On the Number of Discrete Chains

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Abstract

We study a generalization of Erdős's unit distances problem to chains of k distances. Given $\mathcal{P}$, a set of n points, and a sequence of distances $(\delta_1, \ldots, \delta_k)$, we study the maximum possible number of tuples of distinct points $(p_1, \ldots, p_{k+1}) \in \mathcal{P}^{k+1}$ satisfying $|p_j p_{j+1}| = \delta_j$ for every $1 \le j \le k$. We study the problem in $\mathbb{R}^2$ and in $\mathbb{R}^3$, and derive upper and lower bounds for this family of problems.

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1 Introduction

Erdős' unit distances problem is one of the main open problems of Discrete Geometry. To quote the book *Research Problems in Discrete Geometry* [3], this is "possibly the best known (and simplest to explain) problem in combinatorial geometry". The problem simply asks: In a set of n points in $\mathbb{R}^2$, what is the maximum number of pairs of point at a distance of one? We denote this maximum value as $u(n)$. In 1946, Erdős [4] constructed a configuration of n points that span $n^{c\sqrt{\log n}}$ unit distances, for some constant c. While over 70 years have passed, this remains the current best lower bound for $u(n)$. In 1984, Spencer, Szemerédi, and Trotter [14] derived the current best upper bound $u(n) = O(n^{4/3})$.

Although the unit distances problem is a central open problem, with many people studying it and with connections to many other problems, no new bounds has been obtained for $u(n)$ since 1984. As is often the case in such situations of stagnation, researchers began studying variants of the problem. For example, Matoušek [10] showed that when replacing the Euclidean distance norm with a generic norm, the maximum number of unit distances becomes significantly smaller. Thus, to solve the unit distances problem, one probably has to rely on a property that is special to the Euclidean norm. For other variants of the problem, see for example [3].

In this note, we study the following generalization of the unit distances problem. We have a set $\mathcal{P}$ of n points in $\mathbb{R}^2$, a positive integer k, and sequence of distances $(\delta_1, \delta_2, \ldots, \delta_k)$. Denote the distance between two points $p, q \in \mathbb{R}^2$ as $|pq|$. We define a *k-chain* to be a $(k+1)$-tuple of distinct points $(p_1, \ldots, p_{k+1}) \in \mathcal{P}^{k+1}$ such that for every $1 \le j \le k$ we have $|p_j p_{j+1}| = \delta_j$. We are interested in the maximum possible value of such chains. By considering the case of $k = 1$, we note that the chains problem is indeed a generalization of the unit distances problem.

The discrete k-chains problem has also been studied by Frankl and Kupavskii, who have improved upon some of the results from this note in [6]. More general repeated configurations have been studied by Gunter, Rhodes, and the first two listed authors; see [7] and the references contained therein. In the related pursuit of distinct configurations, Rudnev [13] derived a lower bound for the minimum number of distinct 2-chains that n points in $\mathbb{R}^2$ can span. This was recently generalized to k-chains by Passant, in [12]. A continuous d-dimensional variant of the repeated chain problem was previously studied by Bennett, Iosevich, and Taylor in [2], and more recently by Ou and Taylor in [11].

Our results. For a positive integer k, we denote as $\mathcal{C}_k(n)$ the maximum number of k-chains that can be spanned by n points in $\mathbb{R}^2$, for any sequence of distances $(\delta_1, \delta_2, \ldots, \delta_k)$. We only consider the case where k is a constant that does not depend on n. We trivially have $\mathcal{C}_0(n) = n$, and we also have $\mathcal{C}_1(n) = u(n) = O(n^{4/3})$. While obtaining a tight bound for $\mathcal{C}_1(n)$ is equivalent to a notoriously difficult open problem, it is surprisingly easy to show that $\mathcal{C}_2(n) = \Theta(n^2)$. See Lemma 3.1 below.

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The following theorem is our main result in $\mathbb{R}^2$.

Theorem 1.1. *For every integer $k \geq 3$, we have*

$$\mathcal{C}_k(n) = O\left(n^{2k/5+1+\gamma(k)}\right),$$

where

$$\gamma(k) = \begin{cases} \frac{1}{75}\cdot\left(4-4\cdot(-1/4)^{k/4}\right) & \text{if } k \equiv 0 \pmod 4, \\ \frac{1}{75}\cdot\left(4-9\cdot(-1/4)^{\lfloor k/4\rfloor}\right) & \text{if } k \equiv 1 \pmod 4, \\ \frac{1}{75}\cdot\left(4+11\cdot(-1/4)^{\lfloor k/4\rfloor}\right) & \text{if } k \equiv 2 \pmod 4, \\ \frac{1}{75}\cdot\left(4-\frac{13}{2}\cdot(-1/4)^{\lfloor k/4\rfloor}\right) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

When $k \geq 3$ we have that $\gamma(k) \leq 1/12$. As $k \to \infty$ we have that $\gamma(k) \to \frac{4}{75}$.

In the other direction, we derive the following lower bounds for $\mathcal{C}_k(n)$.

Proposition 1.2 (Lauren Childs' construction)**.** *For any integer $k \geq 0$, we have*

$$\mathcal{C}_k(n) = \begin{cases} \Omega\left(n^{k/3+1}\right) & \text{if } k \equiv 0 \pmod 3, \\ \Omega\left(n^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ \Omega\left(n^{(k+1)/3+1}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Note that there is still a polynomial gap between the bounds of Theorem 1.1 and Proposition 1.2. We believe that the lower bounds are tight up to subpolynomial terms. Indeed, when assuming the unit distances conjecture $u(n) = \Theta\left(n^{c\sqrt{\log n}}\right)$, one can show that the bounds of Proposition 1.2 are tight up to subpolynomial factors.

Proposition 1.3. *For every $k \geq 3$, we have*

$$\mathcal{C}_k(n) = \begin{cases} O\left(n \cdot u_2(n)^{k/3}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(u_2(n)^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^2 \cdot u_2(n)^{(k-2)/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Chains in three dimensions. The unit distances problem has also been studied in $\mathbb{R}^3$ for many decades. Let $u_3(n)$ denote the maximum number of unit distances that can be spanned by n points in $\mathbb{R}^3$. In 1960 Erdős [5] derived the bound $u_3(n) = \Omega(n^{4/3}\log\log n)$, and this remains the current best lower bound. Unlike the study of $u(n)$, in recent years there have been several small improvements in the upper bound for $u_3(n)$. The current best bound, by Zahl [16], states that for any $\varepsilon > 0$ we have $u_3(n) = O\left(n^{\frac{295}{197}+\epsilon}\right)$.

For a positive integer k, we denote as $\mathcal{C}_k^{(3)}(n)$ the maximum number of k-chains that can be spanned by n points in $\mathbb{R}^3$, for any sequence of distances $(\delta_1, \delta_2, \ldots, \delta_k)$. We only consider the case where k is a constant that does not depend on n. We trivially have $\mathcal{C}_0^{(3)}(n) = n$. For any $\varepsilon > 0$, we have that $\mathcal{C}_1^{(3)}(n) = u_3(n) = O\left(n^{\frac{295}{197}+\epsilon}\right)$. The following is our main result in $\mathbb{R}^3$.

Theorem 1.4. *For every $k \geq 2$ and any $\varepsilon > 0$, we have*

$$\mathcal{C}_k^{(3)}(n) = \begin{cases} O\left(n^{2k/3+1}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(n^{2k/3+23/33+\varepsilon}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^{2k/3+2/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

In particular, note that $\mathcal{C}_2^{(3)}(n) = \mathcal{C}_2(n) = \Theta(n^2)$.

In the case of $\mathcal{C}_3^{(3)}(n)$, one can obtain a slight improvement over Theorem 1.4 with a simple use of Zahl's bound $u_3(n) = O\left(n^{\frac{295}{197}+\epsilon}\right)$. Given a sequence of distances $(\delta_1, \delta_2, \delta_3)$ and a point set $\mathcal{P}$, we look for 3-chains $(p_1, p_2, p_3, p_4) \in \mathcal{P}^4$. There are at most $u_3(n)$ options for choosing the pair $(p_1, p_2) \in \mathcal{P}^2$ and at most $u_3(n)$ options for choosing $(p_3, p_4) \in \mathcal{P}^2$. This implies that $\mathcal{C}_3^{(3)}(n) = O\left((u_3(n))^2\right) = O\left(n^{2.995}\right)$. The same approach does not yield improved bounds for any larger values of k.

With respect to lower bounds, Erdős's unit distances bound implies $\mathcal{C}_1^{(3)}(n) = u_3(n) = \Omega(n^{4/3}\log\log n)$. For longer chains, we derive the following bounds.

Proposition 1.5. *For any integer $k \geq 2$, we have*

$$\mathcal{C}_k^{(3)}(n) = \begin{cases} \Omega\left(n^{(k+1)/2}\right) & \text{if } k \text{ is odd,} \\ \Omega\left(n^{k/2+1}\right) & \text{if } k \text{ is even.} \end{cases}$$

In $\mathbb{R}^d$ with $d \geq 4$, the unit distances problem becomes significantly simpler. In particular, in this case it is possible to have $\Theta(n^2)$ pairs of points at a distance of one (see [9]). The same construction immediately implies that one can have $\Theta(n^{k+1})$ chains of length k, for any $k \geq 1$. Thus, the chains problem is trivial in this case.

In Section 2 we introduce geometric incidence results that we require for our proofs. In Section 3 we derive our bounds in $\mathbb{R}^2$. Finally, in Section 4 we derive our bounds in $\mathbb{R}^3$.

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2 Preliminaries: Geometric incidences

Given a set $\mathcal{P}$ of points and a set Γ of circles, both in $\mathbb{R}^2$, an *incidence* is a pair $(p, \gamma) \in \mathcal{P} \times \Gamma$ such that the point p is contained in the circle γ. We denote by $I(\mathcal{P}, \Gamma)$ the number of incidences in $\mathcal{P} \times \Gamma$. Aronov and Sharir [1] proved the following result.

Theorem 2.1. *Consider a set $\mathcal{P}$ of m points and a set Γ of n circles, both in $\mathbb{R}^2$. For every $\varepsilon > 0$, we have*

$$I(\mathcal{P}, \Gamma) = O\left(m^{9/11+\varepsilon} n^{6/11} + m^{2/3} n^{2/3} + m + n\right).$$

This bound also holds when assuming that $\mathcal{P}$ and Γ lie on a common sphere in $\mathbb{R}^3$.

There are many similar incidence problems where we have a set of points $\mathcal{P}$ and a set of "objects" Γ in $\mathbb{R}^d$, and are looking for the maximum number of incidences. In every such problem we use $I(\mathcal{P}, \Gamma)$ to denote the number of incidences in $\mathcal{P} \times \Gamma$. The following result was proved by Zahl [17] and independently by Kaplan, Matoušek, Safernová, and Sharir [8].

Theorem 2.2. *Consider a set $\mathcal{P}$ of m points and a set $\mathcal{S}$ of n spheres of the same radii, both in $\mathbb{R}^3$. For every $\varepsilon > 0$, we have*

$$I(\mathcal{P}, \mathcal{S}) = O\left(m^{3/4} n^{3/4} + m + n\right).$$

Let $\mathcal{S}$ be a set of spheres of the same radii in $\mathbb{R}^3$. For an integer $r \geq 2$, we say that a point $p \in \mathbb{R}^2$ is *r-rich* if p is incident to at least r spheres of $\mathcal{S}$. Upper bounds for incidence problems such as Theorems 2.1 and 2.2 have *dual formulations* in terms of r-rich points. These results are dual in the sense that there are short simple ways of getting from each upper bound to the other (for example, see [15]). The following is the dual formulation of Theorem 2.2.

Theorem 2.3. *Consider a set $\mathcal{S}$ of n spheres of the same radii in $\mathbb{R}^3$. For every integer $r \geq 2$, the number of r-rich points is*

$$O\left(\frac{n^3}{r^4} + \frac{n}{r}\right).$$

The following is the dual form of the upper bound for the number of incidences with circles of the same radii in $\mathbb{R}^2$. In other words, it is the dual of the current best upper bound for $u(n)$.

Theorem 2.4. *Consider a set Γ of n circles of the same radii in $\mathbb{R}^2$. For every integer $r \geq 2$, the number of r-rich points is*

$$O\left(\frac{n^2}{r^3} + \frac{n}{r}\right).$$

3 Bounds for the number of k-chains in $\mathbb{R}^2$

In this section we prove our bounds for the maximum number of k-chains in $\mathbb{R}^2$. We refer to a 2-chain as a *hinge*. As a warm-up, we first derive a tight bound for the case of hinges. We will also require this bound to prove Theorem 1.1.

Lemma 3.1. $\mathcal{C}_2(n) = \Theta\left(n^2\right)$.

Proof. Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$. Consider a sequence of distances (δ_1, δ_2), and denote a hinge as $(p_1, p_2, p_3) \in \mathcal{P}^3$. There are n choices for p_1, and then at most $n-1$ choices for p_3. After these two points are chosen, p_2 must be on a circle of radius δ_1 centered at p_1 and also on a circle of radius δ_2 centered at p_2. These two circles intersect in at most two points, so for every choice of p_1 and p_3 there are at most two valid options for p_2. This immediately implies that $\mathcal{C}_2(n) = O\left(n^2\right)$.

To see that the above upper bound is tight, consider a sequence of distances (δ_1, δ_2). Let C_1 and C_2 be circles centered at the origin of respective radii δ_1 and δ_2. We construct a set $\mathcal{P}$ by taking the origin, $\lfloor (n-1)/2 \rfloor$ points from C_1, and $\lceil (n-1)/2 \rceil$ points from C_2. It is not difficult to verify that $\mathcal{P}$ is a set of n points that span $\Theta(n^2)$ hinges. $\square$

We now move to derive upper bounds for $\mathcal{C}_k(n)$ when $k \geq 3$. We first recall the statement of the Theorem 1.1.

Theorem 1.1. *For every $k \geq 3$, we have*

$$\mathcal{C}_k(n) = O\left(n^{2k/5+1+\gamma(k)}\right),$$

where

$$\gamma(k) = \begin{cases} \frac{1}{75} \cdot \left(4 - 4 \cdot (-1/4)^{k/4}\right) & \text{if } k \equiv 0 \pmod 4, \\ \frac{1}{75} \cdot \left(4 - 9 \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 1 \pmod 4, \\ \frac{1}{75} \cdot \left(4 + 11 \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 2 \pmod 4, \\ \frac{1}{75} \cdot \left(4 - \frac{13}{2} \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Proof. The proof consists of two parts. In the first part we derive a recurrence relation for upper bounds on $\mathcal{C}_k(n)$. In the second part we solve this relation.

Deriving a recurrence relation. Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$. Consider a sequence of distances $(\delta_1, \dots, \delta_k)$, and denote a k-chain as $(p_1, \dots, p_{k+1}) \in \mathcal{P}^{k+1}$. Let $Q \subset \mathcal{P}^{k+1}$ denote the set of k-chains that correspond to the sequence of distances.

For a point $p_2 \in \mathbb{R}^2$, we denote by $\alpha_1(p_2)$ the number of points $p_1 \in \mathcal{P}$ that satisfy $|p_1p_2| = \delta_1$. Similarly, for a point $p_k \in \mathcal{P}$ we denote by $\alpha_{k+1}(p_k)$ the number of points $p_{k+1} \in \mathcal{P}$ that satisfy $|p_kp_{k+1}| = \delta_k$. We partition Q into two disjoint sets Q' and Q'', as follows. Let Q' be the set of k-chains $(p_1, \dots, p_{k+1}) \in Q$ satisfying $\alpha_1(p_2) \geq \alpha_{k+1}(p_k)$. Let $Q'' = Q \setminus Q'$ be the set of k-chains $(p_1, \dots, p_{k+1}) \in Q$ satisfying $\alpha_1(p_2) < \alpha_{k+1}(p_k)$.

Without loss of generality, assume that $|Q''| \geq |Q'|$. In this case, we have that $|Q| \leq 2|Q''|$. For $0 \leq j < \lceil \log_2 n \rceil$, let Q''_j be the set of k-chains $(p_1, \dots, p_{k+1}) \in Q''$ that satisfy $2^{j-1} \leq \alpha_{k+1}(p_k) < 2^j$. In other words, we dyadically decompose Q'' into $\lceil \log_2 n \rceil$ subsets, each consisting of k-chains $(p_1, \dots, p_{k+1})$ having approximately the same value of $\alpha_{k+1}(p_k)$. Note that $|Q''| = \sum_{j=0}^{\lceil \log_2 n \rceil} |Q''_j|$.

For a fixed $0 \leq j < \lceil \log_2 n \rceil$, we now derive an upper bound on the number of k-chains in Q''_j. When placing a circle of radius δ_k around every point of $\mathcal{P}$, a point p_k that satisfies $\alpha_{k+1}(p_k) \geq 2^{j-1}$ is also a 2^{j-1}-rich point (as defined in Section 2). Thus, Theorem 2.4 implies that the number of points p_k that satisfy $\alpha_{k+1}(p_k) \geq 2^{j-1}$ is

$$O\left(\frac{n^2}{2^{3j}} + \frac{n}{2^j}\right).$$

The above is an upper bound for the number of ways to choose p_k. After choosing a specific p_k, there are fewer than 2^j choices for p_{k+1}. There are at most $\mathcal{C}_{k-3}(n)$ choices for $(p_1, \dots, p_{k-2}) \in \mathcal{P}^{k-2}$. Finally, after choosing $(p_1, \dots, p_{k-2})$ and p_k there are at most two choices for p_2, since this point lies on the intersection of two circles (as in the proof of Lemma 3.1). We conclude that

$$|Q''_j| \leq O\left(\frac{n^2}{2^{3j}} + \frac{n}{2^j}\right) \cdot 2^j \cdot \mathcal{C}_{k-3}(n) \cdot 2 = O\left(\frac{n^2 \cdot \mathcal{C}_{k-3}(n)}{2^{2j}} + n \cdot \mathcal{C}_{k-3}(n)\right). \tag{1}$$

The bound of (1) is reasonable for large values of j, but weak for small values of j. We thus derive another bound for the number of k-chains in Q''_j. There are at most $\mathcal{C}_{k-2}(n)$ choices of $(p_2, \ldots, p_k) \in \mathcal{P}^{k-1}$. As before, there are fewer than 2^j choices for p_{k+1}. Since we are only counting quadruples from Q'', there are also fewer than 2^j choices for p_1. Combining these observations leads to

$$|Q''_j| \le \mathcal{C}_{k-2}(n) \cdot 2^{2j}. \tag{2}$$

Set $\beta = \left\lfloor \log_2 \left(\frac{n^2 \cdot \mathcal{C}_{k-3}(n)}{\mathcal{C}_{k-2}(n)} \right)^{1/4} \right\rfloor$. By combining (1) and (2), we obtain

$$\begin{aligned}
|Q| \le 2|Q''| = 2\sum_{j=0}^{\lg n} |Q''_j| &= 2\sum_{j=0}^{\beta} |Q''_j| + 2\sum_{j=\beta+1}^{\lceil \log_2 n \rceil} |Q''_j| \\
&= O\left(\sum_{j=0}^{\beta} \mathcal{C}_{k-2}(n) \cdot 2^{2j} + \sum_{j=\beta+1}^{\lceil \log_2 n \rceil} \left(\frac{n^2 \cdot \mathcal{C}_{k-3}(n)}{2^{2j}} + n \cdot \mathcal{C}_{k-3}(n) \right) \right) \\
&= O\left(n \cdot \sqrt{\mathcal{C}_{k-2}(n) \cdot \mathcal{C}_{k-3}(n)} + n \log n \cdot \mathcal{C}_{k-3}(n) \right).
\end{aligned}$$

The first term in this bound dominates when $\mathcal{C}_{k-2}(n) = \Omega\left(\mathcal{C}_{k-3}(n) \log^2 n\right)$. This will be the case for all of our bounds, so we may ignore the second term and consider the bound

$$\mathcal{C}_k(n) = O\left(n \cdot \sqrt{\mathcal{C}_{k-2}(n) \cdot \mathcal{C}_{k-3}(n)} \right). \tag{3}$$

Solving the recurrence relation. Recall that $\mathcal{C}_0(n) = n$ and that $\mathcal{C}_1(n) = u(n) = O(n^{4/3})$. Combining these with (3) yields $\mathcal{C}_3(n) = O\left(n^{13/6}\right)$. We can similarly derive an upper bound for $\mathcal{C}_k(n)$ with any $k \ge 3$.

Let a_k to be the current best exponent of n in the upper bound for $\mathcal{C}_k(n)$. As already mentioned above, we have $a_0 = 1, a_1 = 4/3$, and $a_2 = 2$. By (3), we have that

$$a_k = \frac{1}{2}a_{k-3} + \frac{1}{2}a_{k-2} + 1.$$

Solving this relation and initial values yields

$$a_k = \frac{79}{75} + \left(-\frac{2}{75} - \frac{11}{75}i\right)\left(-\frac{1}{2} - \frac{i}{2}\right)^k + \left(-\frac{2}{75} + \frac{11}{75}i\right)\left(-\frac{1}{2} + \frac{i}{2}\right)^k + \frac{2}{5}k.$$

Simplifying this leads to

$$a_k = \begin{cases} 1 + \frac{2}{5}k + \frac{1}{75}\left(4 - 4 \cdot (-1/4)^{k/4}\right) & \text{if } k \equiv 0 \pmod 4, \\ 1 + \frac{2}{5}k + \frac{1}{75}\left(4 - 9 \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 1 \pmod 4, \\ 1 + \frac{2}{5}k + \frac{1}{75}\left(4 + 11 \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 2 \pmod 4, \\ 1 + \frac{2}{5}k + \frac{1}{75}\left(4 - \frac{13}{2} \cdot (-1/4)^{\lfloor k/4 \rfloor}\right) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

□

We now derive our lower bounds for $\mathcal{C}_k(n)$. We first recall the statement of the corresponding proposition.

Proposition 1.2. *For any integer $k \ge 0$, we have*

$$\mathcal{C}_k(n) = \begin{cases} \Omega\left(n^{k/3+1}\right) & \text{if } k \equiv 0 \pmod 3, \\ \Omega\left(n^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ \Omega\left(n^{(k+1)/3+1}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

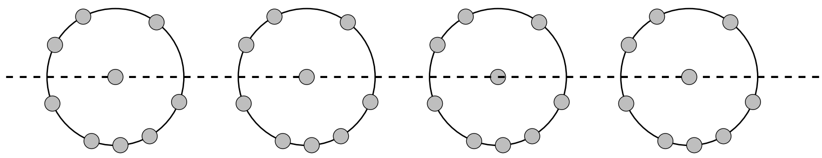

Figure 1: $(k+1)/3$ translated copies of the same circle configuration.

Proof. First assume that $k \equiv 2 \pmod 3$. We choose two arbitrary distances $\delta_1, \delta_2 > 0$, and consider the sequence of distances

$$(\delta_1, \delta_1, \delta_2, \delta_1, \delta_1, \delta_2, \delta_1, \delta_1, \delta_2, \ldots, \delta_2, \delta_1, \delta_1).$$

Set $m = \lfloor 3n/(k+1) \rfloor$. Let γ be a circle of radii δ_1, place $m-1$ on γ, and place one additional point at the center of γ. Denote this configuration of m points as A. We create a second copy of A, translated a distance of δ_2 in the x direction. That is, we now have two circles and $2m$ points. We keep creating more copies of A, each translated a distance of δ_3 in the x-direction from the preceding one. After having $(k+1)/3$ copies of A, we denote the resulting set of $m(k+1)/3 \le n$ points as $\mathcal{P}$. Such a configuration is illustrated in Figure 1.

To obtain a chain, we first choose p_1 to be a point on the first circle and set p_2 to be the center of that circle. We have $m-1 = \Theta(n)$ options for choosing p_1, and a single choice for p_2. We then choose p_3 to be another point on the first circle, set p_4 to be the point in the second copy of A that corresponds to p_3, and set p_5 to be the center of the second copy of A. There are $m-2 = \Theta(n)$ choices for p_3, a single option for p_4, and a single option for p_5. We repeat this step $(k+1)/3 - 1$ times: Starting from a center of a circle, choosing a point on the circle, moving to the corresponding point in the next circle, and moving to the center of that next circle. At each step we determine three vertices of the k-chain and have $m-2 = \Theta(n)$ choices.

When the above process ends, we obtain a k-chain that corresponds to our sequence of distances. This process consists of $(k+1)/3+1$ steps where we have at least $m-2 = \Theta(n)$ choices. Thus, the number of k-chains that correspond to the above sequence of distances is $\Theta\left(n^{(k+1)/3+1}\right)$.

Next, consider the case when $k \equiv 1 \pmod 3$. In this case, we repeat the above construction with $m = \lfloor 3n/(k+2) \rfloor$ and have $(k+2)/3$ copies of A. When creating a k-chain as before, we only take two points from the rightmost circle. Thus, the sequence of distances ends with $(\ldots, \delta_2, \delta_1)$. This implies that we have only $(k+2)/3$ steps with $\Theta(n)$ choices. That is, the number of k-chains is $\Theta\left(n^{(k+2)/3}\right)$.

The case of $k \equiv 0 \pmod 3$ is handled symmetrically. We repeat the above construction with $m = \lfloor 3n/(k+3) \rfloor$, and have $(k+3)/3$ circles. When creating a k-chain, we only take one points from the rightmost circle. Thus, the sequence of distances ends with $(\ldots, \delta_2)$. This implies that we have only $(k+3)/3 = k/3+1$ steps with $\Theta(n)$ choices. That is, the number of k-chains is $\Theta\left(n^{k/3+1}\right)$. $\square$

We conclude this section with a proof of Proposition 1.3. We first recall the statement of this proposition.

Proposition 1.3. *For every $k \ge 3$, we have*

$$\mathcal{C}_k(n) = \begin{cases} O\left(n \cdot u_2(n)^{k/3}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(u_2(n)^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^2 \cdot u_2(n)^{(k-2)/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$. Consider a sequence of distances $(\delta_1, \ldots, \delta_k)$, and denote a k-chain as $(p_1, \ldots, p_{k+1}) \in \mathcal{P}^{k+1}$. The proof is split into cases according to $k \pmod 3$.

First, we consider the case of $k \equiv 0 \pmod 3$. There are n possible choices for p_{k+1}. We split the first k points into triples of the form $(p_{3j+1}, p_{3j+2}, p_{3j+3})$, for every $0 \le j < k/3$. In each of these $\frac{k}{3}$ triples, we have at most $u_2(n)$ choices for the pair (p_{3j+1}, p_{3j+2}). After choosing the first pair of points in each triple, there remain $k/3$ points that were not yet chosen in the chain. For every $0 \le j < k/3$, since we already chose both p_{3j+2} and p_{3j+4}, there are at most two valid choices for p_{3j+3} (using the same argument as in the proof of Lemma 3.1). We conclude that

$$\mathcal{C}_k(n) = O\left(n \cdot u_2(n)^{k/3}\right).$$

We move to consider the case of $k \equiv 1 \pmod 3$. There are at most $u_2(n)$ choices for the pair of points $(p_k, p_{k+1}) \in \mathcal{P}^2$. We split the first $k-1$ points into triples of the form $(p_{3j+1}, p_{3j+2}, p_{3j+3})$, for every $0 \leq j < (k-1)/3$. By handling these $(k-1)/3$ triples as in the previous case, we obtain

$$\mathcal{C}_k(n) = O\left(u(n)^{(k+2)/3}\right).$$

Finally, we consider the case of $k \equiv 2 \pmod 3$. Since (p_{k-1}, p_k, p_{k+1}) form a hinge, by Lemma 3.1 there are $O(n^2)$ choices for this triple. We split the remaining $k-2$ points into triples of the form $(p_{3j+1}, p_{3j+2}, p_{3j+3})$, for every $0 \leq j < (k-2)/3$. By handling these $(k-2)/3$ triples as in the previous cases, we obtain

$$\mathcal{C}_k(n) = O\left(n^2 \cdot u(n)^{(k-2)/3}\right).$$

□

4 Bounds for the number of k-chains in $\mathbb{R}^3$

In this section we derive our bounds for $\mathcal{C}_k^{(3)}(n)$. We begin by recalling the statement of Theorem 1.4.

Theorem 1.4. *For every $k \geq 2$ and any $\varepsilon > 0$, we have*

$$\mathcal{C}_k^{(3)}(n) = \begin{cases} O\left(n^{2k/3+1}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(n^{2k/3+23/33+\varepsilon}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^{2k/3+2/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Moreover, $\mathcal{C}_3^{(3)}(n) = O((u_3(n))^2)$.

Proof. The proof has two steps. We first prove that $\mathcal{C}_2^{(3)}(n) = \Theta(n^2)$, and then rely on this bound to study large values of k. Recall that we refer to 2-chains as hinges.

Deriving a bound for the number of hinges and 3-chains. The lower bound $\mathcal{C}_2^{(3)}(n) = \Omega(n^2)$ is obtained in the same way as in Lemma 3.1. It remains to derive a matching upper bound.

Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^3$. Consider a sequence of distances (δ_1, δ_2), and denote the hinges corresponding to this sequence as $(p_1, p_2, p_3) \in \mathcal{P}^3$. For an integer $0 \leq j < \lceil \log_2 n \rceil$, let $\mathcal{P}_j$ be the set of points $p_2 \in \mathcal{P}$ such that there exist at least 2^{j-1} points $p_1 \in \mathcal{P}$ satisfying $|p_1p_2| = \delta_1$, and less than 2^j such points. When placing a sphere of radius δ_1 around every point of $\mathcal{P}$, every point $p_2 \in \mathcal{P}_j$ is also a 2^{j-1}-rich point. Thus, by Theorem 2.3

$$|\mathcal{P}_j| = O\left(\frac{n^3}{2^{4j}} + \frac{n}{2^j}\right).$$

For a fixed $0 \leq j < \lceil \log_2 n \rceil$, we now bound the number of hinges with $p_2 \in \mathcal{P}_j$. We know that for every $p_2 \in \mathcal{P}_j$, there are $\Theta(2^j)$ choices for p_1. Let $\mathcal{S}$ be a set of $|\mathcal{P}_j|$ spheres of radius δ_2 centered around the points of $\mathcal{P}_j$. Every incidence in $\mathcal{P}_j \times \mathcal{S}$ corresponds to a pair $(p_2, p_3) \in \mathcal{P}_j \times \mathcal{P}$ that appear together in a hinge. By Theorem 2.2, the number of hinges with $p_2 \in \mathcal{P}_j$ is

$$\begin{aligned} O\left(2^j\right) \cdot I(\mathcal{P}, \mathcal{S}) &= O\left(2^j \cdot \left(n^{3/4}|\mathcal{P}_j|^{3/4} + n + |\mathcal{P}_j|\right)\right) \\ &= O\left(2^j \cdot \left(n^{3/4}\left(\frac{n^3}{2^{4j}} + \frac{n}{2^j}\right)^{3/4} + n + \left(\frac{n^3}{2^{4j}} + \frac{n}{2^j}\right)\right)\right) \\ &= O\left(\frac{n^3}{2^{2j}} + n^{3/2}2^{j/4} + n \cdot 2^j\right). \end{aligned}$$

Summing the above bound over every j from $\lceil \log_2 n \rceil/2$ up $\lceil \log_2 n \rceil - 1$ leads to $O(n^2)$. That is, the number of hinges with $p_2 \in \mathcal{P}_j$ for any $\lceil \log_2 n \rceil/2 \leq j \leq \lceil \log_2 n \rceil$ is $O(n^2)$. It remains to bound the number of hinges with $p_2 \in \mathcal{P}_j$ for any $0 \leq j < \lceil \log_2 n \rceil/2$.

By definition, we also have $|\mathcal{P}_j| \leq |\mathcal{P}| = n$. Repeating the above argument with this bound implies that the number of hinges with $p_2 \in \mathcal{P}_j$ is

$$O\left(2^j\right) \cdot I(\mathcal{P}, \mathcal{S}) = O\left(2^j \cdot \left(n^{3/4}|\mathcal{P}_j|^{3/4} + n + |\mathcal{P}_j|\right)\right) = O\left(2^j n^{3/2}\right).$$

Summing the above bound over every j from 0 up to $\lceil \log_2 n \rceil/2 - 1$ leads to $O(n^2)$. We conclude that the total number of hinges in $\mathcal{P}$ is $O(n^2)$.

To bound the number of 3-chains, we simply notice that there are no more than $u_3(n)$ ways to choose p_1 and p_2 such that $|p_1p_2| = \delta_1$, and no more than $u_3(n)$ ways to choose p_3 and p_4 such that $|p_3p_4| = \delta_3$.

Deriving bounds for larger values of k. Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^3$. Consider a sequence of distances $(\delta_1, \ldots, \delta_k)$ and denote a k-chain as $(p_1, \ldots, p_{k+1}) \in \mathcal{P}^{k+1}$.

First assume that $k \equiv 2 \pmod 3$. For $0 \le j < (k+1)/3$, by the above bound for $\mathcal{C}_2^{(3)}(n)$ there are $O(n^2)$ ways for choosing $(p_{3j+1}, p_{3j+2}, p_{3j+3})$. Thus, in this case the number of k-chains is $O(n^{2(k+1)/3})$. Next, we assume that $k \equiv 0 \pmod 3$. For $0 \le j < k/3$, there are $O(n^2)$ ways for choosing $(p_{3j+1}, p_{3j+2}, p_{3j+3})$. Since there are n ways for choosing p_{k+1}, we get a total of $O(n^{2k/3+1})$ chains.

Finally, consider the case of $k \equiv 1 \pmod 3$. For $0 \le j < (k-1)/3$, there are $O(n^2)$ ways for choosing $(p_{3j+1}, p_{3j+2}, p_{3j+3})$. Consider such a fixed choice of $p_1, \ldots, p_{k-1}$. Let S be the sphere centered at p_{k-1} and of radius δ_{k-1}. There are n ways for choosing p_{k+1}, and for every such choice p_k must be on a specific circle on S. In particular, this circle is the intersection of S with the sphere centered at p_{k+1} and of radius δ_k. Let Γ denote the set of circles that are obtained in this way.

A specific circle in Γ can originate from at most two values of p_{k+1}, since at most two spheres of radius δ_k can contain a given circle. This implies that $|\Gamma| = \Theta(n)$. For a fixed p_{k+1}, the number of choices for p_k is the number of points on the corresponding circle. Thus, the total number of choices for both p_{k+1} and p_k is $I(\mathcal{P}, \Gamma)$. Theorem 2.1 implies that $I(\mathcal{P}, \Gamma) = O(n^{15/11+\varepsilon})$ for any $\varepsilon > 0$. We conclude that, in the case of $k \equiv 1 \pmod 3$, the number of k-chains is

$$O\left(n^{2(k-1)/3} \cdot n^{15/11+\varepsilon}\right) = O\left(n^{2k/3+23/33+\varepsilon}\right).$$

□

We now prove Proposition 1.5. We begin by recalling the statement of this proposition.

Proposition 1.5. *For any integer $k \ge 2$, we have*

$$\mathcal{C}_k^{(3)}(n) = \begin{cases} \Omega\left(n^{(k+1)/2}\right) & \text{if } k \text{ is odd,} \\ \Omega\left(n^{k/2+1}\right) & \text{if } k \text{ is even.} \end{cases}$$

Proof. The proof is a variant of the proof of Proposition 1.2, taking advantage of the extra dimension that is available in this case.

First assume that k is even. We choose two arbitrary distances $0 < \delta_1 < \delta_2$, and consider the sequence of distances

$$(\delta_1, \delta_2, \delta_1, \delta_2, \delta_1, \delta_2, \ldots, \delta_2, \delta_1).$$

Set $m = \lfloor 2n/k \rfloor$. Let γ be a circle of radius δ_1 centered at the origin of $\mathbb{R}^3$ and contained in the plane defined by $x = 0$. We place $m - 1$ points on γ, and one additional point at the center of γ. Denote this configuration of m points as A. We create a second copy of A, translated in the x direction such that the distance between the origin and every point on the translated copy of γ is δ_2. We now have two circles and $2m$ points. We keep creating more copies of A, each translated the same distance in the x-direction from the preceding one. After having $k/2$ copies of A, we denote the resulting set of $mk/2 \le n$ points as $\mathcal{P}$.

To obtain a chain, we first choose p_1 to be a point on the first circle and set p_2 to be the center of that circle. We have $m - 1 = \Theta(n)$ options for choosing p_1, and a single way to choose p_2. We then choose p_3 to be a point on the second circle, and p_4 to be the center of the second circle. There are $m - 1 = \Theta(n)$ options for choosing p_3 and a single option for p_4. We repeat this step another $k/2 - 2$ times: Starting from a center of a circle, choosing a point on the next circle, and moving to the center of that circle. At each step we determine two vertices of the k-chain and have $m - 1 = \Theta(n)$ choices.

When the above process ends, we obtain a k-chain that corresponds to our sequence of distances. This process consists of $k/2+1$ steps where we have $m-1 = \Theta(n)$ choices. Thus, the number of k-chains that correspond to the above sequence of distances is $\Theta\left(n^{k/2+1}\right)$.

Next, consider the case when k is odd. In this case, we repeat the above construction with $m = \lfloor 2n/(k-1) \rfloor$ and have $(k-1)/2$ circles. Then we add one more point p_{k+1} as the center of yet another circle, but with no points on the circle around it. When creating a k-chain as before, we always end with this fixed final point p_{k+1}. Thus, the sequence of distances ends with δ_2 in this case. This implies that we have $(k+1)/2$ steps with $\Theta(n)$ choices. That is, the number of k-chains is $\Theta\left(n^{(k+1)/2}\right)$. □

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