

LINEARIZED WAVE-DAMPING STRUCTURE OF VLASOV–POISSON IN $\mathbb{R}^{3*}$

JACOB BEDROSSIAN[†], NADER MASMOUDI[‡], AND CLÉMENT MOUHOT[§]

Abstract. In this paper we study the linearized Vlasov–Poisson equation for localized disturbances of an infinite, homogeneous Maxwellian background distribution in $\mathbb{R}_x^3 \times \mathbb{R}_v^3$. In contrast to the confined case $\mathbb{T}_x^d \times \mathbb{R}_v^d$, or the unconfined case $\mathbb{R}_x^d \times \mathbb{R}_v^d$ with screening, the dynamics of the disturbance are not scattering towards free transport as $t \rightarrow \pm\infty$: we show that the electric field decomposes into a very weakly damped Klein–Gordon-type evolution for long waves and a Landau-damped evolution. The Klein–Gordon-type waves solve, to leading order, the compressible Euler–Poisson equations linearized about a constant density state, despite the fact that our model is collisionless, i.e., there is no trend to local or global thermalization of the distribution function in strong topologies. We prove dispersive estimates on the Klein–Gordon part of the dynamics. The Landau damping part of the electric field decays faster than free transport at low frequencies and damps as in the confined case at high frequencies; in fact, it decays at the same rate as in the screened case. As such, neither contribution to the electric field behaves as in the vacuum case.

Key words. Landau damping, Vlasov–Poisson, Klein–Gordon, dispersive, linearized, long-time dynamics

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1. Introduction.

1.1. The problem at hand. One of the fundamental equations in the kinetic theory of plasmas is the Vlasov–Poisson equations for an infinitely extended plasma (see, e.g., [12, 7]),

$$(1.1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + E(t, x) \cdot \nabla_v f = 0, \\ E(t, x) = -\nabla_x W * \rho(t, x), \\ \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv - n_0, \\ f(t = 0, x, v) = f_{in}(x, v), \end{cases}$$

for the time-dependent probability density function $f(t, x, v) \geq 0$ of the electrons in the phase space $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, where n_0 is the number density of the constant ion

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[†]Department of Mathematics, University of Maryland, College Park, MD 20742-3289 USA (jacob@math.umd.edu).

[‡]NYUAD Research Institute, New York University Abu Dhabi, Abu Dhabi, United Arab Emirates, and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012 USA (masmoudi@cims.nyu.edu).

[§]Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0WA, United Kingdom (c.mouhot@dpmms.cam.ac.uk).

background, and W is the kernel of the Coulomb interaction,¹

$$W(x) = \frac{q^2}{4\pi\epsilon_0 m_e |x|},$$

with q the electron charge, m_e the electron mass, and ϵ_0 the vacuum permittivity. We will consider (1.1) linearized about the homogeneous Maxwellian background with fixed temperature T ,

$$(1.2) \quad f^0(v) := \frac{n_0}{(2\pi T)^{3/2}} e^{-\frac{m_e |v|^2}{2T}}.$$

In the 1930s and 1940s, Vlasov [39, 38] suggested neglecting collisions and derived the so-called Vlasov–Poisson equation for long-range interactions, independently from Jeans’s derivation [23] in stellar dynamics. Motivated by the mathematical understanding of the plasma oscillations previously theorized in particular by Langmuir, he studied the linearized approximation of (1.1) (see (1.6) below) formally searching for eigenmodes in the form of planar waves $e^{-i\omega t + ikx} F(v)$, given some velocity distribution F , and computed various dispersion relations (see, for instance, equation (50) in [39]). He asserted—not quite correctly but *almost correctly*, as we shall see and clarify in this paper—that in the long-wave limit $|k| \ll 1$, where k is the Fourier variable in space, one has the dispersion relation

$$(1.3) \quad \omega^2 = \omega_p^2 + \frac{3T}{m_e} |k|^2 + \mathcal{O}(|k|^4) \quad \text{as } k \rightarrow 0, \quad \text{where} \quad \omega_p := \frac{q^2 n_0}{\epsilon_0 m_e}$$

is the *cold plasma frequency*. The relation $\omega^2 = \omega_p^2 + \frac{3T}{m_e} |k|^2$ is called the *Bohm–Gross dispersion relation*² and arises from the following compressible Euler–Poisson model for the spatial density $n(t, x) \geq 0$ and macroscopic velocity $u(t, x) \in \mathbb{R}^3$:

$$(1.4) \quad \begin{cases} \partial_t n + \nabla_x \cdot (nu) = 0, \\ m_e n [\partial_t u + (u \cdot \nabla_x) u] = qnE(t, x) - 3T \nabla_x n, \\ \epsilon_0 \nabla_x \cdot E(t, x) = qn; \end{cases}$$

when linearized around a constant density state $n_0 > 0$, $u_0 = 0$:

$$(1.5) \quad \begin{cases} \partial_t n + n_0 \nabla_x \cdot u = 0, \\ m_e n_0 \partial_t u = qn_0 E(t, x) - 3T \nabla_x n, \\ \epsilon_0 \nabla_x \cdot E(t, x) = qn. \end{cases}$$

This model is sometimes referred to as a *warm plasma model* in the physics literature; see, e.g., [12, Chapter 16] for more details. Vlasov’s prediction was shown to be incomplete by Landau in 1946 [25], who showed that in fact the linearized electric field decays to zero as $t \rightarrow \infty$, a phenomenon now known as *Landau damping*, and

¹Note that we are slightly abusing notation, as E in (1.1) actually denotes the acceleration of the electrons due to the electric field, not the electric field itself. We use this convention for the remainder of the paper.

²Technically, the relation seems to have first appeared in Vlasov’s earlier work [39] before appearing in [6]; however, it is likely that access to Vlasov’s work was difficult at that time. See also, e.g., page 260 of [12].

in particular that nontrivial planar waves never satisfy the dynamics. The damping arises due to *mixing/filamentation* in phase space, not unlike a scalar quantity being stirred in a fluid (as first pointed out in [37]). Landau damping was later observed in experiments [26, 27] and is considered one of the most important properties of collisionless plasmas [32, 12, 35].

On $\mathbb{T}^3 \times \mathbb{R}^3$, the linearized electric field decays rapidly, provided the initial data is regular; specifically, its decay in time is comparable to the decay in the Fourier variable associated with v of the initial data [28, 4]. Such rapid decay estimates do not hold at low frequencies in the electric field on $\mathbb{R}^3 \times \mathbb{R}^3$ if one uses a model with Debye shielding (this arises when studying the ion distribution function), i.e., where W is the fundamental solution to the elliptic problem $-\Delta W + \alpha W = \delta$ for some constant $\alpha > 0$ (see [5, 19] and the references therein). However, one nevertheless obtains a rather rapid Landau damping rate of $\|E(t)\|_{L^\infty} \lesssim \langle t \rangle^{-4}$ (in three dimensions), which is the same rate as one would obtain from the free transport equation. Indeed, both the shielded problem on $\mathbb{R}^3 \times \mathbb{R}^3$ and (1.6) posed on $\mathbb{T}^3 \times \mathbb{R}^3$ “scatter” to the free transport dynamics in a very strong sense. However, such scattering and rapid decay estimates on the linearized electric field are false in $\mathbb{R}^3 \times \mathbb{R}^3$ for Vlasov–Poisson. Landau himself predicted an extremely slow Landau damping of long waves approximately solving the dispersion relation (1.3) (see also, e.g., [12, 35] for modern exposition in the physics literature), something that we will make much more precise below. This lack of significant Landau damping was proved rigorously by Glassey and Schaeffer [10, 11], though a precise characterization of the dynamics was still lacking.

In this work, we make precise the linearized dynamics and prove that the linearized electric field can be split into two contributions: one contribution at long spatial waves $k \sim 0$ that is a Klein–Gordon-like propagation matching (1.3) to leading order with very weak Landau damping with rate “ $\mathcal{O}(|k|^\infty)$,” and another contribution, which is properly Landau damping and decays at a rate *faster* than kinetic free transport for long waves $k \sim 0$, i.e., faster than the Vlasov–Poisson equation linearized around $f^0 = 0$, $n_0 = 0$. In particular, our work shows that the hydrodynamic description (linearized Euler–Poisson) in fact is the leading order description of the electric field at long waves, despite the lack of collisional effects. As such, we remark that Vlasov was essentially correct to leading order in his prediction of the long-wave dynamics of (1.6). For long waves, we show that the distribution function decomposes (to leading order) into two pieces: a term that factorizes as $\tilde{E}(t, x) \cdot \nabla_v f^0(v)$ (where f^0 is the Maxwellian background), where $\tilde{E}$ solves a Klein–Gordon-type equation (including an additional tiny damping), and a separate contribution that scatters to free transport in $L^p_{x,v}$ for $p > 6$. Hence, the hydrodynamic behavior arises from a large-scale collective motion of the plasma that is insensitive to the filamentation in phase space normally associated with Landau damping.

Understanding the relationship between Landau damping (or other kinetic effects) and the observed large-scale hydrodynamic behavior in collisionless plasmas has been an area of research in the physics literature for some time (see, e.g., [17, 16, 33, 20] and the references therein). Our work provides a precise description of the hydrodynamic behavior and its leading order corrections due to Landau damping and other kinetic effects, for the simple linearized problem; studies of more physical settings (e.g., external and/or self-consistent magnetic fields, inclusion of ions, inhomogeneous backgrounds, etc.) and/or the inclusion of nonlinear effects is an interesting direction for further research.

Our work is only linear, but we remark that much progress has been made at the nonlinear level in recent years in other settings. After the earlier work of [8]

(see also [21]), the major breakthrough came in [28], when Landau damping for the nonlinear problem was shown on $\mathbb{T}^3 \times \mathbb{R}^3$ for all sufficiently small and smooth initial perturbation of Landau-stable stationary solutions, the actual smoothness required being Gevrey or analytic. We also refer the reader to [4, 13] for simplified proofs and [3] for a study of the problem with collisions. The work [2] shows that the results therein do not hold in finite regularity (see also [14]). Our previous work [5] studied the nonlinear problem with shielding on $\mathbb{R}^3 \times \mathbb{R}^3$; see also [19, 29] for an alternative approach and some refinements. Other works, old and new, have studied the nonlinear Cauchy problem near the vacuum state, that is, without the presence of a nonzero spatially homogeneous background equilibrium f^0 ; see, e.g., [1, 22]. As can be seen from our results below, the dynamics are significantly different in this case.

1.2. Main results. We linearize the Vlasov–Poisson equation (1.1) on $\mathbb{R}_x^3 \times \mathbb{R}_v^3$ around an infinitely extended, homogeneous background $f^0(v) \geq 0$. This models a spatially localized disturbance of an infinite plasma in which collisions can be neglected, a fundamental problem in the kinetic theory of plasmas [12, 7, 35]. For a localized disturbance $h := f^0 - f$, the *linearized* Vlasov–Poisson equations for the electron distribution function are given by

$$(1.6) \quad \begin{cases} \partial_t h + v \cdot \nabla_x h + E(t, x) \cdot \nabla_v f^0 = 0, \\ E(t, x) = -\nabla_x W *_x \rho(t, x), \\ \rho(t, x) = \int_{\mathbb{R}^3} h(t, x, v) dv, \\ h(t = 0, x, v) = h_{in}(x, v), \end{cases}$$

where we take the charge neutrality assumption $\int_{\mathbb{R}^3 \times \mathbb{R}^3} h_{in}(x, v) dx dv = 0$. Define the standard plasma constants (number density, plasma frequency, and temperature),

$$n_0 := \widehat{f^0}(0), \quad \omega_p^2 := \frac{q^2 n_0}{\epsilon_0 m_e}, \quad T := \frac{m_e}{3n_0} \int_{\mathbb{R}^3} |v|^2 f^0(v) dv,$$

and as stated above, we assume the Maxwellian distribution background (1.2).

Our main results are asymptotic decomposition of the electric field and the distribution function; we give slightly simplified statements for readability and refer the reader to the main body of the paper for more detailed expansions. See subsection 1.3 for the notation $\langle x \rangle$, $\langle x, y \rangle$, $\langle \nabla \rangle$, and $W_w^{k+0,p}$.

We remark that another paper proving similar results with somewhat different methods [18] has recently been completed as well. These works have been completed totally independently.

THEOREM 1. *Suppose $\int_{\mathbb{R}^3 \times \mathbb{R}^3} h_{in} dx dv = 0$, and let h and E solve (1.6). There are $\delta_0 > 0$ and a decomposition of the electric field $E = E_{KG} + E_{LD}$ between “Klein–Gordon” and “Landau damped” parts, with E_{KG} supported in spatial frequencies $|k| < \nu_0$, and the following hold:*

- E_{LD} satisfies the following Landau damping–type decay estimates for any $\sigma \in \mathbb{N}$:

$$(1.7) \quad \begin{aligned} \|\langle \nabla_x, t \nabla_x \rangle^\sigma E_{LD}(t)\|_{L_x^\infty} &\lesssim \frac{1}{\langle t \rangle^{5/2}} \|h_{in}\|_{W_0^{\sigma+\frac{3}{2}+0,1}}, \\ \|\langle \nabla_x, t \nabla_x \rangle^\sigma E_{LD}(t)\|_{L_x^\infty} &\lesssim \frac{1}{\langle t \rangle^4} \|h_{in}\|_{W_0^{\sigma+3+0,1}}. \end{aligned}$$

- E_{KG} further decomposes as $E_{KG} = E_{KG}^{(1)} + E_{KG}^{(2)}$ with the pointwise-in-time estimates:

$$\begin{aligned} \|E_{KG}^{(1)}(t)\|_{L_x^2} &\lesssim \|E_{in}\|_{L^2} + \|vh_{in}\|_{L_{x,v}^2} + \|h_{in}\|_{W_4^{0,1}}, \\ \|E_{KG}^{(2)}(t)\|_{L_x^2} &\lesssim \|h_{in}\|_{W_5^{0,1}}, \\ \|E_{KG}^{(2)}(t)\|_{L_x^\infty} &\lesssim \langle t \rangle^{-3/2} \|h_{in}\|_{W_5^{3/2,1}}, \\ \|E_{KG}^{(1)}(t)\|_{L_x^p} &\lesssim_p \langle t \rangle^{-3(\frac{1}{2}-\frac{1}{p})} \|\langle x \rangle h_{in}\|_{W_4^{0,1}} \quad (\forall 2 \leq p < \infty), \\ \|\nabla_x E_{KG}^{(1)}(t)\|_{L_x^\infty} &\lesssim \langle t \rangle^{-3/2} \|h_{in}\|_{W_4^{0,1}}. \end{aligned}$$

- $E_{KG}^{(1)}$ solves a weakly damped Klein-Gordon-type equation in the following sense: there are bounded, smooth functions $\lambda, \Omega, k \in B(0, \delta_0)$ such that (for errors independent of t)

$$\begin{aligned} \widehat{E_{KG}^{(1)}}(t, k) &= \widehat{E_{in}}(k) e^{-\lambda(k)t} \cos(\Omega(k)t) + e^{-\lambda(k)t} \frac{ik}{|k|^2} \left(k \cdot \nabla_\eta \widehat{h_{in}}(k, 0) \right) \frac{\sin(\Omega(k)t)}{\Omega(k)} \\ (1.8) \quad &+ \mathcal{O}(|k|^2) e^{-\lambda(k)t + i\Omega(k)t} + \mathcal{O}(|k|^2) e^{-\lambda(k)t - i\Omega(k)t}, \\ \text{where } \lambda(k) &> 0, \quad \lambda(k) = \mathcal{O}(|k|^\infty), \quad \Omega^2(k) = \omega_p^2 + \frac{3T}{m_e} |k|^2 + \mathcal{O}(|k|^4) \\ &\text{as } k \rightarrow 0. \end{aligned}$$

Furthermore, there holds, for all $2 < p \leq \infty$,

$$\|(-\Delta_x)^{-1} E_{KG}^{(2)}(t)\|_{L^p} \lesssim \langle t \rangle^{-3(\frac{1}{2}-\frac{1}{p})} \|h_{in}\|_{W_5^{3/2,1}},$$

which further emphasizes that for small k , $E_{KG}^{(1)}$ is much larger than $E_{KG}^{(2)}$.

Remark 1. The $E_{KG}^{(2)}$ electric field is essentially a Klein-Gordon-type evolution subjected to a Landau damping external forcing, whereas $E_{KG}^{(1)}$ is to leading order the solution to a Klein-Gordon-type initial value problem with initial conditions determined by the initial density and momentum, as can be seen from (1.8) (and higher order expansions are similar but depend on higher moments of h_{in}). Hence, the motivations for separating these fields as distinct contributions. See section 2 for more details on the decomposition. Note also that $E_{KG}^{(2)}$ is significantly smaller when $k \rightarrow 0$ than $E_{KG}^{(1)}$, and thus the asymptotic expansion of E_{KG} as $k \rightarrow 0$ is given by the corresponding leading order terms in $E_{KG}^{(1)}$.

Remark 2. For frequencies bounded away from zero, i.e., for any $\delta > 0$, $P_{\geq \delta} E$, Landau damps at a polynomial rate $\langle t \rangle^{-\sigma}$, provided the initial data is $W^{\sigma,1}$; hence the extremely slow Landau damping of E_{KG} manifests only at $k = 0$. As expected, for frequencies bounded from zero, the electric field (both E_{LD} and E_{KG}) damps exponentially fast if the initial data is analytic (and analogously $e^{-\langle kt \rangle^s}$ for Gevrey initial data).

Remark 3. As the electric field is not shielded, it is too restrictive to assume that $E_{in} \in L^1$, regardless of how well localized the initial h_{in} is. If $xh_{in} \in L^1$ and $\int_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} h_{in} dx dv = 0$, then $E_{in}(x) \approx |x|^{-3}$ as $x \rightarrow \infty$ (hence, $E_{in} \in L^p$ for $p > 1$ but

not $p = 1$), but one cannot obtain faster decay at infinity unless one has higher zero-moment conditions, such as $\int_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} x^\alpha h_{in} dx dv = 0$, and the latter are not propagated by the semigroup.

Remark 4. In the case of the kinetic free transport, the electric field decays only as $\|E_{FT}(t)\|_{L_x^\infty} \lesssim \langle t \rangle^{-2}$, which is two full powers of t slower than (1.7). For frequencies bounded away from zero, both the free transport electric field E_{FT} and our E_{LD} are Landau damped exponentially fast for analytic data (and polynomial for Sobolev data); the difference in decay rates comes from the contribution of long waves.

Remark 5. There is a clear distinction between Landau damping and dispersive decay above: for the Landau damping contributions, each derivative buys one power of time (a structure that can be guessed from free transport), whereas for the Klein–Gordon-like contributions, there is no such behavior.

Remark 6. We have not endeavored to get the sharpest dependence on the initial data that could be possible. This could be important for nonlinear extensions.

We deduce a similar decomposition at the level of the distribution function.

THEOREM 2. *The solution to (1.6) decomposes as $h = h_{KG} + h_{LD}$ with $h_{KG} = \tilde{E}_{KG}(t, x) \cdot \nabla_v f^0(v)$ for some effective electric field $\tilde{E}_{KG}$, with the following estimates:*

- *for all $m \geq 0$, all $r \in [1, \infty]$, and all $p \in [2, \infty)$ ($\tilde{E}_{KG}$ is essentially a phase-shifted version of $E_{KG}^{(1)}$ and satisfies the same estimates),*

$$(1.9) \quad \begin{aligned} \|\langle v \rangle^m h_{KG}(t)\|_{L_x^2 L_v^r} &\lesssim_m \|h_{in}\|_{W_4^{0,1}}, \\ \|\langle v \rangle^m h_{KG}(t)\|_{L_x^p L_v^r} &\lesssim_{p,m} \langle t \rangle^{-3(\frac{1}{2} - \frac{1}{p})} \|\langle x \rangle h_{in}\|_{W_4^{0,1}}, \\ \|\langle v \rangle^m \nabla_x h_{KG}(t)\|_{L_x^\infty L_v^r} &\lesssim_m \langle t \rangle^{-3/2} \|h_{in}\|_{W_4^{0,1}}; \end{aligned}$$

- *h_{LD} scatters to free transport in all $L_{x,v}^p$, $p > 6$ (provided one has enough initial regularity). In particular, if $\langle \nabla \rangle^\sigma \langle v \rangle^m h_{in}$ is integrable for $\sigma, m \in \mathbb{N}$ large enough, then for all $p > 6$, there exists an $h_\infty \in L_{x,v}^p$ such that*

$$\left\| \langle v \rangle^m \left[h_{LD}(t, x - vt, v) - h_\infty(x, v) \right] \right\|_{L_{x,v}^p} \xrightarrow{t \rightarrow +\infty} 0.$$

Remark 7. Using the Strichartz estimates [9] for the transport equation, one obtains

$$\|h_{LD}\|_{L_t^q L_x^p L_v^r} \lesssim \|h_{in}\|_{W_m^{\sigma,1}}$$

for all (p, q, r, a) satisfying

$$\frac{1}{r} - \frac{1}{n} < \frac{1}{p} \leq \frac{1}{r} \leq 1, \quad 1 \leq \frac{1}{p} + \frac{1}{r}, \quad \frac{2}{q} = n \left(\frac{1}{r} - \frac{1}{p} \right), \quad \frac{1}{a} = \frac{1}{2} \left(\frac{1}{r} + \frac{1}{p} \right), \quad a > 6.$$

Theorem 2 shows that the leading order term in the asymptotics of the distribution h factorizes between a function of (t, x) and a fixed function of v , which is reminiscent of a hydrodynamical limit *even though the equations are collisionless*. One can actually push this intuition further: a long-wave rescaling of the electric field converges weakly (in negative Sobolev spaces) to the electric field, solving the linearized Euler–Poisson system (1.5). See subsection 2.5 for a proof.

THEOREM 3. Consider an initial data $h_{in}(x, v) = \epsilon^3 \mathcal{H}_0(\epsilon x, v)$ such that $\mathcal{H}_0$ has zero average and $\mathcal{H}_0 \in W_5^{5+0,1}$, and denote

$$E_0 = \frac{q^2 n_0}{\epsilon_0 m_e} \nabla_x (-\Delta_x)^{-1} \int_{\mathbb{R}^3} \mathcal{H}_0(\cdot, v) dv.$$

Let $\mathcal{E}$ be the unique solution in the natural energy space $\mathcal{E} \in L_t^\infty H_x^1$ and $\partial_t \mathcal{E} \in L_t^\infty L_x^2$ to the following Klein–Gordon equation:

$$(1.10) \quad \begin{cases} \partial_t^2 \mathcal{E} + \left(\omega_p^2 - \frac{3T}{m_e} \Delta \right) \mathcal{E} = 0, \\ \mathcal{E}(0, x) = E_0(x), \\ \partial_t \mathcal{E}(0, x) = -n_0 \nabla_x \cdot \left(\int v h_{in} dv \right). \end{cases}$$

Then, for all $s \in (5/2, 7/2)$, $0 < \epsilon \ll 1$, for all $0 < t < \epsilon^{-N}$, there holds

$$(1.11) \quad \left\| E\left(t, \frac{\cdot}{\epsilon}\right) - \mathcal{E}\left(t, \frac{\cdot}{\epsilon}\right) \right\|_{H^{-s}} \lesssim \epsilon^2 \left(\epsilon^2 + \epsilon^{s-\frac{3}{2}-0} \langle t \rangle \right) \|\mathcal{H}_0\|_{W_5^{2,1}}.$$

Moreover, given such initial data with appropriate scaling, both the electric fields in the linearized Vlasov–Poisson and the linearized Euler–Poisson systems are asymptotic to the Bohm–Gross dispersion relation at large scale: $E(t, x) = E_+ + E_-$ and $\mathcal{E}(t, x) = \mathcal{E}_+ + \mathcal{E}_-$ with the following weak L_x^2 limit on $t \in [-T, T]$ for any fixed finite T :

$$\begin{aligned} \frac{1}{\epsilon^2} e^{\mp i \omega_p \frac{t}{\epsilon^2}} E_\pm \left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon} \right) &\xrightarrow[L_x^2]{weak} \frac{1}{2} e^{\pm \frac{3T}{m_e} i \Delta t} E_0, \\ \frac{1}{\epsilon^2} e^{\mp i \omega_p \frac{t}{\epsilon^2}} \mathcal{E}_\pm \left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon} \right) &\xrightarrow[L_x^2]{weak} \frac{1}{2} e^{\pm \frac{3T}{m_e} i \Delta t} \mathcal{E}_0. \end{aligned}$$

Remark 8. The Klein–Gordon equation arises as the linearization of many non-linear PDEs around homogeneous equilibria. The specific link with the linearized Euler–Poisson is seen by keeping track of the physical parameters in (1.5) and seeing that the results of Theorems 1–3 show that (1.6) reduces at low frequencies to leading order specifically to (1.5), and not to some other Klein–Gordon equation (see also (1.10) and (1.11) of Theorem 3). However, a more complete linking between (1.4) and (1.1) would have to come from a weakly nonlinear study.

1.3. Notation. Denote $\langle x \rangle = (1 + |x|^2)^{1/2}$ and by $\langle \nabla \rangle$ the Fourier multiplier defined by

$$\widehat{\langle \nabla \rangle f}(\xi) = \langle \xi \rangle \widehat{f}(\xi);$$

we define other Fourier multipliers in an analogous way. We define $P_{\leq N}$ to be the Littlewood–Paley low-pass filter in x ; in particular, we define $\chi \in C_c^\infty(B(0, 2))$ with $\chi(x) = 1$ for $|x| \leq 1$, and define

$$\widehat{P_{\leq N} f}(k) = \chi\left(\frac{k}{N}\right) \widehat{f}(k).$$

Denote the weighted Sobolev spaces (note that the weight index refers to the variable v) as follows:

$$\|f\|_{L_{x,v}^p} := \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(x,v)|^p dx dv \right)^{1/p}, \quad \|\rho\|_{L_x^p} := \left(\int_{\mathbb{R}^3} |\rho(x)|^p dx \right)^{1/p},$$

$$\|f\|_{W_m^{\sigma,p}} := \|\langle v \rangle^m \langle \nabla_{x,v} \rangle^\sigma f\|_{L^p}.$$

For inequalities involving norms $W_m^{\sigma+0,p}$ we use the notation $+0$ if we mean that the estimate holds for $\sigma + \delta$ with $\delta \in (0, 1)$ with a constant that depends on δ . For $p \in [1, \infty]$ we denote $p' = \frac{p}{p-1}$ as the Hölder conjugate. Let $f : [0, \infty) \rightarrow \mathbb{C}$ satisfy $e^{-\mu t} f(t) \in L^1$ for some $\mu \in \mathbb{R}$. Then for all complex numbers $\Re z \geq \mu$, we can define the Fourier–Laplace transform via the (absolutely convergent) integral

$$(1.12) \quad \hat{f}(z) := \frac{1}{2\pi} \int_0^\infty e^{-zt} f(t) dt.$$

This transform is inverted by integrating the $\tilde{f}$ along a so-called Bromwich contour via the inverse Fourier–Laplace transform: let $\gamma > \mu$ and define

$$\check{f}(t) := \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} f(z) dz.$$

2. Decomposition of the electric field.

2.1. Volterra equation. The most important property of the linearized Vlasov equations is that one can reduce the problem to a Volterra equation for the density separately for each spatial frequency. We now recall how this is done. Writing a Duhamel representation along free transport gives

$$h(t, x, v) = h_{in}(x - vt, v) + \int_0^t (\nabla_x W *_x \rho)(t, x - v(t-s)) \cdot \nabla_v f^0(v) ds.$$

Taking the Fourier transform in x and integrating in v gets (with $w_0 := \omega_p^2 n_0^{-1}$)

$$(2.1) \quad \hat{\rho}(t, k) = \widehat{h_{in}}(k, kt) - w_0 \int_0^t (t - \tau) \widehat{f^0}(k(t - \tau)) \hat{\rho}(\tau, k) d\tau.$$

Taking the Fourier–Laplace transform (1.12) in time for $\Re z$ sufficiently large gives

$$(2.2) \quad \tilde{\rho}(z, k) = H(z, k) + \mathcal{L}(z, k) \tilde{\rho}(z, k),$$

where $H(z, k)$ is the Fourier–Laplace transform of $t \mapsto \widehat{h_{in}}(k, kt)$ and the *dispersion function* is

$$(2.3) \quad \mathcal{L}(z, k) := -w_0 \int_0^{+\infty} t \widehat{f^0}(kt) e^{-zt} dt = -\frac{w_0}{|k|^2} \int_0^{+\infty} e^{-\frac{z}{|k|} s} \widehat{f^0}(\hat{k}s) ds,$$

with the standard notation $\hat{k} = k/|k|$. This change of variable allows for the function $s \mapsto s \widehat{f^0}(\hat{k}s)$ to have regularity bounds independently of the size of k . Our assumption (1.2) implies that $z \mapsto \mathcal{L}(z, k)$ is an entire function for all $k \neq 0$. Moreover, we see that away from $z = 0$ or $k = 0$, $\mathcal{L}(z, k)$ is also a smooth function of k . We will see that analyzing the behavior near $(z, k) = (z, 0)$ and $(z, k) = (0, 0)$ is an important and subtle aspect of our analysis (see Lemma 2.3 below).

2.2. Asymptotic expansions and lower bounds on the dispersion function. Solving (2.2) for ρ works, formally at least, except where $\mathcal{L}(z, k)$ gets close to one. In the case of not-so-small spatial frequencies k , the treatment is similar to that of the periodic domain $x \in \mathbb{T}^3$, and so we merely sketch the argument here; see [28, 4] for more details.

LEMMA 2.1 (resolvent estimates for nonsmall frequencies). *There exists a $\lambda > 0$ such that for any $\nu_0 > 0$, $\exists \kappa > 0$ (depending on ν_0) such that*

$$(2.4) \quad \forall |k| > \nu_0, \quad \inf_{\Re z > -\lambda|k|} |1 - \mathcal{L}(z, k)| > \kappa.$$

Furthermore, the following estimate holds uniformly on the critical vertical line of this region:

$$(2.5) \quad \forall |k| > \nu_0, \quad \omega \in \mathbb{R}, \quad |\mathcal{L}(-\lambda|k| + i\omega, k)| \lesssim_\lambda \frac{1}{1 + |k|^2 + \omega^2}.$$

Proof. The estimate (2.4) is proved in, e.g., [28, 4]. To see (2.5), we use integration by parts and the analyticity of f^0 to get

$$\begin{aligned} |\mathcal{L}(-\lambda|k| + i\omega, k)| &= \frac{w_0}{|k|^2} \left| \int_0^\infty e^{\lambda s - i \frac{\omega}{|k|} s} s \widehat{f^0}(\hat{k}s) ds \right| \\ &= \frac{w_0}{|k|^2} \left| \int_0^\infty \frac{1}{(\lambda - i \frac{\omega}{|k|})^2} \partial_s^2 \left(e^{\lambda s - i \frac{\omega}{|k|} s} \right) s \widehat{f^0}(\hat{k}s) ds \right| \\ &\leq \frac{w_0}{(\lambda^2 |k|^2 + \omega^2)} \left[|n_0| + \int_0^\infty \left(2 \left| \partial_s \widehat{f^0}(\hat{k}s) \right| + s \left| \partial_s^2 \widehat{f^0}(\hat{k}s) \right| \right) ds \right] \\ &\lesssim_\lambda \frac{1}{1 + |k|^2 + \omega^2}, \end{aligned}$$

which completes the proof. $\square$

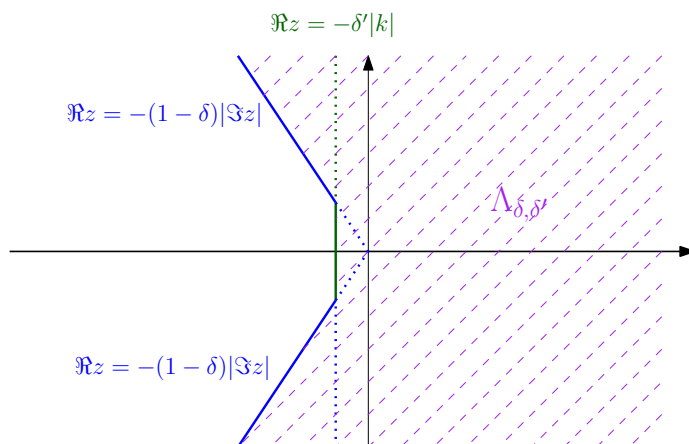
Next we turn to the low frequency estimates, which are more challenging, and contain the long-wave dispersive structure. For $\delta, \delta' > 0$, define the following region in the complex plane (see Figure 1 for a diagram):

$$\Lambda_{\delta, \delta'} := \left\{ z = \lambda + i\omega \in \mathbb{C} : \lambda > -\min \left[(1 - \delta) |\omega|, \delta' |k| \right] \right\}.$$

We will next show that $\mathcal{L}(z, k)$ stays uniformly away from one in the region $\Lambda_{\delta, \delta'} \setminus \{|z \pm i\omega_p| < \epsilon\}$, where ω_p is the cold plasma frequency. The proof relies on two representations: (1) an expansion obtained by the stationary phase method (i.e., successive integrations by parts in time) meaningful for large values of $\frac{z}{|k|}$, and (2) an approximation argument using the explicit formula obtained from the Plemelj formula at the imaginary line $\Re z = 0$ that provides estimates near this line. The first representation is given by the following lemma. This expansion is ultimately the source of the Bohm–Gross dispersion relation (1.3).

LEMMA 2.2 (asymptotic expansion of $\mathcal{L}$). *For any $\delta \in (0, 1)$ and any $\delta' > 0$ there holds*

$$(2.6) \quad \forall z \in \Lambda_{\delta, \delta'}, \quad \mathcal{L}(z, k) = -\frac{\omega_p^2}{z^2} \left[1 + \frac{3T|k|^2}{m_e z^2} + \mathcal{O} \left(\frac{|k|^4}{|z|^4} \right) \right], \quad \text{as } \frac{|z|}{|k|} \rightarrow \infty.$$

FIG. 1. The region $\Lambda_{\delta, \delta'}$.

Note that this expansion only contains information for frequencies $|k| \ll |z|$. The expansion (2.6) gives the natural continuous extension of $\mathcal{L}(z, k)$ at $(z, 0)$ for $z \in \mathbb{C} \setminus \{0\}$:

$$\mathcal{L}(z, 0) := -\frac{\omega_p^2}{z^2}.$$

We will see below that this extension makes $\mathcal{L}(z, k)$ smooth in k and analytic in z away from $(z, k) = (0, 0)$.

Remark 9. Notice that this expansion holds everywhere in $\Lambda_{\delta, \delta'}$, and in particular only requires $\Re z \geq -\delta' |k|$ near $z = 0$. This observation is important for our analysis of the behavior near $(z, k) = (\pm i\omega_p, 0)$, which is well separated from the singularities.

Proof. Since $s \mapsto s\widehat{f^0}(\hat{k}s)$ is odd, observe that $\partial_s^j(s\widehat{f^0}(\hat{k}s))|_{s=0} = 0$ for all even j . Therefore, integrating by parts repeatedly in s in (2.3) gives

$$\begin{aligned} \mathcal{L}(z, k) &= -\frac{w_0}{z^2} \widehat{f^0}(0) - \frac{w_0}{z^2} \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^2 \left(s\widehat{f^0}(\hat{k}s) \right) ds \\ &= -\frac{w_0}{z^2} \widehat{f^0}(0) - \frac{3w_0 |k|^2}{z^4} \left[\hat{k} \otimes \hat{k} : \nabla^2 \widehat{f^0}(0) \right] - \frac{w_0 |k|^2}{z^4} \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^4 \left(s\widehat{f^0}(\hat{k}s) \right) ds \\ &=: -\frac{w_0}{z^2} \widehat{f^0}(0) - \frac{3w_0 |k|^2}{z^4} \left[\hat{k} \otimes \hat{k} : \nabla^2 \widehat{f^0}(0) \right] - \frac{w_0 |k|^2}{z^4} \zeta(z, k). \end{aligned}$$

Since $w_0 \widehat{f^0}(0) = w_0 n_0 = \omega_p^2$ and $3n_0 T := m_e \hat{k} \otimes \hat{k} : \nabla^2 \widehat{f^0}(0)$ gives the leading order terms in (2.6). Note that we have

$$\zeta(z, k) := \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^4 \left(s\widehat{f^0}(\hat{k}s) \right) ds.$$

It remains to show that for $z \in \Lambda_{\delta, \delta'}$ there holds

$$|\zeta(z, k)| \lesssim \frac{|k|^2}{|z|^2}.$$

First consider the region $\Re z \geq -\delta'|k|$: integrating by parts two more times, we obtain

$$\forall \Re z \geq -\delta'|k|, \quad |\zeta(z, k)| \lesssim \frac{|k|^2}{|z|^2} \left(1 + \int_0^\infty e^{\delta' s} \left| \partial_s^6 \left(s \widehat{f^0}(ks) \right) \right| ds \right) \lesssim_{\delta'} \frac{|k|^2}{|z|^2},$$

where the last line follows by the analyticity of f^0 .

Turn next to the region $\Re z < -\delta'|k|$ with $\Re z > -(1-\delta)|\Im z|$. Observe then $\arg z^2 \in [\frac{\pi}{2} + \beta, \frac{3\pi}{2} - \beta]$ for a small $\beta > 0$ depending on δ . Write

$$\zeta(z, k) = \int_{-\infty}^\infty e^{-\frac{z}{|k|}s} \partial_s^4 \left(s \widehat{f^0}(ks) \right) ds - \int_{-\infty}^0 e^{-\frac{z}{|k|}s} \partial_s^4 \left(s \widehat{f^0}(ks) \right) ds =: \zeta_1(z, k) + \zeta_2(z, k).$$

On the one hand, ζ_2 is bounded as in the region $\Re z \geq -\delta'|k|$ due to the now advantageous sign of the exponent. On the other hand, ζ_1 is a true Fourier-Laplace transform, and due to (1.2),

$$\begin{aligned} \zeta_1(z, k) &= \int_{-\infty}^\infty e^{-\frac{z}{|k|}s} \partial_s^4 \left(s \widehat{f^0}(ks) \right) ds = \frac{z^4}{|k|^4} \int_{-\infty}^\infty e^{-\frac{z}{|k|}s} \widehat{f^0}(ks) ds \\ &= \frac{n_0}{m_e^{3/2}} \frac{z^4}{|k|^4} \int_{-\infty}^\infty e^{-\frac{z}{|k|}s - s^2 \frac{T}{2m_e}} ds = \frac{n_0}{m_e^{3/2}} \frac{z^4}{|k|^4} e^{\frac{m_e z^2}{2T|k|^2}}. \end{aligned}$$

Due to $\arg z^2 \in [\frac{\pi}{2} + \beta, \frac{3\pi}{2} - \beta]$, it holds that $\Re z^2 \lesssim_\delta -|\Im z|^2 \lesssim_\delta -|z|^2$, and this term vanishes (to infinite order) in terms of $\frac{|z|}{|k|} \rightarrow \infty$. This completes the proof. $\square$

Lemma 2.2 suffices to estimate the resolvent in much of the areas of interest. The next lemma estimates the resolvent for low frequencies k in the half-plane $\Re z \geq -\delta'|k|$ and away from the cold plasma frequencies $\pm i\omega_p$. Given $\epsilon > 0$, define the following region (see Figure 2 for a diagram):

$$\mathbb{H}_{\epsilon, \delta'} := \{z = \lambda + i\omega \in \mathbb{C} : \lambda > -\delta'|k| \text{ and } |z \pm i\omega_p| \geq \epsilon\}.$$

LEMMA 2.3 (low frequency resolvent estimates). *Given $\epsilon, \delta' > 0$, there are $\nu_0, \kappa > 0$ such that*

$$\forall |k| < \nu_0, \quad \forall z \in \mathbb{H}_{\epsilon, \delta'}, \quad |1 - \mathcal{L}(z, k)| \geq \kappa.$$

Remark 10. One of the central difficulties of this lemma is dealing with the region near $(z, k) = (0, 0)$, which is explicitly excluded in Lemma 2.2.

Proof. Let $R > 0$ be fixed large depending on f^0 but independent of k .

Case 1: $|z| > R|k|$. In this region the estimate follows from (2.6) taking $R > 0$ sufficiently large.

Case 2: $|z| \leq R|k|$. In this region the asymptotic expansion (2.6) is no longer useful and we use the *Plemelj formula* instead. Writing $z = \lambda + i\omega$, it is classical that for $\lambda = 0$, one has (see, e.g., [31, 28] for explanations)

$$\mathcal{L}(i\omega, k) = \frac{w_0}{|k|^2} \int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r - \frac{\omega}{|k|}} dr + i \frac{w_0 \pi}{|k|^2} (f_k^0)' \left(\frac{\omega}{|k|} \right),$$

where, for any $k \neq 0$, the partial hyperplane average is defined as

$$\forall r \in \mathbb{R}, \quad f_k^0(r) := \int_{\frac{k}{|k|}r + k_\perp} f^0(v_*) dv_*.$$

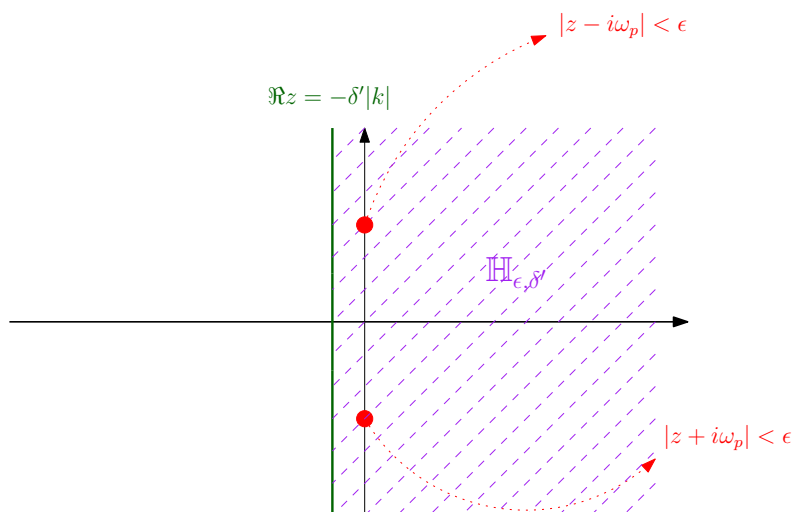


FIG. 2. The region $\mathbb{H}_{\epsilon, \delta'}$. The bold line denotes the line $\Re z = -\delta' |k|$, while the real and imaginary axes are denoted by thin black lines with arrows.

Moreover, observe that for any $z \in \mathbb{C}$ such that $|z| \leq R|k|$ and $\Re z \in (-\delta' |k|, 0]$,

$$(2.7) \quad |\partial_z \mathcal{L}(z, k)| \lesssim \frac{w_0}{|k|^3} \int_0^\infty s e^{\delta' s} \left| \widehat{f^0}(\hat{k}s) \right| ds \lesssim_{\delta'} \frac{1}{|k|^3},$$

where we have used $\delta' > 0$ and the Gaussian decay of $\widehat{f^0}$.

Subcase 2.1: $|z| \leq R|k|$ and $c|k| \leq |\omega| \leq R|k|$. Given any $c > 0$, we deduce from decay, smoothness, radial symmetry, and monotonicity of f^0 that

$$\inf_{c < \frac{|\omega|}{|k|} < R} \frac{w_0 \pi}{|k|^2} \left| (f_k^0)' \left(\frac{\omega}{|k|} \right) \right| \gtrsim_{c, R} \frac{1}{|k|^2},$$

and therefore, using (2.7), $|\Im \mathcal{L}(z, k)| \gtrsim 1$ holds uniformly for all $\Re z \in (-\delta' |k|, 0)$.

Subcase 2.2: $|z| \leq R|k|$ and $|\omega| < c|k|$. Observe that $\frac{(f_k^0)'(r)}{r}$ is integrable since $(f_k^0)'(0) = 0$ and f^0 is smooth, and if we denote

$$\int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r} dr = \mathfrak{c}_0 \neq 0,$$

we deduce that if $c \leq \frac{|\mathfrak{c}_0|}{2}$ is small enough, then

$$\left| \int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r - \frac{\omega}{|k|}} dr - \mathfrak{c}_0 \right| = \left| \frac{\omega}{|k|} \int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r \left(r - \frac{\omega}{|k|} \right)} dr \right| \lesssim \frac{|\omega|}{|k|} \leq \frac{|\mathfrak{c}_0|}{2},$$

and thus

$$|1 - \mathcal{L}(i\omega, k)| \gtrsim \frac{1}{|k|^2}.$$

Using again (2.7), for δ' small enough we deduce that for $\Re z \in (-\delta'|k|, 0]$,

$$|1 - \mathcal{L}(z, k)| \gtrsim 1.$$

These different cases above together prove that $1/|1 - \mathcal{L}|$ is bounded from above on the strip $\Re z \in (-\delta'|k|, 0]$ and outside $B(0, R|k|) \cap \{\Re z > 0\}$, and since there are no poles within the remaining region $B(0, R|k|) \cap \{\Re z > 0\}$, the function is holomorphic in this region and the upper bound is also valid there by the maximum principle, which completes the proof. $\square$

2.3. Construction of the branches of poles. From Lemma 2.2, we have a pole at $|k| = 0$ at the cold plasma frequency: $\mathcal{L}(\pm i\omega_p, 0) = 1$. It follows from Rouché's theorem that if $|k|$ is small enough, exactly two poles persist in respective neighborhoods of $\pm i\omega_p$: Given $\epsilon > 0$, the two functions $F(z) := 1 - \mathcal{L}(z, 0)$ and $G(z) := \mathcal{L}(z, k) - \mathcal{L}(z, 0)$ are holomorphic on the set $|z \mp i\omega_p| = \epsilon$, and Lemma 2.2 implies $|1 - \mathcal{L}(z, 0)| \gtrsim \epsilon$ and $|\mathcal{L}(z, 0) - \mathcal{L}(z, k)| \lesssim |k|^2$ on $|z \mp i\omega_p| = \epsilon$. Therefore, $F(z) = 1 - \mathcal{L}(z, 0)$ and $F(z) - G(z) = 1 - \mathcal{L}(z, k)$ have the same number of poles in $|z \mp i\omega_p| < \epsilon$, provided that $|k|$ is sufficiently small relative to ϵ .

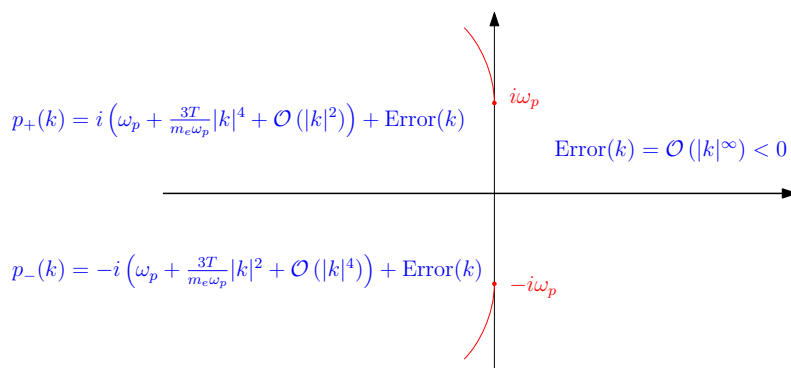


FIG. 3. The branches of poles $k \mapsto p_{\pm}(k)$.

However, knowing just the approximate location of the poles is not enough to deduce dispersive estimates. We next use the implicit function theorem to construct the branches of solutions $p_{\pm}(k)$ (see Figure 3 for a diagram of the branches).

LEMMA 2.4. *There are $\epsilon, \nu_0 > 0$ such that for all $|k| < \nu_0$, there are unique $p_{\pm}(k) \in \mathbb{C}$ solutions to $\mathcal{L}(p_{\pm}(k), k) = 1$ in $\{|z \mp i\omega_p| < \epsilon\}$ and $k \mapsto p_{\pm}(k) =: -\lambda(k) \pm i\Omega(k)$ are smooth (but not analytic) and satisfy $\lambda(k) > 0$ and $p_{\pm}(k) \sim_{k \rightarrow 0} \pm i\omega_p$ with the following expansions as $k \rightarrow 0$:*

$$(2.8) \quad \Omega(k)^2 = \omega_p^2 + \frac{3T}{m_e} |k|^2 + \mathcal{O}(|k|^4),$$

$$(2.9) \quad \nabla \Omega(k) = i \frac{3T}{m_e \omega_p} k + \mathcal{O}(|k|^3),$$

$$(2.10) \quad \nabla^2 \Omega(k) = i \frac{3T}{m_e \omega_p} \text{Id} + \mathcal{O}(|k|^2),$$

$$(2.11) \quad |\nabla^j \lambda(k)| \lesssim_{j,N} |k|^N \quad \text{for any } j, N \in \mathbb{N}.$$

Remark 11. This expansion of $\Im p_{\pm}(k) = \pm\Omega(k)$ provides the rigorous justification for the *Bohm–Gross dispersion* relation in kinetic theory. Regarding the real part $\Re p_{\pm}(k) = -\lambda(k)$, physicists assert that (see, e.g., [12, p. 419])

$$\lambda(k) \approx \sqrt{\pi} \frac{m_e^{3/2} \omega_p^4}{|k|^3 T^{3/2}} \exp\left(-\frac{m_e \omega_p^2}{4|k|^2 T}\right);$$

however, at this time we lack a mathematically rigorous explanation for this exact prediction.

Proof. Since $\widehat{f^0}$ is real, $p_+ = \overline{p_-}$ and it is enough to build the branch near $+i\omega_p$. By the implicit function theorem applied to the function $\mathcal{L}$ of $(z, k) \in \mathbb{C} \times \mathbb{R}^d$, the result will follow by verifying $\partial_z \mathcal{L}(i\omega_p, 0) \neq 0$ along with observing that $\mathcal{L}$ is smooth in (z, k) and analytic in z in this neighborhood. Roughly speaking we want to take derivatives of the expansion (2.6). From (2.3) and integrating by parts as in the proof of Lemma 2.2,

$$\begin{aligned} \partial_z \mathcal{L}(z, k) &= \frac{w_0}{|k|^3} \int_0^\infty e^{-\frac{z}{|k|}s} s^2 \widehat{f^0}(\hat{k}s) ds = \frac{w_0}{z^2 |k|} \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^2 \left(s^2 \widehat{f^0}(\hat{k}s) \right) ds \\ &= \frac{2w_0 n_0}{z^3} + \frac{w_0}{z^3} \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^3 \left(s^2 \widehat{f^0}(\hat{k}s) \right) ds \\ &= \frac{2w_0 n_0}{z^3} + \frac{6w_0 |k|}{z^4} \left[\hat{k} \cdot \nabla \widehat{f^0}(0) \right] + \frac{w_0 |k|}{z^4} \int_0^\infty e^{-\frac{z}{|k|}s} \partial_s^4 \left(s^2 \widehat{f^0}(\hat{k}s) \right) ds \\ &= \frac{2w_0 n_0}{z^3} + \frac{w_0 |k|}{z^4} \zeta'(z, k), \end{aligned}$$

with $\zeta'(z, k)$ that is uniformly bounded for $|z - i\omega_p| < \epsilon$ and $|k| < \nu_0$ (using the analyticity of f^0). From the above calculation we can also observe that $\mathcal{L}(z, k)$ is analytic in z in this neighborhood. Moreover, $\partial_z \mathcal{L}(i\omega_p, 0) = \frac{2\omega_p^2}{(i\omega_p)^3} = \frac{2i}{\omega_p} \neq 0$. By the same integration by parts method used in Lemma 2.2, we obtain

$$(2.12) \quad \nabla_k \mathcal{L}(z, k) = -\frac{w_0}{|k|^3} \int_0^\infty e^{-\frac{z}{|k|}s} s^2 \nabla \widehat{f^0}(\hat{k}s) ds = \frac{2w_0 k}{z^4} \left[\hat{k} \otimes \hat{k} : \nabla^2 \widehat{f^0}(0) \right] + \mathcal{O}(|k|^3).$$

Similar, but lengthier, calculations verify that near $z = \pm i\omega_p$, $\mathcal{L}$ is also smooth in k . The implicit function theorem then implies the existence of a unique smooth solution $k \mapsto p_+(k)$ to the equation $\mathcal{L}(p_+(k), k) = 1$ in a neighborhood of $i\omega_p$.

To get more precise information on the behavior of the poles near $k = 0$, we need to compute derivatives in k as well as $\mathcal{L}$. Expansion (2.8) immediately follows from Lemma 2.2:

$$p_+^2 = p_+^2 \mathcal{L}(p_+, k) = \omega_p^2 - \frac{3T\omega_p^2}{m_e p_+^2} |k|^2 + \mathcal{O}(|k|^4) = \omega_p^2 + \frac{3T}{m_e} |k|^2 + \mathcal{O}(|k|^4).$$

Next, observe that

$$(2.13) \quad \nabla p_+ = -\frac{(\nabla_k \mathcal{L})(p_+, k)}{(\partial_z \mathcal{L})(p_+, k)},$$

$$(2.14) \quad \nabla^2 p_+ = -(\partial_z \mathcal{L})(p_+, k)^{-1} \left[(\nabla_k^2 \mathcal{L})(p_+, k) + (\nabla_k \partial_z \mathcal{L})(p_+, k) \cdot \nabla p_+ + (\partial_z^2 \mathcal{L})(p_+, k) |\nabla p_+|^2 \right].$$

Using $\nabla^2 \widehat{f^0}(0) = 3n_0 \frac{T}{m_e} \text{Id}$ and $w_0 = \omega_p^2 n_0^{-1}$ in (2.12), we obtain

$$\nabla_k \mathcal{L}(p_+(k), k) = \frac{6T}{m_e \omega_p^2} k + \mathcal{O}(|k|^3),$$

which implies (2.9). The proof of (2.10) is similar, using now (2.14) instead of (2.13); the lengthier calculations are omitted for the sake of brevity. In these calculations a clear pattern emerges: in all derivatives $\nabla_k^m \mathcal{L}(z, k)$ (resp., $\nabla_k^m \partial_z \mathcal{L}(z, k)$), for $m \in \mathbb{N}$, the leading order as $k \rightarrow 0$ is an even (resp., odd) power of z^{-1} , and thus at $z = \pm i\omega_p$ all derivatives $\nabla_k^m \mathcal{L}(i\omega_p, 0)$ (resp., $\nabla_k^m \partial_z \mathcal{L}(i\omega_p, 0)$) are purely real (resp., purely imaginary). This proves all derivatives $\nabla^m p_+(0)$ are purely imaginary and thus implies (2.11), i.e., $\Re p_+ = -\lambda$ vanishes to infinite order at $k = 0$. Observe that since nevertheless $\Re p_+ < 0$ as $k \sim 0$ (by Lemma 2.1), the function $k \mapsto p_+(k)$ differs from its Taylor series at $k = 0$ and is therefore not analytic. $\square$

2.4. Spectral surgery and extraction of Klein–Gordon waves. Through (2.2), the solution to the Volterra equation (2.1) is classically [15] given formally as

$$(2.15) \quad \hat{\rho}(t, k) = \widehat{h_{in}}(k, kt) + \int_0^t \mathcal{R}(t - \tau, k) \widehat{h_{in}}(k, k\tau) d\tau,$$

where the *resolvent kernel* $\mathcal{R}$ is given by the inverse Laplace transform

$$\mathcal{R}(t, k) = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)} dz$$

for a suitable Bromwich contour such that $z \mapsto \frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)}$ is holomorphic for $\Re z > \gamma - 0$.

The calculations in subsections 2.2 and 2.3 show that for $|k| < \nu_0$ sufficiently small, $\frac{\mathcal{L}(\cdot, k)}{1 - \mathcal{L}(\cdot, k)}$ is holomorphic in the region $\mathbb{H}_{\epsilon, \delta'}$ represented in Figure 2 (the half-plane $\Re z \leq -\delta'|k|$ minus ϵ -discs around for the two poles), with one isolated pole $p_{\pm}(k)$ in each disc, depending on k as studied in the last subsection. Therefore, by Cauchy's residue theorem,

$$\begin{aligned} \mathcal{R}(t, k) &= \left(\frac{e^{p_+(k)t}}{-\partial_z \mathcal{L}(p_+(k), k)} + \frac{e^{p_-(k)t}}{-\partial_z \mathcal{L}(p_-(k), k)} \right) + \frac{1}{2i\pi} \int_{\gamma' - i\infty}^{\gamma' + i\infty} e^{zt} \frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)} dz \\ &=: \underbrace{\mathcal{R}_{KG}^+(t, k) + \mathcal{R}_{KG}^-(t, k)}_{\mathcal{R}_{KG}(t, k)} + \mathcal{R}_{RFT}(t, k) \end{aligned}$$

for some $\gamma' \in (-\delta'|k|, 0)$ so that the vertical line is to the *left* of the poles $p_{\pm}(k)$ but still in $\mathbb{H}_{\epsilon, \delta'}$, where $\mathcal{L}/(1 - \mathcal{L})$ is meromorphic (always possible for $|k|$ sufficiently small since $|\Re p_{\pm}(k)| \lesssim |k|^{-\infty}$). This decomposes the resolvent $\mathcal{R} = \mathcal{R}_{KG} + \mathcal{R}_{RFT}$ into a *Klein–Gordon* part and a *remainder free transport* part, the former named thus because the poles satisfy to leading order (1.3) and due to the Klein–Gordon-type E_{KG} contribution in Theorem 1. This yields a corresponding decomposition of the density $\hat{\rho}(t, k)$ through (2.15):

$$\begin{aligned} \hat{\rho}(t, k) &= \widehat{h_{in}}(k, kt) + \int_0^t \mathcal{R}_{KG}(t - \tau, k) \widehat{h_{in}}(k, k\tau) d\tau + \int_0^t \mathcal{R}_{RFT}(t - \tau, k) \widehat{h_{in}}(k, k\tau) d\tau \\ &=: \hat{\rho}_{FT}(t, k) + \hat{\rho}_{KG}^+(t, k) + \hat{\rho}_{KG}^-(t, k) + \hat{\rho}_{RFT}(t, k). \end{aligned}$$

We first prove a general expansion of $\hat{\rho}_{KG}^\pm(t, k)$ by successive integrations in time (producing additional powers of k). Terms in this expansion are either comparable to solutions to free transport (with additional Fourier multipliers) or to solutions to a Klein–Gordon-like evolution equation.

LEMMA 2.5 (expansion of the Klein–Gordon density). *For all $|k| < \nu_0$ and all $\ell \in \mathbb{N}$, we have*

$$(2.16) \quad \begin{aligned} \hat{\rho}_{KG}^\pm(t, k) &= \sum_{j=0}^{\ell} e^{p_\pm(k)t} A_j^\pm(k) \left[k^{\otimes j} : \nabla_\eta^j \widehat{h_{in}}(k, 0) \right] - \sum_{j=0}^{\ell} A_j^\pm(k) \left[k^{\otimes j} : \nabla_\eta^j \widehat{h_{in}}(k, kt) \right] \\ &\quad + \int_0^t \mathcal{R}_{KG}(t - \tau, k) A_{\ell+1}^\pm(k) \left[k^{\otimes(\ell+1)} : \nabla_\eta^{\ell+1} \widehat{h_{in}}(k, k\tau) \right] d\tau, \end{aligned}$$

where $\nabla_\eta \widehat{h_{in}}(k, \eta)$ is the differential in the second Fourier variable, and with the notation

$$A_j^\pm(k) := -\frac{J_\pm(k)}{p_\pm(k)^{j+1}} \quad \text{and} \quad J_\pm(k) := -\frac{1}{\partial_z \mathcal{L}(p_\pm(k), k)}.$$

Remark 12. Note that the Fourier multipliers $A_j^\pm$ are smooth and bounded for $|k| < \nu_0$.

Proof. Integrating by parts in time gives

$$\begin{aligned} \hat{\rho}_{KG}^\pm(t, k) &= \int_0^t \mathcal{R}_{KG}^\pm(t - \tau, k) \widehat{h_{in}}(k, k\tau) d\tau \\ &= \int_0^t J_\pm(k) e^{p_\pm(k)(t-\tau)} \widehat{h_{in}}(k, k\tau) d\tau \\ &= - \int_0^t \frac{J_\pm(k)}{p_\pm(k)} \partial_\tau \left(e^{-\lambda(k)(t-\tau) \pm i\Omega(k)(t-\tau)} \right) \widehat{h_{in}}(k, k\tau) d\tau \\ &= \frac{J_\pm(k)}{p_\pm(k)} H(k, kt) - \frac{J_\pm(k)}{p_\pm(k)} e^{p_\pm(k)t} \widehat{h_{in}}(k, 0) + \int_0^t \frac{J_\pm(k)}{p_\pm(k)} e^{p_\pm(k)(t-\tau)} \left[k \cdot \nabla_\eta \widehat{h_{in}}(k, k\tau) \right] d\tau, \end{aligned}$$

and iterating finitely many times yields the result. $\square$

Note that by symmetry $J_+(k) = \overline{J_-(k)}$ and $A_j^-(k) = \overline{A_j^+(k)}$, and the calculations of Lemma 2.4 give the expansion $J_\pm(k) = \mp \frac{\omega_p}{2i} + \mathcal{O}(|k|^2)$, which allows us to expand the coefficients $A_j^\pm(k)$ in (2.16). Denoting $p_\pm(k) = -\lambda(k) \pm i\Omega(k)$ with $\lambda(k) > 0$ and $\Omega(k) = \omega_p + \mathcal{O}(|k|^2)$, it immediately implies the following lemma.

LEMMA 2.6 (Klein–Gordon coefficients at low frequencies). *One has, as $k \rightarrow 0$,*

$$(2.17) \quad \begin{aligned} A_0^+(k) + A_0^-(k) &= 1 + \mathcal{O}(|k|^2), \\ A_1^+(k) + A_1^-(k) &= \mathcal{O}(|k|^2), \\ e^{p_+(k)t} A_0^+(k) + e^{p_-(k)t} A_0^-(k) &= e^{-\lambda(k)t} \left[\cos[\Omega(k)t] + \mathcal{O}(|k|^2) e^{i\Omega(k)t} + \mathcal{O}(|k|^2) e^{-i\Omega(k)t} \right], \\ e^{p_+(k)t} A_1^+(k) + e^{p_-(k)t} A_1^-(k) &= e^{-\lambda(k)t} \left[\frac{\sin[\Omega(k)t]}{\Omega(k)} + \mathcal{O}(|k|^2) e^{i\Omega(k)t} + \mathcal{O}(|k|^2) e^{-i\Omega(k)t} \right], \end{aligned}$$

where the $\mathcal{O}(|k|^2)$ in the above represent infinitely differentiable, bounded functions of k which are independent of time.

We are now able to precisely define our decomposition of the electric field.

DEFINITION 2.1. We define the variable precision decomposition,

$$\begin{aligned}
 \widehat{E}_{LD}^{(1;\ell)}(t, k) &= w_0 \frac{ik}{|k|^2} [1 - A_0^+(k) - A_0^-(k)] \widehat{h}_{in}(k, kt) \\
 &\quad - w_0 \frac{ik}{|k|^2} \sum_{j=1}^{\ell} A_j^{\pm}(k) \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h}_{in}(k, kt) \right), \\
 \widehat{E}_{LD}^{(2)}(t, k) &= w_0 \frac{ik}{|k|^2} \int_0^t \mathcal{R}_{RFT}(t - \tau, k) \widehat{h}_{in}(k, k\tau) d\tau, \\
 \widehat{E}_{KG}^{(1;\ell)}(t, k) &= w_0 \frac{ik}{|k|^2} \sum_{j=0}^{\ell} e^{p_{\pm}(k)t} A_j^{\pm}(k) \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h}_{in}(k, 0) \right), \\
 \widehat{E}_{KG}^{(2;\ell)}(t, k) &= w_0 \frac{ik}{|k|^2} \int_0^t \mathcal{R}_{KG}(t - \tau, k) A_{\ell+1}^{\pm}(k) \left[k^{\otimes(\ell+1)} : \nabla_{\eta}^{\ell+1} \widehat{h}_{in}(k, k\tau) \right] d\tau,
 \end{aligned}
 \tag{2.18}$$

and accordingly define the particular decomposition we shall use in what follows (setting $\ell = 4$),

$$\begin{aligned}
 E_{LD}^{(1)} &:= E_{LD}^{(1;4)}, & E_{LD}^{(2)} &:= \text{as above}, \\
 E_{KG}^{(1)} &:= E_{KG}^{(1;4)}, & E_{KG}^{(2)} &:= E_{KG}^{(2;4)}, \\
 E_{LD} &:= E_{LD}^{(1)} + E_{LD}^{(2)}, & E_{KG} &:= E_{KG}^{(1)} + E_{KG}^{(2)}.
 \end{aligned}$$

Remark 13. Note crucially that (2.17) *cancels* the leading order of the free transport evolution for long waves as $k \rightarrow 0$, as can be seen in the first term in (2.18) recalling the expansion (2.17). This ultimately leads to an improved decay of the Landau damping contribution of the electric field.

Remark 14. The choice $\ell = 4$ was chosen large enough so that the expansions can all be seen to be distinct in nature and in decay rates.

Next, we estimate the “remainder free transport part” of the resolvent. The gain in powers of k present in Lemma 2.7 is critical to the high quality decay rate of the Landau damping electric field.

LEMMA 2.7 (remainder free transport resolvent at low frequencies). *There exists $\lambda_0 > 0$ such that for all $|k| < \nu_0$ there holds*

$$\forall |k| < \nu_0, \quad |\mathcal{R}_{RFT}(t, k)| \lesssim |k|^3 e^{-\lambda_0 |k|t}.$$

Proof. We add and subtract by the expected leading order behavior as $k \rightarrow 0$ (using that the integration path is away from $z = 0$), hence defining for $\alpha := 3T\omega_p^2 m_e^{-1}$

$$\mathcal{Q}(z) = \frac{\omega_p^2}{z^2 + \omega_p^2} + \frac{\alpha |k|^2}{(z^2 + \omega_p^2)^2}.$$

The function $z \mapsto e^{zt} \mathcal{Q}(z, k)$ is holomorphic and decaying in the *left* half-plane $\Re z < -\gamma' |k|$, and hence we can deform the contour defined by the standard Bromwich

contour (from $-\gamma' - i\infty$ to $-\gamma' - i\infty$) to a new contour which basically follows the boundary of $\Lambda_{\delta,\delta'}$, the region of validity of the asymptotic expansion in Lemma 2.2. Accordingly, we design the new contour into three natural pieces: a center part being a contribution with $\Re z = -\delta|k|$ with $\Im z \in (-R|k|, R|k|)$ and two “tails” connecting this central contour to infinity along diagonal lines. This contour is visualized below in Figure 4. That is, we have the following:

$$\begin{aligned}\mathcal{R}_{RFT}(t, k) &= \frac{1}{2i\pi} \left(\int_{\Gamma_0} + \int_{\Gamma_+} + \int_{\Gamma_-} \right) e^{zt} \left(\frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)} + \mathcal{Q}(z, k) \right) dz \\ &=: \mathcal{R}_{RFT}^0 + \mathcal{R}_{RFT}^+ + \mathcal{R}_{RFT}^- \quad \text{with} \\ \Gamma_0 &= \left\{ z = \lambda + i\omega : \lambda = -\delta|k|, \omega \in (-R|k|, R|k|) \right\}, \\ \Gamma_+ &= \left\{ z = \lambda - i(1+\delta)\lambda + i[R - (1+\delta)\delta]|k| : \lambda \in (-\infty, -\delta|k|] \right\}, \\ \Gamma_- &= \left\{ z = \lambda + i(1+\delta)\lambda - i[R - (1+\delta)\delta]|k| : \lambda \in (-\infty, -\delta|k|] \right\}.\end{aligned}$$

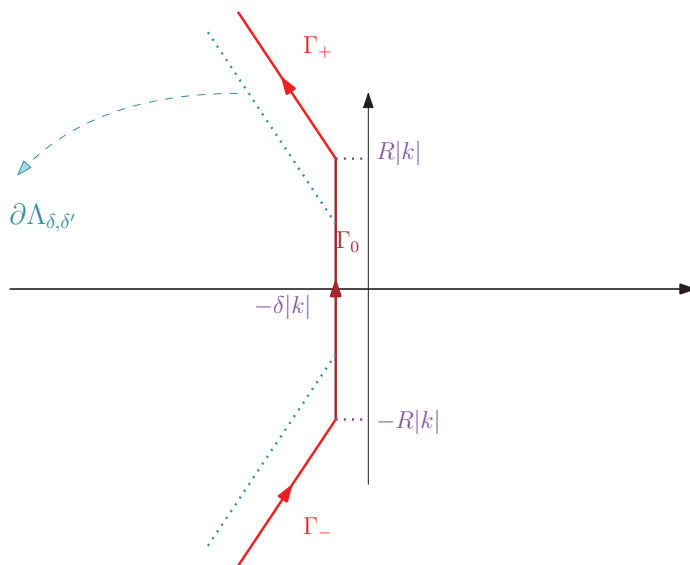


FIG. 4. The contour of integration (note that it is slightly modified as compared to $\partial\Lambda_{\delta,\delta'}$).

We separate cases as in Lemma 2.3. Consider first $z \in \Gamma_0$ with $|\Im z| < \delta'|k|$. As in the proof of Lemma 2.3, in this region there holds the following expansion, valid for δ' sufficiently small:

$$\forall z \in \Gamma_0 \text{ with } |\Im z| < \delta'|k|, \quad \mathcal{L}(\Im z, k) = -\frac{\omega_p^2}{|k|^2} + \mathcal{O}\left(\frac{|\Im z|}{|k|^3}\right);$$

for all $|z| \lesssim |k|$,

$$|\mathcal{L}(z, k)| \lesssim |k|^{-2}, \quad |\partial_z \mathcal{L}(z, k)| \lesssim |k|^{-3}.$$

Therefore, for δ and δ' sufficiently small, we have $\mathcal{L}(z, k) \approx |k|^{-2}$ for $z \in \Gamma_0$ with $|\Im z| < \delta' |k|$. Next consider the case $z \in \Gamma_0$ with $\delta' |k| \leq |\Im z| \leq R |k|$. By arguments similar to those above and in Lemma 2.3, we have $|1 - \mathcal{L}(z, k)| \gtrsim |k|^{-2}$ and $|\mathcal{L}(z, k)| \lesssim |k|^{-2}$. Therefore, on Γ_0 , the integrand is $\mathcal{O}(|k|^2)$, resulting in the estimate

$$\left| \int_{\Gamma_0} e^{zt} \left(\frac{z^2 \mathcal{L}(z, k) + \omega_p^2}{(1 - \mathcal{L}(z, k))(z^2 + \omega_p^2)} + \frac{\alpha |k|^2}{(z^2 + \omega_p^2)^2} \right) dz \right| \lesssim |k|^3 e^{-\delta |k|t}.$$

This completes the estimates on Γ_0 .

We next turn to Γ_+, Γ_- . We need only consider Γ_+ , as Γ_- is analogous. We now use $|z| \gg |k|$ and the decomposition from Lemma 2.2:

$$\mathcal{L}(z, k) = -\frac{\omega_p^2}{z^2} - \frac{3\omega_p^2 T |k|^2}{m_e z^4} + \frac{|k|^4}{z^6} \zeta(z, k),$$

with $\zeta(z, k)$ uniformly bounded on Γ_+ and decaying at infinity. Using this expansion, we have

$$\begin{aligned} \frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)} + \mathcal{Q}(z, k) &= \frac{z^2 \mathcal{L}(z, k) + \omega_p^2}{(1 - \mathcal{L})(z^2 + \omega_p^2)} + \frac{\alpha |k|^2}{(z^2 - \omega_p^2)(z^2 + \omega_p^2)} \\ &= \frac{\alpha k^2 (z^2 \mathcal{L} + \omega_p^2)}{(z^2 - z^2 \mathcal{L})(z^2 + \omega_p^2)^2} + \frac{\zeta(z, k) k^4}{z^2 (z^2 - z^2 \mathcal{L})(z^2 + \omega_p^2)} \\ &= \frac{\alpha k^2 (\alpha k^2 + \zeta(z, k) k^4 z^{-2})}{(z^2 - z^2 \mathcal{L})(z^2 + \omega_p^2)^2} + \frac{\zeta(z, k) k^4}{z^2 (z^2 - z^2 \mathcal{L})(z^2 + \omega_p^2)}. \end{aligned}$$

Using the uniform boundedness of $\zeta(z, k)$, integration gives

$$|\mathcal{R}_{RFT}^+| \lesssim e^{-\delta |k|t} \int_{R|k|}^{\infty} \frac{|k|^4}{x^2} dx = |k|^3 e^{-\delta |k|t}.$$

This completes the proof of Lemma 2.7. $\square$

We finally estimate the whole resolvent at frequencies bounded away from zero, which is simpler.

LEMMA 2.8 (nonsmall frequencies resolvent estimate). *Given any $\nu_0 > 0$ there is $\lambda_1 > 0$ such that*

$$\forall |k| \geq \frac{\nu_0}{2}, \quad \mathcal{R}(t, k) \lesssim \frac{1}{|k|} e^{-\lambda_1 |k|t}.$$

Proof. Choose $\lambda > 0$ as in Lemma 2.1 and deform the contour to get (there are no poles in $\Re z \geq -\lambda |k|$ when $|k| \geq \nu_0/2$)

$$\mathcal{R}(t, k) = \frac{1}{2i\pi} \int_{-\lambda |k| - i\infty}^{-\lambda |k| + i\infty} e^{zt} \frac{\mathcal{L}(z, k)}{1 - \mathcal{L}(z, k)} dz,$$

and using the estimate of Lemma 2.1 gives

$$|\mathcal{R}(t, k)| \lesssim e^{-\lambda |k|t} \int_{-\infty}^{\infty} \frac{1}{|k|^2 + |\omega|^2} d\omega \lesssim \frac{1}{|k|} e^{-\lambda |k|t}. \quad \square$$

2.5. Another look at the long-wave hydrodynamic behavior. In this subsection we prove Theorem 3 on the basis of the decomposition of Definition 2.1 and the previous estimates of this section.

Proof of Theorem 3. We prove the first part of the theorem, as the second part is seen directly from the decomposition of Definition 2.1, together with some basic estimates that are similar to or easier than what is required to prove the first part. For any field F , define

$$F_\epsilon(t, x) := \frac{1}{\epsilon^3} F\left(t, \frac{x}{\epsilon}\right), \quad \text{with Fourier transform} \quad \widehat{F}_\epsilon(t, k) = \widehat{F}(t, \epsilon k).$$

Note that we have defined h_{in} such that for all $\epsilon > 0$, $h_{in, \epsilon}(x, v) = \mathcal{H}_0(x, v)$. Define the initial macroscopic flux (recall that n_0 is the total mass of $f^0(v)$),

$$j_{in}(x) := \frac{1}{n_0} \int_{\mathbb{R}^3} v h_{in} dv \quad \text{with Fourier transform} \quad \widehat{j}_{in}(k) = \frac{1}{n_0} i \nabla_\eta \widehat{h}_{in}(k, 0).$$

Denote

$$\widehat{J}(k) = \frac{1}{n_0} i \nabla_\eta \widehat{\mathcal{H}}_0(k).$$

The rescaled electric field $\mathcal{E}_\epsilon$ in the linearized Euler–Poisson system (1.10) satisfies

$$\epsilon \widehat{\mathcal{E}}_\epsilon(t, k) = \frac{ik}{|k|^2} \widehat{\mathcal{H}}_0(k) \cos \Omega_{KG}(\epsilon k) t + \epsilon \omega_p^2 \frac{k}{|k|^2} \left(k \cdot \widehat{J}(k) \right) \frac{\sin \Omega_{KG}(\epsilon k) t}{\Omega_{KG}(\epsilon k)},$$

where we define the exact Euler–Poisson imaginary phase $\Omega_{KG}(k)$,

$$\Omega_{KG}(k) := \sqrt{\omega_p^2 + \frac{3T}{m} |k|^2}.$$

Consider $E_{KG}^{(1)}$ first defined in the decomposition of Definition 2.1, since it is the contribution which is asymptotic to $\mathcal{E}_\epsilon$. Lemma 2.6 shows that

$$\begin{aligned} \epsilon \widehat{E}_{KG, \epsilon}^{(1)}(t, k) &= e^{-\lambda(\epsilon k)t} \frac{ik}{|k|^2} \widehat{\mathcal{H}}_0(k) \cos [\Omega(\epsilon k)t] + e^{-\lambda(\epsilon k)t} \epsilon \omega_p^2 \frac{ik}{|k|^2} \left(k \cdot \widehat{J}(k) \right) \frac{\sin [\Omega(\epsilon k)t]}{\Omega(\epsilon k)} \\ &\quad + \mathcal{O}(\epsilon |\epsilon k|^2) e^{-\lambda(\epsilon k)t} \|\mathcal{H}_0\|_{W_4^{0,1}}. \end{aligned}$$

Note that the errors depend on time, but in a uniformly bounded way, and that they depend on v -moments of h_{in} up to order 5. By the expansion of Ω in Lemma 2.4, we have

$$|\Omega(\epsilon k) - \Omega_{KG}(\epsilon k)| \lesssim \left| \sqrt{\omega_p^2 + \frac{3T}{m} |\epsilon k|^2} + \mathcal{O}(|\epsilon k|^4) - \sqrt{\omega_p^2 + \frac{3T}{m} |\epsilon k|^2} \right| \lesssim |\epsilon k|^4$$

and thus

$$\begin{aligned} \left| \cos [\Omega(\epsilon k)t] - \cos [\Omega_{KG}(\epsilon k)t] \right| &= \mathcal{O}(|\epsilon k|^4 t), \\ \left| \sin [\Omega(\epsilon k)t] - \sin [\Omega_{KG}(\epsilon k)t] \right| &= \mathcal{O}(|\epsilon k|^4 t), \end{aligned}$$

and for any $N > 0$, $|1 - e^{-\lambda(\epsilon k)t}| = \mathcal{O}(|\epsilon k|^N t)$, and therefore we deduce

$$\epsilon \left| \widehat{E}_{KG,\epsilon}^{(1)} - \widehat{\mathcal{E}}_\epsilon \right| \lesssim \left(|\epsilon k|^4 t + \epsilon |\epsilon k|^2 \right) \|\mathcal{H}_0\|_{W_4^{0,1}},$$

and thus

$$\begin{aligned} \epsilon \left\| E_{KG,\epsilon}^{(1)}(t) - \mathcal{E}_\epsilon(t) \right\|_{H^{-s}} &\lesssim \|\mathcal{H}_0\|_{W_4^{0,1}} \left(\int_{|\epsilon k| \lesssim 1} \frac{\epsilon^2 |\epsilon k|^4 + |\epsilon k|^8 t^2}{\langle k \rangle^{2s}} dk \right)^{\frac{1}{2}} \\ &\lesssim \|\mathcal{H}_0\|_{W_4^{0,1}} \epsilon^{s-\frac{3}{2}-0} \langle t \rangle \end{aligned}$$

for $s \in (\frac{3}{2}, \frac{7}{2})$. Turn next to $\widehat{E}_{KG}^{(2)}$; from its definition

$$\begin{aligned} \epsilon \left\| E_{KG,\epsilon}^{(2)}(t) \right\|_{H^{-s}} &\lesssim \epsilon \int_0^t \left(\int_{|\epsilon k| \lesssim 1} \frac{1}{|\epsilon k|^2 \langle k \rangle^{2s}} |\epsilon k|^{10} \left| \nabla_\eta^5 \widehat{\mathcal{H}}_0(k, k\tau) \right|^2 dk \right)^{1/2} d\tau \\ &\lesssim \|\mathcal{H}_0\|_{W_4^{0,1}} \epsilon^{s-\frac{1}{2}-0} \end{aligned}$$

for $s \in (\frac{3}{2}, \frac{19}{2})$. This completes the treatment of the “Klein–Gordon parts” of the electric field.

We turn next to the Landau damping contributions, which in fact dominate the error. It is convenient to subdivide the Landau damping field as in Definition 2.1. The contribution of $E_{LD}^{(1)}$ is straightforward; indeed,

$$\epsilon \left\| E_{LD,\epsilon}^{(1)}(t) \right\|_{H^{-s}} \lesssim \epsilon \left(\sum_{j=0}^4 \int_{\mathbb{R}^3} \langle k \rangle^{-2s} |\epsilon k|^2 \left| \epsilon k^j \widehat{\nabla_\eta^j \mathcal{H}_0}(k, kt) \right|^2 dk \right)^{\frac{1}{2}} \lesssim \|\mathcal{H}_0\|_{W_4^{2,1}} \frac{\epsilon^2}{\langle t \rangle}$$

for $s > \frac{5}{2}$. Turn finally to $E_{LD}^{(2)}$, which produces the dominant error. Then

$$\begin{aligned} \epsilon \left\| E_{LD,\epsilon}^{(2)}(t) \right\|_{H^{-s}} &\lesssim \epsilon \int_0^t \left(\int_{\mathbb{R}^3} \frac{\min(|\epsilon k|^2, |\epsilon k|^{-1})}{\langle k \rangle^{2s}} e^{-\lambda_0 |\epsilon k|(t-\tau)} \left| \widehat{\mathcal{H}}_0(k, k\tau) \right|^2 dk \right)^{1/2} d\tau \\ &\lesssim \epsilon^2 \|\mathcal{H}_0\|_{W_0^{2,1}} \end{aligned}$$

for $s > \frac{5}{2}$. This completes the proof of Theorem 3. $\square$

3. Electric field estimates.

3.1. Landau damping estimates on the electric field. In this section we provide estimates for E_{LD} . We start with the optimal decay estimates for the density for the kinetic free transport (optimal in terms of time decay, not in the dependence on the initial data). Denote the spatial density of the solution to the free transport equation

$$\mathfrak{H}(t, x) := \int_{\mathbb{R}^3} h_{in}(x - tv, v) dv \quad \text{with Fourier transform} \quad \widehat{\mathfrak{H}}(t, k) = \widehat{h_{in}}(k, kt).$$

LEMMA 3.1. *For all $\sigma \geq 0$,*

$$(3.1) \quad \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma \mathfrak{H}(t, \cdot) \right\|_{L_x^1} \lesssim \|h_{in}\|_{W_0^{\sigma,1}},$$

$$\left\| \langle \nabla_x, t \nabla_x \rangle^\sigma \mathfrak{H}(t, \cdot) \right\|_{L_x^2} \lesssim \langle t \rangle^{-3/2} \|h_{in}\|_{W_0^{\sigma+3/2+0,1}},$$

$$(3.2) \quad \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma \mathfrak{H}(t, \cdot) \right\|_{L_x^\infty} \lesssim \langle t \rangle^{-3} \|h_{in}\|_{W_0^{\sigma+3+0,1}}.$$

More generally, for all $1 \leq p \leq \infty$, $j \geq 0$,

$$(3.3) \quad \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma \left(\nabla_x^{\otimes j} : \int_{\mathbb{R}^3} v^{\otimes j} h_{in}(\cdot - tv, v) dv \right) \right\|_{L_x^p} \lesssim \langle t \rangle^{-j-3(1-\frac{1}{p})} \|h_{in}\|_{W_j^{\sigma+3+j+0,1}}.$$

Proof. Inequality (3.1) is clear. To see (3.2) note that

$$\begin{aligned} \|\nabla_x^\sigma \mathfrak{H}(t, \cdot)\|_{L_x^\infty} &\lesssim \int_{\mathbb{R}^3} |k|^\sigma |\widehat{\mathfrak{H}}(t, k)| dk \lesssim \left(\int_{\mathbb{R}^3} \frac{|k|^\sigma}{\langle k, kt \rangle^{3+\sigma+}} dk \right) \|h_{in}\|_{W_0^{\sigma+3+1}} \\ &\lesssim \langle t \rangle^{-3-\sigma} \|h_{in}\|_{W_0^{\sigma+3+1}}. \end{aligned}$$

The proof of (3.3) follows similarly (using also interpolation). $\square$

Next, we turn to estimates on the damped part of the electric field. By Lemmas 2.6 and 3.1, $E_{LD}^{(1)}$ satisfies the estimates claimed in Theorem 1.

We turn next to obtaining estimates on $E_{LD}^{(2)}$.

LEMMA 3.2. *There holds the estimates*

$$(3.4) \quad \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma E_{LD}^{(2)}(t) \right\|_{L_x^2} \lesssim \frac{1}{\langle t \rangle^{5/2}} \|h_{in}\|_{W_0^{\sigma+2+0,1}},$$

$$(3.5) \quad \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma E_{LD}^{(2)}(t) \right\|_{L_x^\infty} \lesssim \frac{1}{\langle t \rangle^4} \|h_{in}\|_{W_0^{\sigma+3+0,1}}.$$

Proof. Consider first the low spatial frequencies $|k| \lesssim \nu_0$. Compute using Lemma 2.7; for any $a > 0$,

$$\begin{aligned} \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma P_{\leq \nu_0} E_{LD}^{(2)}(t) \right\|_{L_x^\infty} &\leq \int_0^t \left\| \langle \nabla_x, t \nabla_x \rangle^\sigma |\nabla_x|^{-1} P_{\leq \nu_0} R_{RFT}(t-\tau) *_x \mathfrak{H}(\tau, \cdot) \right\|_{L_x^\infty} d\tau \\ &\lesssim \int_0^t \int_{\mathbb{R}^3} \chi\left(\frac{k}{\nu_0}\right) \langle k, (t-\tau)k \rangle^\sigma |k|^{-1} |\mathcal{R}_{RFT}(t-\tau, k)| \langle k, \tau k \rangle^\sigma |\widehat{h_{in}}(k, \tau k)| dk d\tau \\ &\lesssim \left[\int_0^t \left(\int_{\mathbb{R}^3} |k|^2 \chi\left(\frac{k}{\nu_0}\right) \langle \tau k \rangle^{-3-a} \langle (t-\tau)k \rangle^{-3-a} dk \right) d\tau \right] \|h_{in}\|_{W_0^{\sigma+3+a,1}}. \end{aligned}$$

We split the integral

$$\begin{aligned} &\int_0^t \left(\int_{\mathbb{R}^3} |k|^2 \chi\left(\frac{k}{\nu_0}\right) \langle \tau k \rangle^{-3-a} \langle (t-\tau)k \rangle^{-3-a} dk \right) d\tau \\ &= \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \left(\int_{\mathbb{R}^3} |k|^2 \chi\left(\frac{k}{\nu_0}\right) \langle \tau k \rangle^{-3-a} \langle (t-\tau)k \rangle^{-3-a} dk \right) d\tau \end{aligned}$$

and change variables $k' = \tau k$ or $k' = (t-\tau)k$ in each one to obtain

$$\int_0^t \left(\int_{\mathbb{R}^3} \chi\left(\frac{k}{\nu_0}\right) |k|^2 \langle \tau k \rangle^{-3-a} \langle (t-\tau)k \rangle^{-3-a} dk \right) d\tau \lesssim t^{-4}.$$

Note that the $|k|^2$ in the numerator is crucial for obtaining this sharp rate. This concludes the proof of (3.5) at low frequencies. At frequencies bounded away from zero, i.e., the corresponding estimate on $P_{\geq \nu_0} E_{LD}^{(2)}$, the estimate is similar though more straightforward and with less loss of regularity. The L^2 case (3.4) follows similarly and is omitted for brevity. $\square$

3.2. Dispersive estimates of the electric field. We now consider the “Klein–Gordon part” of the electric field in Definition 2.1.

Consider first the following useful dispersive estimates.

LEMMA 3.3 (dispersive estimates for weakly damped poles). *We have $1 \leq p \leq 2$ and the following holds for $t \geq 0$ and $\mathbf{f} = \mathbf{f}(x) \in L^p(\mathbb{R}_x^3)$:*

$$(3.6) \quad \left\| e^{p\pm(\nabla)t} P_{\leq \nu_0} \mathbf{f} \right\|_{L^{p'}} \lesssim t^{-3(\frac{1}{p}-\frac{1}{2})} \|\mathbf{f}\|_{L^p}.$$

Remark 15. The linear propagator $e^{p\pm(\nabla)t}$ is not a unitary operator, and the standard TT^* argument as in, e.g., [24, 36] do not apply. Hence, it is not as trivial to obtain the homogeneous Strichartz estimates or the full range of expected inhomogeneous Strichartz estimates (although some inhomogeneous Strichartz estimates follow immediately from (3.6) and the Young–O’Neil convolution inequality [30]).

Proof. It suffices to consider $e^{p+}(\nabla)t$; the proof for $e^{p-}(\nabla)t$ is analogous. The case $p = 2$ is immediate. For the case $p = 1$, write

$$e^{p+}(\nabla)t P_{\leq \nu_0} \mathbf{f} = K(t) * \mathbf{f},$$

with the integral kernel

$$K(t, x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot k + it\Omega(k) - \lambda(k)t} a_{\nu_0}(k) dk,$$

where a_{ν_0} is a Schwartz class function compactly supported in a ball of radius $\leq 2\nu_0$ corresponding to the Littlewood–Paley projection. Hence the $p = 1$ case follows from the pointwise kernel estimate

$$(3.7) \quad \|K(t)\|_{L^\infty} \lesssim t^{-3/2} \quad (\forall t \geq 0).$$

The intermediate exponents $p \in (1, 2)$ are then obtained by the Riesz–Thorin interpolation theorem.

Let us prove (3.7). Despite the complex phase, K is essentially a standard oscillatory integral, and we may easily adapt the standard arguments as in, e.g., [34, Proposition 6, p. 344]. Let us first explain the argument assuming that $\Omega(k) = |k|^2$ (the case $\Omega^2(k) = \omega_p^2 + \frac{3T}{m_e}|k|^2$ can be treated similarly, as discussed below). In this case, we make the change of variables $y = k + \frac{x}{2t}$ and write (for some scale ϵ chosen below)

$$\begin{aligned} K(t, x) &= \frac{e^{i|x|^2/4t^2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{it|y|^2 - \lambda t} \chi\left(\frac{y}{\epsilon}\right) a_{\nu_0}\left(y - \frac{x}{2t}\right) dy \\ &\quad + \frac{e^{i|x|^2/4t^2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{it|y|^2 - \lambda t} \left[1 - \chi\left(\frac{y}{\epsilon}\right)\right] a_{\nu_0}\left(y - \frac{x}{2t}\right) dy \\ &=: K_S(t, x) + K_{NS}(t, x), \end{aligned}$$

where $\chi \in C_c^\infty(B(0, 1))$ with $\chi(z) = 1$ for $|z| \leq 1/2$, and where we have split K into a “stationary” part and a “nonstationary” part. For the stationary part, K_S , we simply bound the integrand and estimate the volume of integration:

$$(3.8) \quad |K_S(t, x)| \lesssim_{\nu_0} \epsilon^3.$$

For the nonstationary part, K_{NS} , we integrate by parts using the nonvanishing of the phase. In particular observe (noting that $|\cdot|^2 = x^*x$ for $x \in \mathbb{C}^n$) that

$$t |2iy - \nabla \lambda|^2 e^{it|y|^2 - \lambda t} = (-2iy - \nabla \lambda) \cdot \nabla_y e^{it|y|^2 - \lambda t}.$$

Therefore, if we define the differential operator

$$Df := \nabla_y \cdot \left(\frac{-2iy - \nabla \lambda}{|2iy - \nabla \lambda|^2} f \right),$$

then by repeated integration by parts we have

$$|K_{NS}| \lesssim \frac{1}{t^N} \left| \int_{\mathbb{R}^3} e^{it|y|^2 - \lambda t} D^N \left(\left(1 - \chi\left(\frac{y}{\epsilon}\right)\right) a_{\nu_0}\left(y - \frac{x}{2t}\right) \right) dy \right|.$$

We next show that the integrand is bounded by $|y|^{-2N}$; for the proof in three dimensions, we only need $N = 1, 2$, though it holds for all N . The case $N = 1$ is easily checked. Indeed,

$$\partial_j \left(\frac{-2iy - \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) = - \frac{-2ie_j - \partial_j \nabla \lambda}{|2iy - \nabla \lambda|^2} = \mathcal{O}(|y|^{-2}),$$

which is sufficient as the terms in which the derivative lands elsewhere are only better (estimates on $\nabla^n \lambda$ are provided by Lemma 2.4). For $N = 2$ we analogously have

$$\begin{aligned} \partial_\ell \left(\left(\frac{-2iy - \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) \frac{-2ie_j - \partial_j \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) &= - \left(\frac{-2ie_\ell - \partial_\ell \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) \left(\frac{-2ie_j - \partial_j \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) \\ &+ \left(\frac{-2iy - \nabla \lambda}{|2iy - \nabla \lambda|^2} \right) \left(\frac{\partial_\ell \nabla \lambda}{|2iy - \nabla \lambda|^2} - \frac{(-2ie_\ell - \partial_\ell \nabla \lambda)(2iy - \nabla \lambda) \cdot (-2ie_j - \partial_j \nabla \lambda)}{|2iy - \nabla \lambda|^4} \right) \\ &= \mathcal{O}(|y|^{-4}), \end{aligned}$$

which is similarly sufficient (note the pattern that selects a particular dominant term, whereas the more complicated error terms are smaller, hence the desired estimates hold for all N). Therefore, provided we choose $N \geq 2$,

$$|K_{NS}| \lesssim \frac{1}{t^N} \int_{|y| \geq \epsilon} \frac{1}{|y|^{2N}} dy \lesssim \frac{1}{t^N} \epsilon^{3-2N}.$$

Hence making the choice $\epsilon \sim t^{-1/2}$ gives the result when combined with (3.8). The case using the true $\Omega(k)$ follows by Morse's lemma due to (2.10) and the other estimates in Lemma 2.4 (see [34, Proposition 6, p. 344] for more details). This completes the main dispersive estimates (3.6). $\square$

The estimates on $E_{KG}^{(1)}$ in Theorem 1 follow immediately from (3.6) and the decomposition in Definition 2.1 upon observing that for $\int_{\mathbb{R}^3} \rho_{in} dx = 0$ we have for all $1 < p$

$$\|\nabla_x (-\Delta_x)^{-1} P_{\leq \nu_0} \rho_{in}\|_{L^p} \lesssim_{p, \nu_0} \|\langle x \rangle \rho_{in}\|_{W_0^{0,1}}.$$

Next, we will prove the pointwise-in-time decay estimates on $E_{KG}^{(2)}$.

LEMMA 3.4. *There holds for all $2 \leq p \leq \infty$*

$$\left\| E_{KG}^{(2)} \right\|_{L_x^p} \lesssim \langle t \rangle^{-3(\frac{1}{2} - \frac{1}{p})} \|h_{in}\|_{W_5^{3/2,1}}.$$

Proof. Define T as above:

$$T(t, x) = \int_{\mathbb{R}^3} v^{\otimes 5} h_{in}(x - tv, v) dv, \quad \widehat{T}(t, k) = i^5 \nabla_\eta^{\otimes 5} \widehat{h_{in}}(k, 0).$$

By Minkowski's inequality (and boundedness of the prefactor multipliers on L^p as they are Schwartz class functions),

$$\left\| E_{KG}^{(2)} \right\|_{L_x^p} \lesssim \sum_{+,-} \int_0^t \left\| e^{p_\pm(\nabla_x)(t-\tau)} P_{\leq \nu_0} |\nabla_x|^{-1} \nabla_x^5 : T \right\|_{L_x^p} d\tau.$$

On the one hand, by Bernstein's inequality, for any $p \geq r' > 2$, it follows from (3.6) that we have

$$\begin{aligned} \left\| e^{p_\pm(\nabla_x)(t-\tau)} |\nabla_x|^{-1} (\nabla_x^{\otimes 5} : T) \right\|_{L_x^p} &\lesssim \left\| e^{p_\pm(\nabla_x)(t-\tau)} P_{\leq \nu_0} |\nabla_x|^{-1} (\nabla_x^{\otimes 5} : T) \right\|_{L_x^{r'}} \\ &\lesssim \frac{1}{|t - \tau|^{3(\frac{1}{r} - \frac{1}{2})}} \left\| P_{\leq \nu_0} |\nabla_x|^{-1} (\nabla_x^{\otimes 5} T) \right\|_{L_x^r}. \end{aligned}$$

On the the other hand, we similarly have

$$\left\| e^{p_\pm(\nabla_x)(t-\tau)} |\nabla_x|^{-1} (\nabla_x^{\otimes 5} : T) \right\|_{L_x^p} \lesssim \left\| P_{\leq \nu_0} |\nabla_x|^{-1} (\nabla_x^5 : T) \right\|_{L_x^2}.$$

Therefore, by Lemma 3.1, we have for $r > 1$

$$\left\| E_{KG}^{(2)} \right\|_{L_x^p} \lesssim \|h_{in}\|_{W_5^{3/2,1}} \sum_{+,-} \int_0^t \min \left(\langle \tau \rangle^{-3/2-3/2}, |t - \tau|^{-3(\frac{1}{r} - \frac{1}{2})} \langle \tau \rangle^{-3(1 - \frac{1}{r}) - \frac{3}{2}} \right) d\tau,$$

which integrates to imply the stated result. $\square$

4. Decomposition and scattering for the distribution function. In this section we prove Theorem 2. Denote the solution in the free transport moving frame as

$$g(t, x, v) := h(t, x + vt, v),$$

which satisfies $\partial_t g = -E(t, x + tv) \cdot \nabla_v f^0(v)$. Therefore we have on the Fourier side (note $\widehat{h}(t, k, \eta) = \widehat{g}(t, k, \eta + kt)$)

$$\widehat{g}(t, k, \eta) = \widehat{h_{in}}(k, \eta) - \int_0^t \widehat{E}(\tau, k) \cdot \widehat{\nabla_v f^0}(\eta - k\tau) d\tau.$$

Consider the contribution $E_{KG}^{(1)}$:

$$\begin{aligned}
& \int_0^t \widehat{E}_{KG}^{(1)}(\tau, k) \cdot \widehat{\nabla_v f^0}(\eta - k\tau) d\tau \\
&= w_0 \frac{ik}{|k|^2} \sum_{j=0}^{\ell} \sum_{+,-} \int_0^t e^{p_{\pm}(k)\tau} A_j^{\pm}(k) \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h_{in}}(k, 0) \right) \cdot \widehat{\nabla_v f^0}(\eta - k\tau) d\tau \\
&= w_0 \frac{ik}{|k|^2} \sum_{j=0}^{\ell} \sum_{+,-} e^{p_{\pm}(k)t} \frac{A_j^{\pm}(k)}{p_{\pm}(k)} \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h_{in}}(k, 0) \right) \cdot \widehat{\nabla_v f^0}(\eta - kt) \\
&\quad - w_0 \frac{ik}{|k|^2} \sum_{j=0}^{\ell} \sum_{+,-} \frac{A_j^{\pm}(k)}{p_{\pm}(k)} \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h_{in}}(k, 0) \right) \cdot \widehat{\nabla_v f^0}(\eta) \\
&\quad - w_0 \frac{ik}{|k|^2} \sum_{j=0}^{\ell} \int_0^t e^{p_{\pm}(k)\tau} \frac{A_j^{\pm}(k)}{p_{\pm}(k)} \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h_{in}}(k, 0) \right) \cdot \nabla_{\eta} \widehat{\nabla_v f^0}(\eta - k\tau) k d\tau \\
&=: \widehat{g_{KG}} + \widehat{g_2} + \widehat{g_3},
\end{aligned}$$

with h_{KG} , h_2 , and h_3 defined analogously. Note that since $\widehat{h_{KG}}(t, k, \eta) = \widehat{g_{KG}}(t, k, \eta + kt)$,

$$\widehat{h_{KG}}(t, k, \eta) = \tilde{E}_{KG}(t, k) \cdot \widehat{\nabla_v f^0}(\eta),$$

where we define

$$\tilde{E}_{KG}(t, k) := w_0 \frac{ik}{|k|^2} \sum_{j=0}^4 \sum_{+,-} e^{p_{\pm}(k)t} \frac{A_j^{\pm}(k)}{p_{\pm}(k)} \left(k^{\otimes j} : \nabla_{\eta}^j \widehat{h_{in}}(k, 0) \right).$$

Arguing as for the $E_{KG}^{(1)}$ estimates in Theorem 1, h_{KG} satisfies (1.9). Similarly, we define

$$\begin{aligned}
(4.1) \quad \widehat{h_{LD}}(k, \eta) &= \widehat{h_{in}}(k, \eta) + \widehat{g_2}(k, \eta) + \widehat{g_3}(t, k, \eta) \\
&\quad - \int_0^t \left(\widehat{E_{KG}}^{(2)} + \widehat{E_{LD}} \right) (\tau, k) \cdot \widehat{\nabla_v f^0}(\eta - k\tau) d\tau.
\end{aligned}$$

The term g_2 is constant in time and in L^p for all $p \geq 2$ by the assumptions on the initial data. The term g_3 on the physical side is written in the form

$$h_3(t, x, v) = \int_0^t \nabla_x \tilde{E}_3(\tau, x + \tau v) : (v \otimes \nabla_v f^0(v)) d\tau$$

for a suitable $\tilde{E}_3$. By straightforward variations of the arguments used to estimate $E_{KG}^{(1)}$ above, we have for any $6 < p$

$$\begin{aligned}
\int_0^t \left\| \nabla_x \tilde{E}_3(\tau, x + \tau v) : (v \otimes \nabla_v f^0(v)) \right\|_{L_{x,v}^p} d\tau &\lesssim \int_0^t \left\| \nabla_x \tilde{E}_3(\tau, \cdot) \right\|_{L_x^p} d\tau \\
&\lesssim \|h_{in}\|_{W_5^{3+0,1}} \int_0^t \langle \tau \rangle^{-3(\frac{1}{2} - \frac{1}{p})} d\tau,
\end{aligned}$$

and hence h_3 converges in $L_{x,v}^p$ for all $p > 6$ as $t \rightarrow \infty$. Due to the decay estimates in Lemma 3.3, the term in (4.1) involving $E_{KG}^{(2)}$ similarly converges in $L_{x,v}^p$ for all $p > 6$.

Next we prove that h_{LD} converges in $L^p_{x,v}$ for $p > 6$. The easiest contribution is from E_{LD} . Recall the decomposition $E_{LD} = E_{LD}^{(1)} + E_{LD}^{(2)}$ from subsection 3.1. From a straightforward variant of Lemma 3.1, we have for all $2 \leq p \leq \infty$

$$\|E_{LD}(t)\|_{L^p} \lesssim \langle t \rangle^{-4+\frac{3}{p}} \|h_{in}\|_{W_0^{4+0,1}},$$

and hence, for $p \geq 2$,

$$\int_0^t \|E_{LD}(\tau, x + \tau v) \cdot \nabla_v f^0(v)\|_{L^p_{x,v}} d\tau \leq \int_0^t \|E_{LD}(\tau)\|_{L^p_x} \|\nabla_v f^0\|_{L^p_v} d\tau \lesssim \|h_{in}\|_{W_0^{4+0,1}},$$

and so the corresponding contribution converges as $t \rightarrow \infty$.

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