Emergent dual holographic description as a nonperturbative generalization of the Wilsonian renormalization group

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In holographic duality, dynamics along the emergent extra-dimensional space describes a renormalization group (RG) flow of the corresponding quantum field theory (QFT). Following this idea, we develop an emergent holographic description of a QFT, where not only the information of the RG flow is introduced into an IR holographic dual effective field theory (HDEFT), but also the UV information of the QFT is encoded in the HDEFT through the IR boundary condition. In particular, we argue that this dual holographic construction is self-consistent within the assumption of bulk locality, showing the following two aspects: The solution of the Hamilton-Jacobi equation is given by the IR boundary effective action, and the Ward identity involving the QFT energy-momentum tensor current is satisfied naturally. We discuss the role of the RG β-function in the bulk effective dynamics of the metric tensor near a conformally invariant fixed point. We claim that this emergent dual gravity theory generalizes the perturbative Wilsonian RG framework into a nonperturbative way.

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I. INTRODUCTION

Nonperturbative approaches to quantum field theory (QFT) are highly sought after to study strongly-coupled problems. Specifically, we need a framework in which relevant quantum corrections to self-energies and vertices are self-consistently resummed in the infinite order in the renormalization group (RG) sense. The $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$ duality conjecture [1–4] has been regarded as a nonperturbative theoretical framework, where a nonperturbative RG flow of a UV field theory is realized through the evolution along the extra-dimensional space [5–7]. Here, D is the spacetime dimension.

The holographic approach has been applied to various strongly-coupled problems, such as confinement and chiral symmetry breaking in quantum chromodynamics [2,3] and superconductivity and non-Fermi liquids in condensed matter physics [8–17], and provided remarkable solutions, e.g., emergent physics of effective hydrodynamics [18–21]. These results, in view of universality, can in principle be applicable to a wide class of problems. However, there still exists an unsatisfactory point; it is not entirely clear how to relate UV microscopic degrees of freedom with IR emergent macroscopic observables. To overcome this difficulty, various approaches have been tried to derive an effective holographic dual field theory based on RG transformations [22–59].

In this study, we continue to follow these lines of approach and develop further an emergent holographic description of a QFT. In this framework, not only the information of the RG flow is included into an IR holographic dual effective field theory, but also the UV information of the QFT is encoded through the IR boundary condition. In particular, we argue that, within the assumption of bulk locality, this dual holographic construction is self-consistent, showing the following two aspects: The solution of the Hamilton-Jacobi equation is given by the IR boundary effective action, and the Ward identity involving the QFT energy-momentum tensor current is satisfied naturally. We discuss the role of the RG β-function in the bulk effective dynamics of the metric tensor near a conformally-invariant fixed point.

Recently, it has been clarified that the Wess-Zumino consistency condition for the local RG flow of a QFT can be translated into the Hamilton-Jacobi formulation of a holographic dual effective field theory [46–49]. The present study takes into account this internal consistency for the emergent dual holographic description. The resulting holographic dual effective field theory generalizes the previous construction, where an IR boundary condition is introduced as the solution of the Hamilton-Jacobi equation. This IR

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boundary condition makes it manifest how to encode the UV information of the QFT into the IR holographic dual effective field theory even away from quantum criticality. This framework thus extends the $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$ duality conjecture to systems away from criticality. More generally, we claim that this emergent dual gravity theory is a nonperturbative generalization of the perturbative Wilsonian RG framework.

II. EMERGENT DUAL HOLOGRAPHY AS A RENORMALIZATION GROUP FLOW

A. Construction of holographic dual effective field theory

The starting point of our analysis is the following Euclidean path integral in D spacetime dimensions,

$$Z = \int D\psi_\alpha(x) D g_B^{\mu\nu}(x) \delta(g_B^{\mu\nu}(x) - \delta^{\mu\nu}) \exp\left[- \int d^D x \sqrt{g_B(x)} \left\{ \mathcal{L}[\psi_\alpha(x), g_B^{\mu\nu}(x)] + \frac{\lambda}{2N} T^{\mu\nu}(x) \mathcal{G}^B_{\mu\nu\rho\gamma}(x) T^{\rho\gamma}(x) \right\} \right]. \quad (1)$$

Here, $\psi_\alpha(x)$ $(\alpha = 1, \ldots, N)$ is a matter field with its dynamics described by the Lagrangian $\mathcal{L}[\psi_\alpha(x), g_B^{\mu\nu}(x)]$. In this study, we focus only on relativistic invariant theories. $g_B^{\mu\nu}(x)$ is a formally introduced background metric, and enforced to be the flat metric, $g_B^{\mu\nu}(x) = \delta^{\mu\nu}$, by the delta functional. The Lagrangian $\mathcal{L}[\psi_\alpha(x), g_B^{\mu\nu}(x)]$ is deformed by an effective interaction that is quadratic in the energy-momentum tensor (the $T\bar{T}$ deformation in D spacetime dimensions) [60], where $T^{\mu\nu}(x) = \frac{2}{\sqrt{g_B(x)}} \frac{\delta}{\delta g_{\mu\nu}^B(x)} (\sqrt{g_B(x)} \mathcal{L}[\psi_\alpha(x), g_B^{\mu\nu}(x)])$, $\lambda \geq 0$ is the coupling constant, and $\mathcal{G}^B_{\mu\nu\rho\gamma}(x) \equiv \frac{1}{2} g^B_{\mu\rho}(x) g^B_{\nu\gamma}(x) + \frac{1}{2} g^B_{\nu\rho}(x) g^B_{\mu\gamma}(x) - \frac{1}{D-1} g^B_{\mu\nu}(x) g^B_{\rho\gamma}(x)$ is the DeWitt supermetric [61], taking into account transverseness.

The type of theory in (1) can be studied by using the "recursive RG transformations"; Performing the functional RG transformation (with the Hubbard-Stratonovich transformation) in a recursive way, one can construct an IR holographic dual effective field theory, which describes the evolution of the metric tensor in the RG (energy) scale. The resulting theory takes the form of emergent gravity, with emergent extra dimension representing the RG scale of the problem [43–45,50–53]. In the following, we coordinatize the extra dimension by z, and $z = 0$, and z_f by convention represent the UV and IR energy scales, respectively.

The details of the steps to derive the holographic theory starting from specific UV quantum field theories with double trace interactions can be found in [43–45,50–53]. In general, the functional RG transformations give rise to nonlocal effective interactions. Such emergent nonlocal interactions can however be "localized" at the cost of introducing higher-spin fields to decompose them in a local fashion based on the corresponding group structure [62–66]. In other words, integrating over such higher-spin fields gives rise to an effective gravity theory including only up to spin two fields, but in the presence of effective nonlocal interactions between gravitons. In most cases, we will work with a proper local truncation of these RG-generated nonlocal terms [67], keeping the original form of the effective Lagrangian as in the conventional RG transformation [50,51].

With the locality assumption in mind, in this paper, we propose a generic dual holographic effective theory resulting from the recursive RG transformations. It is given by

$$\begin{aligned} Z = \int & D\psi_\alpha(x) D g_{\mu\nu}(x,z) D\pi^{\mu\nu}(x,z) D\mathcal{N}(x,z) D\mathcal{N}_\mu(x,z) \delta(g^{\mu\nu}(x,0) - g_B^{\mu\nu}(x)) \\ & \times \exp\left[- \int d^D x \sqrt{g(x,z_f)} \mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)] \right. \\ & \left. - N \int_0^{z_f} dz \int d^D x \{ \pi^{\mu\nu}(x,z) \partial_z g_{\mu\nu}(x,z) + \mathcal{N}(x,z) \mathcal{H} + \mathcal{N}_\mu(x,z) \mathcal{H}^\mu \} \right]. \end{aligned} \quad (2)$$

Here, the emergent bulk dynamical metric tensor is given by

$$\begin{aligned} ds^2 = & (\mathcal{N}^2(x,z) + \mathcal{N}_\mu(x,z) \mathcal{N}^\mu(x,z)) dz^2 \\ & + 2\mathcal{N}_\mu(x,z) dx^\mu dz + g_{\mu\nu}(x,z) dx^\mu dx^\nu. \end{aligned} \quad (3)$$

$\mathcal{N}(x,z)$ and $\mathcal{N}^\mu(x,z)$ are the lapse function and the shift vector, respectively, and $g_{\mu\nu}(x,z)$ is the D-dimensional metric tensor in the Arnowitt-Deser-Misner (ADM) decomposition [68]. The dynamics of the metric tensor is governed by the effective Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{\lambda}{2} \frac{1}{\sqrt{g(x,z)}} \pi^{\mu\nu}(x,z) \mathcal{G}_{\mu\nu\rho\gamma}(x,z) \pi^{\rho\gamma}(x,z) \\ & + \mathcal{H}_\beta + \mathcal{H}_g, \end{aligned} \quad (4)$$

that can be regarded as a generator of the RG transformation along the z direction. Here, $\pi^{\mu\nu}(x,z)$ is the momentum that is canonically conjugate to the metric tensor $g_{\mu\nu}(x,z)$, and $\mathcal{G}_{\mu\nu\rho\gamma}(x,z)$ is the bulk supermetric tensor. The first term in this bulk effective Hamiltonian results from the energy-momentum tensor deformation at UV [60].

The last part of this effective Hamiltonian originates from quantum fluctuations of matter fields in the RG transformation, expressed by the vacuum-energy functional of the renormalized effective Lagrangian at a given RG scale z,

$$\int d^D x\, \mathcal{H}_g = (-1)^F \ln \int D\psi^h_\alpha(x) \exp\left\{ - \int d^D x \sqrt{g(x,z)}\mathcal{L}[\psi^h_\alpha(x), g^{\mu\nu}(x,z)] \right\} \equiv (-1)^F \mathrm{tr}\, \ln \left\langle \frac{\partial^2}{\partial\psi_\alpha(x)\partial\psi_\beta(x')} (\sqrt{g(x,z)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z)]) \right\rangle. \tag{5}$$

Here, $\int D\psi^h_\alpha(x)$ represents to take high-energy quantum fluctuations of matter fields at a given RG scale z, and $F=0$ ($F=1$) when the matter fields are bosonic (fermionic). Performing the gradient expansion for the metric tensor, one finds the Einstein-Hilbert action $\mathcal{H}_g = \frac{\sqrt{g(x,z)}}{2\kappa}(R(x,z)-2\Lambda)$, referred to as induced gravity [69,70], where higher-curvature terms are not taken into account [71]. Here, both the cosmological constant Λ and the effective gravitational one κ can in principle be determined by performing the gradient expansion on a general curved spacetime manifold explicitly, while it can be demanding in practice due to renormalization effects. In this study we regard them as input parameters.

The second part of this effective Hamiltonian is given by

$$\mathcal{H}_\beta = -\pi^{\mu\nu}(x,z)\beta^g_{\mu\nu}[g_{\mu\nu}(x,z)], \tag{6}$$

where $\beta^g_{\mu\nu}[g_{\mu\nu}(x,z)]$ is the RG β-function of the metric tensor and given by

$$\beta^g_{\mu\nu}[g_{\mu\nu}(x,z)] = -\frac{\mathcal{C}_g}{N}\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\langle T^{\rho\gamma}(x,z)\rangle. \tag{7}$$

Here, $\mathcal{C}_g$ is a numerical constant of order one and

$$\langle T^{\rho\gamma}(x,z)\rangle = \frac{2}{\sqrt{g(x,z)}}\frac{\delta}{\delta g_{\rho\gamma}(x,z)} \times \langle \sqrt{g(x,z)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z)]\rangle \tag{8}$$

is the energy-momentum tensor defined in terms of the effective Lagrangian for the matter field at a given RG-transformation slice z.

Finally, the last bulk term of the holographic dual effective field theory (2) is given by

$$\mathcal{H}^\mu = 2\mathcal{D}_\nu \pi^{\mu\nu}(x,z), \tag{9}$$

where $\mathcal{D}_\nu$ is the covariant derivative in the ADM decomposition. This is the generator for diffeomorphism of the D-dimensional spacetime. Performing the path integral for $\mathcal{N}_\mu(x,z)$, we obtain the constraint $\mathcal{D}_\nu\pi^{\mu\nu}(x,z)=0$ [72]. This corresponds to the Ward identity involved with the D-dimensional QFT energy-momentum tensor current at a given z. We will show that the canonical momentum tensor $\pi^{\mu\nu}(x,z_f)$ is given by the energy-momentum tensor of the renormalized IR QFT at the IR boundary $z=z_f$.

Once again, the holographic dual effective field theory (2) can in principle be derived, starting from a given UV field theory, by following the recursive RG procedures in [43–45,50–53]. We expect to end up with the holographic dual effective field theory (2). Instead of pursuing the top-down approach, we will verify, in the next Sec. II B, that the effective holographic theory (2), once discretized, leads to the recursive RG transformations. In Sec. II C, we further discuss self-consistency of the holographic dual effective theory, in particular, the compatibility of the Callan-Symanzik equation $d\ln Z/dz_f = 0$ and the Hamilton (Hamilton-Jacobi) equation of motion derived from (2).

Before proceeding to these discussions, we make a few brief comments here.

First, we note that if we start from the UV boundary theory which is conformal as in $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$, the beta function vanishes and we do not have the $\mathcal{H}_\beta$ term that is linear in $\pi^{\mu\nu}(x,z)$. The holographic dual effective theory (2) is more generic, and incorporate the effect of the nonzero beta function.

Second, compared with the holographic effective theories considered in [43–45,50–53], Eq. (2) is written in a covariant way by incorporating the lapse function and the shift vector. Taking the limit of $\lambda\to 0$ with gauge fixing $\mathcal{N}(x,z)=1$ and $\mathcal{N}_\mu(x,z)=0$, we obtain the RG flow of the metric tensor, $\partial_z g_{\mu\nu}(x,z) = \beta^g_{\mu\nu}[g_{\mu\nu}(x,z)]$ after the path integral over $\pi^{\mu\nu}(x,z)$. Solving the RG equation for the metric with a suitable boundary condition, we obtain the renormalized metric $g_{\mu\nu}(x,z_f)$ in IR, which in turn enters in the IR effective Lagrangian $\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]$ and determines the IR boundary condition. $g_{\mu\nu}(x,z)$ thus needs to be determined self-consistently [50–53]. Both the RG β-function and the IR boundary condition complete the UV-IR mapping manifestly, which will be more clarified below.

B. The holographic dual effective field theory and recursive RG transformations

To verify the above construction and make a contact with the recursive RG transformations, we perform the path integral with respect to the canonical momentum $\pi^{\mu\nu}(x,z)$ and obtain the Lagrangian formulation as follows:

$$
\begin{aligned}
Z = \int D\psi_\alpha(x) Dg_{\mu\nu}(x,z)\delta(g^{\mu\nu}(x,0)-\delta^{\mu\nu})\exp\Big[-\int d^Dx\sqrt{g(x,z_f)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}(x,z_f)\Big]\\
-N\int_0^{z_f}dz\Big\{-\int d^Dx\frac{\sqrt{g(x,z)}}{2\lambda}\mathcal{G}^{\mu\nu\rho\gamma}(x,z)\Big(\partial_z g_{\mu\nu}(x,z)+\frac{2\mathcal{C}_g}{\sqrt{g(x,z)}}\mathcal{G}_{\mu\nu\alpha\beta}(x,z)\frac{\delta\langle\sqrt{g(x,z)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}(x,z)]\rangle}{\delta g_{\alpha\beta}(x,z)}\Big)\\
\times\Big(\partial_z g_{\rho\gamma}(x,z)+\frac{2\mathcal{C}_g}{\sqrt{g(x,z)}}\mathcal{G}_{\rho\gamma\alpha'\beta'}(x,z)\frac{\delta\langle\sqrt{g(x,z)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}(x,z)]\rangle}{\delta g_{\alpha'\beta'}(x,z)}\Big)\\
+(-1)^F\mathrm{tr}\,\ln\Big\langle\frac{\partial^2}{\partial\psi_\alpha(x)\partial\psi_\beta(x')}(\sqrt{g(x,z)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}(x,z)])\Big\rangle\Big\}\Big].
\end{aligned}
\tag{10}
$$

Here, the normal coordinate system of $ds^2(x,z)=dz^2+g_{\mu\nu}(x,z)dx^\mu dx^\nu$ has been considered with gauge fixing $\mathcal{N}(x,z)=1$ and $\mathcal{N}_\mu(x,z)=0$. We emphasize that all the essential information of the bulk effective action is given by the effective renormalized UV field theory $\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}(x,z)]$ at a given RG scale z in a self-consistent way as it should.

Now, we make the extradimensional space z discrete and introduce the discrete coordinate k that represents the RG transformation step,

$$
\begin{aligned}
Z = \int D\psi_\alpha(x) Dg^{(k)}_{\mu\nu}(x)\delta(g^{\mu\nu}_{(0)}(x)-\delta^{\mu\nu})\exp\Big[-\int d^Dx\sqrt{g_{(f)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(f)}(x)]\\
-N(dz)\sum_{k=1}^{f}\Big\{-\int d^Dx\frac{\sqrt{g_{(k-1)}(x)}}{2\lambda}\mathcal{G}^{\mu\nu\rho\gamma}_{(k-1)}(x)\Big(g^{(k)}_{\mu\nu}(x)-g^{(k-1)}_{\mu\nu}(x)\\
+\frac{2\mathcal{C}_g}{\sqrt{g_{(k-1)}(x)}}\mathcal{G}^{(k-1)}_{\mu\nu\alpha\beta}(x)\frac{\delta}{\delta g^{(k-1)}_{\alpha\beta}(x)}\Big\langle\sqrt{g_{(k-1)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(k-1)}(x)]\Big\rangle\Big)\Big(g^{(k)}_{\rho\gamma}(x)-g^{(k-1)}_{\rho\gamma}(x)\\
+\frac{2\mathcal{C}_g}{\sqrt{g_{(k-1)}(x)}}\mathcal{G}^{(k-1)}_{\rho\gamma\alpha'\beta'}(x)\frac{\delta}{\delta g^{(k-1)}_{\alpha'\beta'}(x)}\Big\langle\sqrt{g_{(k-1)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(k-1)}(x)]\Big\rangle\Big)\\
+(-1)^F\mathrm{tr}\ln\Big\langle\frac{\partial^2}{\partial\psi_\alpha(x)\partial\psi_\beta(x')}\Big(\sqrt{g_{(k-1)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(k-1)}(x)]\Big)\Big\rangle\Big\}\Big].
\end{aligned}
\tag{11}
$$

Here, dz is an energy scale for the RG transformation, and $z_f=fdz$ is the energy scale where quantum fluctuations of matter fields are integrated out. The above discrete expression is consistent with the recursive RG-transformation method of our previous studies [50–53].

Focusing on the first iteration of the RG transformation, we consider $k=1$ and obtain

$$
\begin{aligned}
Z = \int D\psi_\alpha(x) Dg^{(0)}_{\mu\nu}(x) Dg^{(1)}_{\mu\nu}(x)\delta(g^{\mu\nu}_{(0)}(x)-\delta^{\mu\nu})\exp\Big[-\int d^Dx\sqrt{g_{(1)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(1)}(x)]\\
-N(dz)\Big\{-\int d^Dx\frac{\sqrt{g_{(0)}(x)}}{2\lambda}\mathcal{G}^{\mu\nu\rho\gamma}_{(0)}(x)\Big(g^{(1)}_{\mu\nu}(x)-g^{(0)}_{\mu\nu}(x)\\
+\frac{2\mathcal{C}_g}{\sqrt{g_{(0)}(x)}}\mathcal{G}^{(0)}_{\mu\nu\alpha\beta}(x)\frac{\delta}{\delta g^{(0)}_{\alpha\beta}(x)}\langle\sqrt{g_{(0)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(0)}(x)]\rangle\Big)\Big(g^{(1)}_{\rho\gamma}(x)-g^{(0)}_{\rho\gamma}(x)\\
+\frac{2\mathcal{C}_g}{\sqrt{g_{(0)}(x)}}\mathcal{G}^{(0)}_{\rho\gamma\alpha'\beta'}(x)\frac{\delta}{\delta g^{(0)}_{\alpha'\beta'}(x)}\langle\sqrt{g_{(0)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(0)}(x)]\rangle\Big)\\
+(-1)^F\mathrm{tr}\ln\Big\langle\frac{\partial^2}{\partial\psi_\alpha(x)\partial\psi_\beta(x')}(\sqrt{g_{(0)}(x)}\mathcal{L}[\psi_\alpha(x),g^{\mu\nu}_{(0)}(x)])\Big\rangle\Big\}\Big].
\end{aligned}
\tag{12}
$$

Taking the $\lambda \to 0$ limit, we find the RG flow of the metric tensor is given by $g^{(1)}_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) - \frac{2\mathcal{C}_g}{\sqrt{g_{(0)}(x)}}\mathcal{G}^{(0)}_{\mu\nu\alpha\beta}(x)\frac{\delta}{\delta g^{(0)}_{\alpha\beta}(x)}\langle\sqrt{g_{(0)}(x)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}_{(0)}(x)]\rangle$. This turns out to be identical with the RG transformation at the one-loop level [50–53]. The next iteration step is to separate slow and fast degrees of freedom for both $g^{(1)}_{\mu\nu}(x)$ and $\psi_\alpha(x)$, and to perform the path integral with respect to all fast degrees of freedom. As a result, $g^{(1)}_{\mu\nu}(x)$ renormalizes into $g^{(2)}_{\mu\nu}(x)$, where the RG β function is now given by $g^{(1)}_{\mu\nu}(x)$. The renormalized metric tensor $g^{(2)}_{\mu\nu}(x)$ appears into the renormalized effective Lagrangian. Repeating this RG-transformation procedure, we obtain Eq. (11) in a discrete form and Eq. (10) in a continuum expression.

We recall that this RG-transformation procedure is analogous to the numerical renormalization-group (NRG) method [73,74]. First, we perform exact diagonalization in the so-called Wilson chain, and truncate the resulting Hilbert space into its low-energy subspace, regarded to be coarse graining. Then, we increase the system size, adding one site into the Wilson chain, and repeat the RG procedure until it converges. The present recursive RG-transformation method implements the NRG philosophy in an analytic way, representing renormalization effects of coupling functions as an RG flow of the metric tensor through the emergent extra-dimensional space with a single-trace deformation of the energy-momentum tensor current. This above demonstrates how the present holographic dual effective field theory takes into account quantum corrections in a nonperturbative way, i.e., all-loop order resummed through the RG flow in the extra-dimensional space [54].

C. Self-consistency of the holographic dual effective field theory in the Hamilton-Jacobi formulation

We now study the self-consistency of the holographic dual effective field theory, including the case of non-conformal theories, $\beta^g_{\mu\nu} \neq 0$. In particular, we examine the Hamilton-Jacobi equation. As before, we gauge-fix and consider the normal coordinate system, $ds^2(x,z) = dz^2 + g_{\mu\nu}(x,z)dx^\mu dx^\nu$, i.e., $\mathcal{N}^\mu = 0$ and $\mathcal{N} = 1$. Furthermore, we assume that vacuum fluctuations of high-energy matter fields, performing the gradient expansion, are approximated by the Einstein-Hilbert action, $\mathcal{H}_g = \frac{1}{2\kappa}\sqrt{g}(R - 2\Lambda)$. (For the explicit evaluation of $\mathcal{H}_g$ for specific models, see [50–53].) The holographic dual effective field theory is then given by

$$Z = \int D\psi_\alpha(x) Dg_{\mu\nu}(x,z) D\pi^{\mu\nu}(x,z)\delta(g^{\mu\nu}(x,0) - g^{\mu\nu}_B(x)) \exp\left[-\int d^Dx\sqrt{g}\mathcal{L}[\psi_\alpha, g^{\mu\nu}]\Big|_{z=z_f}\right.$$
$$\left. - N\int_0^{z_f} dz \int d^Dx\left\{\pi^{\mu\nu}(\partial_z g_{\mu\nu} - \beta^g_{\mu\nu}[g]) + \frac{\lambda}{2}\frac{1}{\sqrt{g}}\pi^{\mu\nu}\mathcal{G}_{\mu\nu\rho\gamma}\pi^{\rho\gamma} + \frac{1}{2\kappa}\sqrt{g}(R - 2\Lambda)\right\}\right]. \tag{13}$$

It is straightforward to find the bulk Hamilton's equation of motion at $z < z_f$ for metric and its conjugate momenta:

$$\partial_z g_{\mu\nu} - \beta^g_{\mu\nu} = -\frac{\lambda}{\sqrt{g}}\mathcal{G}_{\mu\nu\rho\sigma}\pi^{\rho\sigma},$$
$$g_{\mu\rho}g_{\nu\sigma}\partial_z\pi^{\rho\sigma} = -\pi^{\rho\sigma}\frac{\delta\beta_{\rho\sigma}}{\delta g^{\mu\nu}} + \frac{\lambda}{4\sqrt{g}}g_{\mu\nu}\pi^{\kappa\lambda}\mathcal{G}_{\kappa\lambda\rho\sigma}\pi^{\rho\sigma} - \pi_{\mu\sigma}\pi_\nu{}^\sigma$$
$$+\frac{1}{D-1}\pi_{\mu\nu}\pi^\rho{}_\rho + \frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\right). \tag{14}$$

Note that the boundary equations of motion at $z = z_f$ is given by the variation of $\delta g_{\mu\nu}(x, z_f)$

$$-\int d^Dx\delta g_{\mu\nu}(x,z_f)\left[\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{g}\mathcal{L}[\psi_\alpha, g^{\mu\nu}]) + N\pi^{\mu\nu}\right]_{z=z_f} = 0, \tag{15}$$

where the second term arises from the variation of $g_{\mu\nu}$ in the second line of (13). Using the definition of the energy-momentum tensor in (8), we get the boundary condition for $\pi^{\mu\nu}(x, z_f)$

$$\pi^{\mu\nu}(x,z_f) = -\frac{\sqrt{g}}{2N}\langle T^{\mu\nu}(x,z_f)\rangle. \tag{16}$$

If we substitute this relation into the first equation of (14), we have

$$\partial_z g_{\mu\nu}(x,z_f) - \beta^g_{\mu\nu}(x,z_f) = \frac{\lambda}{2N}\mathcal{G}_{\mu\nu\rho\gamma}(x,z_f)\langle T^{\rho\gamma}(x,z_f)\rangle. \tag{17}$$

We point out that this holographic dual effective field theory is reduced to that of the $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$ duality conjecture when $\beta^g_{\mu\nu}[g_{\mu\nu}(x,z)] = 0$ regardless of z. This indicates that $\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]$ remains at its conformally invariant fixed point under the RG transformation, which corresponds to a special case.

This set of the Hamilton's equation of motion can also be reformulated as an Euler-Lagrange equation of motion. Performing the path integral with respect to the canonical momentum $\pi^{\mu\nu}(x,z)$, we obtain an effective Lagrangian as follows:

$$Z = \int D\psi_\alpha(x) Dg^{\mu\nu}(x,z)\delta(g^{\mu\nu}(x,0) - g_B^{\mu\nu}(x)) \exp\Big[- \int d^D x \sqrt{g(x,z_f)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)] - N\int_0^{z_f} dz \int d^D x \sqrt{g(x,z)}\Big\{ -\frac{1}{2\lambda} V_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\gamma}(x,z)V_{\rho\gamma}(x,z) + \frac{1}{2\kappa}(R(x,z) - 2\Lambda)\Big\}\Big], \tag{18}$$

where $\mathcal{G}^{\mu\nu\rho\sigma}$ is the inverse DeWitt metric satisfying $\mathcal{G}_{\alpha\beta\gamma\delta}\mathcal{G}^{\gamma\delta\mu\nu} = \delta_{(\alpha}{}^{\mu}\delta_{\beta)}{}^{\nu}$

$$\mathcal{G}^{\mu\nu\rho\sigma} = \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} + \frac{1}{2} g^{\nu\rho} g^{\mu\sigma} - g^{\mu\nu} g^{\rho\sigma}, \tag{19}$$

and we introduced an auxiliary field $V_{\mu\nu}$,

$$V_{\mu\nu}(x,z) = \partial_z g_{\mu\nu}(x,z) - \beta^g_{\mu\nu}[g_{\mu\nu}(x,z)], \tag{20}$$

to lighten notations.

Accordingly, the Euler-Lagrange equation is given by

$$\frac{g_{\mu\alpha}g_{\nu\beta}}{\sqrt{g}}\partial_z\left(\frac{\sqrt{g}}{\lambda}\mathcal{G}^{\alpha\beta\gamma\delta}V_{\gamma\delta}\right) - \left(\frac{g_{\mu\nu}V_{\alpha\beta}}{4\lambda} - \frac{\eta_{\mu\nu\alpha\beta}}{\lambda}\right)\mathcal{G}^{\alpha\beta\gamma\delta}V_{\gamma\delta} + \frac{g^{\rho\sigma}}{\lambda}(V_{\mu\rho}V_{\nu\sigma} - V_{\mu\nu}V_{\rho\sigma}) + \frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) = 0. \tag{21}$$

Here, we introduce

$$\eta_{\mu\nu\alpha\beta}(x,z) \equiv \frac{\delta}{\delta g^{\mu\nu}(x,z)}\beta^g_{\alpha\beta}[g_{\alpha\beta}(x,z)]. \tag{22}$$

We recall that the RG β-function for the metric tensor is given by the energy-momentum tensor of the renormalized QFT at a given RG scale z. It is natural to call $\eta_{\mu\nu\alpha\beta}(x,z)$ the viscosity tensor, as it is given by the derivative of the energy-momentum tensor with respect to the metric tensor. The role of this viscosity tensor in the dynamics of metric fluctuations will be discussed below.

We now discuss the self-consistency of the present formulation, in particular, the consistency of the RG invariance, and the Hamilton (Hamilton-Jacobi) equations [75]. The generating functional has to be invariant under the RG transformation in the sense that

$$\frac{d}{dz_f}\ln Z = 0. \tag{23}$$

This holds true if the following condition is satisfied

$$(\partial_{z_f} g_{\mu\nu}(x,z_f))\frac{1}{N}\frac{\partial}{\partial g_{\mu\nu}(x,z_f)}\langle\sqrt{g(x,z_f)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]\rangle + \frac{1}{N}\partial_{z_f}\langle\sqrt{g(x,z_f)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]\rangle + \pi^{\mu\nu}(x,z_f)(\partial_{z_f} g_{\mu\nu}(x,z_f) - \beta^g_{\mu\nu}[g_{\mu\nu}(x,z_f)]) + \frac{\lambda}{2}\frac{1}{\sqrt{g(x,z_f)}}\pi^{\mu\nu}(x,z_f)\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\pi^{\rho\gamma}(x,z_f) + \frac{1}{2\kappa}\sqrt{g(x,z_f)}(R(x,z_f) - 2\Lambda) = 0. \tag{24}$$

One may regard this equation as the Callan-Symanzik equation [76] for the free-energy functional in the large N limit. We recall the IR boundary canonical momentum tensor is given by Eq. (15). In addition, it is natural to assume $\partial_{z_f}\langle\sqrt{g(x,z_f)}\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]\rangle = 0$, i.e., the IR boundary Lagrangian does not depend explicitly on the boundary coordinate z_f, since all the cutoff dependence is through the running of the coupling constants as a function of z. As a result, the RG invariance of the free energy (24) is reduced to the Hamilton-Jacobi equation

$$0 = \frac{\lambda}{2}\frac{1}{\sqrt{g(x,z_f)}}\pi^{\mu\nu}(x,z_f)\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\pi^{\rho\gamma}(x,z_f) - \pi^{\mu\nu}(x,z_f)\beta^g_{\mu\nu}[g_{\mu\nu}(x,z_f)] + \frac{1}{2\kappa}\sqrt{g(x,z_f)}(R(x,z_f) - 2\Lambda), \tag{25}$$

where $\pi^{\mu\nu} = \delta S/\delta g_{\mu\nu}$. We emphasize that the solution of this Hamilton-Jacobi equation is given by the IR boundary condition [Eq. (15)], where $\mathcal{L}[\psi_\alpha(x), g^{\mu\nu}(x,z_f)]$ is the IR boundary effective Lagrangian determined self-consistently. Again, this Hamilton-Jacobi equation becomes that of the AdS_{D+1}/CFT_D duality conjecture when $\beta^g_{\mu\nu}[g_{\mu\nu}(x,z)] = 0$.

As a further consistency check, we point out that the holographic dual effective field theory will follow the constraint. Inserting the Hamilton's equation for the metric tensor into the constraint $\mathcal{D}_\nu\pi^{\mu\nu}(x,z) = 0$, it is natural to expect that the covariant derivative for the metric tensor would vanish [72], regarded to be a part of full equations of motion [77], and that for the RG β-function also becomes zero, nothing but the energy-momentum tensor-current conservation law. It seems to be a natural generalization to introduce the RG β-function of the metric tensor into the

bulk effective action for gravity, expected to work away from quantum criticality.

D. Discussion

1. Entanglement entropy perspectives

The present RG-reformulated dual gravity action may be reinterpreted in perspectives of entanglement entropy [78–82]. The entanglement entropy is given by

$$S_{EE}^{\rm UV} = S_{EE}^{GR}(z_f) + S_{EE}^{M}(z_f) \tag{26}$$

in our holographic dual effective field theory. Here, $S_{EE}^{\rm UV}$ is the entanglement entropy of the UV effective QFT defined at the UV boundary $z = 0$. $S_{EE}^{GR}(z_f)$ is the entanglement entropy of the emergent dual gravity bulk action with the IR boundary $z = z_f$. $S_{EE}^{M}(z_f)$ is that of the IR boundary action at $z = z_f$. This seemingly natural formula can be derived from the holographic dual effective field theory [50] using the replica trick [83,84].

Since the entanglement entropy of the UV effective QFT does not depend on the IR boundary coordinate z_f, we obtain the following Callan-Symanzik equation for the entanglement entropy as

$$0 = \partial_{z_f} S_{EE}^{GR}(z_f) + \partial_{z_f} S_{EE}^{M}(z_f). \tag{27}$$

Resorting to the replica trick for the gravitational effective action [84], one may argue that $S_{EE}^{GR}(z_f)$ is given by an area of the Ryu-Takayanagi minimal surface [85,86]. Since we did not address the role of the RG β function in the Ryu-Takayanagi minimal surface yet, we used the term of "may argue". Essentially the same replica trick gives rise to the area law of $S_{EE}^{M}(z_f)$ [83] at the IR boundary. Here, we represent both entanglement entropies as follows:

$$S_{EE}^{GR}(z_f) = \frac{\mathcal{A}_{RT}(z_f)}{4G_{D+1}}, \qquad S_{EE}^{M}(z_f) = \frac{\mathcal{A}_{QFT}(z_f)}{4G_D}. \tag{28}$$

$\mathcal{A}_{RT}(z_f)$ is a $(D-1)$-dimensional Ryu-Takayanagi minimal-surface area at $z = z_f$, and G_{D+1} is $(D+1)$-dimensional Newton constant in $S_{EE}^{GR}(z_f)$. $\mathcal{A}_{QFT}(z_f)$ is a $(D-2)$-dimensional surface area of the QFT with renormalization at $z = z_f$, and G_D is D-dimensional Newton constant in $S_{EE}^{M}(z_f)$.

As a result, we obtain

$$0 = \frac{\partial_{z_f}\mathcal{A}_{RT}(z_f)}{4G_{D+1}} + \frac{\partial_{z_f}\mathcal{A}_{QFT}(z_f)}{4G_D}. \tag{29}$$

This area formulation interprets the appearance of the RG-reformulated dual gravity action in a geometrical way. The decrease of the $(D-2)$-dimensional surface area of the QFT, representing the decrease of the entanglement entropy in the QFT, gives rise to the increase of the $(D-1)$-dimensional Ryu-Takayanagi minimal-surface area, describing the increase of the entanglement entropy in the bulk gravity theory, where the RG transformation is performed at $z = z_f$. It would be interesting to show this relation explicitly.

2. Role of the viscosity tensor in the dynamics of metric fluctuations

Finally, we discuss the role of the RG β-function in the bulk dynamics of metric fluctuations. It is not easy to solve Eq. (21) and find the RG flow of the metric tensor because the RG β-function gives rise to higher-curvature corrections in the D-dimensional Einstein-Hilbert action. Performing the gradient expansion in Eq. (7) with Eq. (8) [83], one can express the average of the energy-momentum tensor in terms of curvature tensors, which results in higher-curvature terms in the case of $\lambda \neq 0$. Here, we consider a near fixed-point solution of the metric tensor, which allows us to investigate the bulk dynamics of metric fluctuations in a linearized fashion around the fixed-point background geometry.

We recall the IR boundary condition (17). Taking the $z_f \to \infty$ limit, quantum fluctuations of matter fields are integrated out completely. As a result, the average of the energy-momentum tensor cannot but vanish. Since the RG β-function also vanishes, the resulting IR boundary condition is given by

$$\lim_{z_f\to\infty} \partial_z g_{\mu\nu}(x,z)|_{z=z_f} = 0. \tag{30}$$

In this limit, the RG flow equation of the metric tensor is reduced to

$$\begin{aligned}&\partial_z V_{\mu\nu}(x,z) - g_{\mu\nu}(x,z) g^{\rho\sigma}(x,z)\partial_z V_{\rho\sigma}(x,z)\\ &= \frac{\lambda}{2\kappa\sqrt{g(x,z)}}\left(R_{\mu\nu}(x,z) - \frac{1}{2}R(x,z)g_{\mu\nu}(x,z) + \Lambda g_{\mu\nu}(x,z)\right).\end{aligned} \tag{31}$$

We recall $V_{\mu\nu}(x,z) = \partial_z g_{\mu\nu}(x,z) - \beta^g_{\mu\nu}[g_{\mu\nu}(x,z)]$. This equation is identical to that of the conventional holography if $\lim_{z_f\to\infty} \partial_z \beta^g_{\mu\nu}[g_{\mu\nu}(x,z)]|_{z=z_f} = 0$ is assumed near an IR fixed point. Here, gauge fixing is assumed as discussed before. The background solution is an (thermal) AdS_{D+1} geometry at zero temperature (below the Hawking-Page transition temperature) and an AdS_{D+1} black hole above the Hawking-Page transition temperature [87].

Considering small fluctuations around this background geometry $\tilde{g}_{\mu\nu}$ as

$$g_{\mu\nu}(x,z) = \tilde{g}_{\mu\nu}(x,z) + h_{\mu\nu}(x,z), \tag{32}$$

the linearized "Einstein" equation for the metric tensor is given by

$$\frac{\tilde{g}_{\mu a}\tilde{g}_{\nu\beta}}{\sqrt{\tilde{g}}}\partial_z\left[\frac{\sqrt{\tilde{g}}}{\lambda}\left(\frac{1}{2}\tilde{g}^{\rho\sigma}h_{\rho\sigma}\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}\tilde{V}_{\gamma\delta}+(\delta_h\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta})\tilde{V}_{\gamma\delta}+\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}(\delta_h\tilde{V}_{\gamma\delta})\right)\right]+\frac{2h_{(\mu|\alpha|}\tilde{g}_{\nu)\beta}}{\sqrt{\tilde{g}}}\partial_z\left[\frac{\sqrt{\tilde{g}}}{\lambda}\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}\tilde{V}_{\gamma\delta}\right]$$
$$-\left(\frac{\tilde{g}_{\mu\nu}\tilde{V}_{\alpha\beta}}{2\lambda}-\frac{\tilde{\eta}_{\mu\nu\alpha\beta}}{\lambda}\right)\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}\delta_h\tilde{V}_{\gamma\delta}-\left(\frac{\tilde{g}_{\mu\nu}}{4\lambda}\tilde{V}_{\alpha\beta}-\frac{\tilde{\eta}_{\mu\nu\alpha\beta}}{\lambda}\right)(\delta_h\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta})\tilde{V}_{\gamma\delta}-\frac{h_{\mu\nu}}{4\lambda}\tilde{V}_{\alpha\beta}\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}\tilde{V}_{\gamma\delta}$$
$$-\frac{h^{\rho\sigma}}{\lambda}(\tilde{V}_{\mu\rho}\tilde{V}_{\nu\sigma}-\tilde{V}_{\mu\nu}\tilde{V}_{\rho\sigma})+\frac{\tilde{g}^{\rho\sigma}}{\lambda}(\tilde{V}_{\mu\rho}(\delta_h\tilde{V}_{\nu\rho})+(\delta_h\tilde{V}_{\mu\rho})\tilde{V}_{\nu\rho}-\tilde{V}_{\mu\nu}(\delta_h\tilde{V}_{\rho\sigma})-(\delta_h\tilde{V}_{\mu\nu})\tilde{V}_{\rho\sigma})$$
$$+\frac{1}{4\kappa}[2h_{(\mu}{}^{\alpha}\tilde{R}_{\nu)\alpha}-2h^{\alpha\beta}\tilde{R}_{\mu\alpha\nu\beta}+\tilde{\nabla}^{\rho}(\tilde{\nabla}_{\mu}h_{\rho\nu}+\tilde{\nabla}_{\nu}h_{\rho\mu}-\tilde{\nabla}_{\rho}h_{\mu\nu})-\tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu}h^{\rho}{}_{\rho}$$
$$+\tilde{g}_{\mu\nu}(h^{\rho\sigma}\tilde{R}_{\rho\sigma}-\tilde{\nabla}_{\rho}\tilde{\nabla}_{\sigma}h^{\rho\sigma}+\tilde{\nabla}^{\rho}\tilde{\nabla}_{\rho}h^{\sigma}{}_{\sigma})-h_{\mu\nu}(\tilde{R}-2\Lambda)]=0, \tag{33}$$

where $\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}$ and $\tilde{V}_{\mu\nu}$ are the background quantities and

$$\delta_h\tilde{\mathcal{G}}^{\alpha\beta\gamma\delta}=-(h^{\alpha(\gamma}\tilde{g}^{\delta)\beta}+h^{\beta(\gamma}\tilde{g}^{\delta)\alpha}-h^{\alpha\beta}\tilde{g}^{\gamma\delta}-\tilde{g}^{\alpha\beta}h^{\gamma\delta}),$$
$$\delta_h\tilde{V}_{\alpha\beta}=\partial_z h_{\alpha\beta}+h^{\rho\sigma}\tilde{\eta}_{\rho\sigma\alpha\beta}. \tag{34}$$

Here, $\tilde{\eta}_{\mu\nu\alpha\beta}(x,z)$ is the viscosity tensor with a background black hole geometry, $\tilde{\eta}_{\mu\nu\alpha\beta}(x,z)\equiv\frac{\delta}{\delta g^{\mu\nu}(x,z)}\beta^{g}_{\alpha\beta}[g_{\alpha\beta}(x,z)]|_{g^{\mu\nu}(x,z)=\tilde{g}^{\mu\nu}(x,z)}$ [see (22)]. This may not vanish near the fixed point $z_f\to\infty$ while the RG β function itself becomes zero. It is interesting to observe that this viscosity tensor can result in instability of metric fluctuations near the fixed-point background geometry. We speculate that this potential instability originates from higher-curvature corrections to the Einstein-Hilbert action [71]. More generally, we suspect that the RG β-function of the metric tensor may encode the so-called Ricci flow [88–93]. The Ricci flow equation is to describe the deformation of a Riemannian metric $g_{\mu\nu}(x,z)$ with an extra-dimensional space coordinate z, which plays the same role as time. This evolution equation may be regarded as an analog of the diffusion equation for geometries, given by a parabolic partial differential equation. The deformation is governed by the Ricci curvature, and leads to homogeneity of geometry. In principle, one may consider that this Ricci flow equation arises from the gradient expansion of the Green's function with respect to the mass parameter [50]. Actually, this instability of the background geometry may be interpreted as a runaway RG flow toward a fixed point different from the present one. It would be interesting to study how the universal lower bound of the ratio between the shear viscosity and the entropy [21] is modified by this viscosity tensor [94].

III. CONCLUSION

We proposed a prescription for an emergent dual holographic description of a quantum field theory, expected to work even away from quantum criticality. Although we invoke the bulk locality assumption and do not include higher-spin fields, the holographic dual effective field theory takes into account quantum corrections in a nonperturbative way through a nonperturbative RG flow in the emergent extra-dimensional space. Self-consistency of this nonperturbative framework was claimed based on the Hamilton-Jacobi equation, the solution of which is given by the IR boundary effective action.

We would like to mention that the emergent holographic duality between the quantum field theory and emergent gravity partition functions holds beyond the large N limit. However, the self-consistency discussed in Sec. II C and the equation of motion shown in Sec. II D are on the level of an effective on-shell action, based on the duality relation to hold in the large N limit.

Before closing, we point out that it is straightforward to generalize the present dual holographic description to the case with additional effective interactions. For example, one may consider either spontaneous chiral symmetry breaking or effective interactions between U(1) conserved currents. Such interactions are responsible for appearance of dual scalar fields and U(1) gauge fields, respectively, in the corresponding holographic dual effective field theory.

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