Well-Posedness in Gevrey Function Space for 3D Prandtl Equations without Structural Assumption

WEI-XI LI
Wuhan University

NADER MASMOUDI
Courant Institute and NYU Abu-Dhabi

AND

TONG YANG
City University of Hong Kong

Abstract

We establish the well-posedness in Gevrey function space with optimal class of regularity 2 for the three-dimensional Prandtl system without any structural assumption. The proof combines in a novel way a new cancellation in the system with some of the old ideas to overcome the difficulty of the loss of derivatives in the system. This shows that the three-dimensional instabilities in the system leading to ill-posedness are not worse than the two-dimensional ones.

1 Introduction and Main Results

As the foundational system of boundary layer theories, Prandtl equation was derived by Prandtl in 1904 from the incompressible Navier-Stokes equation with no-slip boundary condition for the description of the behavior of fluid motion near the boundary when viscosity vanishes. In fact, in this viscous to inviscid limit process, there exists a boundary layer where the majority of the drag experienced by the solid body can be modeled by a "simplified" system derived from the incompressible Navier-Stokes equations for balancing the inertial and frictional forces. Outside this layer, the viscosity can be basically neglected as it has no significant effect on the fluid so that the fluid motion can be modeled by the Euler equation. Even though there are fruitful mathematical theories developed since the seminal works by Oleinik in 1960s, most of the well-posedness theories are limited to the two space dimensions under Oleinik's monotonicity condition except the classical work by Sammartino-Caflisch in 1998 in the framework of analytic functions and some recent work in Gevrey function spaces.

The Prandtl equation can be viewed as a typical example of partial differential equations with rich structure that includes mix-type and degeneracy in dissipation.

Communications on Pure and Applied Mathematics, Vol. LXXV, 1755–1797 (2022)

Hence, it provides many challenging mathematical problems and many of them remain unsolved after more than one hundred years from its derivation.

This paper aims to establish the well-posedness theory for the three-dimensional Prandtl equation in Gevrey spaces with the optimal class of regularity 2 that is implied by the instability results; cf. [22, 24]. Compared with the recent result in two space dimensions [9], our new approach is more direct and robust to take care of the loss of derivative in the two tangential directions. In particular, it gives a simpler proof to the result in two dimensions [9]. Hence, this paper is a complete answer to the well-posedness theory without any structual assumption in the three-dimensional setting and also shows the optimality of the ill-posedness theories.

Denote $\mathbb{R}^3_+ = \big\{(x,y,z) \in \mathbb{R}^3;\ z > 0\big\}$ and let (u, v) be the tangential component and w be the vertical component of the velocity field. Then the three-dimensional Prandtl system in $\mathbb{R}^3_+$ reads

$$
\begin{cases}
(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2)u + \partial_x p = 0, \quad t > 0,\ (x,y,z) \in \mathbb{R}^3_+,\\
(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2)v + \partial_y p = 0, \quad t > 0,\ (x,y,z) \in \mathbb{R}^3_+,\\
\partial_x u + \partial_y v + \partial_z w = 0, \quad t > 0,\ (x,y,z) \in \mathbb{R}^3_+,\\
u|_{z=0} = v|_{z=0} = w|_{z=0} = 0, \quad \lim_{z\to+\infty}(u,v) = \big(U(t,x,y), V(t,x,y)\big),\\
u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad (x,y,z) \in \mathbb{R}^3_+,
\end{cases}
\tag{1.1}
$$

where $(U(t,x,y), V(t,x,y))$ and $p(t,x,y)$ are the boundary traces of the tangential velocity field and pressure of the outer flow, satisfying

$$
\begin{cases}
\partial_t U + U\partial_x U + V\partial_y U + \partial_x p = 0,\\
\partial_t V + U\partial_x V + V\partial_y V + \partial_y p = 0.
\end{cases}
$$

Here, p, U, V are given functions determined by the Euler flow. Note that (1.1) is a degenerate parabolic system losing one order derivative in the tangential variable. We refer to [26, 29, 30] for the background and mathematical presentation of this fundamental system.

So far, the well-posedness theories for the Prandtl equation are basically limited to the two space dimensions except the works by Sammartino-Caflisch [31] in analytic function space and some recent works in Gevrey function space. In the two-dimensional case, under Oleinik's monotonicity condition, there are mainly two analytic techniques for the well-posedness theories, one referred to as coordinate transformations and the second one referred to as cancellations. Precisely, the Crocco transformation was used by Oleinik [29] for the unsteady layer to transfer the two-dimensional Prandtl equation into a degenerate parabolic equation. The cancellations in the convection terms were observed in recent years by two research groups independently [1, 28] to overcome the difficulty of the loss of derivatives in the system. However, these two powerful analytic techniques are limited to the two space dimensions so far. For three dimensions, much less is known in the well-posedness theories in Sobolev spaces. Let us also mention the work [33] on the global existence of weak solutions under an additional favorable pressure condition.

In two space dimensions, without the monotonicity condition, boundary layer separation is well expected, and there are a lot of studies of the instability phenomena. Here we only mention the works [10] about the construction of blowup solutions (see also [5,6]), [17] on the unstable Euler shear flow that yields instability of Prandtl equation, [12, 16, 20] about the instability around a shear flow with a nondegenerate critical point, [18] on the instabilility even for Rayleigh's stable shear flow, [24] about three-dimensional perturbation of shear flow when the initial data satisfy $U(z) \not\equiv kV(z)$ with a constant k. In fact, the instability result in [12] implies that the critical Gevrey index for well-posedness without structural condition is 2, and this is proved in two-dimensional space [9]. The well-posedness theories in function spaces of smooth functions were proved in [31, 32] with justification of the Prandtl ansatz when the data is analytic; then it was then studied in [15] for two space dimensions with Gevrey index = $\frac{7}{4}$, which was improved in [23] to the Gevrey index in $(1, 2]$ with nonmonotonic flow and then finalized in two space dimensions without any structural condition in [9]. In three-dimensional space, we also have some work recently without monotonicity assumption. In addition, recently the separation singularity for stationary Prandtl system was studied in [8] which justifies the Goldstein singularity.

All these results are in fact related to the high Reynolds number limit for viscous fluid systems, which is important in both mathematics and physics. Without boundary effect, the mathematical theories are now satisfactory (see, for instance, [7, 27] and references therein). The case with boundary is more complicated and interesting. For this, Kato in 1984 gave a necessary and sufficient condition for weak convergence of viscous fluid to inviscid fluid in terms of the vanishing energy dissipation rate in the region near the boundary. Recently there has been a series of works [3, 4] on such limit with relation to the Onsager conjecture. As for the Prandtl boundary layer, [25] gave a proof when the initial vorticity is supported away from the boundary for two-dimensional flow that was generalized to three dimensions in [11]; recently, there have also been some interesting works on the limit to steady flow in [21] over a moving plate and in [19] over a small distance, and in [15] about the Sobolev stability of steady shear flow in two-dimensional space. For the time-dependent problem, the stability of Prandtl expansion in two space dimensions in Gevrey function space was studied in [14].

Without loss of generality we will assume that $(U, V) \equiv 0$. Extending our result to the case of a general outer flow requires using some nontrivial weights similar to those in [9].

Then for the zero outer flow, the Prandtl system (1.1) can be written as

$$
\begin{cases}
(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2)u = 0, \quad t > 0,\ (x, y, z) \in \mathbb{R}^3_+, \\
(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2)v = 0, \quad t > 0,\ (x, y, z) \in \mathbb{R}^3_+, \\
(u, v)|_{z=0} = (0, 0), \quad \lim_{z\to+\infty}(u, v) = \big(0, 0\big), \\
(u, v)|_{t=0} = (u_0, v_0), \quad (x, y, z) \in \mathbb{R}^3_+,
\end{cases}
\tag{1.2}
$$

with

$$w(t,x,y,z) = -\int_0^z \partial_x u(t,x,y,\tilde{z})d\tilde{z} - \int_0^z \partial_y v(t,x,y,\tilde{z})d\tilde{z}.$$

Before stating our main result concerning the well-posedness of the Prandtl system (1.2), we first list some notations to be used frequently throughout the paper and then introduce the Gevrey function space.

Notations. Throughout the paper we will use without confusion $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\mathbb{R}^3_+)$, and use the notations $\|\cdot\|_{L^2(\mathbb{R}^2_{x,y})}$ and $(\cdot,\cdot)_{L^2(\mathbb{R}^2_{x,y})}$ when the variables are specified, similarly for L^∞. Moreover, we also use $L^\infty_{x,y}(L^2_z) = L^\infty(\mathbb{R}^2; L^2(\mathbb{R}_+))$ to stand for the classical Sobolev space, as does the Sobolev space $L^2_{x,y}(L^\infty_z)$. In the following discussion, by ∂^α we always mean $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ with each multi-index $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2_+$.

DEFINITION 1.1. Let $\ell > 1/2$ be a given number. With each pair (ρ,σ), $\rho > 0$ and $\sigma \geq 1$, a Banach space $X_{\rho,\sigma}$ consists of all smooth vector-valued functions (u,v) such that the Gevrey norm $\|(u,v)\|_{\rho,\sigma} < +\infty$, where $\|\cdot\|_{\rho,\sigma}$ is defined below. Recalling $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ we define

$$\begin{aligned}
&\|(u,v)\|_{\rho,\sigma} \\
&\quad = \sup_{\substack{0\leq j\leq 5\\ |\alpha|+j\geq 7}} \frac{\rho^{|\alpha|+j-7}}{[(|\alpha|+j-7)!]^\sigma}\left(\left\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u\right\|_{L^2} + \left\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j v\right\|_{L^2}\right) \\
&\qquad + \sup_{\substack{0\leq j\leq 5\\ |\alpha|+j\leq 6}} \left(\left\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u\right\|_{L^2} + \left\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j v\right\|_{L^2}\right),
\end{aligned}$$

where here and throughout the paper $\langle z\rangle = (1+|z|^2)^{1/2}$. We call σ the Gevrey index.

Remark 1.2. Note that $X_{\rho,\sigma}$ is a partial Gevrey function space. By partial Gevrey function space, we mean it consists of functions that are of Gevrey class in tangential variables x, y and lie in Sobolev space for a normal variable z.

We will look for the solutions to (1.2) in the Gevrey function space $X_{\rho,\sigma}$. For this, the initial data (u_0, v_0) satisfy the following compatibility conditions:

$$\begin{cases} (u_0,v_0)|_{z=0} = (0,0),\ \lim_{z\to+\infty}(u_0,v_0)| = (0,0),\ (\partial_z^2 u_0, \partial_z^2 v_0)|_{z=0} = (0,0), \\ \partial_z^4 u_0\big|_{z=0} = (\partial_z u_0)(\partial_x\partial_z u_0 - \partial_y\partial_z v_0)|_{z=0} + 2(\partial_z v_0)\partial_y\partial_z u_0|_{z=0}, \\ \partial_z^4 v_0\big|_{z=0} = (\partial_z v_0)(\partial_y\partial_z v_0 - \partial_x\partial_z u_0)|_{z=0} + 2(\partial_z u_0)\partial_x\partial_z u_0|_{z=0}. \end{cases} \tag{1.3}$$

The main result can be stated as follows.

THEOREM 1.3. *Let* $1 < \sigma \leq 2$. *Suppose the initial datum* (u_0, v_0) *belongs to* $X_{2\rho_0,\sigma}$ *for some* $\rho_0 > 0$ *and satisfies the compatibility condition* (1.3). *Then the system* (1.2) *admits a unique solution* $(u,v) \in L^\infty([0,T];\ X_{\rho,\sigma})$ *for some* $T > 0$ *and some* $0 < \rho < 2\rho_0$.

The rest of the paper is organized as follows. We will prove in Sections 2–5 a priori estimates. The proof of the well-posedness for the Prandtl system is given in the last section.

2 A Priori Estimate

Suppose $(u, v) \in L^\infty([0, T];\ X_{\rho_0,\sigma})$ is a solution to the Prandtl system (1.2) with initial datum $(u_0, v_0) \in X_{2\rho_0,\sigma}$. This section and Sections 3-5 are to derive an a priori estimate for (u, v).

2.1 Methodologies for a toy model

Our argument is inspired by the abstract Cauchy-Kowalewski theorem, whose statement in general Banach scale can be found in [2] and the references therein; see [23,31] as well for its application to the well-posedness theory of Prandtl equations in analytic or Gevrey spaces . Let $(Y_\rho, |\cdot|_\rho)$, $0 < \rho \le \rho_0$, be a Banach scale, which means $Y_\rho, 0 < \rho \le \rho_0$ is a family of Banach spaces with norm $|\cdot|_\rho$ such that $Y_{\rho_1} \subset Y_{\rho_2}$ for $0 < \rho_2 \le \rho_1 \le \rho_0$. Consider the initial value problem, with F a given function,

$$\partial_t \vec{u} = F(t, \nabla\vec{u}), \quad \vec{u}|_{t=0} = \vec{u}_0,$$

where the unknown $\vec{u} = \vec{u}(t, p)$ is a vector-valued function and $\nabla = \nabla_p$. Note this equation loses one order derivative, and its existence theory follows from the classical Cauchy-Kowalewski theorem provided F is an analytic function; meanwhile we can apply the abstract Cauchy-Kowalewski theorem to derive its existence in the Banach scale of *analytic* functions (see [31] for instance) and the key part is to find an inequality of the following type:

$$|\vec{u}(t)|_\rho \le C \int_0^t \frac{|\vec{u}(s)|_{\tilde{\rho}}}{\tilde{\rho} - \rho} ds + \text{l.o.t.} + \text{initial data}, \tag{2.1}$$

with $\rho < \tilde{\rho}$, where here and below l.o.t. refers to lower-order terms that are easier to control and initial data refers to terms that are controlled by the initial data. The intrinsic idea behind the abstract Cauchy-Kowalewski theorem is to overcome the loss of derivatives by shrinking the radius ρ. More generally, the existence theory can be extended to a Banach scale of Gevrey space rather than of analytic space when considering the following:

$$\partial_t^2 \vec{u} = F(t, \nabla\vec{u}), \quad \vec{u}|_{t=0} = \vec{u}_0,\ \partial_t\vec{u}|_{t=0} = \vec{v}_0. \tag{2.2}$$

In fact, the above equation is equivalent to

$$\begin{cases} \partial_t \vec{u} = \vec{v}, \\ \partial_t \vec{v} = F(t, \nabla\vec{u}), \\ \vec{u}|_{t=0} = \vec{u}_0,\ \vec{v}|_{t=0} = \vec{v}_0, \end{cases} \tag{2.3}$$

and roughly speaking we will lose only 1/2- rather than 1-order derivatives in each equation of (2.3) if $\vec{v}$ behaves like the $1/2$-order derivative of $\vec{u}$. Then following the argument used in the analytic case, the estimate (2.1) still holds with the analytic norm therein replaced by a Gevrey norm with index ≤ 2. We will explain it in detail in the next paragraph.

Similarly to (2.2), we replace ∂_t there by a linear operator, and consider a toy model of the Prandtl equation:

$$\left(\partial_t + \partial_x + \partial_y + \partial_z - \partial_z^2\right)^2 \varphi = G, \tag{2.4}$$

with G being the linear combinations of the following types:

$$\partial^\beta \varphi, \quad |\beta| \leq 1,$$

with $\partial^\beta = \partial_x^{\beta_1}\partial_y^{\beta_2}$.

Moreover, as explained in the previous paragraph, we rewrite (2.4) as

$$\begin{cases} \left(\partial_t + \partial_x + \partial_y + \partial_z - \partial_z^2\right)\varphi = \xi, \\ \left(\partial_t + \partial_x + \partial_y + \partial_z - \partial_z^2\right)\xi = G, \end{cases} \tag{2.5}$$

and consider ξ to behave like the $1/2$-order derivative of φ; that is, $\xi \sim \Lambda_{x,y}^{1/2}\varphi$ where we denote by $\Lambda_{x,y}$ the Fourier multiplier of symbol $(\xi^2 + \eta^2)^{1/2}$ with (ξ, η) the dual variable of (x, y). Now let $\varphi \in Y_\rho$ with Y_ρ being the Gevrey space of index 2, that is,

$$\|\partial^\alpha \varphi\|_{L^2} \leq C(|\alpha|!)^2/\rho^{|\alpha|}$$

with C independent of $\alpha \in \mathbb{Z}_+^2$. Then the quantity $\frac{\rho^{|\alpha|}}{(|\alpha|!)^2}\|\partial^\alpha\varphi\|_{L^2}$ is uniformly bounded with respect to $\alpha \in \mathbb{Z}_+^2$. Then by the interpolation inequality we have, supposing $\rho \leq 1$ without loss of generality,

$$\begin{aligned} \left\|\Lambda_{x,y}^{1/2}\partial^\alpha\varphi\right\|_{L^2} &\leq C\left[\left((|\alpha|+1)\right)^2/\rho^{|\alpha|+1}\right]^{1/2}[(|\alpha|!)^2/\rho^{|\alpha|}]^{1/2} \\ &\leq \widetilde{C}\,\frac{((|\alpha|+1)!)^2}{\rho^{|\alpha|+1}}\,\frac{1}{|\alpha|} \end{aligned} \tag{2.6}$$

with $\widetilde{C}$ independent of α. This motivates us to define $|\cdot|_\rho$ for each $\rho > 0$ by

$$|\vec{b}|_\rho = \sup_{|\alpha|\geq 0}\frac{\rho^{|\alpha|}}{(|\alpha|!)^2}\|\partial^\alpha\varphi\|_{L^2} + \sup_{|\alpha|\geq 0}\frac{\rho^{|\alpha|+1}}{[(|\alpha|+1)!]^2}|\alpha|\|\partial^\alpha\xi\|_{L^2}, \tag{2.7}$$

where we use the notation $\vec{b} = (\varphi, \xi)$ with ξ given in (2.5) and $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$. Note in the above definition there is an additional factor $|\alpha|$ before the norms $\|\partial^\alpha\xi\|_{L^2}$, which follows from (2.6) since $\xi \sim \Lambda_{x,y}^{1/2}\varphi$.

Next we will derive the estimate for the norm $|\vec{b}|_\rho$ defined above. Suppose $\partial_z\varphi|_{z=0} = \xi|_{z=0} = 0$. Then applying the standard energy method to (2.5) we have, for any $|\alpha| \geq 1$ and any $0 < \rho < 1$,

$$\begin{aligned}
&\frac{\rho^{2|\alpha|}}{(|\alpha|!)^4}\|\partial^\alpha \varphi(t)\|_{L^2}^2 + \frac{\rho^{2(|\alpha|+1)}}{[(|\alpha|+1)!]^4}|\alpha|^2\|\partial^\alpha \xi(t)\|_{L^2}^2 \\
&\quad + \int_0^t \frac{\rho^{2|\alpha|}}{(|\alpha|!)^4}\|\partial_z\partial^\alpha \varphi(s)\|_{L^2}^2\, ds + \int_0^t \frac{\rho^{2(|\alpha|+1)}}{[(|\alpha|+1)!]^4}|\alpha|^2\|\partial_z\partial^\alpha \xi(s)\|_{L^2}^2\, ds \\
&\quad \leq 2\frac{\rho^{2|\alpha|}}{(|\alpha|!)^4}\int_0^t \|\partial^\alpha \xi(s)\|_{L^2}\|\partial^\alpha \varphi(s)\|_{L^2}\, ds \\
&\qquad + C|\alpha|^2\frac{\rho^{2(|\alpha|+1)}}{[(|\alpha|+1)!]^4}\int_0^t \sum_{|\beta|=1}\|\partial^{\alpha+\beta}\varphi(s)\|_{L^2}\|\partial^\alpha \xi(s)\|_{L^2}\, ds \\
&\qquad + \text{l.o.t.} + \text{initial data}.
\end{aligned}$$

From the definition of $|\cdot|_r$ given in (2.7), it follows that

$$\forall r > 0, \forall j \geq 1, \|\partial^j \varphi\|_{L^2} \leq \frac{(j!)^2}{r^j}|\vec{b}|_r \text{ and } \|\partial^j \xi\|_{L^2} \leq \frac{1}{j}\frac{[(j+1)!]^2}{r^{j+1}}|\vec{b}|_r.$$

We use the above estimates to compute, for any $\widetilde{\rho}$ with $0 < \rho < \widetilde{\rho} \leq 1$,

$$\begin{aligned}
&\frac{\rho^{2|\alpha|}}{(|\alpha|!)^4}\int_0^t \|\partial^\alpha \xi(s)\|_{L^2}\|\partial^\alpha \varphi(s)\|_{L^2}\, ds \\
&\quad \leq \frac{\rho^{2|\alpha|}}{(|\alpha|!)^4}\int_0^t \frac{1}{|\alpha|}\frac{[(|\alpha|+1)!]^2}{\widetilde{\rho}^{|\alpha|+1}}\frac{(|\alpha|!)^2}{\widetilde{\rho}^{|\alpha|}}|\vec{b}(s)|_{\widetilde{\rho}}^2\, ds \\
&\quad \leq 4\int_0^t \frac{|\alpha|}{\widetilde{\rho}}\frac{\rho^{2|\alpha|}}{\widetilde{\rho}^{2|\alpha|}}|\vec{b}(s)|_{\widetilde{\rho}}^2\, ds \leq 4\int_0^t \frac{|\alpha|}{\widetilde{\rho}}\Big(\frac{\rho}{\widetilde{\rho}}\Big)^{|\alpha|}|\vec{b}(s)|_{\widetilde{\rho}}^2\, ds \leq 4\int_0^t \frac{|\vec{b}(s)|_{\widetilde{\rho}}^2}{\widetilde{\rho}-\rho}\, ds,
\end{aligned}$$

the last inequality using the fact that for any integer $k \geq 1$ and for any pair $(\rho, \widetilde{\rho})$ with $0 < \rho < \widetilde{\rho} \leq 1$,

$$k\Big(\frac{\rho}{\widetilde{\rho}}\Big)^k \leq \frac{k}{\widetilde{\rho}}\Big(\frac{\rho}{\widetilde{\rho}}\Big)^k \leq \frac{1}{\widetilde{\rho}-\rho}. \tag{2.8}$$

Note that the first inequality in (2.8) is obvious since $\widetilde{\rho} \leq 1$, and the second one follows from the fact that

$$\frac{1}{1-\frac{\rho}{\widetilde{\rho}}} = \sum_{j=0}^{\infty}\Big(\frac{\rho}{\widetilde{\rho}}\Big)^j \geq k\Big(\frac{\rho}{\widetilde{\rho}}\Big)^k.$$

Applying a similar argument to that above also gives

$$\begin{aligned}
&|\alpha|^2\frac{\rho^{2(|\alpha|+1)}}{[(|\alpha|+1)!]^4}\int_0^t \sum_{|\beta|=1}\|\partial^{\alpha+\beta}\varphi(s)\|_{L^2}\|\partial^\alpha \xi(s)\|_{L^2}\, ds \\
&\quad \leq 2\int_0^t |\alpha|\frac{\rho^{2(|\alpha|+1)}}{\widetilde{\rho}^{2(|\alpha|+1)}}|\vec{b}(s)|_{\widetilde{\rho}}^2\, ds \leq 2\int_0^t \frac{|\vec{b}(s)|_{\widetilde{\rho}}^2}{\widetilde{\rho}-\rho}\, ds.
\end{aligned}$$

Then combining the above inequalities we conclude

$$|\vec{b}(t)|_{\rho}^{2} \leq C \int_{0}^{t} \frac{|\vec{b}(s)|_{\widetilde{\rho}}^{2}}{\widetilde{\rho}-\rho}\, ds + \text{l.o.t.} + \text{initial data}.$$

This estimate enables us to follow the argument for proving the abstract Cauchy-Kowalewski theorem. We see, for instance, [23, Section 8] and Section 6 below for the detailed discussion, to obtain the existence of the solution to (2.5).

Finally, we remark that in the above argument we do not use the diffusion property of the system (2.5) in the normal variable z when dealing with the tangential derivatives of φ and ξ.

2.2 Auxiliary functions and statement of the a priori estimate

Inspired by Dietert and Gérard-Varet's work [9] let $\mathcal{U}$ be a solution to the linear initial boundary problem

$$\text{(2.9)}\quad \begin{cases} \left(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\right)\int_0^z \mathcal{U}(t,x,y,\tilde{z})d\tilde{z} = -\partial_x w(t,x,y,z), \\ \mathcal{U}|_{t=0} = 0, \quad \partial_z\mathcal{U}|_{z=0} = \mathcal{U}|_{z\to+\infty} = 0. \end{cases}$$

The existence of $\mathcal{U}$ just follows from the standard parabolic theory. In fact, we first construct a solution f to the following:

$$\text{(2.10)}\quad \begin{cases} \left(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\right) f = -\partial_x w, \\ f|_{t=0} = 0, \quad f|_{z=0} = \partial_z f|_{z\to+\infty} = 0, \end{cases}$$

and then define $\mathcal{U} = \partial_z f$, which will solve (2.9). Similarly, let $\widetilde{\mathcal{U}}$ solve

$$\text{(2.11)}\quad \begin{cases} \left(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\right)\int_0^z \widetilde{\mathcal{U}}(t,x,y,\tilde{z})d\tilde{z} \\ \quad = -\partial_y w(t,x,y,z), \\ \widetilde{\mathcal{U}}|_{t=0} = 0, \quad \partial_z\widetilde{\mathcal{U}}|_{z=0} = \widetilde{\mathcal{U}}|_{z\to+\infty} = 0. \end{cases}$$

Moreover, we define λ, δ and $\widetilde{\lambda}, \widetilde{\delta}$ as follows:

$$\text{(2.12)}\quad \begin{cases} \lambda = \partial_x u - (\partial_z u)\int_0^z \mathcal{U}\, d\tilde{z}, \quad \widetilde{\lambda} = \partial_y u - (\partial_z u)\int_0^z \widetilde{\mathcal{U}}\, d\tilde{z}, \\ \delta = \partial_x v - (\partial_z v)\int_0^z \mathcal{U}\, d\tilde{z}, \quad \widetilde{\delta} = \partial_y v - (\partial_z v)\int_0^z \widetilde{\mathcal{U}}\, d\tilde{z}, \end{cases}$$

that are to be used to derive the estimate on $\mathcal{U}$ and $\widetilde{\mathcal{U}}$. As to be seen later a new type of cancellation will be applied when deriving the equation for λ and this enables us to eliminate the bad term involving $\partial_x w$ that loses one order derivative. The idea of observing cancellation mechanism to overcome the lost of derivative was initiated independently by Alexandre-Wang-Xu-Yang [1] and Masmoudi-Wong [28], where they considered the two-dimensional case and introduced the good-unknown of the type $\partial_x^m\partial_z u - \left(\partial_z^2 u/\partial_z u\right)\partial_x^m u$ for $m \geq 1$, under the Oleinik's monotonicity condition $\partial_z u \neq 0$; see also [15, 23] for other type of cancellations when exploiting the well-posedness theory for Prandtl equation without analyticity or monotonicity. Note that we can not apply directly the above good-unknown in our case since

$\partial_z u$ may vanish, and the novelty here is the introduction of the auxiliary functions $\lambda, \delta, \cdots$, in (2.12), which are the generalized case of the good-unknown aforementioned. Our argument will combine a new cancellation for these auxiliary functions with the idea of introducing $\mathcal{U}$ initiated by Dietert and Gérard-Varet [9]. In fact these auxiliary functions play an import role when performing the energy estimate for $\mathcal{U}$. Precisely, if we apply ∂_z to (2.9) then we have an evolution equation for $\mathcal{U}$ with λ and δ as source terms that lead to the loss of one order derivatives. Our observation is that we only lose half rather than one order derivative in the equation for $\mathcal{U}$ since we have additionally evolution equations for λ and δ that do not lose derivatives anymore. This enables us to close the energy estimates for $\mathcal{U}$ in the Gevrey space of index up to 2 rather than in analytic space. We will explain in more detail in the next paragraph. By virtue of these functions in (2.12) we can apply the idea in the previous subsection to derive the desired estimate for $\mathcal{U}$; this is essentially different from the treatment presented in Dietert and Gérard-Varet's work [9].

Next we will explain the main difficulties and the new ideas introduced in this paper. We first estimate u. Applying ∂_x to the first equation in (1.2) yields

$$(2.13)\quad \big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x u = -(\partial_x w)\partial_z u - (\partial_x u)\partial_x u - (\partial_x v)\partial_y u.$$

Note we lose one order tangential derivatives in $\partial_x w$ which is the main difficulty for the existence theory of Prandtl equation. To overcome the loss of derivatives we introduce a new cancellation in the system. Multiplying the equation (2.9) by $\partial_z u$ and then subtracting the resulting equation by (2.13); this eliminates the term $(\partial_x w)\partial_z u$ that loses derivatives and yields

$$(2.14)\quad \begin{aligned} &\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\overbrace{\Big[\partial_x u - (\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z}\Big]}^{=\lambda} \\ &\quad = -(\partial_x u)\partial_x u - (\partial_x v)\partial_y u \\ &\qquad - \big[(\partial_y v)\partial_z u - (\partial_y u)\partial_z v\big]\int_0^z \mathcal{U}\,d\tilde{z} + 2\big(\partial_z^2 u\big)\mathcal{U}. \end{aligned}$$

Note the above equation for λ doesn't lose derivatives if considering λ has the same order as that of $\partial_x u$ and $\mathcal{U}$. Thus we can derive the estimate for λ from the equation (2.14) above and as a result the estimate on $\partial_x u$ will follow provided we can control $(\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z}$.

To control $(\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z}$ we can not perform the energy estimate from its equation (2.9) since we lose one order derivatives caused by the source term $\partial_x w$. Instead we will control $(\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z}$ in terms of $\mathcal{U}$ which solves the following equation, applying ∂_z to (2.9),

$$
\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\mathcal{U}\\
&\quad = \underbrace{\partial_x^2 u - \partial_x\Big[(\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z}\Big]}_{=\partial_x\lambda}\\
&\qquad + \underbrace{\partial_x\partial_y v - \partial_y\Big[(\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z}\Big]}_{=\partial_y\delta} + \text{l.o.t.},
\end{aligned}
\tag{2.15}
$$

recalling λ and δ are given by (2.12). Furthermore, it follows from (2.14) that

$$
\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x\lambda\\
&\quad = -\big[(\partial_x u)\partial_x\lambda + (\partial_x v)\partial_y\lambda + (\partial_x w)\partial_z\lambda\big]\\
&\qquad - \partial_x\bigg[(\partial_x u)\partial_x u + (\partial_x v)\partial_y u\\
&\qquad\qquad + [(\partial_y v)\partial_z u - (\partial_y u)\partial_z v]\int_0^z \mathcal{U}\,d\tilde{z} - 2\big(\partial_z^2 u\big)\mathcal{U}\bigg],
\end{aligned}
$$

and similarly for $\partial_y\delta$. Then combining the two equations above gives

$$
\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)^2\mathcal{U}\\
&\qquad = \text{terms involving the second-order derivatives} + \text{l.o.t.},
\end{aligned}
\tag{2.16}
$$

Then the situation is similar to that for the model equation (2.4) or (2.5), with φ and ξ therein corresponding to $\mathcal{U}$ and $\partial_x\lambda + \partial_y\delta$, respectively. Inspired by the treatment of the model equation (2.4) or (2.5) and the definition in (2.7), it is natural to consider the uniform upper bound with respect to α of the following norm:

$$
\frac{\rho^{|\alpha|-6}}{[(|\alpha|-6)!]^\sigma}\big\|\partial^\alpha\mathcal{U}\big\|_{L^2} + \frac{\rho^{(|\alpha|+1)-6}}{[(|\alpha|+1-6)!]^\sigma}(|\alpha|+1)\big(\|\partial^\alpha\partial_x\lambda\|_{L^2} + \|\partial^\alpha\partial_y\delta\|_{L^2}\big)
$$

or its equivalence

$$
\frac{\rho^{|\alpha|-6}}{[(|\alpha|-6)!]^\sigma}\big\|\partial^\alpha\mathcal{U}\big\|_{L^2} + \frac{\rho^{|\alpha|-6}}{[(|\alpha|-6)!]^\sigma}|\alpha|\big(\|\partial^\alpha\lambda\|_{L^2} + \|\partial^\alpha\delta\|_{L^2}\big)
$$

with $\sigma \le 2$, where we recall that $\mathcal{U}$ has the same order of $\partial_x u$. Precisely, we will define $|\cdot|_{\rho,\sigma}$ as below, similar to the definition in (2.7). We use the notation

$$
\vec{a} = (u, v, \mathcal{U}, \widetilde{\mathcal{U}}, \lambda, \widetilde{\lambda}, \delta, \widetilde{\delta})
$$

Recall $\mathcal{U}, \widetilde{\mathcal{U}}$ are given by (2.9) and (2.11), and $\lambda, \widetilde{\lambda}, \delta, \widetilde{\delta}$ are defined by (2.12).

DEFINITION 2.1. Let $\|(u,v)\|_{\rho,\sigma}$ be given in Definition 1.1. With the notation $\vec{a}$ above, we define $|\vec{a}|_{\rho,\sigma}$ by setting

$$\begin{aligned}|\vec{a}|_{\rho,\sigma} = \|(u,v)\|_{\rho,\sigma} &+ \sup_{|\alpha|\geq 6}\frac{\rho^{|\alpha|-6}}{[(|\alpha|-6)!]^{\sigma}}\big(\|\partial^\alpha\mathcal{U}\|_{L^2}+\|\partial^\alpha\widetilde{\mathcal{U}}\|_{L^2}\big)\\ &+\sup_{|\alpha|\leq 5}\big(\|\partial^\alpha\mathcal{U}\|_{L^2}+\|\partial^\alpha\widetilde{\mathcal{U}}\|_{L^2}\big)\\ &+\sup_{|\alpha|\geq 6}\frac{\rho^{|\alpha|-6}}{[(|\alpha|-6)!]^{\sigma}}|\alpha|\big(\|\partial^\alpha\lambda\|_{L^2}+\|\partial^\alpha\delta\|_{L^2}+\|\partial^\alpha\widetilde{\lambda}\|_{L^2}+\|\partial^\alpha\widetilde{\delta}\|_{L^2}\big)\\ &+\sup_{|\alpha|\leq 5}|\alpha|\big(\|\partial^\alpha\lambda\|_{L^2}+\big\|\partial^\alpha\delta\big\|_{L^2}+\|\partial^\alpha\widetilde{\lambda}\|_{L^2}+\|\partial^\alpha\widetilde{\delta}\|_{L^2}\big).\end{aligned}$$

Note the additional factor $|\alpha|$ before the L^2-norms of $\partial^\alpha\lambda$, $\partial^\alpha\delta$ and $\partial^\alpha\widetilde{\lambda}$, $\partial^\alpha\widetilde{\delta}$.

Remark 2.2. The auxiliary functions $\mathcal{U},\lambda,\delta$ are introduced for treating the derivatives ∂_x^m, while $\widetilde{\mathcal{U}},\widetilde{\lambda},\widetilde{\delta}$ are for ∂_y^m. Then the estimate for the general $\partial^\alpha=\partial_x^{\alpha_1}\partial_y^{\alpha_2}$ will follow as well using the relationship

$$\forall\alpha\in\mathbb{Z}_+^2,\ \forall F\in H^\infty,\quad \|\partial^\alpha F\|_{L^2}\leq\big\|\partial_x^{|\alpha|}F\big\|_{L^2}+\big\|\partial_y^{|\alpha|}F\big\|_{L^2}. \tag{2.17}$$

In this paper we will focus on performing only the estimates for ∂_x^m, since the estimates for ∂_y^m can be treated symmetrically in the same way.

Now we are ready to state the main a priori estimate. We will present in detail the proof of Theorem 1.3 for $\sigma\in[3/2,2]$. Note that the constraint $\sigma\geq 3/2$ is not essential, and indeed it is just a technical assumption for clear presentation. We refer to [22, sec. 8] to explain how to modify the proof for the case when $1<\sigma<3/2$. We make the following low-regularity assumption that will be checked in the last section of the paper:

ASSUMPTION 2.3. *Let $X_{\rho,\sigma}$ be the Gevrey function space equipped with the norm $\|\cdot\|_{\rho,\sigma}$ given in Definition* 1.1. *Suppose $(u,v)\in L^\infty([0,T];\ X_{\rho_0,\sigma})$ for some $0<\rho_0\leq 1$ and $\sigma\in[3/2,2]$ is a solution to the Prandtl system* (1.2) *with initial datum $(u_0,v_0)\in X_{2\rho_0,\sigma}$. Without loss of generality we may assume $T\leq 1$. Moreover we suppose that there exists a constant C_* such that for any $t\in[0,T]$,*

$$\sup_{\substack{0\leq j\leq 5\\ |\alpha|+j\leq 10}}\Big(\big\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u(t)\big\|_{L^2}+\big\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j v(t)\big\|_{L^2}\Big)\leq C_*, \tag{2.18}$$

where the constant $C_*\geq 1$ depends only on the Sobolev embedding constants, $\|(u_0,v_0)\|_{2\rho_0,\sigma}$, and the numbers ρ_0,σ,ℓ that are given in Definition 1.1.

THEOREM 2.4 (A priori estimate in Gevrey space). *Under Assumption* 2.3 *above, we can find two constants* $C_1, C_2 \geq 1$ *such that the estimate*

$$|\vec{a}(t)|_{\rho,\sigma}^2 \leq C_1 \|(u_0, v_0)\|_{2\rho_0,\sigma}^2 + e^{C_2 C_*^2} \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4)\, ds + e^{C_2 C_*^2} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}\, ds$$

holds for any pair $(\rho, \tilde{\rho})$ *with* $0 < \rho < \tilde{\rho} < \rho_0$ *and any* $t \in [0, T]$, *where the constant* C_1 *can be computed explicitly and the constant* C_2 *depends only on the Sobolev embedding constants and the numbers* ρ_0, σ, ℓ *given in Definition* 1.1. *Both* C_1 *and* C_2 *are independent of the constant* C_* *given in* (2.18).

3 Estimate on $\partial^\alpha \mathcal{U}$ and $\partial^\alpha \widetilde{\mathcal{U}}$

To prove the a priori estimate stated in Theorem 2.4, we will proceed through this section and Sections 4–5 to derive the upper bound for the terms involved in Definition 2.1 of $|\vec{a}|_{\rho,\sigma}$. For the argument presented in Sections 3–5 we always suppose Assumption 2.3 is fulfilled by $(u, v) \in L^\infty([0, T];\ X_{\rho_0,\sigma})$.

To simplify the notation, we use from now on the two capital letters C_1, C to denote some generic constant that may vary from line to line, both depending only on the Sobolev embedding constants and the numbers ρ_0, σ, ℓ given in Definition 1.1 but *independent of* the constant C_* in (2.18) and the order of derivatives denoted by m.

In this part we will derive the upper bound for the terms involving $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ in Definition 2.1 of $|\vec{a}|_{\rho,\sigma}$. Recall that $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ solve, respectively, the equations (2.9) and (2.11).

PROPOSITION 3.1. *Under Assumption* 2.3 *we have, for any* $t \in [0, T]$ *and for any pair* $(\rho, \tilde{\rho})$ *with* $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,

$$\sup_{|\alpha| \geq 6} \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha| - 6)!]^{2\sigma}} \|\partial^\alpha \mathcal{U}(t)\|_{L^2}^2 + \sup_{|\alpha| \leq 5} \|\partial^\alpha \mathcal{U}(t)\|_{L^2}^2 \leq C C_* \left(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}\, ds \right),$$

where $C_* \geq 1$ *is the constant given in* (2.18). *Symmetrically, the same upper bound also holds with* $\mathcal{U}$ *replaced by* $\widetilde{\mathcal{U}}$.

We first derive the evolution equation for $\partial_x^m \mathcal{U}$. Applying ∂_z to (2.9) yields

$$\begin{aligned} &\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\mathcal{U} \\ &\quad = \partial_x^2 u + \partial_y \partial_x v - (\partial_z u)\partial_x \int_0^z \mathcal{U}\, d\tilde{z} \\ &\qquad - (\partial_z v)\partial_y \int_0^z \mathcal{U}\, d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}, \end{aligned} \tag{3.1}$$

and thus, using the representation of λ and δ given in (2.12),

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\mathcal{U}\\&\quad = \partial_x\lambda + \partial_y\delta + (\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}.\end{aligned}$$

Then, applying ∂_x^m to the above equation we get

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m\mathcal{U}\\&\quad = -\sum_{j=1}^m \binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U} + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U} + \big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\mathcal{U}\big]\\&\qquad + \partial_x^m\big(\partial_x\lambda + \partial_y\delta\big) + \partial_x^m\bigg[(\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}\bigg].\end{aligned}$$

Taking the scalar product with $\partial_x^m\mathcal{U}$ and observing $\mathcal{U}|_{t=0} = \partial_z\mathcal{U}|_{z=0} = 0$ gives

$$\begin{aligned}&\frac{1}{2}\big\|\partial_x^m\mathcal{U}(t)\big\|_{L^2}^2 + \int_0^t \big\|\partial_z\partial_x^m\mathcal{U}(s)\big\|_{L^2}^2\,ds\\&\quad = \int_0^t \big(\big(\partial_s + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m\mathcal{U},\ \partial_x^m\mathcal{U}\big)_{L^2}\,ds\\&\quad = -\int_0^t \left(\sum_{j=1}^m \binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U} + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U}\big],\ \partial_x^m\mathcal{U}\right)_{L^2} ds\\&\qquad - \int_0^t \left(\sum_{j=1}^m \binom{m}{j}\big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\mathcal{U}, \partial_x^m\mathcal{U}\right)_{L^2} ds\\&\qquad + \int_0^t \big(\partial_x^m(\partial_x\lambda + \partial_y\delta), \partial_x^m\mathcal{U}\big)_{L^2} ds\\&\qquad + \int_0^t \left(\partial_x^m\bigg[(\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}\bigg],\ \partial_x^m\mathcal{U}\right)_{L^2} ds.\end{aligned}\tag{3.2}$$

Next we derive the upper bound for the terms on the right-hand side through the following three lemmas. (2.9) yields

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\mathcal{U}\\&\quad = \partial_x^2 u + \partial_y\partial_x v - (\partial_z u)\partial_x\int_0^z \mathcal{U}\,d\tilde{z}\\&\qquad - (\partial_z v)\partial_y\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U},\end{aligned}\tag{3.3}$$

and thus, using the representation of λ and δ given in (2.12),

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\mathcal{U}\\&\quad = \partial_x\lambda + \partial_y\delta + (\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}.\end{aligned}$$

Then, applying ∂_x^m to the above equation we get

$$
\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m\mathcal{U}\\
&\quad = -\sum_{j=1}^{m}\binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U} + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U} + \big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\mathcal{U}\big]\\
&\qquad + \partial_x^m\big(\partial_x\lambda + \partial_y\delta\big) + \partial_x^m\bigg[(\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}\bigg].
\end{aligned}
$$

Taking the scalar product with $\partial_x^m\mathcal{U}$ and observing $\mathcal{U}|_{t=0} = \partial_z\mathcal{U}|_{z=0} = 0$ gives

$$
\begin{aligned}
&\frac{1}{2}\big\|\partial_x^m\mathcal{U}(t)\big\|_{L^2}^2 + \int_0^t \big\|\partial_z\partial_x^m\mathcal{U}(s)\big\|_{L^2}^2\,ds\\
&\quad = \int_0^t \big(\big(\partial_s + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m\mathcal{U},\ \partial_x^m\mathcal{U}\big)_{L^2}\,ds\\
&\quad = -\int_0^t \bigg(\sum_{j=1}^{m}\binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U} + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U}\big],\ \partial_x^m\mathcal{U}\bigg)_{L^2}ds\\
&\qquad -\int_0^t \bigg(\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j w)\partial_x^{m-j}\partial_z\mathcal{U}, \partial_x^m\mathcal{U}\bigg)_{L^2}ds\\
&\qquad + \int_0^t \big(\partial_x^m(\partial_x\lambda + \partial_y\delta), \partial_x^m\mathcal{U}\big)_{L^2}ds\\
&\qquad + \int_0^t \bigg(\partial_x^m\bigg[(\partial_x\partial_z u + \partial_y\partial_z v)\int_0^z \mathcal{U}\,d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}\bigg],\\
&\qquad\qquad \partial_x^m\mathcal{U}\bigg)_{L^2}ds.
\end{aligned}
\tag{3.4}
$$

Next we derive the upper bound for the terms on the right-hand side by using the following three lemmas.

LEMMA 3.2. *Under the same assumptions as in Proposition* 3.1 *we have, for any* $m \geq 6$, *any* $t \in [0, T]$, *and for any pair* $(\rho, \tilde{\rho})$ *with* $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,

$$
\int_0^t \big(\partial_x^m(\partial_x\lambda + \partial_y\delta),\ \partial_x^m\mathcal{U}\big)_{L^2}\,dt \leq C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}\,ds.
$$

PROOF. It follows from Definition 2.1 of $|\vec{a}|_{r,\sigma}$ that, for any $\alpha \in \mathbb{Z}_+^2$ and for any $r > 0$,

$$
|\alpha|\big(\|\partial^\alpha\lambda\|_{L^2} + \|\partial^\alpha\delta\|_{L^2}\big) \leq \begin{cases} \frac{[(|\alpha|-6)!]^\sigma}{r^{(|\alpha|-6)}}|\vec{a}|_{r,\sigma} & \text{if } |\alpha| \geq 6,\\ |\vec{a}|_{r,\sigma} & \text{if } |\alpha| \leq 5, \end{cases}
\tag{3.5}
$$

and that, observing $\ell > 1/2$,

$$\begin{aligned}&\left\|\langle z\rangle^{-\ell-\frac{1}{2}}\int_0^z \partial^\alpha\mathcal{U}\,d\tilde{z}\right\|_{L^2}+\left\|\langle z\rangle^{-1/2}\int_0^z \partial^\alpha\mathcal{U}\,d\tilde{z}\right\|_{L^2_{x,y}(L^\infty_z)}\\ &\quad\le C\|\partial^\alpha\mathcal{U}\|_{L^2}\le\begin{cases}C\frac{[(|\alpha|-6)!]^\sigma}{r^{|\alpha|-6}}|\vec{a}|_{r,\sigma} & \text{if } |\alpha|\ge 6,\\ C|\vec{a}|_{r,\sigma}, & \text{if } |\alpha|\le 5.\end{cases}\end{aligned}\tag{3.6}$$

Using the above estimates we compute

$$\begin{aligned}&\int_0^t \big(\partial_x^m\big(\partial_x\lambda+\partial_y\delta\big),\ \partial_x^m\mathcal{U}\big)_{L^2}dt\\ &\quad\le\int_0^t\frac{1}{m+1}\frac{[(m-5)!]^\sigma}{\tilde{\rho}^{m-5}}\frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}}|\vec{a}(s)|^2_{\tilde{\rho},\sigma}\,ds\\ &\quad\le C\int_0^t\frac{m^{\sigma-1}}{\tilde{\rho}}\frac{[(m-6)!]^{2\sigma}}{\tilde{\rho}^{2(m-6)}}|\vec{a}(s)|^2_{\tilde{\rho},\sigma}\,ds\le\frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t\frac{|\vec{a}(s)|^2_{\tilde{\rho},\sigma}}{\tilde{\rho}-\rho}ds,\end{aligned}$$

the last inequality holding because $\sigma\le 2$ and

$$\begin{aligned}\frac{m}{\tilde{\rho}}\frac{1}{\tilde{\rho}^{2(m-6)}}&=\frac{1}{\rho^{2(m-6)}}\frac{m}{\tilde{\rho}}\frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}}\\ &\le C\frac{1}{\rho^{2(m-6)}}\frac{m-6}{\tilde{\rho}}\left(\frac{\rho}{\tilde{\rho}}\right)^{m-6}\le C\frac{1}{\rho^{2(m-6)}}\frac{1}{\tilde{\rho}-\rho}\end{aligned}$$

due to (2.8). The proof of Lemma 3.2 is completed. □

LEMMA 3.3. *Under the same assumption as in Proposition* 3.1 *we have, for any* $m\ge 6$, *any* $t\in[0,T]$, *and any pair* $(\rho,\tilde{\rho})$ *with* $0<\rho<\tilde{\rho}<\rho_0\le 1$,

$$\begin{aligned}&-\int_0^t\left(\sum_{j=1}^m\binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U}+\big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U}\big],\ \partial_x^m\mathcal{U}\right)_{L^2}ds\\ &\quad-\int_0^t\left(\sum_{j=1}^m\binom{m}{j}\big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\mathcal{U},\ \partial_x^m\mathcal{U}\right)_{L^2}ds\\ &\qquad\le\frac{1}{2}\int_0^t\|\partial_z\partial_x^m\mathcal{U}\|^2_{L^2}\,ds+C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\left(\int_0^t\big(|\vec{a}(s)|^3_{\rho,\sigma}+|\vec{a}(s)|^4_{\rho,\sigma}\big)ds\right)\\ &\qquad\quad+CC_*\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t\frac{|\vec{a}(s)|^2_{\tilde{\rho},\sigma}}{\tilde{\rho}-\rho}\,ds,\end{aligned}$$

where C_* *is the constant in* (2.18).

PROOF. We treat the first term on the left side and write

$$
\begin{aligned}
&\sum_{j=1}^{m}\binom{m}{j}\big\|\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U}\big\|_{L^2}\\
&\quad\le \sum_{j=1}^{[m/2]}\binom{m}{j}\big\|\partial_x^j u\big\|_{L^\infty}\big\|\partial_x^{m-j+1}\mathcal{U}\big\|_{L^2}\\
&\qquad+\sum_{j=[m/2]+1}^{m}\binom{m}{j}\big\|\partial_x^j u\big\|_{L^2_{x,y}(L^\infty_z)}\big\|\partial_x^{m-j+1}\mathcal{U}\big\|_{L^\infty_{x,y}(L^2_z)},
\end{aligned}
\tag{3.7}
$$

where $[p]$ denotes the largest integer less than or equal to p. We need the following Sobolev embedding inequalities:

$$
\begin{aligned}
\|F\|_{L^\infty(\mathbb{R}^2_{x,y})}&\le\sqrt{2}\big(\|F\|_{L^2_{x,y}}+\|\partial_x F\|_{L^2_{x,y}}+\|\partial_y F\|_{L^2_{x,y}}\\
&\qquad+\|\partial_x\partial_y F\|_{L^2_{x,y}}\big),\\
\|F\|_{L^\infty}&\le 2(\|F\|_{L^2}+\|\partial_x F\|_{L^2}+\|\partial_y F\|_{L^2}+\|\partial_z F\|_{L^2})\\
&\qquad+2(\|\partial_x\partial_y F\|_{L^2}+\|\partial_x\partial_z F\|_{L^2}+\|\partial_y\partial_z F\|_{L^2}\\
&\qquad+\|\partial_x\partial_y\partial_z F\|_{L^2}\big).
\end{aligned}
\tag{3.8}
$$

Moreover, it follows from the definition of $|\vec{a}|_{r,\sigma}$ and from Assumption 2.3 that, for any $\alpha\in\mathbb{Z}^2_+$, any $0\le j\le 5$, and any $r>0$,

$$
\begin{aligned}
&\big\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u\big\|_{L^2}+\big\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j v\big\|_{L^2}\\
&\quad\le\begin{cases}\frac{[(|\alpha|+j-7)!]^\sigma}{r^{(|\alpha|+j-7)}}|\vec{a}|_{r,\sigma} & \text{if } |\alpha|+j\ge 7,\\ \min\{|\vec{a}|_{r,\sigma},\ C_*\} & \text{if } |\alpha|+j\le 6,\end{cases}
\end{aligned}
\tag{3.9}
$$

where C_* is the constant given in (2.18). Consequently, we use the above estimates and (3.6) to compute

$$
\begin{aligned}
&\sum_{j=1}^{[m/2]}\binom{m}{j}\|\partial_x^j u\|_{L^\infty}\|\partial_x^{m-j+1}\mathcal{U}\|_{L^2}\\
&\quad\le C\sum_{j=4}^{[m/2]}\frac{m!}{j!(m-j)!}\frac{[(j-4)!]^\sigma}{\rho^{j-4}}\frac{[(m-j-5)!]^\sigma}{\rho^{m-j-5}}|\vec{a}|^2_{\rho,\sigma}\\
&\qquad+CC_*\sum_{1\le j\le 3}\frac{m!}{j!(m-j)!}\frac{[(m-j-5)!]^\sigma}{\tilde{\rho}^{m-j-5}}|\vec{a}|_{\tilde{\rho},\sigma}.
\end{aligned}
\tag{3.10}
$$

Direct verification shows that

$$
\sum_{1\le j\le 3}\frac{m!}{j!(m-j)!}\frac{[(m-j-5)!]^\sigma}{\tilde{\rho}^{m-j-5}}|\vec{a}|_{\tilde{\rho},\sigma}\le Cm\frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}}|\vec{a}|_{\tilde{\rho},\sigma},
$$

and

$$\begin{aligned}
&\sum_{j=4}^{[m/2]} \frac{m!}{j!(m-j)!}\frac{[(j-4)!]^\sigma}{\rho^{j-4}}\frac{[(m-j-5)!]^\sigma}{\rho^{m-j-5}}|\vec{a}|^2_{\rho,\sigma}\\
&\le C\frac{|\vec{a}|^2_{\rho,\sigma}}{\rho^{m-6}}\sum_{j=4}^{[m/2]}\frac{m![(j-4)!]^{\sigma-1}[(m-j-5)!]^{\sigma-1}}{j^4(m-j)^5}\\
&\le C\frac{|\vec{a}|^2_{\rho,\sigma}}{\rho^{m-6}}\sum_{j=4}^{[m/2]}\frac{(m-6)!m^6}{j^4m^5}[(m-9)!]^{\sigma-1}\\
&\le C\frac{[(m-6)!]^\sigma|\vec{a}|^2_{\rho,\sigma}}{\rho^{m-6}}\frac{m}{m^{3(\sigma-1)}}\sum_{j=4}^{[m/2]}\frac{1}{j^4}\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec{a}|^2_{\rho,\sigma},
\end{aligned}$$

the last inequality using the fact that $\sigma \in [3/2, 2]$. Combining the above inequalities with (3.10) gives

$$\begin{aligned}
&\sum_{j=1}^{[m/2]}\binom{m}{j}\|\partial_x^j u\|_{L^\infty}\|\partial_x^{m-j+1}\mathcal{U}\|_{L^2}\\
&\quad\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec{a}|^2_{\rho,\sigma}+CC_*m\frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}}|\vec{a}|_{\tilde{\rho},\sigma}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{j=[m/2]+1}^{m}\binom{m}{j}\left\|\partial_x^j u\right\|_{L^2_{x,y}(L^\infty_z)}\left\|\partial_x^{m-j+1}\mathcal{U}\right\|_{L^\infty_{x,y}(L^2_z)}\\
&\le\sum_{j=[m/2]+1}^{m-3}\frac{m!}{j!(m-j)!}\frac{[(j-6)!]^\sigma}{\rho^{j-6}}\frac{[(m-j-3)!]^\sigma}{\rho^{m-j-3}}|\vec{a}|^2_{\rho,\sigma}\\
&\quad+\sum_{j=m-2}^{m}\frac{m!}{j!(m-j)!}\frac{[(j-6)!]^\sigma}{\rho^{j-6}}|\vec{a}|^2_{\rho,\sigma}\\
&\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec{a}|^2_{\rho,\sigma}.
\end{aligned}$$

Putting these inequalities into (3.7) we get

$$\begin{aligned}
&\sum_{j=1}^{m}\binom{m}{j}\left\|\left(\partial_x^j u\right)\partial_x^{m-j+1}\mathcal{U}\right\|_{L^2}\\
&\quad\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec{a}|^2_{\rho,\sigma}+CC_*m\frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}}|\vec{a}|_{\tilde{\rho},\sigma}.
\end{aligned}\tag{3.11}$$

The above estimate also holds with $(\partial_x^j u)\partial_x^{m-j+1}\mathcal{U}$ replaced by $(\partial_x^j v)\partial_x^{m-j}\partial_y\mathcal{U}$. This gives, using (3.6) and (2.8),

$$-\int_0^t \left(\sum_{j=1}^m \binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\mathcal{U} + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\mathcal{U}\big],\ \partial_x^m\mathcal{U}\right)_{L^2} ds$$
$$\leq C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\left(\int_0^t |\vec{a}(s)|_{\rho,\sigma}^3\,ds + C_*\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\right).$$

The assertion in Lemma 3.3 will follow if we have

$$\begin{aligned} &-\int_0^t \left(\sum_{j=1}^m \binom{m}{j}(\partial_x^j w)\partial_x^{m-j}\partial_z\mathcal{U},\ \partial_x^m\mathcal{U}\right)_{L^2} ds \\ &\leq \frac{1}{2}\int_0^t \left\|\partial_z\partial_x^m\mathcal{U}\right\|_{L^2}^2 ds \\ &\quad + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^3 + |\vec{a}(s)|_{\rho,\sigma}^4\big)ds. \end{aligned} \tag{3.12}$$

It follows from integration by parts that

$$-\int_0^t \left(\sum_{j=1}^m \binom{m}{j}(\partial_x^j w)\partial_x^{m-j}\partial_z\mathcal{U},\ \partial_x^m\mathcal{U}\right)_{L^2} ds \leq J_1 + J_2, \tag{3.13}$$

with

$$J_1 = \int_0^t \sum_{j=1}^m \binom{m}{j}\left\|\big(\partial_x^j w\big)\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\left\|\partial_z\partial_x^m\mathcal{U}\right\|_{L^2} ds$$
$$J_2 = \int_0^t \sum_{j=1}^m \binom{m}{j}\left\|\big(\partial_x^{j+1}u + \partial_x^j\partial_y v\big)\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\left\|\partial_x^m\mathcal{U}\right\|_{L^2} ds.$$

From $\|\partial^\alpha w\|_{L_z^\infty} \leq C(\|\langle z\rangle^\ell\partial_x\partial^\alpha u\|_{L_z^2} + \|\langle z\rangle^\ell\partial_y\partial^\alpha v\|_{L_z^2})$ for $\ell > 1/2$, it follows (3.9) that

$$\|\langle z\rangle^{-\ell}\partial^\alpha w\|_{L^2} + \|\partial^\alpha w\|_{L_{x,y}^2(L_z^\infty)} \leq \begin{cases} C\frac{[(|\alpha|-6)!]^\sigma}{r^{(|\alpha|-6)}}|\vec{a}|_{r,\sigma} & \text{if } |\alpha| \geq 6, \\ C\min\{|\vec{a}|_{r,\sigma},\ C_*\} & \text{if } |\alpha| \leq 5. \end{cases} \tag{3.14}$$

Then applying a similar argument for proving (3.11), we have

$$\sum_{j=1}^m \binom{m}{j}\left\|\big(\partial_x^j w\big)\partial_x^{m-j}\mathcal{U}\right\|_{L^2} + \sum_{j=1}^m \binom{m}{j}\left\|\big(\partial_x^{j+1}u + \partial_x^j\partial_y v\big)\partial_x^{m-j}\mathcal{U}\right\|_{L^2}$$
$$\leq C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec{a}|_{\rho,\sigma}^2,$$

and thus, using the above inequality and (3.6), we get

$$J_1 + J_2 \le \frac{1}{2}\int_0^t \|\partial_z \partial_x^m \mathcal{U}\|_{L^2}^2\, ds + C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^3 + |\vec{a}(s)|_{\rho,\sigma}^4\big)\, ds.$$

This with (3.13) yields (3.12). The proof of Lemma 3.3 is completed. □

LEMMA 3.4. *Under the same assumption as in Proposition* 3.1 *we have, for any* $m \ge 6$, *any* $t \in [0,T]$, *and any pair* $(\rho, \tilde{\rho})$ *with* $0 < \rho < \tilde{\rho} < \rho_0 \le 1$,

$$\begin{aligned}&\int_0^t \left(\partial_x^m\left[(\partial_x\partial_z u + \partial_y \partial_z v)\int_0^z \mathcal{U}\, d\tilde{z} + (\partial_x u + \partial_y v)\mathcal{U}\right], \partial_x^m \mathcal{U}\right)_{L^2} ds\\ &\quad\le \frac{1}{2}\int_0^t \|\partial_z \partial_x^m \mathcal{U}\|_{L^2}^2\, ds + C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^3 + |\vec{a}(s)|_{\rho,\sigma}^4\big)\, ds.\end{aligned}$$

PROOF. We only need to treat the first term on the left side and use the Leibniz formula and integration by parts to write

$$\begin{aligned}&\int_0^t \left(\partial_x^m\left[(\partial_x\partial_z u)\int_0^z \mathcal{U}\, d\tilde{z}\right],\ \partial_x^m \mathcal{U}\right)_{L^2} ds\\ &\quad= -\int_0^t \left(\partial_x^m\left[(\partial_x u)\int_0^z \mathcal{U}\, d\tilde{z}\right],\ \partial_z\partial_x^m \mathcal{U}\right)_{L^2} ds\\ &\qquad - \int_0^t \left(\partial_x^m\big[(\partial_x u)\mathcal{U}\big],\ \partial_x^m \mathcal{U}\right)_{L^2} ds \le I_1 + I_2,\end{aligned}$$

with

$$\begin{aligned}I_1 &= \int_0^t \sum_{j=1}^{[m/2]} \binom{m}{j} \|\langle z\rangle^{\ell}\partial_x^j \partial_x u\|_{L^\infty_{x,y}(L^2_z)} \left\|\langle z\rangle^{-\ell}\int_0^z \partial_x^{m-j}\mathcal{U}\, d\tilde{z}\right\|_{L^2_{x,y}(L^\infty_z)}\\ &\qquad\cdot \|\partial_z\partial_x^m \mathcal{U}\|_{L^2}\, ds\\ &\quad + \int_0^t \sum_{j=[m/2]+1}^{m} \binom{m}{j} \|\langle z\rangle^{\ell}\partial_x^j \partial_x u\|_{L^2} \left\|\langle z\rangle^{-\ell}\int_0^z \partial_x^{m-j}\mathcal{U}\, d\tilde{z}\right\|_{L^\infty}\\ &\qquad\cdot \|\partial_z\partial_x^m \mathcal{U}\|_{L^2}\, ds,\end{aligned}$$

$$\begin{aligned}I_2 &= \int_0^t \sum_{0\le j\le [m/2]} \binom{m}{j} \|\partial_x^j\partial_x u\|_{L^\infty_{x,y}(L^2_z)} \|\partial_x^{m-j}\mathcal{U}\|_{L^2} \|\partial_x^m \mathcal{U}\|_{L^2_{x,y}(L^\infty_z)} ds\\ &\quad + \int_0^t \sum_{[m/2]+1\le j\le m} \binom{m}{j} \|\partial_x^j\partial_x u\|_{L^2} \|\partial_x^{m-j}\mathcal{U}\|_{L^\infty_{x,y}(L^2_z)} \|\partial_x^m \mathcal{U}\|_{L^2_{x,y}(L^\infty_z)} ds.\end{aligned}$$

Now we follow the similar argument as in the proof of Lemma 3.3, using the estimates (3.6) and (3.9) as well as the Sobolev inequality (3.8), to compute

$$\begin{aligned}
&\sum_{0\le j\le[m/2]}\binom{m}{j}\big\|\langle z\rangle^{\ell}\partial_x^j\partial_x u\big\|_{L^\infty_{x,y}(L^2_z)}\Big\|\langle z\rangle^{-\ell}\int_0^z\partial_x^{m-j}\mathcal{U}\,d\tilde z\Big\|_{L^2_{x,y}(L^\infty_z)}\\
&\le C\sum_{j=4}^{[m/2]}\frac{m!}{j!(m-j)!}\frac{[(j-4)!]^\sigma}{\rho^{j-4}}|\vec a|_{\rho,\sigma}\times\frac{[(m-j-6)!]^\sigma}{\rho^{m-j-6}}|\vec a|_{\rho,\sigma}\\
&\quad+C\sum_{0\le j\le 3}\frac{m!}{j!(m-j)!}|\vec a|_{\rho,\sigma}\times\frac{[(m-j-6)!]^\sigma}{\rho^{m-j-6}}|\vec a|_{\rho,\sigma}\le\\
&\le C\frac{|\vec a|^2_{\rho,\sigma}}{\rho^{m-6}}\sum_{j=4}^{[m/2]}\frac{(m-6)!m^6[(j-4)!]^{\sigma-1}[(m-j-6)!]^{\sigma-1}}{j^4(m-j)^6}\\
&\quad+C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec a|^2_{\rho,\sigma}\\
&\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec a|^2_{\rho,\sigma}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{[m/2]+1\le j\le m}\binom{m}{j}\big\|\langle z\rangle^{\ell}\partial_x^j\partial_x u\big\|_{L^2}\Big\|\langle z\rangle^{-\ell}\int_0^z\partial_x^{m-j}\mathcal{U}\,d\tilde z\Big\|_{L^\infty}\\
&\quad\le C\frac{[(m-6)!]^\sigma}{\rho^{m-6}}|\vec a|^2_{\rho,\sigma}.
\end{aligned}$$

Thus

$$I_1\le\frac{1}{8}\int_0^t\big\|\partial_z\partial_x^m\mathcal{U}\big\|^2_{L^2}\,ds+C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t|\vec a(s)|^4_{\rho,\sigma}\,ds.$$

Observe $\|\partial_x^m\mathcal{U}\|_{L^2_{x,y}(L^\infty_z)}\le C\|\partial_x^m\mathcal{U}\|_2+C\|\partial_z\partial_x^m\mathcal{U}\|_{L^2}$. Then following the argument for treating I_1, we also have

$$I_2\le\frac{1}{8}\int_0^t\big\|\partial_z\partial_x^m\mathcal{U}\big\|^2_{L^2}\,ds+C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t\big(|\vec a(s)|^3_{\rho,\sigma}+|\vec a(s)|^4_{\rho,\sigma}\big)ds.$$

Then

$$\begin{aligned}
&\int_0^t\left(\partial_x^m\Big[(\partial_x\partial_z u)\int_0^z\mathcal{U}\,d\tilde z\Big],\ \partial_x^m\mathcal{U}\right)_{L^2}ds\le I_1+I_2\\
&\quad\le\frac{1}{4}\int_0^t\big\|\partial_z\partial_x^m\mathcal{U}\big\|^2_{L^2}\,ds+C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t\big(|\vec a(s)|^3_{\rho,\sigma}+|\vec a(s)|^4_{\rho,\sigma}\big)ds.
\end{aligned}$$

Just following the argument above with slight modification, the other terms can be controlled by the same upper bound as above, The proof of Lemma 3.4 is completed. □

COMPLETION OF THE PROOF OF PROPOSITION 3.1. We put the estimates in Lemmas 3.2–3.4 into (3.4) to obtain, for any $m \geq 6$, any $t \in [0, T]$, and any pair $(\rho, \widetilde{\rho})$ with $0 < \rho < \widetilde{\rho} < \rho_0 \leq 1$,

$$\begin{aligned}&\|\partial_x^m \mathcal{U}(t)\|_{L^2}^2\\&\quad \leq C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\bigg(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big)ds + C_*\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\bigg).\end{aligned}$$

Similarly the above estimate also holds with ∂_x^m replaced by ∂_y^m. Thus by (2.17) we have, for any $t \in [0, T]$ and for any pair $(\rho, \widetilde{\rho})$ with $0 < \rho < \widetilde{\rho} < \rho_0 \leq 1$,

$$\begin{aligned}&\sup_{|\alpha|\geq 6}\frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-6)!]^{2\sigma}}\|\partial^\alpha \mathcal{U}(t)\|_{L^2}^2\\&\quad \leq C\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big)ds + CC_*\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds.\end{aligned}$$

It can be checked straightforwardly that the same upper bound holds for $\|\partial^\alpha \mathcal{U}\|_{L^2}$ with $|\alpha| \leq 5$. Then the desired estimate for $\partial^\alpha \mathcal{U}$ in Proposition 3.1 follows, and similarly for $\partial^\alpha \widetilde{\mathcal{U}}$. The proof of Proposition 3.1 is thus completed. □

4 Estimate on $\|(u, v)\|_{\rho,\sigma}$

The main estimate on $\|(u, v)\|_{\rho,\sigma}$ can be stated as follows, recalling $\|(u, v)\|_{\rho,\sigma}$ is given in Definition 1.1.

PROPOSITION 4.1. *Under Assumption* 2.3 *we have, for any* $t \in [0, T]$ *and any pair* $(\rho, \widetilde{\rho})$ *with* $0 < \rho < \widetilde{\rho} < \rho_0 \leq 1$,

$$\begin{aligned}\|\big(u(t), v(t)\big)\|_{\rho,\sigma}^2 &\leq C_1\|(u_0, v_0)\|_{2\rho_0,\sigma}^2\\&\quad + CC_*^3\bigg(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big)ds + \int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\bigg),\end{aligned}$$

where $C_* \geq 1$ *is the constant given in* (2.18).

In view of Definition 1.1 of $\|(u, v)\|_{\rho,\sigma}$, the above proposition will follow from the two lemmas as below.

LEMMA 4.2 (Estimate on the tangential derivatives). *Under the same assumption as in Proposition* 4.1 *we have, recalling* $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$,

$$\sup_{|\alpha|\geq 7}\frac{\rho^{2(|\alpha|-7)}}{[(|\alpha|-7)!]^{2\sigma}}\|\langle z\rangle^\ell \partial^\alpha u(t)\|_{L^2}^2 + \sup_{|\alpha|\leq 6}\|\langle z\rangle^\ell \partial^\alpha u(t)\|_{L^2}^2 \leq$$

$$\le C_1 \|(u_0, v_0)\|_{2\rho_0,\sigma}^2 + CC_*^3\Big(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big) ds + \int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\, ds\Big).$$

Similarly, the upper bound still holds with $\partial^\alpha u$ replaced by $\partial^\alpha v$.

PROOF. We need only to estimate u since v can be treated in the same way. Applying ∂_x^m to the first equation in (1.2) gives

$$\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m u = -\big(\partial_x^m w\big)\partial_z u + F_m \tag{4.1}$$

with

$$F_m = -\sum_{j=1}^m \binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1} u + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y u\big] - \sum_{j=1}^{m-1}\binom{m}{j}\big(\partial_x^j w\big)\partial_x^{m-j}\partial_z u.$$

On the other hand, applying $(\partial_z u)\partial_x^{m-1}$ to (2.9) we have

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)(\partial_z u)\int_0^z \partial_x^{m-1}\mathcal{U}\, d\tilde{z}\\ &\quad= -\big(\partial_x^m w\big)\partial_z u + L_m\end{aligned} \tag{4.2}$$

with

$$\begin{aligned}L_m = &-(\partial_z u)\sum_{j=1}^{m-1}\binom{m-1}{j}\Big[\big(\partial_x^j u\big)\int_0^z \partial_x^{m-j}\mathcal{U}\, d\tilde{z} + \big(\partial_x^j v\big)\int_0^z \partial_x^{m-1-j}\partial_y \mathcal{U}\, d\tilde{z}\Big]\\ &- (\partial_z u)\sum_{j=1}^{m-1}\binom{m-1}{j}(\partial_x^j w)\partial_x^{m-1-j}\mathcal{U}\\ &+ \Big[(\partial_y v)\partial_z u - (\partial_y u)\partial_z v\Big]\int_0^z \partial_x^{m-1}\mathcal{U} d\tilde{z} - 2\big(\partial_z^2 u\big)\partial_x^{m-1}\mathcal{U},\end{aligned}$$

where we have used the fact that, denoting by $[T_1, T_2] = T_1T_2 - T_2T_1$ the commutator of two operators T_1, T_2,

$$\begin{aligned}&\big[\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2,\ (\partial_z u)\big]\\ &\quad= \big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_z u - 2(\partial_z^2 u)\partial_z\\ &\quad= (\partial_y v)\partial_z u - (\partial_y u)\partial_z v - 2(\partial_z^2 u)\partial_z.\end{aligned}$$

Now we subtract the equation (4.2) by (4.1) to eliminate the higher-order term $(\partial_x^m w)\partial_z u$ and this leads to the equation for

$$\psi_m = \partial_x^m u - (\partial_z u)\int_0^z \partial_x^{m-1}\mathcal{U}\, d\tilde{z}; \tag{4.3}$$

that is,

$$\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\psi_m = F_m - L_m, \tag{4.4}$$

and thus

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\langle z\rangle^\ell \psi_m\\&\quad= \langle z\rangle^\ell F_m - \langle z\rangle^\ell L_m + w(\partial_z\langle z\rangle^\ell)\psi_m - (\partial_z^2\langle z\rangle^\ell)\psi_m - 2(\partial_z\langle z\rangle^\ell)\partial_z\psi_m.\end{aligned}$$

Then we take the scalar product with $\langle z\rangle^\ell \psi_m$ and observe $\langle z\rangle^\ell \psi_m|_{z=0} = 0$, to obtain

$$\begin{aligned}&\frac{1}{2}\|\langle z\rangle^\ell \psi_m(t)\|_{L^2}^2 - \frac{1}{2}\|\langle z\rangle^\ell \psi_m(0)\|_{L^2}^2 + \int_0^t \big\|\partial_z\big[\langle z\rangle^\ell \psi_m(s)\big]\big\|_{L^2}^2\, ds\\&\quad= \int_0^t \Big(\big(\partial_s + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\langle z\rangle^\ell \psi_m,\ \langle z\rangle^\ell \psi_m\Big)_{L^2} ds\\&\quad= \int_0^t \Big(\langle z\rangle^\ell F_m,\ \langle z\rangle^\ell \psi_m\Big)_{L^2} ds - \int_0^t \Big(\langle z\rangle^\ell L_m,\ \langle z\rangle^\ell \psi_m\Big)_{L^2} ds\\&\qquad+ \int_0^t \Big(w(\partial_z\langle z\rangle^\ell)\psi_m - (\partial_z^2\langle z\rangle^\ell)\psi_m - 2(\partial_z\langle z\rangle^\ell)\partial_z\psi_m,\ \langle z\rangle^\ell \psi_m\Big)_{L^2} ds.\end{aligned} \tag{4.5}$$

Note that for any $0 < r \le \rho_0$ we have, observing $C_* \ge 1$,

$$\begin{aligned}\|\langle z\rangle^\ell \psi_m\|_{L^2} &\le \|\langle z\rangle^\ell \partial_x^m u\|_{L^2} + CC_*\|\partial_x^{m-1}\mathcal{U}\|_{L^2}\\&\le CC_*\frac{[(m-7)!]^\sigma}{r^{m-7}}|\vec{a}|_{r,\sigma}\end{aligned} \tag{4.6}$$

due to the definition of ψ_m given by (4.3) as well as (3.9), (3.6), and (2.18). Then, in view of (3.14),

$$\begin{aligned}&\int_0^t \big(w(\partial_z\langle z\rangle^\ell)\psi_m - (\partial_z^2\langle z\rangle^\ell)\psi_m - 2(\partial_z\langle z\rangle^\ell)\partial_z\psi_m,\ \langle z\rangle^\ell \psi_m\big)_{L^2}\, ds\\&\quad\le CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\int_0^t |\vec{a}(s)|_{\rho,\sigma}^2\, ds.\end{aligned} \tag{4.7}$$

Note F_m is given in (4.1), and we apply similar computation to that in the proof of Lemma 3.3, using (3.9) instead of (3.6); this yields

$$\begin{aligned}&\int_0^t \Big(\langle z\rangle^\ell F_m,\ \langle z\rangle^\ell \psi_m\Big)_{L^2} ds\\&\quad\le CC_*^2\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\bigg(\int_0^t |\vec{a}(s)|_{\rho,\sigma}^3\, ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho} ds\bigg).\end{aligned} \tag{4.8}$$

Recall L_m is given in (4.2). Then

$$
\begin{aligned}
&\|\langle z\rangle^{\ell} L_m\| \\
&\quad\le \|\langle z\rangle^{\ell+1}\partial_z u\|_{L^\infty}\sum_{j=1}^{m-1}\binom{m-1}{j}\Big\|\langle z\rangle^{-1}\big(\partial_x^j u\big)\int_0^z \partial_x^{m-j}\mathcal{U}\,d\tilde z\Big\|_{L^2}\\
&\qquad+\|\langle z\rangle^{\ell+1}\partial_z u\|_{L^\infty}\sum_{j=1}^{m-1}\binom{m-1}{j}\Big\|\langle z\rangle^{-1}\big(\partial_x^j v\big)\int_0^z \partial_x^{m-1-j}\partial_y\mathcal{U}\,d\tilde z\Big\|_{L^2}\\
&\qquad+\|\langle z\rangle^{\ell+1}\partial_z u\|_{L^\infty}\sum_{j=1}^{m-1}\binom{m-1}{j}\big\|\langle z\rangle^{-1}\big(\partial_x^j w\big)\partial_x^{m-1-j}\mathcal{U}\big\|_{L^2}+\\
&\qquad+\big\|\langle z\rangle^{2\ell+1}[(\partial_y v)\partial_z u-(\partial_y u)\partial_z v]\big\|_{L^\infty}\Big\|\langle z\rangle^{-\ell-1}\int_0^z \partial_x^{m-1}\mathcal{U}\,d\tilde z\Big\|_{L^2}\\
&\qquad+2\big\|\langle z\rangle^{\ell}\partial_z^2 u\big\|_{L^\infty}\big\|\partial_x^{m-1}\mathcal{U}\big\|_{L^2}.
\end{aligned}
$$

Thus we apply again the argument for proving Lemma 3.3 to obtain, observing (2.18) and using (3.6), (3.9), and (3.14),

$$
\|\langle z\rangle^{\ell} L_m\|\le C\frac{[(m-7)!]^{\sigma}}{\rho^{m-7}}\big(|\vec a|_{\rho,\sigma}^2+|\vec a|_{\rho,\sigma}^3\big)+CC_*^2 m\frac{[(m-7)!]^{\sigma}}{\tilde\rho^{m-7}}|\vec a|_{\tilde\rho,\sigma},
$$

and thus, with (4.6) and (2.8),

$$
\begin{aligned}
&\int_0^t \big(\langle z\rangle^{\ell} L_m,\ \langle z\rangle^{\ell}\psi_m\big)_{L^2}\,ds\\
&\qquad\le CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\bigg(\int_0^t\big(|\vec a|_{\rho,\sigma}^3+|\vec a|_{\rho,\sigma}^4\big)ds+\int_0^t\frac{|\vec a(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds\bigg).
\end{aligned}
$$

Putting the above estimate and the estimates (4.7) and (4.8) into (4.5) yields

$$
\begin{aligned}
&\big\|\langle z\rangle^{\ell}\psi_m(t)\big\|_{L^2}^2+\int_0^t\big\|\partial_z\big(\langle z\rangle^{\ell}\psi_m(s)\big)\big\|_{L^2}^2\,dt\\
&\quad\le \big\|\langle z\rangle^{\ell}\psi_m(0)\big\|_{L^2}^2\\
&\qquad+CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\bigg(\int_0^t\big(|\vec a(s)|_{\rho,\sigma}^2+|\vec a(s)|_{\rho,\sigma}^4\big)ds+\int_0^t\frac{|\vec a(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}ds\bigg).
\end{aligned}
$$

Moreover, observe $\langle z\rangle^{\ell}\psi_m|_{t=0}=\langle z\rangle^{\ell}\partial_x^m u_0$ and thus

$$
\big\|\langle z\rangle^{\ell}\psi_m(0)\big\|_{L^2}^2\le\frac{[(m-7)!]^{2\sigma}}{(2\rho_0)^{2(m-7)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2\le\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2.
$$

Then we obtain

$$
\begin{aligned}
&\left\|\langle z\rangle^{\ell}\psi_m(t)\right\|_{L^2}^2+\int_0^t\left\|\partial_z\big(\langle z\rangle^{\ell}\psi_m(s)\big)\right\|_{L^2}^2dt\\
&\quad\le\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2\\
&\qquad+CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\\
&\qquad\quad\cdot\left(\int_0^t\big(|\vec a(s)|_{\rho,\sigma}^2+|\vec a(s)|_{\rho,\sigma}^4\big)ds+\int_0^t\frac{|\vec a(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds\right).
\end{aligned}\tag{4.9}
$$

Consequently, this inequality, along with the estimate

$$
\left\|\langle z\rangle^{\ell}\partial_x^m u\right\|_{L^2}^2\le 2\left\|\langle z\rangle^{\ell}\psi_m\right\|_{L^2}^2+2\left\|\langle z\rangle^{\ell}(\partial_z u)\int_0^z\partial_x^{m-1}\mathcal U\,d\tilde z\right\|_{L^2}^2
$$

that is from definition (4.3) of ψ_m, and the fact that

$$
\begin{aligned}
&\left\|\langle z\rangle^{\ell}(\partial_z u(t))\int_0^z\partial_x^{m-1}\mathcal U(t)d\tilde z\right\|_{L^2}^2\\
&\le\left\|\langle z\rangle^{\ell+1}\partial_z u(t)\right\|_{L^\infty_{x,y}(L^2_z)}^2\left\|\langle z\rangle^{-1}\int_0^z\partial_x^{m-1}\mathcal U(t)d\tilde z\right\|_{L^2_{x,y}(L^\infty_z)}^2\\
&\le CC_*^2\left\|\partial_x^{m-1}\mathcal U(t)\right\|_{L^2}^2\\
&\le CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\left(\int_0^t\big(|\vec a(s)|_{\rho,\sigma}^2+|\vec a(s)|_{\rho,\sigma}^4\big)ds+\int_0^t\frac{|\vec a(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds\right)
\end{aligned}
$$

due to (2.18) and Proposition 3.1, yields that for any $m\ge 7$ and any $t\in[0,T]$,

$$
\begin{aligned}
\left\|\langle z\rangle^{\ell}\partial_x^m u(t)\right\|_{L^2}^2&\le 2\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2\\
&\quad+CC_*^3\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\left(\int_0^t\big(|\vec a(s)|_{\rho,\sigma}^2+|\vec a(s)|_{\rho,\sigma}^4\big)ds+\int_0^t\frac{|\vec a(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}ds\right).
\end{aligned}
$$

We have proven the assertion for $\partial^\alpha=\partial_x^m$ with $m\ge 7$. By direct verification we can get the desired estimate for $m\le 6$. Then we have obtained the estimate as desired for $\partial^\alpha=\partial_x^m$. Moreover, the above estimates also hold with ∂_x^m replaced by ∂_y^m, following a similar argument. Thus the desired estimate for general $\partial^\alpha u$ follows in view of (2.17), and similarly for $\partial^\alpha v$. The proof of Lemma 4.2 is completed. □

LEMMA 4.3 (Estimate on the mixed derivatives). *Under the same assumption as in Proposition* 4.1, *we have*

$$\sup_{\substack{1\leq j\leq 5\\ |\alpha|+j\geq 7}} \frac{\rho^{2(|\alpha|+j-7)}}{[(|\alpha|+j-7)!]^{2\sigma}} \|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u(t)\|_{L^2}^2 + \sup_{\substack{1\leq j\leq 5\\ |\alpha|+j\leq 6}} \|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u(t)\|_{L^2}^2$$
$$\leq C_1\|(u_0,v_0)\|_{2\rho_0,\sigma}^2 + C\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big)\,ds,$$

and similarly for $\partial^\alpha v$.

PROOF. The upper bound for $\|\langle z\rangle^{\ell+j}\partial^\alpha\partial_z^j u(t)\|_{L^2}$ with $|\alpha|+j\leq 6$ and $1\leq j\leq 5$ is straightforward. So we only need to consider the case of $|\alpha|+j\geq 7$ with $1\leq j\leq 5$. As before, it suffices to estimate $\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u$ since $\langle z\rangle^{\ell+j}\partial_y^m\partial_z^j u$ can be treated in the same way.

We apply $\langle z\rangle^{\ell+j}\partial_z^j$ to equation (4.1) to get

$$\begin{aligned}&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\\ &\quad= -\langle z\rangle^{\ell+j}\partial_z^j\big[\big(\partial_x^m w\big)\partial_z u\big]\\ &\qquad+ \langle z\rangle^{\ell+j}\partial_z^j F_m + \big[u\partial_x + v\partial_y + w\partial_z - \partial_z^2,\ \langle z\rangle^{\ell+j}\partial_z^j\big]\partial_x^m u,\end{aligned}$$

where F_m is defined in (4.1) and $[T_1,T_2]=T_1T_2-T_2T_1$ stands for the commutator of two operators T_1, T_2. Thus,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big\|_{L^2}^2 + \big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2\\ &\quad+\int_{\mathbb{R}^2}\big(\partial_x^m\partial_z^j u\big)\big(\partial_x^m\partial_z^{j+1}u\big)\big|_{z=0}\,dx\,dy\\ &\quad= \big(\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\ &\quad= \big(-\langle z\rangle^{\ell+j}\partial_z^j\big[\big(\partial_x^m w\big)\partial_z u\big] + \langle z\rangle^{\ell+j}\partial_z^j F_m,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\ &\qquad+ \big(\big[u\partial_x + v\partial_y + w\partial_z - \partial_z^2,\ \langle z\rangle^{\ell+j}\partial_z^j\big]\partial_x^m u,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2},\end{aligned}\tag{4.10}$$

where we used the fact that

$$\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big|_{z=0} = \big(\partial_x^m\partial_z^j u\big)\big(\partial_x^m\partial_z^{j+1}u\big)\big|_{z=0}.$$

As for the terms on the right side of (4.10) we use the argument for proving Lemma 3.3 to get, recalling F_m is given in (4.1),

$$\begin{aligned}&\big(\langle z\rangle^{\ell+j}\partial_z^j F_m,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\ &\quad\leq \frac{1}{8}\big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2 + C\big\|\langle z\rangle^{\ell+j}\partial_z^{j-1}F_m\big\|_{L^2}^2\\ &\qquad+ C\big\|\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big\|_{L^2}^2\\ &\quad\leq \frac{1}{8}\big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2 + C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}\big(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4\big).\end{aligned}$$

Moreover, direct verification shows

$$\begin{aligned}&\big(-\langle z\rangle^{\ell+j}\partial_z^j\big[\big(\partial_x^m w\big)\partial_z u\big]+\big[u\partial_x+v\partial_y+w\partial_z-\partial_z^2,\langle z\rangle^{\ell+j}\partial_z^j\big]\partial_x^m u,\\&\quad\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\&\qquad\le\frac{1}{8}\big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2+C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}(|\vec a|_{\rho,\sigma}^2+|\vec a|_{\rho,\sigma}^4).\end{aligned}$$

By the two inequalities above we get the upper bound for the terms on the right side of (4.10), that is,

$$\begin{aligned}&\big(-\langle z\rangle^{\ell+j}\partial_z^j\big[\big(\partial_x^m w\big)\partial_z u\big]+\langle z\rangle^{\ell+j}\partial_z^j F_m,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\&\quad+\big(\big[u\partial_x+v\partial_y+w\partial_z-\partial_z^2,\ \langle z\rangle^{\ell+j}\partial_z^j\big]\partial_x^m u,\ \langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)_{L^2}\\&\qquad\le\frac{1}{4}\big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2+C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}\big(|\vec a|_{\rho,\sigma}^2+|\vec a|_{\rho,\sigma}^4\big).\end{aligned}$$

This with (4.10) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big\|_{L^2}^2+\frac{3}{4}\big\|\partial_z\big(\langle z\rangle^{\ell+j}\partial_x^m\partial_z^j u\big)\big\|_{L^2}^2\\&\quad\le\left|\int_{\mathbb{R}^2}\big(\partial_x^m\partial_z^j u\big)\big(\partial_x^m\partial_z^{j+1}u\big)\big|_{z=0}dx\,dy\right|\\&\qquad+C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}\big(|\vec a|_{\rho,\sigma}^2+|\vec a|_{\rho,\sigma}^4\big).\end{aligned}\tag{4.11}$$

Next we handle the first term on the right of (4.11). We claim the following estimate holds for $j=1,2,3,5$:

$$\left|\int_{\mathbb{R}^2}\big(\partial_x^m\partial_z^j u\big)\big(\partial_x^m\partial_z^{j+1}u\big)\big|_{z=0}\,dx\,dy\right|\tag{4.12}$$

$$\le\frac{1}{4}\big\|\partial_x^m\partial_z^{j+1}u\big\|_{L^2}^2+C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}\big(|\vec a|_{\rho,\sigma}^2+|\vec a|_{\rho,\sigma}^4\big).\tag{4.13}$$

Observe that $(\partial_z^2 u,\partial_z^2 v)|_{z=0}=(0,0)$, which follows after taking the trace for the equations in (1.2). Moreover, applying ∂_z^2 to the first equation in (1.2) gives

$$\partial_t\partial_z^2 u+\partial_z^2(u\partial_x u+v\partial_y u+w\partial_z u)-\partial_z^4 u=0,$$

and thus

$$\begin{aligned}\partial_z^4 u|_{z=0}&=\partial_z^2(u\partial_x u+v\partial_y u+w\partial_z u)|_{z=0}\\&=(\partial_z u)(\partial_x\partial_z u-\partial_y\partial_z v)|_{z=0}+2(\partial_z v)\partial_y\partial_z u|_{z=0}.\end{aligned}\tag{4.14}$$

Moreover, the above relationship, along with the equation

$$\partial_t\partial_z^4 u+\partial_z^4(u\partial_x u+v\partial_y u+w\partial_z u)-\partial_z^6 u=0,$$

yields

$$\partial_z^6 u|_{z=0} = -(\partial_z^3 u)(\partial_x \partial_z u + \partial_y \partial_z v)|_{z=0} + 4(\partial_z u)\partial_x \partial_z^3 u|_{z=0} + 4(\partial_z v)\partial_y \partial_z^3 u|_{z=0}.$$

Consequently, we apply a similar computation to that in the proof of Lemma 3.4 to get, using the Sobolev inequality and the estimate (3.9),

$$\begin{aligned}\|(\partial_x^m \partial_z^6 u|_{z=0})\|_{L^2_{x,y}} &\leq C \sum_{0\leq j\leq 1} \|\partial_z^j \partial_x^m[(\partial_z^3 u)(\partial_x \partial_z u + \partial_y \partial_z v)]\|_{L^2} \\ &\quad + C \sum_{0\leq j\leq 1} \|\partial_z^j \partial_x^m[(\partial_z u)\partial_x \partial_z^3 u]\|_{L^2} \\ &\quad + C \sum_{0\leq j\leq 1} \|\partial_z^j \partial_x^m[(\partial_z v)\partial_y \partial_z^3 u]\|_{L^2} \\ &\leq C \frac{[(m+5-7)!]^\sigma}{\rho^{m+5-7}} |\vec{a}|_{\rho,\sigma}^2,\end{aligned}$$

and similarly by (4.14),

$$\|(\partial_x^m \partial_z^4 u|_{z=0})\|_{L^2_{x,y}} \leq C \frac{[(m+3-7)!]^\sigma}{\rho^{m+3-7}} |\vec{a}|_{\rho,\sigma}^2. \tag{4.15}$$

Thus

$$\begin{aligned}&\left| \int_{\mathbb{R}^2} (\partial_x^m \partial_z^5 u)(\partial_x^m \partial_z^6 u)\big|_{z=0} dx\, dy \right| \\ &\quad \leq \frac{1}{4} \|\partial_z \partial_x^m \partial_z^5 u)\|_{L^2}^2 + C\|\partial_x^m \partial_z^5 u)\|_{L^2}^2 + C\|(\partial_x^m \partial_z^6 u|_{z=0})\|_{L^2_{x,y}}^2 \\ &\quad \leq \frac{1}{4}\|\partial_x^m \partial_z^6 u\|_{L^2}^2 + C\frac{[(m+5-7)!]^{2\sigma}}{\rho^{2(m+5-7)}}(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4)\end{aligned}$$

and

$$\begin{aligned}&\left| \int_{\mathbb{R}^2} (\partial_x^m \partial_z^3 u)(\partial_x^m \partial_z^4 u)\big|_{z=0}\, dx\, dy \right| \\ &\quad \leq \frac{1}{4}\|\partial_x^m \partial_z^4 u\|_{L^2}^2 + C\frac{[(m+3-7)!]^{2\sigma}}{\rho^{2(m+3-7)}}(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4).\end{aligned}$$

This gives the validity of (4.12) for $j = 3, 5$. Note that (4.12) obviously holds for $j = 1, 2$ since $\partial_z^2 u|_{z=0} = 0$. Thus (4.12) is valid for $j = 1, 2, 3, 5$, and this with (4.11) yields

$$\begin{aligned}&\|\langle z\rangle^{\ell+j}\partial_x^m \partial_z^j u(t)\|_{L^2}^2 + \int_0^t \|\partial_z(\langle z\rangle^{\ell+j}\partial_x^m \partial_z^j u(s))\|_{L^2}^2\, ds \\ &\quad \leq C\frac{[(m+j-7)!]^{2\sigma}}{\rho^{2(m+j-7)}}\left[\|(u_0, v_0)\|_{2\rho_0,\sigma}^2 + \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds\right]\end{aligned} \tag{4.16}$$

for $j = 1, 2, 3, 5$.

It remains to prove the validity of (4.16) for the case of $j=4$. By the Sobolev inequality, we compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\big(\partial_x^m\partial_z^4 u\big)\big(\partial_x^m\partial_z^5 u\big)\big|_{z=0}\,dx\,dy\right|\\
&\le \frac{\rho^2}{m^{2\sigma}}\big\|\big(\partial_x^m\partial_z^5 u|_{z=0}\big)\big\|_{L^2_{x,y}}^2+\frac{m^{2\sigma}}{\rho^2}\big\|\big(\partial_x^m\partial_z^4 u|_{z=0}\big)\big\|_{L^2_{x,y}}^2\\
&\le C\frac{\rho^2}{m^{2\sigma}}\big(\big\|\partial_x^m\partial_z^5 u\big\|_{L^2}^2+\big\|\partial_x^m\partial_z^6 u\big\|_{L^2}^2\big)+C\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}|\vec{a}|_{\rho,\sigma}^4,
\end{aligned}$$

where the last inequality uses (4.15). This, with (4.11) for $j=4$, yields

$$\begin{aligned}
&\big\|\langle z\rangle^{\ell+4}\partial_x^m\partial_z^4 u(t)\big\|_{L^2}^2+\frac{3}{4}\int_0^t\big\|\partial_z\big(\langle z\rangle^{\ell+4}\partial_x^m\partial_z^4 u(s)\big)\big\|_{L^2}^2\,dt\\
&\quad\le\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}\|u_0,v_0\|_{2\rho_0,\sigma}^2\\
&\qquad+C\frac{\rho^2}{m^{2\sigma}}\int_0^t\big(\big\|\partial_x^m\partial_z^5 u(s)\big\|_{L^2}^2+\big\|\partial_x^m\partial_z^6 u(s)\big\|_{L^2}^2\big)ds\\
&\qquad+C\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}\int_0^t\big(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\big)ds.
\end{aligned}$$

Moreover, we use (3.9) to get

$$\frac{\rho^2}{m^{2\sigma}}\int_0^t\big\|\partial_x^m\partial_z^5 u(s)\big\|_{L^2}^2\,ds\le C\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}\int_0^t|\vec{a}(s)|_{\rho,\sigma}^2\,ds,$$

Observe that we have proven (4.16) holds for $j=5$, and this implies

$$\begin{aligned}
&\frac{\rho^2}{m^{2\sigma}}\int_0^t\big\|\partial_x^m\partial_z^6 u(s)\big\|_{L^2}^2\,ds\\
&\quad\le 2\frac{\rho^2}{m^{2\sigma}}\int_0^t\big\|\partial_z\big(\langle z\rangle^{\ell+5}\partial_x^m\partial_z^5 u(s)\big)\big\|_{L^2}^2\,ds\\
&\qquad+C\frac{\rho^2}{m^{2\sigma}}\int_0^t\big\|\langle z\rangle^{\ell+5}\partial_x^m\partial_z^5 u(s)\big\|_{L^2}^2\,ds\\
&\quad\le 2\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2\\
&\qquad+C\frac{[(m+4-7)!]^{2\sigma}}{\rho^{2(m+4-7)}}\int_0^t\big(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\big)ds.
\end{aligned}$$

Combining the above inequalities we obtain the validity of (4.16) for $j=4$. The proof of Lemma 4.3 is thus completed. □

5 Estimate on $\partial^\alpha\lambda, \partial^\alpha\delta$ and $\partial^\alpha\widetilde{\lambda}, \partial^\alpha\widetilde{\delta}$

Recall λ, δ and $\widetilde{\lambda}, \widetilde{\delta}$ are the functions given by (2.12). This section is devoted to treating the terms involving these functions in the representation of $|\vec{a}|_{\rho,\sigma}$ (see Definition 2.1).

PROPOSITION 5.1. *Under Assumption* 2.3 *we have, for any* $t \in [0, T]$ *and for any pair* $(\rho, \widetilde{\rho})$ *with* $0 < \rho < \widetilde{\rho} < \rho_0 \leq 1$,

$$\begin{aligned}
&\sup_{|\alpha|\geq 6} \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-6)!]^{2\sigma}} \big(|\alpha|^2 \|\partial^\alpha\lambda(t)\|_{L^2}^2 + |\alpha|^2\|\partial^\alpha\delta(t)\|_{L^2}^2\big)\\
&\quad + \sup_{|\alpha|\leq 5} \big(|\alpha|^2 \|\partial^\alpha\lambda(t)\|_{L^2}^2 + |\alpha|^2\|\partial^\alpha\delta(t)\|_{L^2}^2\big)\\
&\quad \leq C_1\|(u_0, v_0)\|_{2\rho_0,\sigma}^2\\
&\qquad + e^{CC_*^2}\bigg(\int_0^t \Big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\Big)ds + \int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\bigg),
\end{aligned}$$

where $C_* \geq 1$ *is the constant given in* (2.18). *Similarly the above estimate still holds with* $\partial^\alpha\lambda$ *and* $\partial^\alpha\delta$ *replaced, respectively, by* $\partial^\alpha\widetilde{\lambda}$ *and* $\partial^\alpha\widetilde{\delta}$.

To prove the above proposition, we first derive the equation solved by λ. Note that

$$\lambda = \partial_x u - (\partial_z u)\int_0^z \mathcal{U}\,d\tilde{z} = \psi_1,$$

with ψ_1 defined by (4.3). Then using (4.4) for $m = 1$ we obtain the equation for λ:

$$\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\lambda\\
&\quad = -(\partial_x u)^2 - (\partial_x v)\partial_y u - [(\partial_y v)\partial_z u - (\partial_y u)\partial_z v]\int_0^z \mathcal{U}\,d\tilde{z} + 2\big(\partial_z^2 u\big)\mathcal{U}.
\end{aligned}$$

Now for any $m \geq 6$ we apply ∂_x^m to the above equation; this gives

$$\begin{aligned}
&\big(\partial_t + u\partial_x + v\partial_y + w\partial_z - \partial_z^2\big)\partial_x^m\lambda\\
&\quad = -\sum_{j=1}^m \binom{m}{j}\big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\lambda + \big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\lambda + \big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\lambda\big]\\
&\qquad - \partial_x^m\bigg[(\partial_x u)^2 + (\partial_x v)\partial_y u + [(\partial_y v)\partial_z u - (\partial_y u)\partial_z v]\int_0^z \mathcal{U}\,d\tilde{z} - 2\big(\partial_z^2 u\big)\mathcal{U}\bigg].
\end{aligned}$$

Thus taking the scalar product with $m^2\partial_x^m\lambda$ and observing $\lambda|_{z=0}=0$ and $\lambda|_{t=0}=\partial_x u_0$, we have

$$
\begin{aligned}
&\frac{m^2}{2}\big(\|\partial_x^m\lambda(t)\|_{L^2}^2-\|\partial_x^{m+1}u_0\|_{L^2}^2\big)+m^2\int_0^t\big\|\partial_z\partial_x^m\lambda(s)\big\|_{L^2}^2\,ds\\
&\quad=m^2\int_0^t\big(\big(\partial_s+u\partial_x+v\partial_y+w\partial_z-\partial_z^2\big)\partial_x^m\lambda,\ \partial_x^m\lambda\big)_{L^2}\,ds\\
&\quad=K_1+K_2+K_3,
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{aligned}
K_1&=-m^2\int_0^t\Big(\sum_{j=1}^m\binom{m}{j}\Big[\big(\partial_x^j u\big)\partial_x^{m-j+1}\lambda+\big(\partial_x^j v\big)\partial_x^{m-j}\partial_y\lambda\Big],\ \partial_x^m\lambda\Big)_{L^2}ds\\
&\quad-m^2\int_0^t\Big(\sum_{j=1}^m\binom{m}{j}\big(\partial_x^j w\big)\partial_x^{m-j}\partial_z\lambda,\ \partial_x^m\lambda\Big)_{L^2}ds,\\
K_2&=-m^2\int_0^t\Big(\partial_x^m\Big[(\partial_x u)^2+(\partial_x v)\partial_y u+\big[(\partial_y v)\partial_z u-(\partial_y u)\partial_z v\big]\\
&\qquad\cdot\int_0^z\mathcal{U}\,d\tilde{z}\Big],\partial_x^m\lambda\Big)_{L^2}ds,\\
K_3&=2m^2\int_0^t\big(\partial_x^m\big[(\partial_z^2 u)\mathcal{U}\big],\ \partial_x^m\lambda\big)_{L^2}\,ds.
\end{aligned}
$$

To estimate the above $K_j, 1\le j\le 3$, we need the upper bounds of $\int_0^z\mathcal{U}\,dz$ and $\mathcal{U}$ similar to those in (2.18), which is stated in the following:

LEMMA 5.2. *Under the condition* (2.18) *we have, denoting* $\partial^\beta=\partial_x^{\beta_1}\partial_y^{\beta_2}$ *and recalling* $T\le 1$,

$$
\begin{aligned}
\forall t\in[0,T],\quad &\sum_{|\beta|\le 9}\Big\|\langle z\rangle^{-\ell}\int_0^z\partial^\beta\mathcal{U}(t)dz\Big\|_{L^2}\\
&+\sum_{\substack{|\beta|+j\le 8\\0\le j\le 2}}\big\|\partial^\beta\partial_z^j\mathcal{U}(t)\big\|_{L^2}\le e^{CC_*^2},
\end{aligned}
$$

and

$$
\forall t\in[0,T],\quad\sum_{\substack{|\beta|+j\le 8\\0\le j\le 2}}\|\partial^\beta\partial_z^j\lambda\|_{L^2}\le e^{CC_*^2},
$$

where $C_*\ge 1$ *is the constant in* (2.18), *and* C *is a constant depending only on the Sobolev embedding constants and the numbers* ρ_0,σ,ℓ *given in Definition* 1.1.

PROOF. This just follows from direct computation. Precisely we use the standard energy method for the equation (2.10) solved by $f=\int_0^z\mathcal{U}\,dz$, applying

$\langle z\rangle^{-\ell}\partial^\beta = \langle z\rangle^{-\ell}\partial_x^{\beta_1}\partial_y^{\beta_2}$, $|\beta|\leq 9$, to the equation (2.10) and then taking the scalar product with $\langle z\rangle^{-\ell}\partial^\beta f$; this with Sobolev inequality (3.8) and the condition (2.18) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{|\beta|\leq 9}\|\langle z\rangle^{-\ell}\partial^\beta f\|_{L^2}^2 &+ \sum_{|\beta|\leq 9}\|\partial_z(\langle z\rangle^{-\ell}\partial^\beta f)\|_{L^2}^2\\ &\leq CC_*^2\sum_{|\beta|\leq 9}\|\langle z\rangle^{-\ell}\partial^\beta f\|_{L^2}^2 + CC_*\sum_{|\beta|\leq 9}\|\langle z\rangle^{-\ell}\partial^\beta f\|_{L^2},\end{aligned}$$

where $C_*\geq 1$ is just the constant given in (2.18). Moreover, we apply again the energy method for the equation (3.3) solved by $\mathcal{U}$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|+j\leq 8\\ 0\leq j\leq 2}}\left\|\partial^\beta\partial_z^j\mathcal{U}\right\|_{L^2}^2 &+ \sum_{\substack{|\beta|+j\leq 8\\ 0\leq j\leq 2}}\left\|\partial_z\partial^\beta\partial_z^j\mathcal{U}\right\|_{L^2}^2\\ &\leq CC_*^2\sum_{\substack{|\beta|+j\leq 8\\ 0\leq j\leq 2}}\|\partial^\beta\partial_z^j\mathcal{U}\|_{L^2}^2 + CC_*\sum_{\substack{|\beta|+j\leq 8\\ 0\leq j\leq 2}}\|\partial^\beta\partial_z^j\mathcal{U}\|_{L^2}\\ &\quad + CC_*\sum_{|\beta|\leq 9}\|\langle z\rangle^{-\ell}\partial^\beta f\|_{L^2}^2.\end{aligned}$$

As a result, by using Gronwall's inequality we obtain the first estimate as desired in Lemma 5.2, which with the representation of λ given in (2.12) as well as (2.18), yields the second 1. The proof of Lemma 5.2 is completed. □

Now we continue the proof of Proposition 5.1. Recall that K_1 is given in (5.1). By the second estimate in Lemma 5.2, we can apply a similar argument for proving Lemma 3.3 to compute, using (3.5) here instead of (3.6) and observing that there is a factor m before $\|\partial_x^m\lambda\|_{L^2}$ in (3.5),

$$\begin{aligned}K_1 \leq{}& \frac{m^2}{4}\int_0^t\|\partial_z\partial_x^m\lambda(s)\|_{L^2}^2\,ds\\ &+ C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\bigg(\int_0^t\left(|\vec{a}(s)|_{\rho,\sigma}^3 + |\vec{a}(s)|_{\rho,\sigma}^4\right)ds\\ &\qquad + e^{CC_*^2}\int_0^t\frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}\,ds\bigg)\end{aligned}\tag{5.2}$$

and

$$K_2 \leq C\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\bigg(\int_0^t|\vec{a}(s)|_{\rho,\sigma}^3\,ds + e^{CC_*^2}\int_0^t\frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}\,ds\bigg).\tag{5.3}$$

It remains to treat K_3 given in (5.1). We write

$$K_3 = K_{3,1} + K_{3,2} + K_{3,3},$$

with

$$K_{3,1} = 2m^2 \int_0^t \left(\sum_{j=0}^{[m/2]} \binom{m}{j} \big(\partial_x^j \partial_z^2 u\big) \partial_x^{m-j} \mathcal{U},\ \partial_x^m \lambda \right)_{L^2} ds,$$

$$K_{3,2} = 2m^2 \int_0^t \left(\sum_{j=[m/2]+1}^{m-5} \binom{m}{j} \big(\partial_x^j \partial_z^2 u\big) \partial_x^{m-j} \mathcal{U},\ \partial_x^m \lambda \right)_{L^2} ds,$$

$$K_{3,3} = 2m^2 \int_0^t \left(\sum_{j=m-4}^{m} \binom{m}{j} \big(\partial_x^j \partial_z^2 u\big) \partial_x^{m-j} \mathcal{U},\ \partial_x^m \lambda \right)_{L^2} ds.$$

The remainder of this section is devoted to estimating the above terms $K_{3,1}$, $K_{3,2}$, and $K_{3,3}$. Using (3.9), (3.6), and (3.5) as well as (2.18) and recalling $\sigma \geq 3/2$, we follow a similar computation to that used in the proof of Lemma 3.3 to obtain

$$\begin{aligned} K_{3,1} &\leq 2m^2 \left[\sum_{0\leq j\leq 1} + \sum_{j=2}^{[m/2]} \right] \binom{m}{j} \int_0^t \big\| (\partial_x^j \partial_z^2 u \big\|_{L^\infty} \big\| \partial_x^{m-j} \mathcal{U} \big\|_{L^2} \big\| \partial_x^m \lambda \big\|_{L^2} ds \\ &\leq C C_* \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\, ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t |\vec{a}(s)|_{\rho,\sigma}^3\, ds, \end{aligned}$$

and

$$\begin{aligned} &K_{3,2} \\ &\leq 2m^2 \sum_{j=[m/2]+1}^{m-5} \binom{m}{j} \int_0^t \big\| \big(\partial_x^j \partial_z^2 u\big) \big\|_{L^2} \big\| \partial_x^{m-j} \mathcal{U} \big\|_{L^\infty_{x,y}(L_z^2)} \big\| \partial_x^m \lambda \big\|_{L^2_{x,y}(L_z^\infty)} ds \\ &\leq C \int_0^t \frac{[(m-6)!]^{\sigma}}{\rho^{m-6}} |\vec{a}(s)|_{\rho,\sigma}^2 \big(m \|\partial_x^m \lambda\|_{L^2} + m \|\partial_z \partial_x^m \lambda\|_{L^2} \big) ds \\ &\leq \frac{m^2}{8} \int_0^t \big\| \partial_z \partial_x^m \lambda(s) \big\|_{L^2}^2\, ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^3 + |\vec{a}(s)|_{\rho,\sigma}^4 \big) ds. \end{aligned}$$

Finally, integration by parts gives

$$\begin{aligned} K_{3,3} = &-2m^2 \int_0^t \left(\sum_{j=m-4}^{m} \binom{m}{j} \big(\partial_x^j \partial_z u\big) \partial_x^{m-j} \mathcal{U},\ \partial_z \partial_x^m \lambda \right)_{L^2} ds \\ &- 2m^2 \int_0^t \left(\sum_{j=m-4}^{m} \binom{m}{j} \big(\partial_x^j \partial_z u\big) \partial_z \partial_x^{m-j} \mathcal{U},\ \partial_x^m \lambda \right)_{L^2} ds \leq \end{aligned}$$

$$
\begin{aligned}
&\le \frac{m^2}{8}\int_0^t \big\|\partial_z\partial_x^m\lambda(s)\big\|_{L^2}^2\,dt \\
&\quad + C\int_0^t \Big[m\sum_{j=m-4}^{m}\binom{m}{j}\big\|\big(\partial_x^j\partial_z u\big)\partial_x^{m-j}\mathcal{U}\big\|_{L^2}\Big]^2\,ds \\
&\quad + 2m^2\sum_{j=m-4}^{m}\binom{m}{j}\int_0^t \big\|\partial_x^j\partial_z u\big\|_{L^2}\big\|\partial_z\partial_x^{m-j}\mathcal{U}\big\|_{L^\infty}\big\|\partial_x^m\lambda\big\|_{L^2}\,ds.
\end{aligned}
$$

As for the last term on the right side, we use (3.5), (3.9), and the first assertion in Lemma 5.2 to compute

$$
\begin{aligned}
&2m^2\sum_{j=m-4}^{m}\binom{m}{j}\int_0^t \big\|\partial_x^j\partial_z u\big\|_{L^2}\big\|\partial_z\partial_x^{m-j}\mathcal{U}\big\|_{L^\infty}\big\|\partial_x^m\lambda\big\|_{L^2}\,ds \\
&\quad \le e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds.
\end{aligned}
$$

Meanwhile we claim

$$
\begin{aligned}
&\int_0^t \Big[m\sum_{j=m-4}^{m}\binom{m}{j}\big\|\big(\partial_x^{m-j}\mathcal{U}\big)\partial_x^j\partial_z u\big\|_{L^2}\Big]^2\,ds \\
&\quad \le e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\Big(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\big)ds+\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds\Big).
\end{aligned} \tag{5.4}
$$

The proof of (5.4) is postponed to the end of this section. Thus we combine the above three inequalities to get

$$
\begin{aligned}
K_{3,3} &\le \frac{m^2}{8}\int_0^t \big\|\partial_z\partial_x^m\lambda\big\|_{L^2}^2\,ds \\
&\quad + e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \\
&\quad\cdot\Big(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\big)ds+\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}\,ds\Big),
\end{aligned}
$$

which with upper bounds of $K_{3,1}$ and $K_{3,2}$ yields

$$
\begin{aligned}
K_3 &\le \frac{m^2}{4}\int_0^t \big\|\partial_z\partial_x^m\lambda\big\|_{L^2}^2\,ds \\
&\quad + e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\Big(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\big)ds+\int_0^t \frac{|\vec{a}(s)|_{\tilde\rho,\sigma}^2}{\tilde\rho-\rho}ds\Big).
\end{aligned}
$$

Now we put the above estimate and the estimates (5.2)–(5.3) on K_1, K_2 into (5.1) to obtain

$$
\begin{aligned}
m^2\left\|\partial_x^m\lambda(t)\right\|_{L^2}^2 &+ m^2\int_0^t \left\|\partial_z\partial_x^m\lambda(s)\right\|_{L^2}^2\,ds\\
&\leq m^2\left\|\partial_x^{m+1}u_0\right\|_{L^2}^2\\
&\quad + e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\left(\int_0^t\left(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\right)ds+\int_0^t\frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}\,ds\right),
\end{aligned}
$$

which with the fact that

$$
\begin{aligned}
m^2\left\|\partial_x^{m+1}u_0\right\|_{L^2}^2 &\leq m^2\frac{[(m-6)!]^{2\sigma}}{(2\rho_0)^{2(m-6)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2\\
&\leq 4^6\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\|(u_0,v_0)\|_{2\rho_0,\sigma}^2
\end{aligned}
$$

since $\rho<\rho_0$, gives the desired upper bound for $\partial_x^m\lambda$ with $m\geq 6$.

The estimate for $m\leq 5$ is straightforward. The above estimate still holds with $\partial_x^m\lambda$ replaced by $\partial_y^m\lambda$, which can be treated in the same way. Thus in view of (2.17) the desired estimate on $\partial^\alpha\lambda$ follows. We can apply a similar argument to get the upper bounds of $\partial^\alpha\delta$, $\partial^\alpha\tilde{\lambda}$, and $\partial^\alpha\tilde{\delta}$. Thus the proof of Proposition 5.1 will be completed if the assertion (5.4) holds.

PROOF OF THE ASSERTION (5.4). We write

$$
\begin{aligned}
\int_0^t &\left[m\sum_{j=m-4}^m\binom{m}{j}\left\|(\partial_x^j\partial_z u)\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\right]^2 ds\\
&\leq Cm^2\int_0^t\left\|\mathcal{U}\partial_x^m\partial_z u\right\|_{L^2}^2\,ds+Cm^4\int_0^t\left\|\left(\partial_x\mathcal{U}\right)\partial_x^{m-1}\partial_z u\right\|_{L^2}^2\,ds\\
&\quad + C\sum_{j=m-4}^{m-2}m^{2(m-j+1)}\int_0^t\left\|\partial_x^j\partial_z u\right\|_{L^2}^2\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L^\infty}^2\,ds.
\end{aligned}
$$

As for the last term on the right side, we use the first inequality in Lemma 5.2 and the fact that $\sigma\geq 3/2$ to compute directly 5.2,

$$
\begin{aligned}
\sum_{j=m-4}^{m-2} & m^{2(m-j+1)}\int_0^t\left\|\partial_x^j\partial_z u\right\|_{L^2}^2\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L^\infty}^2\,ds\\
&\leq e^{CC_*^2}m^6m^{-4\sigma}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t|\vec{a}(s)|^2\,ds\\
&\leq e^{CC_*^2}\frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}}\int_0^t|\vec{a}(s)|^2\,ds.
\end{aligned}
$$

Thus the desired (5.4) will follow if we can show that

$$
\begin{aligned}
&\int_0^t \left\|\mathcal{U}\partial_z\partial_x^m u\right\|_{L^2}^2 ds \\
&\leq e^{CC_*^2}\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\right)ds+\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}ds\right)
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
&\int_0^t \left\|(\partial_x\mathcal{U})\partial_z\partial_x^{m-1} u\right\|_{L^2}^2 ds \\
&\leq e^{CC_*^2}\frac{[(m-8)!]^{2\sigma}}{\rho^{2(m-8)}}\left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4\right)ds+\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\right).
\end{aligned}
\tag{5.6}
$$

The argument is quite similar as that for proving Lemma 4.2. In fact, recalling ψ_m is defined in (4.3) and multiplying both sides of (4.4) by $\mathcal{U}$ instead of $\langle z\rangle^\ell$ therein, we obtain

$$
\begin{aligned}
&\left(\partial_t+u\partial_x+v\partial_y+w\partial_z-\partial_z^2\right)\mathcal{U}\psi_m \\
&\quad=\mathcal{U}(F_m-L_m)-2(\partial_z\mathcal{U})\partial_z\psi_m \\
&\qquad+\left[\partial_x^2u+\partial_y\partial_x v-(\partial_z u)\partial_x\int_0^z\mathcal{U}\,d\widetilde{z}\right. \\
&\qquad\quad\left.-(\partial_z v)\partial_y\int_0^z\mathcal{U}\,d\widetilde{z}+(\partial_x u+\partial_y v)\mathcal{U}\right]\psi_m,
\end{aligned}
$$

where we used the equation (3.3). In view of the first assertion in Lemma 5.2, we repeat the argument for proving (4.9) with slight modification to conclude, observing $\mathcal{U}\psi_m|_{t=0}=0$,

$$
\begin{aligned}
&\|\mathcal{U}(t)\psi_m(t)\|_{L^2}^2+\int_0^t \left\|\partial_z\left[\mathcal{U}(s)\psi_m(s)\right]\right\|_{L^2}^2 ds \\
&\quad\leq e^{CC_*^2}\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2+|\vec{a}(s)|_{\rho,\sigma}^4)ds+\int_0^t \frac{|\vec{a}(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho}-\rho}\,ds\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^t \left\|\mathcal{U}\partial_z\partial_x^m u\right\|_{L^2}^2 ds \\
&\quad\leq 2\int_0^t \left\|\partial_z\left[\mathcal{U}\partial_x^m u\right]\right\|_{L^2}^2 ds+C\int_0^t \left\|\left(\partial_z\mathcal{U}\right)\partial_x^m u\right\|_{L^2}^2 ds \\
&\quad\leq 4\int_0^t \left(\|\partial_z[\mathcal{U}\psi_m]\|_{L^2}^2+\left\|\partial_z\left[\mathcal{U}(\partial_z u)\int_0^z \partial_x^{m-1}\mathcal{U}\,d\widetilde{z}\right]\right\|_{L^2}^2\right)ds \\
&\qquad+C\int_0^t \left\|\left(\partial_z\mathcal{U}\right)\partial_x^m u\right\|_{L^2}^2 ds\leq
\end{aligned}
$$

$$\leq 4\int_0^t \|\partial_z[\mathcal{U}\psi_m]\|_{L^2}^2\,ds + e^{CC_*^2}\int_0^t \|\partial_x^{m-1}\mathcal{U}\|_{L^2}^2\,ds + e^{CC_*^2}\int_0^t \|\partial_x^m u\|_{L^2}^2\,ds,$$

the second inequality following from (4.3) and the last inequality using Lemma 5.2 and the assumption (2.18). Combining the above inequalities we conclude for any $m \geq 7$, using again (3.6) and (3.9),

$$\int_0^t \big\|\mathcal{U}(s)\partial_z\partial_x^m u(s)\big\|_{L^2}^2\,ds \leq e^{CC_*^2}\frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}}\bigg(\int_0^t \big(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4\big)ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}-\rho}\,ds\bigg).$$

We have proven (5.5), and in a similar manner for (5.6). Thus the proof of (5.4) is complete. □

6 Proof of the Main Result

We will prove in this section the main result on the existence and uniqueness for the Prandtl system (1.2). Since the proof is similar to that in the 2D case once we have the a priori estimate, we will only give a sketch, and refer to [23, secs. 7 and 8] for the detailed discussion.

PROOF OF THEOREM 1.3. The proof relies on the a priori estimates given in Theorems 2.4. In order to obtain the existence of solutions to the Prandtl equations (1.2), there are two main ingredients, and one is to investigate the existence of approximate solutions to the regularized Prandtl system:

$$(6.1)\qquad \begin{cases} \partial_t u_\varepsilon + \big(u_\varepsilon\partial_x + v_\varepsilon\partial_y + w_\varepsilon\partial_z\big)u_\varepsilon - \partial_z^2 u_\varepsilon - \varepsilon\partial_x^2 u_\varepsilon - \varepsilon\partial_y^2 u_\varepsilon = 0,\\ \partial_t v_\varepsilon + \big(u_\varepsilon\partial_x + v_\varepsilon\partial_y + w_\varepsilon\partial_z\big)v_\varepsilon - \partial_z^2 v_\varepsilon - \varepsilon\partial_x^2 v_\varepsilon - \varepsilon\partial_y^2 v_\varepsilon = 0,\\ (u_\varepsilon, v_\varepsilon)|_{z=0} = (0,0), \quad \lim_{z\to+\infty}(u_\varepsilon, v_\varepsilon) = (0,0),\\ (u_\varepsilon, v_\varepsilon)|_{t=0} = (u_0, v_0), \end{cases}$$

with $w_\varepsilon = -\int_0^z(\partial_x u_\varepsilon + \partial_y v_\varepsilon)d\tilde{z}$. We remark that the regularized equations above share the same compatibility condition (1.3) as the original system (1.2). Another ingredient is to derive a uniform estimate with respect to ε for the approximate solutions $(u_\varepsilon, v_\varepsilon)$.

The existence for the parabolic system (6.1) is standard. Indeed, suppose that $(u_0, v_0) \in X_{2\rho_0,\sigma}$. Then we can construct, following a similar scheme to that in [23, sec. 7], a solution $(u_\varepsilon, v_\varepsilon) \in L^\infty([0, \widetilde{T}_\varepsilon]; X_{3\rho_0/2,\sigma})$ to (6.1) for some $\widetilde{T}_\varepsilon > 0$ that may depend on ε.

It remains to derive a uniform estimate for the approximate solutions $(u_\varepsilon, v_\varepsilon)$, so that we can remove the ε-dependence of the lifespan $\widetilde{T}_\varepsilon$. To do so we define as in Section 2.2 the auxiliary functions $\mathcal{U}_\varepsilon, \lambda_\varepsilon, \delta_\varepsilon$ in the similar way as that for $\mathcal{U}, \lambda, \delta$

given in Section 2.2, with (u, v, w) and the Prandtl operator therein replaced, respectively, by $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ and the regularized Prandtl operator given above. The argument is similar to that for $\widetilde{\mathcal{U}}_\varepsilon, \widetilde{\lambda}_\varepsilon, \widetilde{\delta}_\varepsilon$. Accordingly, denote

$$\vec{a}_\varepsilon = (u_\varepsilon, v_\varepsilon, \mathcal{U}_\varepsilon, \widetilde{\mathcal{U}}_\varepsilon, \lambda_\varepsilon, \widetilde{\lambda}_\varepsilon, \delta_\varepsilon, \widetilde{\delta}_\varepsilon)$$

and define $|\vec{a}_\varepsilon|_{\rho,\sigma}$ similarly as that of $|\vec{a}|_{\rho,\sigma}$ (see Definition 2.1). Note that

$$\vec{a}_\varepsilon|_{t=0} = (u_0, v_0, 0, 0, \partial_x u_0, \partial_y u_0, \partial_x v_0, \partial_y v_0).$$

Then we can verify directly that

$$\forall \rho \le \rho_0 \quad |\vec{a}_\varepsilon(0)|_{\rho,\sigma} \le C_{\rho_0,\sigma} \|(u_0, v_0)\|_{2\rho_0,\sigma}, \tag{6.2}$$

with $C_{\rho_0,\sigma}$ a constant depending only on ρ_0 and σ.

Let $\tau > 1$ be a fixed number to be determined later. We define

$$|||\vec{a}_\varepsilon|||_{(\tau)} \stackrel{\text{def}}{=} \sup_{\rho,t} \left(\frac{\rho_0 - \rho - \tau t}{\rho_0 - \rho} \right)^{1/2} |\vec{a}_\varepsilon(t)|_{\rho,\sigma}, \tag{6.3}$$

where the supremum is taken over all pairs (ρ, t) such that $\rho > 0$, $0 \le t \le \rho_0/(4\tau)$, and $\rho + \tau t < \rho_0$. Letting $C_{\rho_0,\sigma}$ be the constant given in (6.2) and letting $C_1 \ge 1$ be the constant given in Theorem 2.4, which depends only on ρ_0, σ and the Sobolev embedding constants, we denote

$$C_0 = 2(C_{\rho,\sigma} + C_1)\|(u_0, v_0)\|_{2\rho_0,\sigma} + 1. \tag{6.4}$$

In the following discussion, we will use the bootstrap argument to prove the assertion that

$$|||\vec{a}_\varepsilon|||_{(\tau)} \le C_0/2 \tag{6.5}$$

for some τ large enough, if the condition

$$|||\vec{a}_\varepsilon|||_{(\tau)} \le C_0 \tag{6.6}$$

is fulfilled.

Step 1. Observe, for any $t \in [0, \rho_0/(4\tau)]$,

$$\begin{aligned} \frac{\sqrt{2}}{2} \|(u_\varepsilon(t), v_\varepsilon(t))\|_{\frac{\rho_0}{2},\sigma} &\le \frac{\sqrt{2}}{2} |\vec{a}_\varepsilon(t)|_{\frac{\rho_0}{2},\sigma} \\ &\le \left(\frac{\rho_0 - \frac{\rho_0}{2} - \tau t}{\rho_0 - \frac{\rho_0}{2}} \right)^{1/2} |\vec{a}_\varepsilon(t)|_{\frac{\rho_0}{2},\sigma} \le |||\vec{a}_\varepsilon|||_{(\tau)}. \end{aligned} \tag{6.7}$$

Thus under the condition (6.6), we have $\|(u_\varepsilon(t), v_\varepsilon(t))\|_{\rho_0/2,\sigma} \le \sqrt{2} C_0$ for any $t \in [0,\ \rho_0/(4\tau)]$, and thus it follows from the definition of $\|(u_\varepsilon(t), v_\varepsilon(t))\|_{\rho_0/2,\sigma}$ that, for any $t \in [0,\ \rho_0/(4\tau)]$,

$$\sup_{0 \le j \le 5 |\alpha| + j \le 10} \left(\left\| \langle z \rangle^{\ell+j} \partial^\alpha \partial_z^j u_\varepsilon(t) \right\|_{L^2} + \left\| \langle z \rangle^{\ell+j} \partial^\alpha \partial_z^j v_\varepsilon(t) \right\|_{L^2} \right) \le \widetilde{C}_{\rho_0,\sigma} C_0$$

with $\widetilde{C}_{\rho_0,\sigma} \geq 1$ a constant depending only on ρ_0 and σ. Then the condition in Assumption 2.3 is fulfilled by $(u_\varepsilon, v_\varepsilon)$, and thus similarly to Theorem 2.4 we can repeat the argument in Sections 3–5 with minor modifications to obtain the following assertion: for any $t \in [0, \ \rho_0/(4\tau)]$ and any pair $(\rho, \widetilde{\rho})$ with $0 < \rho < \widetilde{\rho} < \rho_0 \leq 1$,

$$
\begin{aligned}
&|\vec{a}_\varepsilon(t)|_{\rho,\sigma}^2 \\
&\quad \leq C_1 \|(u_0, v_0)\|_{2\rho_0,\sigma}^2 \\
&\qquad + e^{C_2 C_0^2} \left(\int_0^t (|\vec{a}_\varepsilon(s)|_{\rho,\sigma}^2 + |\vec{a}_\varepsilon(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}_\varepsilon(s)|_{\widetilde{\rho},\sigma}^2}{\widetilde{\rho} - \rho} \, ds \right),
\end{aligned} \tag{6.8}
$$

where $C_2 > 0$ is a constant depending only on the numbers ρ_0, σ and the Sobolev embedding constants but independent of ε, and the constant $C_1 \geq 1$ is just the one given in Theorem 2.4.

Step 2. We let (ρ, t) be an arbitrary pair that satisfies that $\rho > 0$, $t \in [0, \rho_0/(4\tau)]$, and $\rho + \tau t < \rho_0$. Then it follows from the definition (6.3) of $|||\vec{a}_\varepsilon|||_{(\tau)}$ that

$$
\forall\, 0 \leq s \leq t, \quad |\vec{a}_\varepsilon(s)|_{\rho,\sigma} \leq |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{\rho_0 - \rho}{\rho_0 - \rho - \tau s} \right)^{1/2}. \tag{6.9}
$$

Furthermore, we take in particular such a $\widetilde{\rho}(s)$ that

$$
\widetilde{\rho}(s) = \frac{\rho_0 + \rho - \tau s}{2}.
$$

Then direct calculation shows that

$$
\forall\, 0 \leq s \leq t \quad \rho < \widetilde{\rho}(s) \quad \text{and} \quad \widetilde{\rho}(s) + \tau s < \rho_0, \tag{6.10}
$$

and

$$
\forall\, 0 \leq s \leq t \quad \widetilde{\rho}(s) - \rho = \frac{\rho_0 - \rho - \tau s}{2} = \rho_0 - \widetilde{\rho}(s) - \tau s. \tag{6.11}
$$

By the inequalities in (6.10) and the second equality in (6.11), it follows that, for any $0 \leq s \leq t$,

$$
|\vec{a}_\varepsilon(s)|_{\widetilde{\rho}(s),\sigma} \leq |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{\rho_0 - \widetilde{\rho}(s)}{\rho_0 - \widetilde{\rho}(s) - \tau s} \right)^{\frac{1}{2}} \leq |||\vec{a}_\varepsilon|||_{(\tau)} \left(\frac{2(\rho_0 - \rho)}{\rho_0 - \rho - \tau s} \right)^{\frac{1}{2}}. \tag{6.12}
$$

Putting (6.9) and (6.12) into the estimate (6.8) and using the first equality in (6.11), we have

$$
\begin{aligned}
&|\vec{a}_\varepsilon(t)|_{\rho,\sigma}^2 \\
&\quad \leq C_1 \|(u_0, v_0)\|_{2\rho_0,\sigma}^2 + e^{C_2 C_0^2} |||\vec{a}_\varepsilon|||_{(\tau)}^2 \int_0^t \frac{\rho_0 - \rho}{\rho_0 - \rho - \tau s} \, ds \\
&\qquad + e^{C_2 C_0^2} |||\vec{a}_\varepsilon|||_{(\tau)}^2 \\
&\qquad\quad \cdot \left(|||\vec{a}_\varepsilon|||_{(\tau)}^2 \int_0^t \frac{(\rho_0 - \rho)^2}{(\rho_0 - \rho - \tau s)^2} \, ds + \int_0^t \frac{2^2(\rho_0 - \rho)}{(\rho_0 - \rho - \tau s)^2} \, ds \right) \leq
\end{aligned}
$$

$$\leq C_1\|(u_0,v_0)\|^2_{2\rho_0,\sigma}+\frac{e^{C_2C_0^2}(5+C_0^2)}{\tau}|||\vec{a}_\varepsilon|||^2_{(\tau)}\frac{\rho_0-\rho}{\rho_0-\rho-\tau t},$$

where in the last inequality we have used the condition (6.6) and the fact that

$$\frac{\rho_0-\rho}{\rho_0-\rho-\tau s}\leq\frac{(\rho_0-\rho)^2}{(\rho_0-\rho-\tau s)^2}\leq\frac{\rho_0-\rho}{(\rho_0-\rho-\tau s)^2}.$$

Thus we multiply both sides by the fact $(\rho_0-\rho-\tau t)/(\rho_0-\rho)$ and observe (ρ,t) is an arbitrary pair with $\rho>0$, $t\in[0,\rho_0/(4\tau)]$, and $\rho+\tau t<\rho_0$; this with $C_1\geq 1$ gives

$$|||\vec{a}_\varepsilon|||_{(\tau)}\leq C_1\|(u_0,v_0)\|_{2\rho_0,\sigma}+\frac{\sqrt{e^{C_2C_0^2}\left(5+C_0^2\right)}}{\sqrt{\tau}}|||\vec{a}_\varepsilon|||_{(\tau)}. \tag{6.13}$$

Now we choose such a τ that

$$1-\frac{\sqrt{e^{C_2C_0^2}\left(5+C_0^2\right)}}{\sqrt{\tau}}=\frac{C_1}{C_1+C_{\rho,\sigma}}. \tag{6.14}$$

Then it follows from (6.13) that

$$|||\vec{a}_\varepsilon|||_{(\tau)}\leq(C_1+C_{\rho,\sigma})\|(u_0,v_0)\|_{2\rho_0,\sigma}\leq C_0/2,$$

recalling that C_0 is given by (6.4). This gives the desired assertion (6.5) provided (6.6) holds. Thus by the bootstrap argument we conclude, with τ defined by (6.14),

$$|||\vec{a}_\varepsilon|||_{(\tau)}\leq(C_1+C_{\rho,\sigma})\|(u_0,v_0)\|_{2\rho_0,\sigma}+1/2,$$

which with (6.7) yields

$$\forall t\in[0,\rho_0/(4\tau)],$$
$$\|(u_\varepsilon(t),v_\varepsilon(t))\|_{\rho_0/2,\sigma}\leq\sqrt{2}(C_1+C_{\rho,\sigma})\|(u_0,v_0)\|_{2\rho_0,\sigma}+\frac{\sqrt{2}}{2}.$$

Now letting $\varepsilon\to 0$ we have, by compactness arguments, that the limit u of u_ε solves the equation (1.2). We complete the existence part of Theorem 1.3. The uniqueness will follow from a similar argument as in [23, sec. 8.2] so we omit it here for brevity. Thus the proof of Theorem 1.3 is completed.

□

Acknowledgment. The research of the first author was supported by NSFC (nos. 11871054, 11771342, and 11961160716) and the Natural Science Foundation of Hubei Province (2019CFA007). The second author is partially supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE (Center for Stability, Instability, and Turbulence at NYUAD) and by NSF-DMS 1716466. The research of the third author was supported by the General Research Fund of Hong Kong, CityU No.11302518, and the Fundamental Research Funds for the Central Universities No.2019CDJCYJ001.

Bibliography

[1] Alexandre, R.; Wang, Y.-G.; Xu, C.-J.; Yang, T. Well-posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.* **28** (2015), no. 3, 745–784. doi:10.1090/S0894-0347-2014-00813-4

[2] Asano, K. A note on the abstract Cauchy-Kowalewski theorem. *Proc. Japan Acad. Ser. A Math. Sci*. **64** (1988), no. 4, 102–105.

[3] Bardos, Claude W.; Titi, Edriss S. Mathematics and turbulence: where do we stand? *J. Turbul.* **14** (2013), no. 3, 42–76. doi:10.1080/14685248.2013.771838

[4] Bardos, C.; Titi, E. S.; Wiedemann E. Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. *Comm. Math. Phys.* **370** (2019), no. 1, 291-–310. doi:10.1007/s00220-019-03493-6

[5] Collot, C.; Ghoul, T.-E.; Ibrahim, S.; Masmoudi, N. On singularity formation for the two dimensional unsteady Prandtl's system. Preprint, 2018. arXiv:1808.05967 [math.AP]

[6] Collot, C.; Ghoul, T.-E; Masmoudi, N. Unsteady separation for the inviscid two-dimensional Prandtl's system. Preprint, 2019. arXiv:1903.08244 [math.AP]

[7] Constantin, P. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. *Comm. Math. Phys.* **104** (1986), no. 2, 311–326.

[8] Dalibard, A.-L.; Masmoudi, N. Separation for the stationary Prandtl equation. *Publ. Math. Inst. Hautes Études Sci.* **130** (2019), 187-–297. doi:10.1007/s10240-019-00110-z

[9] Dietert, H.; Gérard-Varet, D. Well-posedness of the Prandtl equations without any structural assumption. *Ann. PDE* **5** (2019), no. 1, Paper no. 8, 51 pp. doi:10.1007/s40818-019-0063-6

[10] E, W.; Engquist, B. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.* **50** (1997), no. 12, 1287–1293. doi:10.1002/(SICI)1097-0312(199712)50:12<1287::AID-CPA4>3.0.CO;2-4

[11] Fei, M.; Tao, T.; Zhang, Z.-F. On the zero-viscosity limit of the Navier-Stokes equations in the half-space. Preprint, arXiv:1609.03778.

[12] Gérard-Varet, D.; Dormy, E. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.* **23** (2010), no. 2, 591–609. doi:10.1090/S0894-0347-09-00652-3

[13] Gérard-Varet, D.; Maekawa, Y. Sobolev stability of Prandtl expansions for the steady Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **233** (2019), no. 3, 1319–1382. doi:10.1007/s00205-019-01380-x

[14] Gérard-Varet, D.; Maekawa, Y.; Masmoudi, N. Gevrey stability of Prandtl expansions for 2-dimensional Navier-Stokes flows. *Duke Math. J.* **167** (2018), no. 13, 2531–2631. doi:10.1215/00127094-2018-0020

[15] Gérard-Varet, D.; Masmoudi, N. Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér. (4)* **48** (2015), no. 6, 1273–1325. doi:10.24033/asens.2270

[16] Gérard-Varet, D.; Nguyen, T. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77** (2012), no. 1-2, 71–88.

[17] Grenier, E. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53** (2000), no. 9, 1067–1091. doi:10.1002/1097-0312(200009)53:9<1067::AID-CPA1>3.3.CO;2-H

[18] Grenier, E.; Guo, Y.; Nguyen, Toan T. Spectral instability of general symmetric shear flows in a two-dimensional channel. *Adv. Math.* . **292** (2016), 52–110. doi:10.1016/j.aim.2016.01.007

[19] Guo, Y.; Iyer, S. Validity of steady Prandtl layer expansions. Preprint, 2018. arXiv:1805.05891 [math.AP]

[20] Guo, Y.; Nguyen, T. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.* **64** (2011), no. 10, 1416–1438. doi:10.1002/cpa.20377

[21] Guo, Y.; Nguyen, Toan T. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. *Ann. PDE* **3** (2017), no. 1, Paper no. 10, 58 pp. doi:10.1007/s40818-016-0020-6

[22] Li, W.-X.; Yang, T. Well-posedness in Gevrey function space for the three-dimensional Prandtl equations. Preprint, 2017. arXiv:1708.08217 [math.AP]

[23] Li, W.-X.; Yang, T. Well-posedness in Gevrey function spaces for the Prandtl equations with non-degenerate critical points. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 3, 717–775. doi:10.4171/jems/931

[24] Liu, C.-J.; Wang, Y.-G.; Yang, T. On the ill-posedness of the Prandtl equations in three-dimensional space. *Arch. Ration. Mech. Anal.* **220** (2016), no. 1, 83–108. doi:10.1007/s00205-015-0927-1

[25] Maekawa, Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* **67** (2014), no. 7, 1045–1128. doi:10.1002/cpa.21516

[26] Masmoudi, N. Examples of singular limits in hydrodynamics. *Handbook of differential equations: evolutionary equations. Vol. III*, 195–275. Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2007. doi:10.1016/S1874-5717(07)80006-5

[27] Masmoudi, N. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.* **270** (2007), no. 3, 777–788. doi:10.1007/s00220-006-0171-5

[28] Masmoudi, N.; Wong, T. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.* **68** (2015), no. 10, 1683–1741.

[29] Oleinik, O. A.; Samokhin, V. N. Mathematical models in boundary layer theory. Applied Mathematics and Mathematical Computation, 15. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[30] Prandtl, L. Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. *Verhandlungen des III. Internationalen Mathematiker-Kongresses (Heidelberg 1904)*, 484–491. Teubner, 1905.

[31] Sammartino, M; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), no. 2, 433–461. doi:10.1007/s002200050304

[32] Sammartino, M; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), no. 2, 463–491. doi:10.1007/s002200050305

[33] Xin, Z.; Zhang, L. On the global existence of solutions to the Prandtl's system. *Adv. Math.* **181** (2004), no. 1, 88–133. doi:10.1016/S0001-8708(03)00046-X

WEI-XI LI
School of Mathematics and Statistics
and
Hubei Key Laboratory
of Computational Science
Wuhan University
Wuhan 430072
CHINA
E-mail: wei-xi.li@whu.edu.cn

TONG YANG
Department of Mathematics
City University of Hong Kong
Tat Chee Avenue
Kowloon
HONG KONG
E-mail: matyang@cityu.edu.hk

NADER MASMOUDI
Courant Institute
251 Mercer St.
New York, NY 10012
USA
and
NYUD Research Institute
Abu-Dhabi
UAE
E-mail: masmoudi@cims.nyu.edu

Received January 2020.