SINGULARITY FORMATION FOR BURGERS EQUATION WITH TRANSVERSE VISCOSITY

CHARLES COLLOT, TEJ-EDDINE GHOUL, AND NADER MASMOUDI

Abstract. We consider Burgers equation with transverse viscosity

$$\partial_t u + u\partial_x u - \partial_{yy} u = 0, \quad (x,y)\in\mathbb{R}^2, \quad u:[0,T)\times\mathbb{R}^2\to\mathbb{R}.$$

We construct and describe precisely a family of solutions which become singular in finite time by having their gradient becoming unbounded. To leading order, the solution is given by a backward self-similar solution of Burgers equation along the x variable, whose scaling parameters evolve according to parabolic equations along the y variable, one of them being the quadratic semi-linear heat equation. We develop a new framework adapted to this mixed hyperbolic/parabolic blow-up problem, revisit the construction of flat blow-up profiles for the semi-linear heat equation, and the self-similarity in singularities of the inviscid Burgers equation.

1. Introduction

1.1. Setting of the problem and motivations

We consider Burgers equation with transverse viscosity

$$\begin{cases} \partial_t u + u\partial_x u - \partial_{yy} u = 0, & (x,y)\in\mathbb{R}^2, \\ u_{t=0} = u_0, \end{cases} \tag{1.1}$$

eq:burgers

for $u:[0,T)\times\mathbb{R}^2\to\mathbb{R}$. The present study is motivated by the following. This model reduces to the classical inviscid Burgers equation for solutions that are independent of the transverse variable $u(t,x,y)=U(t,x)$, which is a classical example of a nonlinear hyperbolic equation for which initially smooth solutions can become singular in finite time, see for example [11, 27]. The effects of viscosity in the streamwise direction, namely the equation $\partial_t u + u\partial_x u - \epsilon\partial_{xx} u = 0$, have been extensively studied, see [18, 19] and references therein. This work aims at understanding precisely the consequence of an additional effect, here the transverse viscosity, on a blow-up dynamics that it does not prevent. Moreover, this new effect changes the nature of the equation which is of a mixed hyperbolic/parabolic type. Handling these two issues, our result then extends known ones for blow-ups in a new direction, as well as raising new interesting problems, see the comments after the main Theorem 3.

More importantly, the study of (1.1) is motivated by fluid dynamics, from the fact that it is a simplified version of Prandtl's boundary layer equations. Solutions to Prandtl's equations might blow up in finite time [8, 12, 21] but a precise description of the singularity formation is still lacking. The present work is a step towards that goal and this issue will be investigated in a forthcoming work. Finally, let us mention that there has been recent progress on other models

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for singular solutions in fluid dynamics, see [5, 6, 28] and references therein.

The existence of smooth enough solutions to (1.1) follows from classical arguments. For example, relying on a fixed point argument and energy estimates, one can show that the equation is locally well-posed in $H^s(\mathbb{R}^2)$ for $s\geq 3$. There then holds the following blow-up criterion (again from energy estimates because of the identity $|\int uvv_x|\lesssim \|u_x\|_{L^\infty}\int v^2$): the solution u blows up at time $T>0$ if and only if

$$\limsup_{t\uparrow T}\|\partial_x u\|_{L^\infty(\mathbb{R}^2)}=+\infty. \tag{1.2}$$

eq:critere ex

The existence of global kinetic solutions $u\in L^\infty([0,+\infty),L^1(\mathbb{R}^2))$ has been showed by Chen and Perthame [4] following the framework of Lions, Perthame and Tadmor [22]. We refer to [27] for an introduction on kinetic solutions for scalar conservation laws. We expect singularities for such low regularity solutions to be different than the solutions in the present paper, as regularity plays a key role in the blow-up mechanism we describe. Before stating the main theorem, let us give the structure of the singularities of Burgers equation, and of the ones of the parabolic system encoding the effects of the transverse viscosity.

1.2. Self-similarity in shocks for Burgers equation

ubsec:Burgers

Burgers equation

$$\begin{cases}\partial_t U+U\partial_x U=0, & x\in\mathbb{R},\\ U_{t=0}=U_0,\end{cases} \tag{1.3}$$

eq:burgers2

admits solutions becoming singular in finite time in a self-similar way:

$$U(t,x)=\mu^{-1}(T-t)^{\frac{1}{2i}}\Psi_i\left(\mu\frac{x}{(T-t)^{1+\frac{1}{2i}}}\right)$$

where $(\Psi_i)_{i\in\mathbb{N}^*}$ is a family of analytic profiles (see [13] for example), and where $\mu>0$ is a free parameter. They are at the heart of the shock formation, a fact that is rarely emphasised, which lead us to give a precise and concise study in Section 2. Self-similar and discretely self-similar blow-up profiles for Burgers equation are classified in Proposition 4. Different scaling laws are thus possible, depending on the initial condition via its behaviour near the characteristic where the shock will form, which has to do with the fact that the scaling group of (1.3) is two-dimensional, see Section (2). This possibility of several scaling exponents is referred to as self-similarity of the second kind [1]. For each $i\in\mathbb{N}^*$, Ψ_i, defined in Proposition 5, is an odd decreasing profile, which is nonnegative and concave on $(-\infty,0]$ and such that $\partial_X\Psi_i$ is minimal at the origin with asymptotic $\Psi_i(X)=-X+X^{2i+1}$ as $X\to 0$. One has in particular the formula

$$\Psi_1(X):=\left(-\frac{X}{2}+\left(\frac{1}{27}+\frac{X^2}{4}\right)^{\frac12}\right)^{\frac13}+\left(-\frac{X}{2}-\left(\frac{1}{27}+\frac{X^2}{4}\right)^{\frac12}\right)^{\frac13}, \tag{1.4}$$

eq:def Psi1

for the fundamental one [9]. As in the above formula, all these profiles are unbounded at infinity but they emerge nonetheless from well localised initial data. A precise description of these profiles is given in Proposition 5. Any regular enough non-degenerate solution v to (1.3) that forms a shock at (T,x_0) is equivalent to leading order near the singularity to a self-similar profile Ψ_i up to the symmetries of the equation

$$U(t,x)\sim(T-t)^{\frac{1}{2i}}\mu^{-1}\Psi_i\left(\mu\frac{x-(x_0-c(T-t))}{(T-t)^{1+\frac{1}{2i}}}\right)+c \ \text{ as } (t,x)\to(T,x_0), \tag{1.5}$$

eq:def selfs

see Proposition 9. The blow-up dynamics involving the concentration of Ψ_1 is a stable one for smooth enough solutions. The scenario corresponding to the concentration of Ψ_i for $i\geq 2$ is an unstable one. For a suitable topological functional space, the set of initial conditions leading to the concentration of Ψ_i for $i \geq 2$ is located at the boundary of the set of initial condition leading to the concentration of Ψ_1, and admits $2(i-1)$ instability directions yielding one or several shocks formed by Ψ_j for $j < i$. The linearised dynamics is described in Proposition 8.

1.3. Blow-up for the reduced parabolic system

For a solution u to (1.1) that is odd in x, the behaviour on the transverse axis $\{x = 0\}$ is encoded by a closed system, which is the motivation for this symmetry assumption. It admits solutions blowing up simultaneously with a precise behaviour. Indeed, assume $\partial_x^j u_0(0, y) = 0$ for all $y \in \mathbb{R}$ for $2 \leq j \leq 2i$ for some integer $i \in \mathbb{N}$. This remains true for later times and the trace of the derivatives

$$\xi(t, y) := -\partial_x u(t, 0, y) \quad \text{and} \quad \zeta(t, y) := \partial_x^{2i+1} u(t, 0, y) \tag{1.6}$$

def:xi

solve the parabolic system

$$\begin{cases} (NLH) & \xi_t - \xi^2 - \partial_{yy}\xi = 0, \\ (LFH) & \zeta_t - (2i+2)\xi\zeta - \partial_{yy}\zeta = 0. \end{cases} \tag{1.7}$$

eq:NLH

Solutions to the nonlinear heat equation (NLH) might blow up in finite time, a dynamics that can be detailed precisely, see [26] for an overview. There exists a stable fundamental blow-up [2, 3, 17, 24], and a countable number of unstable "flatter" blow-ups [3, 15], all driven to leading order by the ODE $f' = f^2$. For the present work, we had to show additional weighted estimates than those showed in these articles. In particular, we revisited the proof in [3, 15, 24] and obtained a true improvement for the "flat" unstable blow-ups, see the comment below. For the solutions ξ to (NLH) below the solution to the linearly forced heat equation (LFH) may also blow-up in finite time with precise asymptotic that we detail later on.

n:NLHinstable

Theorem 1. *Let $J \in \mathbb{N}$. There exists an open set for a suitable topology of even solutions to (NLH) blowing up in finite time $T > 0$ with*

$$\xi(t, y) = \frac{1}{T-t} \, \frac{1}{1 + \frac{y^2}{8(T-t)|\log(T-t)|}} + \tilde{\xi},$$

where the remainder $\tilde{\xi}$ satisfies for $0 \leq j \leq J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{\xi}| \leq \frac{C}{(T-t)|\log(T-t)|^{\frac{1}{4}}} \, \frac{1}{\left(1 + \frac{y^2}{8(T-t)|\log(T-t)|}\right)^{\frac{3}{4}}} \frac{1}{\left(\sqrt{(T-t)|\log(T-t)|} + |y|\right)^j}.$$

For any $k \in \mathbb{N}$, $k \geq 2$, $a > 0$, there exists $T^* > 0$, such that for any $0 < T < T^*$ there exists an even solution to (1.7) blowing up at time T with

$$\xi(t, y) = \frac{1}{T - t + a y^{2k}} + \tilde{\xi}, \tag{1.8}$$

id:decomposi

where the remainders $\tilde{\xi}$ satisfies for $j = 0, ..., J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{\xi}| \leq C \left((T-t)^{\frac{1}{2k}} + |y| \right)^{\frac{1}{2} - (2k+j)}. \tag{1.9}$$

bd:remainder

Comments on the result.

The even assumption is not necessary, it is here to fit the even assumption on (1.1). The construction that we give here for the second case of the unstable blow-ups is not a copy of the seminal previous ones [3, 15, 24], but a bit simpler and more precise. In particular, we extensively use the fact that these profiles are perturbations of the smooth unstable self-similar profiles of the quadratic equation $f_t = f^2$, and that away from the origin in self-similar variables the problem is a perturbation of the renormalised quadratic equation $f_s + f - (Z/2k) f_Z - f^2 = 0$. We avoid the use of maximum principle as in [15] or of Feynman-Kac formula as in [3, 24], and obtain a sharp estimate. Namely, the convergence of the solution to the blow-up profile holds in a spatial region that is of size one in original y variables which is the estimate (1.9). For example, this estimate directly implies the existence of a profile at blow-up time $u(t,y) \to U^*(y)$ as $t \to T$ for $y \neq 0$, and that it satisfies $U^*(y) \sim (ay^{2k})^{-1}$ as $y \to 0$ (this fact would not be obtained directly in previous works).

Proof of Theorem 1. The second part, concerning the unstable blow-ups, is proved in Section 4. The proof of the first part for the stable blow-up is very similar, and though our method is a bit simpler than [3, 24] it does not yield truly improved estimates, hence we just sketch the proof in Section 5.

□

1.4. Statement of the result

The main result of this paper shows how, in a case with symmetries, the viscosity affects the shock formation of Burgers equation, resulting in a concentration of a self-similar shock Ψ_i along the vertical axis $\{x = 0\}$, with scaling parameters that are related to the solution of the parabolic system $(NLH) - (LFH)$. As a consequence, any blow-up solutions to the two one-dimensional equations can be combined to obtain a two-dimensional blow-up. Note that the solutions below can be chosen initially with compact support, and that we are only able to construct them around an initially concentrated blow-up profile. The first theorem involves the stable blow-up of (NLH). The blow-up pattern is stable in a Banach space $\mathcal{B}$ of C^4 regularity with polynomial weight associated to the norm:

$$\|u\|_{\mathcal{B}} := \sum_{j_1+j_2=0}^{4} \left\| \frac{\langle x\rangle^{j_1} \langle y\rangle^{j_2} \partial_x^{j_1} \partial_y^{j_2} u}{\langle x\rangle^4 (\langle y\rangle^3 + \langle x\rangle)^{-\frac{11}{3}}} \right\|_{L^\infty(\mathbb{R}^2)}. \tag{1.10}$$ bd:def mathca

th:mainstable **Theorem 2.** *For any $i \in \mathbb{N}^*$ and $b > 0$, there exists a Schwartz class solution u to (1.1), blowing up at time T with*

$$u(t,x,y) = b^{-1} \lambda^{\frac{1}{2i+1}}(t,y) \Psi_i \left(b \frac{x}{\lambda(t,y)} \right) + \tilde{u}(t,x,y)$$

where Ψ_i is defined by (2.5) and the transverse scale satisfies

$$\lambda(t,y) = (T-t)^{1+\frac{1}{2i}} \left(1 + \frac{y^2}{8(T-t)|\log(T-t)|} \right)^{1+\frac{1}{2i}}$$

and one has the convergence in self-similar variables (X,Z)

$$(T-t)^{-\frac{1}{2i}} u\left((T-t)^{1+\frac{1}{2i}} X, \sqrt{(T-t)|\log(T-t)|} Z \right) \to b^{-1} (1 + Z^2/8)^{\frac{1}{2i}} \Psi_i \left(b \frac{X}{(1+Z^2/8)^{1+\frac{1}{2i}}} \right) \tag{1.11}$$ main:bdtildeu

in C^1 on compacts sets and for some constants $C > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \leq C (T-t)^{-1} |\log(T-t)|^{-\frac{1}{4}}. \tag{1.12}$$ main:bdtildeu

For $i = 1$, there exists a ball in $\mathcal{B}$ around $u(t=0)$ such that any other solution with initial datum in that set blows up with the same behaviour.

For $i=1$, we did not pursue optimality for the weighted space $\mathcal{B}$. Other choices for the weight may be possible, but the C^4 regularity is essential and could only be lowered to C^{3+} by adapting the proof. Importantly, $\mathcal{B}$ allows for unbounded perturbations[1], highlighting the fact that only the control of derivatives is of importance in this problem. The second theorem involves the unstable "flat" blow-ups of (NLH).

th:main **Theorem 3.** *For any $k,i\in\mathbb{N}^*$, $k\geq 2$, $a,b>0$, there exists $T^*>0$, such that for any $0<T<T^*$ there exists a Schwartz class solution u to (1.1) odd in x and even in y, blowing up at time T with*

$$u(t,x,y)=b^{-1}\lambda^{\frac{1}{2i+1}}(t,y)\Psi_i\left(b\frac{x}{\lambda(t,y)}\right)+\tilde{u}(t,x,y) \qquad (1.13)$$ main:decompo

where $\lambda(t,y)=(T-t+ay^{2k})^{1+\frac{1}{2i}}$ and one has the convergence in self-similar variables (X,Z)

$$(T-t)^{-\frac{1}{2i}}u\left((T-t)^{1+\frac{1}{2i}}X,(T-t)^{\frac{1}{2k}}Z\right)\to b^{-1}(1+aZ^{2k})^{\frac12}\Psi_i\left(b\frac{X}{(1+aZ^{2k})^{\frac32}}\right) \qquad (1.14)$$ main:bdtildeu

in C^1 on compact sets and for some constants $C,\eta>0$ the remainder satisfies

$$\|\partial_x\tilde{u}\|_{L^\infty}\leq C(T-t)^{-1+\eta}. \qquad (1.15)$$ main:bdtildeu

Proof. Theorem 3 is proved in Section 3. It is a consequence of Proposition 16 and Lemma 18. The proof of Theorem 2 is very similar, and is just sketched in Section 5. □

1.5. Comments on the result and open problems

1. Stability/Instability in the symmetric case. The solutions of Theorem 2 with $i\geq 2$, or that of Theorem 3 are instable within the class of odd in x and even in y solutions. For $i\geq 2$, these solutions are such that $\partial_x^3 u_{|x=0}=0$, and we expect a generic perturbation of this third order derivative on the axis to lead to the blow-up behaviour described in Proposition 10 for $k\geq 2$ and (5.3) for $k=1$. The sign of such perturbation is important as we believe $b<0$ would create shocks outside the vertical axis. For $k\geq 2$, these solutions are such that $\partial_x u_{|x=0}$ is an instable blow-up solution of (NLH) and generic perturbations lead to the stable blow-up $k=1$ [16]. We do not believe that solutions of Theorem 2 with $i\geq 1$, though being stable, provide a generic blow-up behaviour in this symmetry class: the blow-up might occur at another point than the origin where such symmetry around that point would fail.

2. Symmetry breaking. We expect all solutions to Theorem 2 and 3 to be instable by symmetry breaking. Formally, our blow-up profile (3.13) admits non-symmetrical analogues, but the anisotropic viscosity make them fail to be approximate solutions, so our Ansatz does not adapt. The symmetry class we use allow for a control of the viscosity effects. But we wonder wether outside this symmetry class, viscosity, via some kind of hypo-elliptic effect, might prevent blow-up. This can be seen formally by considering solutions of the form $u=V(t,x-\epsilon y)$. For $\epsilon=0$, V solves Burgers equation (2.1) and u might blow-up. By tilting slightly the symmetry axis $\epsilon>0$, V solves the viscous Burgers equation $V_t+VV_x-\epsilon^2V_{xx}=0$ and is global. Investigating the generic behaviour is thus an interesting open problem.

3. Anisotropy. Very few results concerning a precise description of anisotropic singularity formation exist, despite its fundamental relevance in fluid dynamics. We see that here a wide

[1]The local well-posedness for equation (1.3) in $\mathcal{B}$ follows from local well-posedness for localised data, using the finite speed of propagation of the equation along the x variable.

range of different scaling laws in the x and y variables are possible. The formation of shocks for two-dimensional extensions of Burgers equation is studied in [25]. Let us also mention that in [10, 23] anistropic blow-ups were constructed for the energy supercritical semi-linear heat equation.

4. Connections between self-similar blow-ups. (1.1) appears to be a good candidate to study connexions between self-similar profiles. As the concentration of Ψ_i for $i \geq 2$ for Burgers is unstable with instabilities yielding to the concentration of Ψ_j for $j < i$, and as the same should hold for the unstable blow-ups of (NLH) (see [16] for the genericity result), one interesting result would be to prove rigorously that solutions to (1.1) concentrating the i-th profile of Burgers and the k-th of (NLH) are unstable with instabilities yielding to the concentration of the j-th profile of Burgers and the ℓ-th of (NLH) for $(j,\ell) < (i,k)$.

5. Continuation after blow-up. The inviscid Burgers equation possesses global weak solutions that can be obtained using a viscous approximation and that are unique under a suitable entropy condition. The investigation of the analogous problem for (1.1) is natural. In particular, if the solution can be continued and has jumps, what is the set of points with discontinuities and its dynamics?

1.6. Ideas of the proof and Organisation of the paper

The result relies on the extension of a lower-dimensional blow-up along a new spatial direction, as in [10, 23]. Self-similar blow-up in Burgers equation is completely studied via direct computations, without technical difficulties. It is an easy setting to understand properties of blow-ups, for example regularity and stability issues and discretely self-similar singularities. The extension along the transverse direction is studied through modulation equations (1.7), which for the first time are non-trivial PDEs. To obtain weighted estimates for (NLH) we adapt [24] and use a new exterior Lyapunov functional in Lemma 33, see the comments below Theorem 1. The blow-up of the solution to (NLF) can then be studied in the same analytical framework. The core of the paper is the 2-d analysis. The ideas are somewhat similar to those used in other contexts of blow-up through a prescribed profile, but are specific to the problem at hand and we hope that they will have other applications in transport and mixed hyperbolic/parabolic problems. We derive a blow-up profile with well-understood properties and linearisation, and build an approximate blow-up profile using modulation to neutralise growing modes. We then construct a solution in its vicinity via a bootstrap argument. We use solely weighted energy estimates, which are robust and reminiscent of a duality method for the asymptotic linear operator, and derivatives are taken along adapted vector fields to commute well with the equation.

The paper contains two independent sections devoted to Burgers equation and the modulation system, and another one proving the main theorem which can also be read separately as it uses their results as a black box. Section 2 concerns the self-similarity in the blow-ups of Burgers equation. Section 3 is devoted to the proof of Theorem 3, assuming some results for the derivatives on the vertical axis, Theorem 1 and Proposition 10. The blow-up profile and the linearised dynamics are studied in Lemma 11 and Proposition 12, and the heart of the proof is a bootstrap argument in Proposition 16. Section 4 deals with the two Propositions 1 and 10 admitted in Section 3, and concerns in particular the flat blow-ups for the semi-linear heat equation. Finally in Section 5 we sketch how the proof of Theorem 3 can be adapted to prove Theorem 2.

1.7. Notations

We use the Japanese bracket notation

$$\langle Y\rangle = (1+Y^2)^{\frac12}.$$

For functions having in argument a rescaling of the variable X, we use the general notation $\tilde X$ for their variable, as in $(\tilde X+\tilde X^2)(cX) = cX+(cX)^2$ for example. Depending on the context, $\tilde X$ will also refer to the main renormalised variable

$$\tilde X = \frac{X}{(1+Z^{2k})^{\frac32}} \tag{1.16}$$

eq:deftildeX

and there should not be confusions. We write $a\lesssim b$ if there exists a constant C independent of the other constants of the problem such that $a\le Cb$. We write $a\approx b$ if $a\lesssim b$ and $b\lesssim a$. Generally, C will denote a constant that is independent of the parameters used in the proof, whose value can change from one line to another. When its value depends on some parameter p, we will specify it by the notation $C(p)$. To perform localisations, the function χ is a smooth nonnegative cut-off function, $\chi=1$ on $[-1,1]$ and $\chi=0$ outside $[-2,2]$.

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2. Self similarity in shocks for Burgers equation

sec:burgers

This section is devoted to the formation of shocks for Burgers equation

$$U_t+UU_x=0 \tag{2.1}$$

eq:burgersusu

This simple equation appears as a toy model for blow-up issues involving self-similar behaviours. However, we did not find works in which this was emphasised apart from [13] (though implicit in some other works) where the existence of smooth self-similar singularities and their linearised dynamics are briefly studied, the usual point of view being geometrical [7]. Everything is explicit, which is convenient as the picture described in Subsection 1.2 shares many similarities with other equations. In particular one sees the link between the regularity of the solution and its blow-up behaviours (this issue appearing in other hyperbolic equations as in [20]).

2.1. Invariances

If $U(t,x)$ is a solution to (2.1), then the following function is again a solution by time and space translation, galilean transformation, space and time scaling invariances:

$$\frac{\mu}{\lambda}U\left(\frac{t-t_0}{\lambda},\frac{x-x_0-ct}{\mu}\right)+c.$$

In particular for $\lambda\in\mathbb{R}_+^*$ and $\alpha\in\mathbb{R}$, $\lambda^{\alpha-1}U(t/\lambda,x/\lambda^\alpha)$ is also a solution. The associated infinitesimal generators of the above transformations are[2]

$$\Lambda_\mu := \mathrm{Id}-x\partial_x,\ \ \tilde\Lambda_\lambda^{(\alpha)} := -(1-\alpha)\mathrm{Id}-t\partial_t-\alpha x\partial_x,\ \ \tilde\Lambda_c := -t\partial_x+1,\ \ \Lambda_{x_0} := -\partial_x,\ \ \Lambda_{t_0} := -\partial_t \tag{2.2}$$

burgers:def:

and there holds the commutators relations

$$[\Lambda_\mu,\tilde\Lambda_\lambda^{(\alpha)}]=0,\ \ [\tilde\Lambda_c,\tilde\Lambda_\lambda^{(\alpha)}]=-(\alpha-1)\tilde\Lambda_c+1,\ \ [\Lambda_{x_0},\tilde\Lambda_\lambda^{(\alpha)}]=-\alpha\Lambda_{x_0},\ \ [\Lambda_{t_0},\tilde\Lambda_\lambda^{(\alpha)}]=-\Lambda_{t_0}. \tag{2.3}$$

id:comutator

[2]Here Id stands for the identity and 1 for the function with constant value 1.

The tilde comes from the fact that we will use their spatial counterparts:

$$\Lambda_\alpha := (1-\alpha)\mathrm{Id} + \alpha X\partial_X, \quad \Lambda_c := \partial_X + 1, \quad \Lambda := -1 + X\partial_X \tag{2.4}$$ burgers:def:

2.2. Self-similar and discretely self-similar solutions

Important solutions are those who constantly reproduce themselves to smaller and smaller scales. To measure their regularity, let us define the following Hölder spaces. For $i \in \mathbb{N}$ one takes $C^i(\mathbb{R})$ to be the usual space of real-valued functions i times continuously differentiable on $\mathbb{R}$. For $i \in \mathbb{N}$ and $\delta \in (0,1)$, $C^{i+\delta}$ is the set of functions $f \in C^i(\Omega)$ such that

$$\lim_{x\uparrow x_0} \frac{\partial_x^i f(x) - \partial_x^i f(x_0)}{|x-x_0|^\delta} \quad \text{and} \quad \lim_{x\downarrow x_0} \frac{\partial_x^i f(x) - \partial_x^i f(x_0)}{|x-x_0|^\delta}$$

are well-defined for all $x_0 \in \mathbb{R}$. We then use the notation $C^{r+} = \cup_{r'>r} C^{r'}$ and $C^{r-} = \cup_{r'<r} C^{r'}$. Assume U is a C^1 solution to Burgers equation becoming singular at a singularity point (t_0, x_0). Then one can always use gauge invariance to map it to a solution defined on some domain $(T,0)\times\mathbb{R}$ with $T<0$, that becomes singular at $(0,0)$ and such that $U(t,0)=0$ for all $t\in(T,0)$. In particular, $U_x(t,\cdot)$ is minimal at the origin with $U_x(t,0) = -1/t$. The subgroup of the invariances $\mathbb{R}^3 \times (\mathbb{R}^*)^2$ preserving these properties is

$$g = (\lambda,\mu) \in \mathcal{G} := (0,+\infty)^2, \quad g.U : (t,x) \mapsto \frac{\mu}{\lambda} U\left(\frac{t}{\lambda}, \frac{x}{\mu}\right).$$

Let $\Omega := (-\infty,0)\times\mathbb{R}$. The stabiliser of $U \in C^1(\Omega)$ is the subgroup $\mathcal{G}_s(U) := \{g \in \mathcal{G},\ g.U = U\}$. Solutions with invariances can be classified according to their regularity.

pr:clas **Proposition 4** (Classification of self-similar solutions)**.** *Let $U \in C^1(\Omega)$ be a solution to (2.1) with $U(-1,0)=0$, $\inf_{\mathbb{R}} U_x(-1,\cdot) = U_x(-1,0) = -1$ and such that $\mathcal{G}_s$ is nontrivial. Then three scenarios only are possible and exist, the profiles $\Psi \in C^1(\mathbb{R})$ below being defined in Propositions 5 and 6 and in (2.13).*

- Analytic self-similarity: *U is analytic and there exists $i \in \mathbb{N}$ and $\mu > 0$ such that*

$$U(t,x) = \mu^{-1}(-t)^{\frac{1}{2i}} \Psi_i\left(\mu \frac{x}{(-t)^{1+\frac{1}{2i}}}\right),$$

or $U(t,x) = \Psi_\infty(x/(-t)) = x/t$.

- Non-smooth self-similarity: *There exists $i,\mu,\mu' > 0$ with $i \notin \mathbb{N}$ and $\mu = \mu'$ (resp. $i > 0$ and $\mu \neq \mu'$) such that*

$$U(t,x) = (-t)^{\frac{1}{2i}} \Psi_{(i,\mu,\mu')}\left(\frac{x}{(-t)^{1+\frac{1}{2i}}}\right).$$

where $\Psi_{(i,\mu,\mu')}$ is defined by (2.13), and $\Psi_{(i,\mu,\mu')} \in C^{1+2i}(\mathbb{R})$, $\Psi_{(i,\mu,\mu')} \notin C^{1+2i+}(\mathbb{R})$ (resp. $\Psi_{(i,\mu,\mu')} \in C^{1+2i-}(\mathbb{R})$, $\Psi_{(i,\mu,\mu')} \notin C^{1+2i}(\mathbb{R})$).

- Non-smooth discrete self-similarity: *There exists $i > 0$ and $\lambda > 1$ such that $U \notin C^{1+2i}(\Omega)$ (there exist such solutions with any regularity bewteen C^1 and C^{1+2i-}), that for all $k \in \mathbb{Z}$:*

$$U(t,x) = \lambda^{\frac{k}{2i}} U\left(\frac{t}{\lambda^k}, \frac{x}{\lambda^{k(1+\frac{1}{2i})}}\right),$$

and that there exists $(t,x) \in \Omega$ such that $U(t,x) \neq (-t)^{1/(2i)} U(-1, x/(-t)^{1+1/(2i)})$.

Before proving the above Proposition 4, let us present the self-similar and discretely self-similar solutions.

smoothselfsim **Proposition 5** (Self-similar solutions [13]). *There exists a set* $\{\Psi_i,\ i \in \mathbb{N}^*\} \cup \{\Psi_\infty\}$ *of analytic functions on* $\mathbb{R}$ *with the following properties. One has* $\Psi_\infty(X) = -X$. *For* $i \in \mathbb{N}^*$, *the function* Ψ_i *is odd, decreasing, and concave on* $(-\infty, 0]$, *satisfy the implicit equation*

$$X = -\Psi_i(X) - (\Psi_i(X))^{1+2i} \tag{2.5}$$ id:implicit

and have the following asymptotic expansions:

$$\Psi_i^{(i)}(X) = -X + X^{2i+1} + \sum_{k=2}^{+\infty} c_{i,k} X^{2ki+1} \quad \text{as } X \to 0, \tag{2.6}$$ id:dvptUic

$$\Psi_i(X) = -sgn(X)|X|^{\frac{1}{1+2i}} + sgn(X)\frac{|X|^{-1+\frac{2}{2i+1}}}{2i+1} + O(|X|^{-2+\frac{3}{2i+1}}) \quad \text{as } |X| \to +\infty. \tag{2.7}$$ id:asUiinfty

Moreover, it solves the equation

$$-\frac{1}{2i}\Psi_i + \frac{2i+1}{2i} X\partial_X \Psi_i + \Psi_i \partial_X \Psi_i = 0 \tag{2.8}$$ eq:smoothsel

and any other globally defined C^1 *solution is of the form* $\Psi = \mu^{-1}\Psi_i(\mu X)$ *for some* $\mu > 0$ *or is* $-X$ *or* 0. *The functions* $U^{(\infty)}(t,x) = x/t$ *and* $U^{(i,\mu)}(t,x) = \mu^{-1}(-t)^{1/(2i)}\Psi_i(\mu x/(-t)^{1+1/(2i)})$ *are solutions to* (2.1).

Proof. Consider the function $\phi(\Psi) = -\Psi - \Psi^{2i+1}$ which is an analytic diffeomorphism on $\mathbb{R}$. Its inverse $\Psi_i := \phi^{-1}$ satisfies (2.5), (2.6), (2.7) and the other properties of the proposition from direct computations. Since

$$\begin{aligned}
&\frac{2i}{(\phi^{-1})'(X)}\left(-\frac{1}{2i}\phi^{-1}(X) + \frac{2i+1}{2i}X(\phi^{-1})'(X) + \phi^{-1}(X)(\phi^{-1})'(X)\right) \\
&\qquad = -\phi^{-1}(X)\phi'(\phi^{-1}(X)) + (2i+1)X + 2i\phi^{-1}(X) \\
&\qquad = -\phi^{-1}(X)(-1-(2i+1)(\phi^{-1}(X))^{2i}) + (2i+1)X + 2i\phi^{-1}(X) \\
&\qquad = -(2i+1)(-\phi^{-1}(X) - (\phi^{-1}(X))^{2i+1}) + (2i+1)X = 0,
\end{aligned}$$

it solves the equation (2.8). Since it solves this equation, $U(t,x) = (-t)^{1/(2i)}\Psi_i(x/(-t)^{1+1/(2i)})$ solves (2.1) since introducing $\alpha_i = 1 + 1/(2i)$:

$$\begin{aligned}
U_t + UU_x &= -(\alpha_i - 1)(-t)^{\alpha_i - 2}\Psi_i\left(\frac{x}{(-t)^{\alpha_i}}\right) + \alpha_i(-t)^{\alpha_i-2}\frac{x}{(-t)^{\alpha_i}}\partial_X\Psi_i\left(\frac{x}{(-t)^{\alpha_i}}\right) \\
&\quad + (-t)^{\alpha_i-1}\Psi_i\left(\frac{x}{(-t)^{\alpha_i}}\right)(-t)^{-1}\partial_X\Psi_i\left(\frac{x}{(-t)^{\alpha_i}}\right) \\
&= (-t)^{\alpha_i-2}\left(-(\alpha_i-1)\Psi_i + \alpha_i y U_y^{(i)} + \Psi_i\partial_X\Psi_i\right)\left(\frac{x}{(-t)^{\alpha_i}}\right) = 0.
\end{aligned}$$

The same reasoning applies for $\mu^{-1}\Psi_i(\mu X)$ since (2.8) is invariant under the transformation $\Psi \mapsto \mu^{-1}\Psi(\mu X)$. If Ψ is another solution to (2.8) with $-1 < \Psi(1) < 0$ then using this invariance $\Psi = \mu^{-1}\Psi(\mu X)$ for some $\mu > 0$. If $\Psi(1) < -1$ or $\Psi(1) > 0$ it is easy to check that the solution is not globally defined.

□

There also exist solutions reproducing themselves to a smaller scale, but in a somewhat periodic manner, unlike self-similar solutions. They have a fractal behaviour near the origin and are never smooth.

roughdselfsim **Proposition 6** (Non-smooth discretely self-similar blow-up)**.** *Let $\alpha > 1$, $\lambda > 1$, $X_0, X_1 \in (-\infty, 0)$ with $\lambda^{1-\alpha} X_0 < X_1 < \lambda^{-\alpha} X_0$ and consider a function $V \in C^1([X_0, X_1), \mathbb{R})$ satisfying*[3]

$$X_1 = \lambda^{-\alpha} X_0 + (\lambda^{-\alpha} - \lambda^{1-\alpha}) V(X_0),$$
$$V(X) \in (0, -X) \quad \textit{and} \quad V_X(X) \in (-1, 0) \quad \textit{on} \quad [X_0, X_1),$$

and

$$\lim_{X \to X_1} V(X) = \lambda^{1-\alpha} V(X_0), \quad \lim_{X \to X_1} V_X(X) = \frac{\lambda V_X(X_0)}{1 - (\lambda - 1) V_X(X_0)}. \tag{2.9}$$

id:cond v

Then there exists a unique odd function $W \in \mathcal{C}^1(\mathbb{R})$ such that for all $X \in \mathbb{R}$,

$$W(X) = \lambda^{1-\alpha} W\left(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X)\right) \tag{2.10}$$

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and $W = V$ on $[X_0, X_1)$. One has $W(X) \in (0, -X)$ and $W_X(X) \in (-1, 0)$ for all $X \in (-\infty, 0)$, and its derivative is minimal at the origin with value $W_X(0) = -1$. Let $i = 1/(2(\alpha - 1))$. Then

$$0 < \liminf_{X \uparrow 0} \frac{-W(X) - X}{|X|^{1+2i}} \leq \limsup_{X \uparrow 0} \frac{-W(X) - X}{|X|^{1+2i}} < +\infty$$

with equality if and only if $W(X) = \mu^{-1} \Psi_i(\mu X)$ for some $\mu > 0$ where Ψ_i is given by (2.5). Therefore, unless $W = \mu^{-1} \Psi_i(\mu X)$ one has $W \notin C^{2i+1}$. There exist such solutions of regularity $C^{2i+1-\epsilon}$ for any $\epsilon > 0$. Moreover, the solution U defined on $(-\infty, 0) \times \mathbb{R}$ as the solution to (2.1) with $U(-1, x) = W(x)$ satisfies

$$U(t, X) = \frac{1}{\lambda^{k(1-\alpha)}} U\left(\frac{t}{\lambda^k}, \frac{X}{\lambda^{k\alpha}}\right) \tag{2.11}$$

id:dss

for all $(t, X, k) \in (-\infty, 0) \times \mathbb{R} \times \mathbb{Z}$.

Remark 7. *If $(1-\alpha) W + \alpha X W_X + W W_X \neq 0$, then U is not of the form $U = (-t)^{\alpha-1} W(x/(-t)^\alpha)$, implying that the set of all $k \in \mathbb{R}$ such that (2.11) hold is isomorphic to $\mathbb{Z}$ and that the solution is not continuously self-similar.*

Proof. We proceed in two steps. First we extend V in a periodic manner, and then we show the regularity properties.

Step 1 *Construction.* Consider the mapping $\phi : [X_0, X_1) \to \mathbb{R}$ defined by

$$\phi(X) = \lambda^{-\alpha} X + (\lambda^{-\alpha} - \lambda^{1-\alpha}) V(X).$$

One has $\phi(X_0) = X_1$ and since $\lambda, \alpha > 1$ and $V_X \in (-1, 0)$ one computes

$$\phi_X(X) = \lambda^{-\alpha} + (\lambda^{-\alpha} - \lambda^{1-\alpha}) V_X(X) > \lambda^{-\alpha} > 0$$

and hence ϕ is a C^1 diffeomorphism onto its image. Define

$$X_2 = \lim_{X \to X_1} \phi(X) = \lambda^{-\alpha} X_1 + (\lambda^{-\alpha} - \lambda^{1-\alpha}) \lambda^{1-\alpha} V(X_0).$$

and for $X \in [X_1, X_2)$ extend V by

$$W(X) = \lambda^{1-\alpha} V(\phi^{-1}(X)).$$

Claim: One has $X_1 < X_2 < 0$, that W is C^1 on $[X_0, X_2)$ and that restricted to $[X_1, X_2)$ it satisfies the condition of the proposition. Moreover for all $X \in [X_1, X_2)$, $\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X) \in [X_0, X_1)$ and

$$W(X) = \lambda^{1-\alpha} W\left(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X)\right).$$

[3]The set of such functions is non empty and it contains profiles which do not satisfy $(1-\alpha)V + \alpha X V_X + V V_X = 0$.

The proof of this claim involves only basic computations that we omit here.

From the Claim we see that we can repeat the construction a countable number of time. If $(X_k)_{k\in\mathbb{N}}$ denotes the set of points coming from the construction then by induction

$$X_k = \lambda^{-k\alpha} X_0 + (\lambda^{-k\alpha} - \lambda^{k(1-\alpha)}) W(X_0)$$

hence $X_k \to 0$. The construction then provides a C^1 extension W of V on $(X_0, 0)$ such that for all X in this set, $0 < W(X) < -X$, $-1 < W_X < 0$, and for all $X_1 \le X < 0$, $\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X) \in [X_0, 0)$ and

$$W(X) = \lambda^{1-\alpha} W\left(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X)\right).$$

Step 2: *Properties.* From the definition of the extensions one has

$$\sup_{X\in[X_{k+1},X_{k+2}]} |W(X)| = \lambda^{1-\alpha} \sup_{X\in[X_k,X_{k+1}]} |W(X)|$$

and therefore $\lim_{X\to 0} W(X) = 0$. From (2.10) one sees that

$$\partial_X W\left(\lambda^{-\alpha} X + (\lambda^{-\alpha} - \lambda^{1-\alpha}) W(X)\right) = f(\partial_X W(X)), \quad f(a) = \frac{\lambda a}{1 - (1-\lambda) a}.$$

f has two fixed points -1 and 0, is increasing on $(-1, 0)$ with $-1 < f(a) < a$. Therefore,

$$-1 < \inf_{[X_{k+1},X_{k+2}]} \partial_X W = f(\inf_{[X_k,X_{k+1}]} \partial_X W) \le \sup_{[X_{k+1},X_{k+2}]} \partial_X W = f(\sup_{[X_k,X_{k+1}]} \partial_X W) \to -1$$

implying that $\partial_X W(X) \to -1$ as $X \to 0$, and in particular $\partial_X W$ is minimal at the origin with $\partial_X W(0) = -1$. We now prove the absence of regularity at the origin. Take any $z_0 \in [X_0, X_1)$ and define the sequence z_k by induction following

$$z_k = \lambda^{-\alpha} z_{k-1} + (\lambda^{-\alpha} - \lambda^{1-\alpha}) W(z_{k-1}).$$

It follows that $W(z_k) = \lambda^{1-\alpha} W(z_{k-1})$. By induction,

$$z_k = \lambda^{-k\alpha} z_0 + (\lambda^{-k\alpha} - \lambda^{k(1-\alpha)}) W(z_0) = -\lambda^{k(1-\alpha)} W(z_0)(1 + O(\lambda^{-k}))$$

as $k \to +\infty$ since $\lambda, \alpha > 1$, with the constant in the O uniform in $z_0 \in [X_0, X_1)$. By induction,

$$-z_{k+1} - W(z_{k+1}) = \lambda^{-k\alpha}(-z_0 - W(z_0)).$$

Therefore,

$$\frac{-z_k - W(z_k)}{|z_k|^{\frac{\alpha}{\alpha-1}}} \to \frac{-z_0 - W(z_0))}{|W(z_0)|^{\frac{\alpha}{\alpha-1}}} > 0$$

as $k \to +\infty$. One then deduces that since the convergence is uniform for z_0 taken in $[X_0, X_1)$,

$$0 < \liminf_{X\to 0} \frac{-X - W(X)}{|X|^{\frac{\alpha}{\alpha-1}}} \le \limsup_{X\to 0} \frac{-X - W(X)}{|X|^{\frac{\alpha}{\alpha-1}}} < +\infty.$$

Therefore the solution is not $C^{\frac{\alpha}{\alpha-1}}$ if the equality does not hold. Assume now the equality. This means that there exist a constant $c > 0$ such that for any $X \in [X_0, X_1)$ one has

$$\frac{-X_0 - W(X_0))}{|W(X_0)|^{\frac{\alpha}{\alpha-1}}} = c, \quad \text{i.e.} \quad X = -W - c|W|^{\frac{\alpha}{\alpha-1}}.$$

W is then the self-similar profile[4] of Proposition 5, and is not discretely self-similar.

[4]For $\alpha \neq 1 + 1/(2i)$ for $i \in \mathbb{N}$ the profiles $\Psi_{1/(2(\alpha-1))}$ defined in Proposition 5 still exist and have all the corresponding properties, they just are no longer smooth.

One can apply the same extension technique to define W on the other side $(-\infty, X_0)$. The uniqueness of the extension follows from an induction, using the fact that if W is given on some $[X_k, X_{k+1})$ then it has to be given on $[X_{k+1}, X_{k+2})$ by the construction we provided. We leave to the reader to prove that if $V \in C^\gamma$ for some $1 < \gamma < \alpha/(\alpha-1)$ then so is W.

□

Self-similar and discretely self-similar solutions having been presented in Propositions 5 and 6, we can now give the proof of the classification Proposition 4.

Proof of Proposition 4. We only sketch the proof, since either the computations involved are rather easy or they are very similar to what can be found in the proofs of Proposition 5 and 6. The stabiliser of U is closed in $\mathcal{G}$ from the regularity of U. One identifies $\mathcal{G}_s$ to a closed subgroup of $\mathbb{R}^2$ via $(z_1, z_2) = (\log(\lambda), \log(\mu))$, and recall that closed subgroups of $\mathbb{R}^2$ are isomorphic to one of the following groups: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{R}\times\mathbb{Z}$ or $\mathbb{R}^2$.

Case 1, $\mathcal{G}_s \simeq \mathbb{Z}$: in that case $\mathcal{G}_s = \{(\lambda^k, \mu^k),\ k \in \mathbb{Z}\}$ for $(\lambda,\mu) \neq (1,1)$ meaning that

$$U(t,x) = \frac{\mu^k}{\lambda^k} U\left(\frac{t}{\lambda^k}, \frac{x}{\mu^k}\right), \quad \forall k \in \mathbb{Z}.$$

One can check that if $\lambda = 1$ then $u = c(t)x$, and if $\mu = 1$ then $U = 0$, which are contradictions. Hence $\lambda, \mu \neq 1$ and we define $\alpha \in \mathbb{R}$, by $\mu = \lambda^\alpha$ giving $\mathcal{G}_s = \{(\lambda^k, \lambda^{k\alpha}),\ k \in \mathbb{Z}\}$. For all $k \in \mathbb{Z}$,

$$U(t,x) = \frac{1}{\lambda^{(1-\alpha)k}} U\left(\frac{t}{\lambda^k}, \frac{x}{\lambda^{\alpha k}}\right)$$

and since $\mathcal{G}_s \not\simeq \mathbb{R}$ there exists (t,x) such that $U(t,x) \neq (-t)^{\alpha-1} U(-1, x/(-t)^\alpha)$. One can always take $\lambda > 1$. We take $t = -1$, $k = 1$, to obtain

$$U\left(\frac{-1}{\lambda}, x\right) = \lambda^{1-\alpha} U(-1, \lambda^\alpha x).$$

From the relation on characteristics

$$U(-1,x) = U\left(-\frac{1}{\lambda}, x + \left(1 - \frac{1}{\lambda}\right) U(-1,x)\right).$$

Introducing the profile $W(X) := U(-1, X)$ one deduces that it satisfies

$$W(X) = \lambda^{1-\alpha} W\left(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X)\right) \tag{2.12}$$

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and that W is C^1 with $W(0) = 0$ and W_X minimal at zero with $W_y(0) = -1$. We claim that if $\alpha > 1$ then W is a function as described in Proposition 6 in which the above functional equation was studied. We claim that the case $\alpha = 1$ is impossible and that if $\alpha < 1$ the function is not C^1 by looking at its behaviour at the origin. Therefore Case 1 corresponds to Proposition 6.

Case 2, $\mathcal{G}_s \simeq \mathbb{R}$: in that case $\mathcal{G}_s = \{(\lambda^a, \mu^a),\ a \in \mathbb{R}\}$ for $(\lambda,\mu) \neq (1,1)$ meaning that

$$U(t,x) = \frac{\mu^a}{\lambda^a} U\left(\frac{t}{\lambda^a}, \frac{x}{\mu^a}\right), \quad \forall a \in \mathbb{R}.$$

This group of transformation contains the cases $a \in \mathbb{Z}$, and we have seen in case 1 that one cannot have $\lambda = 1$ or $\mu = 1$. Hence $\lambda \neq 1$ and $\mu \neq 1$. Define α by $\mu = \lambda^\alpha$ giving (up to an abuse of notation) $\mathcal{G}_s = \{(\lambda, \lambda^\alpha),\ \lambda > 0\}$ and that for all $\lambda > 0$,

$$U(t,x) = \frac{1}{\lambda^{(1-\alpha)}} U\left(\frac{t}{\lambda}, \frac{x}{\lambda^\alpha}\right).$$

In particular, u is invariant by the transformation

$$U(t,x)=\frac{1}{\lambda^{k(1-\alpha)}}U\left(\frac{t}{\lambda^k},\frac{x}{\lambda^{k\alpha}}\right).$$

for any fixed $\lambda>1$ and $k\in\mathbb{Z}$. We have seen in the study of Case 1 that one cannot have $\alpha<1$ for such an invariance, hence $\alpha>1$. We now write

$$U(t,x)=\frac{1}{(-t)^{(1-\alpha)}}U\left(-1,\frac{x}{(-t)^{\alpha}}\right).$$

Hence the profile $W(X)=U(-1,X)$ satisfies the equation

$$(1-\alpha)W+\alpha XW_X+WW_X=0.$$

Solutions to this equation with $W(0)=0$, W_X minimal at 0 with $W_X(0)=-1$ have been classified when $1/(2(\alpha-1))\in\mathbb{N}^*$ in Proposition 5. It is straightforward to check that if $1/(2(\alpha-1))\notin\mathbb{N}^*$ the profiles $\Psi_{1/(2(\alpha-1))}$ defined in Proposition 5 exist, have all the corresponding properties, and are $C^{\alpha/(\alpha-1)}$. Any self-similar shocks can then be written in the form

$$\Psi_{(i,\mu,\mu')}(X)=\begin{cases}\mu^{-1}\Psi_i(\mu X) & \text{if } X\leq 0,\\ {\mu'}^{-1}\Psi_i(\mu' X) & \text{if } X\geq 0,\end{cases}\tag{2.13}$$

burgers:def::

for $i\in\mathbb{R}$, $i>0$ where Ψ_i is given by (2.5). When $\alpha=1$, the only solution to $XW_X+WW_X=0$ with $W(0)=0$ and $W_X(0)=-1$ is $W(X)=-X$ which is a contradiction.

Case 3 If $\mathcal{G}_s\simeq\mathbb{Z}^2$ or $\mathcal{G}_s\simeq\mathbb{Z}\times\mathbb{R}$, in this case that there exists a subgroup of $\mathcal{G}_s$ of the form $\{(\lambda^k,\lambda^{k\alpha}),\ k\in\mathbb{Z}\}$ with $\lambda>1$ and $\alpha<0$. Indeed, the mapping $(\lambda,\mu)\mapsto(\log\lambda,\log\mu)$ transforms $\mathcal{G}_s$ into a subgroup of $\mathbb{R}^2$. This new subgroup is also isomorphic to $\mathbb{Z}^2$ or $\mathbb{Z}\times\mathbb{R}$. Any such subgroup must contains a point (z_1,z_2) in the bottom right quadrant $z_1>0$ and $z_2<0$, then $\lambda=e^{z_1}$ and $\alpha=\frac{z_2}{z_1}$ gives the desired subgroup. From the study of Case 1, such an invariance is impossible. If $\mathcal{G}_s\simeq\mathbb{R}^2$, one can check that $u(t,x)=x/t$.

□

2.3. Stability and convergence at blow-up to self-similar solutions

The suitable framework for the stability of Ψ_i is that of self-similar variables where the linearised operator is

$$H_X:=\Lambda_{\alpha_i}+\Psi_i\partial_X+\partial_X\Psi_i=(1-\alpha_i)+\partial_X\Psi_i+(\alpha_iX+\Psi_i)\partial_X.\tag{2.14}$$

def:H

pr:H **Proposition 8** (Spectral properties of H_X [13]). *The point spectrum of H_X on smooth functions is*

$$\Upsilon(H_X)=\left\{\frac{j-2i-1}{2i},\quad j\in\mathbb{N}\right\}.\tag{2.15}$$

def:nuk

The eigenfunctions related to symmetries are

$$H_X\Lambda_{x_0}\Psi_i=-\alpha_i\Lambda_{x_0}\Psi_i,\ \ H_X(\Lambda_{\alpha_i}\Psi_i)=-\Lambda_{\alpha_i}\Psi_i,\ \ H_X(\Lambda_c\Psi_i)=-(\alpha_i-1)(\Lambda_c\Psi_i),\ \ H_X\Lambda_\mu\Psi_i=0.\tag{2.16}$$

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More generally, the eigenfunctions are given by the formula:

$$H_X(\phi_{X,j})=\frac{j-2i-1}{2i}\phi_{X,j},\quad \phi_{X,j}:=\frac{(-1)^k\Psi_i^j}{1+(2i+1)\Psi_i^{2i}}.\tag{2.17}$$

eq:def phiXj

They have the following asymptotic behaviour:

$$\phi_{X,j}(X)=X^j-(j+2i+1)X^{j+2i}+O(X^{j+4i})\quad as\quad X\to 0,\tag{2.18}$$

id:as phik0

$$\phi_{X,j}(X)=\frac{1}{2i+1}|X|^{\frac{j-2i}{2i+1}}+O(|X|^{\frac{jf-2i}{2i+1}-\frac{2i}{2i+1}})\quad as\quad X\to+\infty. \qquad (2.19)$$ id:as phikin

Proof. **Step 1** *Proof of* (2.16). Let $U(t,x):=(-t)^{\alpha_i-1}\Psi_i(x/(-t)^{\alpha_i})$ which solves (2.1) and by invariance, $(\tau_c^{(3)}U)_t=-(\tau_c^{(3)}U)\partial_x(\tau_c^{(3)}U)$ for any $c\in\mathbb{R}$. Differentiating with respect to c one obtains $(\tilde\Lambda_cU)_t=-\tilde\Lambda_cU\partial_xU-U\partial_x(\tilde\Lambda_cU)$, which evaluated at $t=-1$ yields:

$$\partial_t\left(\tilde\Lambda_cU\right)(-1,\cdot)=-\Psi_i\partial_X(\Lambda_c\Psi_i)-\partial_X\Psi_i\Lambda_c\Psi_i. \qquad (2.20)$$ burgers:id:e

Self-similarity implies from (2.2) that $\tilde\Lambda_\lambda^{(\alpha_i)}u=0$ hence $\tilde\Lambda_\lambda^{(\alpha_i)}\tilde\Lambda_cu+[\tilde\Lambda_c,\tilde\Lambda_\lambda^{(\alpha_i)}]u=0$. This identity reads from the commutator relation (2.3):

$$\tilde\Lambda_\lambda^{(\alpha_i)}\tilde\Lambda_cu=(\alpha_i-1)\tilde\Lambda_cu.$$

At time $t=-1$ the above identity yields from (2.2) and (2.4):

$$\partial_t(\tilde\Lambda_cu)(-1,\cdot)-(1-\alpha_i)\Lambda_c\Psi_i-\alpha_iX\partial_X\Lambda_c\Psi_i=(\alpha_i-1)\Lambda_c\Psi_i.$$

From (2.20) the left hand side in this identity is $-H_X\Lambda_c\Psi_i$, ending the proof of (2.16). The proof for the eigenfunctions related to the other symmetries (2.16) is exactly the same.

Step 2 *Proof of* (2.15) *and* (2.17). Assume f solves $H_Xf=\nu f$. Then using the implicit equation (2.5) one obtains:

$$\frac{\partial f}{\partial\Psi_i}=f\left[\frac{\alpha_i+\nu+(\alpha_i-1+\nu)(2i+1)\Psi_i^{2i}}{(\alpha_i-1)\Psi_i+\alpha_i\Psi_i^{2i+1}}\right]=f\left[\frac{2i+1+2i\nu+(1+2i\nu)(2i+1)\Psi_i^{2i}}{\Psi_i+(2i+1)\Psi_i^{2i+1}}\right]$$

whose solution is of the form

$$f\in\text{Span}\left(\frac{\Psi_i^{2i+1+2i\nu}}{1+(2i+1)\Psi_i^{2i}}\right)$$

From (2.6) the above formula defines a smooth function if and only if $\nu=(j-2i-1)/(2i)$ for some $j\in\mathbb{N}$.

□

The smooth self-similar profiles are the asymptotic attractors of all smooth and non-degenerate shocks in the following sense.

pr:nbassin **Proposition 9.** *Let $U_0\in C^\infty(\mathbb{R})$ be such that ∂_xU_0 is minimal at x_0 with*

$$U_0(x_0)=c,\quad \partial_xU_0(x_0)<0,\quad \partial_x^jU_0(x_0)=0\ \text{ for } j=2,...,2i,\quad\text{and}\quad \partial_x^{2i+1}U_0(x_0)>0 \qquad (2.21)$$ n:id:cond in

for some $i\in\mathbb{N}^*$. Then u blows up at time $T=-1/U_x(x_0)$ at the point $x_\infty=x_0+cT$ with:

$$U(t,x)=\mu^{-1}(T-t)^{\frac{1}{2i}}\Psi_i\left(\mu\frac{x-x_0-ct}{(T-t)^{1+\frac{1}{2i}}}\right)+c+w(t,x)$$

where Ψ_i is defined by Proposition 5, where $\mu=\left(\frac{\partial_x^{2i+1}U(x_0)}{(2i+1)!(-\partial_xU(x_0))^{2i+2}}\right)^{\frac{1}{2i}}$ and where

$$\frac{w}{(T-t)^{\frac{1}{2i}}\Psi_i\left(\mu\frac{x-x_0-ct}{(T-t)^{1+\frac{1}{2i}}}\right)}\to0\quad as\ (x,t)\to(x_\infty,T). \qquad (2.22)$$ burgers:cvse

Proof. Without loss of generality, up to the symmetries of the equation we consider the case $x_0=0$, $U(0)=0$, $U_x(0)=-1$ and $\partial_x^{2i+1}U_0(0)=(2i+1)!$, i.e. $T=1=b$, $c=0$. For $0\leq t<1$ and $x\in\mathbb{R}$ we have the formula using characteristics for $|y|\leq 1$:

$$U(x,t)=U_0(\phi_t^{-1}(x)),\ \ \phi_t(y)=y+tU_0(y)=(1-t)y+y^{2i+1}+O(y^{2i+2})+O(|y|^{2i+1}|1-t|), \quad (2.23)$$ burgers:chara

ϕ_t defining a diffeomorphism on $\mathbb{R}$ for all $0\leq t<1$. Given (t,x) close to $(1,0)$ we look for an inverse $\phi_t^{-1}(x)$ of the form $-(1-t)^{1/(2i)}\Psi_i(x(1+h)/(1-t)^{1+1/(2i)})$. Since $|\Psi_i(x)|\lesssim|x|^{1/(2i+1)}$ for $x\in\mathbb{R}$, we compute using (2.5):

$$\begin{aligned}
&\phi_t\left(-(1-t)^{\frac{1}{2i}}\Psi_i\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)\right)\\
=&-(1-t)^{1+\frac{1}{2i}}\left(\Psi_i+\Psi_i^{2i+1}\right)\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)+O\left((1-t)^{1+\frac{2}{2i}}\Psi_i^{2i+2}\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)\right)\\
&+O\left((1-t)^{2+\frac{1}{2i}}|\Psi_i^{2i+1}|\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)\right)\\
=&\ x(1+h)+O\left((1-t)^{1+\frac{2}{2i}}\left(\frac{|x||1+h|}{(1-t)^{1+1/(2i)}}\right)^{1+\frac{1}{2i+1}}\right)+O\left((1-t)^{2+\frac{1}{2i}}\frac{|x||1+h|}{(1-t)^{1+1/(2i)}}\right)\\
=&\ x(1+h)+O(|x|^{1+\frac{1}{2i+1}}|1+h|)+O((1-t)|x||1+h|).
\end{aligned}$$

From the intermediate values theorem, there exists $h=O(|x|^{1/(2i+1)}+(1-t))$ such that there holds the inverse formula

$$\phi_t\left(-(1-t)^{\frac{1}{2i}}\Psi_i\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)\right)=x \quad (2.24)$$ burgers:id:ir

and there holds

$$\begin{aligned}
\Psi_i\left(\frac{x(1+h)}{(1-t)^{1+1/(2i)}}\right)&=\Psi_i\left(\frac{x}{(1-t)^{1+1/(2i)}}\right)+\int_{\mu=1}^{1+h}\frac{d\mu}{\mu}(\tilde{X}\partial_{\tilde{X}}\Psi_i)\left(\frac{x\mu}{(1-t)^{1+1/(2i)}}\right)\\
&=\Psi_i\left(\frac{x}{(1-t)^{1+1/(2i)}}\right)+O\left(|h|\left|\Psi_i\left(\frac{x}{(1-t)^{1+1/(2i)}}\right)\right|\right).
\end{aligned}$$

Injecting $U_0(y)=-y+O(|y|^{2i+1})$ in (2.23), using (2.24), the bound on h, the above bound and $(1-t)^{1/(2i)}|\Psi_i|(x/(1-t)^{1+1/(2i)})\lesssim|x|^{1/(2i+1)}$, one obtains (2.22).

□

sec:main

3. Proof of the main Theorem 3

To ease notations we consider the case $i=1$ corresponding to the Ψ_1 profile for Burgers, the proof being the same for $i\geq 2$. Recall the notation for the derivatives on the transverse axis (1.6) and the corresponding system (1.7) that they solve under the odd in x and even in y symmetry assumption. Solutions to (NLH) in (1.7) might blow up according to a dynamic described in Theorem 1. The following proposition then describes how the singularity formation for ξ makes some solutions to the other equation (LFH) in (1.7) blow up in finite time with a precise behaviour. Its proof and that of Theorem 1 are relegated to Section 4 and we prove here Theorem 3 admitting them.

pr:LHinstable **Proposition 10.** *Let $i=1$. For any $k\in\mathbb{N}$ with $k\geq 2$, $a,b>0$ and $J\in\mathbb{N}$, there exists $T^*>0$ such that for any $0<T<T^*$, there exists ξ a solution to (1.7) satisfying (1.8) and (1.9), and*

ζ_0 *such that the corresponding solution to* (LFH) *blows up at time* T *with*

$$\zeta = \frac{b}{(T-t+ay^{2k})^4} + \tilde{\zeta},$$

where the remainders $\tilde{\zeta}$ *satisfy for* $j = 0, ..., J$ *for some constant* $C(a,b) > 0$*:*

$$|\partial_y^j \tilde{\zeta}| \leq C \left((T-t)^{\frac{1}{2k}} + |y| \right)^{\frac{1}{2}-(8k+j)}. \qquad (3.1)$$ bd:remainder

Proof of Theorem 1 and of Proposition 10. Section 4 is devoted to their proof. Proposition 29 and the estimates (4.31) for ξ, and Proposition 36 and the estimates (4.51) for ζ, indeed imply that Theorem 1 and Proposition 10 hold for one particular value of $a > 0$ and of $b > 0$. One then obtains the general result for any value of a and b by using the symmetries of the equation. Namely, (NLH) and (LFH) are invariant by time translation, (1.7) is invariant by the scaling transformation $\xi \mapsto \lambda^2 \xi(\lambda^2 t, \lambda y)$ for any $\lambda > 0$ and (LFH) is invariant by homothety since it is linear.

□

We assume throughout the section that ξ and ζ satisfy the conclusions of Proposition 10.

3.1. Self-similar variables

First, the behaviour as $t \to T$ of ξ and ζ, in Proposition 10, suggests that the typical scale along the y variable is $|y| \sim (T-t)^{1/(2k)}$. The typical scale for diffusive effects for a blow-up at the origin at time T is $|y| \sim \sqrt{T-t}$. Formally, since $k \geq 2$ the diffusive effects are negligible. As $|\xi| \sim (T-t)^{-1}$ and $|\zeta| \sim (T-t)^{-4}$, this suggests the scale $|x| \sim (T-t)^{3/2}$. We introduce:

$$X := \sqrt{\frac{b}{6}} \frac{x}{(T-t)^{\frac{3}{2}}}, \quad Y := a^{\frac{1}{2k}} \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := e^{-\frac{k-1}{2k}s} Y = \frac{a^{\frac{1}{2k}} y}{(T-t)^{\frac{1}{2k}}} \qquad (3.2)$$ eq:def XYZ

and

$$u(t,x,y) = \sqrt{\frac{6}{b}} (T-t)^{\frac{1}{2}} v(s, X, Y)$$

where the renormalisation factors $\sqrt{b/6}$ and $a^{1/2k}$ will simplify notations. To ease the analysis, since the value of a and b will never play a role in this section, we take

$$a = 1 = b \qquad (3.3)$$ eq:def a b

without loss of generality for the argument. Then v solves from the choice (3.3):

$$v_s - \frac{1}{2} v + \frac{3}{2} X \partial_X v + \frac{1}{2} Y \partial_Y v + v \partial_X v - \partial_{YY} v = 0. \qquad (3.4)$$ main:eqvauto

We define accordingly

$$f(s,Y) := -\partial_X v(s,0,Y) = (T-t)\xi(t,y), \quad g(s,Y) := \partial_X^3 v(s,0,Y) = (T-t)^4 \frac{6}{b} \zeta(t,y), \qquad (3.5)$$ main:eq:def f

which from Theorem 1 satisfy:

$$f(s,Y) = F_k(Z) + \tilde{f}, \quad F_k(Z) := \frac{1}{1+Z^{2k}}, \quad |\partial_Z^j \tilde{f}| \lesssim e^{-\frac{1}{4k}s} (1+|Z|)^{\frac{1}{2}-2k-j}, \quad j = 0, ..., J, \qquad (3.6)$$ eq:def tilde

$$g(s,Y) = G_k + \tilde{g}, \quad G_k := \frac{6}{(1+Z^{2k})^4}, \quad |\partial_Z^j \tilde{g}| \lesssim e^{-\frac{1}{4k}s} (1+|Z|)^{\frac{1}{2}-8k-j}, \quad j = 0, ..., J, \qquad (3.7)$$ eq:def tilde

and solve the system from (1.7) and (3.3):

$$\begin{cases} f_s + f + \frac{Y}{2} \partial_Y f - f^2 - \partial_{YY} f = 0, \\ g_s + 4g + \frac{Y}{2} \partial_Y g - 4fg - \partial_{YY} g = 0. \end{cases} \qquad (3.8)$$ eq:f

In (3.2), the variable Y is adapted to the viscosity effects whereas the variable that is adapted to the blow-up profile is Z. The renormalised function $w(s,X,Z)=v(s,X,Y)$ solves in fact

$$w_s-\frac{1}{2}w+\frac{3}{2}X\partial_X w+\frac{1}{2k}Z\partial_Z w+w\partial_X w-e^{-\frac{k-1}{k}s}\partial_{ZZ}w=0. \tag{3.9}$$

eq:w

3.2. 2D Blow-up profile and spectral analysis

The infinitesimal behaviour near the origin along the transverse axis being understood by (3.6) and (3.7), we need to "extend" it along the x variable. A reasonable guess is that the blow-up of a solution to (1.1) is given by a shock of Burgers equation $\lambda^{1/2}\mu\Psi_1(\lambda^{-3/2}\mu^{-1}x)$ whose two parameters are dictated by (1.7). Let us first give additional properties of Ψ_1 than those contained in Subsection 1.2. From Proposition 5 it solves

$$-\frac{1}{2}\Psi_1+\frac{3}{2}X\partial_X\Psi_1+\Psi_1\partial_X\Psi_1=0 \tag{3.10}$$

main:eq:Psi

and has the asymptotic behaviour

$$\Psi_1(X)\underset{X\to 0}{=}-X+X^3+O(X^5),\quad \Psi_1(X)\underset{|X|\to+\infty}{=}-\text{sgn}(X)|X|^{\frac{1}{3}}+O(|X|^{-\frac{1}{3}}). \tag{3.11}$$

main:eqPsi1o

Since w is a global solution to (3.9) whose derivatives up to third order on the axis $\{X=0\}$ converge to some fixed profiles from (3.6) and (3.7) one can believe that w converges as $s\to+\infty$ to a profile w_∞ which then has to solve the asymptotic stationary self-similar[5] equation

$$-\frac{1}{2}w_\infty+\frac{3}{2}X\partial_X w_\infty+\frac{1}{2k}Z\partial_Z w_\infty+w_\infty\partial_X w_\infty=0. \tag{3.12}$$

eq:Q

lem:Psi1

Lemma 11. *For any $a,b>0$, equation (3.12) admits the following solution that is odd in X and even in Z:*

$$\Theta[a,b](X,Z):=b^{-1}F_k^{-\frac{1}{2}}(aZ)\Psi_1\left(bF_k^{\frac{3}{2}}(aZ)X\right)$$

Proof. This is a direct computation. The equation is invariant by the scaling $z\mapsto aZ$, $x\mapsto bX$ and $w_\infty\mapsto b^{-1}w_\infty$, so that we take $a=b=1$ without loss of generality. From (3.6) and (3.10):

$$\begin{aligned}
&-\frac{1}{2}\Theta[1,1]+\frac{3}{2}\tilde{X}\partial_X\Theta[1,1]+\frac{1}{2k}Z\partial_Z\Theta[1,1]+\Theta[1,1]\partial_X\Theta[1,1]\\
=\;& F_k^{-\frac{1}{2}}(Z)\left(-\frac{1}{2}\Psi_1+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1\right)(F_k^{\frac{3}{2}}(Z)X)\\
&+\frac{1}{2k}Z\partial_Z F_k(Z)F_k^{-\frac{3}{2}}(Z)\left(-\frac{1}{2}\Psi_1+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1\right)(F_k^{\frac{3}{2}}(Z)X)+F_k^{\frac{1}{2}}(Z)(\Psi_1\partial_{\tilde{X}}\Psi_1)(F_k^{\frac{3}{2}}(Z)X)\\
=\;& F_k^{-\frac{1}{2}}\left(-\frac{1}{2}\Psi_1+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1\right)(F_k^{\frac{3}{2}}X)\\
&+(F_k^{\frac{1}{2}}-F_k^{-\frac{1}{2}})\left(-\frac{1}{2}\Psi_1+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1\right)(F_k^{\frac{3}{2}}X)-F_k^{\frac{1}{2}}\left(-\frac{1}{2}\Psi_1+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1\right)(F_k^{\frac{3}{2}}X)=0.
\end{aligned}$$

□

The choice (3.3) implies that the good candidate for (3.12) is

$$\Theta(X,Z):=\Theta[1,1](X,Z)=F_k^{-\frac{1}{2}}(Z)\Psi_1\left(F_k^{\frac{3}{2}}(Z)X\right). \tag{3.13}$$

eq:def Theta

[5]Self-similarity is here with respect to the equation (1.1) without viscosity.

The linearised operator corresponding to (3.9) near Θ neglecting the transversal viscosity is

$$\begin{aligned}\mathcal{L}_Z &:= -\frac{1}{2}+\frac{3}{2}X\partial_X+\frac{1}{2k}Z\partial_Z+\Theta\partial_X+\partial_X\Theta\\ &= -\frac{1}{2}+\frac{3}{2}X\partial_X+\frac{1}{2k}Z\partial_Z+F_k^{-\frac{1}{2}}\Psi_1\left(F_k^{\frac{3}{2}}(Z)X\right)\partial_X+F_k(Z)\partial_X\Psi_1\left(F_k^{\frac{3}{2}}(Z)X\right).\end{aligned}$$

We claim that its spectral structure can be understood trough the spectral analysis of two linearised operators, H_X for Burgers equation studied in Proposition 8 and H_Z for the semi-linear heat equation studied in Proposition 24.

pr:mathcalL **Proposition 12.** *Let $k\in\mathbb{N}$, $k\geq 2$. For any $(j,\ell)\in\mathbb{N}^2$, $(j-3)/2+\ell/(2k)$ is an eigenvalue of the operator $\mathcal{L}_Z:\mathcal{C}^1(\mathbb{R}^2)\to\mathcal{C}^0(\mathbb{R}^2)$ associated to the eigenfunction*

$$\varphi_{j,\ell}(X,Z)=\phi_{Z,\ell}(Z)F_k^{-1-\frac{j}{2}}(Z)\phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right)=Z^\ell F_k^{1-\frac{j}{2}}(Z)\times\frac{(-1)^j\Psi_1^j\left(F_k^{\frac{3}{2}}(Z)X\right)}{1+3\Psi_1^2\left(F_k^{\frac{3}{2}}(Z)X\right)},\quad (3.14)$$ id:phij0

where $\phi_{X,j}$ and $\phi_{Z,\ell}$ are defined by (2.17) and (4.3).

Proof. This is a direct computation. From (4.2), (2.17) and (4.3) one has:

$$\begin{aligned}\mathcal{L}_Z\varphi_{j,k} &= \phi_{Z,\ell}F_k^{-1-\frac{j}{2}}\left(-\frac{1}{2}\phi_{X,j}+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\phi_{X,j}\right)(F_k^{\frac{3}{2}}X)+\frac{1}{2k}Z\partial_Z\phi_{Z,\ell}F_k^{-1-\frac{j}{2}}\phi_{X,j}(F_k^{\frac{3}{2}}X)\\ &\quad+\frac{1}{2k}Z\partial_ZF_k\phi_{Z,\ell}F_k^{-2-\frac{j}{2}}\left((-1-\frac{j}{2})\phi_{X,j}+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\phi_{X,j}\right)(F_k^{\frac{3}{2}}X)\\ &\quad+\phi_{Z,\ell}F_k^{-\frac{j}{2}}(\partial_{\tilde{X}}\Psi_1\phi_{X,j}+\Psi_1\partial_X\phi_{X,j})(F_k^{\frac{3}{2}}X)\\ &= \phi_{Z,\ell}F_k^{-1-\frac{j}{2}}\left(-\frac{1}{2}\phi_{X,j}+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\phi_{X,j}\right)(F_k^{\frac{3}{2}}X)\\ &\quad+\left((\frac{\ell-2k}{2k}-1)\phi_{Z,\ell}+2F_k\phi_{Z,\ell}\right)F_k^{-1-\frac{j}{2}}\phi_{X,j}(F_k^{\frac{3}{2}}X)\\ &\quad+\phi_{Z,\ell}(F_k^{-\frac{j}{2}}-F_k^{-1-\frac{j}{2}})\left((-1-\frac{j}{2})\phi_{X,j}+\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\phi_{X,j}\right)(F_k^{\frac{3}{2}}X)\\ &\quad+\phi_{Z,k}F_k^{-\frac{j}{2}}\left((\frac{j-3}{2}+\frac{1}{2})\phi_{X,j}-\frac{3}{2}\tilde{X}\partial_{\tilde{X}}\phi_{X,j}\right)(F_k^{\frac{3}{2}}X)\\ &= \left(\frac{j-3}{2}+\frac{\ell}{2k}\right)\phi_{Z,\ell}F_k^{-1-\frac{j}{2}}\phi_{X,j}(F_k^{\frac{3}{2}}X).\end{aligned}$$

□

3.3. Linear estimates

A maximum principle holds for the linear transport operator $\mathcal{L}_Z$. Also, we will not use this estimate as is, we believe it is of interest.

Lemma 13. *Assume $\varepsilon_0\in L^\infty_{loc}(\mathbb{R}^2)$ is such that $|\varepsilon_0|\leq C|\varphi_{j,\ell}|$ on $\mathbb{R}^2\backslash\{X=0\}$, for some $C>0$ and $(j,\ell)\in\mathbb{N}^2$. Then the solution to $\partial_s\varepsilon+\mathcal{L}_Z\varepsilon=0$ with initial datum ε_0 satisfies:*

$$\left\|\frac{\varepsilon}{\varphi_{j,\ell}}\right\|_{L^\infty(\mathbb{R}^2\backslash\{X=0\})}\leq e^{-(\frac{j-3}{2}+\frac{\ell}{2k})s}\left\|\frac{\varepsilon_0}{\varphi_{j,\ell}}\right\|_{L^\infty(\mathbb{R}^2\backslash\{X=0\})}.\quad (3.15)$$ main:bd:estim

Proof. This is a straightforward computation along the characteristics, that we omit. □

We now investigate the linear dynamics in the presence of dissipation, for the full operator:

$$\mathcal{L} := -\frac{1}{2} + \partial_X \Theta + \left(\frac{3}{2}X + \Theta\right)\partial_X + \frac{1}{2}Y\partial_Y - \partial_{YY} = \mathcal{L}_Z + \frac{k-1}{2k} Z\partial_Z - \partial_{YY},$$

and find an energy estimate that mimics (3.15). We will use eigenfunctions of the form $\phi_{j,\ell}$ with $j > 3$ and $\ell = 0$ to ensure time decay, and because the choice $\ell > 0$ would produce a singularity near $Z = 0$ that is incompatible with the viscosity. We replace the L^∞ norm by a weighted L^q norm with q large enough, also in order to be compatible with the viscous term.

lem:lineaire **Lemma 14.** *Let $0 \le j < i_0$. For any $\kappa > 0$, there exists $q^* \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ with $q \ge q^*$ there exists $s^* \ge 0$ such that the following holds. For any $s^* \le s_0 < s_1$, if ε and Ξ are in the Schwartz class and satisfy on $[s_0, s_1]$ that:*

$$\varepsilon_s + \mathcal{L}\varepsilon = \Xi, \tag{3.16}$$

eq:evolinee

that for $i = 0, ..., i_0$ one has the cancellation on the axis $\{X = 0\}$

$$\partial_X^i \varepsilon(s, 0, Y) = 0 \quad \text{and} \quad \partial_X^i \Xi(s, 0, Y) = 0, \tag{3.17}$$

eq:condevoli

then the following energy identity holds:

$$\begin{aligned} &\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right) \qquad (3.18)\\ &\le -\left(\frac{j-3}{2} - \frac{\kappa}{2}\right)\int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} - \frac{2q-1}{q^2}\int \frac{|\partial_Y(\varepsilon^q)|^2}{\varphi_{j,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + \int \frac{\varepsilon^{2q-1}\Xi}{\varphi_{j,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}. \end{aligned}$$

id:estimatio

Proof. This is a direct computation. One computes from the evolution equation (3.16), performing integration by parts,

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)=\int_{\mathbb{R}^2}\frac{\varepsilon^{2q-1}\partial_s\varepsilon}{\varphi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}\partial_s\varphi_{j,0}(X,Z)}{\varphi_{j,0}^{2q+1}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
=&\int\frac{\varepsilon^{2q-1}}{\varphi_{j,0}^{2q}(X,Z)}\Big[\frac12\varepsilon-\frac32X\partial_X\varepsilon-\frac12Y\partial_Y\varepsilon-F_k^{-\frac12}\Psi_1\left(F_k^{\frac32}(Z)X\right)\partial_X\varepsilon+\partial_{YY}\varepsilon\\
&-F_k(Z)\partial_X\Psi_1\left(F_k^{\frac32}(Z)X\right)\varepsilon+\Xi\Big]\frac{dXdY}{|X|\langle Y\rangle}+\frac{k-1}{2k}\int\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q+1}(X,Z)}Z\partial_Z\varphi_{j,0}(X,Z)\frac{dXdY}{|X|\langle Y\rangle}\\
=&\int\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q+1}(X,Z)}\Big[\frac12\varphi_{j,0}-\frac32X\partial_X\varphi_{j,0}-\frac{1}{2k}Z\partial_Z\varphi_{j,0}-F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)\partial_X\varphi_{j,0}\\
&-F_k(Z)\partial_X\Psi_1(F_k^{\frac32}(Z)X)\varphi_{j,0}\Big]\frac{dXdY}{|X|\langle Y\rangle}\\
&+\frac{1}{2q}\int\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}}\left(\frac12\partial_Y\left(\frac{Y}{\langle Y\rangle}\right)\frac{1}{|X|}+\partial_X\left(\frac{F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}{X}\right)\frac{1}{\langle Y\rangle}\right)dXdY\\
&-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi_{j,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}+\frac{1}{2q}\int\varepsilon^{2q}\partial_{YY}\left(\frac{1}{\varphi_{j,0}^{2q}}\frac{1}{\langle Y\rangle}\right)\frac{dXdY}{|X|}+\int\frac{\varepsilon^{2q-1}\Xi}{\varphi_{j,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\
=&-\frac{j-3}{2}\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi_{j,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}+\int\frac{\varepsilon^{2q-1}\Xi}{\varphi_{j,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\
&+\frac{1}{2q}\int\frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}}\left(\frac12\partial_Y\left(\frac{Y}{\langle Y\rangle}\right)\frac{1}{|X|}+\partial_X\left(\frac{F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}{|X|}\right)\frac{1}{\langle Y\rangle}\right)dXdY\\
&+\frac{1}{2q}\int\varepsilon^{2q}\partial_{YY}\left(\frac{1}{\varphi_{j,0}^{2q}}\frac{1}{\langle Y\rangle}\right)\frac{dXdY}{|X|}
\end{aligned}
$$

where we used Proposition (12). The integrations by parts are legitimate near the axis $\{X=0\}$ because of the cancellation (3.17) and since $\Psi_1(X)\sim -X$ and $\varphi_{j,0}\sim X^j$ as $X\to 0$ from (3.21). The last terms are lower order ones. Indeed, one has:

$$\left|\partial_Y\left(\frac{Y}{\langle Y\rangle}\right)\right|=\frac{1}{\langle Y\rangle^3}$$

and

$$\partial_X\left(\frac{F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}{|X|}\right)=\frac{F_k(Z)\partial_X\Psi_1(F_k^{\frac32}(Z)X)}{|X|}+\frac{F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}{X|X|}.$$

For the first term in the above identity, one has that $|F_k(Z)|=(1+Z^{2k})^{-1}\leq 1$ and that $|\partial_X\Psi_1|=|1/(1+3\Psi_1^2)|\leq 1$. For the second, one has that $|\Psi_1(X)|\leq |X|$. Therefore,

$$\left|\partial_X\left(\frac{F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}{|X|}\right)\right|\leq\frac{2}{|X|}.$$

Next, since $|\partial_Z^j \varphi_{4,0}(X,Z)| \lesssim (1+|Z|)^{-j}|\varphi_{4,0}(X,Z)|$ from (3.14) and $\partial_Y = e^{-(k-1)s/(2k)}\partial_Z$:

$$\left|\partial_{YY}\left(\frac{1}{\varphi_{4,0}^{2q}}\frac{1}{\langle Y\rangle}\right)\right| \lesssim \frac{(1+q^2 e^{-\frac{k-1}{k}s})}{\varphi_{4,0}^{2q}}\frac{1}{\langle Y\rangle}.$$

Therefore,

$$\Big|\frac{1}{2q}\int \frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}}\left(\frac{1}{2}\partial_Y\left(\frac{Y}{\langle Y\rangle}\right)\frac{1}{|X|} + \partial_X\left(\frac{F_k^{-\frac{1}{2}}(Z)\Psi_1(F_k^{\frac{3}{2}}(Z)X)}{|X|}\right)\frac{1}{\langle Y\rangle}\right)dXdY$$
$$+\frac{1}{2q}\int \varepsilon^{2q}\partial_{YY}\left(\frac{1}{\varphi_{j,0}^{2q}}\frac{1}{\langle Y\rangle}\right)\frac{dXdY}{|X|}\Big| \leq \frac{C}{q}(1+q^2e^{-\frac{k-1}{k}s})\int \frac{\varepsilon^{2q}}{\varphi_{j,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

which, injected in the previous energy identity yields the desired result upon choosing q large enough and then s^* large enough.

□

3.4. Bootstrap analysis

We are now ready to prove Theorems 2 and 3. Throughout the analysis, the functions F_k, Ψ_1, Θ and $\varphi_{j,0}$ will be extensively used. In particular, from (3.13), a relevant variable for the stream direction is $\tilde{X}$ defined by (1.16) with:

$$|\tilde{X}| \approx |X|(1+|Z|)^{-3k},$$

and from (3.6), (3.13) and (3.14) their size is encoded by the following estimates (which adapt to derivatives)

$$F_k(Z) \approx (1+|Z|)^{-2k}, \quad |\Psi_1(X)| \approx |X|(1+|X|)^{\frac{1}{3}-1}, \tag{3.19}$$ main:sizeFkPs

$$|\Theta(X,Z)| \approx |X|\left((1+|Z|)^{3k}+|X|\right)^{\frac{1}{3}-1} \approx (1+|Z|)^k|\tilde{X}|(1+|\tilde{X}|)^{\frac{1}{3}-1} \tag{3.20}$$ main:sizeThe

$$|\varphi_{j,0}(X,Z)| \approx |X|^j\left((1+|Z|)^{3k}+|X|\right)^{\frac{j-2}{3}-j} \approx (1+|Z|)^{k(j-2)}|\tilde{X}|^j(1+|\tilde{X}|)^{\frac{j-2}{3}-j}, \tag{3.21}$$ main:sizephi

The strategy is to show that there exists global solutions to (3.4) converging to Θ defined by (3.13) as $s \to +\infty$. We will use an approximate blow-up profile, i.e. refine Θ to show this. To obtain decay in the linear estimate (3.18), one needs $j > 3$ which from (3.21) in turn requires that $\varepsilon = O(|X|^j)$ as $X \to 0$. The linearised operator $\mathcal{L}$ has then a damping effect on functions vanishing up to order 3 on the vertical axis. Consequently, we use the profile $\mu^{-1}\lambda^{-1/2}\Psi_1(\mu\lambda^{3/2}X)$ at each line $\{Y = Cte\}$, to match the solution at order 1 and 3 near the vertical axis $\{X = 0\}$. Far away, such a decomposition ceases to make sense since we are no more in the blow-up zone, and the appropriate profile is 0 rather than Ψ_1. We set for $d > 0$ a cut-off function (note that $|Y| \leq de^{s/2}$ is equivalent to $|y| \leq d$ and $|Z| \leq e^{s/2k}$),

$$\chi_d(s,Y) := \chi\left(\frac{Y}{de^{\frac{s}{2}}}\right)$$

and then decompose our solution to (3.4) according to:

$$v(s,X,Y) = Q + \varepsilon, \qquad Q = \chi_d(s,Y)\tilde{\Theta} + (1-\chi_d(s,Y))\Theta_e \tag{3.22}$$ def:e

where $\tilde{\Theta}$ is the approximate blow-up profile in the interior zone

$$\tilde{\Theta}(s,X,Y) := \mu^{-1}(s,Y)f^{-\frac{1}{2}}(s,Y)\Psi_1\left(f^{\frac{3}{2}}(s,Y)\mu(s,Y)X\right) = \sqrt{6}g^{-\frac{1}{2}}f^{\frac{3}{2}}\Psi_1\left(\frac{g^{\frac{1}{2}}f^{-\frac{1}{2}}}{\sqrt{6}}X\right) \tag{3.23}$$ def:tildeThe

where f and g are defined in (3.5) and

$$\mu(s,Y):=\left(\frac{g(s,Y)}{6f^4(s,Y)}\right)^{\frac12}, \qquad (3.24)$$ eq:def b

(notice that for d small enough and for Y in the support of $\chi_d(s,\cdot)$, the functions f and g do not vanish from (3.6) and (3.7), and hence μ and μ^{-1} are well-defined), and where Θ_e is the profile for the external zone

$$\Theta_e(s,X,Y):=\left(-Xf(s,Y)+X^3\frac{g(s,Y)}{6}\right)e^{-\tilde X^4}. \qquad (3.25)$$ def:Thetae

The profile Θ_e matches with v up to third order near the vertical axis, what allows for a unified control of the remainder ε, inside and outside the blow-up zone simultaneously (3.27) and (3.28). It is not a very precise approximation, what does not matter since the weights we use penalise the exterior zone. Other choices for Θ_e are thus possible.

To estimate the remainder ε, we will use weighted Sobolev norms, and to control its derivatives we will use vector fields that commute well with $\partial_s+\mathcal{L}_Z$:

$$A:=\left(\frac32 X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)\right)\partial_X,\quad \partial_Z \quad \text{and} \quad Z\partial_Z \qquad (3.26)$$ main:def:A

and that are equivalent to usual vector fields, see Lemma 46.

Definition 15 (Trapped solutions)**.** *Let constants $\kappa,d>0$, $q\in\mathbb{N}$, $K_{j_1,j_2}>0$ for nonnegative integers j_1,j_2 with $0\leq j_1+j_2\leq 2$, and $\tilde K_{j_1,j_2}>0$ for nonnegative integers j_1,j_2 with $0\leq j_1+j_2\leq 2$ and $j_2\geq 1$, and $s_0<s_1$. We say a solution to (3.4) is trapped on $[s_0,s_1]$, if, decomposed according to (3.22), it satisfies on $[s_0,s_1]$:*

$$\left(\int_{\mathbb{R}^2}\frac{((\partial_Z^{j_1}A^{j_2}\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\leq K_{j_1,j_2}e^{-(\frac12-\kappa)s}, \qquad (3.27)$$ main:weightec

and for $0\leq j_1+j_2\leq 2$ and $j_2\geq 1$:

$$\left(\int_{\mathbb{R}^2}\frac{(((Y\partial_Y)^{j_1}A^{j_2}\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\leq \tilde K_{j_1,j_2}e^{-(\frac12-\kappa)s}, \qquad (3.28)$$ main:weightec

and write $u\in\mathcal{T}(\kappa,q,s_0,s_1,(K_{j_1,j_2})_{0\leq j_1+j_2\leq 2},(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2,1\leq j_1})$ for the set of such solutions.

We claim that ε decays thanks to the following bootstrap argument, which is the heart of our proof of Theorem 3.

pr:bootstrap **Proposition 16.** *Let ξ and ζ be given by Proposition 10. Let $0<\kappa<1/2$ and $q\in\mathbb{N}^*$ such that Lemma 14 holds true. Then there exist positive constants d, $(K_{j_1,j_2})_{0\leq j_1+j_2\leq 2}$ and $(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2,1\leq j_1}$, and $s^*\geq 0$, such that if at anytime $s_0\geq s^*$ the solution is given by (3.22) with $\varepsilon(s_0)=\varepsilon_0$ satisfies*

$$\sum_{0\leq j_1+j_2\leq 1}\int_{\mathbb{R}^2}\frac{(\partial_Z^{j_1}A^{j_2}\varepsilon_0)^{2q}+((Y\partial_Y)^{j_1}A^{j_2}\varepsilon_0)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\leq e^{-2q(\frac12-\kappa)s_0} \qquad (3.29)$$ main:weightec

then the solution to (3.4) is global and is trapped on $[s_0,+\infty)$:

$$u\in\mathcal{T}(\kappa,q,s_0,+\infty,(K_{j_1,j_2})_{0\leq j_1+j_2\leq 2},(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2,1\leq j_1}).$$

rk:constants **Remark 17** (On the constants in the bootstrap analysis)**.** *To track dependencies of the constants in the proof we use the following:*

- *The functions ξ and ζ are chosen in advance as solutions to (1.7) satisfying (1.9) and (3.1). The constants in these estimates are considered as universal and are independent of the bootstrap constants.*
- *The parameter $d>0$ is fixed so that one has for s large enough for $|Z|\leq 2e^{s/2k}$:*

$$\left|\frac{f(s,Z)}{F_k(Z)}-1\right|\leq \frac12 \qquad \text{and} \qquad \left|\frac{g(s,Z)}{G_k(Z)}-1\right|\leq \frac12 \tag{3.30}$$ id:choiced

which is always possible from (3.6) and (3.7).
- *κ and q are fixed, and s^* is large enough, so that Lemma 14 holds.*
- *Thus, the constants which are not fixed at this stage are K_{j_1,j_2}, $\tilde K_{j_1,j_2}$ and s^*. Their choice is made in the following order. $K_{0,0}$ is chosen bigger than a universal constant see Lemma 22, then $K_{0,1}$ is chosen bigger than a constant depending only on $K_{0,0}$ see (3.49), then $K_{1,0}$ is chosen bigger than a constant depending only on $K_{0,0}$ and $K_{0,1}$, then $\tilde K_{1,0}$ is chosen depending on $(K_{0,0},K_{0,1})$, and so on. This is first because a given derivative of ε sees lower order derivatives as forcing terms from Leibniz formula. Second, A commutes with the full transport operator in $\partial_s+\mathcal L_Z$ (3.48), while ∂_Z and $Y\partial_Y$ commute with the ∂_Z of this transport operator but not with the ∂_X part. Hence, we first control ε, then to take derivatives we first control $A\varepsilon$, then $\partial_Z\varepsilon$ and $Y\partial_Y\varepsilon$, and then we move to higher order derivatives and so on.*
- *s^* is chosen last.*
- *Constants C in forthcoming estimates stand for constants that are independent of the K's and $\tilde K$'s constants, unless explicitly mentioned. We shall write $A\lesssim B$ if $A\leq CB$ for such a constant C.*

The proof of the above Proposition 16 follows a classical bootstrap reasoning. Namely, throughout the remaining part of this section we assume that v is a solution to (3.4) defined on $[s_0,s_1]$ and such that the decomposition (3.22) satisfies (3.29), (3.27) and (3.28). All the results below will show that (3.27) and (3.28) are in fact strict at time s_1, what will allow us to conclude the proof of Proposition 16 by a continuity argument at the end of this section.

First, notice that the bounds of Proposition 16 imply pointwise control by weighted Sobolev embedding.

n:pointwise e **Lemma 18.** *There holds on $[s_0,s_1]$ with constants in the inequalities depending on the bootstrap constants K_{j_1,j_2} and $\tilde K_{j_1,j_2}$:*

$$|\varepsilon|\lesssim e^{-(\frac12-\kappa)s}(1+|Z|)^{2k}|\tilde X|^4(1+|\tilde X|)^{\frac23-4}\lesssim e^{-(\frac12-\kappa)s}|X|^4((1+|Z|)^{3k}+|X|)^{\frac23-4}\lesssim e^{-(\frac12-\kappa)s}|X|, \tag{3.31}$$ main:bde

$$|\partial_X\varepsilon|\lesssim e^{-(\frac12-\kappa)s}(1+|Z|)^{-k}|\tilde X|^3(1+|\tilde X|)^{\frac23-4}\lesssim e^{-(\frac12-\kappa)s}|X|^3((1+|Z|)^{3k}+|X|)^{\frac23-4}\lesssim e^{-(\frac12-\kappa)s} \tag{3.32}$$ main:bde2

$$\begin{aligned} |\partial_Z\varepsilon| &\lesssim e^{-(\frac12-\kappa)s}(1+|Z|)^{2k-1}|\tilde X|^4(1+|\tilde X|)^{\frac23-4} \\ &\lesssim e^{-(\frac12-\kappa)s}(1+|Z|)^{-1}|X|^3((1+|Z|)^{3k}+|X|)^{\frac23-4} \\ &\lesssim e^{-(\frac12-\kappa)s}|X|(1+|Z|)^{-1} \end{aligned} \tag{3.33}$$

Proof. Recall that $Y\partial_Y=Z\partial_Z$. **Step 1** *Proof of* (3.31). From the identity

$$|\langle Y\rangle\partial_Y\varepsilon|\lesssim|\partial_Y\varepsilon|+|Y\partial_Y\varepsilon|=e^{-\frac{k-1}{2k}s}|\partial_Z\varepsilon|+|Z\partial_Z\varepsilon| \tag{3.34}$$ main:pointwi

and the equivalence between vector fields (B.6) we infer that

$$|\varepsilon|+|X\partial_X\varepsilon|+|\langle Y\rangle\partial_Y\varepsilon|\lesssim|\varepsilon|+|A\varepsilon|+|\partial_Z\varepsilon|+|Z\partial_Z\varepsilon|$$

and therefore the bootstrap bounds (3.27) and (3.28) imply in particular that:

$$\int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(X\partial_X\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(\langle Y\rangle\partial_Y\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} \lesssim e^{-2q(\frac12-\kappa)s}$$

and the weighted Sobolev embedding (B.2) implies that $|\varepsilon| \lesssim e^{-(1/2-\kappa)s}|\varphi_{4,0}|$ which gives (3.31) using (3.21).

Step 2 *Proof of* (3.32). The very same reasoning as in Step 1 show that the bootstrap bounds (3.27) and (3.28) imply $|A\varepsilon| \lesssim e^{-(1/2-\kappa)s}\varphi_{4,0}$. From (B.4) and (B.6) we infer that $|A\varepsilon| \approx |X\partial_X\varepsilon|$, implying then that $|\partial_X\varepsilon| \lesssim e^{-(1/2-\kappa)s}|X|^{-1}|\varphi_{4,0}|$, which yields (3.32) using (3.21).

Step 3 *Proof of* (3.33). Using (3.34) and (B.3) we obtain that

$$|\langle Y\rangle\partial_Y\partial_Z\varepsilon| \lesssim |\partial_Z^2\varepsilon| + |Z\partial_Z^2\varepsilon| \lesssim |\partial_Z^2\varepsilon| + |Z^2\partial_Z^2\varepsilon| \lesssim |\partial_Z^2\varepsilon| + |Z\partial_Z\varepsilon| + |(Z\partial_Z)^2\varepsilon|$$

and from (B.6) that $|X\partial_X\partial_Z\varepsilon| \lesssim |A\varepsilon| + |\partial_Z A\varepsilon|$. Therefore, we infer from (3.27) and (3.28) that

$$\int_{\mathbb{R}^2} \frac{(\partial_Z\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(X\partial_X\partial_Z\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(\langle Y\rangle\partial_Y\partial_Z\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} \lesssim e^{-2q(\frac12-\kappa)s}$$

implying using (B.2) that $|\partial_Z\varepsilon| \lesssim e^{-(1/2-\kappa)s}|\varphi_{4,0}|$. The very same reasoning applies for the function $Z\partial_Z\varepsilon$, yielding $|Z\partial_Z\varepsilon| \lesssim e^{-(1/2-\kappa)s}|\varphi_{4,0}|$. Therefore, $|\partial_Z\varepsilon| \lesssim e^{-(1/2-\kappa)s}(1+|Z|)^{-1}|\varphi_{4,0}|$ which yields (3.33) using (3.21).

□

We start by investigating the infinitesimal behavior of ε near the line $\{X=0\}$. This corresponds to establishing the so-called modulation equation for the parameters describing the blow-up profile Ψ_1 and the external profile Θ_e on each fixed line $\{Y = Cte\}$. We claim that ε vanishes on the axis up to the third order.

Lemma 19. *For all $s \geq 0$ and $Y \in \mathbb{R}$ one has that*

$$\partial_X^j\varepsilon(s,0,Y) = 0, \quad j = 0,1,2,3,4. \tag{3.35}$$

id:modulatio

Proof. This is a direct computation. First since the profile is odd in X and even in Y one has that ε, $\partial_X^2\varepsilon$ and $\partial_X^4\varepsilon$ vanish on the vertical axis $\{X=0\}$. Then, one has by definition (3.5) of f that $\partial_X v(s,0,Y) = -f(s,Y)$ and from (3.22), (3.23) and (3.25) that

$$\partial_X v(s,0,Y) = -\chi_d f - (1-\chi_d)f + \partial_X\varepsilon(s,0,Y) = -f + \partial_X\varepsilon(s,0,Y).$$

Therefore $\partial_X\varepsilon(s,0,Y) = 0$ for all $s \geq s_0$ and $Y \in \mathbb{R}$. Similarly, by (3.5), $\partial_X^3 v(s,0,Y) = g(s,Y)$ and from (3.22), (3.23) and (3.25) one has

$$\partial_X^3 v(s,0,Z) = \chi_d 6b^2 f^4 + (1-\chi_d)g + \partial_X^3\varepsilon(s,0,Z) = g + \partial_X^3\varepsilon(s,0,Z).$$

Therefore $\partial_X^3\varepsilon(s,0,Z) = 0$ for all $s \geq s_0$ and $Y \in \mathbb{R}$ which ends the proof of the lemma.

□

The time evolution of ε is given by:

$$\varepsilon_s + \mathcal{L}\varepsilon + \tilde{\mathcal{L}}\varepsilon + R + \varepsilon\partial_X\varepsilon = 0 \tag{3.36}$$

main:evoluti

where

$$\mathcal{L} := -\frac12 + \partial_X\Theta + \left(\frac32 X + \Theta\right)\partial_X + \frac12 Y\partial_Y - \partial_{YY} = \mathcal{L}_Z + \frac{k-1}{2k}Z\partial_Z - \partial_{YY},$$

$$\tilde{\mathcal{L}}\varepsilon = (Q-\Theta)\partial_X\varepsilon + (\partial_X Q - \partial_X\Theta)\varepsilon, \tag{3.37}$$

main:def:tL

and

$$R = Q_s - \frac{1}{2}Q + \frac{3}{2}X\partial_X Q + \frac{1}{2}Y\partial_Y Q + Q\partial_X Q - \partial_{YY}Q. \tag{3.38}$$ eq:def R

main:lem:R **Lemma 20.** *Let $q \in \mathbb{N}$, $q \geq 1$, and $3 + 1/k \leq j \leq 5 - 2/k$. Then one has the estimate*

$$\sum_{0\leq j_1+j_2\leq 1}\int_{\mathbb{R}^2}\frac{(\partial_Z^{j_1}A^{j_2}R)^{2q} + ((Y\partial_Y)^{j_1}A^{j_2}R)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} \lesssim se^{-qs}. \tag{3.39}$$ main:estimat

Proof. Recall that $Y\partial_Y = Z\partial_Z$. From (3.38), (3.23) and (3.25) we compute:

$$R = R_1 + R_2 + R_3 + R_4 + R_5$$

where

$$R_1 := \chi_d\left(\tilde\Theta_s - \frac{1}{2}\tilde\Theta + \frac{3}{2}X\partial_X\tilde\Theta + \frac{1}{2}Y\partial_Y\tilde\Theta + \tilde\Theta\partial_X\tilde\Theta - \partial_{YY}\tilde\Theta\right),$$

$$R_2 := (1-\chi_d)\left(\partial_s\Theta_e - \frac{1}{2}\Theta_e + \frac{3}{2}X\partial_X\Theta_e + \frac{1}{2}Y\partial_Y\Theta_e + \Theta_e\partial_X\Theta_e - \partial_{YY}\Theta_e\right),$$

$$R_3 := (\tilde\Theta - \Theta_e)(\partial_s\chi_d + \frac{1}{2}Y\partial_Y\chi_d - \partial_{YY}\chi_d), \quad R_4 := -2\partial_Y\chi_d\partial_Y(\tilde\Theta - \Theta_e),$$

$$R_5 := \chi_d(1-\chi_d)(\tilde\Theta - \Theta_e)\partial_X(\tilde\Theta - \Theta_e).$$

We now prove the corresponding bounds for all terms R_i.

Step 1 *Estimate for R_1.* All the computations are performed in the domain of χ_d, $|Y| \lesssim 2de^{s/2}$, where $f, g > 0$. We compute from (3.23), (3.24), (3.8) and Lemma 11 that only some viscosity terms remain in R_1:

$$\begin{aligned}
&\tilde\Theta_s - \frac{1}{2}\tilde\Theta + \frac{3}{2}X\partial_X\tilde\Theta + \frac{1}{2}Y\partial_Y\tilde\Theta + \tilde\Theta\partial_X\tilde\Theta - \partial_{YY}\tilde\Theta \\
= \;& -\sqrt{6}(\partial_Y g)^2 g^{-\frac{5}{2}} f^{\frac{3}{2}}\left[(-\frac{3}{2} + \frac{1}{2}\tilde X\partial_{\tilde X})(-\frac{1}{2}\Psi_1 + \frac{1}{2}\tilde X\partial_{\tilde X}\Psi_1)\right]\left(\frac{g^{\frac{1}{2}}f^{-\frac{1}{2}}}{\sqrt{6}}X\right) \\
& -\sqrt{6}\partial_Y g\partial_Y f g^{-\frac{3}{2}} f^{\frac{1}{2}}\left[-\frac{3}{2}\Psi_1 + \frac{3}{2}\tilde X\partial_{\tilde X}\Psi_1 - \frac{1}{2}\tilde X^2\partial_{\tilde X}^2\Psi_1\right]\left(\frac{g^{\frac{1}{2}}f^{-\frac{1}{2}}}{\sqrt{6}}X\right) \\
& -\sqrt{6}g^{-\frac{1}{2}}(\partial_Y f)^2 f^{-\frac{1}{2}}\left[(\frac{1}{2} - \frac{1}{2}\tilde X\partial_{\tilde X})(\frac{3}{2} - \frac{1}{2}\tilde X\partial_{\tilde X})\Psi_1\right]\left(\frac{g^{\frac{1}{2}}f^{-\frac{1}{2}}}{\sqrt{6}}X\right)
\end{aligned}$$

We only treat the first term, the proof being the same for the others. First, from (3.6) and (3.7):

$$(1+|Z|)^{-3k} \lesssim \left|g^{\frac{1}{2}}(s,Y)f^{-\frac{1}{2}}(s,Y)\right| \lesssim (1+|Z|)^{-3k}, \quad |(Y\partial_Y)^j\chi_d| \lesssim 1$$

on the support of χ_d. For any $j \in \mathbb{N}$, from (3.11) one has that

$$\left|\left[(\tilde X\partial_{\tilde X})^j(-\frac{3}{2} + \frac{1}{2}\tilde X\partial_{\tilde X})(-\frac{1}{2} + \frac{1}{2}\tilde X\partial_{\tilde X})\Psi_1\right](\tilde X)\right| \lesssim |\tilde X|^5(1+|\tilde X|)^{\frac{1}{3}-5}$$

and from (3.6) and (3.7) and for $j_1 + j_1' \leq J$:

$$\frac{|(Y\partial_Y)^j\partial_Z^{j'}(g^{\frac{1}{2}}f^{-\frac{1}{2}})|}{g^{\frac{1}{2}}f^{-\frac{1}{2}}} \lesssim (1+|Z|)^{-j'}, \quad |(Y\partial_Y)^j\partial_Z^{j'}((\partial_Y g)^2 g^{-\frac{5}{2}} f^{\frac{3}{2}})| \lesssim e^{-\frac{k-1}{k}s}(1+|Z|)^{k-2-j'}$$

since $\partial_Y = e^{-(k-1)s/(2k)}\partial_Z$. We therefore infer that:

$$\begin{aligned}
&\left| (Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left(\left[(-\frac32+\frac12\tilde X\partial_{\tilde X})(-\frac12+\frac12\tilde X\partial_{\tilde X})\Psi_1\right]\left(\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right)\right)\right|\\
\lesssim &\left|\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right|^5\left(1+\left|\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right|\right)^{\frac13-5}\lesssim|\tilde X|^5(1+|\tilde X|)^{\frac13-5}\lesssim\frac{|X|^5((1+|Z|)^{3k}+|X|)^{\frac13-5}}{(1+|Z|)^k},
\end{aligned}$$

and in turn, using (3.21):

$$\begin{aligned}
&\frac{\left|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left(\left[(\partial_Yg)^2g^{-\frac52}f^{\frac32}(-\frac32+\frac12\tilde X\partial_{\tilde X})(-\frac12+\frac12\tilde X\partial_{\tilde X})\Psi_1\right]\left(\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right)\right)\right|}{|\varphi_{j,0}(X,Z)|}\\
\lesssim\ & e^{-\frac{k-1}{k}s}|X|^{5-j}(1+|Z|)^{-2}((1+|Z|)^{3k}+|X|)^{j-4-\frac j3}.
\end{aligned}$$

Therefore, one has the following estimate, performing two changes of variables, the first one being $X=(1+Z^{2k})^{3/2}\tilde X$ and the second one $Z=e^{-(k-1)s/(2k)}Y$, with $dX/|X|=d\tilde X/|\tilde X|$, $dY/|Y|=dZ/|Z|$, since $|Y|/\langle Y\rangle\le1$ and $|(Y\partial_Y)^j\partial_Z^{j'}\chi_d|\lesssim1$:

$$\begin{aligned}
&\int_{\mathbb R^2}\left(\frac{(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\chi_d(\partial_Yg)^2g^{-\frac52}f^{\frac32}(-\frac32X+\frac12X\partial_X)(-\frac12+\frac12X\partial_X)\Psi_1\left(\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right)}{|\varphi_{j,0}(X,Z)|}\right)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\\
\lesssim\ & e^{-2q\frac{k-1}{k}s}\int_{\mathbb R^2}\left(|X|^{5-j}(1+|Z|)^{-2}((1+|Z|)^{3k}+|X|)^{j-4-\frac j3}\right)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\\
\lesssim\ & e^{-2q\frac{k-1}{k}s}\int_{\mathbb R}\left(\int_{\mathbb R}\left((1+|Z|)^{3k(1-\frac j3)-2}|\tilde X|^{5-j}(1+|\tilde X|)^{j-4-\frac j3}\right)^{2q}\frac{d\tilde X}{|\tilde X|}\right)\frac{dY}{\langle Y\rangle}\\
\lesssim\ & e^{-2q\frac{k-1}{k}s}\int_{\mathbb R}\left((1+|Z|)^{3k(1-\frac j3)-2}\right)^{2q}\frac{dY}{\langle Y\rangle}\\
\lesssim\ & e^{-2q\frac{k-1}{k}s}\int_{|Y|\le e^{\frac{k-1}{2k}s}}\frac{dY}{\langle Y\rangle}+\int_{|Y|\ge e^{\frac{k-1}{2k}s}}(|Z|^{3k(1-\frac j3)-2})^{2q}\frac{dY}{\langle Y\rangle}\\
\lesssim\ & e^{-2q\frac{k-1}{k}s}\left(s+\int_{|Z|\ge1}(|Z|^{3k(1-\frac j3)-2})^{2q}\frac{dZ}{|Z|}\frac{|Y|}{\langle Y\rangle}\right)\lesssim se^{-2q\frac{k-1}{k}s}
\end{aligned}$$

provided that $3<j<5$. We claim that the very same estimates for the other terms in the expression of R_1 holds, and that they can be proved performing the same computation, thanks to the same fundamental cancellations

$$\left|\left[-\frac32\Psi_1+\frac32\tilde X\partial_{\tilde X}\Psi_1-\frac12\tilde X^2\partial_{\tilde X}^2\Psi_1\right](\tilde X)\right|+\left|\left[(\frac12-\frac12\tilde X\partial_{\tilde X})(\frac32-\frac12\tilde X\partial_{\tilde X})\Psi_1\right](\tilde X)\right|\lesssim|\tilde X|^5(1+|\tilde X|)^{\frac13-5},$$

implying that for $j=4$, since $k\ge2$:

$$\sum_{0\le j_1+j_2\le1}\int_{\mathbb R^2}\frac{((Y\partial_Y)^{j_1}(X\partial_X)^{j_2}R_1)^{2q}+(\partial_Z^{j_1}(X\partial_X)^{j_2}R_1)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim se^{-2q\frac{k-1}{k}s}\lesssim se^{-qs},$$

which can be rewritten using the equivalence (B.3), (B.4) and (B.5):

$$\sum_{0\le j_1+j_2\le1}\int_{\mathbb R^2}\frac{((Y\partial_Y)^{j_1}A^{j_2}R_1)^{2q}+(\partial_Z^{j_1}A^{j_2}R_1)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim se^{-qs}.$$

Step 2 *Estimate for* R_2: Note that $|Z| \geq de^{s/(2k)} \gg 1$ on the support of $1-\chi_d$. We first compute using (3.25), (3.6) and (3.7):

$$
\begin{aligned}
&\partial_s \Theta_e - \frac{1}{2}\Theta_e + \frac{3}{2}X\partial_X \Theta_e + \frac{1}{2}Y\partial_Y\Theta_e + \Theta_e \partial_X \Theta_e - \partial_{YY}\Theta_e \qquad (3.40)\\
= &\frac{1}{12}X^5 g^2 e^{-2\tilde X^4} + (-Xf + X^3\frac{g}{6})(\partial_s + \frac{3}{2}X\partial_X + \frac{1}{2}Y\partial_Y - \partial_{YY})\left(e^{-\tilde X^4}\right)\\
&+(Xf^2 - 4\frac{X^3 g}{6}f)e^{-\tilde X^4}\left(e^{-\tilde X^4}-1\right)\\
&+(-Xf+\frac{X^3}{6}g)^2 e^{-\tilde X^4}\partial_X\left(e^{-\tilde X^4}\right) - 2\partial_Y(-Xf+\frac{X^3}{6}g)\partial_Y\left(e^{-\tilde X^4}\right)
\end{aligned}
\tag{3.41}
$$

In what follows $0<\gamma\ll 1$ denotes a small constant whose value can change from one line to another. For the first term, (3.6) and (3.7) imply that on the support of $1-\chi_d$ for $j_1, j_1', j_2 \in \mathbb{N}$ with $j_1 + j_1' \leq J$:

$$
\left|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left(X^5 g^2 e^{-2\tilde X^4}\right)\right| \lesssim e^{-\frac{1}{2k}s}(1+|Z|)^{1-k-j_1'}|\tilde X|^5 e^{-\gamma \tilde X^4}.
$$

Therefore, using (3.21):

$$
\frac{\left|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left(X^5 g^2 e^{-2\tilde X^4}\right)\right|}{|\varphi_{j,0}(X,Z)|} \lesssim e^{-\frac{1}{2k}s}(1+|Z|)^{1-k(j-3)}|\tilde X|^{5-j}e^{-\gamma\tilde X^4}.
$$

Since $dX/|X| = d\tilde X/|\tilde X|$ and $|Z| \gtrsim e^{\frac{1}{2k}s}$ on the support of $1-\chi_d$ and $|(Y\partial_Y)^j \partial_Z^{j'}\chi_d| \lesssim 1$ one then infers that:

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left((1-\chi_d)X^5 g^2 e^{-2\tilde X^4}\right)}{|\varphi_{j,0}(X,Z)|}\right)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\\
\lesssim\ & e^{-\frac{q}{k}s}\int_{|Z|\geq e^{\frac{1}{2k}s}}\left((1+|Z|)^{1-k(j-3)}|\tilde X|^{5-j}e^{-\gamma\tilde X^4}\right)^{2q}\frac{d\tilde X dZ}{|\tilde X||Z|} \lesssim e^{-\frac{q}{k}s}\int_{|Z|\geq e^{\frac{1}{2k}s}}|Z|^{(1-k(j-3))2q}\frac{dZ}{|Z|}\\
\lesssim\ & e^{-(j-3)qs}
\end{aligned}
$$

provided that $3+1/k<j<5$. We now turn to the second term in the expression of R_2. One has that:

$$
\left|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left((\partial_s + \frac{3}{2}X\partial_X + \frac{1}{2}Y\partial_Y - \partial_{YY})\left(e^{-(\tilde X)^4}\right)\right)\right| \lesssim |\tilde X|^4 e^{-\gamma\tilde X^4}.
$$

Therefore, from (3.6), (3.7) and (3.21) we obtain that:

$$
\begin{aligned}
&\left|\frac{(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left((-Xf+X^3\frac{g}{6})(\partial_s + \frac{3}{2}X\partial_X + \frac{1}{2}Y\partial_Y - \partial_{YY})\left(e^{-(\tilde X)^4}\right)\right)}{\varphi_{4,0}(X,Z)}\right|\\
\lesssim\ & e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}-k(j-3)}|\tilde X|^{5-j}e^{-\gamma\tilde X^4}
\end{aligned}
$$

Since $dX/|X|=d\tilde X/|\tilde X|$, $|Z|\gtrsim e^{\frac{1}{2k}s}$ and $|(Y\partial_Y)^j\chi_d|\lesssim 1$ on the support of $1-\chi_d$ one then infers that:

$$\begin{aligned}
&\int\left|\frac{(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}\left((1-\chi_d)(-Xf+X^3\frac{g}{6})(\partial_s+\frac32 X\partial_X+\frac12 Y\partial_Y-\partial_{YY})\left(e^{-\tilde X^4}\right)\right)}{\varphi_{j,0}(X,Z)}\right|^{2q}\frac{dXdY}{|X|\langle Y\rangle}\\
\lesssim\ & e^{-\frac{q}{2k}s}\int_{|Z|\geq e^{\frac{1}{2k}s}}\left((1+|Z|)^{\frac12-k(j-3)}|\tilde X|^{5-j}e^{-\gamma\tilde X^4}\right)^{2q}\frac{d\tilde XdZ}{|\tilde X||Z|}\\
\lesssim\ & e^{-\frac{q}{2k}s}\int_{|Z|\geq e^{\frac{1}{2k}s}}|Z|^{(\frac12-k(j-3))2q}\frac{dZ}{|Z|}\lesssim e^{-(j-3)qs}
\end{aligned}$$

provided that $3+1/(2k)<j<5$. We claim that all the other remaining terms in (3.40) can be treated verbatim the same way, yielding for $j=4$:

$$\sum_{0\leq j_1+j_2\leq 1}\int_{\mathbb R^2}\frac{((Y\partial_Y)^{j_1}(X\partial_X)^{j_2}R_2)^{2q}+(\partial_Z^{j_1}(X\partial_X)^{j_2}R_2)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim se^{-2q\frac{k-1}{k}s}\lesssim se^{-qs},$$

which can be rewritten using the equivalence (B.3), (B.4) and (B.5):

$$\sum_{0\leq j_1+j_2\leq 1}\int_{\mathbb R^2}\frac{((Y\partial_Y)^{j_1}A^{j_2}R_2)^{2q}+(\partial_Z^{j_1}A^{j_2}R_2)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim e^{-qs}.$$

Step 3 *Estimate for* R_3: one first notices that on the support of this term $de^{s/2}\leq|Y|\leq 2de^{s/2}$, which will then be assumed throughout this step, and that for $j_1,j_1'\in\mathbb N$:

$$|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(\partial_s\chi_d+\frac12 Y\partial_Y\chi_d-\partial_{YY}\chi_d)|\lesssim 1. \tag{3.42}$$

main:intermed

Also, $\partial_X^j\tilde\Theta=\partial_X^j\Theta_e$ for $j=0,...,4$ on the axis $\{X=0\}$. From this, the formulas (3.23) and (3.25), and the estimate (3.6) and (3.7) one obtains that if $j_1+j_1'\leq J$:

$$|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}(\tilde\Theta-\Theta_e)|\lesssim(1+|Z|)^{k-j_1'}|\tilde X|^5(1+|\tilde X|)^{\frac13-5}$$

giving using (3.21) the estimate:

$$\frac{|(Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}(\tilde\Theta-\Theta_e)|}{|\varphi_{j,0}(X,Z)|}\lesssim(1+|Z|)^{-k(j-3)}|\tilde X|^{5-j}(1+|\tilde X|)^{-4+j-\frac j3}$$

The above estimate and (3.42) therefore imply, since $|Z|\sim e^{\frac{1}{2k}s}$ and since $dX/|X|=d\tilde X/|\tilde X|$:

$$\begin{aligned}
&\int\frac{((Y\partial_Y)^{j_1}\partial_Z^{j_1'}(X\partial_X)^{j_2}R_3)^{2q}}{\varphi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
\lesssim\ &\int_{e^{\frac{1}{2k}}\leq|Z|\leq 2e^{\frac{1}{2k}}}\left((1+|Z|)^{-k(j-3)}|\tilde X|^{5-j}(1+|\tilde X|)^{-4+j-\frac j3}\right)^{2q}\frac{d\tilde XdZ}{|\tilde X||Z|}\lesssim e^{-(j-3)qs}
\end{aligned}$$

provided that $3<j<5$. Taking $j=4$ and using (B.3), (B.4) and (B.5) this gives:

$$\sum_{0\leq j_1+j_2\leq 1}\int_{\mathbb R^2}\frac{((Y\partial_Y)^{j_1}A^{j_2}R_3)^{2q}+(\partial_Z^{j_1}A^{j_2}R_3)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim e^{-qs}.$$

Step 4 *Estimate for* R_4 *and* R_5: These estimates can be proved along the very same lines as we just estimated R_1, R_2 and R_3. We leave the proof to the reader in order to keep the present article short.

□

We now estimate the lower order linear term in (3.36).

in:lem:tildeL **Lemma 21.** *There holds on* $[s_0, s_1]$, *for* $j_1 \leq J$ *and* $j_2 \in \mathbb{N}^*$:

$$|\partial_Z^{j_1}(Q-\Theta)| \lesssim e^{-\frac{1}{4k}s}|X|(1+|Z|)^{-j_1} \quad \text{and} \quad |\partial_Z^{j_1}\partial_X^{j_2}(Q-\Theta)| \lesssim e^{-\frac{1}{4k}s}(1+|X|)^{1-j_2}(1+|Z|)^{-j_1}. \tag{3.43}$$

main:Q-Theta

Proof. **Step 1** *Inner estimate.* We first consider the zone $|Y| \leq de^{s/2}$, or equivalently $|Z| \leq de^{s/2k}$. From (3.22) and (3.13):

$$Q - \Theta = \tilde{\Theta} - \Theta = b^{-1}f^{-\frac{1}{2}}\Psi_1\left(bf^{\frac{3}{2}}X\right) - F_k^{-\frac{1}{2}}(Z)\Psi_1\left(F_k^{\frac{3}{2}}(Z)X\right).$$

First, we have using (3.6) and (3.7):

$$|\frac{f}{F_k}-1| = |\frac{\tilde{f}}{F_k}| \lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}}, \qquad |\frac{g}{G_k}-1| = |\frac{\tilde{g}}{G_k}| \lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}}.$$

Since the above right hand sides are also $\leq \frac{1}{2}$ from the choice of d (3.30), and since $G_k/6F_k = 1$, μ defined by (3.24) satisfies:

$$|\mu - 1| \lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}}.$$

Therefore, using (3.6), (3.7), the above inequalities and (3.19):

$$\begin{aligned}
|Q-\Theta| &= \left|\mu^{-1}f^{-\frac{1}{2}}\Psi_1(\mu f^{\frac{3}{2}}X) - f^{-\frac{1}{2}}\Psi_1(f^{\frac{3}{2}}X) + f^{-\frac{1}{2}}\Psi_1(f^{\frac{3}{2}}X) - F_k^{-\frac{1}{2}}(Z)\Psi_1(F_k^{\frac{3}{2}}(Z)X)\right| \\
&= \left|f^{-\frac{1}{2}}\int_{\tilde{\mu}=1}^{\mu}\tilde{\mu}^{-2}(-\Psi_1 + \tilde{X}\partial_{\tilde{X}}\Psi_1)(\tilde{\mu}f^{\frac{3}{2}}X)d\tilde{\mu} + F_k^{-\frac{1}{2}}\int_{\lambda=1}^{f/F_k}\lambda^{-\frac{3}{2}}(-\frac{1}{2}\Psi_1 + \frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1)(F_k^{\frac{3}{2}}\lambda^{\frac{3}{2}}X)d\lambda\right| \\
&\lesssim |f^{-\frac{1}{2}}||\mu-1|\sup_{|\tilde{\mu}|\in[1,\mu]}\left|(-\Psi_1+\tilde{X}\partial_{\tilde{X}}\Psi_1)(\tilde{\mu}f^{\frac{3}{2}}X)\right| \\
&\quad + F_k^{-\frac{1}{2}}\left|\frac{f}{F_k}-1\right|\sup_{\lambda\in[1,f/F_k]}\left|(-\frac{1}{2}\Psi_1 + \frac{3}{2}\tilde{X}\partial_{\tilde{X}}\Psi_1)(F_k^{\frac{3}{2}}\lambda^{\frac{3}{2}}X)\right| \\
&\lesssim \left(|\mu-1| + |\frac{f}{F_k}-1|\right)(1+|Z|)^k|\tilde{X}|(1+|\tilde{X}|)^{\frac{1}{3}-1} \lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{k+\frac{1}{2}}|\tilde{X}|(1+|\tilde{X}|)^{\frac{1}{3}-1} \\
&\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{-2k+\frac{1}{2}}|X|(1+|\tilde{X}|)^{\frac{1}{3}-1} \lesssim e^{-\frac{1}{4k}s}|X|.
\end{aligned}$$

One computes similarly that

$$\begin{aligned}
|\partial_X(Q-\Theta)| &= \left|f\partial_X\Psi_1(\mu f^{\frac{3}{2}}X) - f\partial_X\Psi_1(f^{\frac{3}{2}}X) + f\partial_X\Psi_1(f^{\frac{3}{2}}X) - F_k(Z)\partial_X\Psi_1(F_k^{\frac{3}{2}}(Z)X)\right| \\
&= \left|f\int_{\tilde{\mu}=1}^{\mu}\tilde{\mu}^{-1}(\tilde{X}\partial_{\tilde{X}\tilde{X}}\Psi_1)(\tilde{\mu}f^{\frac{3}{2}}X)d\tilde{\mu} + F_k\int_{\lambda=1}^{f/F_k}\lambda^{-1}(\partial_{\tilde{X}}\Psi_1 + \frac{3}{2}\tilde{X}\partial_{\tilde{X}}^2\Psi_1)(F_k^{\frac{3}{2}}\lambda^{\frac{3}{2}}X)d\lambda\right| \\
&\lesssim \left(|\mu-1| + |\frac{f}{F_k}-1|\right)(1+|Z|)^{-2k}(1+|\tilde{X}|)^{-\frac{2}{3}} \lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}-2k}(1+|\tilde{X}|)^{-\frac{2}{3}} \\
&\lesssim e^{-\frac{1}{4k}s}.
\end{aligned}$$

The proof for higher order derivatives is a direct generalisation of the above computations, that we omit here, giving (3.43) in this zone.

Step 2 *Outer estimate.* Let $0<\gamma\ll 1$ be a small constant whose value can change from one line to another. We now turn to the zone $de^{s/2}\leq |Y|\leq 2de^{s/2}$ or equivalently $de^{s/(2k)}\leq |Z|\leq 2de^{s/(2k)}$. We perform brute force estimates on the identity (3.22) using (3.6) and (3.7):

$$
\begin{aligned}
|Q-\Theta| &= |\chi_d\tilde\Theta+(1-\chi_d)\Theta_e-\Theta|\leq |\Theta|+|\tilde \Theta|+|\Theta_e|\\
&\lesssim (1+|Z|)^k|\tilde X|(1+|\tilde X|)^{\frac13-1}+(1+|Z|)^k|\tilde X|(1+|\tilde X|)^{\frac13-1}+(1+|Z|)^k|\tilde X|e^{-\gamma \tilde X^4}\\
&\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{k+\frac12}|\tilde X|(1+|\tilde X|)^{\frac13-1}\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{-2k+\frac12}|X|(1+|\tilde X|)^{\frac13-1}\\
&\lesssim e^{-\frac{1}{4k}s}|X|.
\end{aligned}
$$

and similarly

$$
\begin{aligned}
|\partial_X(Q-\Theta)| &\leq |\partial_X\Theta|+|\partial_X\tilde\Theta|+|\partial_X\Theta_e|\\
&\lesssim (1+|Z|)^{-2k}(1+|\tilde X|)^{-\frac23}+(1+|Z|)^{-2k}(1+|\tilde X|)^{-\frac23}+(1+|Z|)^{-2k}e^{-\gamma\tilde X^4}\\
&\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{-2k+\frac12}(1+|\tilde X|)^{-\frac23}\lesssim e^{-\frac{1}{4k}s}.
\end{aligned}
$$

Again, the generalisation of this argument for higher order derivatives is direct, yielding (3.43) in this zone.

Step 3 *Outer estimate.* We now turn to the zone $|Z|\geq 2de^{\frac{1}{2k}s}$ where $Q-\Theta=\Theta_e-\Theta$. We perform the very same computations as in Step 2, estimating Θ and Θ_e separately, giving (3.43) in this zone and ending the proof of the Lemma. □

We can now perform energy estimates in the bootstrap regime of Proposition 16 and improve the bootstrap bounds.

n:lem:energy0

Lemma 22. *There exists $K_{0,0}^*>0$ such that for any positive constants $(K_{j_1,j_2})_{0\leq j_1+j_2\leq 2}$ and $(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2,1\leq j_1}$ with $K_{0,0}\geq K_{0,0}^*$, there exists s^* such that if u is trapped on $[s_0,s_1]$ with $s_0\geq s^*$, then:*

$$
\left(\int_{\mathbb R^2}\frac{\varepsilon^{2q}(s_1)}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\leq \frac{K_{0,0}}{2}e^{-(\frac12-\kappa)s_1}. \qquad (3.44)
$$

main:weighted

Proof. We compute from (3.36) and (3.18) (note that this estimate is valid for Schwartz functions so that we implicitly use here a density argument):

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb R^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac12-\frac{\kappa}{2}\right)\int_{\mathbb R^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\
&+\frac{1}{2q}\int\varepsilon^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}-\int\frac{\varepsilon^{2q-1}}{\varphi_{4,0}^{2q}}\left(\varepsilon\varepsilon_X+R+\partial_X(Q-\Theta)\varepsilon\right)\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}
$$

We now estimate the last terms. First,

$$
\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)=\frac{\partial_X(Q-\Theta)}{\varphi_{4,0}^{2q}|X|}-\frac{Q-\Theta}{\varphi_{4,0}^{2q}X|X|}+\frac{Q-\Theta}{|X|}\partial_X\left(\frac{1}{\varphi_{4,0}^{2q}}\right).
$$

One has from (3.14) that $|\partial_X \varphi_{4,0}(Z,X)| \lesssim |\varphi_{4,0}(X,Z)|/|X|$. From this fact, from (3.43) and (3.43) one infers that:

$$\left|\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\right| \lesssim qe^{-\frac{1}{4k}s}\frac{1}{\varphi_{4,0}^{2q}(X,Z)|X|} \tag{3.45}$$

main:bdpaxQT

From the above estimate, (3.32) and (3.43) we infer that:

$$\begin{aligned}
&\left|\frac{1}{2q}\int \varepsilon^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}-\int\frac{\varepsilon^{2q-1}}{\varphi_{4,0}^{2q}}(\varepsilon\varepsilon_X+\partial_X(Q-\Theta)\varepsilon)\frac{dXdY}{|X|\langle Y\rangle}\right| \\
\leq\ & C\left(e^{-\frac{1}{4k}s}+e^{-(\frac12-\kappa)s}\right)\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\leq\frac{\kappa}{2}\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle},
\end{aligned}$$

where above C depends on the boostrap constants except s^*, so that the last inequality is obtained upon taking s^* large enough. Applying Hölder, using (3.27) and (3.39):

$$\left|\int\frac{\varepsilon^{2q-1}}{\varphi_{4,0}^{2q}}R\frac{dXdY}{|X|\langle Y\rangle}\right|\leq\left|\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{R^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\leq CK_{0,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa)s-\kappa s}.$$

We obtain

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac12-\kappa\right)\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa)s-\kappa s}.
\end{aligned}$$

We now reintegrate until the time s_1 the above estimate, yielding from (3.29):

$$\begin{aligned}
\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} &\leq e^{-2q(\frac12-\kappa)(s_1-s_0)}\int_{\mathbb{R}^2}\frac{\varepsilon_0^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,0}^{2q-1}e^{-2q(\frac12-\kappa)s_1}\int_{s_0}^{s}\tilde{s}^{\frac{1}{2q}}e^{-\kappa\tilde{s}}d\tilde{s}\\
&\leq\frac{K_{0,0}^{2q}}{2^{2q}}e^{-2q(\frac12-\kappa)s_1},
\end{aligned}$$

as C is independent of $(K_{j_1,j_2})_{0\leq j_1+j_2\leq 2}$, $(\tilde{K}_{j_1,j_2})_{0\leq j_1+j_2\leq 2, 1\leq j_1}$ and s^*, upon choosing $K_{0,0}$ large enough. This ends the proof of the Lemma.

□

We now perform similar weighted energy estimates for the derivatives of ε.

main:lem:energy1

Lemma 23. *There exists a choice of constants $K_{j_1,j_2}\gg 1$ for $0\leq j_1+j_2\leq 2$ and $(j_1,j_2)\neq(0,0)$, and $\tilde{K}_{j_1,j_2}\gg 1$ for $0\leq j_1+j_2\leq 2$ and $j_1\geq 1$, such that if u is trapped on $[s_0,s_1]$, then at time s_1, for $0\leq j_1+j_2\leq 2$ and $(j_1,j_2)\neq(0,0)$:*

$$\left(\int_{\mathbb{R}^2}\frac{((\partial_Z^{j_1}A^{j_2}\varepsilon(s_1))^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\leq\frac{K_{j_1,j_2}}{2}e^{-(\frac12-\kappa)s_1}, \tag{3.46}$$

main:weighted

and for $0\leq j_1+j_2\leq 2$ and $j_2\geq 1$:

$$\left(\int_{\mathbb{R}^2}\frac{(((Y\partial_Y)^{j_1}A^{j_2}\varepsilon(s_1))^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\leq\frac{\tilde{K}_{j_1,j_2}}{2}e^{-(\frac12-\kappa)s_1}. \tag{3.47}$$

main:weighted

Proof. **Step 1** *Proof for $A\varepsilon$.* We claim that for any $K_{0,0}>0$, there exists $K_{0,1}^*>0$ such that for any positive constants $(K_{j_1,j_2})_{1\leq j_1+j_2\leq 2}$ and $(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2, 1\leq j_1}$ with $K_{0,1}\geq K_{0,1}^*$, there exists s^* such that if $s_0\geq s^*$ then (3.46) holds true for $j_1=0$ and $j_2=1$. We now prove this claim. Recall (3.26) and let $\varepsilon_1:=A\varepsilon$. Then A commutes with the transport part of the flow:

$$\left[A,\partial_s+\left(\frac32 X+\Theta\right)\partial_X+\frac12 Z\partial_Z\right]=0. \quad (3.48)$$

id:commutati

Indeed, we compute the commutator using (4.2):

$$\begin{aligned}
&\left[(\frac32 X+F_k^{-\frac32}(Z)\Theta)\partial_X,\partial_s+\left(\frac32 X+\Theta\right)\partial_X+\frac12 Z\partial_Z\right]\\
=&\ \Big(-\partial_s(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}X))+\left(\frac32 X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}X)\right)\partial_X\left(\frac32 X+F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}X)\right)\\
&-\left(\frac32 X+F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}X)\right)\partial_X\left(\frac32 X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}X)\right)-\frac12 Z\partial_Z\left(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}X)\right)\Big)\partial_X\\
=&\ \frac32\left(-\frac{1}{2k}Z\partial_Z F_kF_k^{-\frac52}+F_k^{-\frac12}-F_k^{-\frac32}\right)(-\Psi_1+\tilde X\partial_{\tilde X}\Psi_1)(F_k^{\frac32}X)\partial_X=0.
\end{aligned}$$

We compute from (3.26), (3.36) and the above cancellation the evolution of ε_1:

$$(\varepsilon_1)_s+\mathcal L\varepsilon_1+\tilde{\mathcal L}\varepsilon_1+(A\partial_X\Theta)\varepsilon+AR+[A,\tilde{\mathcal L}]\varepsilon+\varepsilon_1\varepsilon_X+\varepsilon\partial_X\varepsilon_1+\varepsilon[A,\partial_X]\varepsilon-[A,\partial_{YY}]\varepsilon=0.$$

Using the linear energy identity (3.18) we infer that:

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb R^2}\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq&\ -\left(\frac12-\frac\kappa2\right)\int_{\mathbb R^2}\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(w^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\
&+\frac{1}{2q}\int\varepsilon_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}\\
&-\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\Big((A\partial_X\Theta)\varepsilon+[A,\tilde{\mathcal L}]\varepsilon+AR+\varepsilon_1\varepsilon_X+\varepsilon\partial_X\varepsilon_1+\varepsilon[A,\partial_X]\varepsilon-[A,\partial_{YY}]\varepsilon+\partial_X(Q-\Theta)\varepsilon_1\Big)\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}$$

From (3.45), and (3.43), one has:

$$\left|\frac{1}{2q}\int\varepsilon_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}-\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_X(Q-\Theta)\varepsilon_1\frac{dXdY}{|X|\langle Y\rangle}\right|\lesssim e^{-\frac{1}{4k}s}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

Moreover, since

$$A\partial_X\Theta=\frac32 F_k(Z)(\tilde X\partial_{\tilde X}^2\Psi_1)(F_k^{\frac32}(Z)X)+F_k(Z)(\Psi_1\partial_{\tilde X}^2\Psi_1)(F_k^{\frac32}(Z)X)$$

and since both F_k and $X\partial_X^2\Psi_1$ and $\Psi_1\partial_X^2\Psi_1$ are bounded, one has that

$$|A\partial_X\Theta|\lesssim 1$$

and therefore using the bootstrap bound (3.27) on ε one deduces from Hölder:

$$\left|\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}(A\partial_X\Theta)\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|\leq C\left|\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\leq CK_{0,1}^{2q-1}K_{0,0}e^{-2q(\frac12-\kappa)s}.$$

We compute from (3.26) and (3.37) that

$$
\begin{aligned}
[A,\tilde{\mathcal{L}}]\varepsilon &= \left(A(Q-\Theta)-(Q-\Theta)(\frac{3}{2}+\partial_{\tilde X}\Psi_1(F_k^{\frac32}X))\right)\partial_X\varepsilon+(A\partial_X(Q-\Theta))\varepsilon\\
&= \frac{A(Q-\Theta)-(Q-\Theta)(\frac32+\partial_{\tilde X}\Psi_1(F_k^{\frac32}X))}{\frac32 X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)}\varepsilon_1+(A\partial_X(Q-\Theta))\varepsilon.
\end{aligned}
$$

From (3.26), (B.10), (3.19) and (3.43) we then obtain

$$
|[A,\tilde{\mathcal{L}}]\varepsilon|\leq Ce^{-\frac{1}{4k}s}(|\varepsilon_1|+|\varepsilon|)
$$

for C independent of the bootstrap bounds, which implies using Hölder and (3.27):

$$
\begin{aligned}
&\left|\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}[A,\tilde{\mathcal{L}}]\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|\leq Ce^{-\frac{1}{4k}s}\int\frac{|\varepsilon_1|^{2q-1}}{\varphi_{4,0}^{2q}}(|\varepsilon_1|+|\varepsilon|)\frac{dXdY}{|X|\langle Y\rangle}\\
\leq\ & Ce^{-\frac{1}{4k}s}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}+C\left(\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{2q-1}{2q}}\left(\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\\
\leq\ & Ce^{-\frac{1}{4k}s}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,1}^{2q-1}K_{0,0}e^{-2q(\frac12-\kappa)s}.
\end{aligned}
$$

One then deduces from (B.10), (3.26), (3.39) and Hölder:

$$
\begin{aligned}
&\left|\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}AR\frac{dXdY}{|X|\langle Y\rangle}\right|\leq\left|\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{(AR)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\
\leq\ & K_{0,1}^{\frac{2q-1}{2q}}e^{-\frac{2q-1}{2q}(\frac12-\kappa)s}C\left|\int\frac{(X\partial_XR)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\leq CK_{0,1}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa)s-\kappa s}.
\end{aligned}
$$

Using (3.32) we infer that for a constant C depending on $(K_{j_1,j_2})_{1\leq j_1+j_2\leq 2}$ and $(\tilde K_{j_1,j_2})_{0\leq j_1+j_2\leq 2,1\leq j_1}$:

$$
\left|\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\varepsilon_1\varepsilon_X\frac{dXdY}{|X|\langle Y\rangle}\right|\leq Ce^{-(\frac12-\kappa)s}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\leq\frac{\kappa}{16}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}
$$

upon choosing s^* large enough. Integrating by parts, one has the identity:

$$
\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}(X,Z)}\varepsilon\partial_X\varepsilon_1\frac{dXdY}{|X|\langle Y\rangle}=-\frac{1}{2q}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\partial_X\varepsilon-\frac{1}{2q}\int\varepsilon_1^{2q}\varepsilon\partial_X\left(\frac{1}{\varphi_{4,0}^{2q}(X,Z)|X|}\right)\frac{dXdY}{\langle Y\rangle}.
$$

From (3.21) one has that $|\partial_X\varphi_{4,0}(X,Z)/\varphi_{4,0}(X,Z)|\lesssim|X|^{-1}$. Therefore, using (3.31) we obtain that, again for s^* large enough:

$$
\begin{aligned}
\left|\int\frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}(X,Z)}\varepsilon\partial_X\varepsilon_1\frac{dXdY}{|X|\langle Y\rangle}\right| &\leq C\left(\|\partial_X\varepsilon\|_{L^\infty}+\|\frac{\varepsilon}{|X|}\|_{L^\infty}\right)\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
&\leq \frac{\kappa}{16}\int\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}
$$

Next, from the identity

$$
[A,\partial_X]=-\left(\frac32+\partial_{\tilde X}\Psi_1(F_k^{\frac32}(Z)X)\right)\partial_X=-\frac{\frac32+\partial_{\tilde X}\Psi_1(F_k^{\frac32}(Z)X)}{\frac32X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)}A,
$$

since F_k and $\partial_X \Psi_1$ are uniformly bounded, from (B.10) and (3.31) we obtain that:

$$\left| \int \frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} \varepsilon [A, \partial_X] \varepsilon \frac{dXdY}{|X|\langle Y\rangle} \right| = \left| \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \varepsilon \frac{\frac32 + F_k(Z)\partial_{\tilde X}\Psi_1(F_k^{\frac32}(Z)X)}{\frac32 X + F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)} \frac{dXdY}{|X|\langle Y\rangle} \right|$$
$$\lesssim \quad \|\frac{\varepsilon}{X}\|_{L^\infty} \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle} \lesssim \frac{\kappa}{16} \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle}.$$

Finally, one computes that:

$$[A, \partial_{YY}]\varepsilon = -2\partial_Y \left(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X) \right) \partial_{YX}\varepsilon - \partial_{YY} \left(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X) \right) \partial_X \varepsilon$$
$$= \quad -2\partial_Y \left(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X) \right) \partial_Y \left(\frac{\varepsilon_1}{\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)} \right) - \frac{\partial_{YY}(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))}{\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)} \varepsilon_1$$
$$= \quad F_1\varepsilon_1 + F_2 \partial_Y \varepsilon_1$$

where

$$F_1 := \left(2\frac{\left(\partial_Y(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))\right)^2}{\left(\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)\right)^2} - \frac{\partial_{YY}(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))}{\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)} \right)$$

and

$$F_2 := -2\frac{\partial_Y(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))}{\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))}.$$

One has that $\partial_Z F_k/F_k$ is bounded, and that $|\Psi_1(\tilde X)| + |\tilde X \partial_{\tilde X}\Psi_1(\tilde X)| \lesssim |\tilde X|$. Therefore,

$$\left| \partial_Y(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)) \right| = e^{-\frac{k-1}{2k}s} \left| \partial_Z F_k F_k^{-\frac52}(-\frac12\Psi_1 + \frac32 \tilde X \partial_{\tilde X}\Psi_1)(F_k^{\frac32}(Z)X) \right| \lesssim e^{-\frac{k-1}{2k}s}|X|.$$

The same computation can be performed for the second term in F_1, giving from (B.10):

$$\left| \frac{(\partial_Y(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)))^2}{\left(\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)\right)^2} \right| + \left| \frac{\partial_{YY}(F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X))}{\frac32 X + F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)} \right| \lesssim e^{-\frac{k-1}{k}s}$$

and hence for s^* large enough:

$$\left| \int \frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} F_1\varepsilon_1 \frac{dXdY}{|X|\langle Y\rangle} \right| \lesssim e^{-\frac{k-1}{k}s} \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle} \leq \frac{\kappa}{32} \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle}.$$

From the above estimate one obtains similarly that:

$$|F_2| \lesssim e^{-\frac{k-1}{2k}s}, \quad |\partial_Y F_2| \lesssim e^{-\frac{k-1}{k}s}$$

so that integrating by parts and using (3.21), for s^* large enough:

$$\left| \int \frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} F_2 \partial_Y w \frac{dXdY}{|X|\langle Y\rangle} \right| = \frac{1}{2q} \left| \int \varepsilon_1^{2q} \partial_Y \left(\frac{F_2}{\varphi_{4,0}^{2q}(X,Z)\langle Y\rangle} \right) \frac{dXdY}{|X|} \right| \lesssim \frac{\kappa}{32} \int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle}.$$

One has then proven that

$$\left|\int \frac{\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}[A,\partial_{YY}]\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right| \lesssim \frac{\kappa}{16}\int \frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

From the collection of the above estimates, one infers that:

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac12-\frac34\kappa\right)\int_{\mathbb{R}^2}\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,1}^{2q-1}K_{0,0}e^{-2q(\frac12-\kappa)s}+CK_{0,1}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa+\frac{\kappa}{2q})s}\\
\leq\ & -\left(\frac12-\frac34\kappa\right)\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,1}^{2q-1}K_{0,0}e^{-2q(\frac12-\kappa)s}.
\end{aligned}$$

We reintegrate with time the above estimate, yielding from (3.29):

$$\begin{aligned}
\int_{\mathbb{R}^2}\frac{\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\leq & e^{-2q(\frac12-\frac34\kappa)(s-s_0)}\int_{\mathbb{R}^2}\frac{\varepsilon_1(s_0)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+CK_{0,1}^{2q-1}K_{0,0}e^{-2q(\frac12-\frac34\kappa)s}\int_{s_0}^s e^{\frac{q\kappa}{2}\tilde s}d\tilde s\\
\leq & C(1+K_{0,1}^{2q-1}K_{0,0})e^{-2q(\frac12-\kappa)s}\leq \frac{K_{0,1}^{2q}}{2^{2q}}e^{-2q(\frac12-\kappa)s} \qquad (3.49)
\end{aligned}$$

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where the last inequality holds if $K_{0,1}$ has been chosen large enough depending on $K_{0,0}$.

Step 2 *Proof for ∂_Z.* We claim that for any $K_{0,0},K_{0,1}>0$, there exists $K_{1,0}^*>0$ such that for any choice of the remaining constants K_{j_1,j_2} and $\tilde K_{j_1,j_2}$ with $K_{1,0}\geq K_{1,0}^*$, there exists s^* such that if $s_0\geq s^*$ then (3.46) holds true for $j_1=1$ and $j_2=0$. To prove this claim, define $\tilde\varepsilon_1=\partial_Z\varepsilon=e^{(k-1)s/(2k)}\partial_Y\varepsilon$, which from (3.36) solves:

$$\begin{aligned}
0\ =\ & (\tilde\varepsilon_1)_s+\frac{1}{2k}\tilde\varepsilon_1+\mathcal{L}\varepsilon-\partial_{YY}\tilde\varepsilon_1+\tilde{\mathcal{L}}\tilde\varepsilon_1+\partial_Z\Theta\partial_X\varepsilon+\partial_{ZX}\Theta\varepsilon+\partial_Z(Q-\Theta)\partial_X\varepsilon+\partial_{XZ}(Q-\Theta)\varepsilon\\
&+\partial_ZR+\tilde\varepsilon_1\partial_X\varepsilon+\varepsilon\partial_X\tilde\varepsilon_1,
\end{aligned}$$

and hence obeys the energy identity from (3.18):

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac12+\frac{1}{2k}-\frac{\kappa}{2}\right)\int_{\mathbb{R}^2}\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\tilde\varepsilon_1^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\
&+\frac{1}{2q}\int\tilde\varepsilon_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}\\
&-\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\Big(\partial_Z\Theta\partial_X\varepsilon+\partial_{ZX}\Theta\varepsilon+\partial_Z(Q-\Theta)\partial_X\varepsilon+\partial_{XZ}(Q-\Theta)\varepsilon\\
&+\partial_ZR+\tilde\varepsilon_1\partial_X\varepsilon+\varepsilon\partial_X\tilde\varepsilon_1+\partial_X(Q-\Theta)\tilde\varepsilon_1\Big)\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}$$

Using (3.45), and (3.43) we infer that for s^* large enough:

$$\left|\frac{1}{2q}\int\tilde\varepsilon_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}-\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_X(Q-\Theta)\tilde\varepsilon_1\frac{dXdY}{|X|\langle Y\rangle}\right|\leq\frac{\kappa}{12}\int\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

Next one computes that

$$\partial_Z\Theta\partial_X\varepsilon=\frac{\partial_Z F_k(Z)}{F_k(Z)}\frac{F_k^{-\frac12}(Z)(-\frac12\Psi_1+\frac32\tilde X\partial_{\tilde X}\Psi_1)(F_k^{\frac32}(Z)X)}{\frac32 X+F_k^{-\frac12}(Z)(\Psi_1(F_k^{\frac32}(Z)X))}A\varepsilon.$$

In the above formula, $\partial_Z F_k(Z)/F_k(Z)$ is uniformly bounded, and as F_k is bounded and $|(-1/2\Psi_1+3/2\tilde X\partial_{\tilde X}\Psi_1)(\tilde X)|\lesssim|\tilde X|$ we obtain from (B.10) that:

$$\left|\frac{\partial_Z F_k(Z)}{F_k(Z)}\frac{F_k^{-\frac12}(Z)(-\frac12\Psi_1+\frac32\tilde X\partial_{\tilde X}\Psi_1)(F_k^{\frac32}(Z)X)}{\frac32 X+F_k^{-\frac12}(Z)(\Psi_1(F_k^{\frac32}(Z)X))}\right|\lesssim 1.$$

From Hölder, (3.27) and (3.28) we then infer that:

$$\begin{aligned}\left|\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_Z\Theta\partial_X\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|&\leq C\int\frac{|\tilde\varepsilon_1|^{2q-1}}{\varphi_{4,0}^{2q}}|A\varepsilon|\frac{dXdY}{|X|\langle Y\rangle}\\ \leq\ C\left|\int\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}&\left|\int\frac{(A\varepsilon)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\leq CK_{1,0}K_{0,1}^{2q-1}e^{-2q(\frac12-\kappa)s}.\end{aligned}$$

Similarly, since F_k, $\partial_Z F_k/F_k$, $\partial_{\tilde X}\Psi_1$ and $\tilde X\partial_{\tilde X}^2\Psi_1$ are uniformly bounded,

$$|\partial_{ZX}\Theta|=\left|\frac{\partial_Z F_k(Z)}{F_k(Z)}F_k(\partial_{\tilde X}\Psi_1+\frac32\tilde X\partial^2_{\tilde X}\Psi_1)(F_k^{\frac32}(Z)X)\right|\lesssim 1,$$

and from Hölder, (3.27) and (3.28) we then infer that:

$$\begin{aligned}\left|\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_{ZX}\Theta\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|&\leq C\left|\int\frac{|\tilde\varepsilon_1|^{2q-1}}{\varphi_{4,0}^{2q}}|\varepsilon|\frac{dXdY}{|X|\langle Y\rangle}\right|\\ \leq\ C\left|\int\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}&\left|\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\leq CK_{0,0}K_{1,0}^{2q-1}e^{-2q(\frac12-\kappa)s}.\end{aligned}$$

Then, from (3.43) and (B.10) we infer that:

$$|\partial_Z(Q-\Theta)\partial_X\varepsilon|=\left|\frac{\partial_Z(Q-\Theta)}{\frac32 X+F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}\right||A\varepsilon|\lesssim e^{-\frac{1}{4k}s}|A\varepsilon|$$

and therefore from Hölder, (3.27) and (3.28):

$$\begin{aligned}\left|\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_Z(Q-\Theta)\partial_X\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|&\lesssim e^{-\frac{1}{4k}s}\left|\int\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{(A\varepsilon)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\lesssim C(K_{1,0},K_{0,1})e^{-2q(\frac12-\kappa)s}e^{-\frac{1}{4k}s}.\end{aligned}$$

Using (3.43) one has that

$$|\partial_{ZX}(Q-\Theta)|\lesssim e^{-\frac{1}{4k}s}.$$

Therefore, one infers by Hölder, (3.27) and (3.28):

$$\begin{aligned}\left|\int\frac{\tilde\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_{ZX}(Q-\Theta)\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right|&\lesssim e^{-\frac{1}{4k}s}\left|\int\frac{\tilde\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\lesssim C(K_{0,0},K_{1,0})e^{-2q(\frac12-\kappa)s}e^{-\frac{1}{4k}s}.\end{aligned}$$

Next, from Hölder, (3.39) and (3.27):

$$\left|\int \frac{\tilde{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_Z R \frac{dXdY}{|X|\langle Y\rangle}\right| \leq C\left|\int \frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{(\partial_Z R)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}} \leq CK_{1,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa+\frac{\kappa}{2q})s}.$$

From (3.32) one has that for s^* large enough:

$$\left|\int \frac{\tilde{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}\tilde{\varepsilon}_1\partial_X\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right| \leq \|\partial_X\varepsilon\|_{L^\infty}\int \frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle} \leq \frac{\kappa}{12}\int \frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

We finally perform an integration by parts to obtain:

$$\int \frac{\tilde{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}(X,Z)}\varepsilon\partial_X\tilde{\varepsilon}_1\frac{dXdY}{|X|\langle Y\rangle} = \frac{1}{2q}\int \tilde{\varepsilon}_1^{2q}\partial_X\left(\frac{\varepsilon}{\varphi_{4,0}^{2q}(X,Z)|X|}\right)\frac{dXdY}{\langle Y\rangle}.$$

From (3.31), (3.32), (3.21) one has for C depending on the bootstrap constants except s^*:

$$\left|\partial_X\left(\frac{\varepsilon}{\varphi_{4,0}^{2q}(X,Z)|X|}\right)\right| \leq \frac{Ce^{-(\frac12-\kappa)s}}{\varphi_{4,0}^{2q}(X,Z)|X|} \tag{3.50}$$

bd:pointwise

so that for s^* large enough:

$$\left|\int \frac{\tilde{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}(X,Z)}\varepsilon\partial_X\tilde{\varepsilon}_1\frac{dXdY}{|X|\langle Y\rangle}\right| \leq \frac{\kappa}{12}\int \frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

From the collection of the above estimates, the energy identity becomes:

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac12+\frac{1}{2k}-\frac34\kappa\right)\int_{\mathbb{R}^2}\frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
&+C(K_{0,1}+K_{0,0})K_{1,0}^{2q-1}e^{-2q(\frac12-\kappa)s}+CK_{1,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q\left(\frac12-\kappa+\frac{\kappa}{2q}\right)s}+C(K_{0,0},K_{1,0},K_{0,1})e^{-2q(\frac12-\kappa)s}e^{-\frac{1}{4k}s}\\
\leq\ & -\left(\frac12-\frac34\kappa\right)\int_{\mathbb{R}^2}\frac{\tilde{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+C(K_{0,1}+K_{0,0})K_{1,0}^{2q-1}e^{-2q(\frac12-\kappa)s}
\end{aligned}$$

for s^* large enough. From the initial size (3.29) the above differential inequality yields:

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{\tilde{\varepsilon}_1^{2q}(s_1)}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
\leq\ & e^{-2q(\frac12-\frac34\kappa)(s_1-s_0)}\int_{\mathbb{R}^2}\frac{\tilde{\varepsilon}_1(s=s_0)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+CK_{1,0}^{2q-1}e^{-2q\left(\frac12-\frac34\kappa\right)s}\int_{s_0}^{s}(K_{0,0}+K_{0,1})e^{\frac{q\kappa}{2}\tilde{s}}d\tilde{s}\\
\leq\ & C(1+K_{1,0}^{2q-1}(K_{0,0}+K_{0,1}))e^{-2q\left(\frac12-\kappa\right)s} \leq \frac{K_{1,0}^{2q}}{2^{2q}}e^{-2q\left(\frac12-\kappa\right)s}
\end{aligned}$$

if $K_{1,0}$ has been chosen large enough depending on $K_{0,0}$ and $K_{0,1}$.

Step 3 *Proof for $Y\partial_Y$.* We here claim that for any $K_{0,0},K_{0,1}>0$, there exists $\tilde{K}_{1,0}^*>0$ such that for any choice of the remaining constants K_{j_1,j_2} and $\tilde{K}_{j_1,j_2}$ with $\tilde{K}_{1,0}\geq\tilde{K}_{1,0}^*$, there exists

s^* such that if $s_0 \geq s^*$ then (3.47) holds true for $j_1 = 1$ and $j_2 = 0$. To prove this claim, Let $\hat{\varepsilon}_1 = Z\partial_Z \varepsilon = Y\partial_Y \varepsilon$. From (3.36) one obtain the evolution of w:

$$\begin{aligned} 0 &= (\hat{\varepsilon}_1)_s + \mathcal{L}\varepsilon - \partial_{YY}\hat{\varepsilon}_1 + \tilde{\mathcal{L}}\hat{\varepsilon}_1 + Z\partial_Z\Theta\partial_X\varepsilon + Z\partial_{ZX}\Theta\varepsilon \\ &\quad +2\partial_{YY}\varepsilon + Z\partial_Z(Q-\Theta)\partial_X\varepsilon + Z\partial_{ZX}(Q-\Theta) + Z\partial_Z R + \hat{\varepsilon}_1\partial_X\varepsilon + \varepsilon\partial_X\hat{\varepsilon}_1. \end{aligned}$$

Using (3.18) yields the energy identity:

$$\begin{aligned} &\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\hat{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right) \\ \leq\ & -\left(\frac{1}{2}-\frac{\kappa}{2}\right)\int_{\mathbb{R}^2}\frac{\hat{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} - \frac{2q-1}{q^2}\int\frac{|\partial_Y(\hat{\varepsilon}_1^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle} \\ &+\frac{1}{2q}\int\hat{\varepsilon}_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle} \\ &-\int\frac{\hat{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}\Big(Z\partial_Z\Theta\partial_X\varepsilon + Z\partial_{ZX}\Theta\varepsilon + 2\partial_{YY}\varepsilon + Z\partial_Z(Q-\Theta)\partial_X\varepsilon + \\ &Z\partial_{ZX}(Q-\Theta)\varepsilon + Z\partial_Z R + \hat{\varepsilon}_1\partial_X\varepsilon + \varepsilon\partial_X\hat{\varepsilon}_1 + \partial_X(Q-\Theta)\hat{\varepsilon}_1\Big)\frac{dXdY}{|X|\langle Y\rangle}. \end{aligned}$$

Using (3.45), and (3.43) we infer that for s^* large enough:

$$\left|\frac{1}{2q}\int\hat{\varepsilon}_1^{2q}\partial_X\left(\frac{Q-\Theta}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle} - \int\frac{\hat{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}\partial_X(Q-\Theta)\hat{\varepsilon}_1\frac{dXdY}{|X|\langle Y\rangle}\right| \leq \frac{\kappa}{8}\int\frac{\hat{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}.$$

Next,

$$Z\partial_Z\Theta\partial_X\varepsilon = \frac{Z\partial_Z F_k(Z)}{F_k(Z)}\frac{F_k^{-\frac12}(Z)(-\frac12\Psi_1+\frac32\tilde{X}\partial_{\tilde{X}}\Psi_1)(F_k^{\frac32}(Z)X)}{\frac32 X + F_k^{-\frac12}(Z)(\Psi_1(F_k^{\frac32}(Z)X))}A\varepsilon.$$

In the above formula, $Z\partial_Z F_k(Z)/F_k(Z)$ is uniformly bounded, and as F_k is bounded and $|(-1/2\Psi_1 + 3/2\tilde{X}\partial_{\tilde{X}}\Psi_1)(\tilde{X})| \lesssim |\tilde{X}|$ we obtain from (B.10) that:

$$\left|\frac{Z\partial_Z F_k(Z)}{F_k(Z)}\frac{F_k^{-\frac12}(Z)(-\frac12\Psi_1+\frac32\tilde{X}\partial_{\tilde{X}}\Psi_1)(F_k^{\frac32}(Z)X)}{\frac32 X + F_k^{-\frac12}(Z)(\Psi_1(F_k^{\frac32}(Z)X))}\right| \lesssim 1.$$

From Hölder, (3.27) and (3.28) we then infer that:

$$\begin{aligned} \left|\int\frac{\hat{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}Z\partial_Z\Theta\partial_X\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right| &\leq C\left|\int\frac{\hat{\varepsilon}_1^{2q-1}}{\varphi_{4,0}^{2q}}A\varepsilon\frac{dXdY}{|X|\langle Y\rangle}\right| \leq C\left|\int\frac{\hat{\varepsilon}_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int\frac{(A\varepsilon)^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}} \\ &\leq C\tilde{K}_{1,0}^{2q-1}K_{0,1}e^{-2q(\frac12-\kappa)s}. \end{aligned}$$

Similarly, since F_k, $\partial_Z F_k/F_k$, $\partial_{\tilde{X}}\Psi_1$ and $\tilde{X}\partial_{\tilde{X}}^2\Psi_1$ are uniformly bounded,

$$|Z\partial_{ZX}\Theta| = \left|\frac{Z\partial_Z F_k(Z)}{F_k(Z)}F_k(\partial_{\tilde{X}}\Psi_1 + \frac32\tilde{X}\partial_{\tilde{X}}^2\Psi_1)(F_k^{\frac32}(Z)X)\right| \lesssim 1,$$

and from Hölder, (3.27) and (3.28) we then infer that:

$$\begin{aligned}\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} Z\partial_{ZX}\Theta\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right| &\le C\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right| \le C\left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\le C\tilde K_{1,0}^{2q-1}K_{0,0}e^{-2q(\frac12-\kappa)s}.\end{aligned}$$

We then integrate by parts:

$$\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}(X,Z)}\partial_{YY}\varepsilon \frac{dXdY}{|X|\langle Y\rangle} = -\frac{2q-1}{q}\int \frac{\partial_Y(\hat\varepsilon_1^q)\hat\varepsilon_1^{q-1}}{\varphi_{4,0}^{2q}(X,Z)}\partial_Y\varepsilon \frac{dXdY}{|X|\langle Y\rangle} - \int \hat\varepsilon_1^{2q-1}\partial_Y\varepsilon\partial_Y\left(\frac{1}{\varphi_{4,0}^{2q}(X,Z)\langle Y\rangle}\right)\frac{dXdY}{|X|}.$$

For the first term we use the generalised Hölder inequality, (3.27) and (3.28):

$$\begin{aligned}&\left|\int \frac{\partial_Y(\hat\varepsilon_1^q)\hat\varepsilon_1^{q-1}}{\varphi_{4,0}^{2q}(X,Z)}\partial_Y\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right|\\ \le\ & e^{-\frac{k-1}{2k}s}\left|\int \frac{|\partial_Y(\hat\varepsilon_1^q)|^2}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac12}\left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{q-1}{2q}}\left|\int \frac{(\partial_Z\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ \le\ & \nu\int \frac{|\partial_Y(\hat\varepsilon_1^q)|^2}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle} + C(\tilde K_{1,0},K_{1,0})e^{-2q(\frac12-\kappa)s-\frac{k-1}{2k}s}.\end{aligned}$$

for any ν small enough to be chosen later on. For the second term, from (3.21) we infer that

$$\left|\partial_Y\left(\frac{1}{\varphi_{4,0}^{2q}(X,Z)\langle Y\rangle}\right)\right| \lesssim \frac{1}{\varphi_{4,0}^{2q}(X,Z)\langle Y\rangle},$$

and therefore, from Hölder, (3.27) and (3.28):

$$\begin{aligned}&\left|\int \hat\varepsilon_1^{2q-1}\partial_Y\varepsilon\partial_Y\left(\frac{1}{\varphi_{4,0}^{2q}(X,Z)\langle Y\rangle}\right)\frac{dXdY}{|X|}\right|\\ \lesssim\ & e^{-\frac{k-1}{2k}s}\left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{(\partial_Z\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}} \le C(\tilde K_{1,0},K_{1,0})e^{-2q(\frac12-\kappa)s-\frac{k-1}{2k}s}.\end{aligned}$$

From (B.10) and (3.43):

$$\begin{aligned}|Z\partial_Z(Q-\Theta)\partial_X\varepsilon| &= \left|\frac{Z\partial_Z(Q-\Theta)}{\frac32 X+F_k^{-\frac12}(Z)\Psi_1(F_k^{\frac32}(Z)X)}\right||A\varepsilon|\\ &\lesssim \frac{e^{-\frac{1}{4k}s}|Z|(1+|Z|)^{-2k-\frac12}|X|(1+|\tilde X|)^{\frac13-1}}{|X|}|A\varepsilon| \lesssim e^{-\frac{1}{4k}s}|A\varepsilon|\end{aligned}$$

and therefore from Hölder, (3.27) and (3.28):

$$\begin{aligned}\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} Z\partial_Z(Q-\Theta)\partial_X\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right| &\lesssim e^{-\frac{1}{4k}s}\left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{(A\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\lesssim C(K_{0,1},\tilde K_{1,0})e^{-\frac{1}{4k}s}e^{-2q(\frac12-\kappa)s}.\end{aligned}$$

Similarly, from (3.43):

$$|Z\partial_{XZ}(Q-\Theta)| \lesssim e^{-\frac{1}{4k}s}|Z|(1+|Z|)^{-2k-\frac12}(1+|\tilde X|)^{\frac13-1} \lesssim e^{-\frac{1}{4k}s}$$

and from Hölder, (3.27) and (3.28):

$$\begin{aligned}\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} Z\partial_{ZX}(Q-\Theta)\varepsilon \frac{dXdY}{|X|\langle Y\rangle}\right| &\lesssim e^{-\frac{1}{4k}s}\left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{(\varepsilon)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\lesssim C(K_{0,0},\tilde K_{1,0})e^{-\frac{1}{4k}s}e^{-2q(\frac12-\kappa)s}.\end{aligned}$$

Next, from Hölder, (3.39) and (3.28):

$$\begin{aligned}\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}} Z\partial_Z R\frac{dXdY}{|X|\langle Y\rangle}\right| &\leq \left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{2q-1}{2q}}\left|\int \frac{(Z\partial_Z R)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right|^{\frac{1}{2q}}\\ &\leq C\tilde K_{1,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q(\frac12-\kappa+\frac{\kappa}{2q})s}.\end{aligned}$$

Performing an integration by parts, and then using (3.32) and (3.50) we finally obtain:

$$\begin{aligned}&\left|\int \frac{\hat\varepsilon_1^{2q-1}}{\varphi_{4,0}^{2q}}(\hat\varepsilon_1\partial_X\varepsilon+\varepsilon\partial_X\hat\varepsilon_1)\frac{dXdY}{|X|\langle Y\rangle}\right| = \left|\int \frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\partial_X\varepsilon\frac{dXdY}{|X|\langle Y\rangle}-\frac{1}{2q}\int\hat\varepsilon_1^{2q}\partial_X\left(\frac{\varepsilon}{\varphi_{4,0}^{2q}|X|}\right)\frac{dXdY}{\langle Y\rangle}\right|\\ \lesssim\ & \left(\|\partial_X\varepsilon\|_{L^\infty}+\|X\varphi_{4,0}^{2q}(X,Z)\partial_X\left(\frac{\varepsilon}{\varphi_{4,0}^{2q}|X|}\right)\|_{L^\infty}\right)\int\frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\ \leq\ & \frac{\kappa}{8}\int\frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\end{aligned}$$

for s^* large enough. From the collection of the above estimates, as $(k-1)/(2k)\geq 1/(4k)$ one deduces that:

$$\begin{aligned}&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb R^2}\frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\ \leq\ & -\left(\frac12-\frac34\kappa\right)\int_{\mathbb R^2}\frac{\hat\varepsilon_1^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\left(\frac{2q-1}{q^2}-\nu\right)\int\frac{|\partial_Y(\hat\varepsilon_1^q)|^2}{\varphi_{4,0}^{2q}}\frac{dXdY}{|X|\langle Y\rangle}\\ &+C(K_{0,0},K_{1,0},K_{0,1},\tilde K_{1,0})e^{-2q(\frac12-\kappa)s}e^{-\frac{1}{4k}s}+C\tilde K_{1,0}^{2q-1}(K_{0,0}+K_{0,1})e^{-2q\left(\frac12-\kappa\right)s}\\ &+C\tilde K_{1,0}^{2q-1}s^{\frac{1}{2q}}e^{-2q\left(\frac12-\kappa+\frac{\kappa}{2q}\right)s}\\ \leq\ & -\left(\frac12-\frac{3\kappa}{4}\right)\int_{\mathbb R^2}\frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+C\tilde K_{1,0}^{2q-1}(K_{0,0}+K_{0,1})e^{-2q\left(\frac12-\kappa\right)s}\end{aligned}$$

if ν has been chosen small enough depending only on q, and then s^* has been chosen large enough. From the initial size (3.29) the above differential inequality yields:

$$\begin{aligned}\int_{\mathbb R^2}\frac{\hat\varepsilon_1^{2q}(s_1)}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} &\leq e^{-2q\left(\frac12-\frac34\kappa\right)(s-s_0)}\int_{\mathbb R^2}\frac{\hat\varepsilon_1(s=s_0)^{2q}}{\varphi_{4,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\ &\quad+C\tilde K_{1,0}^{2q-1}(K_{0,0}+K_{0,1})e^{-2q\left(\frac12-\frac34\kappa\right)s}\int_{s_0}^s e^{\frac{q\kappa}{2}\tilde s}d\tilde s\\ &\leq C(1+\tilde K_{1,0}^{2q-1}(K_{0,0}+K_{0,1}))e^{-2q\left(\frac12-\kappa\right)s}\leq\frac{\tilde K_{1,0}^{2q}}{2^{2q}}e^{-2q\left(\frac12-\kappa\right)s}\end{aligned}$$

if $\tilde{K}_{1,0}$ has been chosen large enough depending on $K_{0,1}$ and $K_{0,0}$.

Step 4 *Proof for higher order derivatives.* The proof for higher order derivatives works the very same way and we leave it to the reader.

□

We can now end the proof of Proposition 16.

Proof of Proposition 16. We reason by contradiction. Let v be a solution to (3.4) with initial value $v(s_0)$ at time s_0 that satisfies (3.29) when decomposed according to (3.22). Let s_1 denote the supremum of times $\tilde{s} \geq s_0$ such that v is well defined and that the bounds (3.27) and (3.28) hold on $[s_0, \tilde{s}]$. From the initial bounds (3.29) and a continuity argument, one has that $s_1 > s_0$ is well defined. We then prove Proposition 16 by contradiction and assume that s_1 is finite. If it is the case, then the bounds (3.27) and (3.28) are strict at time s_1 from (3.44), (3.46) and (3.47). Therefore, by a continuity argument there exists $\delta > 0$ such that v is well defined and satisfies (3.27) and (3.28) on $[s_1, s_1 + \delta]$, contradicting the definition of s_1.

□

4. Analysis of the vertical axis

sec:NLH

This section is devoted to the proof of Theorem 1 and Proposition 10. The proof of Theorem 1 follows and refines the works [2, 17, 24], and differs in particular in the way we deal with the problem outside the origin, see Lemma 33. For more comparisons with these works, see the comments after Theorem 1. The proof of Proposition 10 then uses a very similar analytical framework.

4.1. Flat blow-up for the semi-linear heat equation

We prove in this subsection the result in Theorem 1 concerning the solution ξ to (1.7). The strategy is the following. We construct an approximate blow-up profile in self-similar variables and show the existence of a true solution staying in its neighbourhood via a bootstrap argument. This existence result relies on the control of the difference of the two functions via a spectral decomposition at the origin and energy estimates far away, showing the existence of a finite number of instabilities only allowing for the use of a topological argument to control them.

The unstable blow-ups are related to unstable analytic backward self-similar solutions of the quadratic equation

$$\xi_t - \xi^2 = 0. \tag{4.1}$$

eq:quadra

Their properties are the following.

pr:Fk **Proposition 24** (unstable self-similar blow-ups for the quadratic equation). *For $k \in \mathbb{N}$, $k \geq 2$, the functions $F_k(Z) := (1 + Z^{2k})^{-1}$ are such that*

$$\xi(t, y) = \frac{1}{T - t} F_k \left(\frac{ay}{(T-t)^{\frac{1}{2k}}} \right)$$

is a solution of (4.1) for any $T \in \mathbb{R}$ and $a > 0$. For any $a > 0$, $Z \mapsto F_k(aZ)$ is a solution of the stationary self-similar equation

$$F_k + \frac{1}{2k} Z \partial_Z F_k - F_k^2 = 0. \tag{4.2}$$

NLH:eq:Fk

The linearised transport operator $H_Z := 1 + \frac{1}{2k} Z\partial_Z - 2F_k(aZ)$ *acting on* $C^\infty(\mathbb{R})$ *has the point spectrum*

$$\Upsilon(H_Z) = \left\{ \frac{\ell - 2k}{2k}, \quad \ell \in \mathbb{N} \right\}.$$

The associated eigenfunctions are

$$H_Z \phi_{Z,\ell} = \frac{\ell - 2k}{2k} \phi_{Z,\ell}, \quad \phi_{Z,\ell} = \frac{Z^\ell}{(1+(aZ)^{2k})^2}. \tag{4.3}$$ eq:def phiXel

Two of them[6] *are linked to the invariances of the flow:*

$$\phi_0 = F_k(aZ) + \frac{1}{2k} Z\partial_Z F_k(aZ) = \frac{\partial}{\partial_\lambda} \left(\lambda F_k(\lambda^{\frac{1}{2k}} bZ) \right)_{|\lambda=1}, \quad \phi_{2k} = Z\partial_Z F_k(aZ) = \frac{\partial}{\partial_{\tilde a}} \left(F_k(\tilde a a Z) \right)_{|\tilde b = 1}.$$

Proof. The proof is made of straightforward computations that we do not write here. □

We now introduce for ξ a solution to (1.7) for $a > 0$ and $T > 0$ the self-similar variables following [14]

$$Y := \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := \frac{aY}{e^{\frac{k-1}{2k}s}}, \quad f(s,Y) = (T-t)\xi(t,y), \tag{4.4}$$ NLH:id:def a

to zoom at the blow-up location, and f solves the first equation in (3.8). The function that we want to construct here, from (1.8) and (1.9), should converge to 1 in compact sets of the variable Y. Therefore close to the origin the linearised operator is

$$H_\rho := -1 + \frac{Y}{2}\partial_Y - \partial_{YY}.$$

Its spectral structure is well-known on the weighted L^2-based Sobolev spaces

$$H^k_\rho := \left\{ f \in H^k_{\text{loc}}(\mathbb{R}), \quad \sum_{k'=0}^{k} \int_{\mathbb{R}} |\partial_Y^k f|^2 e^{-\frac{Y^2}{4}} dY < +\infty \right\}$$

with norm and scalar product

$$\|f\|^2_{H^k_\rho} := \sum_{k'=0}^{k} \int_{\mathbb{R}} |\partial_Y^k f|^2 e^{-\frac{Y^2}{4}} dY, \quad \langle f, g \rangle_\rho := \int_{\mathbb{R}} f g e^{-\frac{Y^2}{4}} dY, \quad \rho(Y) := e^{-\frac{Y^2}{4}}. \tag{4.5}$$ eq:def L2rho

pr:Lrho **Proposition 25** (Linear structure at the origin (see e.g. [24]))**.** *The operator* H_ρ *is essentially self-adjoint on* $C^2_0(\mathbb{R}) \subset L^2(\rho)$ *with compact resolvant. The space* H^2_ρ *is included in the domain of its unique self-adjoint extension. Its spectrum is*

$$\Upsilon(H_\rho) = \left\{ \frac{\ell - 2}{2}, \ \ell \in \mathbb{N} \right\}.$$

The eigenvalues are all simple and the associated orthonormal basis of eigenfunctions is given by Hermite polynomials:

$$h_\ell(Y) := c_\ell \sum_{n=0}^{[\frac{\ell}{2}]} \frac{\ell!}{n!(\ell-2n)!} (-1)^n Y^{\ell-2n}. \tag{4.6}$$ def:hell

h_ℓ *is orthogonal to any polynomial of degree lower than* $\ell - 1$ *for the* L^2_ρ *scalar product.*

To construct the blow-up solution of Theorem 1, we will use an approximate solution to (3.8) close to $F_k(bZ)$ that is adapted to the linearised dynamics both at the origin and far away.

[6]In fact there is a third one due to translation invariance which is absent here due to even symmetry.

LH:pr:profile **Proposition 26** (Approximate blow-up profile)**.** *Let $k\in\mathbb{N}$ with $k\geq 2$. There exists universal constants $(\bar c_{2\ell})_{0\leq \ell\leq k-1}$ with:*

$$\bar c_{2k-2}:=-2k(2k-1),\quad \bar c_{2\ell}:=-\frac{(2\ell+2)(2\ell+1)}{k-\ell}\bar c_{2\ell+2}\quad \text{for } \ell=0,...,k-2,$$

such that for any $0\leq s_0<s_1$ and $a\in C^1([s_0,s_1],(\frac12,\frac32))$, the profile

$$F[a](s,Y):=F_k(Z)+\sum_{\ell=0}^{k-1}\bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z) \tag{4.7}$$
def:Fb

satisfies the following identity:

$$\partial_s F[a]+F[a]+\frac{Y}{2}\partial_Y F[a]-F^2[a]-\partial_{YY}F[a]=a_s\partial_a F[a]+\Psi,$$

with the error Ψ satisfying for any $j\geq 0$:

$$\|\partial_Z^j\Psi\|_{L^2_\rho}\lesssim e^{-\left(2(k-1)-j\frac{k-1}{2k}\right)s}\quad \text{for}\quad j=0,...,2k,\quad \|\partial_Z^j\Psi\|_{L^2_\rho}\lesssim e^{-\frac{k-1}{k}s}\quad\text{for}\quad j\geq 2k+1, \tag{4.8}$$
bd:psirho

and for $|Y|\geq 1$:

$$|(Z\partial_Z)^j\Psi|\lesssim e^{-\frac{k-1}{k}s}|Z|^{4k-2}(1+|Z|)^{-6k}. \tag{4.9}$$
bd:Psi2

and

$$|\partial_Z^j\Psi|\lesssim e^{-\frac{k-1}{k}s}|Z|^{4k-2-j}(1+|Z|)^{-6k}\ \ \text{for } j=0,...,2k,\ \ |\partial_Z^j\Psi|\lesssim e^{-\frac{k-1}{k}s}(1+|Z|)^{-2k-2-j}\ \ \text{for } j\geq 2k+1, \tag{4.10}$$
bd:Psi3

The variation with respect to a enjoys the following properties:

$$\langle\partial_a F[a],h_{2k}\rangle_\rho=ca^{2k-1}e^{-(k-1)s}\left(1+O(e^{-(k-1)s})\right),\quad c\neq 0, \tag{4.11}$$
id:pabFproje

and for $j\in\mathbb{N}$:

$$\|\partial_Z^j\partial_a F[a]\|_{L^2_\rho}\lesssim e^{-\left((k-1)-j\frac{k-1}{2k}\right)s}\quad\text{for}\quad j=0,...,2k,\quad \|\partial_Z^j\partial_aF[a]\|_{L^2_\rho}\lesssim 1\quad\text{for}\quad j\geq 2k+1, \tag{4.12}$$
bd:pabFrho

and for $|Y|\geq 1$:

$$|(Z\partial_Z)^j\partial_a F[a]|\lesssim |Z|^{2k}(1+|Z|)^{-4k}. \tag{4.13}$$
bd:pabF2

and

$$|\partial_Z^j\partial_aF[a]|\lesssim |Z|^{2k-j}(1+|Z|)^{-4k}\ \ \text{for } j=0,...,2k,\ \ |\partial_Z^j\partial_aF[a]|\lesssim(1+|Z|)^{-2k-j}\ \ \text{for } j\geq 2k+1, \tag{4.14}$$
bd:pabF3

In all the above estimates, the implicit constants in the $\lesssim$ notation depend solely on k and j.

Proof. This is a brute force computation.

Step 1: *Estimates for* Ψ. We first decompose from (4.2) and (4.3):

$$
\begin{aligned}
&\Psi = \partial_s F[a] + F[a] + \frac{Y}{2}\partial_Y F[a] - F^2[a] - \partial_{YY} F[a] - a_s \partial_a F[a] \\
= \;& F_k(Z) + \frac{1}{2k} Z\partial_Z F_k(Z) - F_k^2(Z) - \sum_{\ell=0}^{k-1} \bar c_{2\ell} \frac{k-1}{2k}(2k-2\ell)(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z) \\
& + \sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}(H_Z\phi_{2\ell})(Z) - \sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell+2}(\partial_{ZZ}\phi_{2\ell})(Z) - (ae^{-\frac{k-1}{2k}s})^2 \partial_{ZZ}F_k(Z) \\
& - \left(\sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z)\right)^2 \\
= \;& -\sum_{\ell=0}^{k-1} \bar c_{2\ell}(k-\ell)(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z) - \sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell+2}(\partial_{ZZ}\phi_{2\ell})(Z) - (ae^{-\frac{k-1}{2k}s})^2 \partial_{ZZ}F_k(Z) \\
& - \left(\sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z)\right)^2 \\
= \;& -\sum_{\ell=0}^{k-2} (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left(\bar c_{2\ell}(k-\ell)\phi_{2\ell}(Z) + \bar c_{2\ell+2}\partial_{ZZ}\phi_{2\ell+2}(Z)\right) \\
& -(ae^{-\frac{k-1}{2k}s})^2(\bar c_{2k-2}\phi_{2k-2} - \partial_{ZZ}F_k) - \bar c_0 (ae^{-\frac{k-1}{2k}s})^{2k+2}\partial_{ZZ}\phi_0(Z) - \left(\sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z)\right)^2 \\
= \;& -\sum_{\ell=0}^{k-2} (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \bar c_{2\ell+2} \left(\partial_{ZZ}\phi_{2\ell+2}(Z) - (2\ell+2)(2\ell+1)\phi_{2\ell}(Z)\right) \\
& +(ae^{-\frac{k-1}{2k}s})^2(2k(2k-1)\phi_{2k-2} + \partial_{ZZ}F_k) - \bar c_0 (ae^{-\frac{k-1}{2k}s})^{2k+2}\partial_{ZZ}\phi_0(Z) - \left(\sum_{\ell=0}^{k-1} \bar c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z)\right)^2
\end{aligned}
$$

As

$$
\partial_{ZZ}\phi_{2\ell} = \frac{2\ell(2\ell-1)Z^{2\ell-2}}{(1+Z^{2k})^2} - \frac{4k(4\ell+2k-1)Z^{2\ell+2k-2}}{(1+Z^{2k})^3} + \frac{24k^2 Z^{2\ell+4k-2}}{(1+Z^{2k})^4}
$$

one deduces that for $\ell = 0, ..., k-2$:

$$
\begin{aligned}
& \left| (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left(\partial_{ZZ}\phi_{2\ell+2}(Z) - (2\ell+2)(2\ell+1)\phi_{2\ell}(Z)\right) \right| \\
= \;& \left| (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left(-\frac{4k(4\ell+2k+3)kZ^{2\ell+2k}}{(1+Z^{2k})^3} + \frac{24k^2 Z^{2\ell+4k}}{(1+Z^{2k})^4} \right) \right| \\
\lesssim \;& (e^{-\frac{k-1}{2k}s})^{2k-2\ell} Z^{2\ell+2k}(1+|Z|)^{-6k}.
\end{aligned}
$$

Similarly, as

$$
\partial_{ZZ}F_k = -\frac{2k(2k-1)Z^{2k-2}}{(1+Z^{2k})^2} + \frac{8k^2 Z^{4k-2}}{(1+Z^{2k})^3}
$$

one deduces that

$$\left|(ae^{-\frac{k-1}{2k}s})^2(2k(2k-1)\phi_{2k-2}+\partial_{ZZ}F_k)\right| = \left|(ae^{-\frac{k-1}{2k}s})^2\frac{8k^2Z^{4k-2}}{(1+Z^{2k})^3}\right| \lesssim (e^{-\frac{k-1}{2k}s})^2Z^{4k-2}(1+|Z|)^{-6k}.$$

Eventually,

$$\left|-(ae^{-\frac{k-1}{2k}s})^{2k+2}\partial_{ZZ}\phi_0(Z)\right| = \left|(ae^{-\frac{k-1}{2k}s})^{2k+2}\left(\frac{4k(2k-1)Z^{2k-2}}{(1+Z^{2k})^3}+\frac{24k^2Z^{4k-2}}{(1+Z^{2k})^4}\right)\right| \lesssim (e^{-\frac{k-1}{2k}s})^{2k+2}Z^{2k-2}(1+|Z|)^{-6k}$$

and

$$\left(\sum_{\ell=0}^{k-1}\bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\phi_{2\ell}(Z)\right)^2 \lesssim \sum_{\ell=0}^{k-1}(e^{-\frac{k-1}{2k}s})^{4k-4\ell}Z^{4\ell}(1+|Z|)^{-8k}.$$

From the above identities one deduces that:

$$|\Psi| \lesssim \sum_{\ell=1}^{k+1}(e^{-\frac{k-1}{2k}s})^{2\ell}|Z|^{4k-2\ell}(1+|Z|)^{-6k}+\sum_{\ell=0}^{k-1}(e^{-\frac{k-1}{2k}s})^{4k-4\ell}|Z|^{4\ell}(1+|Z|)^{-8k}. \quad (4.15)$$ eq:boundpoin

For $\ell = 1, ..., k+1$ one computes that

$$\int\left((e^{-\frac{k-1}{2k}s})^{2\ell}Z^{4k-2\ell}(1+|Z|)^{-6k}\right)^2e^{-\frac{Y^2}{4}}dY \lesssim \int(e^{-\frac{k-1}{2k}s})^{4\ell}(Ye^{-\frac{k-1}{2k}s})^{8k-4\ell}e^{-\frac{Y^2}{4}}dY \lesssim e^{-4(k-1)s}$$

and similarly for $\ell = 0, ..., k-1$:

$$\int\left((e^{-\frac{k-1}{2k}s})^{4k-4\ell}Z^{4\ell}(1+|Z|)^{-8k}\right)^2e^{-\frac{Y^2}{4}}dY \lesssim e^{-4(k-1)s}.$$

The above two bounds imply (4.8) for $j=0$. For $|Y|\geq 1$ one has that for $\ell = 1, ..., k+1$:

$$(e^{-\frac{k-1}{2k}s})^{2\ell}Z^{4k-2\ell}(1+|Z|)^{-6k} \lesssim e^{-\frac{k-1}{k}s}(aYe^{-\frac{k-1}{2k}s})^{2\ell-2}Z^{4k-2\ell}(1+|Z|)^{-6k} = e^{-\frac{k-1}{k}s}Z^{4k-2}(1+|Z|)^{-6k}$$

and similarly for $\ell = 0, ..., k-1$:

$$(e^{-\frac{k-1}{2k}s})^{4k-4\ell}Z^{4\ell}(1+|Z|)^{-8k} \lesssim e^{-\frac{k-1}{k}s}Z^{4k-2}(1+|Z|)^{-8k}.$$

The above two bounds yield (4.9) for $j=0$. We claim the the other bounds for $|\Psi|$ can be proved the same way as (4.15) naturally extends to derivatives.

Step 2 *Estimate for $\partial_a F$.* We compute two identities:

$$\begin{aligned} a\partial_a F[a] &= -2k\phi_{2k}(Z)+\sum_{\ell=0}^{k-1}\bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\left((2k-2\ell)\phi_{2\ell}(Z)+Z\partial_Z\phi_{2\ell}(Z)\right)\\ &= -2k\frac{Z^{2k}}{(1+Z^{2k})^2}+\sum_{\ell=0}^{k-1}\bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\left(\frac{2kZ^{2\ell}}{(1+Z^{2k})^2}-\frac{4kZ^{2\ell+2k}}{(1+Z^{2k})^3}\right) \quad (4.16)\\ &= -2kZ^{2k}+\sum_{\ell=0}^{k-1}\bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}2kZ^{2\ell} \quad (4.17)\\ &\quad +2k\frac{2Z^{4k}-Z^{6k}}{(1+Z^{2k})^2}-\sum_{\ell=0}^{k-1}\bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell}\left(\frac{2kZ^{2\ell+2k}}{(1+Z^{2k})^2}+\frac{4kZ^{2\ell+4k}}{(1+Z^{2k})^3}\right) \end{aligned}$$

From the first identity (4.16) we infer that:

$$|\partial_a F[a]| \lesssim \sum_{\ell=0}^{k} (e^{-\frac{k-1}{2k}s})^{2k-2\ell} Z^{2\ell}(1+Z^{2k})^{-2}. \tag{4.18}$$

bd:pointwise

This implies that

$$|\partial_a F[a]| \lesssim e^{-(k-1)s} \sum_{\ell=0}^{k} Y^{2\ell}$$

which yields (4.12) for $j=0$. For $|Y|\geq 1$ one therefore estimates:

$$|\partial_a F[a]| \lesssim \sum_{\ell=0}^{k} (bYe^{-\frac{k-1}{2k}s})^{2k-2\ell} Z^{2\ell}(1+Z^{2k})^{-2} \lesssim Z^{2k}(1+Z^{2k})^{-2}$$

which proves (4.13) for $j=0$. Again, we claim that the other bounds concerning $\partial_a F[a]$ can be proved along the same lines since (4.18) naturally extends to derivatives. We now use the second identity (4.17) and compute since $h_{2k}(Y)$ is orthogonal to any polynomial of degree less or equal than $2k-1$ and $Z=e^{-(k-1)s/(2k)}Y$:

$$\langle -2kZ^{2k} + \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} 2kZ^{2\ell}, h_{2k}\rangle_\rho = \langle -2kZ^{2k}, h_{2k}\rangle_\rho = b^{2k}ce^{-(k-1)s}, \quad c\neq 0.$$

We then get the desired nondegeneracy (4.11) since

$$\| 2k\frac{2Z^{4k}-Z^{6k}}{(1+Z^{2k})^2} - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left(\frac{2kZ^{2\ell+2k}}{(1+Z^{2k})^2} + \frac{4kZ^{2\ell+4k}}{(1+Z^{2k})^3} \right) \|_{L^2_\rho} \lesssim e^{-2(k-1)s}.$$

□

We now fix $k\geq 1$ for the remaining part of the Subsection, and prove Theorem 1 by showing that there exists a global solution to (3.8) converging to $F_k(Z)$ as $s\to+\infty$. To this end, we perform a bootstrap argument near the approximate profile $F[a]$. We decompose the solution in self-similar variables according to (using (4.14)):

$$f = F[a]+\varepsilon, \quad \varepsilon = \sum_{\ell=0}^{k-1} c_{2\ell}h_{2\ell}(Y) + \tilde{\varepsilon}, \quad \tilde{\varepsilon} \perp_\rho h_{2\ell} \ \text{ for } \ \ell=0,...,k, \tag{4.19}$$

id:decomposi

where the $\perp_\rho$ is the orthogonality with respect to the L^2_ρ scalar product. Such a decomposition in the vicinity of $F[a]$ is a consequence of (4.11) and of the implicit function Theorem.

ef:trappedNLH

Definition 27 (Trapped solutions)**.** *Fix $J\in\mathbb{N}$, and let constants $\tilde{K}\geq K_J\geq ...\geq K_0>0$. We say a solution to (NLH) is trapped on $[s_0,s_1]$ if it is even, can be decomposed according to*[7] *(4.19) and if it satisfies for $j=0,...,J$:*

$$\int_{|Y|\geq 1} \frac{|(Y\partial_Y)^j\varepsilon|^2}{\phi^2_{2k+1}(Z)}\frac{dY}{|Y|} \leq K_j e^{-\frac{3}{4k}s}, \tag{4.20}$$

bd:eweightlo

$$\int_{|Y|\geq 1} \frac{|(Y\partial_Y)^j\varepsilon|^2}{\phi^2_{2k+1/2}(Z)}\frac{dY}{|Y|} \leq K_j e^{-\frac{1}{2k}s}, \tag{4.21}$$

bd:eweightsh

[7]Note that for s_0 large enough, due to the bounds for trapped solutions, this decomposition is unique and the associated parameters are C^∞ from a standard application of the implicit function Theorem and of parabolic regularity

$$\int_{|Y|\geq 1}\frac{|\partial_Z^j\varepsilon|^2}{\phi_0^2(Z)}\frac{dY}{|Y|}\leq K_je^{-\frac{1}{2k}s} \quad \textit{for } j\geq 2k+1, \tag{4.22}$$

bd:eweightjg

$$\|\partial_Z^j\tilde\varepsilon\|_{L^2_\rho}\leq \sqrt{K_j}e^{-\left(k-\frac12+\frac{j}{2k}-\frac{j}{2}\right)s} \ \textit{for } j=0,...,2k, \ \ \|\partial_Z^j\tilde\varepsilon\|_{L^2_\rho}\leq \sqrt{K_j}e^{-\frac{1}{2k}s} \ \textit{for } j\geq 2k+1, \tag{4.23}$$

bd:eloc

$$\left(\sum_{\ell=0}^{k-1}|c_{2\ell}|^2\right)^{\frac12}\leq \sqrt{\tilde K}e^{-(k-\frac12+\frac{1}{4k})s} \tag{4.24}$$

bd:cell boots

We will show that one solutions remained trapped forever, and our argument necessitates to adjust the initial values of the projection of the error on the modes $(h_{2\ell})_{0\leq \ell\leq k-1}$. These functions are unbounded but the following technical Lemma provides a harmless localisation.

lem:implicit

Lemma 28. *There exists $M^*,C>0$ such that for all, $M\geq M^*$, $\tilde K>0$, there exists $s^*\geq 0$ such that for $s\geq s^*$ the following holds true introducing $\chi_M(Y):=\chi(Y/M)$. For any $\frac12<\bar a<\frac32$, $(\bar c_{2\ell})_{0\leq\ell\leq k-1}\in B(\tilde K e^{-(k-\frac12+\frac{1}{4k})s})$ (the Euclidean ball) there exist unique parameters $(\bar c_{2\ell})_{0\leq \ell\leq k-1}\in B(2\tilde K e^{-(k-\frac12+\frac{1}{4k})s})$ and a with $|a-\bar a|\leq C\tilde K e^{-(\frac12+\frac{1}{2k})s}$ such that*

$$F[\bar a]+\chi_M\sum_{\ell=0}^{k-1}\bar c_{2\ell}h_{2\ell}(Y)=F[a]+\sum_{\ell=0}^{k-1}c_{2\ell}h_{2\ell}(Y)+\tilde\varepsilon$$

where $\tilde\varepsilon$ satisfies the orthogonality conditions in (4.19). Moreover, with $\bar a$ fixed, the mapping that to $(\bar c_{2\ell})_{0\leq\ell\leq k-1}$ associates $(c_{2\ell})_{0\leq \ell\leq k-1}$ is a smooth diffeomorphism and the preimage of $B(\tilde K e^{-(k-\frac12+\frac{1}{4k})s})$ is contained in $B(2\tilde K e^{-(k-\frac12+\frac{1}{4k})s})$.

Proof. We fix $\bar a\in[\frac12,\frac32]$. We write $a=\bar a+\tilde a e^{(k-1)s}$ and define the mappings $\omega:\mathbb R^k\to\mathbb R^{k+1}$ and $\tilde\omega:\mathbb R^{k+1}\to\mathbb R^{k+1}$ by:

$$\omega\left(\bar c_0,...,\bar c_{2k-2}\right)=(\langle \chi_M\sum_{\ell=0}^{k-1}\bar c_{2\ell}h_{2\ell},h_{2n}\rangle_\rho)_{0\leq n\leq k},$$

$$\tilde\omega(c_0,...,c_{2k-2},\tilde a)=(\langle F[a]-F[\bar a]+\sum_{\ell=0}^{k-1}\bar c_{2\ell}h_{2\ell},h_{2n}\rangle_\rho)_{0\leq n\leq k}.$$

One has the identities $\omega\left(\bar c_0,...,\bar c_{2k-2}\right)=(\bar c_0,...,\bar c_{2k-2},0)+o_{M\to\infty}(|(\bar c_0,...,\bar c_{2k-2})|)$ as the h_ℓ form an orthonormal basis of L^2_ρ of polynomials, and since the weight ρ decreases exponentially fast. One similarly has that $\tilde\omega(c_0,...,c_{2k-2},\tilde a)=(c_0,...,c_{2k-2},ca^{2k-1}\tilde a)+O(\tilde a^2)$ where $c>0$ is defined in (4.11). The Lemma is obtained choosing $(c_0,...,c_{2k-2},(a-\bar a)e^{-(k-1)s})=\tilde\omega^{-1}\circ\omega(\bar c_0,...,\bar c_{2k-2})$. The inversion of $\tilde\omega$, and the desired estimates, are consequences of the aforementioned identities and of the implicit function Theorem. □

pr:bootstrap

Proposition 29. *Fix $J\geq 1$ and $M>0$ so that Lemma 28 holds true. There exists $\tilde K\geq K_J\geq ...\geq K_0>0$ and $s^*\geq 0$ such that the following holds for any $s_0\geq s^*$. For any $\bar\varepsilon_0=\bar\varepsilon(s_0)$ that is even, satisfies the orthogonality conditions in (4.19) and*

$$\sum_{j=0}^J\int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\bar\varepsilon_0|^2}{\phi_{2k+1}^2(Z)}\frac{dY}{|Y|}\leq e^{-\frac{3}{4k}s_0},\quad \sum_{j=0}^J\int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\bar\varepsilon_0|^2}{\phi_{2k+1/2}^2(Z)}\frac{dY}{|Y|}\leq e^{-\frac{1}{2k}s_0}, \tag{4.25}$$

bd:decayextin

$$\sum_{j=2k+1}^J\int_{|Y|\geq 1}\frac{|\partial_Z^j\bar\varepsilon_0|^2}{\phi_0^2(Z)}\frac{dY}{|Y|}\leq e^{-\frac{1}{2k}s_0}, \tag{4.26}$$

bd:decayextin

$$\|\partial_Z^j\bar\varepsilon_0\|_{L^2_\rho}\leq e^{-\left(k-\frac12+\frac{j}{2k}-\frac{j}{2}\right)s_0} \ \ \textit{for } \ j=0,...,2k, \ \ \|\partial_Z^j\bar\varepsilon_0\|_{L^2_\rho}\leq e^{-\frac{1}{2k}s_0} \ \ \textit{for } j\geq 2k+1, \tag{4.27}$$

bd:eL2rhoini

and $3/4 \le a(s_0) \le 5/4$, *there exist* $\bar c_0(s_0), ..., \bar c_{2\ell-2}(s_0)$ *with*

$$\left(\sum_{\ell=0}^{k-1} |\bar c_{2\ell}(s_0)|^2\right)^{\frac12} \le 2\sqrt{\tilde K} e^{-(k-\frac12+\frac{1}{4k})s_0} \qquad (4.28)$$

NLH:eq:bootst

such that the solution f *to* (3.8) *with initial datum*

$$f(s_0) = F[a](s_0) + \chi_M(Y) \sum_{\ell=0}^{k-1} \bar c_{2\ell}(s_0) h_\ell(Y) + \bar\varepsilon_0, \qquad (4.29)$$

bd:initialde

is a global solution to (3.8)*, which is trapped on* $[s_0, +\infty)$*. Moreover, there exists an asymptotic limit* $a^* \in \mathbb R$*, with* $1/2 \le a^* \le 3/2$ *such that*

$$|a - a^*| \lesssim (K_1^2 + K_2^2) e^{-(\frac12+\frac{1}{4k})s}. \qquad (4.30)$$

NLH:bd:a

All parameters are fixed, except the constants $\tilde K$, K_j and s^* that will be fixed in the forthcoming Lemmas. Hence generic constant C will be used when independent of $\tilde K$, K_j and s^*, with associated notation $\lesssim$. We prove Proposition 16 with a classical bootstrap reasoning. The results below will specify the dynamics in the trapped regime and allow to prove Proposition 29 at the end of the subsection.

em:pointwise

Lemma 30. *For any constants* $\tilde K \ge K_J \ge ... \ge K_0 > 0$*, there exists* s^* *large enough such that if* f *is trapped on* $[s_0, s_1]$ *with* $s_0 \ge s^*$*, there holds for* $j = 0, ..., J-1$*, for any* $Z \in \mathbb R$*:*

$$|\partial_Z^j \varepsilon| \lesssim \sqrt{K_J} e^{-\frac{1}{4k}s} (1+|Z|)^{\frac12 - 2k - j}, \qquad (4.31)$$

bd:eLinfty

and for any $C > 0$*, for all* $|Y| \le C$*:*

$$|\partial_Z^j \varepsilon| \lesssim \sqrt{\tilde K} e^{-(k-\frac12+\frac{j}{2k}-\frac{j}{2})s} \ \text{ for } \ j = 0, ..., 2k, \quad |\partial_Z^j \varepsilon| \lesssim \sqrt{\tilde K} e^{-\frac{1}{2k}s} \ \text{ for } j \ge 2k+1. \qquad (4.32)$$

bd:eLinftylo

Proof. **Step 1:** *Proof of* (4.32). From (4.23), and Sobolev embedding one deduces that for $|Y| \le C$:

$$|\partial_Z^j \tilde\varepsilon| \lesssim \sqrt{K_J} e^{-(k-\frac12+\frac{j}{2k}-\frac{j}{2})s}.$$

From (4.24) and (4.6) for $j = 0, ..., 2k-1$ and $|Y| \le C$ one has:

$$|\partial_Z^j (c_\ell h_\ell(Y))| = |e^{\frac{k-1}{2k}js} \partial_Y^j (c_\ell h_\ell(Y))| \lesssim e^{\frac{k-1}{2k}js} |c_\ell| \lesssim \sqrt{\tilde K} e^{-(k-\frac12+\frac{1}{4k}-\frac{j}{2}+\frac{j}{2k})} \lesssim \sqrt{\tilde K} e^{-(k-\frac12-\frac{j}{2}+\frac{j}{2k})}$$

for s_0 large enough. For $j \ge 2k$ and $\ell \le 2k-1$ one notices that $\partial_Z^j h_\ell = 0$. Therefore, one obtains (4.32) from the decomposition (4.19) and the two above bounds.

Step 2: *Proof of* (4.31). We apply (A.2) and use the facts that $Y\partial_Y = Z\partial_Z$ and $|Z\partial_Z \phi_{2k+1/2}| \lesssim |\phi_{2k+1/2}| \lesssim |Z\partial_Z \phi_{2k+1/2}|$:

$$\begin{aligned} \| \frac{Z^j \partial_Z^j \varepsilon}{\phi_{2k+1/2}(Z)} \|^2_{L^\infty(\{|Y|\ge 1\})} &\lesssim \| \frac{Z^j \partial_Z^j \varepsilon}{\phi_{2k+1/2}(Z)} \|^2_{L^2\left(\{|Y|\ge1\}, \frac{dY}{|Y|}\right)} + \| Z\partial_Z \Big(\frac{Z^j \partial_Z^j \varepsilon}{\phi_{2k+1/2}(Z)} \Big) \|^2_{L^2\left(\{|Y|\ge1\}, \frac{dY}{|Y|}\right)} \\ &\lesssim \sum_{k=0}^{j+1} \| \frac{(Z\partial_Z)^j \varepsilon}{\phi_{2k+1/2}(Z)} \|^2_{L^2\left(\{|Y|\ge1\}, \frac{dY}{|Y|}\right)} \lesssim K_J e^{-\frac{1}{2k}s}. \end{aligned}$$

Since $|\phi_{2k+1/2}| \lesssim |Z|^{2k+1/2} (1+|Z|)^{-4k}$ one obtains that for $|Y| \ge 1$

$$|\partial_Z^j \varepsilon| \lesssim \sqrt{K_J} e^{-\frac{1}{4k}s} |Z|^{2k+1/2-j} (1+|Z|)^{-4k}. \qquad (4.33)$$

bd:pointwise

For $j\geq 2k+1$, the fact that $|\phi_{2k+1/2}|(Z)=|Z|^{2k+1}|\phi_0(Z)|$ yields the inequality

$$\frac{|Z|}{|\phi_0(Z)|}\lesssim \frac{1}{|\phi_0(Z)|}+\frac{|Z|^{j+1}}{|\phi_{2k+1/2}|}$$

from which we infer that

$$\int_{|Y|\geq 1}\frac{|Z\partial_Z(\partial_Z^j\varepsilon)|^2}{\phi_0^2(Z)}\frac{dY}{|Y|}\lesssim \int_{|Y|\geq 1}\frac{|\partial_Z^{j+1}\varepsilon|^2}{\phi_0^2(Z)}\frac{dY}{|Y|}+\int_{|Y|\geq 1}\frac{|Z^{j+1}\partial_Z^{j+1}\varepsilon)|^2}{\phi_0^2(Z)}\frac{dY}{|Y|}\lesssim K_Je^{-\frac{1}{2k}s}.$$

The very same reasoning using the above bound and (4.22) gives

$$|\partial_Z^j\varepsilon|\lesssim \sqrt{K_J}e^{-\frac{1}{4k}s}(1+|Z|)^{-4k}.$$

From the two above bounds one infers that for $|Y|\geq 1$:

$$|\partial_Z^j\varepsilon|\lesssim \sqrt{K_J}e^{-\frac{1}{4k}s}(1+|Z|)^{\frac12-2k-j}. \tag{4.34}$$ bd:pointwise

Combining (4.33), (4.34) and (4.32) yields (4.31).

□

The evolution of ε is given by:

$$\varepsilon_s+\frac{Y}{2}\partial_Y\varepsilon+\varepsilon-2F_k(Z)\varepsilon+2(F_k(Z)-F[a])\varepsilon-\partial_{YY}\varepsilon+a_s\partial_aF[a]+\Psi-\varepsilon^2=0 \tag{4.35}$$ eq:evolution

and that of $\tilde\varepsilon$ by:

$$\tilde\varepsilon_s+\mathcal{L}_\rho\tilde\varepsilon+\sum_{\ell=0}^{k-1}(c_{2\ell,s}+\frac{2\ell-2}{2}c_{2\ell})h_{2\ell}+2(1-F[a])\varepsilon+a_s\partial_aF[a]+\Psi-\varepsilon^2=0 \tag{4.36}$$ eq:evolution

em:modulation **Lemma 31** (Modulation equations). *For any constants $\tilde K\geq K_J\geq ...\geq K_0>0$, there exists s^* large enough such that if f is trapped on $[s_0,s_1]$ with $s_0\geq s^*$, the following identities hold for $\ell=0,1,...,k-1$:*

$$|a_s|\lesssim K_Je^{-(\frac12+\frac{1}{4k})s}\quad \textit{and}\quad \left|c_{2\ell,s}+\frac{2\ell-2}{2}c_\ell\right|\lesssim K_Je^{-(k-\frac12+\frac{1}{4k})s} \tag{4.37}$$ eq:modulation

Proof. **Step 1** *Law for a.* We take the scalar product between (4.36) and h_{2k}, yielding, using (4.11) and (4.19):

$$a_sca^{2k-1}e^{-(k-1)s}(1+O(e^{-(k-1)s}))=\langle -2(1-F[a])\varepsilon-\Psi+\varepsilon^2,h_{2k}\rangle_\rho.$$

We now estimate the right hand side. First, since $|F[a]-1|\lesssim e^{-(k-1)s}\sum_0^kY^{2\ell}$, using (4.23) and (4.24) for s^* large enough:

$$|\langle -2(1-F[a])\varepsilon,h_{2k}\rangle_\rho|\lesssim \|\varepsilon\|_{L^2_\rho}\|(1-F[a])h_{2k}\|_{L^2_\rho}\lesssim \sqrt{K_0}e^{-(k-\frac12)s}e^{-(k-1)s}=\sqrt{K_0}e^{-(2k-\frac32)s}.$$

Using the bound on the error (4.8):

$$|\langle\Psi,h_{2k}\rangle_\rho|\lesssim e^{-2(k-1)s}.$$

Finally, using the bounds (4.23), (4.24) and (4.31) for the nonlinear term:

$$\left|\langle\varepsilon^2,h_{2k}\rangle_\rho\right|\lesssim \|\varepsilon\|_{L^2_\rho}\|\varepsilon\|_{L^\infty}\lesssim (\sqrt{K_0}e^{-(k-\frac12)s}+\sqrt{\tilde K}e^{-(k-\frac12+\frac{1}{4k})s})\sqrt{K_J}e^{-\frac{1}{4k}s}\lesssim K_Je^{-(k-\frac12+\frac{1}{4k})s},$$

for s^* large enough. Summing the above identities yields (4.37) for a_s.

Step 2 *Law for c_ℓ.* We take the scalar product between (4.36) and $h_{2\ell}$ for $\ell=0,1,...,k-1$, yielding, using (4.11) and (4.19):

$$(c_{2\ell,s}+\frac{2\ell-2}{2}c_{2\ell})\|h_{2\ell}\|_{L^2_\rho}=-\langle 2(1-F[a])\varepsilon+a_s\frac{\partial}{\partial_a}F[a]+\Psi-\varepsilon^2,h_{2\ell}\rangle$$

Performing the same computations as in Step 1 gives for s^* large enough:

$$\left|\langle 2(1-F[a])\varepsilon+\Psi-\varepsilon^2,h_{2\ell}\rangle\right|\lesssim K_Je^{-(k-\frac12+\frac{1}{4k})s}.$$

Using the bound for a_s (4.37) obtained in Step 1 and (4.12) one obtains:

$$|\langle a_s\partial_aF[a],h_{2\ell}\rangle|\lesssim K_Je^{-(\frac12+\frac{1}{4k})s}e^{-(k-1)s}=K_Je^{-(k-\frac12+\frac{1}{4k})s}.$$

The three previous identities then yield (4.37) for $c_{2\ell}$.

□

lem:energyrho **Lemma 32.** *There exists a choice of constants $K_J\geq...\geq K_0>0$, such that for any $\tilde K\geq K_J$, there exists s^* large enough such that if f is trapped on $[s_0,s_1]$ with $s_0\geq s^*$ and satisfies (4.27), at time s_1 there holds for $j=0,...,J$:*

$$\|\partial_Z^j\tilde\varepsilon(s_1)\|_{L^2_\rho}\leq\frac{\sqrt{K_j}}{2}e^{-(k-\frac12+\frac{j}{2k}-\frac{j}{2})s_1}\ \text{for } j=0,...,2k,\quad \|\partial_Z^j\tilde\varepsilon(s_1)\|_{L^2_\rho}\leq\frac{\sqrt{K_j}}{2}e^{-\frac{1}{2k}s_1}\ \text{for } j\geq 2k+1. \tag{4.38}$$

eq:boundfatig

Proof. Set $\tilde\varepsilon_j=\partial_Z^j\tilde\varepsilon$. Then $\tilde\varepsilon_j$ solves from (4.36):

$$\partial_s\tilde\varepsilon_j+\frac{j}{2k}\tilde\varepsilon_j+H_\rho\tilde\varepsilon_j+\sum_{\ell=0}^{k-1}(c_{2\ell,s}+\frac{2\ell-2}{2}c_{2\ell})\partial_Z^j(h_{2\ell})+2\partial_Z^j((1-F[a])\varepsilon)+a_s\partial_Z^j\partial_aF[a]+\partial_Z^j\Psi-\partial_Z^j\varepsilon^2=0$$

which yields the following expression for the energy identity:

$$\begin{aligned}\frac{d}{ds}\left(\frac12\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY\right) &= -\langle\tilde\varepsilon_j,(H_\rho+\frac{j}{2k})\tilde\varepsilon_j\rangle_\rho-\sum_{\ell=0}^{k-1}(c_{2\ell,s}+\frac{2\ell-2}{2}c_{2\ell})\langle\tilde\varepsilon_j,\partial_Z^j(h_{2\ell})\rangle_\rho\\ &\quad -2\langle\tilde\varepsilon_j,\partial_Z^j((1-F[a])\varepsilon)\rangle_\rho+\langle\tilde\varepsilon_j,a_s\partial_Z^j\partial_aF[a]+\partial_Z^j\Psi\rangle_\rho-\langle\tilde\varepsilon_j,\partial_Z^j\varepsilon^2\rangle_\rho.\end{aligned}$$

For $j=0,...,2k$, by integrating by parts one obtains from (4.19) that $\tilde\varepsilon_j$ is orthogonal to h_ℓ for $\ell=0,...,2k-j$ for the L^2_ρ scalar product and therefore to any polynomial of degree less or equal to $2k-j$. Therefore, from Proposition 25 there holds:

$$-\langle\tilde\varepsilon_j,(H_\rho+\frac{j}{2k})\tilde\varepsilon_j\rangle_\rho\leq\left\{\begin{array}{ll}-\left(k-\frac{j}{2}-\frac12+\frac{j}{2k}\right)\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY & \text{if } j=0,...,2k,\\ -(\frac{j}{2k}-1)\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY & \text{if } j=2k+1,...,J.\end{array}\right.$$

Let $0<\nu\ll1$ be a small constant to be fixed later on. Integrating by parts yields:

$$-\int\tilde\varepsilon_jH_\rho\tilde\varepsilon_je^{-\frac{Y^2}{4}}dY\leq-c\int Y^2\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY+c'\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY$$

for some constants $c,c'>0$. Combining the above estimates one writes:

$$\begin{aligned}-\langle\tilde\varepsilon_j,(H_\rho+\frac{j}{2k})\tilde\varepsilon_j\rangle_\rho &= -(1-e^{-2\nu s})\langle\tilde\varepsilon_j,(H_\rho+\frac{j}{2k})\tilde\varepsilon_j\rangle_\rho-e^{-2\nu s}\langle\tilde\varepsilon_j,(H_\rho+\frac{j}{2k})\tilde\varepsilon_j\rangle_\rho\\ &\leq -ce^{-2\nu s}\int Y^2\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY+O(e^{-2\nu s}\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY)\\ &\quad+\left\{\begin{array}{ll}-\left(k-\frac{j}{2}-\frac12+\frac{j}{2k}\right)\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY & \text{if } j=0,...,2k,\\ -(\frac{j}{2k}-1)\int\tilde\varepsilon_j^2e^{-\frac{Y^2}{4}}dY & \text{if } j=2k+1,...,J.\end{array}\right.\end{aligned}$$

As we said earlier, $\tilde{\varepsilon}_j$ is orthogonal to any polynomial of degree less or equal to $2k-j$ for $j=0,...,2k$. For $j\geq 2k+1$ one notices that $\partial_Z^j h_{2\ell}=0$ for any $\ell=0,...,k-1$. Hence the cancellation for $j=0,...,J$:

$$\sum_{\ell=0}^{k-1}(c_{2\ell,s}+\frac{2\ell-2}{2}c_{2\ell})\langle\tilde{\varepsilon}_j,\partial_Z^j(h_{2\ell})\rangle_\rho=0$$

One has from (4.7) that the lower order linear potential satisfies (as $Z=Ye^{-\frac{k-1}{2k}s}$):

$$|(1-F[a])|\lesssim|Z|^2(1+|Z|)^{-2-2k}\lesssim\min(1,e^{-\frac{k-1}{k}s}Y^2)$$

which adapts to derivatives. Therefore, applying Leibnitz rule and Cauchy-Schwarz one gets that for $j=0,...,J$ using (4.23):

$$\begin{aligned}\left|\langle\tilde{\varepsilon}_j,\partial_Z^j((1-F[a])\varepsilon)\rangle_\rho\right|&\leq C\left(\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}\sum_{i=0}^{j-1}\left(\int|\partial_Z^i\tilde{\varepsilon}|^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}+Ce^{-\frac{k-1}{k}s}\int Y^2\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}dY\\&\leq Ce^{-\frac{k-1}{k}s}\int Y^2\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}dY+\begin{cases}Ce^{-2\left(k-\frac12+\frac{j}{2k}+\nu\right)s}K_j&\text{for } j=0,...,2k+1,\\ C\sqrt{K_j}\sqrt{K_{j-1}}e^{-\frac1k s}&\text{for } j=2k+2,...,J,\end{cases}\end{aligned}$$

where we used that for $i<j$ one has $K_i<K_j$ and $\frac{i}{2k}-\frac{i}{2}\geq\frac{j}{2k}-\frac{j}{2}+2\nu$ for $0<2\nu\leq\frac12-\frac{1}{2k}$. From (4.37), (4.12), (4.8), (4.23) and Cauchy-Schwarz we estimate for $0<\nu<\frac{1}{8k}$:

$$\left|\int\tilde{\varepsilon}_j\partial_Z^j\left(a_s\partial_aF[a]+\Psi\right)e^{-\frac{Y^2}{4}}dY\right|\lesssim\sqrt{K_j}\begin{cases}e^{-2\left(k-\frac12-\frac{j}{2}+\frac{j}{2k}+\nu\right)s}&\text{for } j=0,...,2k,\\ e^{-2\left(\frac{1}{2k}+\nu\right)s}&\text{for } j=2k+1,...,J.\end{cases}$$

Using Leibnitz rule and Cauchy Schwarz, from (4.31), (4.19), (4.23) and (4.24) we infer for the nonlinear term, as $\tilde{K}>K_J>...>K_0$:

$$\begin{aligned}\left|\int\tilde{\varepsilon}_j\partial_Z^j(\varepsilon^2)e^{-\frac{Y^2}{4}}dY\right|&\lesssim\left(\sum_{i=0}^{J-1}\|\partial_Z^i\varepsilon\|_{L^\infty}\right)\left(\sum_{i=0}^{j}\int|\partial_Z^j\varepsilon|^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}\left(\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}\\&\lesssim e^{-\frac{1}{4k}s}\left(\sqrt{\tilde{K}}e^{-(k-\frac12+\frac{1}{4k}-j\frac{k-1}{2k})s}+\left(\sum_{i=0}^{j}\int|\partial_Z^j\tilde{\varepsilon}|^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}\right)\left(\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}dY\right)^{\frac12}\\&\lesssim C(\tilde{K})\begin{cases}e^{-2\left(k-\frac12-\frac{j}{2}+\frac{j}{2k}+\nu\right)s}&\text{for } j=0,...,2k,\\ e^{-2\left(\frac{1}{2k}+\nu\right)s}&\text{for } j=2k+1,...,J.\end{cases}\end{aligned}$$

for $0<2\nu<\frac{1}{4k}$. Putting all the previous estimates together one obtains that there exists $\nu>0$ depending only k such that for s^* large enough, after some signs inspection:

$$\frac{d}{ds}\left(\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}\right)\leq\begin{cases}-2\left(k-\frac{j}{2}-\frac12+\frac{j}{2k}\right)\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}+C(\tilde{K})e^{-2\left(k-\frac{j}{2}-\frac12+\frac{j}{2k}+\nu\right)s}&\text{for } j=0,...,2k\\ -\frac{j-2k}{k}\int\tilde{\varepsilon}_j^2e^{-\frac{Y^2}{4}}+C(\tilde{K})e^{-2\left(\frac{1}{2k}+\nu\right)s}+C(K_{j-1})e^{-\frac1k s}&\text{for } j=2k+2,...,J\end{cases}$$

where we used that $\tilde{K}>K_J>...>K_0$. We claim that integrating over time the above differential inequality shows (4.38). We only show that this is the case for $j=2k+2,...,J$, the proof being the same for $0\leq j\leq 2k+1$. For $j=2k+2,...,J$, one notices that $\frac{j-2k}{k}\geq\frac{2}{k}$, so that from (4.27) and the above inequality one deduces that at time s_1:

$$\begin{aligned}\int \tilde{\varepsilon}_j^2 e^{-\frac{Y^2}{4}}dY &\leq e^{-\frac{2}{k}s_1}\left(\int \tilde{\varepsilon}_j(s_0)^2 e^{-\frac{Y^2}{4}}dY + e^{-\frac{2}{k}s_1}\int_{s_0}^{s_1}\left(C(\tilde{K})e^{\left(\frac{1}{k}-2\nu\right)\tilde{s}} + \sqrt{K_j}\sqrt{K_{j-1}}e^{\frac{1}{k}\tilde{s}}\right)d\tilde{s}\right)\\ &\leq e^{-\frac{2}{k}s_1} + C(\tilde{K})e^{-\frac{1}{k}s}e^{-2\nu s_1} + C\sqrt{K_{j-1}}\sqrt{K_j}e^{-\frac{1}{k}s_1} \leq \frac{K_j}{2}e^{-\frac{1}{k}s_1}\end{aligned}$$

where the last inequality holds true provided $K_j \geq 4$ and $K_j > CK_{j-1}$ for some universal constant C, and that s^* has been chosen large enough depending on $\tilde{K}$. The same inequality holds true for $j=0,...,2k+1$. Therefore, one can choose inductively the constants K_j one after another to satisfy these conditions, ending the proof of the Lemma.

□

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Lemma 33. *There exists a choice of constants $K_J \geq ... \geq K_0 > 0$, such that for any $\tilde{K} \geq K_J$, there exists s^* large enough such that if f is trapped on $[s_0,s_1]$ with $s_0 \geq s^*$ and satisfies (4.25) and (4.26), at time s_1 there holds for $j=0,...,J$:*

$$\int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\varepsilon(s_1)|^2}{\phi_{2k+1}^2(Z)}\frac{dY}{|Y|} \leq \frac{K_j}{2}e^{-\frac{3}{4k}s_1}, \quad \int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\varepsilon(s_1)|^2}{\phi_{2k+1/2}^2(Z)}\frac{dY}{|Y|} \leq \frac{K_j}{2}e^{-\frac{1}{2k}s_1}, \qquad (4.39)$$

bd:lossy impr

$$\int_{|Y|\geq 1}\frac{|\partial_Z^j\varepsilon(s_1)|^2}{\phi_0^2(Z)}\frac{dY}{|Y|} \leq \frac{K_j}{2}e^{-\frac{1}{2k}s_1} \quad \textit{for } j \geq 2k+1. \qquad (4.40)$$

bd:lossy impr

Proof. Note that by even symmetry it suffices to perform the estimates for $Y \geq 1$. We shall then use the identity $Y = |Y|$ in the following.

Step 1: *Proof of* (4.39). Let χ be a smooth cut-off function, with $\chi = 1$ for $Y \geq 2$ and $\chi = 0$ for $Y \leq 1$. Let $\ell = 2k+1$ or $\ell = 2k+1/2$. Set $\hat{\varepsilon}_j = (Y\partial_Y)^j\varepsilon$. Then $\hat{\varepsilon}_j$ solves from (4.35) since $[\partial_{YY}, Y\partial_Y] = 2\partial_{YY}$:

$$\begin{aligned}0 &= \partial_s\hat{\varepsilon}_j + \frac{Y}{2}\partial_Y\hat{\varepsilon}_j + \hat{\varepsilon}_j - 2F_k(Z)\hat{\varepsilon}_j - \partial_{YY}\hat{\varepsilon}_j + 2(F_k\hat{\varepsilon}_j - (Y\partial_Y)^j(F_k\varepsilon)) + 2(Y\partial_Y)^j((F_k(Z)-F[a])\varepsilon)\\ &\quad +2\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}\partial_{YY}(Y\partial_Y)^n\varepsilon + a_s(Y\partial_Y)^j\partial_a F[a] + (Y\partial_Y)^j\Psi - (Y\partial_Y)^j\varepsilon^2\end{aligned}$$

From the above identity we compute, performing integrations by parts, that

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac12\int \chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}\frac{dY}{Y}\right)\\
=&\int \chi\frac{\hat\varepsilon_j}{\phi_\ell(Z)}\frac{1}{Y}\frac{-\hat\varepsilon_j+2F_k\hat\varepsilon_j-\frac{Y}{2}\partial_Y\hat\varepsilon_j+\partial_{YY}\hat\varepsilon_j-2(F_k\hat\varepsilon_j-(Y\partial_Y)^j(F_k\varepsilon))+(Y\partial_Y)^j\varepsilon^2}{\phi_\ell(Z)}dY\\
&+\int \chi\frac{\hat\varepsilon_j}{\phi_\ell(Z)}\frac{1}{Y}\frac{-2(Y\partial_Y)^j((F_k(Z)-F[a])\varepsilon)-2\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}\partial_{YY}(Y\partial_Y)^n\varepsilon-(Y\partial_Y)^j(a_s\partial_aF[a]+\Psi)}{\phi_\ell(Z)}dY\\
&-\int \chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}\frac{1}{Y}\left(\frac{a_s}{a}\frac{(Z\partial_Z\phi_\ell)(Z)}{\phi_\ell(Z)}-\frac{k-1}{2k}\frac{(Z\partial_Z\phi_\ell)(Z)}{\phi_\ell(Z)}\right)dY\\
=&\int \chi\frac{\hat\varepsilon_j^2}{\phi_{2k+1}^2(Z)}\frac{-\phi_\ell(Z)+2F_k(Z)\phi_\ell(Z)-\frac{Z}{2k}\partial_Z\phi_\ell(Z)}{\phi_\ell(Z)}\frac{dY}{Y}-\int\chi\frac{|\partial_Y\hat\varepsilon_j|^2}{\phi_\ell^2(Z)}\frac{dY}{Y}\\
&+\frac14\int\partial_Y\chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}dY+\frac12\int\hat\varepsilon_j^2\partial_{YY}\left(\frac{\chi}{\phi_\ell^2(Z)}\frac1Y\right)dY\\
&-2\int\chi\frac{\hat\varepsilon_j(F_k(Z)\hat\varepsilon_j-(Y\partial_Y)^j(F[a]\varepsilon))}{\phi_\ell^2(Z)}\frac{dY}{Y}-2\int\chi\frac{\hat\varepsilon_j(Y\partial_Y)^j((F_k(Z)-F[a])\varepsilon)}{\phi_\ell^2(Z)}\frac{dY}{Y}\\
&-2\int\chi\frac{\hat\varepsilon_j\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}\partial_{YY}(Y\partial_Y)^n\varepsilon}{\phi_\ell^2(Z)}\frac{dY}{Y}-\frac{a_s}{a}\int\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}\frac{(Z\partial_Z\phi_\ell)(Z)}{\phi_\ell(Z)}\frac{dY}{Y}\\
&-\int\chi\frac{\hat\varepsilon_j}{\phi_\ell(Z)}\frac{(Y\partial_Y)^j(a_s\partial_aF[a]+\Psi)}{\phi_\ell(Z)}\frac{dY}{Y}+\int\chi\frac{\hat\varepsilon_j(Y\partial_Y)^j(\varepsilon^2)}{\phi_\ell^2(Z)}\frac{dY}{Y}
\end{aligned}
$$

where in the last equality, on the first line one has the main order linear effects, on the second their associated boundary terms, one the third and fourth the lower order linear effects, and on the last line the influence of the forcing and of the nonlinear effects. We now estimate all terms. For the first term from (4.3):

$$
\int\chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}\frac{-\phi_\ell(Z)+2F_k[b]\phi_\ell(Z)-\frac{Z}{2k}\partial_Z\phi_\ell(Z)}{\phi_\ell(Z)}\frac{dY}{Y}=-\frac{\ell-2k}{2k}\int\chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}\frac{dY}{Y}.
$$

The second term is dissipative and has a negative sign since χ is positive. For the third term, using (4.19), (4.23) and (4.24):

$$
\begin{aligned}
\left|\frac12\int\partial_Y\chi\frac{\hat\varepsilon_j^2}{\phi_\ell^2(Z)}dY\right| &\lesssim \|\frac{1}{\phi_\ell^2(Z)}\|_{L^\infty(1\le Y\le 2)}\|\hat\varepsilon_j\|^2_{L^2(1\le Y\le2)}\\
&\lesssim \||Z|^{-2\ell}\|_{L^\infty(1\le Y\le2)}\left(\sum_{i=0}^j\|Z^i\partial_Z^i\varepsilon\|^2_{L^2(1\le Y\le2)}\right)\\
&\lesssim \||e^{-\frac{k-1}{2k}s}Y|^{-2\ell}\|_{L^\infty(1\le Y\le2)}\left(\sum_{i=0}^j\|Z^i\partial_Z^i\tilde\varepsilon\|^2_{L^2(1\le Y\le2)}+\sum_{\ell=0}^{k-1}|c_{2\ell}|^2\right)\\
&\lesssim e^{\frac{k-1}{2k}2\ell s}\left(K_je^{-2(k-\frac12)s}+\tilde Ke^{-2(k-\frac12+\frac{1}{4k})s}\right)\lesssim K_je^{-\frac1k s} \qquad (4.41)
\end{aligned}
$$

for s^* large enough. For the fourth term, we first decompose:

$$
\partial_{YY}\left(\frac{\chi}{\phi_\ell^2(Z)Y}\right)=\partial_{YY}\chi\left(\frac{1}{\phi_\ell^2(Z)Y}\right)+2\partial_Y\chi\partial_Y\left(\frac{1}{\phi_\ell^2(Z)Y}\right)+\chi\partial_{YY}\left(\frac{1}{\phi_\ell^2(Z)Y}\right).
$$

Since one has

$$\left|\partial_{YY}\chi\left(\frac{1}{\phi_\ell^2(Z)Y}\right)+2\partial_Y\chi\partial_Y\left(\frac{1}{\phi_\ell^2(Z)Y}\right)\right|\lesssim |Z|^{-2\ell}1_{1\leq Y\leq 2}\lesssim e^{\frac{k-1}{k}\ell s}1_{1\leq Y\leq 2}$$

we claim that one can perform the very same estimate for the first two terms as (4.41), giving:

$$\left|\int \hat{\varepsilon}_j^2\left(\partial_{YY}\chi\left(\frac{1}{\phi_\ell^2(Z)Y}\right)+2\partial_Y\chi\partial_Y\left(\frac{1}{\phi_\ell^2(Z)Y}\right)\right)dY\right|\lesssim K_je^{-\frac1k s}.$$

For the last term, from a direct computation, for $|Y|\geq 1$, one has that:

$$\left|\partial_{YY}\left(\frac{1}{\phi_\ell^2(Z)}\frac{1}{Y}\right)\right|\lesssim \frac{1}{\phi_\ell^2(Z)}\frac{1}{Y^3}.$$

Therefore, if $\ell=2k+1$, we take some $0<\kappa\ll 1$ small enough and split the integral using some $Y^*\gg 1$ large enough, and use (4.19), (4.23) and (4.24):

$$\begin{aligned}
&\left|\int \chi\hat{\varepsilon}_j^2\partial_{YY}\left(\frac{1}{\phi_{2k+1}^2(Z)}\frac{1}{Y}\right)dY\right|\lesssim \int_1^{Y^*}\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}dY+\int_{Y^*}^{+\infty}\hat{\varepsilon}_j^2\frac{1}{\phi_{2k+1}^2(Z)}\frac{1}{|Y|^2}\frac{1}{Y}dY\\
\lesssim\ & \|\frac{1}{\phi_{2k+1}^2(Z)}\|_{L^\infty(1\leq Y\leq Y^*)}\|\hat{\varepsilon}_j\|^2_{L^2(1\leq Y\leq Y^*)}+\kappa\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\\
\lesssim\ & K_je^{\frac{k-1}{k}(2k+1)s}e^{-2(k-\frac12)s}+\kappa\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\lesssim K_je^{-\frac1k s}+\kappa\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}.
\end{aligned}$$

If $\ell=2k+1/2$, one uses the fact that $\phi_{2k+1}(Z)=Z^{1/2}\phi_{2k+1/2}$, so that

$$\frac{1}{\phi_{2k+1/2}^2(Z)}\frac{1}{Y^3}=\frac{1}{\phi_{2k+1}^2(Z)Y^2}e^{-\frac{k-1}{2k}s} \tag{4.42}$$

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to estimate using (4.20):

$$\left|\int \hat{\varepsilon}_j^2\partial_{YY}\left(\frac{1}{\phi_{2k+1/2}^2(Z)}\frac{1}{Y}\right)dY\right|\lesssim e^{-\frac{k-1}{2k}s}\int_{Y\geq 1}\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}dY\lesssim K_Je^{-(\frac12+\frac{1}{4k})s}\lesssim K_Je^{-\frac1k s}.$$

Collecting the above bounds one has proven that:

$$\left|\int \hat{\varepsilon}_j^2\partial_{YY}\left(\frac{\chi}{\phi_\ell^2(Z)}\frac1Y\right)dY\right|\lesssim\begin{cases}K_Je^{-\frac1k s}+\kappa\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y} & \text{for } \ell=2k+1,\\ K_Je^{-\frac1k s} \quad \text{for } \ell=2k+1/2.\end{cases}$$

We turn to the fifth term. We first estimate using Leibniz rule:

$$|F_k(Z)\hat{\varepsilon}_j-(Y\partial_Y)^j(F[a]\varepsilon)|\lesssim\sum_{n=0}^{j-1}|(Z\partial_Z)^{j-n}F[a]||(Y\partial_Y)^n\varepsilon|\lesssim |Z|^{2k}(1+|Z|)^{-4k}\sum_{n=0}^{j-1}|(Y\partial_Y)^n\varepsilon|.$$

If $\ell=2k+1$ we then use Cauchy-Schwarz and (4.20) to obtain, as $K_J>...>K_0$:

$$\begin{aligned}
\left|\int\chi\frac{\hat{\varepsilon}_j(F_k(Z)\hat{\varepsilon}_j-(Y\partial_Y)^j(F[a]\varepsilon))}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\right| &\leq C\left(\int_{Y\geq1}\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\right)^{\frac12}\sum_{n=0}^{j-1}\left(\int_{Y\geq1}\frac{((Y\partial_Y)^n\varepsilon)^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\right)^{\frac12}\\
&\leq \sqrt{K_j}\sqrt{K_{j-1}}e^{-\frac{3}{4k}s}.
\end{aligned}$$

If $\ell=2k+1/2$ we use the fact that

$$\frac{|Z|^{2k}}{\phi_{2k+1/2}^2(Z)(1+|Z|)^{4k}}\leq\frac{1}{\phi_{2k+1}^2(Z)},$$

which, combined with Cauchy-Schwarz and (4.20), gives:

$$\begin{aligned}\left|\int \chi \frac{\hat\varepsilon_j(F_k(Z)\hat\varepsilon_j-(Y\partial_Y)^j(F[a]\varepsilon))}{\phi^2_{2k+1/2}(Z)}\frac{dY}{Y}\right| &\lesssim \left(\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi^2_{2k+1}(Z)}\frac{dY}{Y}\right)^{\frac12}\sum_{n=0}^{j-1}\left(\int_{Y\geq 1}\frac{((Y\partial_Y)^n\varepsilon)^2}{\phi^2_{2k+1}(Z)}\frac{dY}{Y}\right)^{\frac12}\\ &\lesssim \sqrt{K_j}\sqrt{K_{j-1}}e^{-\frac{3}{4k}s}.\end{aligned}$$

One has then proven that:

$$\left|\int \chi \frac{\hat\varepsilon_j(F_k(Z)\hat\varepsilon_j-(Y\partial_Y)^j(F[a]\varepsilon))}{\phi^2_{2k+1/2}(Z)}\frac{dY}{Y}\right| \lesssim \begin{cases}\sqrt{K_{j-1}}\sqrt{K_j}e^{-\frac{3}{4k}s} & \text{for } \ell=2k+1,\\ \sqrt{K_{j-1}}\sqrt{K_j}e^{-\frac{3}{4k}s} & \text{for } \ell=2k+1/2.\end{cases}$$

We turn to the sixth term. Since from (4.7) for any $j\in\mathbb{N}$:

$$|(Z\partial_Z)^j(F_k(Z)-F[a])|\lesssim e^{-\frac{k-1}{k}s},$$

one has using Cauchy-Schwarz that

$$\left|\int \chi \frac{\hat\varepsilon_j(Y\partial_Y)^j((F_k(Z)-F[a])\varepsilon)}{\phi^2_\ell(Z)}\frac{dY}{Y}\right|\lesssim e^{-\frac{k-1}{k}s}\sum_{n=0}^{j}\int_{Y\geq 1}\frac{|(Y\partial_Y)^j\varepsilon|^2}{\phi^2_\ell(Z)}\frac{dY}{Y}$$

which using (4.20) implies that for some $0<\nu\ll 1$ be small enough:

$$\left|\int \chi \frac{\hat\varepsilon_j(Y\partial_Y)^j((F_k(Z)-F[a])\varepsilon)}{\phi^2_\ell(Z)}\frac{dY}{Y}\right|\lesssim \begin{cases}K_Je^{-(\frac{3}{4k}+\nu)s} & \text{for } \ell=2k+1,\\ K_je^{-(\frac{1}{2k}+\nu)s} & \text{for } \ell=2k+1/2.\end{cases}$$

For the seventh term, we use the fact that $\partial_{YY}=((Y\partial_Y)^2-Y\partial_Y)/Y^2$ to decompose:

$$\begin{aligned}\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}\partial_{YY}(Y\partial_Y)^n\varepsilon &= Y^{-1}\partial_Y\hat\varepsilon_j-\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+1}\varepsilon\\ &\quad+\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+2}\varepsilon-Y^{-2}(Y\partial_Y)^{j+1}\varepsilon.\end{aligned}$$

For the first term, we integrate by parts and find:

$$\int\chi\frac{\hat\varepsilon_jY^{-1}\partial_Y\hat\varepsilon_j}{\phi^2_\ell(Z)}\frac{dY}{Y}=\left|-\frac12\int\chi\hat\varepsilon_j^2\partial_Y\left(\frac{1}{\phi^2_\ell(Z)Y^2}\right)dY-\frac12\int\partial_Y\chi\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)}\frac{dY}{Y^2}\right|\lesssim\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)Y^3}dY.$$

If $\ell=2k+1$, we take $0<\kappa\ll 1$ small enough and Y^* large enough so that, using (4.19), (4.23) and (4.24):

$$\begin{aligned}&\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)Y^3}dY\lesssim\int_1^{Y^*}\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)Y^3}dY+\int_{Y^*}^{+\infty}\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)Y^3}dY\\ \lesssim\ &\|\frac{1}{\phi^2_\ell(Z)}\|_{L^\infty(1\leq Y\leq Y^*)}\|\hat\varepsilon_j\|^2_{L^2_\rho}+\kappa\int\chi\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)}\frac{dY}{Y}\lesssim K_Je^{-\frac1k s}+\kappa\int\chi\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)}\frac{dY}{Y}.\end{aligned}$$

If $\ell=2k+1/2$, we use (4.42), (4.20) and obtain for s^* large enough:

$$\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi^2_\ell(Z)Y^3}dY\lesssim e^{-\frac{k-1}{2k}s}\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi^2_{2k+1}(Z)}dY\lesssim K_Je^{-(\frac12+\frac{1}{4k})s}\leq e^{-\frac1k s},$$

If $\ell = 2k+1$, one estimates using (4.20) that

$$\int \chi \frac{\left|\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+1}\varepsilon + \sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+2}\varepsilon - Y^{-2}(Y\partial_Y)^{j+1}\varepsilon\right|^2}{\phi_{2k+1}^2}\frac{dY}{Y}$$

$$\lesssim \sum_{n=0}^{j-1}\int_{Y\geq 1}\frac{((Y\partial_Y)^j\varepsilon)^2}{Y^2\phi_{2k+1}^2}\frac{dY}{Y} \lesssim K_{j-1}e^{-\frac{3}{4k}s}.$$

If $\ell = 2k+1/2$, one estimates using (4.20) and (4.42) that

$$\int \chi \frac{\left|\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+1}\varepsilon + \sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}Y^{-2}(Y\partial_Y)^{n+2}\varepsilon - Y^{-2}(Y\partial_Y)^{j+1}\varepsilon\right|^2}{\phi_{2k+1/2}^2}\frac{dY}{Y}$$

$$\lesssim e^{-(\frac{1}{2}+\frac{1}{4k})s}\sum_{n=0}^{j-1}\int_{Y\geq 1}\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_{2k+1}^2}\frac{dY}{Y} \lesssim K_{j-1}e^{-(\frac{1}{2}+\frac{1}{k})s}.$$

One has therefore proven that, taking s^* large enough:

$$\left|\int \chi \frac{\hat{\varepsilon}_j\sum_{n=0}^{j-1}(Y\partial_Y)^{j-1-n}\partial_{YY}(Y\partial_Y)^n\varepsilon}{\phi_\ell^2(Z)}\frac{dY}{Y}\right|$$

$$\leq \begin{cases} CK_{j-1}e^{-(\frac{3}{4k})s} + \kappa\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y} & \text{for } \ell = 2k+1, \\ e^{-\frac{1}{k}s} \quad \text{for } \ell = 2k+1/2. \end{cases}$$

Since $|Z\partial_Z\phi_\ell/\phi_\ell|$ is bounded, one infers for the eighth term from (4.37), (4.20) and (4.21) that for $\nu > 0$ small enough:

$$\left|\frac{a_s}{a}\int\chi\frac{\hat{\varepsilon}_j^2}{\phi_\ell^2(Z)}\frac{(Z\partial_Z\phi_\ell)(Z)}{\phi_\ell(Z)}\frac{dY}{Y}\right| \lesssim \begin{cases} K_J e^{-(\frac{3}{4k}+\nu)s} & \text{for } \ell = 2k+1, \\ K_j e^{-(\frac{1}{2k}+\nu)s} & \text{for } \ell = 2k+1/2. \end{cases}$$

For the ninth term, from (4.9):

$$\int_{Y\geq 1}\frac{((Z\partial_Z)^j\Psi)^2}{\phi_\ell^2}\frac{dY}{Y} \lesssim e^{-\frac{k-1}{k}2s}\int_{Y\geq 1}\frac{|Z|^{8k-4}(1+|Z|)^{-12k}}{|Z|^{2\ell}(1+|Z|)^{-8k}}\frac{dY}{Y}$$

$$\lesssim e^{-\frac{k-1}{k}2s}\int_{e^{-\frac{k-1}{2k}s}}^{+\infty}|Z|^{8k-4-2\ell}(1+|Z|)^{-4k}\frac{dZ}{Z} \lesssim e^{-\frac{k-1}{k}2s} \lesssim e^{-\frac{3}{2k}s}$$

and from (4.37) and (4.13):

$$\int_{Y\geq 1}\frac{|a_s(Z\partial_Z)^j\partial_a F[a](Z)|^2}{\phi_\ell^2(Z)}\frac{dY}{Y} \lesssim |a_s|^2\int_{Y\geq 1}\chi\frac{|Z|^{4k}(1+|Z|)^{-8k}}{|Z|^{2\ell}(1+|Z|)^{-8k}}\frac{dY}{Y}$$

$$\lesssim e^{-(1+\frac{1}{2k})s}\int_{e^{-\frac{k-1}{2k}s}}^{+\infty}|Z|^{4k-2\ell}\frac{dZ}{Z} \lesssim e^{-\frac{3}{2k}s}.$$

Therefore, using Cauchy Schwarz, (4.20) and (4.21), for $\nu > 0$ small enough

$$\left|\int\chi\frac{\hat{\varepsilon}_j}{\phi_\ell(Z)}\frac{(Y\partial_Y)^j(a_s\partial_a F[a]+\Psi)}{\phi_\ell(Z)}\frac{dY}{Y}\right| \lesssim \begin{cases} K_J e^{-(\frac{3}{4k}+\nu)s} & \text{for } \ell = 2k+1, \\ K_j e^{-(\frac{1}{2k}+\nu)s} & \text{for } \ell = 2k+1/2. \end{cases}$$

Finally, for the last term, using (4.31), (4.20) and (4.21) for $\nu>0$ small enough.

$$\begin{aligned}\left|\int \chi\frac{\hat\varepsilon_j (Y\partial_Y)^j(\varepsilon^2)}{\phi_\ell^2(Z)}\frac{dY}{Y}\right| &\lesssim \left(\int_{Y\geq 1}\frac{\hat\varepsilon_j^2}{\phi_\ell^2}\frac{dY}{Y}\right)^{\frac12}\left(\sum_{n=0}^{j-1}\|(Y\partial_Y)^n\varepsilon\|_{L^\infty}\right)\left(\sum_{n=0}^{j}\left(\int_{Y\geq 1}\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_\ell^2}\frac{dY}{Y}\right)^{\frac12}\right)\\ &\lesssim \begin{cases} K_J e^{-(\frac{3}{4k}+\nu)s} & \text{for } \ell=2k+1,\\ K_j e^{-(\frac{1}{2k}+\nu)s} & \text{for } \ell=2k+1/2.\end{cases}\end{aligned}$$

Combining all the above estimates, for any $\kappa>0$, for some $\nu>0$ small enough depending only on k, and for s^* large enough, then gives the two identities:

$$\frac{d}{ds}\left(\int\chi\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}\right)\leq-\left(\frac1k-\kappa\right)\int\chi\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}+C\sqrt{K_{j-1}}\sqrt{K_j}e^{-\frac{3}{4k}s}$$

$$\frac{d}{ds}\left(\int\chi\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_{2k+1/2}^2(Z)}\frac{dY}{Y}\right)\leq-\frac{1}{2k}\int\chi\frac{((Y\partial_Y)^j\varepsilon)^2}{\phi_{2k+1}^2(Z)}\frac{dY}{Y}+CK_je^{-(\frac{1}{2k}+\nu)s}.$$

with the convention $K_{-1}=1$. After integration in time using (4.25), and (4.23) for the zone $1\leq Y\leq 2$, the above differential inequality shows that, upon choosing the constants K_j inductively one after another, (4.39) holds true.

Step 2: *Proof of* (4.40). Let $j\geq 2k+1$. We claim that this bound can be proved the very same way as in step 1. The main argument is the following. $\bar\varepsilon_j:=\partial_Z^j\varepsilon$ solves from (4.35):

$$\begin{aligned}&\partial_s\bar\varepsilon_j+\frac{j+2k}{2k}\bar\varepsilon_j+\frac{Y}{2}\partial_Y\bar\varepsilon_j-2F_k(Z)\bar\varepsilon_j-\partial_{YY}\bar\varepsilon_j\\ =\ & 2(\partial_Z^j(F_k(Z)\varepsilon)-F_k(Z)\bar\varepsilon_j)-2\partial_Z^j((F_k(Z)-F[a])\varepsilon)-\partial_Z^j(a_s\partial_aF[a]+\Psi).\end{aligned}$$

The function $\phi_0(Z)$ is a stable eigenfunction of the operator without dissipation in the left hand side:

$$\left(\frac{j+2k}{2k}+\frac{Y}{2}\partial_Y-2F_k(Z)\right)\phi_0(Z)=\frac{j-2k}{2k}\phi_0$$

and in particular $(j-2k)/2k\geq 1/(2k)$ since $j\geq 2k+1$. As $1/(2k)>1/(4k)$, one can then prove (4.40) as in Step 1, checking that all the terms in the right hand side of the equation for w are lower order, and that the boundary terms at the origin are controlled by (4.23).

□

We can now end the proof of Proposition 29.

Proof of Proposition 29. We assume that ε_0 is fixed satisfying the orthogonality conditions (4.19) and the initial bounds (4.27), (4.26) and (4.25). We assume $K_J\geq ...\geq K_0>0$ are fixed so that the Lemmas 32 and 33 hold true. We let $\tilde K\geq K_J$ to be fixed at the end of the proof. We use Lemma 28 to relate the initial coefficients $(\bar c_{2\ell}(s_0))_{0\leq\ell\leq k-1}$ and $(c_{2\ell}(s_0))_{0\leq\ell\leq k-1}$. We consider for all possible initial values $(c_{2\ell}(s_0))_{0\leq\ell\leq k-1}\in B(\tilde Ke^{-(k-\frac12+\frac{1}{4k})s_0})$ the corresponding solution to (3.8) with initial datum (4.29).

We let the exit time $s_e\in[s_0,+\infty]$ be the supremum of times $s_1\geq s_0$ such that the solution is trapped on $[s_0,s_1]$. From Definition 27 and Lemmas 32 and 33 and a continuity argument, if $s_e<+\infty$ then necessarily the inequality (4.24) is saturated at time s_e:

$$|(c_{2\ell}(s_e))_{0\leq\ell\leq k-1}|=\tilde Ke^{-(k-\frac12+\frac{1}{4k})s_e}.$$

Hence the exit mapping Φ which to $(c_{2\ell}(s_0))_{0\leq \ell\leq k-1}\in B(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$ such that $s_e<+\infty$ associates

$$\Phi((c_{2\ell}(s_0))_{0\leq \ell\leq k-1})=e^{-(k-\frac12+\frac{1}{4k})(s_0-s_e)}(c_{2\ell}(s_e))_{0\leq \ell\leq k-1}$$

is a mapping whose domain is a subset of the ball $B(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$ and whose range lies in its boundary the sphere $S(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$.

We claim that one can choose $\tilde K\geq K_J$ such that for s^* large enough, Φ is the identity map on the sphere $S(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$. Indeed, for any $\tilde K\geq K_J$ if initially $|(c_{2\ell}(s_0))_{0\leq \ell\leq k-1}|=\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0}$ then for s^* large enough one computes from (4.37) the outgoing flux condition

$$\begin{aligned}
&\partial_s\left(\frac{\sum_{\ell=0}^{2k-1}|c_\ell(s)|^2}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s}}\right)(s_0)=2\left(k-\frac12+\frac{1}{4k}\right)\frac{\sum_{\ell=0}^{2k-1}|c_\ell(s_0)|^2}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s_0}}+2\frac{\sum_{\ell=0}^{2k-1}c_\ell(s_0)\partial_s c_\ell(s_0)}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s_0}}\\
=\ &2\left(k-\frac12+\frac{1}{4k}\right)\frac{\sum_{\ell=0}^{2k-1}|c_\ell(s_0)|^2}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s_0}}+2\frac{\sum_{\ell=0}^{2k-1}c_\ell(s_0)\left(-\frac{\ell-2}{2}c_\ell(s_0)+O(K_Je^{-(k-\frac12+\frac{1}{4k})s_0})\right)}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s_0}}\\
=\ &\frac{2}{\tilde K^2 e^{-2(k-\frac12+\frac{1}{4k})s_0}}\sum_{\ell=0}^{2k-1}\left(k-\frac12+\frac{1}{4k}-\frac{\ell-2}{2}\right)|c_\ell(s_0)|^2+O\left(\frac{K_J}{\tilde K}\right)\\
\geq\ &1+\frac{1}{4k}+O\left(\frac{K_J}{\tilde K}\right)
\end{aligned}$$

One then chooses $\tilde K\geq K_J$ depending on K_J such that the above quantity is ≥ 1. Then in that case the solution leaves the trapped regime immediately at time s_0 since (4.24) fails right after s_0. Hence $s_e=s_0$, and $\Phi((c_{2\ell}(s_0))_{0\leq \ell\leq k-1})=((c_{2\ell}(s_0))_{0\leq \ell\leq k-1})$ is indeed the identity map on the sphere $S(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$.

From the above outgoing flux condition and a standard continuity argument, Φ is continuous on its domain. Φ is thus a continuous mapping from the ball onto its boundary, that is the identity on its boundary. The domain of Φ cannot be the entire ball, as this would contradict Brouwer's fixed point Theorem. This means that there exists one $(c_{2\ell}(s_0))_{0\leq \ell\leq k-1}\in B(\tilde K e^{-(k-\frac12+\frac{1}{4k})s_0})$ such that $s_e=+\infty$, which ends the proof of the Proposition.

□

sec:LFH

4.2. The coupled linear heat equation

We now turn to the proof of Proposition 10. We keep the notations of the previous section. The proof is similar and simpler to the one of the related Theorem 1 concerning ξ. Indeed, there are no nonlinear effects and instabilities. For a solution ζ to (LFH) we start by going again to self-similar variables

$$Y=\frac{y}{\sqrt{T-t}},\quad s=-\log(T-t),\quad g(s,y)=(T-t)^4\zeta(x,t),\quad Z:=\frac{a^*Y}{e^{\frac{k-1}{2k}s}},$$

Then g solves the second equation in (3.8). Throughout this section, we assume that k and J are fixed, and that f is the solution to the first equation in (3.8) satisfying the properties of Proposition 29. In particular, in the current subsection, all the constants appearing in the previous subsection are considered as fixed and universal. Without loss of generality for the argument, since its exact value will never play a role, we fix:

$$a^*=1.$$

In particular since $f = F_k(a^* e^{-(k-1)/(2k)} Y)$ at leading order, and since the dissipation is lower order, the main order equation reads in Z variable:

$$g_s + \mathcal{M}_Z g = 0, \quad \mathcal{M}_Z := 4 - 4F_k(aZ) + \frac{Z}{2k}\partial_Z.$$

pr:mathcalMZ **Proposition 34** (Spectral structure for $\mathcal{M}_Z$)**.** *The operator $\mathcal{M}_Z$ acting on $\mathcal{C}^\infty(\mathbb{R})$ has point spectrum $\Upsilon(\mathcal{M}_Z) = \{\ell/(2k), \quad \ell \in \mathbb{N}\}$ and the associated eigenfunctions are*

$$\psi_\ell := \frac{Z^\ell}{(1+(aZ)^{2k})^4}, \quad \mathcal{M}_Z \psi_\ell = \psi_\ell.$$

Proof. The result comes from a direct computation. □

$\mathcal{M}_Z$ having a nontrivial kernel and non-negative spectrum, we expect formally the solution to approach an element of its kernel as $s \to +\infty$. Near the origin, as $f = F_k(a^* e^{-(k-1)/(2k)} Y)$ at leading order and since $F_k(0) = 1$, the main order equation for g in the zone $|Y| \lesssim 1$ is:

$$g_s + \mathcal{M}_\rho g = 0, \quad \mathcal{M}_\rho := \frac{Y}{2}\partial_Y - \partial_{YY}.$$

pr:Mrho **Proposition 35** (Linear structure at the origin, (see e.g. [24])**.** *The operator $\mathcal{M}_\rho$ is essentially self-adjoint on $C_0^2(\mathbb{R}) \subset L^2(\rho)$ with compact resolvant. The space H^2_ρ is included in the domain of its unique self-adjoint extension. Its spectrum is $\Upsilon(\mathcal{M}_\rho) = \{\ell/2, \ \ell \in \mathbb{N}\}$. The eigenvalues are all simple and the associated orthonormal basis of eigenfunctions is given by the family of Hermite polynomials $(h_\ell)_{\ell\in\mathbb{N}}$ defined by (4.6).*

We now perform a bootstrap analysis and decompose the solution according to:

$$g = b(s) F_k^4(Z^*) + \varepsilon \tag{4.43}$$ eq:decompo v

where $Z^* = a^* e^{-\frac{k-1}{2k}s} Y$ and without loss of generality since the value of a never plays a role we take $a^* = 1$ for simplicity, which fixes $Z^* = e^{-\frac{k-1}{2k}s} Y = Z$, and where b is fixed through the orthogonality condition for the $\langle \cdot, \cdot \rangle_\rho$ scalar product:

$$\varepsilon \perp h_0. \tag{4.44}$$ id:ortho2

r:bootstrap2 **Proposition 36.** *There exist $L_J \geq ... \geq L_0 > 0$ and $s_0 \gg 1$ large enough, such that for $\varepsilon(s_0) = \varepsilon_0$ satisfying the orthogonality condition (4.44) and*

$$\sum_{j=0}^J \int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j \varepsilon_0|^2}{\psi_1^2(Z)} \frac{dY}{|Y|} \leq e^{-\frac{3}{4k}s_0}, \quad \sum_{j=0}^J \int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j \varepsilon_0|^2}{\psi_{1/2}^2(Z)} \frac{dY}{|Y|} \leq e^{-\frac{1}{2k}s_0}, \tag{4.45}$$ bd:bootstrap

$$\sum_{j=1}^J \int_{|Y|\geq 1} \frac{|\partial_Z^j \varepsilon_0|^2}{\psi_0^2(Z)} \frac{dY}{|Y|} \leq e^{-\frac{1}{2k}s_0}, \tag{4.46}$$ bd:bootstrap

$$\|\varepsilon_0\|_{L^2_\rho} \leq e^{-\frac{1}{2}s_0}, \quad \|\partial_Z^j \varepsilon_0\|_{L^2_\rho} \leq e^{-\frac{1}{2k}s_0}, \quad \text{for } j = 1, ..., J, \tag{4.47}$$ bd:bootstrap

and an initial parameter $3/4 \leq b \leq 5/4$ the solution g to the second equation in (3.8) then satisfies for all $s \geq s_0$, for $j = 0, ..., J$:

$$\int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j \varepsilon|^2}{\psi_1^2(Z)} \frac{dY}{|Y|} \leq L_j e^{-\frac{3}{4k}s}, \quad \int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j \varepsilon|^2}{\psi_{1/2}^2(Z)} \frac{dY}{|Y|} \leq L_j e^{-\frac{1}{2k}s}, \tag{4.48}$$ bd:bootstrap

$$\int_{|Y|\geq 1} \frac{|\partial_Z^j \varepsilon|^2}{\psi_0^2(Z)} \frac{dY}{|Y|} \leq L_j e^{-\frac{1}{2k}s}, \tag{4.49}$$ bd:bootstrap

$$\|\varepsilon\|_{L^2_\rho} \le \sqrt{L_0} e^{-\frac{1}{2}s}, \quad \|\partial_Z^j \varepsilon\|_{L^2_\rho} \le L_J e^{-\frac{1}{2k}s}, \quad \textit{for } j=1,...,J, \tag{4.50}$$

bd:bootstrap

and there exists an asymptotic parameter $1/2 \le b^* \le 3/2$ *such that* $|b-b^*| \lesssim e^{-(k-1)s}$.

The rest of the subsection is devoted to the proof of Proposition 36. In what follows we assume that g solves (3.8) and satisfies the bounds of Proposition 36 on some time interval $[s_0, s_1]$, and perform modulation and energy estimates to improve those bounds. Proposition 36 is then proved at the end of the subsection.

Lemma 37. *There holds on* $[s_0, s_1]$ *for* $j=0,...,J-1$*:*

$$|\partial_Z^j \varepsilon| \lesssim L_J e^{-\frac{1}{4k}s}(1+|Z|)^{\frac{1}{2}-8k}. \tag{4.51}$$

2bd:Linfty

Proof. First, since $\partial_Z = e^{\frac{k-1}{2k}s}\partial_Y$, one deduces from (4.50) that $\|\varepsilon\|_{H^j_\rho} \lesssim e^{-s/(2k)}$. Therefore, from Sobolev,

$$|\partial_Z^j \varepsilon| \lesssim L_J e^{-\frac{1}{2k}s} \quad \text{for} \quad |Y| \le 1. \tag{4.52}$$

bd:2Linftyori

Then, applying (A.2), as $|Z\partial_Z \psi_{1/2}| \lesssim |\psi_{1/2}| \lesssim |Z\partial_Z \psi_{1/2}|$:

$$\begin{aligned} \|\frac{Z^j \partial_Z^j \varepsilon}{\psi_{1/2}(Z)}\|^2_{L^\infty(\{|Y|\ge 1\})} &\lesssim \|\frac{Z^j \partial_Z^j \varepsilon}{\psi_{1/2}(Z)}\|^2_{L^2\left(\{|Y|\ge 1\}, \frac{dY}{|Y|}\right)} + \|Z\partial_Z(\frac{Z^j \partial_Z^j \varepsilon}{\psi_{1/2}(Z)})\|^2_{L^2\left(\{|Y|\ge 1\}, \frac{dY}{|Y|}\right)} \\ &\lesssim \sum_{k=0}^{j+1} \|\frac{(Z\partial_Z)^j \varepsilon}{\psi_{1/2}(Z)}\|^2_{L^2\left(\{|Y|\ge 1\}, \frac{dY}{|Y|}\right)} \lesssim L_J e^{-\frac{1}{2k}s} \end{aligned}$$

from (4.48). From the definition of ψ this implies that

$$|\partial_Z^j \varepsilon| \lesssim L_J e^{-\frac{1}{2k}s}|Z|^{\frac{1}{2}-j}(1+|Z|)^{-8k} \quad \text{for} \quad |Y| \ge 1. \tag{4.53}$$

bd:2Linftyawa

Finally, since $\psi_{1/2}(Z) = |Z|^{1/2}\psi_0(Z)$, for $j \ge 1$ one has the inequality

$$\frac{|Z|}{\psi_0(Z)} \lesssim \frac{1}{\psi_0(Z)} + \frac{|Z|^{j+1}}{\psi_{1/2}(Z)}.$$

This implies from (4.48) and (4.49) the estimate for $j \ge 1$:

$$\int_{|Y|\ge 1} \frac{|Z\partial_Z(\partial_Z^j \varepsilon)|^2}{\psi_0^2(Z)} \frac{dY}{|Y|} \lesssim \int_{|Y|\ge 1} \frac{|\partial_Z^{j+1}\varepsilon|^2}{\psi_0^2(Z)} \frac{dY}{|Y|} + \int_{|Y|\ge 1} \frac{|Z^{j+1}\partial_Z^{j+1}\varepsilon)|^2}{\psi_0^2(Z)} \frac{dY}{|Y|} \lesssim L_J e^{-\frac{1}{2k}s}.$$

Thus, from (A.2) one obtains for $j \ge 1$:

$$|\partial_Z^j \varepsilon| \lesssim L_J e^{-\frac{1}{4k}s}(1+|Z|)^{-8k} \quad \text{for } |Y| \ge 1. \tag{4.54}$$

bd:2Linftyawa

The bounds (4.52), (4.53) and (4.54) then imply the desired result (4.51).

□

From (3.8) and (4.43), the evolution of ε is given by the following equation:

$$b_s F_k^4(Z) + \varepsilon_s + \mathcal{M}\varepsilon + \tilde{\mathcal{M}}\varepsilon + \Psi = 0, \tag{4.55}$$

eq:evo e2

where

$$\mathcal{M} := 4 - 4F_k(Z) + \frac{Y}{2}\partial_Y - \partial_{YY}, \quad \tilde{\mathcal{M}} := -4(f - F_k(Z)),$$

and

$$\Psi := -4bF_k^4(Z)(f-F_k(Z)) - b(e^{-\frac{k-1}{2k}s})^2 \partial_{ZZ}(F_k^4)(Z)$$

From the various bounds of Proposition 29 and Lemma 30 we infer the following estimates for the above objects. We recall that the constants of the previous Subsection are fixed and considered as universal.

Lemma 38. *One has the following bounds:*

$$|\partial_Z^j(f-F_k(Z))|\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac12-2k-j}, \tag{4.56}$$ bd:tildemath

$$\|f-F_k(Z)\|_{L^2_\rho}\lesssim e^{-(k-1)s}, \tag{4.57}$$ bd:tildemath

$$|(Z\partial_Z)^j(\Psi)|\lesssim e^{-\frac{1}{4k}s}(1+|Z|)^{\frac12-10k}, \quad j=0,1,2$$

$$\|\Psi\|_{L^2_\rho}\lesssim e^{-(k-1)s}, \quad \|\partial_Z\Psi\|_{L^2_\rho}\lesssim e^{-(k-\frac32+\frac{1}{2k})s} \quad \textit{and} \quad \|\partial_Z^j\Psi\|_{L^2_\rho}\lesssim e^{-\frac{1}{2k}s} \quad \textit{for } j\geq 2, \tag{4.58}$$ 2bd:Psirho

$$\int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\Psi|^2}{\psi_1^2(Z)}\frac{dY}{|Y|}+\int_{|Y|\geq 1}\frac{|(Y\partial_Y)^j\Psi|^2}{\psi_{1/2}^2(Z)}\frac{dY}{|Y|}\lesssim e^{-\frac{3}{4k}s}, \quad j=0,...,J, \tag{4.59}$$ 2bd:PsiZ

$$\int_{|Y|\geq 1}\frac{|\partial_Z^j\Psi|^2}{\psi_0^2(Z)}\frac{dY}{|Y|}\lesssim e^{-\frac{1}{2k}s}, \quad j=1,...,J. \tag{4.60}$$ 2bd:PsiZ2

Proof. These are direct bounds implied by the estimates of Proposition 29, and the behaviour of the corresponding eigenfunctions given by Propositions 24 and 34.

□

We start by computing the evolution of the modulation parameter.

em:modulation **Lemma 39.** *There exists $C>0$ independent of $L_1,...,L_J$ such that for s_0 large enough on $[s_0,s_1]$:*

$$|b_s|\leq Ce^{-(k-1)s}. \tag{4.61}$$ 2bd:mod

Proof. One takes the scalar product between (4.55) and $h_0=1$ in L^2_ρ, yielding using (4.44):

$$b_s\langle F^4(Z^*),h_0\rangle_\rho=\langle -\mathcal{M}\varepsilon-\tilde{\mathcal{M}}\varepsilon-\Psi,h_0\rangle_\rho.$$

First, since $|F_k(Z)-1|\lesssim Z^{2k}(1+|Z|)^{-2k}$ one has the projection in the left hand side is non-degenerate:

$$\langle F^4(Z^*),h_0\rangle_\rho=1+O(e^{-(k-1)s})$$

and that, since $\mathcal{M}=\mathcal{M}_\rho+4(1-F(Z^*))$:

$$\langle\mathcal{M}\varepsilon,h_0\rangle_\rho=0+\langle\varepsilon,4(1-F(Z^*))h_0\rangle_\rho=O(\sqrt{L_0}e^{-(k-\frac12)s}).$$

using (4.44), (4.50) and the fact that $|Z^{2k}|\lesssim e^{-(k-1)s}|Y|^{2k}$. Then, from (4.57) one computes:

$$\langle\tilde{\mathcal{M}}\varepsilon,h_0\rangle_\rho=-4\langle\varepsilon,(f-F(Z^*))h_0\rangle_\rho=O(\sqrt{L_0}e^{-(k-\frac12)s}).$$

Finally, using (4.58):

$$\langle\Psi,h_0\rangle_\rho=O(e^{-(k-1)s}).$$

From the above identities one gets the desired result (4.61).

□

We then perform an energy estimate in the zone $|Y|\lesssim 1$.

em:energyrho **Lemma 40.** *There exist constants $L_J\geq...\geq L_0>0$ such that for s_0 large enough, at time s_1:*

$$\|\varepsilon\|_{L^2_\rho}\leq\frac{\sqrt{L_0}}{2}e^{-\frac12 s_1}, \quad \|\partial_Z^j\varepsilon\|_{L^2_\rho}\leq\frac{L_J}{2}e^{-\frac{1}{2k}s_1}, \quad \textit{for } j=1,...,J. \tag{4.62}$$ bd:bootstrap

Proof. **Step 1** *Estimate for* ε. Let $0<\kappa\ll 1$ be an arbitrarily small constant. Then from (4.55) we infer that:

$$\frac{d}{ds}\frac{1}{2}\int \varepsilon^2 e^{-\frac{Y^2}{4}}dY=\int \varepsilon\left(-b_sF^4(Z^*)-\mathcal{M}\varepsilon-\tilde{\mathcal{M}}\varepsilon-\Psi\right)e^{-\frac{Y^2}{4}}dY.$$

For the first term, from (4.44) and (4.61), using Cauchy-Schwarz:

$$b_s\int \varepsilon F^4(Z^*)e^{-\frac{Y^2}{4}}dY=b_s\int\varepsilon(1-F^4(Z^*))e^{-\frac{Y^2}{4}}dY=O(\sqrt{L_0}e^{-(2(k-1)+\frac12)s}).$$

For the second, as $\mathcal{M}=\mathcal{M}_\rho+4(1-F_k(Z))$, using (4.51):

$$\begin{aligned}&-\int\varepsilon\mathcal{M}\varepsilon e^{-\frac{Y^2}{4}}dY=-\int\varepsilon\mathcal{M}_\rho\varepsilon e^{-\frac{Y^2}{4}}dY+4\int(F_k(Z)-1)\varepsilon^2e^{-\frac{Y^2}{4}}dY\\ \le\ &-\frac12\int\varepsilon^2e^{-\frac{Y^2}{4}}dY+O(\|\varepsilon\|_{L^2_\rho}\|\varepsilon\|_{L^\infty}\|F_k(Z)-1\|_{L^2_\rho})\le-\frac12\int\varepsilon^2e^{-\frac{Y^2}{4}}dY+O(\sqrt{L_0}L_Je^{-(k-\frac12+\frac{1}{4k})s}).\end{aligned}$$

For the third, from (4.56) and (4.50):

$$\left|\int\varepsilon\tilde{\mathcal{M}}\varepsilon e^{-\frac{Y^2}{4}}dY\right|\lesssim\|\varepsilon\|^2_{L^2_\rho}\|f-F(Z^*)\|_{L^\infty}\lesssim e^{-(1+\frac{1}{4k})s}.$$

Finally, for the fourth, from (4.58):

$$\left|\int\varepsilon\Psi e^{-\frac{Y^2}{4}}dY\right|\lesssim\|\varepsilon\|_{L^2_\rho}\|\Psi\|_{L^2_\rho}\lesssim\sqrt{L_0}e^{-(k-\frac12)s}.$$

Combining the above expressions one obtains as $L_J\ge ...\ge L_0>0$, assuming $L_J\ge1$ without loss of generality:

$$\frac{d}{ds}\left(\int\varepsilon^2e^{-\frac{Y^2}{4}}dY\right)\le-\int\varepsilon^2e^{-\frac{Y^2}{4}}dY+CL_je^{-(1+\frac{1}{4k})s}.$$

When reintegrated in time, using (4.47) this gives:

$$\int\varepsilon^2e^{-\frac{Y^2}{4}}dY\le e^{-s}+CL_Je^{-\frac{1}{4k}s_0}e^{-s}\le\frac{L_0^2}{4}e^{-s}$$

provided $L_0>2$ and s_0 has been taken large enough.

Step 2: *Higher order derivatives.* Let $1\le j\le J$ and define $w:=\partial_Z^j\varepsilon$. Then from (4.55) the evolution of w is

$$w_s+\frac{j}{2k}+\mathcal{M}_\rho w+4(1-F_k(Z))w+4(F_k(Z)w-\partial_Z^j(F_k(Z)\varepsilon))-\partial_Z^j((f-F_k(Z))\varepsilon)+\partial_Z^j(\Psi+b_sF_k(Z))=0.$$

From the above equation, we infer that:

$$\begin{aligned}\frac{d}{ds}\frac12\|w\|^2_{L^2_\rho}\ =\ &-\frac{j}{2k}\|w\|^2_{L^2_\rho}-\|\partial_Yw\|^2_{L^2_\rho}+4\langle(F_k(Z)-1)w,w\rangle_\rho+4\langle\partial_Z^j(F_k(Z)\varepsilon)-F_k(Z)w,w\rangle_\rho\\ &+\langle\partial_Z^j((f-F_k(Z))\varepsilon),w\rangle_\rho-\langle\partial_Z^j(\Psi+b_sF_k(Z)),w\rangle_\rho.\end{aligned}$$

Let $0<\nu\ll1$ be a small constant to be fixed later on. We estimate all terms in the right hand side. First, from (A.1) and (4.50) one has:

$$-\|\partial_Yw\|^2_{L^2_\rho}\le-ce^{-\nu s}\|Yw\|^2_{L^2_\rho}+e^{-\nu s}\|w\|^2_{L^2_\rho}\le-ce^{-\nu s}\|Yw\|^2_{L^2_\rho}+CL_je^{-\left(\frac1k+\nu\right)s},\quad c>0.$$

Next, since $|1-F_k(Z)|\lesssim|Z^{2k}|(1+|Z|)^{-2k}\lesssim e^{-(k-1)s/k}|Y|^2$ we infer that

$$|\langle(F_k(Z)-1)w,w\rangle_\rho|\lesssim e^{-\frac{k-1}{k}s}\|Yw\|^2_{L^2_\rho}.$$

From (4.50), as $\partial_Z^j F_k$ is bounded, we infer using Cauchy-Schwarz that:

$$\left|\langle \partial_Z^j (F_k(Z)\varepsilon) - F_k(Z)w, w\rangle_\rho\right| \lesssim \begin{cases} \sqrt{L_0}\sqrt{L_1}e^{-\left(\frac12+\frac{1}{2k}\right)s} & \text{for } j=1, \\ \sqrt{L_{j-1}}L_J e^{-\frac1k s} & \text{for } j\geq 2. \end{cases}$$

Using (4.50) and (4.56) we infer:

$$\left|\langle \partial_Z^j ((f-F_k(Z))\varepsilon), w\rangle_\rho\right| \lesssim \left(\sum_{i=0}^J \|\partial_Z^i \varepsilon\|_{L^2_\rho}^2\right)\left(\sum_{i=0}^J \|\partial_Z^i (f-F_k(Z))\|_{L^\infty}\right) \lesssim L_j e^{-\left(\frac1k+\frac{1}{4k}\right)s}.$$

Finally, from (4.50) and (4.61), (4.58) and Cauchy-Schwarz:

$$\left|\langle \partial_Z^j (\Psi + b_s F_k(Z)), w\rangle_\rho\right| \lesssim \begin{cases} \sqrt{L_1}e^{-\left(\frac12+\frac{1}{k}\right)s} & \text{for } j=1, \\ L_J e^{-\frac1k s} & \text{for } j\geq 2. \end{cases}$$

Collecting the above estimates, one finds finally that there exists $\nu>0$ depending on k, such that for s_0 large enough:

$$\frac{d}{ds}\|w\|_{L^2_\rho} \leq \begin{cases} -\frac1k \|w\|_{L^2_\rho} + O(L_j e^{-\left(\frac1k+\nu\right)s}) & \text{for } j=1, \\ -\frac{j}{k}\|w\|_{L^2_\rho} + \sqrt{L_{j-1}}L_J e^{-\frac1k s} & \text{for } j\geq 2. \end{cases}$$

Reintegrated in time using (4.47), this yields the desired bound (4.50) for $j\geq 1$, upon choosing the constants L_1,...,L_J inductively.

□

energyoutside **Lemma 41.** *There exist constants $K_J \geq ... \geq L_0 > 0$ such that for s_0 large enough, at time s_1, for $j=0,...,J$:*

$$\int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j\varepsilon|^2}{\psi_1^2(Z)}\frac{dY}{|Y|} \leq K_j e^{-\frac{3}{4k}s}, \quad \int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j\varepsilon(s_1)|^2}{\psi_{1/2}^2(Z)}\frac{dY}{|Y|} \leq \frac{K_{j+1}}{2}e^{-\frac{1}{2k}s}, \tag{4.63}$$ bd:2eweightin

$$\int_{|Y|\geq 1} \frac{|\partial_Z^j\varepsilon|^2}{\psi_0^2(Z)}\frac{dY}{|Y|} \leq \frac{K_j}{2}e^{-\frac{1}{2k}s} \quad \textit{for} \quad j\geq 1. \tag{4.64}$$ bd:2eweightin

Proof. We only perform the analysis for $Y\geq 1$ since it is exactly the same for $Y\leq -1$, thus writing $|Y|=Y$.

Step 1: *Bounds for ε.* Let $0<\kappa\ll 1$ be an arbitrarily small constant. Let χ be a smooth and positive cut-off function, $\chi=1$ for $Y\geq 2$ and $\chi=0$ for $Y\leq 1$. Let $\ell=1$ or $\ell=1/2$. We compute first the identity by integrating by parts and using Proposition 34:

$$\begin{aligned} \frac{d}{ds}\frac12\left(\int \chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right) &= -\frac{\ell}{2k}\int\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|} - \int\chi\frac{|\partial_Y\varepsilon|^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|} + \frac14\int\partial_Y\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}dY \\ &+\frac12\int\varepsilon^2\left(\frac{\partial_{YY}\chi}{\psi_\ell^2(Z)Y} + 2\partial_Y\chi\partial_Y\left(\frac{1}{\psi_\ell^2(Z)Y}\right) + \chi\partial_{YY}\left(\frac{1}{\psi_\ell^2(Z)Y}\right)\right)dY \\ &+\int\frac{\varepsilon^2}{\psi_\ell^2(Z)}4(u-F_k(Z))\frac{dY}{|Y|} - \int\chi\frac{\varepsilon}{\psi_\ell^2(Z)}\left(b_sF_k^4(Z)+\Psi\right)\frac{dY}{|Y|}. \end{aligned}$$

We treat the boundary terms using (4.50):

$$\begin{aligned}&\left|\frac14\int\partial_Y\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}dY+\frac12\int\varepsilon^2\left(\frac{\partial_{YY}\chi}{\psi_\ell^2(Z)Y}+2\partial_Y\chi\partial_Y\left(\frac{1}{\psi_\ell^2(Z)Y}\right)\right)dY\right|\\ \lesssim\ &\|\varepsilon\|_{L^2_\rho}^2\left(\|\frac{1}{\psi_\ell^2(Z)}\|_{L^\infty(1\le Y\le 2)}+\|\partial_Y\left(\frac{1}{\psi_\ell^2(Z)}\right)\|_{L^\infty(1\le Y\le 2)}\right)\lesssim e^{-s+\frac{k-1}{k}\ell s}\\ \lesssim\ &\begin{cases}L_0e^{-(\frac{3}{4k}+\frac{1}{4k})s} & \text{if } \ell=1,\\ L_0e^{-(\frac{1}{2k}+\frac12)s} & \text{if } \ell=\frac12.\end{cases}\end{aligned}$$

Next, notice that for $Y\geq 1$,

$$\left|\partial_{YY}\left(\frac{1}{\psi_\ell^2(Z)Y}\right)\right|\lesssim\frac{1}{Y^3\psi_\ell^2(Z)}.$$

If $\ell=1$, we then decompose for Y^* large enough depending on κ:

$$\begin{aligned}\left|\int\varepsilon^2\chi\partial_{YY}\left(\frac{1}{\psi_1^2(Z)Y}\right)dY\right| &\le C\int\chi\frac{\varepsilon^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|}\\ &\le C\int_{Y\le Y^*}\chi\frac{\varepsilon^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|}+C\int_{Y\ge Y^*}\chi\frac{\varepsilon^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|}\\ &\le C\|\varepsilon\|_{L^2_\rho}^2\|\frac{1}{\psi_1^2(Z)}\|_{L^\infty(1\le Y\le Y^*)}+\kappa\int_{Y\ge Y^*}\chi\frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}\\ &\le CL_0e^{-(\frac{3}{4k}+\frac{1}{4k})s}+\kappa\int\chi\frac{\varepsilon^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|}.\end{aligned}$$

If $\ell=1/2$, we use the fact that $1/(Y\psi_{1/2}^2(Z))=e^{-(k-1)/(2k)s}/\psi_1^2(Z)$ to obtain from (4.48):

$$\left|\int\varepsilon^2\chi\partial_{YY}\left(\frac{1}{\psi_{1/2}^2(Z)Y}\right)dY\right|\lesssim\int\chi\frac{\varepsilon^2}{\psi_{1/2}^2(Z)Y^2}\frac{dY}{|Y|}\lesssim e^{-\frac{k-1}{2k}s}\int\chi\frac{\varepsilon^2}{|Y|\psi_1^2(Z)}\frac{dY}{|Y|}\lesssim e^{-(\frac{1}{2k}+\frac{2k-1}{4k})s}.$$

The lower order linear term is estimated via (4.56) and (4.48):

$$\left|\int\frac{\varepsilon^2}{\psi_\ell^2(Z)}4(f-F_k(Z))\frac{dY}{|Y|}\right|\lesssim\|f-F_k(Z)\|_{L^\infty}\int\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\lesssim\begin{cases}L_0e^{-(\frac{3}{4k}+\frac{1}{4k})s} & \text{if } \ell=1,\\ L_0e^{-(\frac{1}{2k}+\frac{1}{4k})s} & \text{if } \ell=\frac12.\end{cases}$$

The error terms are estimated via (4.61), (4.59) and (4.48):

$$\begin{aligned}&\left|\int\chi\frac{\varepsilon}{\psi_\ell^2(Z)}b_sF_k^4(Z)\frac{dY}{|Y|}\right|\lesssim|b_s|\left|\int\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int\chi\frac{F_k^8(Z)}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\\ \lesssim\ &e^{-(k-1)s}\left|\int\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\int_{e^{-\frac{k-1}{2k}s}}^{+\infty}\frac{1}{|Z|^{2\ell}}\frac{dZ}{Z}\lesssim\begin{cases}\sqrt{L_0}e^{-(\frac{3}{4k}+k-2\frac{5}{8k})s} & \text{if } \ell=1,\\ \sqrt{L_0}e^{-(\frac{1}{2k}+k-\frac32+\frac{1}{4k})s} & \text{if } \ell=\frac12,\end{cases}\end{aligned}$$

and via Cauchy-Schwarz using (4.59):

$$\left|\int\chi\frac{\varepsilon}{\psi_\ell^2(Z)}\Psi\frac{dY}{|Y|}\right|\le\left|\int\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int\chi\frac{|\Psi|^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\le\begin{cases}C\sqrt{L_0}e^{-(\frac{3}{4k})s} & \text{if } \ell=1,\\ C\sqrt{L_0}e^{-(\frac{1}{2k}+\frac{1}{8k})s} & \text{if } \ell=\frac12.\end{cases}$$

We now collect all the previous estimates and obtain:

$$\frac{d}{ds}\left(\int\chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right)\quad\le\quad\begin{cases}-\left(\frac1k-\kappa\right)\int\chi\frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}+C\sqrt{L_0}e^{-(\frac{3}{4k})s} & \text{if } \ell=1,\\ -\frac{1}{2k}\int\chi\frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}+CL_0e^{-(\frac{1}{2k}+\frac{1}{8k})s} & \text{if } \ell=\frac12\end{cases}$$

if κ has been chosen small enough, and s_0 large enough. The two above differential inequalities yield the desired results (4.63) when reintegrated in time using (4.45) and (4.50), if L_0 has been chosen large enough independently of the other constants in the bootstrap.

Step 2: *Proof of* (4.63) *for* $Z\partial_Z\varepsilon$. Let $\ell = 1$ or $\ell = 1/2$ and define $w := Z\partial_Z\varepsilon$. It solves from (4.55):

$$w_s + \mathcal{M}_Z w - \partial_{YY}w - 4Z\partial_Z(F_k(Z))\varepsilon + 2\partial_{YY}\varepsilon - Z\partial_Z((f - F_k(Z))\varepsilon) + Z\partial_Z(b_s F_k^4(Z) + \Psi) = 0.$$

Therefore, one infers that

$$\begin{aligned}\frac{d}{ds}\frac{1}{2}\left(\int \chi \frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right) = & -\frac{\ell}{2k}\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|} - \int \chi \frac{|\partial_Y w|^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|} + \frac{1}{4}\int \partial_Y\chi\frac{w^2}{\psi_\ell^2(Z)}dY \\ & + \frac{1}{2}\int w^2\left(\frac{\partial_{YY}\chi}{\psi_\ell^2(Z)Y} + 2\partial_Y\chi\partial_Y\left(\frac{1}{\psi_\ell^2(Z)Y}\right) + \chi\partial_{YY}\left(\frac{1}{\psi_\ell^2(Z)Y}\right)\right)dY \\ & + 4\int \chi\frac{wZ\partial_Z(F_k(Z))\varepsilon}{\psi_\ell^2(Z)}\frac{dY}{|Y|} - \int w^2\frac{1}{Y}\left(\frac{\partial_Y\chi}{\psi_\ell^2(Z)} + \chi\partial_Y\left(\frac{1}{\psi_\ell^2(Z)}\right)\right)\frac{dY}{|Y|} \\ & + 4\int \frac{w}{\psi_\ell^2(Z)}Z\partial_Z((f - F_k(Z))\varepsilon)\frac{dY}{|Y|} - \int \chi\frac{w}{\psi_\ell^2(Z)}Z\partial_Z\left(b_sF_k^4(Z) + \Psi\right)\frac{dY}{|Y|}.\end{aligned}$$

We treat the boundary terms using (4.50) and the fact that $|w| = |Z\partial_Z\varepsilon| \leq e^{-(k-1)s/(2k)}|\partial_Z\varepsilon|$ for $1 \leq Y \leq 2$:

$$\begin{aligned}&\left|\frac{1}{4}\int \partial_Y\chi\frac{w^2}{\psi_\ell^2(Z)}dY + \frac{1}{2}\int w^2\left(\frac{\partial_{YY}\chi}{\psi_\ell^2(Z)Y} + 2\partial_Y\chi\partial_Y\left(\frac{1}{\psi_\ell^2(Z)Y}\right)\right)dY - \int w^2\frac{\partial_Y\chi}{\psi_\ell^2(Z)}\frac{dY}{Y^2}\right| \\ \lesssim\ & e^{-\frac{k-1}{k}s}\|\partial_Z\varepsilon\|_{L^2_\rho}^2\left(\|\frac{1}{\psi_\ell^2(Z)}\|_{L^\infty(1\leq Y\leq 2)} + \|\partial_Y\left(\frac{1}{\psi_\ell^2(Z)}\right)\|_{L^\infty(1\leq Y\leq 2)}\right) \\ \lesssim\ & L_1 e^{-\frac{1}{k}s + \frac{k-1}{k}(\ell-1)s} \lesssim \begin{cases} L_1 e^{-(\frac{3}{4k}+\frac{1}{4k})s} & \text{if } \ell = 1, \\ L_1 e^{-(\frac{1}{2k}+\frac{1}{2})s} & \text{if } \ell = \frac{1}{2}.\end{cases}\end{aligned}$$

As in Step 1, since for $Y \geq 1$, $|\partial_Y^i(1/(\psi_\ell^2(Z)))| \lesssim 1/(\psi_\ell^2(Z)|Y|^\ell))$ one deduces that if $\ell = 1$, for some $Y^* \gg 1$ large enough:

$$\begin{aligned}&\left|\frac{1}{2}\int w^2\chi\partial_{YY}\left(\frac{1}{\psi_1^2(Z)Y}\right)dY - \int w^2\frac{1}{Y}\chi\partial_Y\left(\frac{1}{\psi_1^2(Z)}\right)\frac{dY}{|Y|}\right| \leq C\int \chi\frac{w^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|} \\ \leq\ & C\int_{Y\leq Y^*}\chi\frac{w^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|} + C\int_{Y\geq Y^*}\chi\frac{w^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|} \\ \leq\ & C\|w\|_{L^2_\rho}^2\|\frac{1}{\psi_1^2(Z)}\|_{L^\infty(1\leq Y\leq Y^*)} + \kappa\int_{Y\geq Y^*}\chi\frac{w^2}{\psi_1^2(Z)}\frac{dY}{|Y|} \\ \leq\ & CL_1e^{-(\frac{3}{4k}+\frac{1}{4k})s} + \kappa\int \chi\frac{\varepsilon^2}{\psi_1^2(Z)Y^2}\frac{dY}{|Y|}\end{aligned}$$

using (4.50). If $\ell = 1/2$, we use the fact that $1/(Y\psi_{1/2}^2(Z)) = e^{-(k-1)/(2k)s}/\psi_1^2(Z)$ to obtain from (4.48):

$$\begin{aligned}&\left|\frac{1}{2}\int w^2\chi\partial_{YY}\left(\frac{1}{\psi_{1/2}^2(Z)Y}\right)dY - \int w^2\frac{1}{Y}\chi\partial_Y\left(\frac{1}{\psi_{1/2}^2(Z)}\right)\frac{dY}{|Y|}\right| \\ \leq\ & C\int\chi\frac{w^2}{\psi_{1/2}^2(Z)Y^2}\frac{dY}{|Y|} \lesssim L_1e^{-\frac{k-1}{2k}s}\int\chi\frac{w^2}{|Y|\psi_1^2(Z)}\frac{dY}{|Y|} \lesssim e^{-(\frac{1}{2k}+\frac{2k-1}{4k})s}.\end{aligned}$$

The linear term coming from the commutator between $Z\partial_Z$ and $\mathcal{M}_Z$ is estimated by Cauchy-Schwarz and (4.48) for $\ell = 1$:

$$\left|\int \chi \frac{wZ\partial_Z(F_k(Z))\varepsilon}{\psi_1^2(Z)}\frac{dY}{|Y|}\right| \leq C\left|\int \chi \frac{w^2}{\psi_1^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int \chi \frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}\right|^{\frac12} \leq C\sqrt{K_1}\sqrt{L_0}e^{-\frac{3}{4k}s}$$

as $|Z\partial_Z F_k(Z)| \lesssim |Z|^{2k}(1+|Z|)^{-4k} \lesssim 1$. For $\ell = 1/2$, since

$$\frac{|Z|^{2k}(1+|Z|)^{-4k}}{\psi_{1/2}^2(Z)} \lesssim \frac{1}{\psi_1^2(Z)}$$

as $|\psi_1(Z)| = |Z|^{1/2}|\psi_{1/2}(Z)|$, one uses Cauchy-Schwarz, (4.48):

$$\left|\int \chi \frac{wZ\partial_Z(F_k(Z))\varepsilon}{\psi_{1/2}^2(Z)}\frac{dY}{|Y|}\right| \lesssim \left|\int \chi \frac{w^2}{\psi_1^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int \chi \frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}\right|^{\frac12} \lesssim \sqrt{L_0K_1}e^{-\frac{3}{4k}s}.$$

The lower order linear term is estimated via (4.56), (4.48):

$$\begin{aligned}
&\left|\int \frac{w}{\psi_\ell^2(Z)}Z\partial_Z((f-F_k(Z))\varepsilon)\frac{dY}{|Y|}\right| \\
\lesssim\ & (\|f-F_k(Z)\|_{L^\infty}+\|Z\partial_Z(f-F_k(Z))\|_{L^\infty})\left(\int \chi\frac{\varepsilon^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}+\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right) \\
\lesssim\ & \begin{cases} L_1 e^{-\left(\frac{3}{4k}+\frac{1}{4k}\right)s} & \text{if } \ell=1, \\ L_1 e^{-\left(\frac{1}{2k}+\frac{1}{4k}\right)s} & \text{if } \ell=\frac12. \end{cases}
\end{aligned}$$

The error terms are estimated via (4.61), (4.59), (4.48):

$$\begin{aligned}
&\left|\int \chi\frac{w}{\psi_\ell^2(Z)}b_s Z\partial_Z(F_k^4(Z))\frac{dY}{|Y|}\right| \lesssim |b_s|\left|\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int \chi\frac{|Z|^{4k}(1+|Z|)^{-20k}}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12} \\
\lesssim\ & e^{-(k-1)s}\left|\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\int_{e^{-\frac{k-1}{2k}s}}^{+\infty}|Z|^{4k-\ell}(1+|Z|)^{-4k}\frac{dZ}{Z} \\
\lesssim\ & \begin{cases} \sqrt{L_1}e^{-\left(\frac{3}{4k}+k-1-\frac{3}{8k}\right)s} & \text{if } \ell=1, \\ \sqrt{L_1}e^{-\left(\frac{1}{2k}+k-1-\frac{1}{4k}\right)s} & \text{if } \ell=\frac12, \end{cases}
\end{aligned}$$

and via Cauchy-Schwarz using (4.59):

$$\left|\int \chi\frac{w}{\psi_\ell^2(Z)}Z\partial_Z\Psi\frac{dY}{|Y|}\right| \leq \left|\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12}\left|\int \chi\frac{|Z\partial_Z\Psi|^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right|^{\frac12} \leq \begin{cases} C\sqrt{L_1}e^{-\left(\frac{3}{4k}\right)s} & \text{if } \ell=1, \\ \sqrt{L_1}e^{-\left(\frac{1}{2k}+\frac{1}{8k}\right)s} & \text{if } \ell=\frac12. \end{cases}$$

We now collect all the previous estimates and obtain that for any $\kappa > 0$, for s_0 large enough:

$$\frac{d}{ds}\left(\int \chi\frac{w^2}{\psi_\ell^2(Z)}\frac{dY}{|Y|}\right) \leq \begin{cases} -\left(\frac1k-\kappa\right)\int \chi\frac{w^2}{\psi_1^2(Z)}\frac{dY}{|Y|}+C(\sqrt{L_0}\sqrt{L_1}+\sqrt{L_1})e^{-\left(\frac{3}{4k}\right)s} & \text{if } \ell=1, \\ -\frac{1}{2k}\int \chi\frac{\varepsilon^2}{\psi_1^2(Z)}\frac{dY}{|Y|}+CL_1e^{-\left(\frac{1}{2k}+\frac{1}{8k}\right)s} & \text{if } \ell=\frac12, \end{cases}$$

The two above differential inequalities yield the desired results (4.63) when reintegrated in time using (4.45) and (4.50), if L_1 has been chosen large enough depending on L_0, and s_0 has been chosen large enough.

Step 3: *End of the proof.* We claim that the bounds (4.63) for higher order derivatives, as well as the bound (4.64), can be proved with verbatim the same argument that were used in Step 1 and Step 2. We leave the details to the reader.

□

Proof of Proposition 36. We use a bootstrap argument. Let $s_1 \geq s_0$ be the supremum of times $\tilde{s} \geq s_0$ such that all the bounds of Proposition 36 hold on some time interval $[s_0, \tilde{s}]$. Then (4.45), (4.48) and (4.47) imply $s_1 > s_0$ by a continuity argument. Assume by contradiction that $s_1 < +\infty$. Then the bounds (4.48), (4.49) and (4.50) are strict at time s_1 from (4.50), (4.63) and (4.64). From a continuity argument there exists $\delta > 0$ such that (4.48), (4.49) and (4.50) hold on $[s_1, s_1+\delta]$, contradicting the definition of s_1. Thus $s_1 = +\infty$ and Proposition 36 is proved.

□

5. Proof of Theorem 2

sec:stable

In this section we prove Theorem 2. The proof is the same as the one of Theorem 3, only few details change. Namely, the analysis is now based on the stable blow-up of the self-similar heat equation $\xi_t - \xi^2 - \xi_{yy} = 0$ whose properties are classical [2, 3, 17, 24]. We therefore just sketch the proof, with an emphasise on the differences between this proof and that of Theorem 3. We consider only the case $i = 1$ with the profile Ψ_1 for Burgers equation, the proof being the same for $i \geq 2$. We define the self-similar variables

$$X := \sqrt{\frac{b}{6}}\frac{x}{(T-t)^{\frac{3}{2}}}, \quad Y := \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := \frac{Y}{8\sqrt{s}}, \tag{5.1}$$

def:selfsimva

and

$$u(t,x,y) = \sqrt{\frac{6}{b}}(T-t)^{\frac{1}{2}} v\left(s, X, Y\right),$$

The first step is to obtain precise information for the behaviour of the derivatives of the solution on the transverse axis.

5.1. Analysis on the transverse axis $\{x = 0\}$

sec:1dstable

We start by showing the first part of Theorem 1 and the analogue of Proposition 10. Define for a solution u to (1.1):

$$\xi(t,y) = -u_x(t,0,y), \;\; \xi(t,y) = (T-t)^{-1} f(s,Y), \;\; \zeta(t,y) = \partial_x^3 u(t,0,y), \;\; \zeta(t,y) = (T-t)^{-4} g(s,Y).$$

Then (f,g) solve the system (3.8).

Claim: For $0 < T \ll 1$ small enough, for any $b > 0$ and $J \in \mathbb{N}$, there exists a solution to (3.8) such that for $0 \leq j \leq J$:

$$f(s,Y) = \frac{1}{1+Z^2} + \tilde{f}, \quad |\partial_Z^j \tilde{f}| \lesssim s^{-\frac{1}{4}}(1+|Z|)^{-\frac{3}{2}-j}, \tag{5.2}$$

stable:bd:til

$$g(s,Y) = \frac{b}{(1+Z^2)^4} + \tilde{g}, \quad |\partial_Z^j \tilde{g}| \lesssim s^{-\frac{1}{4}}(1+|Z|)^{-8+\frac{1}{2}-j}. \tag{5.3}$$

stable:bd:til

These estimates are very similar to (3.6) and (3.7), but the smallness of the error is in powers of s^{-1} and not of e^{-s} anymore. This loss will however not be a problem for the sequel.

Sketch of proof of the claim: We adapt the strategy of Section 4. We first construct a solution f to (NLH) satisfying (5.2). We take as an approximate solution to (NLH) the profile

$$F[a](s,Y) := \frac{1}{1+\left(\frac{1}{8s}+a\right)Y^2} + \left(\frac{1}{4s}+2a\right)\frac{1}{\left(1+\left(\frac{1}{8s}+a\right)Y^2\right)^2}.$$

Note that here the corrective parameter a will satisfy $|a| \lesssim |\log(s)s^{-2}|$. The main difference between the stable blow-up and the flat blow-ups for (NLH) is then the following. The scaling parameter in the flat case (4.4) corresponds to leading order to that of the inviscid case and is not affected by the dissipation (4.30), whereas in the stable blow-up case the dissipation has a modulation effect on this parameter, and forces it to tend to 0 through a logarithmic correction.

Following the proof of Proposition 26, the approximate profile satisfies the following identity:

$$\partial_s F[a]+F[a]+\frac{Y}{2}\partial_Y F[a]-F^2[a]-\partial_{YY}F[a]=-\left(\frac{1}{4s^2}+\frac{4a}{s}\right)+\left(-a_s-\frac{2}{s}a\right)h_2+\Psi=R \quad (5.4)$$ stable:Fa

where h_2 is defined by (4.6), and where for a corrective modulation parameter a satisfying the a priori bound $|a|\lesssim|\log(s)|s^{-2}$ and $|a_s|\lesssim|\log(s)|s^{-3}$ the errors Ψ and R satisfy:

$$\|\Psi\|_{L^2_\rho}\lesssim s^{-3},\quad \|\partial_Z^j R\|_{L^2_\rho}\lesssim s^{-3+\frac{j}{2}}\ \text{ for } j=0,1,2 \ \text{ and } \ \|\partial_Z^j R\|_{L^2_\rho}\lesssim s^{-1}\ \text{ for } j\geq 3,$$

$$\int_{|Y|\geq 1}\frac{|(Z\partial_Z)^j R|^2}{|\phi_{\frac52}(Z)|^2}\frac{dY}{|Y|}\lesssim s^{-1} \ \text{ and } \ \int_{|Y|\geq 1}\frac{|\partial_Z^j R|^2}{|\phi_0(Z)|^2}\frac{dY}{|Y|}\lesssim s^{-1}\quad j\geq 3,$$

where ϕ_j denotes the eigenfunction

$$\phi_j(Z)=\frac{Z^j}{(1+Z^2)^2},\quad H_Z\phi_j=\frac{j-2}{2}\phi_j,\quad H_Z=1-\frac{2}{1+Z^2}+\frac{Z}{2}\partial_Z.$$

We then show the existence of a global solution to (NLH) close to $F[a]$ by a bootstrap argument following Proposition 29. We decompose f as

$$f=F[a]+\varepsilon=F[a]+c_1+\tilde\varepsilon,\quad \langle\varepsilon,h_2\rangle_\rho=0,\quad \langle\tilde\varepsilon,1\rangle_\rho=\langle\tilde\varepsilon,h_2\rangle_\rho=0$$

where the orthogonality conditions fix the value of a and c_1 in a unique way. We claim that there exists a global solution to the first equation in (3.8) satisfying for $j=0,...,J+1$:

$$|a(s)|\lesssim s^{-2},\quad |c_1(s)|\lesssim s^{-2},\quad \|\tilde\varepsilon\|_{L^2_\rho}\lesssim s^{-3},\quad \|\partial_Z^j\varepsilon\|_{L^2_\rho}\lesssim s^{-3+\frac{j}{2}}\ j=1,2,\quad \|\partial_Z^j\varepsilon\|_{L^2_\rho}\lesssim s^{-1},\ j\geq 3,$$

$$\int_{|Y|\geq 1}\frac{|(Z\partial_Z)^j\varepsilon|^2}{|\phi_{\frac52}(Z)|^2}\frac{dY}{|Y|}\lesssim s^{-\frac12} \ \text{ and } \ \int_{|Y|\geq 1}\frac{|\partial_Z^j\varepsilon|^2}{|\phi_0(Z)|^2}\frac{dY}{|Y|}\lesssim s^{-\frac12}\ \text{ for } j\geq 3.$$

To prove this fact, one first performs modulation estimates, then energy estimates at the origin with the ρ weight, and then energy estimates outside the origin as in Lemmas 31, 32 and 33. The evolution equation near the origin reads from (5.4)

$$c_{1,s}-c_1-\left(\frac{1}{4s^2}+\frac{4a}{s}\right)-\left(a_s+\frac{2}{s}a\right)h_2+\tilde\varepsilon_s+\mathcal{L}\tilde\varepsilon-2(F[a]-1)\varepsilon-\varepsilon^2+\Psi=0.$$

The modulation estimates are therefore a consequence of the spectral structure of $\mathcal{L}$ in Proposition 25, giving in the bootstrap regime when projecting the above equation on 1 and h_2:

$$\left|a_s+\frac{2}{s}a\right|\lesssim s^{-3},\quad |c_{1,s}-c_1|\lesssim s^{-2}.$$

The first inequality, when reintegrated in time, gives $|a|\lesssim|\log(s)|s^{-2}$. The second inequality shows an instability, and the use of Brouwer's fixed point theorem then implies the existence of a trajectory such that $|c_1(s)|\lesssim s^{-2}$. The orthogonality conditions for $\tilde\varepsilon$ imply the spectral damping $\langle\tilde\varepsilon,\mathcal{L}\tilde\varepsilon\rangle\geq\|\tilde\varepsilon\|^2_{L^2_\rho}$ since $\tilde\varepsilon$ is even, implying the energy identity

$$\frac{d}{ds}\left(\frac12\|\tilde\varepsilon\|^2_{L^2_\rho}\right)\leq-(1-Cs^{-1}-C\|\varepsilon\|_{L^\infty})\|\tilde\varepsilon\|^2_{L^2_\rho}+Cs^{-6}.$$

This yields the desired estimates for $\tilde{\varepsilon}$ when reintegrated with time, and the same technique applies to control its derivatives. In the far field, the analysis is the same as in Lemma 33, the equation for ε reads

$$\varepsilon_s + H_Z\varepsilon - \partial_{YY}\varepsilon - 2\left(F[a] - \frac{1}{1+Z^2}\right)\varepsilon - \varepsilon^2 + R = 0, \quad H_Z = 1 - \frac{2}{1+Z^2} + \frac{Y}{2}\partial_Y.$$

Let χ be a non-negative smooth cut-off function with $\chi = 0$ for $|Y| \leq 1$ and $\chi = 1$ for $|Y| \geq 2$. One obtains from this equation the following energy estimate:

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\int \chi \frac{\varepsilon^2}{|\phi_{\frac{5}{2}}(Z)|^2}\frac{dY}{|Y|}\right) \quad &\leq \quad -\left(\frac{1}{4} - \kappa - \|\varepsilon\|_{L^\infty}\right)\int \chi \frac{\varepsilon^2}{|\phi_{\frac{5}{2}}(Z)|^2}\frac{dY}{|Y|} - \int \chi \frac{(\partial_Y\varepsilon)^2}{|\phi_{\frac{5}{2}}(Z)|^2}\frac{dY}{|Y|} \\ &\quad + O\left(s^{\frac{5}{2}}\|\varepsilon\|^2_{L^2_\rho}\right) + O(s^{-1})\end{aligned}$$

where $0 < \kappa \ll 1$ is an arbitrary small positive number, since $|\phi_{5/2}| \sim |Z|^{5/2} \sim |Y|^{5/2}s^{-\frac{5}{4}}$ on compact sets. Thanks to the damping, this estimate is reintegrated in time and shows the weighted decay outside the origin for ε. The analogous estimates for the derivatives are showed similarly. The strategy that we just explained allows one to close the bootstrap estimates. Using the Sobolev embedding (A.2), this concludes the proof of the existence of a solution f to (NLH) satisfying (5.2).

Once the properties of f are known, the analysis of (LFH) follows very closely the one performed in Subsection 4.2. We decompose g solution to the second equation in (3.8) according to

$$g(s,Y) = b(s)f^4(s,Y) + \bar{\varepsilon}, \quad \langle \bar{\varepsilon}, 1\rangle_\rho = 0$$

where f is the solution (NLH) we just constructed. The evolution equation then reads

$$b_s f^4 + \bar{\varepsilon}_s + 4\bar{\varepsilon} - 4f\bar{\varepsilon} + \frac{Y}{2}\partial_Y\bar{\varepsilon} - \partial_{YY}\bar{\varepsilon} + \bar{R} = 0, \quad \bar{R} = -12b(\partial_Y f)^2 f^2.$$

For $|b| \lesssim 1$ the error satisfies from the properties of f already showed:

$$\|\bar{R}\|_{L^2_\rho} \lesssim s^{-2}, \quad \|\partial_Z^j \bar{R}\|_{L^2_\rho} \lesssim s^{-1} \text{ for } j \geq 1,$$

$$\int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j\bar{R}|}{|\psi_{\frac{1}{2}}(Z)|^2}\frac{dY}{|Y|} \lesssim s^{-1}, \quad \int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j\bar{R}|}{|\psi_0(Z)|^2}\frac{dY}{|Y|} \lesssim s^{-1}$$

where ψ_j denotes the eigenfunction

$$\psi_j(Z) = \frac{Z^j}{(1+Z^2)^4}, \quad \mathcal{M}_Z\psi_j = \frac{j}{2}\psi_j, \quad \mathcal{M}_Z = 4 - \frac{4}{1+Z^2} + \frac{Z}{2}\partial_Z.$$

We claim that there exists a solution satisfying the estimates

$$|b_s| \lesssim s^{-2}, \quad \|\tilde{\varepsilon}\|_{L^2_\rho} \lesssim s^{-2}, \quad \|\partial_Z^j\varepsilon\|_{L^2_\rho} \lesssim s^{-1} \text{ for } j \geq 1,$$

$$\int_{|Y|\geq 1} \frac{|(Z\partial_Z)^j\varepsilon|^2}{|\psi_{\frac{1}{2}}(Z)|^2}\frac{dY}{|Y|} \lesssim s^{-\frac{1}{2}} \text{ and } \int_{|Y|\geq 1} \frac{|\partial_Z^j\varepsilon|^2}{|\psi_0(Z)|^2}\frac{dY}{|Y|} \lesssim s^{-\frac{1}{2}} \text{ for } j \geq 1.$$

Similarly, we prove this property by a bootstrap analysis, following closely the analysis of Lemmas 39, 40 and 41. The equation close to the origin reads

$$b_s f^4 + \bar{\varepsilon}_s + \frac{Y}{2}\partial_Y\bar{\varepsilon} - \partial_{YY}\bar{\varepsilon} + 4(1-f)\bar{\varepsilon} + \bar{R} = 0.$$

Taking the L^2_ρ scalar product against the constant 1 then yields indeed the modulation equation

$$|b_s| \lesssim s^{-2}.$$

Similarly, from the spectral gap $\|\partial_Y \bar\varepsilon\|^2_{L^2_\rho} \geq \|\bar\varepsilon\|^2_{L^2_\rho}$ one deduces the energy identity

$$\frac{d}{ds}\left(\frac12 \|\bar\varepsilon\|^2_{L^2_\rho}\right) \leq -(1-Cs^{-\frac14})\|\tilde\varepsilon\|^2_{L^2_\rho} + Cs^{-4}$$

which yields the corresponding estimate $\|\varepsilon\|_{L^2_\rho} \lesssim s^{-2}$ when reintegrated with time. The corresponding estimates for higher order derivatives are showed the same way. In the far field the evolution equation reads

$$b_s f^4 + \bar\varepsilon_s + 4\bar\varepsilon - \frac{4}{1+Z^2}\bar\varepsilon + \frac{Y}{2}\partial_Y\bar\varepsilon - \partial_{YY}\bar\varepsilon + 4\tilde f\bar\varepsilon + \bar R = 0.$$

This equation enjoys the following energy estimate for an arbitrary constant $0<\kappa\ll 1$:

$$\begin{aligned}\frac{d}{ds}\left(\frac12\int \chi\frac{\bar\varepsilon^2}{|\psi_{\frac12}(Z)|^2}\frac{dY}{|Y|}\right) \quad\leq\quad & -\left(\frac14-\kappa\right)\int\chi\frac{\bar\varepsilon^2}{|\psi_{\frac12}(Z)|^2}\frac{dY}{|Y|} - \int \chi\frac{(\partial_Y\bar\varepsilon)^2}{|\psi_{\frac12}(Z)|^2}\frac{dY}{|Y|}\\ & + O\left(s^{\frac12}\|\bar\varepsilon\|^2_{L^2_\rho}\right) + O(s^{-1})\end{aligned}$$

This estimate is reintegrated in time and shows the expected weighted decay outside the origin for $\bar\varepsilon$. The estimates for the derivatives are showed the same way. Using the Sobolev embedding (A.2), one then obtained the existence of a solution g to (LFH) satisfying (5.3).

5.2. Analysis of the full 2-d problem

We now follow the analysis of Section 3. Let f and g be the solutions to (NLH) and (LFH) satisfying (5.2) and (5.3). For simplicity we fix $b=6$, so that $X = x/(T-t)^{3/2}$. We take the same blow-up profile as in the proof of Theorem 3, adjusting the cut-off between the inner and outer zones. We set for $0<d\ll 1$ a cut-off function $\chi_d(s,Y) := \chi\left(Y/(ds)\right)$ where χ is a smooth nonnegative function with $\chi(Y)=1$ for $|Y|\leq 1$ and $\chi(Y)=0$ for $|Y|\geq 2$. We decompose our solution to (3.4) according to:

$$v(s,X,Y) = Q+\varepsilon, \quad Q = \chi_d(s,Y)\tilde\Theta + (1-\chi_d(s,Y))\Theta_e \tag{5.5}$$ stable:decomp

where (for d small enough f and g do not vanish from (5.2) and (5.3))

$$\tilde\Theta(s,X,Y) = \sqrt6 g^{-\frac12} f^{\frac32}\Psi_1\left(\frac{g^{\frac12}f^{-\frac12}}{\sqrt6}X\right)$$

and where Θ_e is the exterior profile (recall that $\tilde X = X/(1+Z^2)^{3/2}$):

$$\Theta_e(s,X,Y) = \left(-Xf(s,Y) + X^3\frac{g(s,Y)}{6}\right)e^{-\tilde X^4}.$$

We adjust our initial datum $v(s_0)$ such that $-\partial_X v(s_0,0,Y) = f(s_0,Y)$ and $\partial_X^3 v(s_0,0,Y) = g(s_0,Y)$. This way, since v odd in x and even y, one has that $\partial_X^j\varepsilon(s,0,Y)=0$ on the transverse axis for $j=0,1,2,3,4$ for all times $s\geq s_0$. The time evolution of the remainder ε is:

$$\varepsilon_s + \mathcal L\varepsilon + \tilde{\mathcal L}\varepsilon + R + \varepsilon\partial_X\varepsilon = 0$$

where

$$\mathcal L = -\frac12 + \partial_X\Theta + \left(\frac32 X + \Theta\right)\partial_X + \frac12 Y\partial_Y - \partial_{YY},$$

$$\Theta(s,X,Y)=(1+Z^2)^{\frac12}\Psi_1\left(\frac{X}{(1+Z^2)^{\frac32}}\right),\quad \tilde{\mathcal{L}}\varepsilon=(Q-\Theta)\partial_X\varepsilon+(\partial_XQ-\partial_X\Theta)\varepsilon,$$

and

$$R=Q_s-\frac12 Q+\frac32 X\partial_XQ+\frac12 Y\partial_YQ+Q\partial_XQ-\partial_{YY}Q.$$

From Proposition 12 the inviscid linearised operator has eigenvalues of the form $(j+\ell-3)/2$ for $(j,\ell)\in\mathbb{N}$ with associated eigenfunction

$$\varphi_{j,\ell}(X,Z)=Z^\ell F_k^{1-\frac{j}{2}}(Z)\times\frac{(-1)^j\Psi_1^j\left(F_k^{\frac32}(Z)X\right)}{1+3\Psi_1^2\left(F_k^{\frac32}(Z)X\right)}.$$

The sizes of the important objects are

$$|\Theta(X,Z)|\approx|X|\left((1+|Z|)^3+|X|\right)^{\frac13-1}\approx(1+|Z|)|\tilde X|(1+|\tilde X|)^{\frac13-1}$$

$$|\varphi_{\frac72,0}(X,Z)|\approx|X|^{\frac72}\left((1+|Z|)^3+|X|\right)^{\frac12-\frac72}\approx(1+|Z|)^{\frac32}|\tilde X|^{\frac72}(1+|\tilde X|)^{\frac12-\frac72}.$$

In the previous Section, the weight $\varphi_{4,0}$ was used. Any weight $\varphi_{\alpha,0}$ with $\alpha\in(3,4]$ is suitable since it provides linear decay and suitable weighted Sobolev estimates for the error term R as well. We take $\alpha=\frac72$ here to make the weight slightly less singular at the origin, so that the integrals below are well defined for C^4 function. We claim that thanks to (5.2) and (5.3) one has the following estimates for the error, which can be proved as in the proof of Lemma 20:

$$\sum_{0\le j_1+j_2\le 2}\int_{\mathbb{R}^2}\frac{(\partial_Z^{j_1}A^{j_2}R)^{2q}+((Y\partial_Y)^{j_1}A^{j_2}R)^{2q}}{\varphi^{2q}_{\frac72,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\lesssim s^{-q} \tag{5.6}$$ bd:stableR

where A is given by (3.26). From (5.2) and (5.3) one also deduces the following estimates for the lower order linear term

$$|\partial_Z^{j_1}(Q-\Theta)|\lesssim s^{-\frac14}|X|(1+|Z|)^{-j_1},$$

$$|\partial_Z^{j_1}\partial_X^{j_2}(Q-\Theta)|\lesssim s^{-\frac14}(1+|X|)^{1-j_2}(1+|Z|)^{-j_1}.$$

which can be proved as in the proof of Lemma 21. We can therefore perform the same energy estimates as in Lemmas 22 and 23. Indeed, the leading order linear estimate (3.18) holds also true in that case, and with the above control on the lower order linear term and of the error R one obtains that for any $\kappa>0$ for $q\ge 2$ large enough:

$$\begin{aligned}&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi^{2q}_{\frac72,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\ \le\ &-\left(\frac14-\kappa-Cs^{-\frac14}-C\|\partial_X\varepsilon\|_{L^\infty}\right)\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}+Cs^{-q}.\end{aligned}$$

The same type of energy estimates hold when applying A, Z or $Z\partial_Z$, up to terms involving lower order derivatives. For exemple, one can derive the following estimate:

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{(A\varepsilon)^{2q}}{\varphi^{2q}_{\frac72,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac14-\kappa-Cs^{-\frac14}-C\|\partial_X\varepsilon\|_{L^\infty}-C\|\frac{\varepsilon}{|X|}\|_{L^\infty}\right)\int_{\mathbb{R}^2}\frac{(A\varepsilon)^{2q}}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(\varepsilon^q)|^2}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}\\
&+Cs^{-q}+C\int_{\mathbb{R}^2}\frac{\varepsilon^{2q}}{\varphi^{2q}_{\frac72,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}$$

This implies that the analogue of Proposition 16 holds, i.e. that we can bootstrap the following estimates for the remainder ε:

$$\sum_{0\leq j_1+j_2\leq 2}\left(\int_{\mathbb{R}^2}\frac{((\partial_Z^{j_1}A^{j_2}\varepsilon)^{2q}}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}+\left(\int_{\mathbb{R}^2}\frac{(((Y\partial_Y)^{j_1}A^{j_2}\varepsilon)^{2q}}{\varphi^{2q}_{\frac72,0}}\frac{dXdY}{|X|\langle Y\rangle}\right)^{\frac{1}{2q}}\lesssim\frac{1}{\sqrt{s}}. \quad (5.7)$$

bd:varepsilon

This estimate, together with the weighted Sobolev embedding (B.2), gives the following pointwise estimates on ε as in Lemma 18:

$$\begin{aligned}
|\varepsilon|&\lesssim s^{-\frac12}(1+|Z|)^{\frac32}|\tilde X|^{\frac72}(1+|\tilde X|)^{\frac12-\frac72}\lesssim s^{-\frac12}|X|,\\
|\partial_X\varepsilon|&\lesssim s^{-\frac12}(1+|Z|)^{-\frac32}|\tilde X|^{\frac52}(1+|\tilde X|)^{\frac12-\frac72}\lesssim s^{-\frac12},\\
|\partial_Z\varepsilon|&\lesssim s^{-\frac12}(1+|Z|)^{\frac12}|\tilde X|^{\frac72}(1+|\tilde X|)^{\frac12-\frac72}.
\end{aligned}$$

By using the above estimate in the decomposition 5.5, combined with (5.2) and (5.3), we get that on compact sets in the variables X and Z there holds the estimate:

$$v=\Theta+O_{C^1}(s^{-\frac14}).$$

This then ends the existence part of the proof of Theorem 2.

The stability of this blow-up pattern in $\mathcal{B}$ defined by (1.10) is a direct consequence of the stability of the underlying blow-up for (NLH) proved by Merle and Zaag [24]. Indeed, assume u with initial datum u_0 is a solution that is constructed in this section belonging in addition to the Schwartz class, and u_0' is such that $\|u_0'-u_0\|_{\mathcal{B}}\leq\delta$ for some $\delta>0$. It is proved in [24] that there exists T' with $T'\to T$ as $\delta\to 0$ such that $\xi'=-\partial_x u'_{|x=0}$ blows up at time T', and that in the self-similar variables (5.1) with $s'=-\log(T'-t)$ the renormalised solution f' is global and satisfies all the bounds of Subsection (5.1) on $[s_0',+\infty)$.

Let now $t_0>0$. For t_0 small enough, and then for δ small enough, via standard parabolic regularising effects, f' satisfies all the bounds of Subsection (5.1) on $[s'(t_0),+\infty)$, in particular (5.2) for derivatives up to order $J=3$. The same argument applies for $\zeta'=\partial_x^3u'_{|x=0}$, as we proved that the regime described in the corresponding part of Subsection (5.1) is stable: g' satisfies all the bounds of this Subsection on $[s'(t_0),+\infty)$, in particular (5.2) for $J=3$.

On $[s'(t_0),+\infty)$, the bounds (5.2) and (5.3) with $J=3$ for f' and g' implies that the bound (5.6) on R holds true. Moreover, the bound (5.7) holds true at time $s'(t_0)$ for t_0 small enough, because of the continuity of the flow of the equation in $\mathcal{B}$. Thus, from the analysis of the current Subsection, the solution v remains in the bootstrap regime on $[s'(t_0),+\infty)$. The solution u' in original variables thus blows up with the same behaviour than u at time T'.

A. One-dimensional functional analysis results

Lemma 42 (Poincaré inequality in L^2_ρ). *For any $f \in H^1_\rho$ defined by (4.5) one has that $Yf \in L^2_\rho$ with*

$$\|Yf\|_{L^2_\rho} \lesssim \|f\|_{H^1_\rho}. \tag{A.1}$$

eq:Poincare

Proof. We prove (A.1) for smooth and compactly supported functions, and its extension to H^1_ρ follows by a density argument. Performing an integration by parts one first finds

$$\int Y\varepsilon\partial_Y\varepsilon e^{-\frac{Y^2}{4}}dY = \frac14\int Y^2\varepsilon^2 e^{-\frac{Y^2}{4}}dY - \frac12\int \varepsilon^2 e^{-\frac{Y^2}{4}}dY.$$

Therefore, using Cauchy-Schwarz and Young's inequalities one obtains:

$$\begin{aligned}\int Y^2\varepsilon^2 e^{-\frac{Y^2}{4}}dY &= 4\int Y\varepsilon\partial_Y\varepsilon e^{-\frac{Y^2}{4}}dY + 2\int \varepsilon^2 e^{-\frac{Y^2}{4}}dY\\ &\leq 2\epsilon\int Y^2\varepsilon^2 e^{-\frac{Y^2}{4}}ddY + \frac{2}{\epsilon}\int |\partial_Y\varepsilon|^2 e^{-\frac{Y^2}{4}}dY + 2\int \varepsilon^2 e^{-\frac{Y^2}{4}}dY.\end{aligned}$$

Taking $0<\epsilon<1/2$ yields the desired result. □

Lemma 43. *If $\varepsilon \in H^1_{loc}\{|Y|\geq 1\}$ is such that $\int_{|Y|\geq 1}(\varepsilon^2+(Y\partial_Y\varepsilon)^2)|Y|^{-1}dY$ is finite, then ε is bounded with:*

$$\|\varepsilon\|_{L^\infty(\{|Y|\geq 1\})} \lesssim \|\varepsilon\|_{L^2\left(\{|Y|\geq 1\},\frac{dY}{|Y|}\right)} + \|Y\partial_Y\varepsilon\|_{L^2\left(\{|Y|\geq 1\},\frac{dY}{|Y|}\right)} \tag{A.2}$$

bd:Sobolev

Proof. Assume that the right hand side of (A.2) is finite. Let $A\geq 1$ and $v(Z)=\varepsilon(AZ)$. Then, changing variables and using Sobolev embedding gives for some C independent on A:

$$\begin{aligned}&\|\varepsilon\|^2_{L^\infty([A,2A])} = \|v\|^2_{L^\infty([1,2])} \leq C\left(\int_1^2 v^2(Z)dZ + \int_1^2 |\partial_Z v|^2(Z)dZ\right)\\ \leq\ & C\left(\int_A^{2A} v^2(Y)\frac{dY}{A} + \int_1^2 A|\partial_Y v|^2(Y)dY\right) \leq C\left(\int_A^{2A} v^2(Y)\frac{dY}{|Y|} + \int_A^{2A}|Y\partial_Y v|^2(Y)\frac{dY}{|Y|}\right)\\ \leq\ & C\left(\int_{|Y|\geq 1} v^2(Y)\frac{dY}{|Y|} + \int_{|Y|\geq 1}|Y\partial_Y v|^2(Y)\frac{dY}{|Y|}\right)\end{aligned}$$

Taking the supremum with respect to A in the above estimate yields (A.2). □

B. Two-dimensional functional analysis results

Lemma 44. *Let $q\in\mathbb{N}^*$. Then for any $u\in W^{1,2q}_{loc}(\mathbb{R}^2)$ one has:*

$$\|u\|^{2q}_{L^\infty(\mathbb{R}^2)} \leq C(q)\left(\int_{\mathbb{R}^2} u^{2q}\frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2}(X\partial_X u)^{2q}\frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2}(\langle Y\rangle\partial_Y u)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\right) \tag{B.1}$$

eq:Sobo

Proof. The result follows from the classical Sobolev embedding and a scaling argument. Assume that the right hand side of (B.1) is finite. Let $A\in\mathbb{R}$ and change variables $\tilde X = X/A$ and

$u(X,Y)=v(\tilde X,Y)$. From Sobolev embedding one has that

$$\begin{aligned} & \|u\|^{2q}_{L^\infty(\{A\leq |X|\leq 2A,\ |Y|\leq 1\})} = \|v\|^{2q}_{L^\infty(\{1\leq |\tilde X|\leq 2,\ |Y|\leq 1\})} \\ \leq & \ C(q)\int_{1\leq |\tilde X|\leq 2,\ |Y|\leq 1}\left(v^{2q}+(\partial_{\tilde X}v)^{2q}+(\partial_Y v)^{2q}\right)d\tilde X dY \\ \leq & \ C(q)\int_{A\leq |X|\leq 2A,\ |Y|\leq 1}\left(u^{2q}+(A\partial_X u)^{2q}+(\partial_Y u)^{2q}\right)\frac{dXdY}{A} \\ \leq & \ C(q)\left(\int_{\mathbb{R}^2}u^{2q}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}(X\partial_X u)^{2q}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}(\langle Y\rangle\partial_Y u)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\right). \end{aligned}$$

Now let $A>0$ and $B\geq 1$ and change variables $\tilde X=X/A$, $\tilde Y=Y/B$ and $u(X,Y)=v(\tilde X,\tilde Y)$. Then again from Sobolev:

$$\begin{aligned} & \|u\|^{2q}_{L^\infty(\{A\leq |X|\leq 2A,\ B\leq |Y|\leq 2B\})} = \|v\|^{2q}_{L^\infty(\{1\leq |\tilde X|\leq 2,\ 1\leq |\tilde Y|\leq 2\})} \\ \leq & \ C(q)\int_{1\leq |\tilde X|\leq 2,\ 1\leq |\tilde Y|\leq 2}\left(v^{2q}+(\partial_{\tilde X}v)^{2q}+(\partial_{\tilde Y} v)^{2q}\right)d\tilde X d\tilde Y \\ \leq & \ C(q)\Big(\int_{A\leq |X|\leq 2A,\ B\leq |Y|\leq 2B}\left(u^{2q}+(A\partial_X u)^{2q}+(B\partial_Y u)^{2q}\right)\frac{dXdY}{AB} \\ \leq & \ C(q)\left(\int_{\mathbb{R}^2}u^{2q}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}(X\partial_X u)^{2q}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}(\langle Y\rangle\partial_Y u)^{2q}\frac{dXdY}{|X|\langle Y\rangle}\right). \end{aligned}$$

Combing the two above inequalities, as the constant in the second one does not depend on A and B, yields (B.1) but for the quantity $\|u\|^{2q}_{L^\infty(\mathbb{R}^2\setminus\{X=0\})}$. This in turn yields (B.1) by continuity. □

Corollary 45. *Let $q\in\mathbb{N}^*$. Then for any $u\in W^{1,2q}_{loc}(\mathbb{R}^2)$ one has:*

$$\left\|\frac{u}{\phi_{j,0}(X,Z)}\right\|^{2q}_{L^\infty(\mathbb{R}^2)}\leq C(q)\left(\int_{\mathbb{R}^2}\frac{u^{2q}}{\phi^{2q}_{j,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}\frac{(X\partial_X u)^{2q}}{\phi^{2q}_{j,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}+\int_{\mathbb{R}^2}\frac{(\langle Y\rangle\partial_Y u)^{2q}}{\phi^{2q}_{j,0}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right) \tag{B.2}$$

eq:weightedS

Proof. Assume that the right hand side of (B.2) is finite. First, notice from (3.21) that

$$|X\partial_X\phi_{X,j}|\sim|\phi_{X,j}|$$

From this and (3.14) one deduces that:

$$\begin{aligned} & |X\partial_X\phi_{j,0}(X,Z)| = \left|X\partial_X\left(\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\phi_{X,j}\left(F_k^{\frac32}(Z)X\right)\right)\right| \\ = & \ \left|\left(1+Z^{2k}\right)^{\frac{j}{2}-1}(X\partial_X\phi_{X,j})\left(F_k^{\frac32}(Z)X\right)\right|\sim\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\left|\phi_{X,j})\left(F_k^{\frac32}(Z)X\right)\right| = |\phi_{j,0}(X,Z)|. \end{aligned}$$

This implies that

$$\left|X\partial_X\left(\frac{1}{\phi_{j,0}(X,Z)}\right)\right| = \left|\frac{X\partial_X\phi_{j,0}(X,Z)}{\phi^2_{j,0}(X,Z)}\right|\sim\left|\frac{1}{\phi_{j,0}(X,Z)}\right|.$$

Assume now that $|Y| \leq 1$. Since $\partial_Y = e^{-(k-1)/(2k)s}\partial_Z$ then

$$\begin{aligned}
|\partial_Y \phi_{j,0}(X,Z)| &= \left| e^{-\frac{k-1}{2k}s}\partial_Z \left(\left(1+Z^{2k}\right)^{\frac{j}{2}-1} \phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right)\right)\right| \\
\leq & \left| \partial_Z \left(\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\right) \phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right)\right| + \frac{3}{2}\left| \left(1+Z^{2k}\right)^{\frac{j}{2}-1} \frac{\partial_Z F_k(Z)}{F_k(Z)}(X\phi_{X,j})\left(F_k^{\frac{3}{2}}(Z)X\right)\right| \\
\leq & \ C\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right) = C|\phi_{j,0}(X,Z)|
\end{aligned}$$

since $|\partial_Z F_k| \leq F_k$. If $|Y| \geq 1$. Since $Y\partial_Y = Z\partial_Z$ then similarly

$$\begin{aligned}
|Y\partial_Y \phi_{j,0}(X,Z)| &= \left| Z\partial_Z \left(\left(1+Z^{2k}\right)^{\frac{j}{2}-1} \phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right)\right)\right| \\
\lesssim & \left| Z\partial_Z \left(\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\right) \phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right)\right| + \left| \left(1+Z^{2k}\right)^{\frac{j}{2}-1} Z\frac{\partial_Z F_k(Z)}{F_k(Z)}(X\phi_{X,j})\left(F_k^{\frac{3}{2}}(Z)X\right)\right| \\
\leq & \ C\left(1+Z^{2k}\right)^{\frac{j}{2}-1}\phi_{X,j}\left(F_k^{\frac{3}{2}}(Z)X\right) = C|\phi_{j,0}(X,Z)|
\end{aligned}$$

since $|Z\partial_Z F_k| \leq F_k$. From the two above inequalities one deduces that for any $Y \in \mathbb{R}$:

$$|\langle Y\rangle \partial_Y \phi_{j,0}(X,Z)| \leq C|\phi_{j,0}(X,Z)|.$$

This implies again that

$$\left| \langle Y\rangle \partial_Y \left(\frac{1}{\phi_{j,0}(X,Z)}\right)\right| \leq C\left|\frac{1}{\phi_{j,0}(X,Z)}\right|.$$

From this one deduces that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\frac{u^{2q}}{\phi_{j,0}^{2q}(X,Z)} + \left(X\partial_X\left(\frac{u}{\phi_{j,0}(X,Z)}\right)\right)^{2q} + \left(\langle Y\rangle \partial_Y \left(\frac{u}{\phi_{j,0}(X,Z)}\right)\right)^{2q}\right)\frac{dXdY}{|X|\langle Y\rangle} \\
\leq & \ C(q)\left(\int_{\mathbb{R}^2} \frac{u^{2q}}{\phi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(X\partial_X u)^{2q}}{\phi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle} + \int_{\mathbb{R}^2} \frac{(\langle Y\rangle\partial_Y u)^{2q}}{\phi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right).
\end{aligned}$$

We apply (B.1) to $u/\phi_{j,0}(X,Z)$ and use the above inequality to get the desired result (B.2). □

an:lem:equiv **Lemma 46.** *Let $j_1, j_2 \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any function $u \in C^{j_1+j_2}(\mathbb{R}^2)$, for A defined by (3.26) there holds:*

$$\frac{1}{C}\sum_{j_1'=0}^{j_1}\sum_{j_2'=0}^{j_2} |Z|^{j_1'}|X|^{j_2'}|\partial_Z^{j_1'}\partial_X^{j_2'}u| \leq \sum_{j_1'=0}^{j_1}\sum_{j_2'=0}^{j_2} |(Z\partial_Z)^{j_1}(X\partial_X)^{j_2}u| \leq C\sum_{j_1'=0}^{j_1}\sum_{j_2'=0}^{j_2} |Z|^{j_1'}|X|^{j_2'}|\partial_Z^{j_1'}\partial_X^{j_2'}u|, \tag{B.3}$$

an:equivalen

and for $j_2 \geq 1$:

$$|\partial_Z^{j_1}A^{j_2}u| \leq C\sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2} (1+|Z|)^{-(j_1-j_1')}|X|^{j_2'}|\partial_Z^{j_1'}\partial_X^{j_2'}u|, \tag{B.4}$$

an:equivalen

$$|(Z\partial_Z)^{j_1}A^{j_2}u| \leq C\sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2} |Z|^{j_1'}|X|^{j_2'}|\partial_Z^{j_1'}\partial_X^{j_2'}u|, \tag{B.5}$$

an:equivalen

$$|X|^{j_2}|\partial_Z^{j_1}\partial_X^{j_2}u|\leq C\sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}(1+|Z|)^{-(j_1-j_1')}|\partial_Z^{j_1'}A^{j_2'}u|, \quad (B.6)$$ an:equivalen

$$|Z|^{j_1}|X|^{j_2}|\partial_Z^{j_1}\partial_X^{j_2}u|\leq C\sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|(Z\partial_Z)^{j_1'}A^{j_2'}u|. \quad (B.7)$$ an:equivalen

Proof. (B.3) follows from an easy induction argument that we leave to the reader.

Step 1 *Proof of* (B.4). We first claim that there exists a family of profiles $(f_{j_2,j_2'})_{j_2'\leq j_2}$ such that

$$A^{j_2}u=\sum_{j_2'=1}^{j_2}f_{j_2,j_2'}\partial_X^{j_2'}u, \quad (B.8)$$ an:idAj2

and satisfying for any $k_1,k_2\in\mathbb{N}$:

$$|\partial_Z^{k_1}\partial_X^{k_2}f_{j_2,j_2'}|\lesssim(1+|Z|)^{-k_1}(1+|X|)^{\min(-(k_2-j_2'),0)}|X|^{\max(j_2'-k_2,0)}. \quad (B.9)$$ an:bdfj2j2'

We prove this fact by induction on $j_2\in\mathbb{N}^*$. From (3.26), (B.8) holds for $j_2=1$ with $f_{1,1}=3X/2+F_k^{-3/2}(Z)\Psi_1(F_k^{3/2}(Z)X)$. Since from Proposition 5, $\partial_X\Psi_1\leq 0$ and is minimal at the origin we infer that

$$\frac12|X|\leq\left|\frac32X+F_k^{-\frac32}(Z)\Psi_1(F_k^{\frac32}(Z)X)\right|\leq\frac32|X| \quad (B.10)$$ main:eq:estim

and from (3.19) we infer that for $k_1,k_2\in\mathbb{N}$:

$$\left|\partial_Z^{k_1}\partial_X^{k_2}\left(F_k^{-3/2}(Z)\Psi_1(F_k^{3/2}(Z)X)\right)\right|\lesssim\begin{cases}(1+|Z|)^{-k_1}|X| & \text{if } k_2=0,\\ (1+|Z|)^{-k_1}(1+|X|)^{-(k_2-1)} & \text{if } k_2\geq 1.\end{cases}$$

which proves (B.9) for $j_2=1$, and thus the claim is true for $j_2=1$. Assume now that the claim is true for some $j_2\in\mathbb{N}^*$. Then, using (B.8) for the integers j_2 and 1:

$$\begin{aligned}A^{j_2+1}u &= f_{1,1}\partial_X\left(\sum_{j_2'=1}^{j_2}f_{j_2,j_2'}\partial_X^{j_2'}u\right)=\sum_{j_2'=1}^{j_2}f_{1,1}\partial_X(f_{j_2,j_2'})\partial_X^{j_2'}u+f_{1,1}f_{j_2,j_2'}\partial_X^{j_2'+1}u\\ &= \sum_{j_2'=1}^{j_2}f_{1,1}\partial_X(f_{j_2,j_2'})\partial_X^{j_2'}u+\sum_{j_2'=2}^{j_2+1}f_{1,1}f_{j_2,j_2'-1}\partial_X^{j_2'}u\end{aligned}$$

and the claim is true from the bounds (B.9) for the integers j_2 and 1. Hence it is true for all $j_2\in\mathbb{N}^*$. We now apply Leibniz formula and obtain from (B.8):

$$\partial_Z^{j_1}A^{j_2}u=\sum_{j_2'=1}^{j_2}\sum_{j_1'=0}^{j_1}C_{j_1'}^{j_1}(\partial_Z^{j_1-j_1'}f_{j_2,j_2'})\partial_Z^{j_1'}\partial_X^{j_2'}u,\quad |\partial_Z^{j_1-j_1'}f_{j_2,j_2'}|\lesssim(1+|Z|)^{-(j_1-j_1')}|X|^{j_2'}$$

where the second estimate comes from (B.9), and (B.4) is proven.

Step 2 *Proof of* (B.5). This is a direct consequence of (B.3) and (B.4):

$$
\begin{aligned}
|(Z\partial_Z)^{j_1}A^{j_2}u| &\lesssim \sum_{j_1'=0}^{j_1}|Z|^{j_1'}|\partial_Z^{j_1'}A^{j_2}u| \lesssim \sum_{j_1'=0}^{j_1}\sum_{\tilde{j}_1'=0}^{j_1'}\sum_{j_2'=1}^{j_2}|Z|^{j_1'}(1+|Z|)^{-(j_1'-\tilde{j}_1')}|X|^{j_2'}|\partial_Z^{\tilde{j}_1'}(X\partial_X)^{j_2'}u|,\\
&\lesssim \sum_{j_1'=0}^{j_1}\sum_{\tilde{j}_1'=0}^{j_1'}\sum_{j_2'=1}^{j_2}|Z|^{\tilde{j}_1'}|X|^{j_2'}|\partial_Z^{\tilde{j}_1'}(X\partial_X)^{j_2'}u| \lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|Z|^{j_1'}|X|^{j_2'}|\partial_Z^{j_1'}(X\partial_X)^{j_2'}u|.
\end{aligned}
$$

Step 3 *Proof of* (B.6). First, from (B.3) one has:

$$
|X|^{j_2}|\partial_Z^{j_1}\partial_X^{j_2}| \lesssim \sum_{j_2'=1}^{j_2}|\partial_Z^{j_1}(X\partial_X)^{j_2'}u|. \tag{B.11}
$$

an:interequiv

We then claim that there exists a family of profiles $(g_{j_2,j_2'})_{j_2'\leq j_2}$ such that

$$
(X\partial_X)^{j_2}u = \sum_{j_2'=1}^{j_2} g_{j_2,j_2'}A^{j_2'}u, \tag{B.12}
$$

an:idXpaXj2

and satisfying for any $k_1,k_2\in\mathbb{N}$:

$$
|\partial_Z^{k_1}\partial_X^{k_2}g_{j_2,j_2'}| \lesssim (1+|Z|)^{-k_1}(1+|X|)^{-k_2}. \tag{B.13}
$$

an:bdgj2j2'

From (3.26), (B.12) holds for $j_2=1$ with

$$
g_{1,1} = \frac{X}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1\left(F_k^{\frac{3}{2}}(Z)X\right)} = \left(\frac{\tilde{X}}{\frac{3}{2}\tilde{X}+\Psi_1}\right)(F_k^{\frac{3}{2}}(Z)X).
$$

From (3.19) and (B.10) one has that for any $k_2\in\mathbb{N}$,

$$
\left|\partial_{\tilde{X}}\left(\frac{\tilde{X}}{\frac{3}{2}\tilde{X}+\Psi_1(\tilde{X})}\right)\right| \lesssim (1+|\tilde{X}|)^{-k_2}
$$

and hence from (3.19) the estimate (B.13) holds for $g_{1,1}$. The claim can then be proven by induction on $j_2\in\mathbb{N}^*$ by the same techniques as in Step 1, and we do not give the details here. Using (B.12), (B.13) and Leibniz formula then yields that

$$
|\partial_Z^{j_1}(X\partial_X)^{j_2}u| \lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|\partial_Z^{j_1-j_1'}g_{j_2,j_2'}\partial_Z^{j_1'}A^{j_2'}u| \lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}(1+|Z|)^{-(j_1-j_1')}|\partial_Z^{j_1'}A^{j_2'}u|.
$$

This implies (B.6) using (B.11).

Step 4 *Proof of* (B.7). This is a direct consequence of (B.3) and (B.6):

$$
\begin{aligned}
|Z|^{j_1}|X|^{j_2}|\partial_Z^{j_1}\partial_X^{j_2}u| &\lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|Z|^{j_1}(1+|Z|)^{-(j_1-j_1')}|\partial_Z^{j_1'}A^{j_2'}u| \lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|Z|^{j_1'}|\partial_Z^{j_1'}A^{j_2'}u|,\\
&\lesssim \sum_{j_1'=0}^{j_1}\sum_{j_2'=1}^{j_2}|(Z\partial_Z)^{j_1'}A^{j_2'}u|.
\end{aligned}
$$

□

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