k, \eta) - \tilde{F}(k, \xi)| + \frac{|\eta|}{|n|^3} \left| \arctan \left(t - \frac{|\eta|}{|n|} \right) \right| \right)
 \end{aligned}$$

$$\begin{aligned}
& - \arctan \left(\frac{|\eta|}{|n|(2|n|-1)} \right) \Big| + \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(t - \frac{|\xi|}{|n-1|} \right) \right. \right. \\
& \left. \left. - \arctan \left(\frac{|\xi|}{|n-1|(2n-3)} \right) \right) + \tilde{F}(n-1, \xi) \right| \times \max \left\{ \frac{g(t, \xi)}{g(t, \eta)}, \frac{g(t, \eta)}{g(t, \xi)} \right\}.
\end{aligned} \tag{3.28}$$

The first four terms can be estimated as in (3.23), (3.24) and (3.25).

We also have in the time regime (3.27),

$$\begin{aligned}
& \left| \arctan \left(t - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{|\eta|}{|n|(2|n|-1)} \right) \right| \\
& \lesssim \left| \arctan \left(\frac{2|\xi|}{2|n|-1} - \frac{|\eta|}{|n|} \right) - \arctan \left(\frac{2|\eta|}{2|n|-1} - \frac{|\eta|}{|n|} \right) \right| \\
& \lesssim \frac{|\xi - \eta|}{|n|} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{1}{3}} + |\eta|^{\frac{1}{3}}}.
\end{aligned}$$

Now, using (3.11), we have

$$|\tilde{F}(n-1, \eta) - \tilde{F}(n-1, \xi)| \lesssim \frac{|\eta - \xi|}{(|n|-1)^3} \lesssim \frac{|\eta - \xi|}{|\xi|} \lesssim \frac{\langle \xi - \eta \rangle}{|\xi|^{\frac{1}{3}} + |\eta|^{\frac{1}{3}}}.$$

Now, we need to estimate the last term in (3.28).

Recalling (3.8), we have

$$\begin{aligned}
& \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(t - \frac{|\xi|}{|n-1|} \right) - \arctan \left(\frac{|\xi|}{(|n|-1)(2n-3)} \right) \right) + \tilde{F}(n-1, \xi) \right| \\
& = \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(t - \frac{|\xi|}{|n-1|} \right) - \arctan \left(\frac{|\xi|}{|n-1|(2n-3)} \right) \right. \right. \\
& \quad \left. \left. + \arctan \left(\frac{|\xi|}{|n-1|(2n-1)} \right) + \arctan \left(\frac{|\xi|}{|n-1|(2n-3)} \right) \right) \right| \\
& = \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(t - \frac{|\xi|}{|n-1|} \right) + \arctan \left(\frac{|\xi|}{|n-1|(2n-1)} \right) \right) \right|.
\end{aligned}$$

We have, by exploiting (3.27),

$$\begin{aligned}
& \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(t - \frac{|\xi|}{|n-1|} \right) + \arctan \left(\frac{|\xi|}{|n-1|(2n-1)} \right) \right) \right| \\
& = \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(\frac{|\xi|}{|n-1|} - \frac{2|\xi|}{2n-1} \right) - \arctan \left(\frac{|\xi|}{|n-1|} - t \right) \right) \right| \\
& \lesssim \left| \frac{|\xi|}{|n-1|^3} \left(\arctan \left(\frac{|\xi|}{|n-1|} - \frac{2|\xi|}{2n-1} \right) - \arctan \left(\frac{|\xi|}{|n-1|} - \frac{2|\eta|}{2|n|-1} \right) \right) \right| \\
& \lesssim \frac{|\xi|}{|n-1|^3} \frac{|\eta - \xi|}{|n|} \lesssim \frac{\langle \eta - \xi \rangle}{|\xi|^{1/3} + |\eta|^{1/3}}.
\end{aligned}$$

Putting the estimates together, we have proved the lemma. $\square$

Lemma 3.9. Let $t \leq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}$. Then

$$\left| \frac{J_k(t, \eta)}{J_l(t, \xi)} - 1 \right| \lesssim \frac{\langle k-l, \xi-\eta \rangle}{(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}}} e^{C\mu|k-l, \xi-\eta|^{\frac{1}{3}}}.$$

Proof. Keeping (3.3) in mind and due to our assumption on the range of t , then it holds that $J_k(t, \eta) = J_k(t_{E(|\eta|^{\frac{1}{3}}), \eta}, \eta) = J_k(0, \eta)$ and $J_k(t, \xi) = J_k(t_{E(|\xi|^{\frac{1}{3}}), \xi}, \xi) = J_k(0, \xi)$. First, we have from Lemma 3.6:

$$\left| \frac{J_k(t, \eta)}{J_l(t, \xi)} \right| \lesssim e^{\mu|\eta-\xi|^{\frac{1}{3}}} + e^{\mu|k-l|^{\frac{1}{3}}} \lesssim e^{C\mu|\eta-\xi, k-l|^{\frac{1}{3}}}. \quad (3.29)$$

Hence, we obtain

$$\left| \frac{J_k(t, \eta)}{J_l(t, \xi)} - 1 \right| \leq \left| \frac{J_k(t, \eta)}{J_l(t, \xi)} \right| + 1.$$

Hence, the lemma holds for $(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}} \lesssim |k-l| + |\xi-\eta|$. Also, if $(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}} \leq 1$, then Lemma 3.9 holds since due to (3.29) by allowing the constant C in the exponent to be large enough. So, from now on we restrict to the case $|k|^{\frac{2}{3}} + |l|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}} + |\xi|^{\frac{2}{3}} \gtrsim 1$.

We assume now that

$$|k|^{\frac{2}{3}} + |l|^{\frac{2}{3}} + |\eta|^{\frac{2}{3}} + |\xi|^{\frac{2}{3}} \geq 100(|k-l| + |\xi-\eta|)$$

That is, we have in this case

$$|k| + |l| + |\eta| + |\xi| \geq 100(|k-l| + |\xi-\eta|). \quad (3.30)$$

We define the multiplier $\tilde{J}_k(t, \eta)$ as

$$\tilde{J}_k(t, \eta) = \frac{e^{\mu|\eta|^{\frac{1}{3}}}}{\Theta_k(t, \eta)}.$$

Case 1. If $|k, l| \approx |\eta, \xi|$. That is if for instance: $\frac{1}{10}(|k| + |l|) \leq |\eta| + |\xi| \leq 10(|k| + |l|)$. Then this together with (3.30) implies that

$$|k-l, \xi-\eta| \lesssim |k| \approx |l| \approx |\xi| \approx |\eta|.$$

Hence, we have

$$\left| \frac{J_k(t, \eta)}{J_l(t, \xi)} - 1 \right| \leq \left| \frac{\tilde{J}_k(t, \eta) - \tilde{J}_l(t, \xi)}{J_l(t, \xi)} \right| + \left| \frac{e^{\mu|k|^{\frac{1}{3}}} - e^{\mu|l|^{\frac{1}{3}}}}{J_l(t, \xi)} \right|. \quad (3.31)$$

The second term on the right-hand side of (3.31) can be estimate as

$$\begin{aligned} \left| \frac{e^{\mu|k|^{\frac{1}{3}}} - e^{\mu|l|^{\frac{1}{3}}}}{J_l(t, \xi)} \right| &\lesssim |e^{\mu(|k|^{\frac{1}{3}} - |l|^{\frac{1}{3}})} - 1| \\ &\lesssim \frac{|k-l|}{|k|^{2/3} + |k|^{1/3}|l|^{1/3} + |l|^{2/3}} e^{\mu|k-l|^{1/3}} \\ &\lesssim \frac{\langle k-l, \eta-\xi \rangle}{(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}}} e^{\mu|k-l, \eta-\xi|^{\frac{1}{3}}}. \end{aligned}$$

For the first term, we have

$$\begin{aligned} \left| \frac{\tilde{J}_k(t, \eta) - \tilde{J}_l(t, \xi)}{J_l(t, \xi)} \right| &\leq \left| \frac{\tilde{J}_k(t, \eta) - \tilde{J}_l(t, \xi)}{\tilde{J}_l(t, \xi)} \right| \\ &\lesssim \frac{\Theta_l(0, \xi)}{\Theta_k(0, \eta)} |e^{\mu(|\eta|^{1/3} - |\xi|^{1/3})} - 1| + \left| \frac{\Theta_l(0, \xi)}{\Theta_k(0, \eta)} - 1 \right|. \end{aligned} \quad (3.32)$$

Then, we control the first term as, by using the mean value theorem

$$\begin{aligned} |e^{\mu(|\eta|^{1/3} - |\xi|^{1/3})} - 1| &\leq \mu ||\eta|^{1/3} - |\xi|^{1/3}| e^{\mu(|\eta|^{1/3} - |\xi|^{1/3})} \\ &\lesssim \frac{|\eta - \xi|}{|\eta|^{2/3} + |\eta|^{1/3}|\xi|^{1/3} + |\xi|^{2/3}} e^{\mu(|\eta|^{1/3} - |\xi|^{1/3})} \\ &\lesssim \frac{\langle k - l, \eta - \xi \rangle}{(|k| + |l| + |\eta| + |\xi|)^{\frac{2}{3}}} e^{\mu|k-l, \eta-\xi|^{\frac{1}{3}}}, \end{aligned}$$

which together with Lemma 3.6 implies the control of the first term in (3.32).

Our goal now is to control the second term on the right-hand side of (3.32). Due to (3.30) and the fact that $|k| + |l| \approx |\eta| + |\xi|$, it holds that $|\mathbb{E}(\eta) - \mathbb{E}(\xi)| \leq 1$.

Keeping in mind (3.4) and if $\mathbb{E}(|\eta|^{1/3}) = \mathbb{E}(|\xi|^{1/3})$, then we have

$$\left| \frac{\Theta_l(0, \xi)}{\Theta_k(0, \eta)} - 1 \right| = \left| \left(\frac{|\eta|}{|\xi|} \right)^{c\mathbb{E}(|\eta|^{1/3})} - 1 \right|$$

with $2C\kappa + 1$. This implies

$$\left| \left(\frac{|\eta|}{|\xi|} \right)^{c\mathbb{E}(|\eta|^{1/3})} - 1 \right| \leq \left| \left(1 + \frac{|\eta - \xi|}{|\xi|} \right)^{c\mathbb{E}(|\eta|^{1/3})} - 1 \right|.$$

Using the inequality

$$\left(1 + \frac{a}{b^3} \right)^b - 1 \leq C \frac{|a|}{b^2}, \quad |a| < 1, \quad b > 1,$$

we obtain

$$\left| \left(1 + \frac{|\eta - \xi|}{|\xi|} \right)^{c\mathbb{E}(|\eta|^{1/3})} - 1 \right| \lesssim \frac{|\eta - \xi|}{|\xi|^{\frac{2}{3}}}.$$

Next, if $\mathbb{E}(|\eta|^{1/3}) = \mathbb{E}(|\xi|^{1/3}) + 1$, we have $|\xi|^{1/3} \leq \mathbb{E}(|\eta|^{1/3}) \leq |\eta|^{1/3}$

$$\begin{aligned} \left| \frac{\Theta_l(0, \xi)}{\Theta_k(0, \eta)} - 1 \right| &= \left| \left(\frac{|\eta|}{|\xi|} \right)^{c\mathbb{E}(|\xi|^{1/3})} \left(\frac{|\eta|}{(\mathbb{E}(|\eta|^{1/3}))^3} \right)^c - 1 \right| \\ &\lesssim \frac{|\eta - \xi|}{|\xi|^{\frac{2}{3}}} + \left| \left(\frac{|\eta|}{(\mathbb{E}(|\eta|^{1/3}))^3} \right)^c - 1 \right| \end{aligned}$$

Since,

$$\left| \left(\frac{|\eta|}{(\mathbb{E}(|\eta|^{1/3}))^3} \right)^c - 1 \right| \leq \left(\frac{|\eta|}{|\xi|} \right)^c - 1 \lesssim \frac{\langle \eta - \xi \rangle}{|\xi|}.$$

We omit the case $\mathbb{E}(|\eta|^{1/3}) = \mathbb{E}(|\xi|^{1/3}) - 1$ since it can be treated similarly.

The other two cases $|\xi| + |\eta| \geq 10(|k| + |\eta|)$ and $|k| + |l| \geq 10(|\xi| + |\eta|)$ can be treated by modifying slightly the above argument and using the fact that the first condition together with (3.30) implies that $|\xi| \geq \frac{989}{200}(|l| + |k|)$ and the second condition together with (3.30) implies that $|\xi| \geq \frac{989}{200}(|l| + |k|)$ and the second condition together with (3.30) implies $|l| \geq \frac{989}{200}(|\xi| + |\eta|)$ $\square$

4. Zero Mode and Coordinate System

In this section, we deal with the coordinate system and prove Proposition 2.3.

4.1. Assistant estimate

In this section, we prove the estimate of ψ_0 . Recall (2.7), by the Duhamel's principle, we have

$$\langle \psi \rangle = e^{t\partial_{yy}} \langle \psi_{in} \rangle - \int_0^t e^{(t-s)\partial_{yy}} \langle \partial_x \psi_{\neq} u_{\neq}^x \rangle (s) ds.$$

Under the bootstrap hypotheses, by the fact that for any $f_1(y) = f_2(v)$ we have for any $k \geq 0$ and $1 \leq p \leq \infty$, $\|\langle \partial_v \rangle^{k+2} h\|_{L^\infty} \lesssim \epsilon$ and then

$$\|f_1\|_{W_y^{k,p}} \stackrel{\text{def}}{=} \sum_{i=0}^k \left(\int_{\mathbf{R}} |\partial_y^i f_1(y)|^p dy \right)^{\frac{1}{p}} \approx \|f_2\|_{W_v^{k,p}} \stackrel{\text{def}}{=} \sum_{i=0}^k \left(\int_{\mathbf{R}} |\partial_v^i f_2(v)|^p dv \right)^{\frac{1}{p}} \quad (4.1)$$

and for any $\gamma \in (0, 1)$

$$\begin{aligned} \|f_1\|_{\dot{H}_y^\gamma} &\stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^2} \frac{|f_1(y_1) - f_1(y_2)|^2}{|y_1 - y_2|^{1+2\gamma}} dy_1 dy_2 \right)^{\frac{1}{2}} \\ &\approx \left(\int_{\mathbf{R}^2} \frac{|f_2(v_1) - f_2(v_2)|^2}{|v_1 - v_2|^{1+2\gamma}} dv_1 dv_2 \right)^{\frac{1}{2}} \stackrel{\text{def}}{=} \|f_2\|_{\dot{H}_v^\gamma}. \end{aligned} \quad (4.2)$$

which together with the elliptic estimate, we have for any $k \geq 0$

$$\|\langle \partial_z \psi_{\neq} u_{\neq}^x \rangle\|_{W_y^{k,p}} \lesssim \|\langle \partial_z \phi_{\neq} \tilde{u}_{\neq}^x \rangle\|_{W_v^{k,p}} \lesssim \|\phi_{\neq}\|_{H^{k+3}} \|\tilde{u}_{\neq}^x\|_{H^{k+3}} \lesssim \frac{\epsilon^2}{1+t^4}. \quad (4.3)$$

Thus we get that for $0 \leq j \leq 7$

$$\begin{aligned} \|\partial_y^j \langle \psi \rangle\|_{L^p} &\lesssim \frac{1}{\langle t \rangle^{\frac{1}{2}(1-\frac{1}{p})+\frac{j}{2}}} \left(\|\langle \psi_{in} \rangle\|_{L^1} + \|\langle \psi_{in} \rangle\|_{W^{j,\infty}} \right) \\ &\quad + \int_0^t \frac{1}{\langle t-s \rangle^{\frac{1}{2}(1-\frac{1}{p})+\frac{j}{2}}} \frac{\epsilon^2}{1+s^4} ds \\ &\lesssim \frac{\epsilon}{\langle t \rangle^{\frac{1}{2}(1-\frac{1}{p})+\frac{j}{2}}}. \end{aligned}$$

By choosing different j and p , we get $E_{as, \psi_0}(t) \lesssim \epsilon^2$.

By using the fact that $\omega_0 = \partial_{yy} \langle \psi \rangle$ and $\langle u^x \rangle = -\partial_y \langle \psi \rangle$, we have

$$\begin{aligned} \|\omega_0\|_{L^2} &\lesssim \epsilon \langle t \rangle^{-\frac{5}{4}}, \quad \|\partial_y \omega_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}, \\ \|\partial_{yy} \omega_0\|_{L^2} &\lesssim \epsilon \langle t \rangle^{-\frac{9}{4}}, \quad \|\partial_{yy} \langle u^x \rangle\|_{L^\infty} \lesssim \frac{\epsilon}{1+t^2}, \end{aligned}$$

which together with (4.1), (4.3) and (2.6) gives us that

$$\begin{aligned} \|f_0\|_{L^2} &\lesssim \epsilon \langle t \rangle^{-\frac{5}{4}}, \quad \|\partial_v f_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}, \quad \|\partial_t u_0^x\|_{L^\infty} \lesssim \frac{\epsilon}{1+t^2}, \\ \|\partial_t u_0^x\|_{L_y^\infty} &\lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}, \quad \|\partial_t u_0^x\|_{\dot{H}^{\frac{1}{2}-\epsilon_1}}} \lesssim \epsilon \langle t \rangle^{-2+\frac{\epsilon_1}{2}}, \quad \|\partial_t u_0^x\|_{\dot{H}^{\frac{1}{2}+\epsilon_2}}} \lesssim \epsilon \langle t \rangle^{-2-\frac{\epsilon_2}{2}}. \end{aligned}$$

Moreover recall that $K_0(t, v) = \partial_{yy} \omega_0(t, y)$, we get that

$$\|K_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{9}{4}}. \quad (4.4)$$

Recall that $g(t, v) = \partial_t v(t, y)$ and $h(t, v) = \partial_y v(t, y) - 1$ and that $\partial_t v(t, y) = \frac{1}{t^2} \int_0^t s \partial_t u_0^x(s, y) ds$ and $\partial_y v(t, y) - 1 = -\frac{1}{t} \int_0^t \omega_0(s, y) ds$, we get that

$$\begin{aligned} \|\partial_t v(t, y)\|_{L_y^\infty} &\lesssim \frac{1}{\langle t \rangle^2} \int_0^t s \|\partial_t u_0^x\|_{L_y^\infty} ds \lesssim \epsilon (\ln \langle t \rangle + 1) \langle t \rangle^{-2}, \\ \|\partial_t v(t, y)\|_{L^2} &\lesssim \frac{1}{\langle t \rangle^2} \int_0^t s \|\partial_t u_0^x\|_{L_y^2} ds \lesssim \epsilon \langle t \rangle^{-\frac{7}{4}}, \\ \|\partial_t v(t, y)\|_{\dot{H}^{\frac{1}{2}-\epsilon_1}}} &\lesssim \frac{\epsilon}{\epsilon_1} \langle t \rangle^{-2+\frac{\epsilon_1}{2}}, \quad \|\partial_t v(t, y)\|_{\dot{H}^{\frac{1}{2}+\epsilon_2}}} \\ &\lesssim \frac{1}{\langle t \rangle^2} \int_0^t \frac{\epsilon}{\langle s \rangle^{1+\frac{\epsilon_2}{2}}} ds \lesssim \frac{\epsilon}{\epsilon_2} \langle t \rangle^{-2} \end{aligned}$$

and

$$\|\partial_y v(t, y) - 1\|_{L_y^2} \lesssim \frac{1}{\langle t \rangle} \int_0^t \|\omega_0\|_{L^2} ds \lesssim \frac{\epsilon}{\langle t \rangle},$$

Thus by (4.1) and (4.2), we get that

$$\|h\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle}, \quad (4.5)$$

and $E_{as, g}(t) \lesssim \epsilon^2$.

4.2. Estimate of $g(t, y)$

Proof. Recall (2.5c). Under the bootstrap hypotheses we have

$$\|AK_0\|_2 \leq 5\epsilon, \quad \int_1^t \|\partial_v AK_0\|_2^2 ds \leq 8\epsilon^2$$

By the fact that $K_0 = \partial_{vv} f_0 + h(h+2)\partial_{vv} f_0 + \partial_v h(1+h)\partial_v f_0$, we have

$$\begin{aligned} \|A\partial_{vv} f_0\|_2 &\lesssim \|AK_0\|_2 + \|Ah\|_2(2 + \|Ah\|_2)\|A\partial_{vv} f_0\|_2 \\ &\quad + \|A\partial_v h\|_2(1 + \|Ah\|_2)\|A\partial_v f_0\|_2 \\ &\lesssim \|AK_0\|_2 + \epsilon\|A\partial_{vv} f_0\|_2 + \epsilon(\|\partial_v f_0\|_{L^2} + \|A\partial_{vv} f_0\|_2), \end{aligned}$$

which gives us that

$$\|A\partial_{vv} f_0\|_2 \lesssim \|AK_0\|_2 + \epsilon^2 \langle t \rangle^{-\frac{7}{4}} \lesssim \epsilon. \quad (4.6)$$

By the fact that

$$\|AK_0\|_2 \leq \|K_0\|_{L^2} + \|\partial_v AK_0\|_{L^2}, \quad (4.7)$$

we have by (4.4),

$$\int_1^t \|A\partial_{vv} f_0\|_2^2 ds \lesssim \epsilon^2.$$

Thus we get that

$$t\|\partial_{vv} A\bar{h}\|_{L^2} \lesssim \|A\partial_{vv} f_0\|_2 + \|A\partial_{vv} h\|_{L^2} \lesssim \epsilon.$$

and

$$\int_1^t s\|\partial_{vv} A\bar{h}(s)\|_{L^2}^2 ds \lesssim \int_1^t \|A\partial_{vv} f_0(s)\|_2^2 ds + \int_1^t \|A\partial_{vv} h(s)\|_{L^2}^2 ds \lesssim \epsilon^2.$$

Note that

$$\begin{aligned} \|A\partial_{vvv} g\|_2 &\leq \|A\partial_{vv} \bar{h}\|_2 + \|A\partial_{vv}(h\partial_v g)\|_2 \\ &\lesssim \|A\partial_{vv} \bar{h}\|_2 + \|Ah\|_2\|A\partial_{vvv} g\|_2 + \|A\partial_{vv} h\|_2\|A\partial_v g\|_2 \\ &\lesssim \|A\partial_{vv} \bar{h}\|_2 + \|Ah\|_2\|A\partial_{vvv} g\|_2 + \|A\partial_{vv} h\|_2(\|A\partial_{vvv} g\|_2 + \|g\|_{L^2}), \end{aligned}$$

which together with the bootstrap assumption implies that

$$\|A\partial_{vvv} g\|_2 \lesssim \|A\partial_{vv} \bar{h}\|_2 + \epsilon^2 \langle t \rangle^{-\frac{7}{4}},$$

which implies the first inequality in Proposition 2.3.

Moreover by the fact that $\|f_0\|_{L^2} \lesssim \epsilon \langle t \rangle^{-\frac{5}{4}}$ and $\|h\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle}$, we have $\|\bar{h}\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle}$, thus under the bootstrap hypotheses, we get that

$$\|\partial_v g\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle^2}. \quad (4.8)$$

□

4.3. Low Gevrey norm estimate

In this section we prove the rest parts of Proposition 2.3. It is natural to compute the time evolution of E_{l_0, f_0} and $E_{l_0, h}$. Recall that f_0 satisfies (2.5a) and h satisfies (2.5b). We then have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\sum_{k=0}^3 \frac{t^k}{4^k} \|\partial_v^k f_0\|_{\mathcal{G}^{\lambda, \beta; s}}^2 \right) - \dot{\lambda}(t) \sum_{k=0}^3 \frac{t^k}{4^k} \|\partial_v\|^{\frac{s}{2}} \partial_v^k f_0 \|_{\mathcal{G}^{\lambda, \beta; s}}^2 + D_{l_0, f_0}(t) \\
&= \sum_{k=1}^3 \frac{k}{2} \frac{t^{k-1}}{4^k} \|\partial_v^k f_0\|_{\mathcal{G}^{\lambda, \beta; s}}^2 + \sum_{k=0}^3 \frac{t^k}{4^k} \langle \partial_v^k f_0, v' \cdot \nabla^\perp \phi_{\neq} \cdot \nabla f_{\neq} \rangle_{\mathcal{G}^{\lambda, \beta; s}} \\
&\quad - \langle f_0, g \partial_v f_0 \rangle_{\mathcal{G}^{\lambda, \beta; s}} + \langle f_0, ((v')^2 - 1) \partial_{vv} f_0 \rangle_{\mathcal{G}^{\lambda, \beta; s}} \\
&\quad + \langle f_0, v'' \partial_v f_0 \rangle_{\mathcal{G}^{\lambda, \beta; s}} + \sum_{k=1}^3 \left(-\frac{t^k}{4^k} \langle \partial_v^k f_0, \partial_v^k (g \partial_v f_0) \rangle_{\mathcal{G}^{\lambda, \beta; s}} \right. \\
&\quad \left. + \frac{t^k}{4^k} \langle \partial_v^k f_0, \partial_v^k ((v')^2 - 1) \partial_{vv} f_0 \rangle_{\mathcal{G}^{\lambda, \beta; s}} \right. \\
&\quad \left. + \frac{t^k}{4^k} \langle \partial_v^k f_0, \partial_v^k (v'' \partial_v f_0) \rangle_{\mathcal{G}^{\lambda, \beta; s}} \right) = \sum_{j=1}^8 \Pi_{f_0, j},
\end{aligned}$$

where

$$D_{l_0, f_0}(t) = \sum_{k=0}^3 \frac{t^k}{4^k} \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda, \beta; s}}^2,$$

and the inner product is defined as follows:

$$\langle f_1, f_2 \rangle_{\mathcal{G}^{\lambda, \beta; s}} = \frac{1}{2\pi} \int_{\mathbf{R}} \langle \eta \rangle^{2\beta} e^{2\lambda(t)|\eta|^s} \overline{\hat{f}_1(\eta)} \hat{f}_2(\eta) d\eta.$$

We also have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (t^2 \|\partial_v^2 h\|_{\mathcal{G}^{\lambda, \beta; s}}^2) - \dot{\lambda}(t) t^2 \|\partial_v^2 h\|_{\mathcal{G}^{\lambda, \beta; s}}^2 \\
&= -t^2 \langle \partial_{vv} h, \partial_{vv} (g \partial_v h) \rangle_{\mathcal{G}^{\lambda, \beta; s}} - t \langle \partial_{vv} f_0, \partial_{vv} h \rangle_{\mathcal{G}^{\lambda, \beta; s}} = \Pi_{h, 1} + \Pi_{h, 2}.
\end{aligned}$$

By (4.5), it is easy to check that

$$t \|h\|_{\mathcal{G}^{\lambda, \beta+2; s}} \lesssim \epsilon + E_{l_0, h}^{\frac{1}{2}}$$

and by (2.5c), (4.8) and using the same argument as above section, we have

$$t^2 \|\partial_v g\|_{\mathcal{G}^{\lambda, \beta+2; s}} \leq \epsilon + E_{l_0, g}^{\frac{1}{2}}, \quad E_{l_0, g} \lesssim E_{l_0, f_0} + E_{l_0, h}$$

It is easy to check that

$$|\Pi_{f_0, 1}| \leq \frac{3}{8} D_{l_0, f_0}(t).$$

By the elliptic estimate in Lemma 5.1, we have

$$|\Pi_{f_0,2}| \lesssim t^k \|\partial_v^k f_0\|_{\mathcal{G}^{\lambda,\beta;s}} (1 + \|h\|_{\mathcal{G}^{\lambda,\beta;s}}) \|\nabla P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} \|\nabla f_{\neq}\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \frac{\epsilon^2}{\langle t \rangle^2} E_{lo,f_0}^{\frac{1}{2}}.$$

By the fact that $\|g\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle^{\frac{7}{4}}}$, we get that

$$\begin{aligned} |\Pi_{f_0,3}| &\lesssim \|f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \|g\|_{\mathcal{G}^{\lambda,\beta;s}} \|\partial_v f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \|f_0\|_{\mathcal{G}^{\lambda,\beta;s}} (\|g\|_{L^2} + \|\partial_v g\|_{\mathcal{G}^{\lambda,\beta;s}}) \|\partial_v f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \frac{\epsilon}{\langle t \rangle^2} E_{lo,f_0} + \frac{1}{\langle t \rangle^2} E_{lo,f_0} E_{lo,g}^{\frac{1}{2}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\Pi_{f_0,4}| &\leq \|f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \|h\|_{\mathcal{G}^{\lambda,\beta;s}} (2 + \|h\|_{\mathcal{G}^{\lambda,\beta;s}}) \|\partial_{vv} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \frac{\epsilon}{\langle t \rangle^2} E_{lo,f_0} + \frac{1}{\langle t \rangle^2} E_{lo,f_0} E_{lo,h}^{\frac{1}{2}}, \\ |\Pi_{f_0,5}| &\lesssim \|f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \|\partial_v h\|_{\mathcal{G}^{\lambda,\beta;s}} (1 + \|h\|_{\mathcal{G}^{\lambda,\beta;s}}) \|\partial_v f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \frac{\epsilon}{\langle t \rangle^{\frac{3}{2}}} E_{lo,f_0} + \frac{1}{\langle t \rangle^{\frac{3}{2}}} E_{lo,f_0} E_{lo,h}^{\frac{1}{2}}. \end{aligned}$$

By using the fact that $\langle f_1, \partial_v f_2 \rangle_{\mathcal{G}^{\lambda,\beta;s}} = - \langle \partial_v f_1, f_2 \rangle_{\mathcal{G}^{\lambda,\beta;s}}$, we have

$$\begin{aligned} |\Pi_{f_0,6}| &\lesssim \sum_{k=1}^3 \left(t^k \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \|g\|_{\mathcal{G}^{\lambda,\beta;s}} \|\partial_v^k f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \right. \\ &\quad \left. + t^k \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \|\partial_v^2 g\|_{\mathcal{G}^{\lambda,\beta;s}} \|\partial_v f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \right) \\ &\lesssim D_{lo,f_0}^{\frac{1}{2}} E_{lo,f_0}^{\frac{1}{2}} \left(\frac{\epsilon}{\langle t \rangle^{\frac{7}{4}}} + \frac{1}{\langle t \rangle^2} E_{lo,g}^{\frac{1}{2}} \right) + D_{lo,f_0}^{\frac{1}{2}} E_{lo,f_0}^{\frac{1}{2}} \left(\frac{\epsilon}{\langle t \rangle} + \frac{E_{lo,g}^{\frac{1}{2}}}{\langle t \rangle} \right), \\ |\Pi_{f_0,7}| &\lesssim \sum_{k=1}^3 \left(\epsilon t^k \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda,\beta;s}}^2 + t^{\frac{k}{2}} \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} t \|h\|_{\mathcal{G}^{\lambda,\beta+2;s}} t^{\frac{1}{2}} \|\partial_{vv} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \right) \\ &\lesssim \epsilon D_{lo,f_0} + E_{lo,h}^{\frac{1}{2}} D_{lo,f_0}. \end{aligned}$$

Now we deal with $\Pi_{f_0,8}$. For $k = 3$, we have

$$\begin{aligned} |t^3 \langle \partial_v^3 f_0, \partial_v^3 (\partial_v ((v')^2 - 1) \partial_v f_0) \rangle_{\mathcal{G}^{\lambda,\beta;s}}| \\ &\lesssim t^{\frac{3}{2}} \|\partial_v^4 f_0\|_{\mathcal{G}^{\lambda,\beta;s}} t^{\frac{3}{2}} \left\| \partial_v^2 (\partial_v ((v')^2 - 1) \partial_v f_0) \right\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim D_{lo,f_0}^{\frac{1}{2}} \sqrt{t} \|h\|_{\mathcal{G}^{\lambda,\beta+1;s}} t \|\partial_v^3 f_0\|_{\mathcal{G}^{\lambda,\beta;s}} + D_{lo,f_0}^{\frac{1}{2}} \sqrt{t} \|\partial_v^2 h\|_{\mathcal{G}^{\lambda,\beta+1;s}} t \|\partial_v f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \epsilon D_{lo,f_0} + E_{lo,h}^{\frac{1}{2}} D_{lo,f_0} + D_{lo,f_0}^{\frac{1}{2}} \sqrt{t} \|A \partial_v^2 h\|_{2t} \left(\|\partial_v f_0\|_{L^2} + \|\partial_{vv} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} \right) \\ &\lesssim \epsilon D_{lo,f_0} + E_{lo,h}^{\frac{1}{2}} D_{lo,f_0} + \left(\epsilon + E_{lo,f_0}^{\frac{1}{2}} \right) \left(D_{lo,f_0} + \langle t \rangle \|A \partial_v^2 h\|_{2t}^2 \right), \end{aligned}$$

and for $k = 1, 2$, we have

$$\begin{aligned} t^k | < \partial_v^k f_0, \partial_v^k (\partial_v((v')^2 - 1) \partial_v f_0) >_{\mathcal{G}^{\lambda, \beta; s}} | \\ &\lesssim t^{\frac{k}{2}} \|\partial_v^{k+1} f_0\|_{\mathcal{G}^{\lambda, \beta; s}} t \|h\|_{\mathcal{G}^{\lambda, \beta+2; s}} \left(\|\partial_v f_0\|_{\mathcal{G}^{\lambda, \beta; s}} + \|\partial_{vv} f_0\|_{\mathcal{G}^{\lambda, \beta; s}} \right) \\ &\lesssim \epsilon D_{l_0, f_0} + E_{l_0, h}^{\frac{1}{2}} D_{l_0, f_0}. \end{aligned}$$

Next we turn to $\Pi_{h,1}$ and $\Pi_{h,2}$. We get that

$$\begin{aligned} |\Pi_{h,1}| &\lesssim t \|\partial_{vv} h\|_{\mathcal{G}^{\lambda, \beta; s}} t \|g\|_{\mathcal{G}^{\lambda, \beta; s}} \|\partial_{vv} h\|_{\mathcal{G}^{\lambda, \beta+1; s}} \\ &\quad + t \|\partial_{vv} h\|_{\mathcal{G}^{\lambda, \beta; s}} t \|\partial_v^3 g\|_{\mathcal{G}^{\lambda, \beta; s}} \|h\|_{\mathcal{G}^{\lambda, \beta; s}} \\ &\lesssim E_{l_0, h}^{\frac{1}{2}} \sqrt{t} (\|g\|_{L^2} + \|\partial_v^3 g\|_{\mathcal{G}^{\lambda, \beta; s}}) \sqrt{t} \|A \partial_{vv} h\|_2 \\ &\quad + \frac{1}{\langle t \rangle} E_{l_0, h}^{\frac{1}{2}} E_{l_0, g}^{\frac{1}{2}} (\|h\|_{L^2} + \|\partial_{vv} h\|_{\mathcal{G}^{\lambda, \beta; s}}) \\ &\lesssim \frac{\epsilon + E_{l_0, g}^{\frac{1}{2}}}{\langle t \rangle^{\frac{5}{4}}} (E_{l_0, h} + \epsilon) + \frac{\epsilon + E_{l_0, h}^{\frac{1}{2}}}{\langle t \rangle^2} E_{l_0, h}^{\frac{1}{2}} E_{l_0, g}^{\frac{1}{2}}. \end{aligned}$$

Thus we conclude by taking ϵ small enough that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (E_{l_0, f} + E_{l_0, h}) + \frac{1}{2} D_{l_0, f_0} \\ &\lesssim \frac{\epsilon^3}{\langle t \rangle^{\frac{5}{4}}} + \frac{1}{\langle t \rangle^{\frac{5}{4}}} (E_{l_0, f} + E_{l_0, h})^{\frac{3}{2}} + (E_{l_0, f} + E_{l_0, h})^{\frac{1}{2}} D_{l_0, f_0} \\ &\quad + \epsilon \langle t \rangle \|A \partial_v^2 h\|_2^2 + E_{l_0, f_0}^{\frac{1}{2}} \langle t \rangle \|A \partial_v^2 h\|_2^2, \end{aligned}$$

which implies that $E_{l_0, g} \lesssim \epsilon^2$ and

$$(E_{l_0, f} + E_{l_0, h}) + \frac{1}{3} \int_0^t D_{l_0, f_0}(s) ds \lesssim \epsilon^2.$$

We have

$$\begin{aligned} \|g\|_{\mathcal{G}_1^{\lambda, \beta; s}} &\lesssim \int_{|\xi| \leq 1} |\hat{g}(t, \xi)| d\xi + \|\partial_{vv} g\|_{\mathcal{G}_1^{\lambda, \beta; s}} \\ &\lesssim \left\| |\xi|^{-\frac{1}{2} + \epsilon_1} \right\|_{L^2} \|g\|_{\dot{H}^{\frac{1}{2} - \epsilon_1}} + \epsilon \langle t \rangle^{-2} \\ &\lesssim \frac{1}{\sqrt{\epsilon_1}} \|g\|_{\dot{H}^{\frac{1}{2} - \epsilon_1}} + \epsilon \langle t \rangle^{-2} \lesssim \frac{\epsilon}{\sqrt{\epsilon_1^3 \langle t \rangle^{2 - \frac{\epsilon_1}{2}}}}. \end{aligned}$$

which gives the proposition.

Remark 4.1. It holds that

$$\|\partial_v g\|_{\mathcal{G}^{\lambda, \beta+2; s}} \lesssim \frac{\epsilon}{\langle t \rangle^2}.$$

5. Elliptic Estimate

In this section, we study the elliptic estimate. We give the proofs of Propositions 2.11-2.10. Before giving the proof of these propositions, we start with fundamental estimate on the stream function ϕ in a lower norm. The regularity gap between the higher regularity norms and this lower regularity norm allows us to trade the regularity of f in higher norms for the decay of the stream function in lower norms. In other word, we prove the following lemma.

Lemma 5.1. *Under the bootstrap hypothesis and for ϵ sufficiently small, it holds that*

$$\|P_{\neq} \partial_z^{-1} \Delta_L \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} + \langle t \rangle^2 \|P_{\neq} \partial_z^{-1} \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-4; s}} \lesssim \|\partial_z^{-1} P_{\neq} f(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \quad (5.1)$$

and

$$\|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, \sigma; s}} + \langle t \rangle^2 \|\partial_z^{-1} P_{\neq} f(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \lesssim \epsilon. \quad (5.2)$$

Proof. We have for any φ and $\sigma' > 0$

$$\begin{aligned} \|P_{\neq} \partial_z^{-1} \varphi(t)\|_{\mathcal{G}^{\lambda, \sigma'; s}}^2 &= \sum_{k \neq 0} \int_{\eta} e^{2\lambda|k, \eta|^s} \langle k, \eta \rangle^{2\sigma'} |\widehat{\partial_z^{-1} \varphi}(k, \eta)|^2 d\eta \\ &= \sum_{k \neq 0} \int_{\eta} e^{2\lambda|k, \eta|^s} \frac{\langle k, \eta \rangle^{2\sigma'+4}}{\langle k, \eta \rangle^4 (k^2 + (\eta - kt)^2)^2} \\ &\quad \times (k^2 + (\eta - kt)^2)^2 |\widehat{\partial_z^{-1} \varphi}(k, \eta)|^2 d\eta \\ &\lesssim \frac{1}{\langle t \rangle^4} \|P_{\neq} \Delta_L \partial_z^{-1} \varphi(t)\|_{\mathcal{G}^{\lambda, \sigma'+2; s}}^2, \end{aligned}$$

On the other hand, we have

$$\Delta_L P_{\neq} \phi = P_{\neq} f + (1 - (v')^2)(\partial_v - t \partial_z)^2 P_{\neq} \phi - v''(\partial_v - t \partial_z) P_{\neq} \phi.$$

and

$$\Delta_L f = \gamma^2 \partial_z \rho + K - ((v')^2 - 1)(\partial_v - t \partial_z)^2 f - v''(\partial_v - t \partial_z) f. \quad (5.3)$$

Hence, it holds that by using the algebra property of Gevrey spaces together with the bootstrap assumption

$$\begin{aligned} \|P_{\neq} \partial_z^{-1} \Delta_L \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} &\lesssim \|P_{\neq} \partial_z^{-1} f(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \\ &\quad + \|1 - (v')^2\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \|P_{\neq} \partial_z^{-1} \Delta_L \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \\ &\quad + \|v''\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \|P_{\neq} \partial_z^{-1} \Delta_L \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} \\ &\lesssim \|P_{\neq} \partial_z^{-1} f(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}} + \epsilon \|P_{\neq} \partial_z^{-1} \Delta_L \phi(t)\|_{\mathcal{G}^{\lambda, \sigma-2; s}}. \end{aligned}$$

Thus by taking $\varphi = \phi$, $\sigma' = \sigma - 4$ and ϵ sufficiently small, we get (5.1).

Similarly, we have from (5.3)

$$\begin{aligned} \|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, \sigma; s}} &\lesssim \|AP_{\neq} \rho\|_2 + \|A \partial_z^{-1} P_{\neq} K\|_2 + \|G_1\|_{\mathcal{G}^{\lambda, \sigma; s}} \|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, \sigma; s}} \\ &\quad + \|\partial_v G_1\|_{\mathcal{G}^{\lambda, \sigma; s}} \|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, \sigma; s}} \\ &\lesssim \epsilon + \epsilon \|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, \sigma; s}}. \end{aligned}$$

Therefore by taking $\varphi = f$, $\sigma' = \sigma - 2$ and ϵ sufficiently small, we get (5.2). $\square$

5.1. Proof of Proposition 2.9

Proof. It is also easy to check that

$$\begin{aligned} \|AP_{\neq} \partial_z^{-1} \Delta_L f\|_2 &\leq \gamma^2 \|AP_{\neq} \rho\|_2 + \|A \partial_z^{-1} P_{\neq} K\|_2 \\ &\quad + \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{1,f}\|_2 + \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{2,f}\|_2. \end{aligned} \quad (5.4)$$

where

$$\mathcal{M}_{1,f} = ((v')^2 - 1)(\partial_v - t \partial_z)^2 f, \quad \mathcal{M}_{2,f} = v''(\partial_v - t \partial_z) f.$$

Hence, by dividing each via a paraproduct decomposition in the v variable only we have that

$$\begin{aligned} \widehat{\mathcal{M}}_{1,f}(t, k, \eta) &= -\frac{1}{2\pi} \sum_{M \geq 8} \int \widehat{G}_1(\xi)_M ((\eta - \xi) - kt)^2 \widehat{f}_k(\eta - \xi)_{<M/8} d\xi \\ &\quad - \frac{1}{2\pi} \sum_{M \geq 8} \int \widehat{G}_1(\eta - \xi)_{<M/8} (\xi - kt)^2 \widehat{f}_k(\xi)_M d\xi \\ &\quad - \frac{1}{2\pi} \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \int \widehat{G}_1(\xi)_{M'} ((\eta - \xi) - kt)^2 \widehat{f}_k(\eta - \xi)_M d\xi \\ &= \widehat{\mathcal{M}}_{1,f;HL} + \widehat{\mathcal{M}}_{1,f;LH} + \widehat{\mathcal{M}}_{1,f;HH}, \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathcal{M}}_{2,f}(t, k, \eta) &= \frac{i}{2\pi} \sum_{M \geq 8} \int \widehat{v}''(\xi)_M ((\eta - \xi) - kt) \widehat{f}_k(\eta - \xi)_{<M/8} d\xi \\ &\quad + \frac{i}{2\pi} \sum_{M \geq 8} \int \widehat{v}''(\eta - \xi)_{<M/8} (\xi - kt) \widehat{f}_k(\xi)_M d\xi \\ &\quad + \frac{i}{2\pi} \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \int \widehat{v}''(\eta - \xi)_{M'} (\xi - kt) \widehat{f}_k(\xi)_M d\xi \\ &= \widehat{\mathcal{M}}_{2,f;HL} + \widehat{\mathcal{M}}_{2,f;LH} + \widehat{\mathcal{M}}_{2,f;HH}. \end{aligned}$$

where $G_1 = (v')^2 - 1$ and $v'' = \frac{1}{2} \partial_v G_1$.

The treatment of $\widehat{\mathcal{M}}_{1,f;\text{LH}}$ and $\widehat{\mathcal{M}}_{2,f;\text{LH}}$ is similar. By the fact that

$$\frac{J_k(\eta)}{J_k(\xi)} \lesssim e^{C|\eta-\xi|^{\frac{1}{3}}}$$

We get that

$$\begin{aligned} \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;\text{LH}}\|_2^2 &\lesssim \sum_{M \geq 8} \|G_1\|_{\mathcal{G}^{\lambda,0;s}}^2 \|A \partial_z^{-1} (\partial_v - t \partial_z)^2 P_{\neq} f_M\|_2^2, \\ &\lesssim \epsilon^2 \|AP_{\neq} \partial_z^{-1} \Delta_L f\|_2^2, \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;\text{LH}}\|_2^2 &\lesssim \sum_{M \geq 8} \|v''\|_{\mathcal{G}^{\lambda,0;s}}^2 \|A \partial_z^{-1} (\partial_v - t \partial_z) P_{\neq} f_M\|_2^2 \\ &\lesssim \epsilon^2 \|AP_{\neq} \partial_z^{-1} \Delta_L f\|_2^2. \end{aligned} \quad (5.6)$$

Next we consider the high-low interaction. The notation is deceptive: the frequency in z could be very large and hence more ‘derivatives’ are appearing on f and we will be in a situation like the low-high interaction. Hence we break into two cases:

$$\begin{aligned} \widehat{\mathcal{M}}_{1,f;\text{HL}} &= -\frac{1}{2\pi} \sum_{M \geq 8} \int [\mathbf{1}_{|k| \geq \frac{1}{16}|\eta|} + \mathbf{1}_{|k| < \frac{1}{16}|\eta|}] \widehat{G}_1(\xi)_M \\ &\quad ((\eta - \xi) - kt)^2 \widehat{f}_k(\eta - \xi)_{<M/8} d\xi \\ &= \widehat{\mathcal{M}}_{1,f;\text{HL}}^z + \widehat{\mathcal{M}}_{1,f;\text{HL}}^v, \\ \widehat{\mathcal{M}}_{2,f;\text{HL}} &= \frac{i}{2\pi} \sum_{M \geq 8} \int [\mathbf{1}_{|k| \geq \frac{1}{16}|\eta|} + \mathbf{1}_{|k| < \frac{1}{16}|\eta|}] \widehat{v}''(\xi)_M \\ &\quad ((\eta - \xi) - kt) \widehat{f}_k(\eta - \xi)_{<M/8} d\xi \\ &= \widehat{\mathcal{M}}_{2,f;\text{HL}}^z + \widehat{\mathcal{M}}_{2,f;\text{HL}}^v. \end{aligned}$$

Let us first treat $\widehat{\mathcal{M}}_{1,f;\text{HL}}^z$ and $\widehat{\mathcal{M}}_{2,f;\text{HL}}^z$. On the support of the integrand, we get that there is some $c \in (0, 1)$ such that,

$$|k, \eta|^s \leq |k, \eta - \xi|^s + c|\xi|^s$$

We also have

$$\frac{J_k(\eta)}{J_k(\xi)} \lesssim e^{C|k, \eta - \xi|^{\frac{1}{3}}}.$$

Thus we get that

$$\begin{aligned} &\|AP_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;\text{HL}}^z\|_2^2 + \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;\text{HL}}^z\|_2^2 \\ &\lesssim \sum_{M \geq 8} (\|G_1\|_{\mathcal{G}^{\lambda,0;s}}^2 + \|v''\|_{\mathcal{G}^{\lambda,0;s}}^2) \|A \partial_z^{-1} \Delta_L P_{\neq} f_M\|_2^2 \lesssim \epsilon^2 \|A \partial_z^{-1} \Delta_L P_{\neq} f\|_2^2, \end{aligned} \quad (5.7)$$

Next we consider $\mathcal{M}_{1,f;\text{HL}}^v$ and $\mathcal{M}_{2,f;\text{HL}}^v$. Due to the fact that G_1 admits two more derivate.

By the definition of $J_k(t, \eta)$ together Lemma 3.6, (3.2) and the fact that $\eta \approx \xi$, we have

$$\frac{J_k(t, \eta)}{J_0(t, \xi)} \lesssim \begin{cases} e^{C|k, \eta - \xi|^{\frac{1}{3}}} & \text{if } t \notin I_{k, \eta} \\ \frac{\eta}{k^3} e^{C|k, \eta - \xi|^{\frac{1}{3}}} & \text{if } t \in I_{k, \eta} \end{cases} \lesssim \langle \xi \rangle e^{C|k, \eta - \xi|^{\frac{1}{3}}}. \quad (5.8)$$

Hence, we have

$$\begin{aligned} & \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;\text{HL}}^v\|_2^2 + \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;\text{HL}}^v\|_2^2 \\ & \lesssim \sum_{M \geq 8} (\|\langle \partial_v \rangle A(v'')_{\text{M}}\|_2^2 + \|\langle \partial_v \rangle A(G_1)_{\text{M}}\|_2^2) \|P_{\neq} \partial_z^{-1} \Delta_L f\|_{\mathcal{G}^{\lambda, 0; s}}^2 \\ & \lesssim \epsilon^4. \end{aligned} \quad (5.9)$$

The high-high interaction is easy to treat. We show the results and omit the proof.

$$\|AP_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;\text{HH}}^v\|_2^2 + \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;\text{HH}}^v\|_2^2 \lesssim \epsilon^4 \quad (5.10)$$

Plugging (5.5), (5.6), (5.7), (5.14) and (5.10) into (5.4) using the bootstrap assumption and taking ϵ small enough, we get the proposition. $\square$

5.2. Proof of Proposition 2.11

Proof. We write

$$\Delta_L \phi = f + (1 - (v')^2)(\partial_v - t \partial_z)^2 \phi - v''(\partial_v - t \partial_z) \phi.$$

This yields by using the fact that $\partial_v(v')^2 = 2\partial_v h(h+1)$ and

$$\begin{aligned} \Delta_L^2 \phi &= \Delta_L f - G_1(\partial_v - t \partial_z)^2 \Delta_L \phi - \partial_v G_1(\partial_v - t \partial_z) \left(\frac{5}{2}(\partial_v - t \partial_z)^2 + \frac{1}{2} \partial_z^2 \right) \phi \\ &\quad - 2\partial_{vv} G_1(\partial_v - t \partial_z)^2 \phi - \frac{1}{2} \partial_{vvv} G_1(\partial_v - t \partial_z) \phi \\ &= \mathcal{M}_{1,\phi} + \mathcal{M}_{2,\phi} + \mathcal{M}_{3,\phi} + \mathcal{M}_{4,\phi}. \end{aligned} \quad (5.11)$$

Similarly, by following the proof of Lemma 5.1, it is easy to check that under the bootstrap assumption, it holds, for $\sigma_0 \leq \sigma - 1$, that

$$\|P_{\neq} \partial_z^{-1} \Delta_L^2 \phi\|_{\mathcal{G}^{\lambda, \sigma_0; s}} \lesssim \epsilon. \quad (5.12)$$

Hence, it holds, by using (2.14), that

$$\begin{aligned} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2 &\lesssim \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \Delta_L f \right\|_2 \\ &\quad + \sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi} \right\|_2 \\ &\lesssim \epsilon + \sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi} \right\|_2. \end{aligned} \quad (5.13)$$

We write

$$\widehat{\mathcal{M}}_{i,\phi}(t, k, \eta) = \widehat{\mathcal{M}}_{i,\phi;\text{HL}} + \widehat{\mathcal{M}}_{i,\phi;\text{LH}} + \widehat{\mathcal{M}}_{i,\phi;\text{HH}}, \quad i = 1, \dots, 4.$$

To estimate $\widehat{\mathcal{M}}_{1,\phi;\text{LH}}$, we proceed as in the estimate involving $\widehat{\mathcal{M}}_{1,f;\text{LH}}$ and using the fact that on the support of the integrand, we have for the $\langle \frac{\eta}{tk} \rangle^{-1} \approx \langle \frac{\xi}{tk} \rangle^{-1}$, which means that we can move this factor to ϕ and obtain as in (5.5)

$$\left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{1,\phi;\text{LH}} \right\|_2 \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2.$$

We also have similarly

$$\left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{2,\phi;\text{LH}} \right\|_2 \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2$$

Both terms can be absorbed by the left-hand side of (5.13) for sufficiently small ϵ .

The other two terms $\widehat{\mathcal{M}}_{3,\phi;\text{LH}}$ and $\widehat{\mathcal{M}}_{4,\phi;\text{LH}}$ can be treated similarly. We omit the details and write the result

$$\begin{aligned} & \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{3,\phi;\text{LH}} \right\|_2 + \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{LH}} \right\|_2 \\ & \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2. \end{aligned}$$

For the high-low interaction, we write

$$\begin{aligned} \widehat{\mathcal{M}}_{1,\phi;\text{HL}} &= -\frac{1}{2\pi} \sum_{M \geq 8} \int \left\langle \frac{\eta}{tk} \right\rangle^{-1} [\mathbf{1}_{|k| \geq \frac{1}{16}|\eta|} + \mathbf{1}_{|k| < \frac{1}{16}|\eta|}] \widehat{G}_1(\xi)_M \\ & \quad ((\eta - \xi) - kt)^2 \widehat{\Delta}_L \phi_k(\eta - \xi)_{< M/8} d\xi \\ &= \widehat{\mathcal{M}}_{1,\phi;\text{HL}}^z + \widehat{\mathcal{M}}_{1,\phi;\text{HL}}^v, \\ \widehat{\mathcal{M}}_{2,\phi;\text{HL}} &= \frac{i}{2\pi} \sum_{M \geq 8} \int \left\langle \frac{\eta}{tk} \right\rangle^{-1} [\mathbf{1}_{|k| \geq \frac{1}{16}|\eta|} + \mathbf{1}_{|k| < \frac{1}{16}|\eta|}] \widehat{v}''(\xi)_M \\ & \quad ((\eta - \xi) - kt) \widehat{\Delta}_L \phi_k(\eta - \xi)_{< M/8} d\xi \\ &= \widehat{\mathcal{M}}_{2,\phi;\text{HL}}^z + \widehat{\mathcal{M}}_{2,\phi;\text{HL}}^v. \end{aligned}$$

As in (5.14), we have the estimate

$$\begin{aligned} & \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{1,\phi;\text{HL}}^z \right\|_2 + \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{2,\phi;\text{HL}}^z \right\|_2 \\ & \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2, \end{aligned} \tag{5.14}$$

which again can be absorbed by the left-hand side of (5.13).

The estimate of the terms $\widehat{\mathcal{M}}_{1,\phi;\text{HL}}^v$ and $\widehat{\mathcal{M}}_{2,\phi;\text{HL}}^v$ can also be done as in (5.14) and by using (5.12), we get

$$\begin{aligned} & \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{1,\phi;\text{HL}}^v \right\|_2^2 + \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{2,\phi;\text{HL}}^v \right\|_2^2 \\ & \lesssim \sum_{M \geq 8} (\|\langle \partial_v \rangle A(v'')_{\text{M}}\|_2^2 + \|\langle \partial_v \rangle A(G_1)_{\text{M}}\|_2^2) \|P_{\neq} \partial_z^{-1} \Delta_L^2 \phi\|_{\mathcal{G}^{\lambda,0;s}}^2 \lesssim \epsilon^4. \end{aligned}$$

Now, we prove estimates for $\widehat{\mathcal{M}}_{4,\phi;\text{HL}}$. The one of $\widehat{\mathcal{M}}_{3,\phi;\text{HL}}$ is easier compared to $\widehat{\mathcal{M}}_{4,\phi;\text{HL}}$. We write as above $\widehat{\mathcal{M}}_{4,\phi;\text{HL}} = \widehat{\mathcal{M}}_{4,\phi;\text{HL}}^z + \widehat{\mathcal{M}}_{4,\phi;\text{HL}}^v$. The term $\widehat{\mathcal{M}}_{4,\phi;\text{HL}}^z$ can be treated as the low-high interaction and we have

$$\left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HL}}^z \right\|_2 \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2$$

thus, the most changeling term is the term $\widehat{\mathcal{M}}_{4,\phi;\text{HL}}^v$ since in this term all the derivatives are landing on the term $\partial_{vv} G_1$ which will have a regularity loss. Here, where we need to use the factor $\langle \xi/(lt) \rangle^{-1}$ to absorb one derivative by paying time decay.

By the definition of $J_k(t, \eta)$ together Lemma 3.6, (3.2) and the fact that $\eta/k \approx t$, we have

$$\frac{J_k(t, \eta)}{J_0(t, \xi)} \lesssim \begin{cases} e^{C|k, \eta - \xi|^{\frac{1}{3}}} & \text{if } t \notin I_{k, \eta} \\ \frac{\eta}{k^3} e^{C|k, \eta - \xi|^{\frac{1}{3}}} & \text{if } t \in I_{k, \eta} \end{cases} \lesssim \langle t \rangle e^{C|k, \eta - \xi|^{\frac{1}{3}}}. \quad (5.15)$$

Then, we get, by using the fact that on the support of the integrand $|\eta| \approx |\xi|$, and since $s > \frac{1}{3}$,

$$\begin{aligned} & \left| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HL}}^v \right| \\ & \lesssim \sum_{M \geq 8} \int \left\langle \frac{\eta}{tk} \right\rangle^{-1} \mathbf{1}_{|k| < \frac{1}{16}|\eta|} A_0(\xi) |\xi|^3 |\widehat{G}_1(\xi)_{\text{M}}| \langle t \rangle |(\eta - \xi)| \\ & \quad - kt |e^{c\lambda|k, \eta - \xi|^s} e^{C|k, \eta - \xi|^{\frac{1}{3}}} |\widehat{\phi}_k(\eta - \xi)_{<M/8}| d\xi \\ & \lesssim \sum_{M \geq 8} \int \mathbf{1}_{|k| < \frac{1}{16}|\eta|} \langle \xi \rangle^2 A_0(\xi) |\widehat{G}_1(\xi)_{\text{M}}| \langle t \rangle^3 \langle k, \eta - \xi \rangle^2 \\ & \quad e^{c\lambda|k, \eta - \xi|^s} e^{C|k, \eta - \xi|^{\frac{1}{3}}} |\widehat{\phi}_k(\eta - \xi)_{<M/8}| d\xi \end{aligned}$$

This yields by using the bootstrap assumption together with Lemma 5.1,

$$\left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HL}}^v \right\|_2^2 \lesssim \langle t \rangle^6 \sum_{M \geq 8} (\|\langle \partial_v \rangle^2 A h_{\text{M}}\|_2^2) \|P_{\neq} \phi\|_{\mathcal{G}^{\lambda,0;s}}^2 \lesssim \epsilon^4.$$

To estimate the terms $\mathcal{M}_{i,\phi;\text{HH}}$, $i = 1, \dots, 4$, we use the fact that on the support of the integrand, we have $|\eta|^s \leq c|\eta - \xi|^s + c|\xi|^s$, $c \in (0, 1)$, which together with

(5.8), yields, after absorbing all the possible loss of derivatives by the Gevery term (since $c < 1$),

$$\sum_{i=1}^4 \|AP_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{HH}}^v\|_2^2 \lesssim \epsilon^4.$$

Collecting all the above estimates and taking ϵ sufficiently small, we get the desired result. $\square$

5.3. Proof of Proposition 2.10

In this section, we prove the estimate in Proposition 2.10.

5.3.1. Proof of (2.15) We have by using (5.3) (see (5.4))

$$\begin{aligned} \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 &\leq \gamma^2 \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \rho \right\|_2^2 \\ &\quad + \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} P_{\neq} K \right\|_2^2 \\ &\quad + \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f} \right\|_2^2 \\ &\quad + \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{2,f} \right\|_2^2. \end{aligned} \quad (5.16)$$

Keeping in mind (2.9), the first two terms in (5.16) can be estimates using $\text{CK}_{\lambda,K}$ and $\text{CK}_{\lambda,\rho}$ as follows:

$$\left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \rho \right\|_2^2 \leq -C_1 \frac{\dot{\lambda}(t)}{t^{3s-2\bar{q}}} \mathbf{1}_{t \geq M_0} \left\| |\nabla|^{\frac{s}{2}} A P_{\neq} \rho \right\|_2^2 \leq C_1 M_0^{2\bar{q}-3s} \text{CK}_{\lambda,\rho}.$$

Similarly, we have

$$\left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} P_{\neq} K \right\|_2^2 \leq C_1 M_0^{2\bar{q}-3s} \text{CK}_{\lambda,K}.$$

The above constant C_1 depends on δ_λ but it is independent of M_0 .

Now, we estimate the last two terms in (5.16). We simply first write

$$\mathcal{M}_{i,f} = \mathcal{M}_{i,f;\text{HL}} + \mathcal{M}_{i,f;\text{LH}} + \mathcal{M}_{i,f;\text{HH}}$$

as in the proof of Proposition 2.9. Following similar ideas as in the proof of Proposition 2.9, we have ((by using the same notation))

$$\begin{aligned} &\sum_{i=1}^2 \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,f;\text{LH}} \right\|_2 + \sum_{i=1}^2 \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,f;\text{HL}}^z \right\|_2 \\ &\lesssim \epsilon \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2 \end{aligned}$$

which will be absorbed by the left-hand side of (5.16), provided that ϵ is sufficiently small.

Next, the terms involving $\mathcal{M}_{i,f;\text{HL}}^v$ can be estimated as (we omit the details)

$$\sum_{i=1}^2 \left\| \mathbf{1}_{t \geq M_0} \frac{|\nabla|^{\frac{5}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,f;\text{HL}}^v \right\|_2^2 \lesssim \epsilon^2 \text{CK}_{\lambda,h}(t).$$

5.3.2. Proof of (2.16) In this section, we prove the estimate (2.16). Using (5.3), we have

$$\begin{aligned} & \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ & \leq \gamma^2 \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} P_{\neq} \rho \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} \partial_z^{-1} P_{\neq} K \right\|_2^2 \\ & \quad + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f} \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} P_{\neq} \partial_z^{-1} \mathcal{M}_{2,f} \right\|_2^2. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} & \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} P_{\neq} \rho \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} \partial_z^{-1} P_{\neq} K \right\|_2^2 \\ & \leq C_2 (\text{CK}_{\Theta,\rho} + \text{CK}_{\Theta,K}) \end{aligned}$$

for some $C_2 > 0$ independent of δ_L .

Similarly we write

$$\mathcal{M}_{i,f} = \mathcal{M}_{i,f;\text{LH}} + \mathcal{M}_{i,f;\text{HL}} + \mathcal{M}_{i,f;\text{HH}}, \quad i = 1, 2.$$

We write

$$\begin{aligned} & \left| \mathcal{F} \left(\sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{\tilde{A}} P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;\text{LH}} \right) \right| \\ & \lesssim \sum_{M \geq 8} \sum_{k \neq 0} \int \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}} \left\langle \frac{\eta}{tk} \right\rangle^{-1} \tilde{\tilde{A}}(\eta) \widehat{G_1}(\eta - \xi)_{< M/8} (\xi - kt)^2 \widehat{\partial_z^{-1} f}(k, \xi)_{\text{M}} d\xi \end{aligned}$$

To simplify our proof, let us take advantage of the $\frac{1}{t}$ decay of G_1 and $\partial_v G_1$. By the fact that on the support of integrand $|\eta|^{\frac{1}{3}} \lesssim t \lesssim |\eta| \approx |\xi|$, $\tilde{\tilde{A}} \leq A$ and $\sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}} \lesssim 1$, we have by using the fact that $1 \leq \langle t \rangle^{2s} \frac{|\eta|^s}{\langle t \rangle^{3s}}$

$$\begin{aligned}
& \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;LH} \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;LH} \right\|_2^2 \\
& \lesssim \langle t \rangle^{2s} (\|G_1\|_{\mathcal{G}^{\lambda,1;s}}^2 + \|\partial_v G_1\|_{\mathcal{G}^{\lambda,1;s}}^2) \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\
& \lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2.
\end{aligned}$$

The high-low and high-high interactions are easy. Again by using the fact $\sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}} \lesssim 1$ and following the proof of Proposition 2.9, we have

$$\begin{aligned}
& \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;HL} \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;HH} \right\|_2^2 \\
& + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{1,f;HH} \right\|_2^2 + \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{2,f;HL} \right\|_2^2 \\
& \lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 + \epsilon^2 (\|\langle \partial_v \rangle A G_1\|_2^2 + \|\langle \partial_v \rangle A \partial_v G_1\|_2^2) \\
& \lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 + \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2.
\end{aligned}$$

Similarly, by using the fact that $\tilde{A} \leq A$ and

$$\sqrt{\frac{b(t, k, \eta) k^2}{k^2 + (\eta - kt)^2}} + \sqrt{\frac{\partial_t \Theta_k(t, \eta)}{\Theta_k(t, \eta)}} \lesssim 1.$$

We get that

$$\begin{aligned}
& \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\sqrt{\frac{b(t, \nabla) \partial_{zz}}{\Delta_L}} A + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L P_{\neq} \mathcal{M}_{1,f} \right\|_2^2 \\
& + \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\sqrt{\frac{b(t, \nabla) \partial_{zz}}{\Delta_L}} A + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L P_{\neq} \mathcal{M}_{2,f} \right\|_2^2 \\
& \lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 + \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2. \tag{5.17}
\end{aligned}$$

Thus by taking ϵ small enough, we proved Proposition 2.10.

5.4. Proof of Proposition 2.12

Recalling (5.11), we have

$$\begin{aligned}
 & \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2 \\
 & \lesssim \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \Delta_L f \right\|_2 + \sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi} \right\|_2 \\
 & \lesssim \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \Delta_L f \right\|_2 \\
 & \quad + \sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \left(\mathcal{M}_{i,\phi;\text{HL}} + \mathcal{M}_{i,\phi;\text{LH}} + \mathcal{M}_{i,\phi;\text{HH}} \right) \right\|_2.
 \end{aligned} \tag{5.18}$$

Applying Proposition 2.10, we estimate the first term in (5.18) as

$$\left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \Delta_L f \right\|_2 \lesssim C_1 (\text{CK}_{\lambda,K} + \text{CK}_{\lambda,\rho}) + C \epsilon \text{CK}_{\lambda,h}.$$

Now, we estimate the second term in (5.18). For the low-high terms and since on the support of the integrand, we have $|k, \eta| \approx |k, \xi|$ and $|\eta| \approx |\xi|$, then we can move the term $\left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq}$ to land on ϕ and get as in the proof of Proposition 2.11, and under the bootstrap assumption

$$\sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{LH}} \right\|_2 \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2.$$

Next, we treat the high-low part. We use the decomposition

$$\widehat{\mathcal{M}}_{i,\phi;\text{LH}} = \widehat{\mathcal{M}}_{i,\phi;\text{LH}}^z + \widehat{\mathcal{M}}_{i,\phi;\text{LH}}^v, \quad i = 1, \dots, 4.$$

The terms involving $\widehat{\mathcal{M}}_{i,\phi;\text{HL}}^z$ can be treated as the low-high part and we have

$$\sum_{i=1}^4 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{LH}}^z \right\|_2 \lesssim \epsilon \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2.$$

Now, we treat the term involving $\widehat{\mathcal{M}}_{4,\phi;\text{LH}}^v$, which is the most challenging terms. We have, by using (5.15), with the fact that on the support of the integrand we have $|\eta| \approx \xi$ and $|k, \eta| \lesssim |\xi|$ and by making use of the bootstrap assumption,

$$\begin{aligned}
 & \left| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{LH}}^v \right| \lesssim \sum_{M \geq 8} \int \langle \xi \rangle^2 \frac{|\xi|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \mathbf{1}_{|k| < \frac{1}{16}|\eta|} \\
 & A_0(\xi) \langle t \rangle^3 |\widehat{G}_1(\xi)_M| |k, \eta - \xi| e^{c\lambda|k, \eta - \xi|^s} e^{C|k, \eta - \xi|^{\frac{1}{3}}} |\widehat{\phi}_k(\eta - \xi)|_{<M/8} d\xi.
 \end{aligned}$$

This implies

$$\left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{LH}}^v \right\|_2^2 \lesssim \epsilon^2 \text{CK}_{\lambda,h}(t).$$

Similarly, we can prove that

$$\sum_{i=1}^3 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{HL}}^v \right\|_2^2 \lesssim \epsilon^2 \text{CK}_{\lambda,h}(t).$$

Hence, collecting all the above estimates, we obtain

$$\left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2 \lesssim C_1 (\text{CK}_{\lambda,K} + \text{CK}_{\lambda,\rho}) + C \epsilon \text{CK}_{\lambda,h}.$$

Next, we treat the Θ -term on the left-hand side of (2.17). The g -term is similar and we omit the details. We have by using (5.11),

$$\begin{aligned} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2 &\lesssim \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ &+ \sum_{i=1}^4 \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi} \right\|_2^2. \end{aligned} \quad (5.19)$$

The first term on the right-hand side of (5.19) has been already estimated in (5.17).

Using the fact that $\tilde{A} \lesssim A$ together with the bootstrap assumption, we have

$$\left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \rho \right\|_2^2 + \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} \partial_z^{-1} P_{\neq} K \right\|_2^2 \lesssim C_2 (\text{CK}_{\Theta,\rho} + \text{CK}_{\Theta,K}).$$

By the fact that on the support of integrand $|\eta|^{\frac{2}{3}} \lesssim t \lesssim |\eta| \approx |\xi|$, $\tilde{A} \leq A$ and $\sqrt{\frac{\partial_t \Theta(t,\eta)}{\Theta(t,\eta)}} \lesssim 1$, we have by using the fact that $1 \leq \langle t \rangle^{2s} \frac{|\eta|^s}{\langle t \rangle^{3s}}$,

$$\begin{aligned} &\sum_{i=1}^4 \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{LH}} \right\|_2^2 \\ &\lesssim \langle t \rangle^{2s} \left(\sum_{j=0}^3 \|\partial_v^j G_1\|_{\mathcal{G}^{\lambda,1;s}}^2 \right) \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2 \\ &\lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2. \end{aligned}$$

Similarly, we have

$$\sum_{i=1}^4 \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{i,\phi;\text{HL}}^z \right\|_2^2 \lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2.$$

For the terms involving $\mathcal{M}_{i,\phi;\text{HL}}^v$, we treat the most problematic term which is the term $\mathcal{M}_{4,\phi;\text{HL}}^v$ which contains the loss of three derivatives. The other terms can be treated by the same method (even easier). We have by the fact that $\frac{J_k(t,\eta)}{J_0(\xi)} \lesssim \langle t \rangle e^{C|k,\eta-\xi|^{\frac{1}{3}}}$ and $|\eta - \xi - kt| \lesssim \langle k, \eta - \xi \rangle \langle t \rangle$

$$\begin{aligned} & \left| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HL}}^v \right| \\ & \lesssim \sum_{M \geq 8} \int \left\langle \frac{\eta}{tk} \right\rangle^{-1} \mathbf{1}_{|k| < \frac{1}{16} |\eta|} A_0(\xi) |\xi|^3 |\widehat{G_1}(\xi)_M| \langle t \rangle \\ & \quad \times \langle k, \eta - \xi \rangle \langle t \rangle^2 e^{c\lambda|k,\eta-\xi|^s} e^{C|k,\eta-\xi|^{\frac{1}{3}}} |\widehat{\phi}_k(\eta - \xi)_{<M/8}| d\xi \\ & \lesssim \sum_{M \geq 8} \int \mathbf{1}_{|k| < \frac{1}{16} |\eta|} A_0(\xi) |\xi|^2 |\widehat{G_1}(\xi)_M| \langle t \rangle \\ & \quad \times \langle k, \eta - \xi \rangle^2 \langle t \rangle^3 e^{c\lambda|k,\eta-\xi|^s} e^{C|k,\eta-\xi|^{\frac{1}{3}}} |\widehat{\phi}_k(\eta - \xi)_{<M/8}| d\xi, \end{aligned}$$

which, together with the fact that $\|P_{\neq} \phi\|_{\mathcal{G}^{\lambda,0;s}} \lesssim \frac{\epsilon}{\langle t \rangle^4}$, implies that

$$\begin{aligned} \left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HL}}^v \right\|_2^2 & \lesssim \|A \langle \partial_v \rangle^2 G_1\|_2^2 \langle t \rangle^6 \|P_{\neq} \phi\|_{\mathcal{G}^{\lambda,0;s}}^2 \\ & \lesssim \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2. \end{aligned}$$

Similarly, we have

$$\left\| \sqrt{\frac{\partial_t \Theta}{\Theta}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} P_{\neq} \partial_z^{-1} \mathcal{M}_{4,\phi;\text{HH}}^v \right\|_2^2 \lesssim \epsilon^2 \|A \langle \partial_v \rangle^2 h\|_2^2.$$

Thus we have proved the proposition.

6. Estimate of NL_ρ and NL_K^1

This section is devoted to the proof of Proposition 2.4 and hence we estimate NL_ρ and NL_K^1 . Let us focus on NL_ρ , the estimate of NL_K^1 can be obtained by easily replacing ρ by K .

We have

$$\begin{aligned} \text{NL}_\rho &= \int A\rho [A(\mathbf{u} \cdot \nabla \rho) - \mathbf{u} \cdot \nabla A\rho] dz dv - \frac{1}{2} \int \nabla \cdot \mathbf{u} |A\rho|^2 dz dv \\ &= \text{NL}_1 - \frac{1}{2} \int \nabla \cdot \mathbf{u} |A\rho|^2 dz dv. \end{aligned} \tag{6.1}$$

The second term in (6.1) can be estimated as,

$$\left| \int \nabla \cdot \mathbf{u} |A\rho|^2 dz dv \right| \leq \|\nabla \mathbf{u}\|_{L^\infty} \|A\rho\|_{L^2}^2.$$

Recall the fact that

$$\mathbf{u}(t, z, v) = (0, g)^T + v' \nabla_{z,v}^\perp P_\neq \phi = (0, g)^T + h \nabla_{z,v}^\perp P_\neq \phi + \nabla_{z,v}^\perp P_\neq \phi, \quad (6.2)$$

then we have

$$\|\nabla \mathbf{u}\|_{L^\infty} \lesssim (\|\partial_v g\|_{L^\infty} + (1 + \|h\|_{H^2}) \|P_\neq \phi\|_{H^4}) \lesssim \frac{\epsilon}{\langle t \rangle^2}.$$

To handle NL_1 , we use a paraproduct decomposition. Precisely, we define three main contributions: *transport* (low-high interaction), *reaction* (high-low interaction) and a *remainder*:

$$\begin{aligned} \text{NL}_1 &= \int A\rho [A(\mathbf{u} \cdot \nabla_{z,v} \rho) - \mathbf{u} \cdot \nabla_{z,v} A\rho] dz dv \\ &= \frac{1}{2\pi} \sum_{N \geq 8} T_{1;N} + \frac{1}{2\pi} \sum_{N \geq 8} R_{1;N} + \frac{1}{2\pi} \mathcal{R}_1, \end{aligned}$$

where

$$\begin{aligned} T_{1;N} &= 2\pi \int A\rho [A(\mathbf{u}_{<N/8} \cdot \nabla_{z,v} \rho_N) - \mathbf{u}_{<N/8} \cdot \nabla_{z,v} A\rho_N] dz dv \\ R_{1;N} &= 2\pi \int A\rho [A(\mathbf{u}_N \cdot \nabla_{z,v} \rho_{<N/8}) - \mathbf{u}_N \cdot \nabla_{z,v} A\rho_{<N/8}] dz dv \\ \mathcal{R}_1 &= 2\pi \sum_{N \in \mathbb{D}} \sum_{\frac{1}{8}N \leq N' \leq 8N} \int A\rho [A(\mathbf{u}_N \cdot \nabla_{z,v} \rho_{N'}) - \mathbf{u}_N \cdot \nabla_{z,v} A\rho_{N'}] dz dv. \end{aligned}$$

6.1. Reaction term $R_{1;N}$

Recall (6.2), we write

$$R_{1;N} = R_{1;N}^1 + R_{1;N}^{\epsilon,1} + R_{1;N}^2 + R_{1;N}^3$$

where

$$\begin{aligned} R_{1;N}^1 &= \sum_{k,l \neq 0} \int_{\eta, \xi} A\bar{\rho}_k(\eta) A_k(\eta) (\eta l - \xi k) \hat{\phi}_l(\xi)_N \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \\ R_{1;N}^{\epsilon,1} &= \sum_{k,l \neq 0} \int_{\eta, \xi} A\bar{\rho}_k(\eta) A_k(\eta) \left[h \widehat{\nabla^\perp \phi_l} \right](\xi)_N \cdot \widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \\ R_{1;N}^2 &= - \sum_k \int_{\eta, \xi} A\bar{\rho}_k(\eta) A_k(\eta) \widehat{g}(\xi)_N \cdot \widehat{\partial_v \rho_k}(\eta - \xi)_{<N/8} d\eta d\xi \\ R_{1;N}^3 &= - \sum_{k,l} \int_{\eta, \xi} A\bar{\rho}_k(\eta) A_{k-l}(\eta - \xi) \widehat{\mathbf{u}}_l(\xi)_N \widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi. \end{aligned}$$

Here ϵ stands for the smallness of the term coefficient h .

In this section we will prove the following proposition.

Proposition 6.1. *Under the bootstrap hypotheses, it holds that,*

$$\begin{aligned} \sum_{N \geq 8} R_{1;N} &\lesssim \frac{\epsilon^2}{\langle t \rangle^2} + \epsilon \langle t \rangle \|A \partial_v g\|_2^2 + \epsilon CK_{\lambda, \rho} + \epsilon CK_{\Theta, \rho} + \epsilon CK_{M, \rho} \\ &\quad + \epsilon \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t \mathfrak{g}}{\mathfrak{g}}} \tilde{\tilde{A}} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2^2. \end{aligned}$$

6.1.1. Main contribution The main contribution comes from $R_{1;N}^1$. We subdivide this integral depending on whether or not (l, ξ) and/or (k, η) are resonant as each combination requires a slightly different treatment.

Define the partition

$$1 = \mathbf{1}_{t \notin I_{k, \eta}, t \notin I_{l, \xi}} + \mathbf{1}_{t \notin I_{k, \eta}, t \in I_{l, \xi}} + \mathbf{1}_{t \in I_{k, \eta}, t \notin I_{l, \xi}} + \mathbf{1}_{t \in I_{k, \eta}, t \in I_{l, \xi}}.$$

Correspondingly, denote

$$\begin{aligned} R_{1;N}^1 &= \sum_{k, l \neq 0} \int_{\eta, \xi} [\mathbf{1}_{t \notin I_{k, \eta}, t \notin I_{l, \xi}} + \mathbf{1}_{t \notin I_{k, \eta}, t \in I_{l, \xi}} + \mathbf{1}_{t \in I_{k, \eta}, t \notin I_{l, \xi}} + \mathbf{1}_{t \in I_{k, \eta}, t \in I_{l, \xi}}] \\ &\quad \times A \widehat{\rho}_k(\eta) A_k(\eta) (\eta l - \xi k) \widehat{\phi}_l(\xi) \widehat{\rho}_{k-l}(\eta - \xi)_{< N/8} d\eta d\xi \\ &= R_{1;N;NR, NR}^1 + R_{1;N;NR, R}^1 + R_{1;N;R, NR}^1 + R_{1;N;R, R}^1. \end{aligned}$$

On the support of the integrand of $R_{1;N}^1$, it holds that

$$||l, \xi| - |k, \eta|| \leq |k - l, \eta - \xi| \leq \frac{6}{32} |l, \xi|. \quad (6.3a)$$

This implies that $|k, \eta| \approx |l, \xi|$ since $|l, \xi| \leq \frac{32}{28} |k, \eta|$ and

$$|k, \eta| \leq |k - l, \eta - \xi| + |l, \xi| \leq \frac{38}{32} |l, \xi|.$$

Treatment of $R_{1;N;NR, NR}^1$

We write first that

$$\begin{aligned} R_{1;N;NR, NR}^1 &= \sum_{k, l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin I_{k, \eta}, t \notin I_{l, \xi}} A \widehat{\rho}_k(\eta) A_k(\eta) \frac{i l (\eta l - \xi k)}{(l^2 + (\xi - l t)^2)^2} \\ &\quad \widehat{\partial_z^{-1} \Delta_L^2 \phi_l(\xi)} \widehat{\rho}_{k-l}(\eta - \xi)_{< N/8} d\eta d\xi. \end{aligned}$$

First, if $l\xi < 0$, we do not have resonances for positive times. In this case, we have

$$\frac{|l||l, \xi|}{(l^2 + |\xi - l t|^2)^2} \approx \frac{|l||l, \xi|}{l^4 \langle t \rangle^4 + \xi^4} \lesssim \frac{1}{\langle t \rangle^2} |k, \eta|^{\frac{s}{2}} |l, \xi|^{\frac{s}{2}} \left\langle \frac{\xi}{l t} \right\rangle^{-1},$$

we have

$$|\mathbf{R}_{1;N;NR,NR}^1| \lesssim \frac{1}{\langle t \rangle^2} \left\| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \right\|_2 \left\| 1_{NR} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} |\nabla|^{\frac{s}{2}} \partial_z^{-1} \Delta_L^2 A P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}, \quad (6.4)$$

which together with the bootstrap hypotheses implies that

$$\sum_{N \geq 8} |\mathbf{R}_{1;N;NR,NR}^1| + |\mathbf{R}_{1;N;R,NR}^1| \lesssim \epsilon CK_{\lambda,\rho} + \epsilon \left\| 1_{NR} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \partial_z^{-1} \Delta_L^2 A P_{\neq \phi} \right\|_2^2.$$

Second, let us assume that $l\xi > 0$. Now we consider the following two cases: $|\xi| \geq 6|l|$ and $|\xi| < 6|l|$ and in each case, we consider several sub-cases, depending on the time regime.

Case 1: $|\xi| \geq 6|l|$, then it holds by (6.3a) that

$$|\eta - \xi| \leq \frac{7}{32}|\xi| \quad \text{and} \quad \frac{25}{32}|\xi| \leq |\eta| \leq \frac{39}{32}|\xi|. \quad (6.5)$$

we obtain

$$\frac{J_k(\eta)}{J_l(\xi)} \leq \frac{\Theta_l(t, \xi)}{\Theta_k(t, \eta)} e^{\mu|\eta-\xi|^{\frac{1}{3}}} + e^{\mu|k-l|^{\frac{1}{3}}} \leq \frac{\Theta_{NR}(t, \xi)}{\Theta_{NR}(t, \eta)} e^{\mu|\eta-\xi|^{\frac{1}{3}}} + e^{\mu|k-l|^{\frac{1}{3}}}.$$

By Lemma 3.6, we have

$$\frac{J_k(\eta)}{J_l(\xi)} \lesssim e^{10\mu|k-l, \eta-\xi|^{\frac{1}{3}}}$$

We also have

$$\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \leq \frac{g(t, \xi)}{g(t, \eta)} e^{4\pi\delta_L^{-1}|\eta-\xi|^{\frac{1}{3}}} + e^{4\pi\delta_L^{-1}|k-l|^{\frac{1}{3}}} \lesssim e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}}.$$

By using (6.3a), we have $|k, \eta| \approx |\xi, l|$, this means that we can freely interchange between (k, η) and (l, ξ) in the Sobolev correction as well as in the Gevrey part in $A_k(t, \eta)$. We get that

$$\begin{aligned} |\mathbf{R}_{1;N;NR,NR}^1| &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} |A \widehat{\rho}_k(\eta)| \frac{J_k(\eta)}{J_l(\xi)} \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} A_l(\xi) \\ &\quad \times \frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} |\partial_z^{-1} \Delta_L^2 \phi_l(\xi)_N| |\widehat{\nabla} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi. \end{aligned} \quad (6.6)$$

Case 1.1: $1 \leq t \leq \max \left\{ \frac{|\eta|}{2E(|\eta|^{\frac{2}{3}})+1}, \frac{|\xi|}{2E(|\xi|^{\frac{2}{3}})+1} \right\}$: We have $t \lesssim |\xi|^{\frac{1}{3}}$ and hence, we obtain for $|\xi| \leq 2lt$,

$$\frac{|\xi|}{l^3(1 + (t - \frac{\xi}{l})^2)^2} \lesssim 1 \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}}$$

and for $\xi \geq 2lt$, then $\frac{\xi}{l} \geq 2t$ which implies $|\frac{\xi}{l} - t| \gtrsim \frac{\xi}{l}$

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \frac{|l|}{|\xi|^3} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1}$$

Consequently, in this case, we get from (6.6),

$$|\mathbf{R}_{1;\mathbf{N};\mathbf{NR},\mathbf{NR}}^1| \lesssim \frac{1}{\langle t \rangle^{3s}} \left\| |\nabla|^{\frac{s}{2}} \mathbf{A} \rho_{\sim \mathbf{N}} \right\|_2 \left\| |\nabla|^{\frac{s}{2}} 1_{\mathbf{NR}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_{\mathbf{N}}} \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Case 1.2: $\max \left\{ \frac{|\eta|}{2E(|\eta|^{\frac{2}{3}})+1}, \frac{|\xi|}{2E(|\xi|^{\frac{2}{3}})+1} \right\} \leq t \leq \frac{|\xi|}{2E(|\xi|^{\frac{1}{3}})+1}$: In this case, Let j and n be such that $t \in \bar{\mathbf{I}}_{j,\eta} \cap \bar{\mathbf{I}}_{n,\xi}$. Then we have from (6.5) that $|\xi|^{\frac{1}{3}} \approx |\eta|^{\frac{1}{3}} \lesssim t \lesssim |\xi|^{\frac{2}{3}} \approx |\eta|^{\frac{2}{3}}$ and if $|\xi| \geq 2lt$, we have $|\frac{\xi}{l} - t| \gtrsim \frac{\xi}{l} \gtrsim t$ which implies that

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \frac{l}{|\xi|^3} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1},$$

and if $|\xi| < 2lt$, then

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \begin{cases} \frac{1}{\langle \xi \rangle} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |l| \geq E(|\xi|^{\frac{2}{3}}) + 1 \\ \frac{|\xi|/n^3}{(1+(t-\frac{\xi}{n})^2)^2} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } E(|\xi|^{\frac{1}{3}}) + 1 \leq |l| = |n| \leq E(|\xi|^{\frac{2}{3}}) \\ \frac{1}{\sqrt{\xi}} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |l| \neq |n| \text{ and } \begin{cases} \text{if } E(|\xi|^{\frac{1}{3}}) + 1 \leq |l| \leq |\xi|^{\frac{1}{2}} \\ \text{if } |\xi|^{\frac{1}{2}} + 1 \leq |l| \leq E(|\xi|^{\frac{2}{3}}) \end{cases} \\ \frac{1}{(1+(t-\frac{\xi}{j})^2)^{\frac{3}{2}}} \lesssim \frac{l^2}{\xi} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } 1 \leq |l| \leq E(|\xi|^{\frac{1}{3}}). \end{cases}$$

By Lemma 3.3, for the second case, we have if $|k| \approx |j| \approx |n| = |l| \gtrsim |\xi|^{\frac{1}{3}}$, $\eta \approx \xi$ then

$$\frac{\xi/n^3}{(1+(t-\frac{\xi}{n})^2)^2} \lesssim \frac{\sqrt{\xi/n^3}}{1+|t-\frac{\xi}{n}|} \frac{\langle \xi - \eta \rangle \sqrt{\eta/j^3}}{1+|t-\frac{\eta}{j}|} + \frac{|\xi|^s}{\langle t \rangle^{3s}} \langle \xi - \eta \rangle^3,$$

In summary, we get that

$$\begin{aligned} |\mathbf{R}_{1;\mathbf{N};\mathbf{NR},\mathbf{NR}}^1| &\lesssim \epsilon \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \mathbf{A} \rho_{\sim \mathbf{N}} \right\|_2 \left\| 1_{\mathbf{NR}} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_{\mathbf{N}}} \right\|_2 \\ &+ \epsilon \left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} \rho_{\sim \mathbf{N}} \right\|_2 \left\| 1_{\mathbf{NR}} \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \tilde{\mathbf{A}} P_{\neq \phi_{\mathbf{N}}} \right\|_2. \end{aligned}$$

Here we use that fact that $|k| \leq |\eta|$, $|l| \leq |\xi|$ and then $A \lesssim \tilde{A}$.

Case 1.2' (if possible): $\frac{|\xi|}{2E(|\xi|^{\frac{1}{3}})+1} \leq t \leq \max \left\{ \frac{|\xi|}{2E(|\xi|^{\frac{1}{3}})+1}, \frac{|\eta|}{2E(|\eta|^{\frac{1}{3}})+1} \right\}$: Let j and n be such that $t \in \bar{I}_{j,\eta} \cap \bar{I}_{n,\xi}$. Then we have $t \approx |\xi|^{\frac{2}{3}} \approx |\eta|^{\frac{2}{3}}$, $n \approx j \approx |\xi|^{\frac{1}{3}} \approx |\eta|^{\frac{1}{3}}$ and

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \begin{cases} \frac{1}{\langle \xi \rangle} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |l| \geq E(|\xi|^{\frac{2}{3}}) + 1 \\ \frac{1}{\sqrt{\xi}} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |l| \neq |n| \text{ and } \begin{cases} \text{if } E(|\xi|^{\frac{1}{3}}) + 1 \leq |l| \leq |\xi|^{\frac{1}{2}} \\ \text{if } |\xi|^{\frac{1}{2}} + 1 \leq |l| \leq E(|\xi|^{\frac{2}{3}}) \end{cases} \\ \frac{1}{(1+(t-\frac{\xi}{n})^2)^2} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |\xi|^{\frac{1}{3}} \approx |l| = |n| \leq E(|\xi|^{\frac{1}{3}}) \\ \frac{|l|^5}{|\xi|^3} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1} & \text{if } |l| \leq E(|\xi|^{\frac{1}{3}}) \text{ and } |n| \neq |l| \end{cases}$$

By Lemma 3.3, for the third case, we have $|j| \approx |k| \approx |n| = |l| \approx |\xi|^{\frac{1}{3}}$, $\eta \approx \xi$ and

$$\frac{1}{(1+(t-\frac{\xi}{n})^2)^2} \lesssim \frac{\sqrt{\xi/n^3}}{1+|t-\frac{\xi}{n}|} \frac{\langle \xi - \eta \rangle \sqrt{\eta/j^3}}{1+|t-\frac{\eta}{j}|} + \frac{|\xi|^s}{\langle t \rangle^{3s}} \langle \xi - \eta \rangle^3,$$

which implies that

$$\begin{aligned} |\mathbf{R}_{1;N;NR,NR}^1| &\lesssim \epsilon \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \rho_{\sim N} \right\|_2 \left\| 1_{NR} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2 \\ &\quad + \epsilon \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| 1_{NR} \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi_N \right\|_2. \end{aligned}$$

Case 1.3: $t \geq \min\{2|\eta|, 2|\xi|\}$: We have $|l|t \gtrsim |t - \xi| \gtrsim |l|t$, and thus by the fact that $g(t, \eta) = g(t, \xi)$ and $\Theta_l(t, \xi) = \Theta_k(t, \eta)$, we obtain that

$$\begin{aligned} |\mathbf{R}_{1;N;NR,NR}^1| &\lesssim \left| \sum_{k,l \neq 0} \int_{\eta, \xi} 1_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} A \bar{\rho}_k(\eta) A_k(\eta) (\eta l - \xi k) \hat{\phi}_l(\xi)_N \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \right| \\ &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} 1_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} |A \bar{\rho}_k(\eta)| |A_l(\xi)| |\xi| |\hat{\phi}_l(\xi)_N| \end{aligned}$$

$$\begin{aligned}
& e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}} |\widehat{\nabla \rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\
& \lesssim \frac{1}{\langle t \rangle^2} \left\| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \right\|_2 \left\| 1_{NR} |\nabla|^{\frac{s}{2}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq \phi_N} \right\|_2 \|\rho\|_{G^s}.
\end{aligned}$$

Case 1.4: $\max \left\{ \frac{|\eta|}{2E(|\eta|^{\frac{1}{3}})+1}, \frac{|\xi|}{2E(|\xi|^{\frac{1}{3}})+1} \right\} < t < \min\{2|\eta|, 2|\xi|\}$: Let j and n be such that $t \in \bar{I}_{n,\eta} \cap \bar{I}_{j,\xi}$ and we may first consider the case $|lt| \geq 2|\xi|$, then we have

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \frac{|\xi||l|}{(lt-\xi)^4} \lesssim \frac{1}{|l|^2|t|^2} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1}.$$

Then we focus on the case $|lt| < 2|\xi|$, which implies $|l| \lesssim |\xi|^{\frac{1}{3}}$, if $|l| \geq E(|\xi|^{\frac{1}{3}})+1$, then $l \neq j$, $\left|t - \frac{\xi}{l}\right| \gtrsim \frac{\xi}{l^2} \gtrsim |\xi|^{\frac{1}{3}}$ and thus

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \frac{1}{|\xi|^{\frac{4}{3}}} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1}.$$

Now we consider the case $|l| \leq E(|\xi|^{\frac{1}{3}})$, and we have for $|l| \leq \frac{1}{10}|j|$,

$$\left| \frac{\xi}{l} - t \right| \gtrsim \frac{\xi}{l} \gtrsim \frac{\xi}{j} \approx t,$$

which implies that

$$\frac{|\xi|}{l^3(1+(t-\frac{\xi}{l})^2)^2} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1}.$$

For $|l| \geq \frac{1}{10}|j|$, we have $|lt| \gtrsim |jt| \approx |\xi|$ and

$$\left| t - \frac{\xi}{l} \right| \geq \frac{\xi}{l^3}, \quad \left| t - \frac{\xi}{j} \right| \leq \left| t - \frac{\xi}{l} \right|, \quad \left| t - \frac{\eta}{n} \right| \leq \left| t - \frac{\eta}{k} \right|,$$

and

$$\frac{|\xi|}{l^3(1+|t-\frac{\xi}{l}|^2)^2} \lesssim \frac{l^3}{\xi} \frac{1}{(1+|t-\frac{\xi}{l}|)^2} \lesssim \frac{1}{(1+|t-\frac{\xi}{j}|)^2} \left\langle \frac{\xi}{lt} \right\rangle^{-1}$$

Case 1.3.1 $j = n$: We have

$$\frac{1}{(1+|t-\frac{\xi}{j}|)^2} \lesssim \frac{1}{(1+|t-\frac{\xi}{j}|)} \frac{1}{(1+|t-\frac{\eta}{j}|)} \langle \xi - \eta \rangle,$$

which implies

$$\begin{aligned}
 & |R_{1;N;NR,NR}^1| \\
 & \lesssim \left| \sum_{k,l \neq 0} \int_{\eta, \xi} 1_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} A_k \bar{\rho}_k(\eta) A_l(\eta) (\eta l - \xi k) \hat{\phi}_l(\xi) \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \right| \\
 & \lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} 1_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} |A_k \bar{\rho}_k(\eta)| \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}} \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} A_l(\xi) |\partial_z^{-1} \Delta_L^2 \hat{\phi}_l(\xi)|_N \\
 & \quad \times e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}} |\widehat{\nabla^2 \rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\
 & \lesssim \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} 1_{NR} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi_N \right\|_2 \|\rho\|_{\mathcal{G}^s}.
 \end{aligned}$$

Case 1.3.2 $|t - \frac{\xi}{j}| \gtrsim \frac{\xi}{j^2}$ and $|t - \frac{\eta}{n}| \gtrsim \frac{\eta}{n^2}$: We have

$$\frac{1}{(1 + |t - \frac{\xi}{j}|^2)} \lesssim \frac{1}{(1 + |t - \frac{\xi}{j}|)} \frac{1}{(1 + |t - \frac{\eta}{n}|)},$$

which gives us that

$$|R_{1;N;NR,NR}^1| \lesssim \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} 1_{NR} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi_N \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Case 1.3.3 $|\xi - \eta| \gtrsim \frac{|\eta|}{n} \approx t$: We have

$$\frac{|\xi|}{l^3(1 + |t - \frac{\xi}{j}|)} \lesssim \frac{\langle \xi - \eta \rangle^2}{\langle t \rangle^2},$$

which gives us that

$$|R_{1;N;NR,NR}^1| \lesssim \frac{1}{\langle t \rangle^2} \|A \rho_{\sim N}\|_2 \left\| 1_{NR} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Case 2: $|\xi| < 6|l|$: We have

$$\frac{|l|}{l^2 + (\xi - lt)^2} \lesssim \frac{1}{\langle t \rangle^2}$$

which implies

$$\begin{aligned}
 & |R_{1;N;NR,NR}^1| \\
 & \lesssim \left| \sum_{k,l \neq 0} \int_{\eta, \xi} 1_{t \notin I_{k,\eta}, t \notin I_{l,\xi}} A_k \bar{\rho}_k(\eta) A_l(\eta) (\eta l - \xi k) \hat{\phi}_l(\xi) \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \right|
 \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin \mathbf{I}_{k,\eta}, t \notin \mathbf{I}_{l,\xi}} |\mathbf{A} \bar{\rho}_k(\eta)| |\mathbf{A}_l(\xi)| |l| |\hat{\phi}_l(\xi)_N| \\
&\quad e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}} |\widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\
&\lesssim \frac{1}{\langle t \rangle^2} \left\| |\nabla|^{\frac{s}{2}} \mathbf{A} \rho_{\sim N} \right\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \mathbf{1}_{\text{NR}} |\nabla|^{\frac{s}{2}} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}.
\end{aligned}$$

Treatment of $\mathbf{R}_{1;N;R, \text{NR}}^1$

For $l\xi < 0$, we have as in (6.4),

$$|\mathbf{R}_{1;N;R, \text{NR}}^1| \lesssim \frac{1}{\langle t \rangle^2} \left\| |\nabla|^{\frac{s}{2}} \mathbf{A} \rho_{\sim N} \right\|_2 \left\| \mathbf{1}_{\text{NR}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} |\nabla|^{\frac{s}{2}} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s},$$

Now, for $l\xi > 0$, we have

$$\begin{aligned}
&|\mathbf{R}_{1;N;R, \text{NR}}^1| \\
&\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \in \mathbf{I}_{k,\eta}, t \notin \mathbf{I}_{l,\xi}} |\mathbf{A} \bar{\rho}_k(\eta)| |\mathbf{A}_k(\eta)| |\eta l - \xi k| |\hat{\phi}_l(\xi)_N| \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} |d\eta d\xi \\
&\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \in \mathbf{I}_{k,\eta}, t \notin \mathbf{I}_{l,\xi}} |\mathbf{A} \bar{\rho}_k(\eta)| \frac{\mathbf{J}_k(t, \eta)}{\mathbf{J}_l(t, \xi)} \frac{\mathcal{M}(t, \eta)}{\mathcal{M}(t, \xi)} \\
&\quad \times |\mathbf{A}_l(\xi)| |\eta l - \xi k| \frac{|\xi|}{|l|^3 (1 + (\frac{\xi}{l} - t)^2)^2} |\partial_z^{-1} \widehat{\Delta_L^2 \phi_l}(\xi)_N| |\widehat{\rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi.
\end{aligned}$$

Note that for $t \in \mathbf{I}_{k,\eta}$, we have

$$\Theta_R(t, \eta) \approx \Theta_{\text{NR}}(t, \eta) \left[\frac{k^3}{|\eta|} \left(1 + |t - \frac{|\eta|}{|k|}| \right) \right],$$

thus we obtain that

$$\begin{aligned}
\frac{\mathbf{J}(t, \eta)}{\mathbf{J}(t, \xi)} &\leq \frac{\Theta_{\text{NR}}(t, \xi)}{\Theta_R(t, \eta)} e^{\mu|\eta-\xi|^{1/3}} + e^{\mu|k-l|^{1/3}} \\
&\leq \frac{\Theta_{\text{NR}}(t, \xi)}{\Theta_{\text{NR}}(t, \eta)} \frac{\Theta_{\text{NR}}(t, \eta)}{\Theta_R(t, \eta)} e^{\mu|\eta-\xi|^{1/3}} + e^{\mu|k-l|^{1/3}} \\
&\lesssim e^{C\mu|\eta-\xi|^{1/3}} \frac{|\eta|}{|k|^3 (1 + |t - \frac{|\eta|}{|k|}|)} + e^{\mu|k-l|^{1/3}} \\
&\lesssim \frac{|\eta|}{|k|^3 (1 + |t - \frac{|\eta|}{|k|}|)} e^{C\mu|\eta-\xi, k-l|^{1/3}}
\end{aligned}$$

Also using the fact that

$$\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \lesssim e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}}$$

Hence, we obtain that

$$\begin{aligned}
 & |\mathbf{R}_{1;N;R,NR}^1| \\
 & \lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \in I_{k,\eta}, t \notin I_{l,\xi}} \mathbf{A} \bar{\rho}_k(\eta) \mathbf{A}_l(\xi) \frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{|\eta|}{|k|}|)} \\
 & \quad \times |\partial_z^{-1} \Delta_L^2 \phi_l(\xi)_N| |k - l, \eta - \xi| e^{c\lambda|\eta - \xi, k - l|^s} |\widehat{\rho}_{k-l}(\eta - \xi)|_{<N/8} d\eta d\xi.
 \end{aligned} \tag{6.7}$$

with $c \in (0, 1)$ and $s > \frac{1}{3}$. Our goal now is to estimate the symbol

$$\frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{|\eta|}{|k|}|)} \tag{6.8}$$

in different time regimes in order to absorb the large factor

$$\frac{|\eta|}{|k|^3(1 + |t - \frac{|\eta|}{|k|}|)}.$$

Case 1. First, for $lt \geq 2|\xi|$ and $|l| \geq 1$, it holds that

$$\left| t - \frac{\xi}{l} \right| \gtrsim \frac{|\xi|}{l} \quad \text{and} \quad \left| t - \frac{\xi}{l} \right| \geq t - \left| \frac{\xi}{l} \right| \geq t - \frac{t}{2} \gtrsim t.$$

Hence, keeping in mind the fact that $|\xi| \approx |\eta|$, we estimate the factor in (6.8) as

$$\frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{|\eta|}{|k|}|)} \lesssim \frac{1}{1 + (\frac{\xi}{l} - t)^2} \lesssim \frac{1}{\langle t \rangle^2} \left\langle \frac{\xi}{lt} \right\rangle^{-1}.$$

Consequently, it holds from (6.7), that

$$|\mathbf{R}_{1;N;R,NR}^1| \lesssim \frac{1}{\langle t \rangle^2} \left\| |\nabla|^{\frac{s}{2}} \mathbf{A} \rho_{\sim N} \right\|_2 \left\| |\nabla|^{\frac{s}{2}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Now we always assume that $|l|t \leq 2|\xi|$.

Case 2. Now, for $\frac{2|\xi|}{2E(|\xi|^{\frac{2}{3}})+1} < t < 2|\xi|$: then there exists n such that $t \in \bar{I}_{n,\xi}$.

Hence, we have $t \in I_{k,\eta} \cap \bar{I}_{n,\xi} \subset \bar{I}_{k,\eta} \cap \bar{I}_{n,\xi}$, $|k| \approx |n| \lesssim |\xi|^{\frac{1}{3}}$ and $\left| t - \frac{\xi}{n} \right| \lesssim \left| t - \frac{\xi}{l} \right|$, $\left| t - \frac{\xi}{n} \right| \leq \frac{\xi}{n^2}$.

If $|l| \leq \frac{1}{10}|n| \lesssim |\xi|^{\frac{1}{3}}$, then $\left| \frac{\xi}{l} - t \right| \gtrsim \frac{\xi}{l} \gtrsim \frac{\xi}{n} \approx t \approx \frac{\eta}{k}$, thus we have

$$\frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{|\eta|}{|k|}|)} \lesssim \frac{|\xi|^s}{\langle t \rangle^{3s}} \left\langle \frac{\xi}{lt} \right\rangle^{-1},$$

which implies that

$$|\mathbf{R}_{1;N;R,NR}^1| \lesssim \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{3s}} \mathbf{A} \rho_{\sim N} \right\|_2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{3s}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Now we focus on the case $|l| \geq \frac{1}{10}|n|$, thus $|lt| \gtrsim nt \approx |\xi|$ which gives us that

$$\left\langle \frac{\xi}{lt} \right\rangle^{-1} \approx 1.$$

We are in a position to apply Lemma 3.3.

Case 2.1 $k = n$: Since $t \notin I_{l,\xi}$ we have the following two cases:

$$|l| \geq E(|\xi|^{\frac{1}{3}}) + 1 \text{ and } |l| \leq E(|\xi|^{\frac{1}{3}}), \quad \text{with} \quad \left| t - \frac{\xi}{l} \right| \geq \frac{\xi}{2l^3}.$$

For the first one, if $t \leq \frac{2|\xi|}{2E(|\xi|^{\frac{1}{3}})+1}$, then by the fact that $t \in I_{k,\eta} \cap \bar{I}_{k,\xi}$, then $t \approx |\eta|^{\frac{2}{3}}$, hence we get $|k| \approx |\eta|^{\frac{1}{3}}$; and if $t \geq \frac{2|\xi|}{2E(|\xi|^{\frac{1}{3}})+1}$ then $|k| = |n| \leq E(|\xi|^{\frac{1}{3}})$, $\left| t - \frac{\xi}{l} \right| \geq \frac{\xi}{l^2} \geq |\xi|^{\frac{1}{3}}$. Therefore in both sub-cases, we get that

$$\begin{aligned} \frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{\eta}{k}|)} &\lesssim \frac{1}{1 + |t - \frac{\xi}{n}|} \frac{1}{1 + |t - \frac{\eta}{k}|} \\ &\lesssim \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}}. \end{aligned}$$

For the second one, and since $t \notin I_{l,\xi}$, then if $|l| = |n|$ which means that $t \in \bar{I}_{n,\xi} \setminus I_{n,\xi}$, recall that $|k| = |n|$, then we get $\left| t - \frac{\xi}{l} \right| \geq \frac{\xi}{2k^3}$; and if $|l| \neq |n|$ thus $\left| t - \frac{\xi}{l} \right| \gtrsim \frac{\xi}{l^2}$. Therefore in both sub-cases, we get that

$$\begin{aligned} \frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{\eta}{k}|)} &\lesssim \frac{1}{1 + |t - \frac{\xi}{n}|} \frac{1}{1 + |t - \frac{\eta}{k}|} \\ &\lesssim \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}}. \end{aligned}$$

Case 2.2 $|t - \frac{\xi}{n}| \gtrsim \frac{\xi}{n^2}$ and $|t - \frac{\eta}{k}| \gtrsim \frac{\xi}{k^2}$: We have by using the inequality $|t - \frac{\xi}{n}| \lesssim |t - \frac{\xi}{l}|$,

$$\begin{aligned} \frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{\eta}{k}|)} &\lesssim \frac{1}{1 + |t - \frac{\xi}{n}|} \frac{1}{1 + |t - \frac{\eta}{n}|} \\ &\lesssim \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}} \end{aligned}$$

Case 2.3 $|\xi - \eta| \gtrsim \frac{\xi}{n} \gtrsim |\xi|^{\frac{2}{3}} \gtrsim \langle t \rangle^{\frac{2}{3}}$, where we have used the fact that $|k| \approx |n| \leq |\xi|^{1/3}$ and $t \leq 2|\xi|$ in the above time regime. Hence, we have

$$\frac{|\xi|}{|l|^3(1 + (\frac{\xi}{l} - t)^2)^2} \frac{|\eta|}{|k|^3(1 + |t - \frac{\eta}{k}|)} \lesssim \langle \eta - \xi \rangle^2 \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}},$$

Consequently, in all the above cases, we have the following estimate:

$$|\mathbf{R}_{1;N;R,NR}^1| \lesssim \left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} \rho_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \mathbf{1}_{NR} \partial_z^{-1} \Delta_L^2 \tilde{\mathbf{A}} P_{\neq} \phi_N \right\|_2 \|\rho\|_{\mathcal{G}^s}$$

Treatment of $\mathbf{R}_{1;N;NR,R}^1$

By Lemmas 3.5 and 3.6, it holds that $1 \leq |l| \leq E(|\xi|^{\frac{1}{3}})$ and

$$\begin{aligned} |\mathbf{R}_{1;N;NR,R}^1| &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin I_{k,\eta}, t \in I_{l,\xi}} |A \tilde{\rho}_k(\eta) A_k(\eta)| |\xi| |\hat{\phi}_l(\xi)_N \widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\ &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin I_{k,\eta}, t \in I_{l,\xi}} |A \tilde{\rho}_k(\eta)| \frac{J(t, \eta)}{J(t, \xi)} \frac{\mathcal{M}(t, \eta)}{\mathcal{M}(t, \xi)} \\ &\quad A_l(\xi) |\xi| |\hat{\phi}_l(\xi)_N \widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\ &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{t \notin I_{k,\eta}, t \in I_{l,\xi}} |A \tilde{\rho}_k(\eta)| \frac{|l|^3}{|\xi|} (1 + |t - \frac{\xi}{l}|) A_l(\xi) |\xi| |\hat{\phi}_l(\xi)_N| \\ &\quad \times e^{C \delta_L |k-l, \xi-\eta|^{\frac{1}{3}}} \widehat{\nabla} \rho_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi. \end{aligned}$$

By (6.3a), we have $t \geq \frac{\eta}{2E(|\eta|^{\frac{2}{3}})+1}$. Suppose $\frac{3}{2}|\xi| \geq t > 2|\eta|$. Then

$$|\xi - \eta| \geq |\xi| - \frac{3}{4}|\xi| \geq \frac{1}{4}|\xi| \approx t,$$

which implies that

$$|\mathbf{R}_{1;N;NR,R}^1| \lesssim \frac{1}{\langle t \rangle^2} \|\nabla|^{\frac{s}{2}} A \rho_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} |\nabla|^{\frac{s}{2}} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Note that in this case it holds that

$$\left\langle \frac{\xi}{lt} \right\rangle^{-1} \approx 1$$

Now let focus on the case $t \leq 2|\eta|$. Let n be such that $t \in \bar{I}_{n,\eta} \cap I_{l,\xi} \subset \bar{I}_{n,\eta} \cap \bar{I}_{l,\xi}$, we have the following cases:

Case $l = n$: We have $(1 + |t - \frac{\eta}{n}|) \lesssim (1 + |t - \frac{\xi}{l}|) \langle \xi - \eta \rangle$.

Case $|t - \frac{\eta}{n}| \gtrsim \frac{\eta}{n^2}$ and $|t - \frac{\xi}{l}| \gtrsim \frac{\xi}{l^2}$: Then by the fact $t \in I_{l,\xi}$ we have $|n| \approx |l| \approx 1$. Thus it still holds that $(1 + |t - \frac{\eta}{n}|) \lesssim (1 + |t - \frac{\xi}{l}|) \langle \xi - \eta \rangle$.

Therefore, we obtain that

$$\begin{aligned}
 |\mathbf{R}_{1;\mathbf{N};\mathbf{NR},\mathbf{R}}^1| &\lesssim \sum_{k,l \neq 0} \int_{\eta,\xi} \mathbf{1}_{t \notin I_{k,\eta}, t \in I_{l,\xi}} \frac{|\mathbf{A} \bar{\rho}_k(\eta)|}{1 + |t - \frac{\eta}{n}|} \frac{|l|^3 (1 + |t - \frac{\xi}{l}|)^3 \mathbf{A}_l(\xi) |\hat{\phi}_l(\xi)_N|}{1 + |t - \frac{\xi}{l}|} \\
 &\quad \times e^{c\lambda|k-l,\xi-\eta|^s} \widehat{\nabla^2 \rho_{k-l}}(\eta - \xi)_{<N/8} |d\eta d\xi| \\
 &\lesssim \left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} \rho_{\sim \mathbf{N}} \right\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \Theta}{\Theta}} \mathbf{1}_{\mathbf{R}} \tilde{\mathbf{A}} \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_{\mathbf{N}} \right\|_2 \|\rho\|_{\mathcal{G}^s}.
 \end{aligned}$$

Case $|\xi - \eta| \gtrsim \frac{\xi}{l}$: We have $|\xi - \eta| \gtrsim |\xi|^{\frac{2}{3}} \gtrsim \langle t \rangle^{\frac{2}{3}}$ and thus

$$|\mathbf{R}_{1;\mathbf{N};\mathbf{NR},\mathbf{R}}^1| \lesssim \frac{1}{\langle t \rangle^2} \|\nabla|^{\frac{s}{2}} \mathbf{A} \rho_{\sim \mathbf{N}}\|_2 \left\| \mathbf{1}_{\mathbf{R}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} |\nabla|^{\frac{s}{2}} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq} \phi_{\mathbf{N}} \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Treatment of $\mathbf{R}_{1;\mathbf{N};\mathbf{R},\mathbf{R}}^1$

By Lemma 3.3, we consider the following three cases:

Case 1: $k = l$, then

$$\begin{aligned}
 |\mathbf{R}_{1;\mathbf{N};\mathbf{R},\mathbf{R}}^1| &= \left| \sum_{l \neq 0} \int_{\eta,\xi} \mathbf{1}_{t \in I_{l,\eta}, t \in I_{l,\xi}} \mathbf{A} \bar{\rho}_k(\eta) \mathbf{A}_k(\eta) (\eta l - \xi l) \hat{\phi}_l(\xi)_N \widehat{\rho}_0(\eta - \xi)_{<N/8} |d\eta d\xi| \right| \\
 &\lesssim \sum_{l \neq 0} \int_{\eta,\xi} \mathbf{1}_{t \in I_{l,\eta}, t \in I_{l,\xi}} \mathbf{A} |\bar{\rho}_k(\eta)| \frac{\mathbf{J}(t, \eta) \mathcal{M}(t, \eta)}{\mathbf{J}(t, \xi) \mathcal{M}(t, \xi)} \\
 &\quad \mathbf{A}_l(\xi) |l| |\hat{\phi}_l(\xi)_N| |\widehat{\nabla \rho}_0(\eta - \xi)_{<N/8}| |d\eta d\xi|
 \end{aligned}$$

By Lemma 3.5 and the fact that

$$\begin{aligned}
 \frac{\mathbf{J}(t, \eta)}{\mathbf{J}(t, \xi)} &\lesssim \frac{\Theta_{\mathbf{NR}}(t, \xi)}{\Theta_{\mathbf{NR}}(t, \eta)} \frac{\Theta_{\mathbf{R}}(t, \xi)}{\Theta_{\mathbf{NR}}(t, \xi)} \frac{\Theta_{\mathbf{NR}}(t, \eta)}{\Theta_{\mathbf{R}}(t, \eta)} e^{\mu|\eta-\xi|^{\frac{1}{3}}} \\
 &\lesssim e^{C\mu|\eta-\xi|^{\frac{1}{3}}} \frac{1 + |t - \frac{\xi}{l}|}{1 + |t - \frac{\eta}{l}|} \lesssim e^{C\mu|\eta-\xi|^{\frac{1}{3}}}
 \end{aligned} \tag{6.9}$$

and that

$$\frac{1}{1 + |t - \frac{\xi}{l}|} \lesssim \frac{\langle \xi - \eta \rangle}{1 + |t - \frac{\eta}{l}|},$$

we obtain that

$$|\mathbf{R}_{1;\mathbf{N};\mathbf{R},\mathbf{R}}^1| \lesssim \left\| \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} \rho_{\sim \mathbf{N}} \right\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \mathbf{1}_{\mathbf{R}} \partial_z^{-1} \Delta_L^2 \tilde{\mathbf{A}} P_{\neq} \phi_{\mathbf{N}} \right\|_2 \|\rho\|_{\mathcal{G}^s}.$$

Case 2: $|t - \frac{\xi}{l}| \gtrsim \frac{\xi}{l^3}$ and $|t - \frac{\eta}{l}| \gtrsim \frac{\eta}{k^3}$: By Lemma 3.5 and (6.9) and the fact that

$$\begin{aligned} \frac{\xi}{l^3(1+|t-\frac{\xi}{l}|^4)} &\lesssim \frac{1}{(1+|t-\frac{\xi}{l}|^3)} \lesssim \frac{l^3/\xi}{(1+|t-\frac{\xi}{l}|)} \frac{\eta/k^3}{(1+|t-\frac{\eta}{k}|)} \\ &\lesssim \sqrt{\frac{\partial_t g(t, \xi)}{g(t, \xi)}} \sqrt{\frac{\partial_t g(t, \eta)}{g(t, \eta)}}. \end{aligned}$$

we obtain that

$$|\mathbf{R}_{1;N;R,R}^1| \lesssim \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} 1_R \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}$$

Case 3: $|\xi - \eta| \gtrsim \frac{|\xi|}{l^2}$: We have $|l| \lesssim |\xi|^{\frac{1}{3}}$, $|\xi|^{\frac{2}{3}} \lesssim \langle t \rangle \lesssim |\xi|$ which gives us that

$$|\xi - \eta| \gtrsim \langle t \rangle^{\frac{1}{3}}.$$

Thus we get that

$$\begin{aligned} |\mathbf{R}_{1;N;R,R}^1| &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{l \in \mathbf{I}_{k,\eta}, t \in \mathbf{I}_{l,\xi}} |A \bar{\rho}_k(\eta) A_k(\eta)| |\eta l - \xi k| |\hat{\phi}_l(\xi)_N \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\ &\lesssim \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{1}_{l \in \mathbf{I}_{k,\eta}, t \in \mathbf{I}_{l,\xi}} |A \bar{\rho}_k(\eta)| e^{C\delta_L |k-l, \eta-\xi|^{\frac{1}{3}}} \\ &\quad A_l(\xi) |\xi| |\hat{\phi}_l(\xi)_N \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\ &\lesssim \frac{1}{\langle t \rangle^2} \sum_{k,l \neq 0} \int_{\eta, \xi} |A \bar{\rho}_k(\eta)| e^{C\delta_L |k-l, \eta-\xi|^{\frac{1}{3}}} \\ &\quad A_l(\xi) |\hat{\phi}_l(\xi)_N \langle \xi - \eta \rangle^{11} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8}| d\eta d\xi \\ &\lesssim \frac{1}{\langle t \rangle^2} \|A \rho_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} 1_R A P_{\neq \phi_N} \right\|_2 \|\rho\|_{\mathcal{G}^s}. \end{aligned}$$

6.1.2. The term $R_{1;N}^{\epsilon,1}$ In this section, we treat the term $R_{1;N}^{\epsilon,1}$. We also use a paraproduct in v to linearize the high frequencies around the low frequencies of the product $h \nabla^\perp \phi_l$. Hence, we have

$$\begin{aligned} R_{1;N}^{\epsilon,1} &= \frac{1}{2\pi} \sum_{M \geq 8} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A \bar{\rho}_k(\eta) A_k(\eta) ((\eta - \xi)l - \xi'(k-l)) \hat{h}(\xi - \xi')_{<M/8} \\ &\quad \times \varphi_N(l, \xi) \hat{\phi}_l(\xi')_M \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi' \\ &\quad + \frac{1}{2\pi} \sum_{M \geq 8} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A \bar{\rho}_k(\eta) A_k(\eta) ((\eta - \xi)l - \xi'(k-l)) \hat{h}(\xi - \xi')_M \\ &\quad \times \varphi_N(l, \xi) \hat{\phi}_l(\xi')_{<M/8} \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi' \\ &\quad + \frac{1}{2\pi} \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A \bar{\rho}_k(\eta) A_k(\eta) ((\eta - \xi)l - \xi'(k-l)) \end{aligned}$$

$$\begin{aligned} & \hat{h}(\xi - \xi')_{M'} \times \varphi_N(l, \xi) \hat{\phi}_l(\xi')_M \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi' \\ &= R_{1;N;LH}^{\epsilon,1} + R_{1;N;HL}^{\epsilon,1} + R_{1;N;HH}^{\epsilon,1}, \end{aligned}$$

where φ_N denotes the cut-off associated to the N -th dyadic shell in $\mathbb{Z} \times \mathbb{R}$.

Let us first treat the term $R_{1;N;LH}^{\epsilon,1}$. Since h is in low frequency, then it is natural to expect that $R_{1;N;LH}^{\epsilon,1}$ behaves somehow like $R_{1;N}^{\epsilon,1}$, since $\hat{h}_{<M/8}$ provides only a modulation of the term $(\hat{\phi}_l)_M$ for large frequencies.

On the support of the integrand we have (see (6.3a)).

$$\begin{aligned} ||l, \xi| - |k, \eta|| &\leq |k - l, \eta - \xi| \leq \frac{6}{32} |l, \xi| \\ ||l, \xi'| - |l, \xi|| &\leq |\xi - \xi'| \leq \frac{6}{32} |l, \xi'| \end{aligned} \quad (6.10)$$

This implies $|k, \eta| \approx |l, \xi| \approx |l, \xi'|$. As in (6.3a), it holds that

$$|(\eta - \xi)l - \xi'(k - l)| \lesssim |l, \xi'| |k - l, \eta - \xi|. \quad (6.11)$$

Hence, the estimates goes the same as in the one of $R_{1;N}^1$ where (l, ξ') will play the role of (l, ξ) . Therefore, when we switch from $A_k(t, \eta)$ to $A_l(t, \xi')$, we will pay a Gevrey-3 regularity for h as shown by the following inequality:

$$e^{\lambda|k, \eta|^s} \lesssim e^{\lambda|l, \xi'|^s + c\lambda|k-l, \eta-\xi|^s + c\lambda|\xi-\xi'|^s} \quad (6.12)$$

with $0 < c < 1$ which will help to absorb the Sobolev exponent in the estimates. Hence, with the estimate (6.12) in hand, we may rewrite $R_{1;N;LH}^{\epsilon,1}$ as

$$\begin{aligned} |R_{1;N;LH}^{\epsilon,1}| &\lesssim \sum_{M \geq 8} \sum_{k, l \neq 0} \int_{\eta, \xi, \xi'} A|\widetilde{\rho}_k(\eta)| J_k(\eta) \mathcal{M}_k(\eta) \langle l, \xi' \rangle^\sigma \\ &\quad \left| (\eta - \xi)l - \xi'(k - l) \right| \varphi_N(l, \xi) \times e^{c\lambda|\xi - \xi'|^s} |\hat{h}(\xi - \xi')_{<M/8}| \\ &\quad e^{\lambda|l, \xi'|^s} \hat{\phi}_l(\xi')_M e^{c\lambda|k-l, \eta-\xi|^s} \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi'. \end{aligned} \quad (6.13)$$

Thus we have

$$\begin{aligned} |R_{1;N;LH}^{\epsilon,1}| &\lesssim \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \rho_{\sim N} \right\|_2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2 \\ &\quad + \epsilon^2 \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi_N \right\|_2 \\ &\quad + \epsilon^2 \left\| \sqrt{\frac{\partial_t g}{g}} \tilde{A} \rho_{\sim N} \right\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t \Theta}{\Theta}} 1_R \tilde{A} \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_N \right\|_2 \end{aligned}$$

Let us now turn to the term $R_{1;N;HL}^{\epsilon,1}$. We use the cut-off $1 = \mathbf{1}_{16|l| \geq |\xi|} + \mathbf{1}_{16|l| \leq |\xi|}$ and split the term $R_{1;N;HL}^{\epsilon,1}$ according to this cut-off and write

$$\begin{aligned} R_{1;N;HL}^{\epsilon,1} &= \frac{1}{2\pi} \sum_{M \geq 8} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A|\widehat{\rho}_k(\eta)| A_k(\eta) ((\eta - \xi)l - \xi'(k - l)) \widehat{h}(\xi - \xi')_M \\ &\quad \times \varphi_N(l, \xi) \widehat{\phi}_l(\xi')_{<M/8} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} (\mathbf{1}_{16|l| \geq |\xi|} + \mathbf{1}_{16|l| \leq |\xi|}) d\eta d\xi d\xi' \\ &= R_{1;N;HL}^{\epsilon,1;z} + R_{1;N;HL}^{\epsilon,1;v}. \end{aligned}$$

To estimate $R_{1;N;HL}^{\epsilon,1;z}$, we have first form (6.10) and from the fact that

$$\frac{M}{2} \leq |\xi - \xi'| \leq \frac{3M}{2} \quad \text{and} \quad \frac{|\xi'|}{M/8} \leq \frac{3}{4}$$

together with $|\xi| \leq 16|l|$, we have

$$\begin{aligned} ||l, \xi| - |k, \eta|| &\leq |k - l, \eta - \xi| \leq \frac{6}{32} |l, \xi| \\ |l, \xi'| - |l, \xi| &\leq |\xi - \xi'| \leq \frac{16}{13} |\xi| \lesssim |l|. \end{aligned} \quad (6.14a)$$

We discuss two cases: $|l| \geq 16|\xi|$ and $\frac{1}{16}|\xi| \leq |l| \leq 16|\xi|$. For the case $|l| \geq 16|\xi|$ First, if $|l| \geq 16|\xi|$, then it holds that $\frac{16}{13}|\xi| \leq \frac{1}{13}|l| \leq \frac{1}{13}|l, \xi|$, hence (6.12) holds.

Now, for $\frac{1}{16}|\xi| \leq |l| \leq 16|\xi|$, then, we have $|\xi - \xi'| \approx |l, \xi|$ and hence, we can obtain

$$e^{\lambda|l, \xi|^s} \leq e^{c(\lambda|l, \xi'| + |\xi - \xi'|)} \leq e^{\lambda|l, \xi'|^s} e^{c\lambda|\xi - \xi'|^s}$$

Hence, in both cases (6.12), holds. Hence, using the fact that $\langle k, \eta \rangle \approx \langle l, \xi' \rangle \leq \langle l \rangle$, we obtain as in (6.13),

$$\begin{aligned} |R_{1;N;HL}^{\epsilon,1;z}| &\lesssim \sum_{M \geq 8} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} \mathbf{1}_{16|l| \geq |\xi|} A|\widehat{\rho}_k(\eta)| J_k(\eta) \mathcal{M}_k(\eta) \langle l \rangle^\sigma \\ &\quad |(\eta - \xi)l - \xi'(k - l)| \varphi_N(l, \xi) \times e^{c\lambda|\xi - \xi'|^s} |\widehat{h}(\xi - \xi')_M| \\ &\quad e^{\lambda|l, \xi'|^s} \widehat{\phi}_l(\xi')_{<M/8} e^{c\lambda|k-l, \eta-\xi|^s} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi'. \end{aligned}$$

Now, in the above integral, we will take advantages for $|l|$ being large to exclude the resonant interval. For this reason, we fix M_0 large enough and split the above term into two terms:

$$\begin{aligned} |R_{1;N;HL}^{\epsilon,1;z}| &\lesssim \left(\sum_{M \geq M_0} + \sum_{M \leq M_0} \right) \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A|\widehat{\rho}_k(\eta)| J_k(\eta) \mathcal{M}_k(\eta) \langle l \rangle^\sigma \\ &\quad |(\eta - \xi)l - \xi'(k - l)| \varphi_N(l, \xi) \times \mathbf{1}_{16|l| \geq |\xi|} e^{c\lambda|\xi - \xi'|^s} |\widehat{h}(\xi - \xi')_M| \\ &\quad e^{\lambda|l, \xi'|^s} \widehat{\phi}_l(\xi')_{<M/8} e^{c\lambda|k-l, \eta-\xi|^s} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi d\xi' \\ &= R_{1;N;HL;H}^{\epsilon,1;z} + R_{1;N;HL;L}^{\epsilon,1;z}. \end{aligned}$$

For the term $R_{1;N;HL;H}^{\epsilon,1;z}$, we have $|l|$ is large compared to $|\xi'|$ and since $|k, \eta| \approx |l, \xi'|$, then both (k, η) and (l, ξ) are both non-resonant. Then, it holds that

$$\frac{J_k(\eta)\mathcal{M}_k(\eta)}{J_l(\xi')\mathcal{M}_l(\xi')} \lesssim e^{C|k-l, \eta-\xi'|^{\frac{1}{3}}} \lesssim e^{C|k-l, \eta-\xi|^{\frac{1}{3}}+C|\xi-\xi'|^{\frac{1}{3}}}.$$

Hence, it holds that by applying (6.11)

$$\begin{aligned} |R_{1;N;HL;H}^{\epsilon,1;z}| &\lesssim \sum_{M \geq M_0} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A|k, \eta|^{\frac{s}{2}} |\widehat{\rho}_k(\eta)| J_l(\xi') \mathcal{M}_l(\xi') \frac{|l, \xi'|}{l^3(1+|t-\frac{\xi'}{l}|^2)^2} \\ &\quad \times \langle l \rangle^\sigma \varphi_N(l, \xi) \mathbf{1}_{16|l| \geq |\xi|} e^{c\lambda|\xi-\xi'|^s} |\widehat{h}(\xi-\xi')_M| \mathbf{1}_{NR} \\ &\quad \times e^{\lambda|l, \xi'|^s} \partial_z^{-1} \Delta_L^2 \phi_l(\xi')_{<M/8} |k-l, \eta-\xi| \\ &\quad e^{c\lambda|k-l, \eta-\xi|^s} \widehat{\rho}_{k-l}(\eta-\xi)_{<N/8} d\eta d\xi d\xi' \end{aligned}$$

On the support of the integrand, we have

$$\frac{|l, \xi'|}{l^3(1+|t-\frac{\xi'}{l}|^2)^2} \lesssim \frac{1}{l^2+|lt-\xi'|^2} \lesssim \frac{1}{l^2\langle t \rangle^2} \left\langle \frac{\xi}{lt} \right\rangle^{-1} \quad (6.15)$$

Consequently, it holds from above that

$$\begin{aligned} |R_{1;N;HL;H}^{\epsilon,1;z}| &\lesssim \frac{1}{\langle t \rangle^2} \|A\rho_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \mathbf{1}_{NR} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2 \\ &\quad \|\rho_{<N/8}\|_{\mathcal{G}^s} \sum_{M \geq M_0} \frac{1}{M} \|h_M\|_{\mathcal{G}^s} \\ &\lesssim \frac{\epsilon^2}{\langle t \rangle^2} \|A\rho_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \mathbf{1}_{NR} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi_N \right\|_2. \end{aligned}$$

The treatment of $R_{1;N;HL;L}^{\epsilon,1;z}$ is easy since $|\xi-\xi'|+|\xi'|+|l| \lesssim 2^{M_0}$. Thus we have

$$\sum_{N \geq 8} |R_{1;N;HL;L}^{\epsilon,1;z}| \lesssim \epsilon^2 C K_{\lambda, \rho} + \epsilon^2 \left\| \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \mathbf{1}_{NR} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi \right\|_2.$$

Now let us treat $R_{1;N;HL}^{\epsilon,1;v}$. On the support of the integrand, (6.14a) holds. In addition, we have

$$\begin{aligned} ||l, \xi| - |k, \eta|| &\leq |k-l, \eta-\xi| \leq \frac{6}{32} |l, \xi| \\ ||\xi-\xi'| - |l, \xi|| &\leq |l, \xi'| \leq \frac{67}{256} |\xi-\xi'|. \end{aligned}$$

Hence, it holds that

$$e^{\lambda|k, \eta|^s} \leq e^{\lambda|\xi-\xi'|^s + c\lambda|l, \xi'|^s + c\lambda|k-l, \eta-\xi|^s}.$$

Also it holds from (6.6) that $\frac{189}{256}|\xi - \xi'| \leq |l, \xi| \leq \frac{323}{256}|\xi - \xi'|$. Hence, this yields, by using (6.11) and the fact that $\langle k, \eta \rangle \approx \langle l, \xi \rangle$

$$\begin{aligned} |\mathbf{R}_{1;\mathbf{N};\mathbf{HL}}^{\epsilon,1;v}| &\lesssim \sum_{M \geq 8} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A \bar{\rho}_k(\eta) \mathbf{J}_k(\eta) \mathcal{M}_k(\eta) \langle \xi - \xi' \rangle^\sigma e^{\lambda|\xi - \xi'|^s} |\hat{h}(\xi - \xi')_{\mathbf{M}}| \\ &\quad \times \varphi_{\mathbf{N}}(l, \xi) e^{c\lambda|l, \xi'|^s} \hat{\phi}_l(\xi')_{<M/8} |l, \xi'| |k - l, \eta - \xi| \\ &\quad \times e^{c\lambda|k-l, \eta-\xi|^s} \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} \mathbf{1}_{16|l| \leq |\xi|} d\eta d\xi d\xi' \end{aligned}$$

Now, we use the estimate

$$\begin{aligned} \frac{\mathbf{J}_k(\eta)}{\mathbf{J}_0(\xi - \xi')} &\lesssim \langle \xi - \xi' \rangle e^{C|k, \eta - \xi - \xi'|^{\frac{1}{3}}} \lesssim \langle \xi - \xi' \rangle e^{C|l, \xi'|^{\frac{1}{3}} + C|k-l, \eta-\xi|^{\frac{1}{3}}} \\ \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_0(\xi - \xi')} &\lesssim e^{C|k, \eta - \xi - \xi'|^{\frac{1}{3}}} \lesssim e^{C|l, \xi'|^{\frac{1}{3}} + C|k-l, \eta-\xi|^{\frac{1}{3}}}, \end{aligned}$$

which gives us that

$$|\mathbf{R}_{1;\mathbf{N};\mathbf{HL}}^{\epsilon,1;v}| \lesssim \|\mathbf{A} \rho_{\sim \mathbf{N}}\|_2 \|P_{\neq} \phi\|_{\mathcal{G}^{\lambda, \beta; s}} \|\mathbf{A}_0 \partial_v h_{\sim \mathbf{N}}\|_2 \|\rho\|_{\mathcal{G}^{\lambda, \beta; s}}.$$

Now, we estimate the term $\mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1}$. We have

$$\begin{aligned} \mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1} &= \frac{1}{2\pi} \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \sum_{k,l \neq 0} \int_{\eta, \xi, \xi'} A \bar{\rho}_k(\eta) \mathbf{A}_k(\eta) \\ &\quad ((\eta - \xi)l - \xi'(k - l)) \hat{h}(\xi - \xi')_{M'} \\ &\quad \times \varphi_{\mathbf{N}}(l, \xi) \hat{\phi}_l(\xi')_{\mathbf{M}} \hat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta (\mathbf{1}_{|l| \geq 100|\xi'|} + \mathbf{1}_{|l| \leq 100|\xi'|}) d\xi d\xi' \\ &= \mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1;z} + \mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1;v}. \end{aligned}$$

As above, we always have

$$||l, \xi| - |k, \eta|| \leq |k - l, \eta - \xi| \leq \frac{6}{32}|l, \xi|.$$

Also on the support of the integrand, we have

$$\frac{M'}{2} \leq |\xi - \xi'| \leq \frac{3M'}{2} \quad \text{and} \quad \frac{M}{2} \leq |\xi'| \leq \frac{3M}{2}$$

This together with $\frac{1}{8}M \leq M' \leq 8M$ implies

$$||l, \xi| - |l, \xi'|| \leq |\xi - \xi'| \leq 24|\xi'| \leq \frac{24}{100}|l, \xi'|.$$

Thus it holds that

$$e^{\lambda|k, \eta|^s} \leq e^{\lambda|l, \xi|^s} e^{c\lambda|k-l, \eta-\xi|} \leq e^{\lambda|l, \xi'|^s + c\lambda|\xi - \xi'|^s + c\lambda|k-l, \eta-\xi|^s}.$$

Hence, applying (6.15), we get

$$\begin{aligned} |\mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1;z}| &\lesssim \frac{1}{\langle t \rangle^2} \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \mathbf{1}_{\mathbf{NR}} \partial_z^{-1} \Delta_L^2 \mathbf{A} P_{\neq} \phi_{\mathbf{N}} \right\|_2 \|\rho_{<\mathbf{N}/8}\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\quad \times \sum_{\mathbf{M}' \in \mathbf{D}} \frac{1}{\mathbf{M}'} \|\hat{h}_{\mathbf{M}'}\|_{\mathcal{G}^{\lambda,\beta;s}}. \end{aligned}$$

The term $\mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1;v}$ is easy to deal with, we have by using the fact that $2\beta \geq \sigma$

$$\begin{aligned} |\mathbf{R}_{1;\mathbf{N};\mathbf{HH}}^{\epsilon,1;v}| &\lesssim \sum_{\mathbf{M} \in \mathbf{D}} \sum_{\mathbf{M} \approx \mathbf{M}'} \sum_{k,l \neq 0} \int_{\eta,\xi,\xi'} \mathbf{A} |\bar{\rho}_k(\eta)| e^{\lambda|\xi-\xi'|^s} |\hat{h}(\xi - \xi')_{\mathbf{M}'}| \\ &\quad \times \varphi_{\mathbf{N}}(l, \xi) e^{\lambda|l,\xi'|^s} |\hat{\phi}_l(\xi')_{\mathbf{M}}| |k-l, \eta-\xi| e^{c\lambda|k-l, \eta-\xi|^s} |\hat{\rho}_{k-l}(\eta-\xi)_{<\mathbf{N}/8}| \\ &\quad \times \mathbf{1}_{|l| \leq 100|\xi'|} d\eta d\xi d\xi' \\ &\lesssim \sum_{\mathbf{M} \in \mathbf{D}} \sum_{\mathbf{M} \approx \mathbf{M}'} \frac{1}{\mathbf{N}} \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 \|P_{\neq} \phi_{\mathbf{M}}\|_{\mathcal{G}^{\lambda,\beta;s}} \|h_{\mathbf{M}'}\|_{\mathcal{G}^{\lambda,\beta;s}} \|\rho\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \frac{1}{\mathbf{N}} \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 \|P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} \|h\|_{\mathcal{G}^{\lambda,\beta;s}} \|\rho\|_{\mathcal{G}^{\lambda,\beta;s}}. \end{aligned}$$

Consequently, collecting the above estimates, we deduce the result of Proposition 6.1.

6.1.3. The term $\mathbf{R}_{1;\mathbf{N}}^2$ On the support of the integrand it holds that

$$\frac{|k| + |\eta - \xi|}{\mathbf{N}/8} \leq \frac{3}{4} \quad \text{and} \quad \frac{\mathbf{N}}{2} \leq |\xi| \leq \frac{3\mathbf{N}}{2}$$

which implies that

$$|k| + |\eta - \xi| \leq \frac{3}{16} |\xi|$$

and $|k| \leq |\xi| \approx |\eta|$.

We obtain on the support of the integrand for $0 < c < 1$

$$\frac{e^{\lambda|k,\eta|^s}}{e^{\lambda|\xi|^s}} \lesssim e^{c\lambda|k,\eta-\xi|^s}, \quad \frac{\mathbf{J}_k(\eta)}{\mathbf{J}_0(\xi)} \lesssim \langle \xi \rangle e^{C\mu|k,\xi-\eta|^{\frac{1}{3}}}, \quad \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_0(\xi)} \lesssim e^{C\delta_L^{-1}|k,\xi-\eta|^{\frac{1}{3}}},$$

Thus we get that

$$|\mathbf{R}_{1;\mathbf{N}}^2| \lesssim \frac{1}{\langle t \rangle} \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 \langle t \rangle \|\mathbf{A}\partial_v g_{\mathbf{N}}\|_2 \|\rho\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \frac{\epsilon}{\langle t \rangle^2} \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2^2 + \epsilon \langle t \rangle^2 \|\mathbf{A}\partial_v^3 g_{\mathbf{N}}\|_2^2.$$

6.1.4. The term $\mathbf{R}_{1;\mathbf{N}}^3$ Now, we estimate the term $\mathbf{R}_{1;\mathbf{N}}^3$. We have by using the fact that $|k-l, \eta-\xi| \lesssim |l, \xi|$ (see (6.3a))

$$\begin{aligned} \mathbf{R}_{1;\mathbf{N}}^3 &\lesssim \sum_{k,l} \int_{\eta,\xi} \mathbf{A}_k(t, \eta) |\bar{\rho}_k(\eta)| |k-l, \eta-\xi| |\widehat{\mathbf{u}}_l(\xi)_{\mathbf{N}}| \\ &\quad \mathbf{A}_{k-l}(\eta-\xi) |\hat{\rho}_{k-l}(\eta-\xi)_{<\mathbf{N}/8}| d\eta d\xi \\ &\lesssim \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 \|\mathbf{u}_{\mathbf{N}}\|_{H^4} \|\mathbf{A}\rho\|_2 \lesssim \epsilon \|\mathbf{A}\rho_{\sim\mathbf{N}}\|_2 (\|\partial_v^3 g_{\mathbf{N}}\|_{H^4} + \|v' P_{\neq} \phi_{\mathbf{N}}\|_{H^4}). \end{aligned}$$

6.2. Transport term $T_{1;N}$

In this section, we deal with the transport term and prove the following proposition:

Proposition 6.2. *Under the bootstrap hypotheses, it holds that,*

$$\sum_{N \geq 8} T_{1;N} \lesssim \sqrt{\epsilon} CK_{\lambda, \rho}.$$

Decompose the difference:

$$\begin{aligned} A_k(\eta) - A_l(\xi) &= A_l(\xi) [e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} - 1] \\ &\quad + A_l(\xi) e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \left[\frac{J_k(\eta)}{J_l(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} \\ &\quad + A_l(\xi) e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} \\ &\quad + A_l(\xi) e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \left[\frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} - 1 \right] \frac{B_k(\eta)}{B_l(\xi)} \\ &\quad + A_l(\xi) e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \left[\frac{B_k(\eta)}{B_l(\xi)} - 1 \right]. \end{aligned}$$

Hence, we write, accordingly,

$$\begin{aligned} T_{1;N} &= i \sum_{k,l} \int_{\eta, \xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{\mathbf{u}}(k - l, \eta - \xi)_{<N/8} \cdot (l, \xi) (A_k(\eta) \\ &\quad - A_l(\xi)) \hat{\rho}_l(\xi)_{\mathbb{N}} d\xi d\eta \\ &= T_{1;N}^1 + T_{1;N}^2 + T_{1;N}^3 + T_{1;N}^4 + T_{1;N}^5, \end{aligned}$$

with

$$\begin{aligned} T_{1;N}^1 &= i \sum_{k,l} \int_{\eta, \xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{\mathbf{u}}(k - l, \eta - \xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_{\mathbb{N}} \\ &\quad \times [e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} - 1] d\xi d\eta \\ T_{1;N}^2 &= i \sum_{k,l} \int_{\eta, \xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{\mathbf{u}}(k - l, \eta - \xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_{\mathbb{N}} e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \\ &\quad \times \left[\frac{J_k(\eta)}{J_l(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta, \\ T_{1;N}^3 &= i \sum_{k,l} \int_{\eta, \xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{\mathbf{u}}(k - l, \eta - \xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_{\mathbb{N}} e^{\lambda|k, \eta|^s - \lambda|l, \xi|^s} \\ &\quad \times \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \end{aligned}$$

and

$$\begin{aligned}
 T_{1;N}^4 &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\hat{\rho}}(k, \eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\
 &\quad \times \left[\frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} - 1 \right] \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta. \\
 T_{1;N}^5 &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\hat{\rho}}(k, \eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\
 &\quad \times \left[\frac{B_k(\eta)}{B_l(\xi)} - 1 \right] d\xi d\eta.
 \end{aligned}$$

6.2.1. Term $T_{1;N}^1$ We get that

$$\begin{aligned}
 |T_{1;N}^1| &\lesssim \sum_{k,l} \int_{\eta,\xi} |A_k(\eta) \bar{\hat{\rho}}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |(l, \xi)| |A_l(\xi) \hat{\rho}_l(\xi)_N| \\
 &\quad \times \lambda \left| |k, \eta|^s - |l, \xi|^s \right| e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} d\xi d\eta \\
 &\lesssim \lambda \sum_{k,l} \int_{\eta,\xi} |A_k(\eta) \bar{\hat{\rho}}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l, \xi| |A_l(\xi) \hat{\rho}_l(\xi)_N| \\
 &\quad \times \frac{||k, \eta| - |l, \xi||}{|k, \eta|^{1-s} + |l, \xi|^{1-s}} e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} d\xi d\eta
 \end{aligned}$$

On the support of the integrand, we have

$$\begin{aligned}
 ||k, \eta| - |l, \xi|| &\leq |k-l, \eta-\xi| \leq \frac{6}{32} |l, \xi| \\
 \frac{26}{32} |l, \xi| &\leq |k, \eta| \leq \frac{38}{32} |l, \xi|.
 \end{aligned} \tag{6.16}$$

Consequently, we obtain

$$\begin{aligned}
 |T_{1;N}^1| &\lesssim \lambda \sum_{k,l} \int_{\eta,\xi} |A_k(\eta) \bar{\hat{\rho}}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |k, \eta|^{\frac{s}{2}} |l, \xi|^{\frac{s}{2}} |A_l(\xi) \hat{\rho}_l(\xi)_N| \\
 &\quad \times e^{c\lambda|k-l, \eta-\xi|^s} d\xi d\eta \\
 &\lesssim \lambda ||\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 ||\nabla|^{\frac{s}{2}} A \rho_N \|_2 \left(\|v' \nabla P_{\neq \phi}\|_{\mathcal{G}^{\lambda, \beta; s}} + \|g\|_{\mathcal{G}_1^{\lambda, \beta; s}} \right) \\
 &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{2-\epsilon_1/2}} ||\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 ||\nabla|^{\frac{s}{2}} A \rho_N \|_2.
 \end{aligned}$$

6.2.2. Term $T_{1;N}^2$ Now, we treat the term $T_{1;N}^2$. Inspired by the estimate in Lemma 3.9, we split $T_{1;N}^2$ as

$$\begin{aligned} T_{1;N}^2 &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\rho}(k, \eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\ &\quad \times \left[\chi^S + \chi^L \right] \left[\frac{J_k(\eta)}{J_l(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\ &= T_{1;N}^{2;S} + T_{1;N}^{2;L}, \end{aligned}$$

where $\chi^S = \mathbf{1}_{t \leq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}}$ and $\chi^L = 1 - \chi^S$.

We now estimate $T_{1;N}^{2;S}$, where Lemma 3.9 plays a crucial role in absorbing $\frac{2}{3}$ -derivatives. Hence, by (6.16) and Lemma 3.5 together with the fact that $\langle k, \eta \rangle \approx \langle l, \xi \rangle$ on the support of the integrand, we obtain

$$\begin{aligned} |T_{1;N}^{2;S}| &\lesssim \sum_{k,l} \int_{\eta,\xi} \chi^S A_k(\eta) |\bar{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l, \xi|^{\frac{1}{3}} \\ &\quad A_l(\xi) |\hat{\rho}_l(\xi)_N| e^{c\lambda|k-l, \eta-\xi|^s} \times \langle k-l, \xi-\eta \rangle e^{C|k-l, \xi-\eta|^{\frac{1}{3}}} d\xi d\eta \\ &\lesssim \sum_{k,l} \int_{\eta,\xi} \chi^S A_k(\eta) |\bar{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| (1 + |l, \xi|^{\frac{s}{2}} |k, \eta|^{\frac{s}{2}}) \\ &\quad \times A_l(\xi) |\hat{\rho}_l(\xi)_N| e^{\lambda|k-l, \eta-\xi|^s} d\xi d\eta \\ &\lesssim \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2 \left(\|v' \nabla P_{\neq} \phi\|_{\mathcal{G}^{\lambda, \beta; s}} + \|g\|_{\mathcal{G}_1^{\lambda, \beta; s}} \right) \\ &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{2-\epsilon_1/2}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2. \end{aligned}$$

Next, we estimate $T_{1;N}^{2;L}$ which corresponds to $t > \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}$. We rewrite $T_{1;N}^{2;L}$ as

$$\begin{aligned} T_{1;N}^{2;L} &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\rho}(k, \eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\ &\quad \times \chi^L (\mathbf{1}_{|l| \leq 100|\xi|} + \mathbf{1}_{|l| \geq 100|\xi|}) \left[\frac{J_k(\eta)}{J_l(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\ &= T_{1;N}^{2;L;v} + T_{1;N}^{2;L;z}. \end{aligned}$$

Let us now estimate $T_{1;N}^{2;L;z}$. Let us first see that on the support of the integrand, we have

$$|\eta| \leq |\xi| + \frac{6}{32} (|l| + |\xi|) \leq \frac{319}{1600} |l|$$

Hence, we have

$$\left| \frac{J_k(\eta)}{J_l(\xi)} - 1 \right| \lesssim \frac{1}{l^{\frac{2}{3}}} + \frac{|k-l|}{|k|^{\frac{2}{3}} + |l|^{\frac{2}{3}}} e^{C\mu|k-l|^{\frac{1}{3}}},$$

Since $|l, \xi| \lesssim |l|$ and also from (6.16), we have on the support of the integrand $\frac{1297}{1600}|l| \leq |k| \leq \frac{1903}{1600}|l|$. Hence, we obtain

$$\begin{aligned}
 |T_{1;N}^{2;L,z}| &\lesssim \sum_{k,l} \int_{\eta,\xi} A_k(\eta) |\tilde{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l|^{\frac{1}{3}} A_l(\xi) |\hat{\rho}_l(\xi)_N| e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\
 &\quad \times \chi^L \mathbf{1}_{|l| \geq 100|\xi|} \langle k-l \rangle e^{C|k-l|^{\frac{1}{3}}} e^{C|k-l,\xi-\eta|^{\frac{1}{3}}} d\xi d\eta \\
 &\lesssim \sum_{k,l} \int_{\eta,\xi} A_k(\eta) |k|^{\frac{s}{2}} |\tilde{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| \chi^L \mathbf{1}_{|l| \geq 100|\xi|} \\
 &\quad \times |l|^{\frac{s}{2}} A_l(\xi) |\hat{\rho}_l(\xi)_N| e^{\lambda|k-l,\eta-\xi|^s} d\xi d\eta \\
 &\lesssim \|\nabla|^{\frac{s}{2}} A_{\rho \sim N}\|_2 \|\nabla|^{\frac{s}{2}} A_{\rho_N}\|_2 \left(\|v' \nabla P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} + \|g\|_{\mathcal{G}_1^{\lambda,\beta;s}} \right) \\
 &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{2-\epsilon_1/2}} \|\nabla|^{\frac{s}{2}} A_{\rho \sim N}\|_2 \|\nabla|^{\frac{s}{2}} A_{\rho_N}\|_2.
 \end{aligned}$$

Now to estimate $T_{1;N}^{2;L,v}$ and in order to use the decay rate of the nonzero mode, we separate the zero mode and the nonzero mode as follows:

$$\begin{aligned}
 T_{1;N}^{2;L,v} &= i \sum_{k \neq l} \int_{\eta,\xi} A_k(\eta) |\tilde{\rho}(k, \eta)| |v' \widehat{\nabla_{z,v}^\perp P_{\neq} \phi}(k-l, \eta-\xi)_{<N/8}| \\
 &\quad |l, \xi| A_l(\xi) |\hat{\rho}_N(\xi)| e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\
 &\quad \times \chi^L \mathbf{1}_{|l| \leq 100|\xi|} \left[\frac{J_k(\eta)}{J_l(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\
 &\quad + i \sum_k \int_{\eta,\xi} A_k(\eta) |\tilde{\rho}(k, \eta)| |g(\eta-\xi)_{<N/8}| |\xi| A_k(\xi) |\hat{\rho}_k(\xi)_N| e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\
 &\quad \times \chi^L \mathbf{1}_{|k| \leq 100|\xi|} \left[\frac{J_k(\eta)}{J_k(\xi)} - 1 \right] \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_k(\xi)} d\xi d\eta \\
 &= T_{1;N}^{2;L,v,\neq} + T_{1;N}^{2;L,v,0}.
 \end{aligned}$$

Now, we estimate $T_{1;N}^{2;L,v,\neq}$. By the argument in the proof of the reaction term, we have

$$\frac{J_k(\eta)}{J_l(\xi)} \lesssim |\xi| e^{C|k-l,\xi-\eta|^{\frac{1}{3}}}. \quad (6.17)$$

We need to pay decay in time in order to gain regularity. We have on the support of the integrand $|l, \xi| \lesssim |\xi| \lesssim t^{\frac{3}{2}}$ which implies, by using (6.16) that

$$|l, \xi|^2 \lesssim |l, \xi|^{\frac{s}{2}} |k, \eta|^{\frac{s}{2}} t^{\frac{3}{2}(2-s)}. \quad (6.18)$$

Hence, we have by making use of (3.18), (6.16) (6.17) and (6.18),

$$\begin{aligned}
|T_{1;N}^{2;L;v,\neq}| &\lesssim \sum_{k,l} \int_{\eta,\xi} A_k(\eta) |\bar{\rho}(k, \eta)| |v' \widehat{\nabla_{z,v}^\perp} P_{\neq} \phi(k-l, \eta-\xi)_{<N/8}| |l, \xi| A_l(\xi) |\hat{\rho}_l(\xi)_N| \\
&\quad \times |\xi| \chi^L \mathbf{1}_{|l| \leq 100|\xi|} \mathbf{1}_{k \neq l} \langle k-l, \xi-\eta \rangle e^{C\delta_L^{-1}|k-l, \xi-\eta|^{\frac{1}{3}}} e^{c\lambda|k-l, \eta-\xi|^s} d\xi d\eta \\
&\lesssim \sum_{k,l} \int_{\eta,\xi} t^{3-\frac{3}{2}s} |k, \eta|^{\frac{s}{2}} A_k(\eta) |\bar{\rho}(k, \eta)| |v' \widehat{\nabla_{z,v}^\perp} P_{\neq} \phi(k-l, \eta-\xi)_{<N/8}| \\
&\quad \times |l, \xi|^{\frac{s}{2}} A_l(\xi) |\hat{\rho}_N(\xi)| e^{\lambda|k-l, \eta-\xi|^s} d\xi d\eta \\
&\lesssim t^{3-\frac{3}{2}s} \|v' \nabla_{z,v}^\perp P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} \| |\nabla|^{\frac{s}{2}} A_{\rho \sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A_{\rho_N} \|_2 \\
&\lesssim \frac{\epsilon}{\langle t \rangle^{\frac{3}{2}s+1}} \| |\nabla|^{\frac{s}{2}} A_{\rho \sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A_{\rho_N} \|_2.
\end{aligned}$$

Our next goal is to estimate $T_{1;N}^{2;L;v,0}$. By the argument in the reaction term, we have for $t \notin I_{k,\eta}$ or $t \in I_{k,\eta} \cap I_{k,\xi}$,

$$\frac{J_k(\eta)}{J_k(\xi)} \lesssim e^{C\mu|\xi-\eta|^{\frac{1}{3}}},$$

and for $t \in I_{k,\eta}$ and $t \notin I_{k,\xi}$,

$$\begin{aligned}
\frac{J_k(\eta)}{J_k(\xi)} &\lesssim \frac{|\eta|}{k^3(1+|t-\frac{\eta}{k}|)} e^{C\mu|\xi-\eta|^{\frac{1}{3}}} \lesssim \frac{\xi \langle \xi - \eta \rangle}{k^3(1+|t-\frac{\xi}{k}|)} e^{C\mu|\xi-\eta|^{\frac{1}{3}}} \\
&\lesssim \langle \xi - \eta \rangle e^{C\mu|\xi-\eta|^{\frac{1}{3}}}.
\end{aligned}$$

By making use of (6.16), (3.18), we obtain by using the fact that on the support of the integrand, we have $|\xi| \lesssim t^{\frac{3}{2}}$ and hence, $|\xi| \leq |\xi|^{\frac{s}{2}} |\eta|^{\frac{s}{2}} t^{\frac{3}{2}(1-s)}$ and

$$\begin{aligned}
|T_{1;N}^{2;L;v,0}| &\lesssim \sum_k \int_{\eta,\xi} A_k(\eta) |\bar{\rho}(k, \eta)| |\hat{g}(\eta-\xi)_{<N/8}| |\xi| A_k(\xi) |\hat{\rho}_k(\xi)_N| e^{\lambda|k, \eta|^s - \lambda|k, \xi|^s} \\
&\quad \times \chi^L \mathbf{1}_{|k| \leq 100|\xi|} \left| \frac{J_k(\eta)}{J_k(\xi)} - 1 \right| \frac{\mathcal{M}_k(\eta) \langle k, \eta \rangle^\sigma}{\mathcal{M}_k(\xi) \langle k, \xi \rangle^\sigma} d\xi d\eta \\
&\lesssim \sum_k \int_{\eta,\xi} t^{\frac{3}{2}(1-s)} A_k(\eta) |k, \eta|^{\frac{s}{2}} |\bar{\rho}(k, \eta)| |\hat{g}(\eta-\xi)_{<N/8}| |\xi|^{\frac{s}{2}} \\
&\quad A_k(\xi) |\hat{\rho}_k(\xi)_N| e^{\lambda|\eta-\xi|^s} \chi^L d\xi d\eta \\
&\lesssim t^{\frac{3}{2}-\frac{3}{2}s} \|g\|_{\mathcal{G}_1^{\lambda,\beta;s}} \| |\nabla|^{\frac{s}{2}} A_{\rho \sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A_{\rho_N} \|_2 \\
&\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{\frac{3}{2}s+\frac{1}{2}-\frac{\epsilon_1}{2}}} \| |\nabla|^{\frac{s}{2}} A_{\rho \sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A_{\rho_N} \|_2.
\end{aligned}$$

6.2.3. Term $T_{1;N}^3$ Now, we turn to the term $T_{1;N}^3$. We have

$$\begin{aligned} T_{1;N}^3 &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\rho}_k(\eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \\ &\quad \times (\tilde{\chi}^S + \tilde{\chi}^L) \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\ &= T_{1;N}^{3;S} + T_{1;N}^{3;L}, \end{aligned}$$

where $\tilde{\chi}^S = \mathbf{1}_{t \leq \frac{1}{2} \min\{|\xi|^{\frac{1}{3}}, |\eta|^{\frac{1}{3}}\}}$ and $\tilde{\chi}^L = 1 - \tilde{\chi}^S$.

Following the same steps as in the estimate of the term $T_{1;N}^{2;S}$, we use Lemma 3.7 to gain $\frac{2}{3}$ -derivatives. Indeed, we have by using Lemma 3.7 and (6.16),

$$\begin{aligned} |T_{1;N}^{3;S}| &\lesssim \sum_{k,l} \int_{\eta,\xi} A_k(\eta) |\bar{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l, \xi|^{\frac{1}{3}} A_l(\xi) |\hat{\rho}_N(l, \xi)| \\ &\quad \times e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \tilde{\chi}^S \langle k-l, \xi-\eta \rangle e^{C\delta_L^{-1}|k-l, \xi-\eta|^{\frac{1}{3}}} d\xi d\eta \\ &\lesssim \sum_{k,l} \int_{\eta,\xi} A_k(\eta) |k, \eta|^{\frac{s}{2}} |\bar{\rho}(k, \eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l, \xi|^{\frac{s}{2}} A_l(\xi) |\hat{\rho}_N(l, \xi)| \\ &\quad \times e^{\lambda|k-l, \eta-\xi|^s} \tilde{\chi}^S d\xi d\eta \\ &\lesssim \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2 \left(\|v' \nabla P_{\neq} \phi\|_{\mathcal{G}^{\lambda, \beta; s}} + \|g\|_{\mathcal{G}_1^{\lambda, \beta; s}} \right) \\ &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{2-\epsilon_1/2}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2. \end{aligned}$$

Next, we estimate $T_{1;N}^{3;L}$, we have as in the estimate of $T_{1;N}^{2;L}$

$$\begin{aligned} T_{1;N}^{3;L} &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\rho}_k(\eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N \\ &\quad \times e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \tilde{\chi}^L (\mathbf{1}_{k \neq l} + \mathbf{1}_{k=l}) \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\ &= T_{1;N}^{3;L, \neq} + T_{1;N}^{3;L, 0}. \end{aligned}$$

Let us start by estimating $T_{1;N}^{3;L, \neq}$. We have

$$\begin{aligned} T_{1;N}^{3;L, \neq} &= i \sum_{k,l} \int_{\eta,\xi} A_k(\eta) \bar{\rho}_k(\eta) v' \widehat{\nabla_{z,v}^\perp P_{\neq} \phi}(k-l, \eta-\xi)_{<N/8} \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N \\ &\quad \times e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \tilde{\chi}^L (\mathbf{1}_{|l| \leq 100|\xi|} + \mathbf{1}_{|l| \geq 100|\xi|}) \\ &\quad \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_l(\xi)} d\xi d\eta \\ &= T_{1;N}^{3;L, \neq, v} + T_{1;N}^{3;L, \neq, z}. \end{aligned}$$

Appling (3.18), together with the fact that on the support of the integrand, it holds that $|l, \xi| \lesssim |\xi| \lesssim t^3$ which implies $|l, \xi| \leq |l, \xi|^{\frac{1}{2}} |k, \eta|^{\frac{1}{2}} t^{3(1-s)}$ we get as in the above estimates

$$\begin{aligned} |T_{1;N}^{3;L,\neq,v}| &\lesssim t^{3-3s} \|v' \nabla_{z,v}^\perp P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2 \\ &\lesssim \frac{\epsilon}{\langle t \rangle^{3s+1}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2. \end{aligned}$$

Next we estimate $T_{1;N}^{3;L,\neq,z}$. We have

$$\left| \frac{\mathcal{M}_k(\eta)}{\mathcal{M}_l(\xi)} - 1 \right| \lesssim \frac{1}{|l|^{\frac{2}{3}}} + \frac{\langle k-l, \eta-\xi \rangle}{|k|^{\frac{2}{3}}} e^{C\delta_L^{-1}|k-l, \eta-\xi|^{\frac{1}{3}}}.$$

Hence, we obtain

$$\begin{aligned} T_{1;N}^{3,\neq;L,z} &\lesssim \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2 \|v' \nabla_{z,v}^\perp P_{\neq} \phi\|_{\mathcal{G}^{\lambda,\beta;s}} \\ &\lesssim \frac{\epsilon}{\langle t \rangle^4} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2. \end{aligned}$$

Let us now turn to the zero mode term $T_{1;N}^{3;L,0}$. We write We write

$$\begin{aligned} T_{1;N}^{3;L,0} &= i \sum_k \int_{\eta,\xi} A_k(\eta) \bar{\rho}_k(\eta) \hat{g}(\eta - \xi)_{<N/8} A_l(\xi) \hat{\rho}_k(\xi)_{\mathcal{N}} e^{\lambda|k,\eta|^s - \lambda|k,\xi|^s} \\ &\quad \times (\chi_0^M + \chi_0^*) \left[\frac{\mathcal{M}_k(\eta)}{\mathcal{M}_k(\xi)} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma} \frac{B_k(\eta)}{B_k(\xi)} d\xi d\eta \\ &= T_{1;N}^{3;L,0;M} + T_{1;N}^{3;L,0;*}, \end{aligned}$$

where

$$\begin{aligned} \chi_0^M &= \tilde{\chi}^L \chi^S = \mathbf{1}_{\frac{1}{2} \min\{|\xi|^{\frac{1}{3}}, |\eta|^{\frac{1}{3}}\} \leq t \leq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}} \quad \text{and} \\ \chi_0^* &= \tilde{\chi}^L \chi^L = \mathbf{1}_{t \geq \frac{1}{2} \min\{|\xi|^{\frac{2}{3}}, |\eta|^{\frac{2}{3}}\}} \end{aligned}$$

We first estimate the term $T_{1;N}^{3;L,0;M}$. On the support of the integrand, we have

$$|\xi|^{\frac{2}{3}} \lesssim |\xi|^{\frac{s}{2}} |\eta|^{\frac{s}{2}} |\xi|^{\frac{2}{3}-s} \lesssim |\xi|^{\frac{s}{2}} |\eta|^{\frac{s}{2}} t^{2-3s}.$$

Hence, we have by using Lemma 3.8 and (6.16),

$$\begin{aligned} |T_{1;N}^{3;L,0;M}| &\lesssim \sum_k \int_{\eta,\xi} t^{2-3s} A_k(\eta) |\eta|^{\frac{s}{2}} |\bar{\rho}_k(\eta)| |\hat{g}(\eta - \xi)_{<N/8}| |\xi|^{\frac{s}{2}} \\ &\quad A_l(\xi) |\hat{\rho}_k(\xi)_{\mathcal{N}}| e^{\lambda|\eta-\xi|^s} \chi_0^M d\xi d\eta \\ &\lesssim t^{2-3s} \|g\|_{\mathcal{G}_1^{\lambda,s;\beta}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2 \\ &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{3s-\frac{\epsilon}{2}}} \| |\nabla|^{\frac{s}{2}} A \rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A \rho_N \|_2. \end{aligned}$$

Now, for the term $T_{1;N}^{3,L,0;*}$ on the support of the integrand, we have $|\xi| \lesssim |\xi|^{\frac{s}{2}} |\eta|^{\frac{s}{2}} t^{\frac{3}{2}(1-s)}$. Hence, as we did above, we get the following estimate

$$\begin{aligned} T_{1;N}^{3,L,0;*} &\lesssim t^{\frac{3}{2}-\frac{3}{2}s} \|g\|_{\mathcal{G}_1^{\lambda,s;\beta}} \| |\nabla|^{\frac{s}{2}} A\rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A\rho_N \|_2 \\ &\lesssim \frac{\sqrt{\epsilon}}{\langle t \rangle^{\frac{1}{2}+\frac{3}{2}s-\frac{\epsilon_1}{2}}} \| |\nabla|^{\frac{s}{2}} A\rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A\rho_N \|_2. \end{aligned}$$

Note here we need $\frac{1}{2} + \frac{3}{2}s - \frac{\epsilon_1}{2} \geq \tilde{q}$.

6.2.4. Term $T_{1;N}^4$ To estimate $T_{1;N}^4$, we get by (6.16) that

$$\left| \frac{\langle k, \eta \rangle^\sigma}{\langle l, \xi \rangle^\sigma} - 1 \right| \lesssim \frac{|k, \eta| - |l, \xi|}{\langle l, \xi \rangle} \lesssim \frac{|k - l, \eta - \xi|}{\langle l, \xi \rangle}.$$

Consequently, using the above estimate, we obtain as above

$$\begin{aligned} |T_{1;N}^4| &\lesssim \sum_{k,l} \int_{\eta,\xi} |A_k(\eta)| |\tilde{\rho}_k(\eta)| |\hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8}| |l, \xi| |A_l(\xi)| \hat{\rho}_l(\xi)_N \\ &\quad \times \frac{|k-l, \eta-\xi|}{\langle l, \xi \rangle} e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} d\xi d\eta \\ &\lesssim \|A\rho_{\sim N}\|_2 \|A\rho_N\|_2 \|\nabla \mathbf{u}\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \frac{\epsilon}{\langle t \rangle^2} \|A\rho_{\sim N}\|_2 \|A\rho_N\|_2. \end{aligned}$$

6.2.5. Term $T_{1;N}^5$ We get that

$$\begin{aligned} T_{1;N}^5 &\lesssim \sum_{k \neq l} \left| \int_{\eta,\xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{\mathbf{u}}(k-l, \eta-\xi)_{<N/8} \right. \\ &\quad \cdot (l, \xi) A_l(\xi) \hat{\rho}_l(\xi)_N e^{\lambda|k,\eta|^s - \lambda|l,\xi|^s} \times \left[\frac{B_k(\eta)}{B_l(\xi)} - 1 \right] d\xi d\eta \Big| \\ &\quad + \sum_k \left| \int_{\eta,\xi} A_k(\eta) \tilde{\rho}(k, \eta) \hat{g}(\eta-\xi)_{<N/8} \cdot \xi A_k(\xi) \hat{\rho}_k(\xi)_N e^{\lambda|k,\eta|^s - \lambda|k,\xi|^s} \right. \\ &\quad \times \left[\frac{B_k(\eta)}{B_k(\xi)} - 1 \right] d\xi d\eta \Big| \\ &= T_{1;N,\neq}^5 + T_{1;N,0}^5. \end{aligned}$$

Let us first treat the term $T_{1;N,\neq}^5$. On the support of the integrand, we have either $|\eta| \geq 3|k|$ or $|\xi| \geq 3|l|$ which implies that $|k, l| \lesssim |\eta, \xi|$, $\min\{|\xi|^{\frac{1}{3}}, |\eta|^{\frac{1}{3}}\} \lesssim t$ and $|\eta| \approx |\xi|$.

Thus we have $|l, \xi| \lesssim |\xi|^s |\xi|^{1-s} \lesssim |\xi|^{\frac{s}{2}} |\eta|^{\frac{s}{2}} t^{3-3s}$, and then

$$\begin{aligned} |T_{1;N,\neq}^5| &\lesssim t^{3-3s} \| |\nabla|^{\frac{s}{2}} A\rho_{\sim N} \|_2 \|v' P_{\neq} \phi\|_{\mathcal{G}^{\lambda,0;s}} \| |\nabla|^{\frac{s}{2}} A\rho_N \|_2 \\ &\lesssim \frac{\epsilon}{\langle t \rangle^{3s+1}} \| |\nabla|^{\frac{s}{2}} A\rho_{\sim N} \|_2 \| |\nabla|^{\frac{s}{2}} A\rho_N \|_2. \end{aligned}$$

Next we focus on $T_{1;N,0}^5$, we get that on the support of the integrand, $t \gtrsim |\eta|^{\frac{1}{3}} \approx |\xi|^{\frac{1}{3}}$, $|k| \gtrsim |\eta|^{\frac{1}{3}}$ and

$$\left| \frac{B_k(t, \eta)}{B_k(t, \xi)} - 1 \right| \lesssim \int_0^t \left| \frac{b(s, k, \eta)}{1 + (s - \frac{\eta}{k})^2} - \frac{b(s, k, \xi)}{1 + (s - \frac{\xi}{k})^2} \right| ds \lesssim \frac{|\xi - \eta|}{|\eta|^{\frac{1}{3}}}.$$

Thus we get that

$$\left| \frac{B_k(t, \eta)}{B_k(t, \xi)} - 1 \right| |\xi| \lesssim |\xi|^{\frac{2}{3}} |\eta - \xi| \lesssim |\xi|^{\frac{5}{3}} |\eta|^{\frac{5}{3}} t^{2-3s} |\eta - \xi|$$

and thus

$$\begin{aligned} |T_{1;N,0}^5| &\lesssim t^{2-3s} \|\partial_v g\|_{G^{\lambda,0;s}} \|\nabla^{\frac{s}{2}} A\rho_{\sim N}\|_2 \|\nabla^{\frac{s}{2}} A\rho_N\|_2 \\ &\lesssim \frac{\epsilon}{\langle t \rangle^{3s}} \|\nabla^{\frac{s}{2}} A\rho_{\sim N}\|_2 \|\nabla^{\frac{s}{2}} A\rho_N\|_2. \end{aligned}$$

6.3. Remainder

The remainder is easy. We give the result and omit the proof.

$$\mathcal{R} \lesssim \frac{\epsilon^3}{\langle t \rangle^2}.$$

7. Estimate of $E_d(t)$

In this section, we deal with the highest energy of the coordinate system $E_d(t)$. We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} t \|A\partial_{vv}h\|_2^2 &= \frac{1}{2} \|A\partial_{vv}h\|_2^2 + t \int A\partial_{vv}\partial_t h A\partial_{vv}h dv \\ &\quad - CK_{\lambda,h} - CK_{\Theta,h} - CK_{M,h} \\ &= -t \int A\partial_{vv}(g\partial_v h) A\partial_{vv}h dv - \int A\partial_{vv}f_0 A\partial_{vv}h dv \\ &\quad - CK_{\lambda,h} - CK_{\Theta,h} - CK_{M,h} - \frac{1}{2} \|A\partial_{vv}h\|_2^2. \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} \int A\partial_{vv}(g\partial_v h) A\partial_{vv}h dv &= -\frac{1}{2} \int \partial_v g |A\partial_{vv}h|^2 dv \\ &\quad + \int A\partial_{vv}h \left[A\partial_{vv}(g\partial_v h) - g\partial_v A\partial_{vv}h \right] dv \end{aligned} \quad (7.1)$$

The first term in (7.1) can be estimated as follows:

$$t \left| \int \partial_v g |A\partial_{vv}h|^2 dv \right| \lesssim \|\partial_v g\|_{H^2} E_d(t).$$

We write by using the paraproduct with respect to v :

$$\int A\partial_{vv}h[A\partial_{vv}(g\partial_vh) - g\partial_vA\partial_{vv}h]dv = \frac{1}{2\pi} \sum_{M \geq 8} T_M^v + \frac{1}{2\pi} \sum_{M \geq 8} R_M^v + \frac{1}{2\pi} \mathcal{R}^v$$

where

$$\begin{aligned} T_M^v &= 2\pi \int A\partial_{vv}h[A\partial_{vv}(g_{<M/8}\partial_vh_M) - g_{<M/8}\partial_vA\partial_{vv}h_M]dv \\ R_M^v &= 2\pi \int A\partial_{vv}h[A\partial_{vv}(g_M\partial_vh_{<M/8}) - g_M\partial_vA\partial_{vv}h_{<M/8}]dv \\ \mathcal{R}^v &= 2\pi \sum_{M \in \mathbb{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \int A\partial_{vv}h[A\partial_{vv}(g_M\partial_vh_{M'}) - g_M\partial_vA\partial_{vv}h_{M'}]dv. \end{aligned}$$

One may easily follow the argument in section 5 and get that

$$\begin{aligned} &\left| t \int A\partial_{vv}h[A\partial_{vv}(g\partial_vh) - g\partial_vA\partial_{vv}h]dv \right| \\ &\lesssim \sqrt{\epsilon}(CK_{\lambda,h} + CK_{w,h} + CK_{M,h}) + \frac{\epsilon^3}{\langle t \rangle^2} + \epsilon \langle t \rangle^2 \|A\partial_{vv}g\|_2^2 \end{aligned}$$

By (4.4), (4.6) and (4.7), we get that

$$\left| \int A\partial_{vv}f_0A\partial_{vv}h dv \right| \leq C_1 \|\partial_vAK_0\|_2^2 + C \frac{\epsilon^2}{\langle t \rangle^2} + \frac{1}{16} \|A\partial_{vv}h\|_2^2,$$

with $C_1 \geq 1$ independent of K_d .

Thus we obtain that

$$\begin{aligned} E_d(t) &+ \frac{1}{4} \int_1^t (CK_{\lambda,h} + CK_{\Theta,h} + CK_{M,h} + \|A\partial_{vv}h\|_2^2)(s)ds \\ &\leq E_d(1) + C\epsilon^2 + C_1 \int_1^t \|\partial_vAK_0\|_2^2(s)ds. \end{aligned}$$

Note that by choosing $K_d \geq 10C_1$, the last term on the right hand side will be absorbed by the dissipation term $\|\nabla_L AK\|_2^2$.

8. Estimate of NL_K^2

In this section, we treat NL_K^2 and prove Proposition 2.5.

8.1. Treatment of $\mathbf{K}_1 = \gamma^2 \int \mathbf{A} \mathbf{K} \mathbf{A} (\gamma^2 v' \nabla_{z,v}^\perp \partial_z P_{\neq} \phi \cdot \nabla_{z,v} \rho) dz dv$

We have

$$\begin{aligned} \mathbf{K}_1 &= \gamma^2 \int \mathbf{A} \mathbf{K} \mathbf{A} \nabla_{z,v}^\perp \partial_z P_{\neq} \phi \cdot \nabla_{z,v} \rho dz dv \\ &\quad \gamma^2 \int \mathbf{A} \mathbf{K} \mathbf{A} (h \nabla_{z,v}^\perp \partial_z P_{\neq} \phi \cdot \nabla_{z,v} \rho) dz dv \\ &= \mathbf{K}_1^1 + \mathbf{K}_1^\epsilon. \end{aligned}$$

As before, we use the paraproduct in (z, v) and write

$$\mathbf{K}_1^1 = \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{1;HL}^{1;N} + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{1;LH}^{1;N} + \frac{1}{2\pi} \sum_{N \in \mathbf{D}} \mathbf{K}_{1;HH}^{1;N}$$

where

$$\begin{aligned} \mathbf{K}_{1;HL}^{1;N} &= - \sum_{k,l \neq 0} i \int_{\eta, \xi} \mathbf{A} \tilde{K}_k(\eta) A_k(\eta) l(\eta l - \xi k) \hat{\phi}_l(\xi)_{N'} \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} d\eta d\xi \\ \mathbf{K}_{1;LH}^{1;N} &= - \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{A} \tilde{K}_k(\eta) A_k(\eta) (k-l) \widehat{\nabla^\perp \phi}_{k-l}(\eta - \xi)_{<N/8} \widehat{\rho}_l(\xi)_{N'} d\eta d\xi \\ \mathbf{K}_{1;HH}^{1;N} &= - \sum_{\frac{1}{8}N \leq N' \leq 8N} \sum_{k,l \neq 0} \int_{\eta, \xi} \mathbf{A} \tilde{K}_k(\eta) A_k(\eta) l \widehat{\nabla^\perp \phi}_l(\xi)_{N'} \\ &\quad (k-l, \eta - \xi) \widehat{\rho}_{k-l}(\eta - \xi)_{N'} d\eta d\xi. \end{aligned}$$

The high-low interaction similar to the reaction term in section 5. The main difference is there is one more derivative in z direction acting on ϕ . Luckily, we can take advantage of the dissipation term $\|\nabla_L \mathbf{A} \mathbf{K}\|_2$. That is

$$|l(\eta l - \xi k)| \leq (|k| + |k-l|)|l, \xi||k-l, \eta - \xi|.$$

Thus by following the argument of reaction term in section 5, we get that

$$\begin{aligned} |\mathbf{K}_{1;HL}^{1;N}| &\lesssim \epsilon \|\partial_z \mathbf{A} \mathbf{K}_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \mathbf{A} + \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{\mathbf{A}} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_N \right\|_2 \\ &\quad + \epsilon \|\mathbf{A}_0 \mathbf{K}_{\sim N}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \mathbf{A} + \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{\mathbf{A}} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_N \right\|_2, \end{aligned}$$

which implies that

$$\begin{aligned} \left| \sum_{N \geq 8} \mathbf{K}_{1;HL}^{1;N} \right| &\lesssim \epsilon \|\nabla_L \mathbf{A} \mathbf{K}\|_2^2 + \frac{\epsilon^3}{\langle t \rangle^{\frac{9}{2}}} \\ &\quad + \epsilon \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \mathbf{A} + \sqrt{\frac{\partial_t \mathbf{g}}{\mathbf{g}}} \tilde{\mathbf{A}} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{\mathbf{A}} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2. \end{aligned}$$

The low-high interaction is different from the transport term in section 5. Again we will take advantage of the dissipation term $\|\nabla_L AK\|_2$. That is

$$|(k-l)l, \xi| \lesssim |k-l||k, \eta| \lesssim |k-l||k, \eta - kt| \langle t \rangle$$

Thus we get that

$$|\mathbf{K}_{1;LH}^{1;N}| \lesssim \langle t \rangle^2 \|\phi_{\neq}\|_{\mathcal{G}^{\lambda,0;s}} \|\mathbf{A}\rho_N\|_2 \|\nabla_L AK_{\sim N}\|_2.$$

Here we use a rough estimate that

$$\frac{J_k(t, \eta)}{J_l(t, \xi)} \lesssim \langle t \rangle e^{C|k-l, \eta-\xi|^\frac{1}{3}}.$$

Thus we get that

$$\left| \sum_{N \geq 8} \mathbf{K}_{1;LH}^{1;N} \right| \lesssim \epsilon \|\nabla_L AK_{\sim N}\|_2^2 + \frac{\epsilon^3}{\langle t \rangle^4}.$$

The high-high interaction is easy to deal with, we get that

$$\left| \sum_{N \in \mathbf{D}} \mathbf{K}_{1;HH}^{1;N} \right| \lesssim \frac{\epsilon^3}{\langle t \rangle^4}.$$

The treatment of $\mathbf{K}_l^\epsilon$ is similar. We use the paraproduct twice and write

$$\mathbf{K}_l^\epsilon = \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{1;HL}^{\epsilon;N} + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{\epsilon;LH}^{1;N} + \frac{1}{2\pi} \sum_{N \in \mathbf{D}} \mathbf{K}_{1;HH}^{\epsilon;N}$$

where $\mathbf{K}_{1;HL}^{\epsilon;N} = \mathbf{K}_{1;HL,HL}^{\epsilon;N} + \mathbf{K}_{1;HL,LH}^{\epsilon;N} + \mathbf{K}_{1;HL,HH}^{\epsilon;N}$ and

$$\begin{aligned} \mathbf{K}_{1;HL,HL}^{\epsilon;N} &= - \sum_{M \geq 8} \sum_{k,l \neq 0} i \int_{\eta, \xi} A \tilde{K}_k(\eta) A_k(\eta) \hat{h}(\xi - \xi')_M \hat{\phi}_l(\xi')_{<M/8} \\ &\quad \times l(\eta l - \xi k) \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} \varphi_N(l, \xi) d\eta d\xi' d\xi \\ \mathbf{K}_{1;HL,LH}^{\epsilon;N} &= - \sum_{M \geq 8} \sum_{k,l \neq 0} i \int_{\eta, \xi} A \tilde{K}_k(\eta) A_k(\eta) \hat{h}(\xi - \xi')_{<M/8} \hat{\phi}_l(\xi')_M \\ &\quad \times l(\eta l - \xi k) \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} \varphi_N(l, \xi) d\eta d\xi' d\xi \\ \mathbf{K}_{1;HL,HH}^{\epsilon;N} &= - \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \sum_{k,l \neq 0} i \int_{\eta, \xi} A \tilde{K}_k(\eta) A_k(\eta) \hat{h}(\xi - \xi')_M \hat{\phi}_l(\xi')_{M'} \\ &\quad \times l(\eta l - \xi k) \widehat{\rho}_{k-l}(\eta - \xi)_{<N/8} \varphi_N(l, \xi) d\eta d\xi' d\xi \\ \mathbf{K}_{1;LH}^{\epsilon;N} &= - \sum_{k,l \neq 0} \int_{\eta, \xi} A \tilde{K}_k(\eta) A_k(\eta) (k-l) \widehat{\nabla^\perp \phi_{k-l}} \\ &\quad (\eta - \xi)_{<N/8} (l, \xi) \widehat{\rho}_l(\xi)_N d\eta d\xi \end{aligned}$$

$$\mathbf{K}_{1;HH}^{\epsilon;N} = - \sum_{\frac{1}{8}N \leq N' \leq 8N} \sum_{k,l \neq 0} \int_{\eta, \xi} A \bar{\hat{K}}_k(\eta) A_k(\eta) \widehat{lh \nabla^\perp \phi_l}(\xi)_{N'} \\ (k-l, \eta-\xi) \widehat{\rho}_{k-l}(\eta-\xi)_N d\eta d\xi.$$

The treatment of $\mathbf{K}_{1;LH}^{\epsilon;N}$ and $\mathbf{K}_{1;HH}^{\epsilon;N}$ are same as $\mathbf{K}_{1;LH}^{1;N}$ and $\mathbf{K}_{1;HH}^{1;N}$ and we get that

$$\left| \sum_{N \geq 8} \mathbf{K}_{1;LH}^{\epsilon;N} \right| + \left| \sum_{N \in \mathbf{D}} \mathbf{K}_{1;HH}^{\epsilon;N} \right| \lesssim \epsilon^2 \|\nabla_L A K_{\sim N}\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^4}.$$

The treatment of $\mathbf{K}_{1;HL,LH}^{\epsilon;N}$ is same as $\mathbf{K}_{1;HL}^{1;N}$ and we get that

$$\left| \sum_{N \geq 8} \mathbf{K}_{1;HL,LH}^{\epsilon;N} \right| \lesssim \epsilon^2 \|\nabla_L A K_{\sim N}\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^{\frac{9}{2}}} \\ + \epsilon^2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t g}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2^2.$$

One can follow the estimate of $R_{1;N;HL}^{\epsilon,1}$ and $R_{1;N;HH}^{\epsilon,1}$ in section 5 and get that

$$\left| \sum_{N \geq 8} \mathbf{K}_{1;HL,HL}^{\epsilon;N} \right| + \left| \sum_{N \geq 8} \mathbf{K}_{1;HL,HH}^{\epsilon;N} \right| \\ \lesssim \epsilon^2 \|\nabla_L A K\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^4} + \epsilon^2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2^2.$$

$$8.2. \text{ Treatment of } \mathbf{K}_2 = -2 \int A K A (h(\partial_v - t \partial_z) \partial_z f) dz dv$$

We have

$$\mathbf{K}_2 = \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{2;HL}^N + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{2;LH}^N + \frac{1}{2\pi} \sum_{N \in \mathbf{D}} \mathbf{K}_{2;HH}^N,$$

with

$$\mathbf{K}_{2;HL}^N = -2 \sum_{k \neq 0} \int A_k(t, \eta) \bar{\hat{K}}_k(t, \eta) A_k(t, \eta) \hat{h}(\xi)_N \widehat{\nabla_L \partial_z f_k}(\eta - \xi)_{<N/8} d\xi d\eta, \\ \mathbf{K}_{2;LH}^N = 2 \sum_{k \neq 0} \int A_k(t, \eta) \bar{\hat{K}}_k(t, \eta) A_k(t, \eta) \hat{h}(\eta - \xi)_{<N/8} (\xi - kt) k \hat{f}_k(\xi)_N d\xi d\eta,$$

$$\mathbf{K}_{2;\text{HH}}^{\text{N}} = 2 \sum_{\frac{1}{8}\text{N} \leq \text{N}' \leq 8\text{N}} \sum_{k \neq 0} \int A_k(t, \eta) \widehat{K}_k(t, \eta) A_k(t, \eta) \widehat{h}(\eta - \xi)_{\text{N}'} \\ (\xi - kt) k \widehat{f}_k(\xi)_{\text{N}} d\xi d\eta.$$

We first treat $\mathbf{K}_{2;\text{HL}}^{\text{N}}$. We have

$$\frac{J_k(t, \eta)}{J_0(t, \xi)} \lesssim \langle \xi \rangle e^{C|k, \eta - \xi|^{\frac{1}{3}}}$$

Thus we get that

$$\left| \sum_{\text{N} \geq 8} \mathbf{K}_{2;\text{HL}}^{\text{N}} \right| \lesssim \sum_{\text{N} \geq 8} \|A P_{\neq} \mathbf{K}_{\sim \text{N}}\|_2 \|\langle \partial_v \rangle A h_{\text{N}}\|_2 \|\nabla_L P_{\neq} f\|_{\mathcal{G}^{\lambda, 0; s}} \\ \lesssim \frac{\epsilon^3}{\langle t \rangle^2} + \epsilon \|\nabla_L A P_{\neq} \mathbf{K}\|_2.$$

By using the fact that $\xi - kt \lesssim |\eta - kt| + |\eta - \xi|$, and

$$\frac{k^2}{k^2 + (\xi - kt)^2} \lesssim \left\langle \frac{\xi}{kt} \right\rangle^{-1}, \quad \frac{A_k(\eta)}{A_k(\xi)} \lesssim e^{C|\eta - \xi|^{\frac{1}{3}}}$$

we have that

$$\left| \sum_{\text{N} \geq 8} \mathbf{K}_{2;\text{LH}}^{\text{N}} \right| \lesssim \sum_{\text{N} \geq 8} \|\nabla_L A P_{\neq} \mathbf{K}_{\sim \text{N}}\|_2 \|\langle \partial_v \rangle h\|_{\mathcal{G}^{\lambda, 0; s}} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f_{\text{N}} \right\|_2 \\ \lesssim \epsilon \|\nabla_L A P_{\neq} \mathbf{K}\|_2^2 + \frac{\epsilon}{\langle t \rangle^2} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2.$$

Similarly using the fact that $\xi - kt \lesssim |\eta - kt| + |\eta - \xi|$, we have

$$\left| \sum_{\text{N} \in \mathbf{D}} \mathbf{K}_{2;\text{HH}}^{\text{N}} \right| \lesssim \epsilon \|\nabla_L P_{\neq} A K\|_2^2 + \frac{\epsilon^3}{\langle t \rangle^2}.$$

8.3. Treatment of $\mathbf{K}_3 = 2 \int A K A (f_0 v' (\partial_v - t \partial_z) \partial_z f) dz dv$

The treatment of $\mathbf{K}_3$ is similar to $\mathbf{K}_2$. Here we show the result and omit the proof. We have

$$|\mathbf{K}_3| \lesssim \epsilon \|\nabla_L A P_{\neq} \mathbf{K}\|_2^2 + \frac{\epsilon}{\langle t \rangle^2} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 + \frac{\epsilon^3}{\langle t \rangle^2}.$$

$$8.4. \text{ Treatment of } \mathbf{K}_4 = -2 \int A K A ((v')^3 (\partial_v - t \partial_z) \nabla_L^\perp \phi_\# \cdot \nabla_L (\partial_v - t \partial_z) f) dz dv \\ \text{and } \mathbf{K}_5 = -2 \int A K A (v' \partial_z \nabla_L^\perp \phi_\# \cdot \nabla_L \partial_z f) dz dv$$

The treatment of $\mathbf{K}_4$ and $\mathbf{K}_5$ are similar. We will give the estimate of $\mathbf{K}_4$ and only show the result of $\mathbf{K}_5$. We have

$$\mathbf{K}_4 = \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{4;HL}^{1,N} + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{4;HL}^{\epsilon,N} + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{4;LH}^N + \frac{1}{2\pi} \sum_{N \in \mathbf{D}} \mathbf{K}_{4;HH}^N,$$

with $\mathbf{K}_{4;HL}^{\epsilon,N} = \mathbf{K}_{4;HL,HL}^{\epsilon,N} + \mathbf{K}_{4;HL,LH}^{\epsilon,N} + \mathbf{K}_{4;HL,HH}^{\epsilon,N}$ and

$$\begin{aligned} \mathbf{K}_{4;HL}^{1,N} &= i \sum_{k,l \neq 0} \int A_k(\eta) \overline{\widehat{K}}_k(\eta) A_k(\eta) (\xi - lt) (l\eta - k\xi) \widehat{\phi}_l(\xi)_N \\ &\quad \mathcal{F}[(\partial_v - t \partial_z) f]_{k-l}(\eta - \xi)_{<N/8} d\xi d\eta \\ \mathbf{K}_{4;HL,HL}^{\epsilon,N} &= \sum_{M \geq 8} i \sum_{k,l \neq 0} \int A_k(\eta) \overline{\widehat{K}}_k(\eta) A_k(\eta) G_3(\xi')_M \widehat{\phi}_l(\xi - \xi')_{<M/8} \varphi_N(l, \xi) \\ &\quad \times (\xi - \xi' - lt) (l\eta - k(\xi - \xi')) \mathcal{F}[(\partial_v - t \partial_z) f]_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta \\ \mathbf{K}_{4;HL,LH}^{\epsilon,N} &= \sum_{M \geq 8} i \sum_{k,l \neq 0} \int A_k(\eta) \overline{\widehat{K}}_k(\eta) A_k(\eta) G_3(\xi - \xi')_{<M/8} \widehat{\phi}_l(\xi')_M \varphi_N(l, \xi) \\ &\quad \times (\xi' - lt) (l\eta - k\xi') \mathcal{F}[(\partial_v - t \partial_z) f]_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta \\ \mathbf{K}_{4;HL,HH}^{\epsilon,N} &= \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} i \sum_{k,l \neq 0} \int A_k(\eta) \overline{\widehat{K}}_k(\eta) A_k(\eta) G_3(\xi - \xi')_{M'} \widehat{\phi}_l(\xi')_M \varphi_N(l, \xi) \\ &\quad \times (\xi' - lt) (l\eta - k\xi') \mathcal{F}[(\partial_v - t \partial_z) f]_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta \\ \mathbf{K}_{4;LH}^N &= \sum_{k \neq l} \int \overline{\widehat{A K}}_k(\eta) A_k(\eta) \mathcal{F}[G_3(\partial_v - t \partial_z) \nabla_L^\perp \phi]_{k-l}(\eta - \xi)_{<N/8} \\ &\quad \cdot (l, \xi - lt) (\xi - lt) \widehat{f}_l(\xi)_N d\xi d\eta \\ \mathbf{K}_{4;HH}^N &= \sum_{\frac{1}{8}N \leq N' \leq 8N} \sum_{k \neq l} \int A_k(\eta) \overline{\widehat{K}}_k(\eta) A_k(\eta) \mathcal{F}[G_3(\partial_v - t \partial_z) \nabla_L^\perp \phi]_{k-l} \\ &\quad (\eta - \xi)_{N'} \cdot (l, \xi - lt) (\xi - lt) \widehat{f}_l(\xi)_N d\xi d\eta. \end{aligned}$$

The key idea of estimate of $\mathbf{K}_{4;HL}^{1,N}$ is to move the derivative $\partial_v - t \partial_z$ on $\phi_\#$ to K and using the dissipation $\|\nabla_L A K\|_2$.

Indeed we have on the support of the integrand, $|\xi - lt| \leq |\eta - kt| + |\eta - \xi - (k - l)t| \lesssim |k, \eta - kt|(\eta - \xi - (k - l)t)$ which gives us that

$$|\mathbf{K}_{4;HL}^{1,N}| = \sum_{k,l \neq 0} \int A_k(\eta) |\overline{\widehat{\nabla_L K}}_k(\eta)| A_k(\eta) |l\eta - k\xi| |\widehat{\phi}_l(\xi)_N| \\ \mathcal{F}[\Delta_L f]_{k-l}(\eta - \xi)_{<N/8} d\xi d\eta.$$

Thus by following the argument of reaction term in section 5, we get that

$$|\mathbf{K}_{4;\text{HL}}^{1;\text{N}}| \lesssim \epsilon \|\nabla_L A K_{\sim \text{N}}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t \mathbf{g}}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_{\text{N}} \right\|_2,$$

where we use the fact that $\|\Delta_L f\|_{\mathcal{G}^{\lambda, \sigma-4; s}} \lesssim \epsilon$.

Similarly, by using the fact that $|\xi' - lt| \leq |\eta - kt| + |\eta - \xi - (k-l)t| + |\xi - \xi'|$, we can still move the derivative $\partial_v - t \partial_z$ on $\phi_{\neq}$ to K . Thus we get that

$$|\mathbf{K}_{4;\text{HL}, \text{LH}}^{\epsilon; \text{N}}| \lesssim \epsilon^2 \|\nabla_L A K_{\sim \text{N}}\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t \mathbf{g}}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi_{\text{N}} \right\|_2,$$

The treatment of $\mathbf{K}_{4;\text{HL}, \text{HL}}^{\epsilon; \text{N}}$ and $\mathbf{K}_{4;\text{HL}, \text{HH}}^{\epsilon; \text{N}}$ is similar to the $\mathbf{R}_{1;\text{N}; \text{HL}}^{\epsilon, 1}$ and $\mathbf{R}_{1;\text{N}; \text{HH}}^{\epsilon, 1}$. We get that

$$\begin{aligned} \sum_{\text{N} \geq 8} |\mathbf{K}_{4;\text{HL}, \text{HL}}^{\epsilon; \text{N}}| + |\mathbf{K}_{4;\text{HL}, \text{HH}}^{\epsilon; \text{N}}| &\lesssim \epsilon^2 \|\nabla_L A K\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^4} \\ &\quad + \epsilon^2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2. \end{aligned}$$

For the low-high interaction, by the fact that $|\xi - lt| \leq |\eta - kt| + |k - l, \eta - \xi|t$, we have

$$\begin{aligned} |\mathbf{K}_{4;\text{LH}}^{\text{N}}| &\leq \sum_{k \neq l, l \neq 0} \int |\widehat{AK}_k(\eta)| A_k(\eta) |\mathcal{F}[G_3(\partial_v - t \partial_z) \nabla_L^\perp \phi]_{k-l}(\eta - \xi)_{< \text{N}/8}| \\ &\quad |l| |\widehat{\partial_z^{-1} \Delta_L f_l}(\xi)|_{\text{N}} d\xi d\eta + \sum_{k \neq 0} \int |\widehat{AK}_k(\eta)| A_k(\eta) \\ &\quad |\mathcal{F}[G_3(\partial_v - t \partial_z) \nabla_L^\perp \phi]_k(\eta - \xi)_{< \text{N}/8}| |\widehat{\partial_{vv} f_0}(\xi)|_{\text{N}} d\xi d\eta \\ &= \mathbf{K}_{4;\text{LH}, 1}^{\text{N}} + \mathbf{K}_{4;\text{LH}, 2}^{\text{N}}. \end{aligned}$$

By using the fact that on the support of integrand,

$$\frac{|l|}{|l| + |\xi - lt|} \lesssim \left\langle \frac{\xi}{lt} \right\rangle^{-1},$$

and

By using the fact that on the support of integrand, $|l| \leq |k| + |k - l|$ and

$$\frac{J_k(\eta)}{J_l(\xi)} \lesssim \langle t \rangle e^{C|k, \eta - \xi|^{\frac{1}{3}}},$$

we have by (4.6) that,

$$\begin{aligned}
 \sum_{N \geq 8} |\mathbf{K}_{4;LH,1}^N| &\lesssim \sum_{N \geq 8} \frac{\epsilon}{\langle t \rangle} (\|\partial_z A K_{\sim N}\|_2 + \|A P_0 K_{\sim N}\|_2) \|A P_{\neq} \partial_z^{-1} \Delta_L f_N\|_2 \\
 &\lesssim \sum_{N \geq 8} \frac{\epsilon}{\langle t \rangle} (\|\partial_z A K_{\sim N}\|_2 + \|\partial_v A P_0 K_{\sim N}\|_2 \\
 &\quad + \|P_0 K_{\sim N}\|_{L^2}) \|A P_{\neq} \partial_z^{-1} \Delta_L f_N\|_2 \\
 &\lesssim \epsilon \|A \nabla_L K\|_2^2 + \frac{\epsilon}{\langle t \rangle^2} \|A P_{\neq} \partial_z^{-1} \Delta_L f\|_2^2.
 \end{aligned}$$

The treatment of $\mathbf{K}_{4;HH}^N$ is easy. We have

$$\sum_{N \in \mathbf{D}} |\mathbf{K}_{4;HH}^N| \lesssim \frac{\epsilon^3}{\langle t \rangle^4} + \epsilon \|\nabla_L A K\|_2^2$$

We conclude that

$$\begin{aligned}
 |\mathbf{K}_4| + |\mathbf{K}_5| &\lesssim \frac{\epsilon^3}{\langle t \rangle^4} + \epsilon \|\nabla_L A K\|_2^2 + \frac{\epsilon}{\langle t \rangle^2} \|A P_{\neq} \partial_z^{-1} \Delta_L f\|_2^2 \\
 &\quad + \epsilon \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{5}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t g}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq \phi} \right\|_2^2.
 \end{aligned}$$

8.5. Estimate of $\mathbf{K}_6$

The treatment of $\mathbf{K}_6$ is similar. Actually we have

$$\begin{aligned}
 \mathbf{K}_6 &= \int \partial_z A K A \left(v'' v' (\partial_v - t \partial_z) \phi_{\neq} (\partial_v - t \partial_z) f \right) dz dv \\
 &= \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{6;HL}^N + \frac{1}{2\pi} \sum_{N \geq 8} \mathbf{K}_{6;LH}^N + \frac{1}{2\pi} \sum_{N \in \mathbf{D}} \mathbf{K}_{6;HH}^N
 \end{aligned}$$

with $\mathbf{K}_{6;HL}^N = \mathbf{K}_{6;HL,LH}^N + \mathbf{K}_{6;HL,HL}^N + \mathbf{K}_{6;HL,HH}^N$, where

$$\begin{aligned}
 \mathbf{K}_{6;HL,LH}^N &= - \sum_{M \geq 8} \sum_{l \neq 0, k \neq 0} \int \overline{\partial_z A K_k(\eta)} A_k(\eta) \widehat{(v'' v')} (\xi - \xi')_{<M/8} (\xi' - tl) \hat{\phi}_l \\
 &\quad (\xi')_M \times \varphi_N(l, \xi) (\eta - \xi - (k-l)t) \hat{f}_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta \\
 \mathbf{K}_{6;HL,HL}^N &= - \sum_{M \geq 8} \sum_{l \neq 0, k \neq 0} \int \overline{\partial_z A K_k(\eta)} A_k(\eta) \widehat{(v'' v')} (\xi - \xi')_M (\xi' - tl) \\
 &\quad \hat{\phi}_l(\xi')_{<M/8} \times \varphi_N(l, \xi) (\eta - \xi - (k-l)t) \hat{f}_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta \\
 \mathbf{K}_{6;HL,HH}^N &= - \sum_{M \in \mathbf{D}} \sum_{\frac{1}{8}M \leq M' \leq 8M} \sum_{l \neq 0, k \neq 0} \int \overline{\partial_z A K_k(\eta)} A_k(\eta) \widehat{(v'' v')} (\xi - \xi')_M (\xi' - tl) \\
 &\quad \hat{\phi}_l(\xi')_{M'} \times \varphi_N(l, \xi) (\eta - \xi - (k-l)t) \hat{f}_{k-l}(\eta - \xi)_{<N/8} d\xi' d\xi d\eta
 \end{aligned}$$

$$\begin{aligned}\mathbf{K}_{6,\text{LH}}^{\text{N}} &= i \sum_{l \neq 0, k \neq 0} \int \overline{\partial_z A K_k}(\eta) A_k(\eta) \mathcal{F}\left((v''v')(\partial_v - t\partial_z)\phi\right)_{k-l} \\ &\quad (\eta - \xi)_{<N/8}(\xi - lt) \hat{f}_l(\xi)_{\text{N}} d\xi d\eta \\ \mathbf{K}_{6,\text{HH}}^{\text{N}} &= i \sum_{\frac{1}{8}N \leq N' \leq 8N} \sum_{l \neq 0, k \neq 0} \int \overline{\partial_z A K_k}(\eta) A_k(\eta) \mathcal{F}\left((v''v')(\partial_v - t\partial_z)\phi\right)_{k-l} \\ &\quad (\eta - \xi)_{<N/8}(\xi - lt) \hat{f}_l(\xi)_{\text{N}} d\xi d\eta.\end{aligned}$$

By using the fact that $\|v''v'\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \frac{\epsilon}{\langle t \rangle}$,

$$\|\nabla_L f\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \|\partial_v f_0\|_{L^2} + \|\partial_{vv} f_0\|_{\mathcal{G}^{\lambda,\beta;s}} + \|\nabla_L P_{\neq} f\|_{\mathcal{G}^{\lambda,\beta;s}} \lesssim \frac{\epsilon}{\langle t \rangle},$$

and $(\xi' - lt) \lesssim |l, \xi'| \langle t \rangle$, and by following the same argument of $\mathbf{K}_{1;\text{HL},\text{LH}}^{\epsilon;\text{N}}$, $\mathbf{K}_{1;\text{HL},\text{HL}}^{\epsilon;\text{N}}$ and $\mathbf{K}_{1;\text{HL},\text{HH}}^{\epsilon;\text{N}}$, we get that

$$\begin{aligned}\left| \sum_{N \geq 8} \mathbf{K}_{6;\text{HL}}^{\text{N}} \right| &\lesssim \epsilon^2 \|\nabla_L A K\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^2} \\ &\quad + \epsilon^2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \left(\frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} A + \sqrt{\frac{\partial_t g}{g}} \tilde{A} + \sqrt{\frac{\partial_t \Theta}{\Theta}} \tilde{A} \right) \partial_z^{-1} \Delta_L^2 P_{\neq} \phi \right\|_2^2.\end{aligned}$$

The estimates of $\mathbf{K}_{6,\text{LH}}^{\text{N}}$ and $\mathbf{K}_{6,\text{HH}}^{\text{N}}$ are obvious. We have

$$\sum_{N \geq 8} |\mathbf{K}_{6,\text{LH}}^{\text{N}}| + \sum_{N \in \mathbf{D}} |\mathbf{K}_{6,\text{HH}}^{\text{N}}| \lesssim \epsilon^2 \|\nabla_L A K\|_2^2 + \frac{\epsilon^2}{\langle t \rangle^2} \|\partial_z^{-1} \Delta_L P_{\neq} f\|_2^2 + \frac{\epsilon^4}{\langle t \rangle^2}.$$

9. Linear Terms

In this section, we deal with the linear terms Π_ρ , E , Π_K^1 and Π_K^2 .

9.1. Estimate of Π_ρ

We have

$$\begin{aligned}\Pi_\rho &= \frac{i}{2\pi} \sum_{k \neq 0} \int A_k(\eta) \bar{\rho}_k(\eta) A_k(\eta) k \hat{\phi}_k(\eta) d\eta \\ &= -\frac{1}{2\pi} \sum_{k \neq 0} \int A_k(\eta) \bar{\rho}_k(\eta) A_k(\eta) \frac{k^2}{(k^2 + (\eta - kt)^2)^2} \partial_z^{-1} \Delta_L^2 \hat{\phi}_k(\eta) d\eta.\end{aligned}$$

For $\eta > \frac{3}{2}kt$, we get that

$$\frac{k^2}{(k^2 + (\eta - kt)^2)^2} \lesssim \frac{k^2}{\eta^4} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^4}.$$

For $\eta < \frac{1}{2}kt$, we have $\left(\frac{\eta}{kt}\right)^{-1} \approx 1$ and

$$\frac{k^2}{(k^2 + (\eta - kt)^2)^2} \lesssim \frac{1}{k^2 t^4} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^4}.$$

Thus in both cases we have

$$|\Pi_\rho| \lesssim \frac{1}{\langle t \rangle^4} \|A\rho\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi \right\|_2.$$

Next we focus on the case $\frac{1}{2}kt \leq \eta \leq \frac{3}{2}kt$, in which case, it holds that $\frac{1}{3} \leq \left\langle \frac{\eta}{kt} \right\rangle^{-1} \leq 1$.

If $t \in \tilde{I}_{k,\eta}$, then

$$\frac{k^2}{(k^2 + (\eta - kt)^2)^2} \leq C \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{k^2(1 + |\frac{\eta}{k} - t|)^2} \leq C \delta_L \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{\partial_t g}{g}$$

which together with the fact that $A \leq 2\tilde{A}$ if $|k| \leq |\eta|$, gives us that

$$|\Pi_\rho| \leq C \delta_L \left\| \sqrt{\frac{\partial_t g}{g}} A \rho \right\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t g}{g}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi \right\|_2.$$

If $t \notin \tilde{I}_{k,\eta}$, then we consider the following two cases: 1. $t \leq t_{E(|\eta|^{\frac{2}{3}}), \eta}$; 2. $t \in [t_{E(|\eta|^{\frac{2}{3}}), \eta}, 2\eta]$ with $|t - \frac{\eta}{k}| \gtrsim \frac{\eta}{k^2}$. For the first case, using the fact that $kt \approx \eta$, we get that $k \gtrsim |\eta|^{\frac{2}{3}} \gtrsim t^2$ and then

$$\frac{k^2}{(k^2 + (\eta - kt)^2)^2} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^4}.$$

For the second case, we have $|\eta - kt| \gtrsim \frac{\eta}{k} \approx t$ and then

$$\frac{k^2}{(k^2 + (\eta - kt)^2)^2} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{k^2}{(k^2 + t^2)^2} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^2}.$$

Thus we conclude that

$$\begin{aligned} |\Pi_\rho| &\leq C \delta_L C K_{M,\rho} + \delta_L \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t g}{g}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi \right\|_2^2 \\ &\quad + \frac{C}{\langle t \rangle^2} \|A\rho\|_2 \left\| \left\langle \frac{\partial_v}{t\partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi \right\|_2. \end{aligned}$$

9.2. The diffusion term E

In this section, we study the diffusion term E. We have

$$\begin{aligned} E &= \int A K A (\Delta_L K) dz dv + \int A K A ((v')^2 - 1) (\partial_v - t \partial_z)^2 K dz dv \\ &\quad + \int A K A (v'' (\partial_v - t \partial_z) K) dz dv \\ &= -\|\nabla_L A K\|_2^2 + E_1 + E_2. \end{aligned}$$

Let $\widehat{G}_1(t, \eta) = ((v')^2 - 1)(t, \eta)$, then by the fact that $|\xi - kt| \leq |\eta - kt| + |\xi - \eta|$, we have

$$\begin{aligned} |E_1| &\lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_k(\eta) |\widehat{K}_k(\eta)| A_k(\eta) |\widehat{\partial_v G}_1(t, \eta - \xi)| |\xi - kt| |\widehat{K}_k(\xi)| d\xi d\eta \\ &\quad + \sum_{k \neq 0} \int_{\xi, \eta} |\eta - kt| A_k(\eta) |\widehat{K}_k(\eta)| A_k(\eta) |\widehat{G}_1(t, \eta - \xi)| |\xi - kt| |\widehat{K}_k(\xi)| d\xi d\eta \\ &\quad + \int_{\xi, \eta} A_0(\eta) |\widehat{K}_0(\eta)| A_0(\eta) |\widehat{\partial_v G}_1(t, \eta - \xi)| |\xi| |\widehat{K}_0(\xi)| d\xi d\eta \\ &\quad + \int_{\xi, \eta} |\eta| A_0(\eta) |\widehat{K}_0(\eta)| A_0(\eta) |\widehat{G}_1(t, \eta - \xi)| |\xi| |\widehat{K}_0(\xi)| d\xi d\eta \\ &\lesssim E_{1,1}^\neq + E_{1,2}^\neq + E_{1,1}^0 + E_{1,2}^0, \end{aligned}$$

and

$$\begin{aligned} |E_2| &\lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_k(\eta) |\widehat{K}_k(\eta)| A_k(\eta) |\widehat{\partial_v G}_1(t, \eta - \xi)| |\xi - kt| |\widehat{K}_k(\xi)| d\xi d\eta \\ &\quad + \int_{\xi, \eta} A_0(\eta) |\widehat{K}_0(\eta)| A_0(\eta) |\widehat{\partial_v G}_1(t, \eta - \xi)| |\xi| |\widehat{K}_0(\xi)| d\xi d\eta \lesssim E_{1,1}^\neq + E_{1,1}^0. \end{aligned}$$

We also have $v'' = \frac{1}{2} \partial_v [(v')^2 - 1] = \frac{1}{2} \partial_v G_1$, which gives us that $|E_2| \lesssim E_{1,1}$. For $k = 0$, we use the fact that the norm defined by A is an algebra when restricted to the zero mode and obtain that

$$\begin{aligned} |E_{1,1}^0| + |E_{1,2}^0| &\lesssim \|A \partial_v G_1\|_2 \|A \partial_v K_0\|_2 \|A K_0\|_2 + \|A G_1\|_2 \|A \partial_v K_0\|_2^2 \\ &\lesssim \|A \partial_v G_1\|_2 \left(\|A \partial_v K_0\|_2^2 + \|K_0\|_{L^2}^2 \right) \lesssim \epsilon \| \partial_v A K_0 \|_2^2 + \frac{\epsilon^3}{\langle t \rangle^2} \end{aligned}$$

By using the fact that

$$A_k(\eta) \lesssim \langle \eta - \xi \rangle A_0(\eta - \xi) A_k(\xi),$$

we get that

$$|E_{1,1}^\neq| + |E_{1,2}^\neq| \lesssim \|\nabla_L A P_{\neq} K\|_2 \|A \langle \partial_v \rangle^2 G_1\|_2 \|\nabla_L A P_{\neq} K\|_2 \lesssim \epsilon \|\nabla_L A P_{\neq} K\|_2^2.$$

Thus we get by taking ϵ small enough that

$$E \leq -\frac{7}{8} \|\nabla_L A K\|_2^2 + \frac{C \epsilon^3}{\langle t \rangle^2}.$$

9.3. Estimate of Π_K^1

One can easily follow the argument of Π_ρ and get that

$$\begin{aligned} |\Pi_K^1| &\leq \frac{C}{\langle t \rangle^2} \|\partial_z A K\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi \right\|_2 \\ &\quad + C \delta_L \|\partial_z A K\|_2 \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t g}{g}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi \right\|_2 \\ &\leq \frac{1}{16} \|\partial_z A K\|_2^2 + \frac{C}{\langle t \rangle^4} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \partial_z^{-1} \Delta_L^2 A P_{\neq} \phi \right\|_2^2 \\ &\quad + C \delta_L \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \sqrt{\frac{\partial_t g}{g}} \partial_z^{-1} \Delta_L^2 \tilde{A} P_{\neq} \phi \right\|_2^2. \end{aligned}$$

9.4. Estimate of Π_K^2

We get that

$$\begin{aligned} \Pi_K^2 &= 2 \int \nabla_L A K A \partial_z^2 \Delta_L^{-1} \partial_z^{-1} \Delta_L P_{\neq} f \, dz dv \\ &= \frac{1}{\pi} \sum_{k \neq 0} \int \overline{\nabla_L A K_k(t, \eta)} A_k(t, \eta) \frac{k^2}{k^2 + (\eta - kt)^2} \widehat{\partial_z^{-1} \Delta_L f_k(\eta)} d\eta. \end{aligned}$$

We get for $\eta > \frac{3}{2}kt$,

$$\frac{k^2}{k^2 + (\eta - kt)^2} \lesssim \frac{k^2}{\eta^2} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^2},$$

and for $\eta < \frac{1}{2}kt$, it holds that $\left\langle \frac{\eta}{kt} \right\rangle^{-1} \approx 1$ and

$$\frac{k^2}{k^2 + (\eta - kt)^2} \lesssim \frac{k^2}{k^2 + k^2 t^2} \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^2}.$$

Next we focus on the case $\frac{1}{2}kt \leq \eta \leq \frac{3}{2}kt$.

Let $M_0 \geq 10$ be large enough. For $t \leq M_0$, we have

$$\frac{k^2}{k^2 + (\eta - kt)^2} \lesssim_{M_0} \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{t^2}.$$

For $t \in [t_{E(|\eta|^{\frac{1}{3}}), \eta}, 2\eta]$, it holds that

$$\frac{k^2}{k^2 + (\eta - kt)^2} \leq C \sqrt{\delta_L} \left\langle \frac{\eta}{kt} \right\rangle^{-1} \sqrt{\frac{\partial_t g}{g}}.$$

For $t \in [M_0, t_{E(|\eta|^{\frac{2}{3}}), \eta}]$, we get that $t \leq 2|\eta|^{\frac{1}{3}}$ and then

$$\frac{k^2}{k^2 + (\eta - kt)^2} \lesssim 1 \lesssim \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{|\eta|^{\frac{5}{2}}}{t^{\frac{3s}{2}}} 1_{t \geq M_0}.$$

Note that here we obtain the small parameter from the fact that for $2\tilde{q} < 3s$,

$$t^{-2\tilde{q}} \leq M_0^{2\tilde{q}-3s} t^{-3s}.$$

Another way to obtain the small parameter is by assuming λ_0, λ' large enough while here we hope our results have no restriction on λ_0, λ' .

Now we turn to the case $t \in [t_{E(|\eta|^{\frac{2}{3}}), \eta}, t_{E(|\eta|^{\frac{1}{3}}), \eta}] \cap [20, +\infty]$ with $\frac{1}{2}kt \leq \eta \leq \frac{3}{2}kt$.

Thus we have

$$\frac{k^2}{k^2 + (\eta - kt)^2} \lesssim \sqrt{\delta_B} \left\langle \frac{\eta}{kt} \right\rangle^{-1} \frac{1}{1 + |t - \frac{\eta}{k}|} \lesssim \sqrt{\delta_B} \sqrt{\frac{b(t, k, \eta)}{1 + |t - \frac{\eta}{k}|^2}} \left\langle \frac{\eta}{kt} \right\rangle^{-1}.$$

Thus we conclude that

$$\begin{aligned} |\Pi_K^2| &\leq \frac{1}{16} \|\nabla_L A K\|_2^2 + C(M_0) \frac{1}{\langle t \rangle^4} \left\| \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ &\quad + C \left\| 1_{t \geq M_0} \frac{|\nabla|^{\frac{s}{2}}}{\langle t \rangle^{\frac{3s}{2}}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ &\quad + C \delta_L \left\| \sqrt{\frac{\partial_t g}{g}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} \tilde{A} \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2 \\ &\quad + C \delta_B \left\| \sqrt{\frac{b(t, \nabla) \partial_{zz}}{\Delta_L}} \left\langle \frac{\partial_v}{t \partial_z} \right\rangle^{-1} A \partial_z^{-1} \Delta_L P_{\neq} f \right\|_2^2. \end{aligned}$$

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Appendix A. Paraproduct Tools

In this section, we introduce the tools and notations that we should use in the Fourier analysis and paraproduct. We first introduce the dyadic partition of unity that we should use throughout the paper. Let $\varkappa(\xi)$ be a real radial bump function which satisfies $\varkappa(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varkappa(\xi) = 0$ for $|\xi| \geq 3/4$. We define $\varphi(\xi) = \varkappa(\xi/2) - \varkappa(\xi)$ supported on the annulus $\{\frac{1}{2} \leq |\xi| \leq \frac{3}{2}\}$. By construction, we have the following partition of unity:

$$1 = \varkappa(\xi) + \sum_{k \in \mathbb{Z}} \varphi(\xi/2^k) = \varkappa(\xi) + \sum_{M \in 2^{\mathbb{N}}} \varphi(M^{-1}\xi)$$

and we define the cut-off $\varphi_M = \varphi(M^{-1}\xi)$, each supported in the annulus $\frac{M}{2} \leq |\xi| \leq \frac{3M}{2}$

For $f \in L^2(\mathbb{R})$, we define

$$\begin{aligned} f_M &= \varphi_M(|\partial_v|)f, \\ f_{\frac{1}{2}} &= \varkappa(|\partial_v|)f, \\ f_{<M} &= f_{\frac{1}{2}} + \sum_{K \in 2^{\mathbb{N}}: K < M} f_{<K} \end{aligned}$$

Hence, we have the decomposition

$$f = f_{\frac{1}{2}} + \sum_{M \in 2^{\mathbb{N}}} f_M.$$

We also have the almost orthogonality property

$$\|f\|_2^2 \approx \sum_{M \in \mathbf{D}} \|f_M\|_2^2$$

and the approximate projection property

$$\|f_M\|_2 \approx \|(f_M)_M\|_2.$$

More generally if $f = \sum_k D_k$ for any D_k with $\frac{1}{C}2^k \subset \text{supp } D_k \subset C2^k$ it follows that

$$\|f\|_2^2 \approx_c \sum_k \|D_k\|_2^2$$

During much of the proof we are also working with Littlewood–Paley decompositions defined in the (z, v) variables, with the notation conventions being analogous. Our convention is to use N to denote Littlewood–Paley projections in (z, v) and M to denote projections only in the v direction.

For any $1 \leq p \leq q \leq \infty$, there exists a constant independent of M such that

$$\|\partial_x^\alpha f_M\|_q + \|\partial_x^\alpha f_{<M/8}\|_q \leq CM^{d(1/p-1/q)+|\alpha|} \|f\|_p.$$

Appendix B. Important Inequalities

If $|x - y| \leq |x|/K$, then it holds that

$$|x^s - y^s| \leq \frac{s}{(K-1)^{s-1}} |x - y|^s, \quad 0 < s < 1. \quad (\text{B.1})$$

In many occasions, we use the following inequality, which is a result of (B.1):

$$e^{\lambda|x|^s} \leq e^{\lambda|y|^s} e^{c\lambda|x-y|^s}$$

for $|x - y| < |x|/K$, with $K > 1$, $s \in (0, 1)$ and $c = c(s) \in (0, 1)$.

If $|y| \leq |x| \leq K|y|$, then it holds that

$$|x + y|^s \leq \left(\frac{K}{K+1}\right)^{1-s} (x^s + y^s).$$

Lemma B.1. ($L^p - L^q$ decay of the heat kernel) *Let u be the solution of the heat equation*

$$u_t - \Delta u = 0, \quad u(t = 0) = \varphi(x), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Let $S(t) = e^{t\Delta}$ being the heat operator. Then it holds that for any $1 \leq q \leq p \leq \infty$

$$\left\| \partial_t^j \partial_x^\alpha u \right\|_p = \left\| \partial_t^j \partial_x^\alpha S(t) \varphi \right\|_p \leq C t^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p}) - j - \frac{|\alpha|}{2}} \|\varphi\|_{L^q}$$

where j is a positive integer and $\alpha = (\alpha_1, \dots, \alpha_d)$, where α_i , $1 \leq i \leq d$ is a positive integer and $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d}$.

We define the Gevrey in the physical space as (see e. g. [53])

$$\|f\|_{\mathcal{G}^{\lambda;s}} \approx \left[\sum_{n=0}^{\infty} \left(\frac{\lambda^n}{(n!)^{\frac{1}{s}}} \|D^n f\|_2 \right)^2 \right]^{\frac{1}{2}} \quad (\text{B.2a})$$

We can also extend (B.2a) define the more general $\ell^q L^p$ based spaces (see [69]) as

$$\|f\|_{\ell^q L^p; \lambda} \approx \left[\sum_{n=0}^{\infty} \left(\frac{\lambda^n}{(n!)^{\frac{1}{s}}} \|D^n f\|_p \right)^q \right]^{\frac{1}{q}} \quad (\text{B.2b})$$

Then it holds that (see [8, Appendix A.]) for $\lambda > \lambda'$ and for $p, q \in [1, \infty]$, we have

$$\begin{aligned} \|f\|_{\ell^p L^q; \lambda'} &\leq \|f\|_{\ell^1 L^q; \lambda} \lesssim_{\lambda-\lambda'} \|f\|_{\ell^p L^q; \lambda} \\ \|f\|_{\ell^2 L^\infty; \lambda'} &\lesssim \|f\|_{\ell^2 L^2; \lambda}. \end{aligned} \quad (\text{B.3})$$

Next, we show the following lemma, which useful to prove the scattering result in Section 2.5.

Lemma B.2. *The following inequality holds:*

$$\|fg\|_{\ell^1 L^2; \lambda} \lesssim \|f\|_{\ell^1 L^2; \lambda} \|g\|_{\ell^1 L^\infty; \lambda}.$$

Proof. We have, by using (B.2b) together with Leibniz's rule, that

$$\begin{aligned} \|fg\|_{\ell^1 L^2; \lambda} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^{1/s}} \|D^n(fg)\|_{L^2} \\ &\lesssim \sum_{n=0}^{\infty} \sum_{k=0}^n \left(\frac{n!}{k!(n-k)!} \right)^{\frac{1}{s}} \frac{\lambda^k \lambda^{n-k}}{(n!)^{1/s}} \|D^k f\|_{L^2} \|D^{n-k} g\|_{L^\infty} \\ &\lesssim \|f\|_{\ell^1 L^2; \lambda} \|g\|_{\ell^1 L^\infty; \lambda}, \end{aligned}$$

which gives the lemma. $\square$

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