

Determinant Maximization via Matroid Intersection Algorithms

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July 12, 2022

Abstract

Determinant maximization problem gives a general framework that models problems arising in as diverse fields as statistics [Puk06], convex geometry [Kha96], fair allocations [AGSS16], combinatorics [AGV18], spectral graph theory [NST19a], network design, and random processes [KT12]. In an instance of a determinant maximization problem, we are given a collection of vectors $U = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$, and a goal is to pick a subset $S \subseteq U$ of given vectors to maximize the determinant of the matrix $\sum_{i \in S} v_i v_i^\top$. Often, the set S of picked vectors must satisfy additional combinatorial constraints such as cardinality constraint ($|S| \leq k$) or matroid constraint (S is a basis of a matroid defined on the vectors).

In this paper, we give a polynomial-time deterministic algorithm that returns a $r^{O(r)}$ -approximation for any matroid of rank $r \leq d$. This improves previous results that give $e^{O(r^2)}$ -approximation algorithms relying on $e^{O(r)}$ -approximate *estimation* algorithms [NS16, AG17, AGV18, MNST20] for any $r \leq d$. All previous results use convex relaxations and their relationship to stable polynomials and strongly log-concave polynomials. In contrast, our algorithm builds on combinatorial algorithms for matroid intersection, which iteratively improve any solution by finding an *alternating negative cycle* in the *exchange graph* defined by the matroids. While the $\det(\cdot)$ function is not linear, we show that taking appropriate linear approximations at each iteration suffice to give the improved approximation algorithm.

1 Introduction

Determinant maximization problem gives a general framework that models problems arising in as diverse fields as statistics [Puk06], convex geometry [Kha96], fair allocations [AGSS16], combinatorics [AGV18], spectral graph theory [NST19a], network design and random processes [KT12]. In an instance of a determinant maximization problem, we are given a collection of vectors $U = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$, and a goal is to pick a subset $S \subseteq U$ of given vectors to maximize the determinant of the matrix $\sum_{i \in S} v_i v_i^\top$. Additionally, the set S of picked vectors must satisfy additional

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combinatorial constraints such as cardinality constraint ($|S| \leq k$) or matroid constraint (S is a basis of a matroid defined on the vectors).

Apart from its modeling strength, from a technical perspective, determinant maximization has brought interesting connections between areas such as combinatorial optimization, convex analysis, geometry of polynomials, graph sparsification and complexity of permanent and other counting problems [ALSW17, AGSS16, AG17, Kha96].

Applications. Observe that when the number of vectors picked is exactly d , the objective is precisely the square of the volume of the parallelepiped spanned by the selected vectors. The problem of finding the largest volume parallelepiped in a collection of given vectors has been studied [Nik15a, Kha96, SEFM15] for over three decades. Another interesting application is the determinantal point processes [KT12], where a probability distribution over subsets of vectors is defined. The probability of selecting a subset is defined to be proportional to the squared volume of the parallelepiped defined by them. These distributions display nice properties of negative correlation. Finding sets with the largest probability mass is exactly the determinant maximization problem. We refer the reader to [NST19a] for applications in experimental design and to [AGSS16] for application to fair allocations.

The computational complexity of the determinant maximization depends crucially on the combinatorial set family which constrains the set of feasible collection of vectors. The simplest constraint being the cardinality constraint, wherein the number of vectors is fixed, has been the most widely studied variant. For this, a variety of methods including convex programming based methods [ALSW17, SEFM15, Nik15b, SX18], combinatorial methods - such as local search and greedy [Kha96, MSTX19, LZ21] - as well as close relationship to graph sparsification [ALSW17] have been exploited to obtain efficient approximation algorithms with very good guarantees. Overall, these results give a very clear understanding of the computational complexity of the problem.

The more general case when the combinatorial constraints are defined by a matroid constraint has recently received extensive focus [NS16, AGSS16, AG17, AGV18, MNST20]. This is especially interesting since some of the applications are naturally modeled as matroid constraints, in particular, as partition constraints. Unfortunately, there is a big gap between estimation algorithms and approximation algorithms in this case! Indeed, one can approximately *estimate* the value of an optimal solution with a good guarantee, however, *finding* such a solution is much more challenging, leading to an exponential loss in the approximation factor. For example, even for the special case of the partition matroid, there is an e^d -approximate estimation algorithm but the best known approximation algorithms return a solution with an approximation factor of $e^{O(d^2)}$, an exponential blow-up¹. A fundamental reason for this gap is the reliance on the relationship between convex programming relaxations for the problem and the theory of stable polynomials and its generalization to strongly log-concave polynomials. Unfortunately, these methods are inherently non-algorithmic and do not give a simple way to obtain efficient algorithms with the same guarantees that match the estimation bounds.

¹Since we are computing the determinant of $d \times d$ matrices, the exponent of d in the approximation factor is appropriate. Indeed, many works even consider the d^{th} -root of the determinant where the approximation factors are also the d^{th} -root of the above bounds.

1.1 Our Results and Contributions

In this work, we introduce new combinatorial methods for determinant maximization under a matroid constraint and give an $O(d^{O(d)})$ -deterministic approximation algorithm. While previous works have used a convex programming approach and the theory of stable polynomials, our approach builds on the classical matroid intersection algorithm. Our first result focuses on the case when the rank of the matroid is exactly d , i.e., the output solution will contain precisely d vectors.

Theorem 1 *There is a polynomial time algorithm which, given a collection of vectors $v_1, \dots, v_n \in \mathbb{R}^d$ and a matroid $\mathcal{M} = ([n], \mathcal{I})$ of rank d , returns a set $S \in \mathcal{I}$ such that*

$$\det \left(\sum_{i \in S} v_i v_i^\top \right) = \Omega \left(\frac{1}{d^{O(d)}} \right) \max_{S^* \in \mathcal{I}} \det \left(\sum_{i \in S^*} v_i v_i^\top \right).$$

Our results improve the $e^{O(d^2)}$ -approximation algorithm which relies on the $e^{O(d)}$ -estimation algorithm [AGV18, AG17, MNST20]. Our algorithm builds on the matroid intersection algorithm and is an iterative algorithm that starts at any feasible solution and improves the objective in each step. To maintain feasibility in the matroid constraint, each step of the algorithm is an exchange of multiple elements as found by an alternating cycle of an appropriately defined exchange graph.

Result for $r \leq d$. We also generalize the result when the rank r of the matroid is at most d . Observe that the solution matrix $\sum_{i \in S} v_i v_i^\top$ is a $d \times d$ matrix of rank at most r and, therefore, the appropriate objective to consider is the product of its largest r eigenvalues, or equivalently, the elementary symmetric function of order r of its eigenvalues. Let $\text{sym}_r(M)$ be the r^{th} elementary symmetric function of the eigenvalues of the $d \times d$ matrix M . Thus, our objective is to maximize $\text{sym}_r \left(\sum_{i \in S} v_i v_i^\top \right)$.

Theorem 2 *There is a polynomial-time algorithm which, given a collection of vectors $v_1, \dots, v_n \in \mathbb{R}^d$ and a matroid $\mathcal{M} = ([n], \mathcal{I})$ of rank $r \leq d$, returns a set $S \in \mathcal{I}$ such that*

$$\text{sym}_r \left(\sum_{i \in S} v_i v_i^\top \right) = \Omega \left(\frac{1}{r^{O(r)}} \right) \max_{S^* \in \mathcal{I}} \text{sym}_r \left(\sum_{i \in S^*} v_i v_i^\top \right).$$

This again improves the best bound of $e^{O(r^2)}$ -approximation algorithm based on $e^{O(r)}$ -approximate estimation algorithms. The proof of Theorem 2 is presented in Appendix B.

Technical Overview. For intuition, let $\text{vol}(S)$ denote the volume of the parallelepiped spanned by the vectors in S . Then $\text{vol}(S)^2 = \det \left(\sum_{i \in S} v_i v_i^\top \right)$, for any $S \subseteq U$ with $|S| = d$, so we can think of $\text{vol}(S)$ as an equivalent objective function. First, observe that the feasibility problem of checking whether there is a set $S \in \mathcal{I}$ such that $\text{vol}(S) > 0$ can be reduced to matroid intersection. Indeed, the feasibility problem is equivalent to checking if there is a common basis of the matroid \mathcal{M} and the linear matroid defined by the vectors $\{v_1, \dots, v_n\}$. Since we aim to maximize $\text{vol}(S)$ over all independent sets S , a natural approach is to use the weighted matroid intersection algorithm. Unfortunately, our weights are not linear, i.e., $\text{vol}(S)$ does not equal $\sum_{i \in S} w_i$ or log-linear $\prod_{i \in S} w_i$ for some weights w on the vectors. Nonetheless, the matroid intersection algorithm forms the backbone of our approach.

Overview of Matroid Intersection. Before we describe our algorithm, let us review a classical algorithm to find a maximum weight common basis of two matroids. Given $U = \{1, \dots, n\}$, a weight function $w : U \rightarrow \mathbb{R}$ and two matroids $\mathcal{M}_1 = (U, \mathcal{I}_1)$ and $\mathcal{M}_2 = (U, \mathcal{I}_2)$, the goal is to find a common basis S of maximum weight $w(S) := \sum_{e \in S} w_e$. We assume that there exists a common basis of the two matroids. Consider the following simple algorithm that also introduces some of the basic ingredients necessary for our algorithm. The algorithm will take as an input a common basis S and either certify that S is a maximum weight common basis or return a new common basis \hat{S} of higher weight. To explain the algorithm, we recall the important concept of the *exchange graph*. Given the set S , we construct a directed bipartite graph $G(S)$ with bipartitions given by $U \setminus S$ and S . For any $u \in U \setminus S$ and $v \in S$, $G(S)$ contains an arc from u to v if $S - v + u$ is a basis in \mathcal{M}_2 and an arc from v to u if $S - v + u$ is a basis in \mathcal{M}_1 . For convenience, we use $S - v + u$ to refer to the set $(S \cup \{u\}) \setminus \{v\}$. Moreover, give each vertex $u \in U \setminus S$ a weight $-w_u$ and each vertex $v \in S$ a weight of w_v . A nice fact from matroid theory is that S is a maximum weight basis if and only if there is no negative weight cycle in this directed graph (Chapter 41, Theorem 41.5 [Sch03]). Moreover, if C is a directed negative weight cycle with minimum hops², then $S \Delta C$ forms a common basis of the two matroids whose weight is strictly larger than the weight of S . Thus, the algorithm finds a maximum weight basis by iteratively finding a negative weight cycle in such an exchange graph.

With the above algorithm as a guiding tool, we describe our algorithm. The two matroids are precisely the constraint matroid \mathcal{M} and the linear matroid defined over the vectors. A first challenge is that the objective function $\text{vol}(S)^2 = \det(\sum_{i \in S} v_i v_i^\top)$ is not linear. Thus it is not possible to define the vertex weights as was done above. But a natural function to work with instead is the function $\log \text{vol}(S)$, which is known to be submodular. While we do not use submodularity explicitly, our algorithm takes linear approximations of this function at each iteration while searching for improvements as in the matroid intersection algorithm. We use the geometric relationship between vol and \det closely. The first new ingredient in our algorithm is to introduce *arc* weights rather than vertex weights in the exchange graph $G(S)$. Indeed for the forward arcs (u, v) for $u \notin S$ and $v \in S$ that correspond to the linear matroid, we introduce a weight of $-\log \frac{\text{vol}(S - v + u)}{\text{vol}(S)}$. We also introduce a weight of 0 for the backward arcs, which correspond to the arcs for the constraint matroid \mathcal{M} . The crucial observation is the following interpretation of the weight $\log \frac{\text{vol}(S - v + u)}{\text{vol}(S)}$: write the vector $u \notin S$ in the basis S , i.e. $u = \sum_{v \in S} a_{uv} v$, for some $a_{uv} \in \mathbb{R}$ for each $v \in S$. Then $\frac{\text{vol}(S - v + u)}{\text{vol}(S)} = |a_{uv}|$ (See Lemma 3). Such relationships between the ratio of volumes and coefficients in expressing the vectors in basis given by S play an important role.

Our first lemma shows that if the volume of the current solution is *much* smaller than the optimal solution, then there must be a cycle such that the sum of weights of the arcs on the cycle is significantly negative.

Lemma 1 (Determinant to Cycle) *Let S be a basis of \mathcal{M} and OPT be the basis of \mathcal{M} maximizing $\text{vol}(OPT)$. If $\text{vol}(OPT) \geq e^{5d \log d} \cdot \text{vol}(S)$, then there exists a directed cycle C of 2ℓ hops for some $\ell > 0$ in the exchange graph $G(S)$ such that*

$$\prod_{(u,v) \in C, u \notin S, v \in S} |a_{uv}| \geq 2(\ell!)^3 =: f(\ell).$$

We call such a cycle an f -violating cycle. Observe that such a cycle can be found as a negative

²Hops here refers to the number of arcs of the cycle.

weight 2ℓ -hop cycle when weights are updated to $w_\ell(u, v) = \frac{1}{\ell} \log f(\ell) - \log |a_{uv}|$ for a forward arc (u, v) where $u \notin S$ and $v \in S$. The lemma relies on the following observation. Abusing notation slightly, let T and S be matrices whose columns are the vectors in OPT and S , respectively. Writing each vector in OPT in the basis given by S we obtain $T = SA^\top$ for some matrix A . The condition in the lemma implies that $\det(A) \geq e^{5d \log d}$. Also observe that the weight of any (u, v) where $u \in OPT$ and $v \in S$ is exactly $-\log |a_{uv}|$ where a_{uv} is the (u, v) th-entry in A . Combining these facts, we can show there exists a cycle satisfying the conditions of the lemma.

The next step in the algorithm is to find an f -violating cycle C and then update the solution to $T = S\Delta C$. Again, we relate the change in objective $\text{vol}(T)$ to the coefficients. While the entries $|a_{uv}|$ of the cycle are large, the objective of the new solution T depends not only on the weight of the edges of the cycle but the weight of all arcs between all vertices in $C \setminus S$ and $C \cap S$. Indeed, consider a square matrix B with rows and columns indexed by $C \setminus S$ and $S \cap C$ respectively with entry (u, v) as a_{uv} . Recall a_{uv} is the coefficient of vector v when u is expressed in basis S . Then $\text{vol}(T) = |\det(B)| \cdot \text{vol}(S)$ (Lemma 6). Thus it remains to lower bound the determinant of B . The entries on the diagonal of the matrix B exactly correspond to entries that define the weights of the forward arcs on the cycle C . Thus Lemma 1 implies that the product of the diagonal entries of B is large. In the next lemma, we show that if the cycle C is the *minimum hop* f -violating cycle, then we can in fact lower bound the determinant of the matrix.

Lemma 2 (Cycle to Determinant) *If C is a minimum hop f -violating cycle in the exchange graph $G(S)$, then $\text{vol}(S\Delta C) \geq 2 \cdot \text{vol}(S)$. Moreover, $S\Delta C$ is also a basis of \mathcal{M} .*

This lemma crucially uses the fact that C is a minimum hop f -violating cycle as in the case for matroid intersection algorithms. Indeed, off-diagonal entries of the matrix B correspond to arcs that form chords of the cycle C . The minimality of C allows us to show upper bounds on all the off-diagonal entries of the matrix B . A technical calculation then allows us to lower bound the determinant.

1.2 Related Work

Determinant Maximization under Cardinality Constraints. Determinant maximization problems under a cardinality constraint have been studied widely [Kha96, SEFM15, Nik15b, SX18, ALSW17, MSTX19]. Currently, the best approximation algorithm for the case $r \leq d$ is an e^r -approximation due to Nikolov [Nik15b] and for $r \geq d$, there is an e^d -approximation [SX18]. It turns out that the problem gets significantly easier when $r \gg d$, and there is a $(1 + \epsilon)^d$ -approximation when $r \geq d + \frac{d}{\epsilon}$ [ALSW17, MSTX19, LZ21]. These results use local search methods and are closely related to the algorithm discussed in this paper, as the cycle improving algorithm will always find a 2-cycle when the matroid is defined by the cardinality constraint.

Determinant Maximization under Matroid Constraints. As mentioned earlier, determinant maximization under a matroid constraint is considerably challenging and the bounds also depend on the rank r of the constraint matroid. There are $e^{O(r)}$ -estimation algorithms when $r \leq d$ [NS16, AG17, ALGV19] and a $\min\{e^{O(r)}, O(d^{O(d)})\}$ -estimation algorithm when $r \geq d$ [MNST20]. The output of these algorithms is a random feasible set whose objective is at least $\min\{e^{O(r)}, O(d^{O(d)})\}$ of the objective of a convex programming relaxation, in expectation. Since the approximation guarantees are exponential, it can happen that the output set has objective zero almost always. To convert them into deterministic algorithms (or randomized algorithms that work with high

probability), additional loss in approximation factor is incurred. These results imply an $e^{O(d^2)}$ -approximation algorithm when $r \leq d$, and a $O(d^{O(d^3)})$ -approximation algorithm [MNST20] for $r \geq d$. Approximation algorithms are also known where the approximation factor is exponential in the size of the ground set for special classes of matroids [ESV17].

Nash Social Welfare and its generalizations. A special case of the determinant maximization problem is the Nash Social Welfare problem [CKM⁺16]. In the Nash Social Welfare problem, we are given m items and d players and there is a valuation function $v_i : 2^{[m]} \rightarrow \mathbb{R}_+$ for each player $i \in [d]$ that specifies value obtained by a player when given a bundle of items. The goal is to find an assignment of items to players to maximize the *geometric mean* of the valuations of each of the players. When the valuation functions are additive, the problem becomes a special case of the determinant maximization and this connection can be utilized to give an e -approximation algorithm [AGSS16]. Other methods including rounding algorithms [CG15, CDG⁺17] as well as primal-dual methods [BKV18a] have been utilized to obtain improved bounds. The problem has been studied when the valuation function is more general [GHM18, BKV18b, AMGV18, GHV21] and a constant-factor approximation is known when the valuation function is submodular [LV22].

Other Spectral Objectives. While we focus on the determinant objective, the problem is also interesting when considering other spectral objectives including minimizing the trace or the maximum eigenvalue of the $(\sum_{i \in S} v_i v_i^\top)^{-1}$. These problems have been studied for the cardinality constraint [ALSW17, NST19b]. For the case of partition matroid, the problem of maximizing the minimum eigenvalue is closely related to the Kadison-Singer problem [MSS15].

2 Algorithm for Partition Matroid

We first show the algorithm and the analysis for a partition matroid with rank d . This allows us to show the basic ideas without going into the details of matroid theory. The generalizations to general matroid are quite standard. We detail them in Section 3.

Consider a partition matroid \mathcal{M} with d partitions $\mathcal{P}_1, \dots, \mathcal{P}_d$, where each \mathcal{P}_i contains n_i vectors $v_{i1}, \dots, v_{in_i} \in \mathbb{R}^d$. Our goal is to find a set S which provides a good approximation to the objective

$$\max \left\{ \det \left(\sum_{v \in S} v v^\top \right) : |S| = d, |S \cap \mathcal{P}_i| = 1 \forall i \right\}.$$

Let OPT denote the optimal solution set. The following theorem is a specialization of Theorem 1 to the case of partition matroid.

Theorem 3 *Given a partition matroid \mathcal{M} with d parts, let OPT be the optimal solution to the determinant maximization problem on \mathcal{M} . Then, there is a polynomial-time deterministic algorithm that outputs a feasible set $S \in \mathcal{M}$ such that*

$$\det \left(\sum_{i \in S} v_i v_i^\top \right) \geq e^{-10d \log(d)} \cdot \det \left(\sum_{i \in OPT} v_i v_i^\top \right).$$

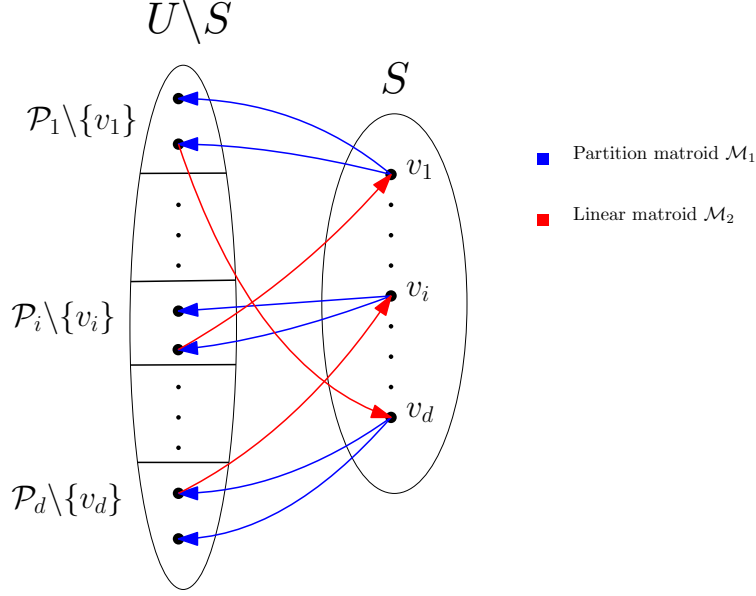


Figure 1: The exchange graph $G(S)$

2.1 Algorithm

We begin by formally defining the exchange graph, the different weight functions, and then the algorithm which helps establish Theorem 1 for the case of partition matroids.

Definition 1 (Exchange Graph) Formally, for a subset of vectors $S = \{v_1, v_2, \dots, v_d\}$ with $v_i \in \mathcal{P}_i$ for all i , we define the exchange graph of S , denoted by $G(S)$ as a bipartite graph, where the right-hand side consists of vectors in S , i.e., $R = \{v_1, v_2, \dots, v_d\}$ and the left-hand side consists of all the vectors $L = \bigcup_{i=1}^d \mathcal{P}_i \setminus \{v_i\}$ (See Figure 1). Each $v_i \in R$ has an edge to every $u \in \mathcal{P}_i \setminus \{v_i\}$, i.e., all the vectors in the same part as v_i . The vertices on the left-hand side have forward edges to every vertex in S .

We define a family of weight functions on the exchange graph. The basic weight function will be denoted by $w_0 : A(G(S)) \rightarrow \mathbb{R}$ and, in addition, we define weight functions w_i for each $1 \leq i \leq d$. To define these weights, we use the function $f : [d] \rightarrow \mathbb{Z}_+$ with $f(i) = 2(i!)^3$ for each $i > 0$.

Definition 2 (Weight functions on the Exchange graph) We first define weight function w_0 . All the backward arcs, from any $v_i \in S$ to every $u_j \in \mathcal{P}_i \setminus \{v_i\}$, have weight 0. For $u_j \in L$, let $u_j = \sum_{i=1}^d a_{ij} \cdot v_i$ be expression for u_j in the basis S where $a_{ij} \in \mathbb{R}$ for each i . Then the forward arc (u_j, v_i) has weight $w_0(u_j, v_i) := -\log(|a_{ij}|)$ for each $i \in [d]$ and each $u_j \in L$.

Now we define the weight function w_ℓ on the arcs for any $1 \leq \ell \leq d$. All backward arcs still have weight 0 but every forward edge (u, v) has weight $w_\ell(u, v) := \frac{\log(f(\ell))}{\ell} + w_0(u, v)$.

The following lemma gives the intuition behind the weight function w_0 defined above. It shows that the weight on arc (u_i, v_j) exactly measures the change in the objective when we replace element v_i with u_j in S . The proof appears in the appendix.

Lemma 3 Let S be a solution with $\text{vol}(S) > 0$ and $u \notin S$. Then for any $v \in S$, we have $w_0(u, v) = -\log \frac{\text{vol}(S+u-v)}{\text{vol}(S)}$.

While we will be specific about which weight function to use, but if it is not specified, then we refer to the weight function w_0 .

Definition 3 (Cycle Weight) *The weight of a cycle C in $G(S)$ is defined as $w_0(C) = \sum_{e \in C} w_0(e)$.*

Observe that the weight of a cycle depends only on the weight of the forward edges as backward edges have a weight 0.

We want to move from the current set S to a set with higher volume by exchanging on cycles in $G(S)$. But we want to exchange only on cycles that satisfy certain nice properties. For this purpose, we define f -Violating Cycles and Minimal f -Violating Cycles. The algorithm will always exchange on a Minimal f -Violating Cycle.

Definition 4 (f -Violating Cycle) *A cycle in $G(S)$ is called an f -violating cycle if*

$$w_0(C) < -\log f(|C|/2),$$

where $|C|$ is the number of arcs in C .

We have the following simple observation regarding f -violating cycle.

Observation 1 *If C is a f -violating cycle then $\prod_{(u,v) \in C: u \in L, v \in R} |a_{uv}| > 2 \left(\left(\frac{|C|}{2} \right)! \right)^3$.*

(See appendix A)

Definition 5 (Minimal f -Violating Cycle) *A cycle C in $G(S)$ is called a minimal f -violating cycle if*

- C is an f -violating cycle, and
- for all cycles C' such that $V(C') \subset V(C)$, C' is not an f -violating cycle.

Note that finding an f -violating cycle with $2i$ arcs is equivalent to finding a negative cycle with $2i$ arcs in $G(S)$ with weights w_i . We use the following simple algorithm to find a minimal f -violating cycle in $G(S)$ (if one exists), where we iterate on the number of arcs in the cycle.

Algorithm 1 Finding minimal f -violating cycle

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for  $i = 1, \dots, d$  do
  if there is a negative cycle  $C$  with exactly  $2i$  arcs in  $G(S)$  with weight function  $w_i$  then
    Return  $C$ 
  end if
end for

```

The following lemma is immediate. A proof appears in the appendix.

Lemma 4 *Algorithm 1 finds the minimal f -violating cycle in $G(S)$.*

After finding a minimal f -violating cycle, C , we modify the current set S to $S\Delta C$ and repeat. Observe that $S\Delta C$ is always a feasible set as it will pick exactly one element from each part. The main idea is that if $\text{vol}(S)$ is small compared to $\text{vol}(OPT)$, i.e., $\text{vol}(S) < \text{vol}(OPT) \cdot e^{-\Omega(d \log(d))}$, then there is always an f -violating cycle in $G(S)$ (see Lemma 5). Moreover, if C is a minimal f -violating cycle, then $\text{vol}(S\Delta C) \geq 2 \cdot \text{vol}(S)$ (see Lemma 7). If we initialize S to any solution with non-zero determinant, then the ratio $\text{vol}(OPT) / \text{vol}(S)$ is at most $2^{4\sigma}$ where σ is the encoding length of our problem input (Chapter 3, Theorem 3.2 [Sch00]). This implies that we need only modify the set S polynomially many times before $\text{vol}(S)$ becomes greater than $\text{vol}(OPT) \cdot e^{-O(d \log(d))}$, which gives Theorem 3. Such an initialization can be obtained by finding a basis of \mathbb{R}^d that picks exactly one vector from each part. As discussed above, this problem can be solved by the matroid intersection algorithm over the partition matroid and the linear matroid defined by the vectors.

Algorithm 2 Algorithm to find an approximation to OPT

$S \leftarrow$ set with $|S| = d$, $|S \cap \mathcal{P}_i| = 1$ for all i , and $\text{vol}(S) > 0$.
while there exists an f -violating cycle in $G(S)$ **do**
 $C =$ minimal f -violating cycle in $G(S)$
 $S = S\Delta C$
end while
Return S

Lemma 5 For any set S with $|S| = d$ and $\text{vol}(S) > 0$, if $\text{vol}(S) < \text{vol}(OPT) \cdot e^{-5d \log(d)}$, then there exists an f -violating cycle in $G(S)$.

Proof Let $OPT = \{u_1, u_2, \dots, u_d\}$ and $S = \{v_1, v_2, \dots, v_d\}$ such that $u_i, v_i \in \mathcal{P}_i$ for all $i \in [d]$. Observe that (v_i, u_i) is an arc in the exchange graph for each i since u_i and v_i belong to the same part³.

Abusing notation slightly, let T and S be matrices whose columns are the vectors in OPT and S , respectively. Let A be the coefficient matrix of T w.r.t. S , i.e., $T = SA^\top$. Then

$$\text{vol}(OPT)^2 = \det(TT^\top) = \det(SA^\top AS^\top) = \det(SS^\top) \cdot |\det(A)|^2.$$

Let $X = OPT \setminus S$, $Y = S \setminus OPT$, and $|X| = |Y| = k$. Without loss of generality, let $Y = \{v_1, \dots, v_k\}$ and $X = \{u_1, \dots, u_k\}$. Then $A = \begin{bmatrix} A_k & A' \\ 0 & I_{d-k} \end{bmatrix}$, where A_k is the sub-matrix of A corresponding to rows in X and columns in Y . Then $\det(A) = \det(A_k)$.

As per the hypothesis in the lemma, we have $\det(SS^\top) < \det(TT^\top) \cdot e^{-10d \log(d)}$. Therefore,

$$|\det(A_k)| > e^{5d \log(d)} \geq e^{5k \log(k)}. \quad (1)$$

By the Leibniz formula, we have $\det(A_k) = \sum_{\sigma \in \mathcal{S}_k} \text{sign}(\sigma) \prod_{i=1}^k a_{i\sigma(i)}$. Taking absolute values gives $|\det(A_k)| \leq \sum_{\sigma \in \mathcal{S}_k} \prod_{i=1}^k |a_{i\sigma(i)}|$. Since $|\mathcal{S}_k| = k! \leq e^{k \log(k)}$, there exists a permutation $\sigma \in \mathcal{S}_k$ such that

$$\prod_{i=1}^k |a_{i\sigma(i)}| > |\det(A_k)| \cdot e^{-k \log(k)} \geq e^{4k \log(k)}. \quad (2)$$

³Given $u_i \neq v_i$

Let the cycle decomposition of this σ be $\sigma = \{C_1, C_2, \dots, C_\ell\}$. Then each C_j corresponds to a unique cycle in $G(S)$ with $2|C_j|$ hops by considering the forward arcs $(u_i, v_{\sigma(i)})$ for each i on the cycle and the backward arcs (v_i, u_i) for each i in C_j . We claim that at least one of these cycles is an f -violating cycle. If not, then by the definition of f -violating cycles, we have $\prod_{i \in C_j} |a_{i\sigma(i)}| \leq 2(|C_j|!)^3$. Multiplying over all cycles in σ gives

$$\prod_{i=1}^k |a_{i\sigma(i)}| = \prod_{j=1}^{\ell} \prod_{i \in C_j} |a_{i\sigma(i)}| < \prod_{j=1}^{\ell} 2(|C_j|!)^3 < 2^k (k!)^3 < e^{4k \log(k)},$$

where the second last inequality follows from $\sum_{j=1}^{\ell} |C_j| = k$. This contradicts eq. (2), so $G(S)$ must contain an f -violating cycle. \square

The requirement in Lemma 5 that $\text{vol}(S) < \text{vol}(OPT) \cdot e^{-5d \log(d)}$ is tight, up to the coefficient in the exponent. Consider the case where d is a power of two (or more generally, any d for which a Hadamard matrix of order d is known to exist), $S = \{e_1, \dots, e_d\}$ consists of the standard basis vectors, and $L = H = \{h_1, \dots, h_d\}$ consists of the columns of the $d \times d$ Hadamard matrix. The entries of H are all ± 1 , and $h_i^\top h_j = 0$ for $i \neq j$. Then $\text{vol}(S) = 1$, and the optimal solution is $OPT = H$, which has objective value

$$\text{vol}(H) = \prod_{i=1}^d \|h_i\| = d^{d/2} = e^{\frac{d}{2} \log(d)} \cdot \text{vol}(S),$$

since the vectors in H are orthogonal. Meanwhile, the exchange matrix in this case is $A = H^\top$. Since all the entries of A are ± 1 , we know that the product of the entries along any cycle will have an absolute value of 1. Thus, we cannot find an f -violating cycle in the same way, despite the fact that $\text{vol}(S) \leq \text{vol}(OPT) \cdot e^{-\frac{d}{2} \log(d)}$.

2.2 Cycle Exchange and Determinant

Now we show that exchanging on a minimal f -violating cycle C increases the objective of the output set by at least a factor of two. The proof relies on two technical lemmas. First, observe that the arc weights given by $w_0(u, v)$ are exactly how much the objective will change if switch from the solution S to $S + u - v$ in the solution. But switching on a cycle will switch multiple elements at the same time. Since our function $\text{vol}(\cdot)$ (or more appropriately $\log \text{vol}(\cdot)$) is not additive, it is not clear what the change in the objective. The following lemma characterizes exactly how the objective changes when we switch a large set.

Consider our current solution S . Let C be the minimal cycle found and $\ell = |C|/2$. Let $X = C \cap L$ and $Y = C \cap S$. Thus the output set $T = (S \cup X) \setminus Y$. We will also abuse notation to let X, Y and S represent the matrices whose columns are the vectors in their respective sets. Note that S is $d \times d$ while both X and Y are $d \times \ell$. Observe that $\text{vol}(S)^2 = \det(SS^\top)$ and $\text{vol}(T)^2 = \det(TT^\top) = \det(SS^\top + XX^\top - YY^\top)$. Crucially, we show that the matrix consisting of coefficients a_{uv} that define the weights on the arcs of the exchange graph for $u \in X$ and $v \in Y$ also defines the change in objective value.

Lemma 6 *Let S be a basis, let X and Y be sets with $|X| = |Y| = \ell$ and $Y \subseteq S$. Let A be the $\ell \times d$ matrix of coefficients so that $X = SA^\top$, and let A_C be the $\ell \times \ell$ submatrix of only the coefficients corresponding to*

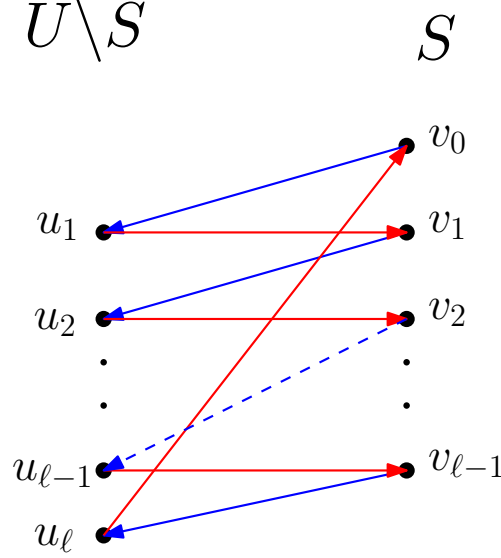


Figure 2: The cycle C

columns in Y . If $T = (S \cup X) \setminus Y$ then $\text{vol}(T)^2 = \text{vol}(S)^2 \cdot \det(A_C A_C^\top)$.

Without loss of generality, let $C = (v_0 \rightarrow u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow \dots \rightarrow u_{\ell} \rightarrow v_0)$ so that $X = \{u_1, \dots, u_{\ell}\}$ and $Y = \{v_1, \dots, v_{\ell-1}, v_0\}$, and order the columns of A_C accordingly so that the ℓ -th column corresponds to v_0 . Observe that diagonal entries of the A_C correspond to coefficient of v_i when expressing u_i in basis of S and thus equals a_{ii} . C being f -violating implies that the product of the diagonal entries $\prod_{i=1}^{\ell} |a_{ii}| > f(\ell)$. To show that the volume of T is large, we need to show $|\det(A_C)|$ is large. To this end, we utilize crucially that C is the *minimal* f -violating cycle. Observe that the off-diagonal entries a_{ij} exactly correspond to the weight on chords of the cycle. Since each chord introduces a cycle with smaller number of arcs, by minimality we know that it is not f -violating. This allows us to prove upper bounds on the off-diagonal entries of the matrix A_C . Finally, a careful argument allows us to give a lower bound on the determinant of any matrix with such bounds on the off-diagonal entries. We now expand on the above outline below.

Lemma 7 *If C is a minimal f -violating cycle in $G(S)$, then $\text{vol}(S\Delta C) \geq 2 \cdot \text{vol}(S)$.*

Proof Let $C = (v_0 \rightarrow u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow \dots \rightarrow u_{\ell} \rightarrow v_0)$ where v_i, u_{i+1} belong to the same part and $v_i \in S$ (See Figure 2).

By the Lemma 6, we know that $\text{vol}(S\Delta C) = \det(A_C A_C^\top)^{1/2} \cdot \text{vol}(S) = |\det(A_C)| \cdot \text{vol}(S)$. We will index the entries of A_C according to the indices of u_i and v_j where the last column corresponds to v_0 . Since C has 2ℓ hops, A_C is an $\ell \times \ell$ matrix.

We now bound each entry of the matrix A_C in terms of its diagonal entries, a_{ii} for $i = 1, \dots, \ell$. We show upper bounds on the absolute value of each entry as a function of the diagonal entries. Consider the i, j -th entry of A_C . For $i > j$, define the cycle $C_{i,j} := (u_i \rightarrow v_j \rightarrow u_{j+1} \rightarrow v_{j+1} \rightarrow \dots \rightarrow v_{i-1} \rightarrow u_i)$. $C_{i,j}$ is a cycle with $2(i-j)$ hops and $V(C_{i,j}) \subset V(C)$. C being a minimal f -violating cycle implies that $C_{i,j}$ is not an f -violating cycle. Therefore, $e^{-w_0(C_{i,j})} = |a_{i,j}| \cdot \prod_{s=j+1}^{i-1} |a_{s,s}| < f(i-j)$

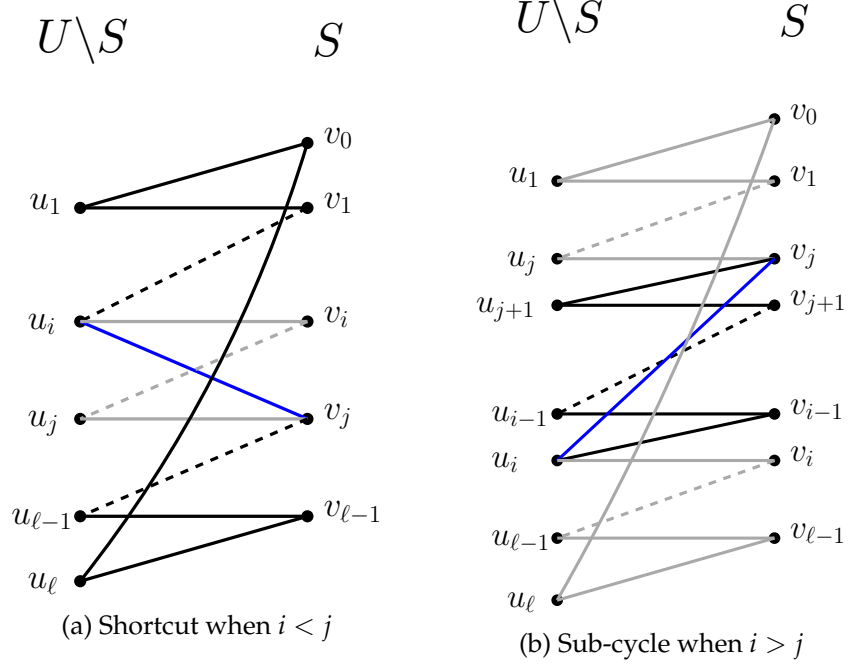


Figure 3: Structure when the edge $u_i \rightarrow v_j$ (in blue) is added

j). This implies

$$|a_{i,j}| < \frac{f(i-j)}{\prod_{s=j+1}^{i-1} |a_{s,s}|}. \quad (3)$$

Similarly for $j = \ell$, we have $|a_{i,\ell}| < \frac{f(i)}{\prod_{s=1}^{i-1} |a_{s,s}|}$.

For $i < j < \ell$, define $C'_{i,j} := (v_0 \rightarrow u_1 \rightarrow v_1 \rightarrow \dots \rightarrow u_i \rightarrow v_j \rightarrow \dots \rightarrow u_\ell \rightarrow v_0)$. Again, $C'_{i,j}$ is a cycle with $2(\ell - j + i)$ hops which is not f -violating. Therefore,

$$|a_{i,j}| \cdot \prod_{s=1}^{i-1} |a_{s,s}| \cdot \prod_{s=j+1}^{\ell} |a_{s,s}| < f(\ell - j + i). \quad (4)$$

Since C is an f -violating cycle, we also have

$$\prod_{s=1}^{\ell} |a_{s,s}| > f(\ell). \quad (5)$$

Combining (4) and (5) gives

$$|a_{i,j}| < \frac{f(\ell - j + i)}{f(\ell)} \cdot \prod_{s=i}^j |a_{s,s}|.$$

Let B_ℓ be the matrix obtained by applying the following operations to A_C

- Multiply the last column by $a_{1,1}$ and for $j < \ell$, divide the j -th column by $\prod_{s=2}^j a_{s,s}$

- Divide the first row by $a_{1,1}$ and for $i > 1$, multiply the i -th row by $\prod_{s=2}^{i-1} a_{s,s}$
- Divide the last column by $f(\ell)$ and, if needed, flip the sign of the last column so that $a_{\ell,\ell} > 0$.

Then $|\det(A_C)| = f(\ell) \cdot |\det(B_\ell)|$, and B_ℓ satisfies the following properties:

- $b_{i,i} = 1$ for all $i \in [\ell - 1]$, $b_{\ell,\ell} \geq 1$,
- $|b_{i,j}| \leq f(i - j)$ for all $j < i \leq \ell$, and
- $|b_{i,j}| \leq f(\ell - j + i) / f(\ell)$ for all $i < j \leq \ell$.

For $\ell \geq 2$, we have the following claim:

Claim 2.1 $\det(B_2) \geq 0.75$ and $\det(B_\ell) > 0.1$ for all $\ell \geq 3$.

With this claim in hand, it implies that $|\det(A_C)| > 0.1 \cdot f(\ell) > 2$ for all $\ell \geq 3$. For $\ell = 2$, $\det(B_2) \geq 0.75$, and $\det(A_C) \geq 0.75 \cdot f(2) > 2$. Therefore, $\text{vol}(S\Delta C) \geq |\det(A_C)| \cdot \text{vol}(S) \geq 2 \cdot \text{vol}(S)$. \square

Proof of Claim 2.1 Consider the following process on B_ℓ :

Algorithm 3 Gaussian Elimination Process (Column Operations)

```

for  $s = 1, \dots, \ell$  do ▷Outer Loop
  for  $j = s + 1, \dots, \ell$  do ▷Inner Loop
     $b_{:,j} = b_{:,j} - b_{:,s} \cdot \frac{b_{s,j}}{b_{s,s}}$ 
  end for
end for

```

Note that $\det(B_2) \geq 0.75$, $\det(B_3) \geq 0.73$, and $\det(B_4) \geq 0.83$ (see the end of the Appendix). From hereafter, we will assume that $\ell \geq 5$.

The output of the Algorithm 3 is a lower triangular matrix. Let $b_{i,j}(s)$ denote the value of $b_{i,j}$ before the s -th iteration of the outer loop of Gaussian Elimination. For example, $b_{i,j}(1) = b_{i,j}$ for all i, j .

For any $i < j$, $b_{i,j}$ becomes 0 at the end of the i -th iteration of the outer loop of the algorithm, and does not change after that. So, the final value of $b_{i,j}$, before it becomes 0, is $b_{i,j}(i)$. Similarly, for $i \geq j$, the value of $b_{i,j}$ does not change after the $(j - 1)$ -th iteration of the outer loop, and therefore the final value of $b_{i,j}$, i.e., $b_{i,j}(\ell)$ satisfies $b_{i,j}(\ell) = b_{i,j}(j)$.

Since this process does not change the determinant of B_ℓ , we have $\det(B_\ell) = \prod_{j=1}^{\ell} b_{j,j}(j)$. By Lemma 8, $b_{j,j}(j) > 1 - 0.92/\ell$ for $j < \ell$ and $b_{\ell,\ell}(\ell) > 0.303$. Therefore,

$$\det(B_\ell) = \prod_{j=1}^{\ell} b_{j,j}(j) \geq \left(1 - \frac{0.92}{\ell}\right)^{\ell-1} \cdot 0.303.$$

The function $\left(1 - \frac{0.92}{\ell}\right)^{\ell-1}$ is a decreasing function of ℓ , but has a horizontal asymptote at ~ 0.39 . Thus, $\left(1 - \frac{0.92}{\ell}\right)^{\ell-1} \geq 0.39$ and this gives

$$\det(B_\ell) > 0.39 \times 0.303 > 0.1.$$

□

Lemma 8 For $\ell \geq 5$, the final values of entries of B_ℓ after Algorithm 3 are bounded as follows:

1. $|b_{i,j}(j)| < \binom{i}{j} \cdot f(i-j)$ for $1 < j < i$,
2. $|b_{i,j}(i)| < 1.5 \cdot f(\ell - j + i)/f(\ell)$ for $i < j < \ell$,
3. $|b_{i,\ell}(i)| < 2.84 \cdot f(i)/f(\ell)$ for $i < \ell$,
4. $b_{j,j}(j) > 1 - \frac{0.92}{\ell}$ for all $j < \ell$,
5. $b_{\ell,\ell}(\ell) > 0.303$.

Proof We will prove the lemma by induction on j , the column index. Note that Algorithm 3 does not change the values of the first column of B_ℓ , and it also does not change the values of the first row of B_ℓ before they become 0. So, the bounds are trivially true for the first column and the first row.

For $i \geq j$,

$$b_{i,j}(j) = b_{i,j}(1) - \sum_{s=1}^{j-1} b_{i,s}(s) \cdot \frac{b_{s,j}(s)}{b_{s,s}(s)}. \quad (6)$$

Taking absolute values gives

$$|b_{i,j}(j) - b_{i,j}(1)| \leq \sum_{s=1}^{j-1} |b_{i,s}(s)| \cdot \frac{|b_{s,j}(s)|}{|b_{s,s}(s)|}. \quad (7)$$

The induction hypothesis implies that for all $s < j$, $|b_{i,s}(s)| < \binom{i}{s} \cdot f(i-s)$, $|b_{s,j}(s)| < 1.5 \cdot f(\ell - j + s)/f(\ell)$, and $b_{s,s}(s) > 1 - 0.92/\ell \geq 0.816$ (since $\ell \geq 5$). Plugging these bounds in (7), we get

$$|b_{i,j}(j) - b_{i,j}(1)| < \frac{1.5}{0.816} \cdot \sum_{s=1}^{j-1} \binom{i}{s} \cdot f(i-s) \cdot \frac{f(\ell - j + s)}{f(\ell)}. \quad (8)$$

Note that

$$\frac{f(i-s) \cdot f(\ell - j + s)}{f(i-j) \cdot f(\ell)} = \frac{((i-s)!)^3 \cdot ((\ell - j + s)!)^3}{((i-j)!)^3 \cdot (\ell!)^3} = \left(\frac{\binom{\ell+i-j}{\ell}}{\binom{\ell+i-j}{i-s}} \right)^3.$$

For any $1 \leq s \leq j-1$, $\frac{\binom{\ell+i-j}{\ell}}{\binom{\ell+i-j}{i-s}} \leq \frac{(i-j+1)}{\ell}$. Therefore,

$$\frac{f(i-s) \cdot f(\ell - j + s)}{f(i-j) \cdot f(\ell)} \leq \frac{\binom{\ell+i-j}{\ell}}{\binom{\ell+i-j}{i-s}} \cdot \frac{(i-j+1)^2}{\ell^2}.$$

Plugging this in (8) gives

$$\begin{aligned}
|b_{i,j}(j) - b_{i,j}(1)| &\leq f(i-j) \left(1.84 \cdot \frac{(i-j+1)^2}{\ell^2} \cdot \sum_{s=1}^{j-1} \binom{i}{s} \cdot \frac{\binom{\ell+i-j}{\ell}}{\binom{\ell+i-j}{i-s}} \right) \\
&\leq f(i-j) \left(1.84 \cdot \frac{(i-j+1)^2}{\ell^2} \cdot \frac{i! \binom{\ell+i-j}{\ell}}{(\ell+i-j)!} \sum_{s=1}^{j-1} \frac{(\ell-j+s)!}{s!} \right) \\
&= f(i-j) \left(1.84 \cdot \frac{(i-j+1)^2}{\ell^2} \cdot \frac{i! \binom{\ell+i-j}{\ell} (\ell-j)!}{(\ell+i-j)!} \sum_{s=1}^{j-1} \binom{\ell-j+s}{\ell-j} \right) \\
&= f(i-j) \left(1.84 \cdot \frac{(i-j+1)^2}{\ell^2} \cdot \frac{i! (\ell-j)!}{\ell! (i-j)!} \sum_{s=1}^{j-1} \binom{\ell-j+s}{\ell-j} \right). \tag{9}
\end{aligned}$$

For positive integers a, b, x with $x \leq a \leq b$,

$$\binom{a}{x} + \binom{a+1}{x} + \binom{a+2}{x} + \dots + \binom{b}{x} = \binom{b+1}{x+1} - \binom{a}{x-1}. \tag{10}$$

Using (10) with $a = \ell - j + 1$, $b = \ell - 1$, and $x = \ell - j$ gives $\sum_{s=1}^{j-1} \binom{\ell-j+s}{\ell-j} \leq \binom{\ell}{\ell-j+1}$ and from (9),

$$\begin{aligned}
|b_{i,j}(j) - b_{i,j}(1)| &\leq f(i-j) \left(1.84 \cdot \frac{(i-j+1)^2}{\ell^2} \cdot \frac{i! (\ell-j)!}{(i-j)! \ell!} \cdot \binom{\ell}{\ell-j+1} \right) \\
&= f(i-j) \left(1.84 \cdot \binom{i}{j} \cdot \frac{(i-j+1)^2 j}{\ell^2 (\ell-j+1)} \right) \tag{11}
\end{aligned}$$

$$\leq f(i-j) \left(1.84 \cdot \binom{i}{j} \cdot \frac{(\ell-j+1)j}{\ell^2} \right). \tag{12}$$

Since $(\ell-j+1)j$ is maximized at $j = (\ell+1)/2$, we have $\frac{(\ell-j+1)j}{\ell^2} \leq \frac{(\ell+1)^2}{4\ell^2} \leq 0.36$ for any $\ell \geq 5$.

Plugging this in (12) gives

$$|b_{i,j}(j)| \leq |b_{i,j}(1)| + f(i-j) \cdot 0.6624 \cdot \binom{i}{j} \leq f(i-j) \left(1 + 0.6624 \cdot \binom{i}{j} \right).$$

Now we will restrict ourselves to the case when $i > j$. For $i = 2$, j can only be 1 and this corresponds to an entry in the first column for which the bounds are trivially true. So, we only need to consider $i \geq 3$. Since $1 \leq j < i$, we have $\binom{i}{j} \geq i$. Furthermore, since $\ell \geq 5$ and $i \geq 3$, we have $1 \leq 0.3376 \cdot i < 0.3376 \binom{i}{j}$. This gives

$$|b_{i,j}(j)| \leq f(i-j) \left(0.3376 \cdot \binom{i}{j} + 0.6624 \cdot \binom{i}{j} \right) \leq f(i-j) \cdot \binom{i}{j}.$$

This concludes the proof of part 1.

For $j > i$, we have

$$|b_{i,j}(i) - b_{i,j}(1)| \leq \sum_{s=1}^{i-1} |b_{i,s}(s)| \cdot \frac{|b_{s,j}(s)|}{|b_{s,s}(s)|}. \tag{13}$$

By the induction hypothesis, $|b_{s,j}(s)| < 1.5 \cdot f(\ell - j + s)/f(\ell)$, $|b_{i,s}(s)| < \binom{i}{s} \cdot f(i - s)$, and $b_{s,s}(s) > 1 - 0.92/\ell \geq 0.816$. Plugging these bounds in (13), we get

$$|b_{i,j}(i) - b_{i,j}(1)| < \frac{1.5}{0.816} \sum_{s=1}^{i-1} \frac{f(\ell - j + s)}{f(\ell)} \cdot \binom{i}{s} \cdot f(i - s). \quad (14)$$

Note that $\frac{f(\ell-j+s) \cdot f(i-s)}{f(\ell-j+i)} = \frac{2((\ell-j+s)!)^3 \cdot ((i-s)!)^3}{((\ell-j+i)!)^3} = 2 \cdot \left(\frac{1}{\binom{\ell-j+i}{i-s}} \right)^3$. For any $1 \leq s \leq i-1$, $\frac{1}{\binom{\ell-j+i}{i-s}} \leq \frac{1}{\ell-j+i}$. Therefore,

$$\frac{f(\ell - j + i - s) \cdot f(s)}{f(\ell - j + i)} \leq \frac{1}{\binom{\ell-j+i}{i-s}} \cdot \frac{2}{(\ell - j + i)^2}.$$

Plugging this in (14) gives

$$\begin{aligned} |b_{i,j}(i) - b_{i,j}(1)| &\leq \frac{f(\ell - j + i)}{f(\ell)} \cdot \frac{3.68}{(\ell - j + i)^2} \cdot \left(\sum_{s=1}^{i-1} \binom{i}{s} \cdot \frac{1}{\binom{\ell-j+i}{i-s}} \right) \\ &= \frac{f(\ell - j + i)}{f(\ell)} \cdot \frac{3.68 \cdot i! (\ell - j)!}{(\ell - j + i)^2 \cdot (\ell - j + i)!} \cdot \sum_{s=1}^{i-1} \binom{\ell - j + s}{\ell - j}. \end{aligned}$$

Using (10) again, we get $\sum_{s=1}^{i-1} \binom{\ell-j+s}{\ell-j} \leq \binom{\ell-j+i}{\ell-j+1}$ and this gives

$$\begin{aligned} |b_{i,j}(i) - b_{i,j}(1)| &\leq \frac{f(\ell - j + i)}{f(\ell)} \cdot \frac{3.68 \cdot i! (\ell - j)!}{(\ell - j + i)^2 \cdot (\ell - j + i)!} \cdot \binom{\ell - j + i}{\ell - j + 1} \\ &= \frac{f(\ell - j + i)}{f(\ell)} \cdot \frac{3.68 \cdot i}{(\ell - j + i)^2 \cdot (\ell - j + 1)}. \end{aligned} \quad (15)$$

The function $\frac{i}{(\ell-j+i)^2}$ is maximized at $i = \ell - j$. So for any $j < \ell$, we have

$$|b_{i,j}(i) - b_{i,j}(1)| \leq \frac{f(\ell - j + i)}{f(\ell)} \cdot \frac{3.68}{4(\ell - j) \cdot (\ell - j + 1)} \leq 0.5 \cdot \frac{f(\ell - j + i)}{f(\ell)}.$$

Using the fact that $|b_{i,j}(1)| \leq f(\ell - j + i)/f(\ell)$, we have $|b_{i,j}(i)| \leq 1.5 \cdot f(\ell - j + i)/f(\ell)$ for $i < j < \ell$.

For $j = \ell$ and $i \geq 2$, equation (15) gives

$$|b_{i,\ell}(i) - b_{i,\ell}(1)| \leq \frac{f(i)}{f(\ell)} \cdot \frac{3.68}{i} \leq 1.84 \cdot \frac{f(i)}{f(\ell)},$$

and therefore $|b_{i,\ell}(i)| \leq 2.84 \cdot \frac{f(i)}{f(\ell)}$. This concludes the proof of parts 2 and 3.

For $i = j$ and $j < \ell$, using (11), we get

$$|b_{j,j}(j) - 1| \leq \frac{1.84 \cdot j}{\ell^2(\ell - j + 1)} \leq \frac{1.84 \cdot (\ell - 1)}{2\ell^2} \leq \frac{0.92}{\ell}.$$

For $i = j = \ell$, by (8) and the induction hypothesis,

$$|b_{\ell,\ell}(\ell) - b_{\ell,\ell}(1)| < \frac{2.84}{1 - 0.92/\ell} \sum_{s=1}^{\ell-1} \binom{\ell}{s} \cdot f(\ell - s) \cdot \frac{f(s)}{f(\ell)}.$$

Following the proof outline of equation (11) gives $|b_{\ell,\ell}(\ell) - b_{\ell,\ell}(1)| \leq \frac{2.84}{0.816} \cdot \frac{1}{\ell} \leq 0.697$. Since $b_{\ell,\ell}(1) \geq 1$, we have $b_{\ell,\ell}(\ell) \geq b_{\ell,\ell}(1) - 0.697 \geq 0.303$. \square

3 Update Step for General Matroids

Consider the case when $\mathcal{M} = ([n], \mathcal{I})$ is a general matroid of rank d . When we exchange on a cycle C and update $S \leftarrow S \Delta C$, the resulting set is guaranteed to be independent in the linear matroid because of the determinant bounds in Lemma 7, but it is not clear that it would be independent in the general constraint matroid, \mathcal{M} , when \mathcal{M} is not a partition matroid. However, by exchanging on a minimal f -violating cycle in our algorithm, we can make the same guarantee.

In this section, we prove the existence of an f -violating cycle for any matroid \mathcal{M} with rank d when the current basis S is sufficiently smaller in volume than the optimal solution OPT . We also prove that exchanging on a minimal f -violating cycle preserves independence in \mathcal{M} .

Theorem 4 *For any basis S with $|S| = d$ and $\text{vol}(S) > 0$, if $\text{vol}(S) < \text{vol}(OPT) \cdot e^{-5d \log(d)}$, then there exists an f -violating cycle in $G(S)$.*

Proof Since S and OPT are independent and $|S| = |OPT|$, there exists a perfect matching between $OPT \setminus S$ and $S \setminus OPT$ using the backward arcs in $G(S)$ (Chapter 39, Corollary 39.12a, [Sch03]). Let $X = OPT \setminus S$, $Y = S \setminus OPT$, and $|X| = |Y| = k$. Without loss of generality, let $Y = \{v_1, \dots, v_k\}$ and $X = \{u_1, \dots, u_k\}$ such that $(v_i \rightarrow u_i)$ is an arc in $G(S)$ for all $i \in [k]$.

Let T and S be matrices whose columns are the vectors in OPT and S , respectively. Let A be the coefficient matrix of T w.r.t. S , i.e., $T = SA^\top$. Then $A = \begin{bmatrix} A_k & A' \\ 0 & I_{d-k} \end{bmatrix}$, where A_k is the sub-matrix of A corresponding to rows in X and columns in Y . Then by the same proof as in 5, there exists a permutation $\sigma \in \mathcal{S}_k$ such that

$$\prod_{i=1}^k |a_{i\sigma(i)}| > |\det(A)| \cdot e^{-k \log(k)} \geq e^{4k \log(k)}. \quad (16)$$

Let the cycle decomposition of σ be $\sigma = \{C_1, C_2, \dots, C_\ell\}$ where $C_i = (i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_j \rightarrow i_1)$. Since there is an edge from $v_{\sigma(j)}$ to $u_{\sigma(j)}$ for all j , every cyclic permutation C_i corresponds to a cycle $(u_{i_1} \rightarrow v_{i_2} \rightarrow u_{i_2} \rightarrow v_{i_3} \dots \rightarrow u_{i_j} \rightarrow v_{i_1} \rightarrow u_{i_1})$ in $G(S)$. We claim that at least one of these cycles is an f -violating cycle. If not, then by the definition of f -violating cycles, we have $\prod_{i \in C_j} |a_{i\sigma(i)}| \leq 2(|C_j|!)^3$ for all $j \leq \ell$. Multiplying over all the cycles in σ gives

$$\prod_{i=1}^k |a_{i\sigma(i)}| = \prod_{j=1}^{\ell} \prod_{i \in C_j} |a_{i\sigma(i)}| \leq \prod_{j=1}^{\ell} 2(|C_j|!)^3 < e^{4k \log(k)},$$

where the last inequality follows from $\sum_{j=1}^{\ell} |C_j| = k$. This contradicts (16), so $G(S)$ must contain an f -violating cycle. \square

Lemma 9 *If C is a minimal f -violating cycle in $G(S)$, then $S\Delta C$ is independent in \mathcal{M} .*

Proof For clarity, let $V(C)$ denote the vertex set of C . Let $T := S\Delta V(C)$ and let $|C| = 2\ell$. Lets consider the graph $G(S)$ with weights w_ℓ , and define $w_\ell(D) := \sum_{e \in D} w_\ell(e)$ for any cycle D . Since C is an f -violating cycle, $w_\ell(C) = w_0(C) + \log(f(\ell)) < 0$.

Let the set of backward arcs in C be N_1 , and the set of forward arcs be N_2 . For the sake of contradiction, assume that $T \notin \mathcal{I}$. Then, there exists a matching N'_1 on $V(C)$ consisting of only backward arcs such that $N_1 \neq N'_1$ (Chapter 39, Theorem 39.13, [Sch03]). Let A be a multiset of arcs consisting of all arcs in N_2 twice and all arcs N_1 and N'_1 (with arcs in $N_1 \cap N'_1$ appearing twice). Consider the directed graph $D = (V(C), A)$. Since $N_1 \neq N'_1$, D contains a directed circuit C_1 with $V(C_1) \subsetneq V(C)$. Every vertex in $V(C)$ has exactly two in-edges and two out-edges in A . Therefore, D is Eulerian, and we can decompose A into directed circuits C_1, \dots, C_k . Since only arcs in N_2 have non-zero weights, we have $\sum_{i=1}^k w_\ell(C_i) = 2w_\ell(C)$.

Because $V(C_1) \subsetneq V(C)$, at most one cycle C_j can have $V(C_j) = V(C)$. If for some j , $V(C) = V(C_j)$, then $w_\ell(C_j) = w_\ell(C)$ as C_j must contain every edge in N_2 . So, $\sum_{i \neq j} w_\ell(C_i) = w_\ell(C) < 0$ and there exists a cycle C_i such that $V(C_i) \subsetneq V(C)$ and $w_\ell(C_i) < 0$. Otherwise $V(C_j) \subsetneq V(C')$ for all j and $\sum_i w_\ell(C_i) = 2w_\ell(C) < 0$. Again, there exists a cycle C_i such that $V(C_i) \subsetneq V(C)$ and $w_\ell(C_i) < 0$.

Let C' be the directed cycle such that $V(C') \subsetneq V(C)$ and $w_\ell(C') \leq w_\ell(C) \leq 0$. Define $y := |C'|/2$. Thus $w_\ell(C') = y \cdot \log(f(\ell))/\ell + w_0(C') < 0$. Since $y < \ell$, $\log(f(y))/y \leq \log(f(\ell))/\ell$. Therefore $w_0(C') \leq -y \cdot \log(f(\ell))/\ell \leq -\log(f(y))$. So C' is an f -violating cycle with $V(C') \subset V(C)$, which contradicts the fact that C is a minimal f -violating cycle. \square

4 Acknowledgements

Adam Brown and Mohit Singh were supported in part by NSF CCF-2106444 and NSF CCF-1910423. Adam Brown was also supported by an ACO-ARC Fellowship. Aditi Laddha was supported in part by a Microsoft fellowship and NSF awards CCF-2007443 and CCF-2134105. Madhusudhan Pittu is supported in part by NSF awards CCF-1955785 and CCF-2006953. Prasad Tetali was supported in part by the NSF grant DMS-2151283.

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A Omitted Proofs

Proof of Lemma 3 Recall the statement of the Lemma: Let S be a solution with $\text{vol}(S) > 0$ and $u \notin S$. Then for any $v \in S$, we have $w_0(u, v) = -\log \frac{\text{vol}(S+u-v)}{\text{vol}(S)}$.

Let $S = \{v_1, \dots, v_d\}$ so that $v = v_1$ and write $u = \sum_{i=2}^d a_i v_i$. We can also write $v = v^\perp + \sum_{i=2}^d b_i v_i$ where v^\perp is orthogonal to $S \setminus \{v\}$. Then $u = a_1 v^\perp + \sum_{i=2}^d (a_1 b_i + a_i) v_i$. For $X \subseteq \mathbb{R}^d$ with $|X| = k \leq d$, let $\text{vol}(X)$ denote the k -dimensional volume of the parallelepiped spanned by X . Then

$$\text{vol}(S) = \text{vol}(S - v) \cdot \|v^\perp\|,$$

while

$$\text{vol}(S + u - v) = \text{vol}(S - v) \cdot |a_1| \|v^\perp\|,$$

since the change in volume from adding a single new vector is proportional to the length of the component of that vector which is orthogonal to our current set. Thus

$$-\log \frac{\text{vol}(S + u - v)}{\text{vol}(S)} = -\log \frac{\text{vol}(S - v) \cdot |a_{uv}| \|v^\perp\|}{\text{vol}(S - v) \cdot \|v^\perp\|} = -\log |a_{uv}|.$$

□

Proof of Observation 1 Recall the statement of the Observation: If C is a f -violating cycle then $\prod_{(u,v) \in C: u \in L, v \in R} |a_{uv}| > 2(|C|/2!)^3$. If C is f -violating then $\ell(C) < -\log f(|C|/2)$, where $\ell(C)$ is the sum of the w_0 edge weights in C , and $f(|C|/2) = 2((|C|/2)!)^3$. Note that $|C \cap R| = |C \cap L| = |C|/2$, so $f(|C|/2) = 2(|C \cap R|!)^3$. By expanding $\ell(C)$ we see that

$$\begin{aligned} \ell(C) &= \sum_{(u,v) \in C: u \in L, v \in R} w_0(u, v) \\ &= -\log \left(\prod_{(u,v) \in C: u \in L, v \in R} |a_{uv}| \right). \end{aligned}$$

Since $\ell(C) < -\log f(|C|/2)$, we can take the exponential to remove the logarithms and attain the desired inequality. □

Proof of Lemma 4 Recall the statement of the Lemma: Algorithm 1 finds the minimal f -violating cycle in $G(S)$, if one exists.

In the i th iteration of Algorithm 1 we determine if there is a negative cycle in $G(S)$ with weights w_i and $2i$ hops, as follows. For each vertex of $G(S)$, we start an instance of Bellman-Ford (See Chapter 8, Section 8.3, [Sch03]) with that vertex as the root, and proceed for $2i$ iterations. For

source u , after $2i$ iterations, we check whether the distance from u to u is negative. If so, we have found a negative cycle with at most $2i$ hops. Note that for weights w_i , any negative cycle with at most $2i$ hops is an f -Violating cycle. Since the $(i - 1)$ -th iteration of the algorithm ensured that there are no f -Violating cycles with at most $2(i - 1)$ hops, a negative cycle in the i -th iteration (if any) must have exactly $2i$ hops.

Suppose there is an f -violating cycle C in $G(S)$, so that $\ell = |C|/2$. Then, with weight w_ℓ , the total weight of the cycle C is

$$w_\ell(C) = \sum_{(u,v) \in C} w_\ell(u,v) = \sum_{(u,v) \in C} \log(f(\ell))/\ell + w_0(u,v) = \log f(\ell) + w_0(C).$$

Since C is f -violating we know that $\log f(\ell) < -w_0(C)$, so the above calculation shows that C has negative total weight with weights w_ℓ . This guarantees that Algorithm 1 will return an f -violating cycle whenever one exists.

Now suppose that C is the cycle returned by Algorithm 1 and we must show that C is minimal f -violating. Let C' be another cycle such that $V(C') \subset V(C)$. Then C' has fewer hops than C , but it was not returned in iteration $|C'|/2$, so we know that C' must not be f -violating. Thus C is indeed minimal. \square

Proof of Lemma 6 Recall the statement of the Lemma 6: Let S be a basis, let X and Y be sets with $|X| = |Y| = \ell$ and $Y \subseteq S$. Let A be the $\ell \times d$ matrix of coefficients so that $X = SA^\top$, and let A_C be the $\ell \times \ell$ submatrix of only the coefficients corresponding to columns in Y . If $T = (S \cup X) \setminus Y$ then $\text{vol}(T)^2 = \text{vol}(S)^2 \cdot \det(A_C A_C^\top)$.

We will abuse notation slightly to let S, X, Y also denote the matrices with columns from their respective sets. Order the columns of S so that Y makes up the first ℓ columns of S . Let A' be the $\ell \times (d - \ell)$ submatrix of A consisting of the remaining columns not already in A_C . Then

$$T = S \begin{bmatrix} A_C & A' \\ 0 & I_{d-\ell} \end{bmatrix}^\top,$$

which implies that

$$\det(T) = \det(S) \cdot \det(A_C).$$

\square

Bounds on $\det(B_2), \det(B_3), \det(B_4)$ mentioned in Proof of Claim 2.1:

For $\ell = 2$, $b_{1,1}(1) = 1$, $|b_{1,2}(1)| \leq 0.125$, $|b_{2,1}(1)| \leq 2$, and $|b_{2,2}(2)| \geq 0.75$.

$$\det(B_2) \geq \prod_{i=1}^2 b_{i,i}(i) \geq 0.75.$$

The bounds on final values of B_3 are:

$b_{1,1}(1) = 1$	$ b_{1,2}(1) \leq 0.0371$	$ b_{1,3}(1) \leq 0.00463$
$ b_{2,1}(1) \leq 2.0$	$b_{2,2}(2) \geq 0.92$	$ b_{2,3}(2) \leq 0.0463$
$ b_{3,1}(1) \leq 16.0$	$ b_{3,2}(2) \leq 2.5926$	$b_{3,3}(3) \geq 0.79$

$$\det(B_3) \geq \prod_{i=1}^3 b_{i,i}(i) \geq 0.73.$$

The bounds on final values of B_4 are:

$b_{1,1}(1) = 1$	$ b_{1,2}(1) \leq 0.015625$	$ b_{1,3}(1) \leq 0.00057871$	$ b_{1,4}(1) \leq 0.000073$
$ b_{2,1}(1) \leq 2.0$	$b_{2,2}(2) \geq 0.96875$	$ b_{2,3}(2) \leq 0.01678241$	$ b_{2,4}(2) \leq 0.00072338$
$ b_{3,1}(1) \leq 16.0$	$ b_{3,2}(2) \leq 2.25$	$b_{3,3}(3) \geq 0.95$	$ b_{3,4}(3) \leq 0.018463$
$ b_{4,1}(1) \leq 432.0$	$ b_{4,2}(2) \leq 22.75$	$ b_{4,3}(3) \leq 2.645$	$b_{4,4}(4) \geq 0.9$

$$\det(B_4) \geq \prod_{i=1}^4 b_{i,i}(i) \geq 0.83.$$

B Rank $r \leq d$

In this section, we prove Theorem 2. Consider a matroid $\mathcal{M} = ([n], \mathcal{I})$ with rank $r \leq d$. Starting with a basis S with non-zero volume, we will use a slight modification of Algorithm 2 to iteratively find a basis with strictly larger volume. However since the set S is not full dimensional, our edge weight functions will be different.

Let $S = \{v_1, v_2, \dots, v_r\}$ be a basis of \mathcal{M} with $\text{vol}(S) > 0$. We can write any vector u_i in \mathcal{M} as

$$u_i = \sum_{j=1}^r a_{i,j} v_j + u_i^\perp,$$

where u_i^\perp is orthogonal to $\text{Span}(S)$.

The change in volume on replacing some $v \in S$ by $u \notin S$ is given by

$$\frac{\text{vol}(S - v + u)}{\text{vol}(S)} = \sqrt{a_{uv}^2 + \frac{\|u^\perp\|^2}{\|v^\perp\|^2}}, \quad (17)$$

where v^\perp is the component of v orthogonal to $\text{Span}(S - v)$. The two terms in equation (17) have geometric meanings. Let us decompose u into $u^\perp + u^\parallel$, where u^\perp is the component of u orthogonal to $\text{Span}(S)$. Then $|a_{uv}|$ is exactly the change in the volume if we project u to $\text{Span}(S)$ before replacing v , i.e., $|a_{uv}| = \frac{\text{vol}(S - v + u^\parallel)}{\text{vol}(S)}$, and $\frac{\|u^\perp\|}{\|v^\perp\|}$ is the change in the volume if we project u orthogo-

nal to $\text{Span}(S)$ before replacing v , i.e., $\frac{\|u^\perp\|}{\|v^\perp\|} = \frac{\text{vol}(S-v+u^\perp)}{\text{vol}(S)}$. So, we augment the exchange graph to reflect this.

Like Lemma 5, when $\text{sym}_r(S) < \text{sym}_r(\text{OPT}) \cdot r^{-\Omega(r)}$, we can find an \tilde{f} -violating cycle (for an appropriate function \tilde{f}) in the augmented exchange graph.

However unlike Lemma 7, the change in the objective induced by a cycle C in the augmented graph is not a simple function of the weights of chords and arcs of C . To get around this issue, we use the geometric relation between sym_r and vol , specifically the subadditivity of vol to relate sym_r to the chord and arc weights of C .

We define the exchange graph $G(S)$ exactly as Definition 1 with $w_0(u_i, v_j) = -\log(|a_{i,j}|)$. Our approach to find a basis with larger volume is to first try to exchange on an f -violating cycle in $G(S)$. Like Algorithm 1, exchanging on an f -violating cycle implies increase in volume. Unlike Algorithm 1, failure to find an f -violating cycle does not imply that the volume of the current solution is close to optimal. So, we move to Stage 2, where we work with an augmented version of the exchange graph defined below.

We decompose every vector u_i in \mathcal{M} as

$$u_i = u_i^\parallel + u_i^\perp,$$

where $u_i^\parallel \in \text{Span}(S)$ and u_i^\perp is orthogonal to $\text{Span}(S)$.

In the augmented exchange graph $\tilde{G}(S)$, for every vector $u_i \in \mathcal{M}$, we create two vertices u_i^\parallel (called a parallel vertex) and u_i^\perp (called a perpendicular vertex) in the left-hand side.

Definition 6 (Augmented Exchange Graph) For a subset of vectors $S = \{v_1, v_2, \dots, v_r\}$, we define the augmented exchange graph of S , denoted by $\tilde{G}(S)$ as a bipartite graph, where the right-hand side consists of vectors in S , i.e., $R = \{v_1, v_2, \dots, v_r\}$ and the left-hand side consists of all the vectors $L = \bigcup_{u \in \mathcal{U} \setminus S} \{u_i^\perp, u_i^\parallel\}$. For each $v_i \in R$, if $S - v_i + u \in \mathcal{I}$, then v_i has an edge to u^\perp and an edge to u^\parallel . The vertices on the left-hand side have forward edges to every vertex in S (See figure Figure 4).

Each vector $v_j \in S$ can be decomposed as $v_j = \sum_{i \neq j} \alpha_{j,i} v_i + v_j^\perp$, where v_j^\perp is orthogonal to the span of $S \setminus \{v_j\}$. We will call v_j^\perp the orthogonal component of v_j and use it to define edge weights in $\tilde{G}(S)$.

Definition 7 (Weight functions on the Augmented Exchange graph) All the arcs from some vertex in R to some vertex in L , have weight 0. The weights of the forward arcs are defined as

- for every parallel vertex $u_i^\parallel \in L$, the weight of the arc $u_i^\parallel \rightarrow v_j$ is

$$\tilde{w}_0(u_i^\parallel \rightarrow v_j) := -\log(|a_{i,j}|),$$

- for every perpendicular vertex $u_i^\perp \in L$, the weight of the arc $u_i^\perp \rightarrow v_j$ is

$$\tilde{w}_0(u_i^\perp \rightarrow v_j) := -\log \left(\frac{\|u_i^\perp\|}{\|v_j^\perp\|} \right).$$

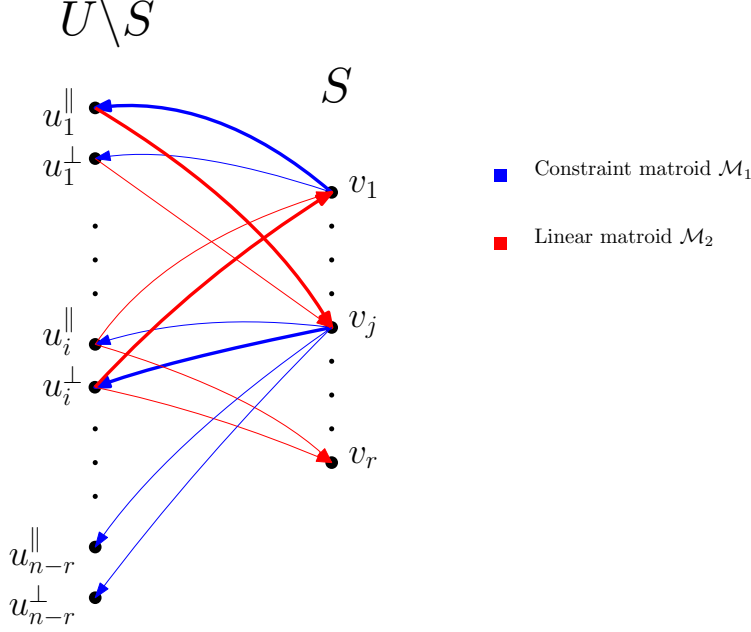


Figure 4: The augmented exchange graph $\tilde{G}(S)$ (An exchange cycle is shown in bold edges)

We define a family of weight functions on the exchange graph. To define these weights we use the new function $\tilde{f}(i) = (i!)^{11}$ if $i \geq 2$ and $\tilde{f}(1) = 2$. Now we define the weight function \tilde{w}_ℓ analogously to w_ℓ , i.e., all backward arcs still have weight 0 but every forward edge (u, v) has weight

$$\tilde{w}_\ell(u, v) := \frac{\log(\tilde{f}(\ell))}{\ell} + \tilde{w}_0(u, v).$$

Observation 2 For a current solution S , let $\tilde{\mathcal{M}}(S)$ be the matroid obtained from \mathcal{M} by adding an element parallel to every $u \in U \setminus S$, and labelling the pair u^\perp, u^\parallel . Then the subgraph of $\tilde{G}(S)$ induced by edges with finite weight is the matroid exchange graph where $\mathcal{M}_1 = \tilde{\mathcal{M}}(S)$, and \mathcal{M}_2 is the linear matroid on $S \cup \bigcup_{u \in U \setminus S} \{u^\perp, u^\parallel\}$.

By construction, no independent set in $\tilde{\mathcal{M}}(S)$ contains both u^\perp and u^\parallel , for any $u \in U \setminus S$. Thus, if $\tilde{\mathcal{I}}$ is an independent set in $\tilde{\mathcal{M}}(S)$ and \mathcal{I} is obtained from $\tilde{\mathcal{I}}$ by replacing each instance of u^\perp , or u^\parallel with the original element $u \in U \setminus S$, then \mathcal{I} is independent in \mathcal{M} .

Similar to Lemma 3, a function of $\tilde{w}_0(u_i^\parallel, v_j)$ and $\tilde{w}_0(u_i^\perp, v_j)$ measures the change in objective when we replace element v_j by u_i .

Lemma 10 Let S be a solution with $\text{vol}(S) > 0$ and $u \notin S$ with $u = u^\parallel + u^\perp$. Then for any $v \in S$, we have $\sqrt{e^{-2\tilde{w}_0(u^\parallel, v)} + e^{-2\tilde{w}_0(u^\perp, v)}} = \frac{\text{vol}(S - u + v)}{\text{vol}(S)}$.

Our algorithm works in 2 stages. In the first stage, we try to find an f -violating cycle in the exchange graph $G(S)$. If the algorithm finds such a cycle, it exchanges on it. If no f -violating cycle is found, we move to the augmented exchange graph $\tilde{G}(S)$, and search for an \tilde{f} -violating cycle in $\tilde{G}(S)$. If no such cycle is found in Stage 2, then Lemma 12 guarantees that $\text{vol}(S) \geq e^{-O(r \log(r))} \cdot \text{vol}(\text{OPT})$.

Algorithm 4 Algorithm to find an approximation to OPT

$S \leftarrow$ basis with $\text{vol}(S) > 0$.
Let $f(i) = 2(i!)^3$ and $\tilde{f}(i) = (i!)^{11}$
while There exists an f -violating cycle in $G(S)$ or an \tilde{f} -violating cycle in $\tilde{G}(S)$ **do**
 if There exists an f -violating cycle in $G(S)$ **then**
 Stage 1:
 $C =$ minimal f -violating cycle in $G(S)$
 $S = S\Delta C$
 else
 Stage 2:
 $C =$ minimal \tilde{f} -violating cycle in $\tilde{G}(S)$
 $\tilde{C} = \{u \in [n] \setminus S : u^\perp \text{ or } u^\parallel \in C\} \cup \{v \in S : v \in C\}$
 $S = S\Delta\tilde{C}$
 end if
end while
Return S

Lemma 11 *If Algorithm 4 finds an f -violating cycle, C , in $G(S)$, then $\text{vol}(S\Delta C) \geq 2 \cdot \text{vol}(S)$.*

Proof Let C' be the projection of C onto $\text{Span}(S)$. By Lemma 20, we know that $\text{vol}(S\Delta C) \geq \text{vol}(S\Delta C')$, and by Lemma 7, we know that $\text{vol}(S\Delta C') \geq 2 \cdot \text{vol}(S)$. Therefore,

$$\text{vol}(S\Delta C) \geq \text{vol}(S\Delta C') \geq 2 \cdot \text{vol}(S),$$

which concludes the proof of the Lemma. \square

From hereafter we analyze the case when Algorithm 4 does not find an f -violating cycle in Stage 1, moves on to Stage 2 and finds an \tilde{f} -violating cycle in $\tilde{G}(S)$.

B.1 Existence of \tilde{f} -violating cycle

To guarantee we make progress at every iteration, we need to ensure there will always be an \tilde{f} -violating cycle, whenever our current volume is far from the optimal volume. Before we prove the existence of an \tilde{f} -violating cycle, we state a couple of useful observations.

For convenience, to specify the volume of a set of vectors $\{v_1, v_2, \dots, v_r\}$, instead of writing $\text{vol}(\{v_1, v_2, \dots, v_r\})$, we use $\text{vol}(v_1, v_2, \dots, v_r)$.

Observation 3 *For any set of vectors $\{v_1^a, v_1^b, v_2, \dots, v_r\}$,*

$$\begin{aligned} \text{vol}(v_1^a + v_1^b, v_2, \dots, v_r) &\leq \text{vol}(v_1^a, v_2, \dots, v_r) + \text{vol}(v_1^b, v_2, \dots, v_r), \\ \text{vol}(v_1^a + v_1^b, v_2, \dots, v_r) &\geq \text{vol}(v_1^a, v_2, \dots, v_r) - \text{vol}(v_1^b, v_2, \dots, v_r). \end{aligned}$$

Proof Let P^\perp be the projection matrix orthogonal to $\text{Span}(v_2, \dots, v_r)$. By triangle inequality, we

have that

$$\begin{aligned} \|P^\perp(v_1^a + v_1^b)\| &\leq \|P^\perp v_1^a\| + \|P^\perp v_1^b\|, \text{ and} \\ \|P^\perp(v_1^a + v_1^b)\| &\geq \|P^\perp v_1^a\| - \|P^\perp v_1^b\|. \end{aligned}$$

Since $\text{vol}(v_1^a, v_2, \dots, v_r) = \|P^\perp v_1^a\| \cdot \text{vol}(v_2, \dots, v_r)$, multiplying both sides by $\text{vol}(v_2, \dots, v_r)$ gives us the required inequalities. \square

Observation 4 If v_i^\perp is the orthogonal projection of v_i onto $\text{Span}(S - v_i)$, then

$$\text{vol}(v_1, \dots, v_k) \cdot \prod_{i=k+1}^r \|v_i^\perp\| \leq \text{vol}(v_1, \dots, v_r)$$

for any $k \in [r]$.

Proof Let $S_i = \{v_1, \dots, v_i\}$ for $i \in [r]$ and $S_0 = \emptyset$. Since $\text{vol}(v_1, \dots, v_r) = \prod_{i=1}^r \frac{\text{vol}(S_i)}{\text{vol}(S_{i-1})}$ where $\text{vol}(\emptyset) = 1$, it suffices to prove that $\frac{\text{vol}(S_i)}{\text{vol}(S_{i-1})} \geq \|v_i^\perp\| = \frac{\text{vol}(S)}{\text{vol}(S \setminus \{v_i\})}$ which follows from the submodularity of $\log \text{vol}(\cdot)$ as $S_{i-1} \subseteq S \setminus \{v_i\}$. \square

The following lemma is an extension to Lemma 5 when the current solution has $r \leq d$ vectors.

Lemma 12 For any basis $S \in \mathcal{I}$ with $\text{vol}(S) > 0$, if $\text{vol}(OPT) > \text{vol}(S) \cdot r^{2r} \cdot \tilde{f}(r)$, then there exists an \tilde{f} -violating cycle in $\tilde{G}(S)$.

Proof Since S and OPT are independent and $|S| = |OPT|$, there exists a perfect matching between $OPT \setminus S$ and $S \setminus OPT$ using the backward arcs in $G(S)$ (Chapter 39, Corollary 39.12a, [Sch03]). Let $X = OPT \setminus S$, $Y = S \setminus OPT$, and $|X| = |Y| = \ell$. Without loss of generality, let $Y = \{v_1, \dots, v_\ell\}$ and $X = \{u_1, \dots, u_\ell\}$ such that $(v_i \rightarrow u_i)$ is an arc in $G(S)$ for all $i \in [\ell]$. Let $Z = OPT \cap S = \{v_{\ell+1}, \dots, v_r\}$ and let us use T instead of OPT for ease of notation.

From the hypothesis of the lemma, we have

$$r^{2r} \cdot \tilde{f}(r) \leq \frac{\text{vol}(T)}{\text{vol}(S)} = \frac{\text{vol}(X \cup Z)}{\text{vol}(Y \cup Z)}. \quad (18)$$

Since $u_i = \sum_{j=1}^r a_{i,j} v_j + u_i^\perp$, we can decompose each vector u_i into a sum of $r + 1$ vectors, i.e., $u_i = \sum_{j=0}^r u_i^{(j)}$ where

- $u_i^{(0)} := u_i^\perp$ for all i , and
- $u_i^{(j)} := a_{i,j} v_j$.

Now using Observation 3, we can expand $\text{vol}(T)$ as

$$\frac{\text{vol}(T)}{\text{vol}(S)} \leq \sum_{i_1, i_2, \dots, i_\ell \in \{0, \dots, r\}} \frac{\text{vol}(\{u_1^{(i_1)}, u_2^{(i_2)}, \dots, u_\ell^{(i_\ell)}\} \cup Z)}{\text{vol}(S)}. \quad (19)$$

If $i_j > \ell$ for any j , then $u_j^{(i_j)}$ and the vectors in set Z are linearly dependent and therefore the volume is 0. So we can restrict ourselves to the case when $i_j \in \{0, \dots, \ell\}$ for all $j \in [\ell]$.

For any permutation $\sigma \in \mathcal{S}_\ell$, define $\mathcal{F}(\sigma) := \{(i_1, i_2, \dots, i_\ell) : i_j \in \{\sigma(j), 0\}\}$. Since $u_{i_1}^{(j)}$ and $u_{i_2}^{(j)}$ are linearly dependent whenever $j > 0$, we can rewrite $\text{vol}(T)$ as

$$\begin{aligned} \text{vol}(T) &\leq \sum_{\tau \in \bigcup_{\sigma} \mathcal{F}(\sigma)} \text{vol}(\{u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z) \\ &\leq \sum_{\sigma \in \mathcal{S}_\ell} \sum_{\tau \in \mathcal{F}(\sigma)} \text{vol}(\{u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z). \end{aligned} \quad (20)$$

We will upper bound $\text{vol}(\{u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)$ for each τ separately. As an illustration for a fixed σ , let us consider $\tau = (0, \dots, 0, \sigma(k+1), \dots, \sigma(\ell))$. The corresponding volume term is equal to

$$\frac{\text{vol}(\{u_1^{(0)}, \dots, u_k^{(0)}, u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)} = \frac{\text{vol}(u_1^{(0)}, \dots, u_k^{(0)}) \cdot \text{vol}(\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)},$$

as the sets of vectors $\{u_1^{(0)}, \dots, u_k^{(0)}\}$ and $\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z$ are orthogonal to each other. Upper bounding $\text{vol}(u_1^{(0)}, \dots, u_k^{(0)})$ by $\prod_{i=1}^k \|u_i^\perp\|$, we get

$$\frac{\text{vol}(\{u_1^{(0)}, \dots, u_k^{(0)}, u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)} \leq \left(\prod_{i=1}^k \|u_i^\perp\| \right) \cdot \frac{\text{vol}(\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)}.$$

To bound $\text{vol}(\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)$, consider

$$\begin{aligned} \frac{\text{vol}(\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)} &= \frac{\text{vol}(\{a_{k+1, \tau_{k+1}} v_{\tau_{k+1}}, \dots, a_{\ell, \tau_\ell} v_{\tau_\ell}\} \cup Z)}{\text{vol}(S)} \\ &= \left(\prod_{i=k+1}^{\ell} |a_{i, \tau_i}| \right) \frac{\text{vol}(\{v_{\sigma(k+1)}, \dots, v_{\sigma(\ell)}\} \cup Z)}{\text{vol}(S)}. \end{aligned} \quad (\text{since } \tau_j = \sigma(j) \text{ for } k < j \leq \ell)$$

Using 4,

$$\frac{\text{vol}(\{v_{\sigma(k+1)}, \dots, v_{\sigma(\ell)}\} \cup Z)}{\text{vol}(S)} \leq \prod_{j \in [\ell] \setminus \{\sigma(k+1), \dots, \sigma(\ell)\}} \frac{1}{\|v_j^\perp\|} = \prod_{j \in \{\sigma(1), \dots, \sigma(k)\}} \frac{1}{\|v_j^\perp\|}.$$

Continuing the above chain of inequalities,

$$\begin{aligned} \frac{\text{vol}(\{u_{k+1}^{(\tau_{k+1})}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)} &\leq \left(\prod_{i=k+1}^{\ell} |a_{i, \tau_i}| \right) \cdot \prod_{j \in \{\sigma(1), \dots, \sigma(k)\}} \frac{1}{\|v_j^\perp\|} \cdot \prod_{i=1}^k \|u_i^\perp\| \\ &= \left(\prod_{i=k+1}^{\ell} |a_{i, \tau_i}| \right) \cdot \left(\prod_{i=1}^k \frac{\|u_i^\perp\|}{\|v_{\sigma(i)}^\perp\|} \right). \end{aligned}$$

Now consider any $\tau \in \mathcal{F}(\sigma)$, and let $I_\tau^0 := \{j : \tau(j) = 0\}$ and $I_\tau^c = [\ell] \setminus I_\tau^0$. Then following a similar chain of proof as above, we get

$$\frac{\text{vol}(\{u_1^{(\tau_1)}, \dots, u_\ell^{(\tau_\ell)}\} \cup Z)}{\text{vol}(S)} \leq \left(\prod_{i \in I_\tau^c} |a_{i\sigma(i)}| \right) \cdot \left(\prod_{i \in I_\tau^0} \frac{\|u_i^\perp\|}{\|v_{\sigma(i)}^\perp\|} \right).$$

Summing over all $\tau \in \mathcal{F}(\sigma)$,

$$\begin{aligned} \sum_{\tau \in \mathcal{F}(\sigma)} \frac{\text{vol}(\{u_1^{(\tau_1)}, \dots, u_r^{(\tau_r)}\} \cup Z)}{\text{vol}(S)} &\leq \sum_{\tau \in \mathcal{F}(\sigma)} \left(\prod_{i \in I_\tau^c} |a_{i\sigma(i)}| \right) \left(\prod_{i \in I_\tau^0} \frac{\|u_i^\perp\|}{\|v_{\sigma(i)}^\perp\|} \right) \\ &= \sum_{\tau \in \mathcal{F}(\sigma)} \left(\prod_{i \in I_\tau^c} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right) \left(\prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \right). \end{aligned}$$

Summing over all permutations,

$$r^{2r} \cdot \tilde{f}(r) \leq \frac{\text{vol}(T)}{\text{vol}(S)} \leq \sum_{\sigma \in \mathcal{S}_\ell} \sum_{\tau \in \mathcal{F}(\sigma)} \left(\prod_{i \in I_\tau^c} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right) \left(\prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \right).$$

The RHS is sum of $2^\ell \ell!$ positive terms. So, there exists some permutation $\sigma \in \mathcal{S}_\ell$ and $\tau \in \mathcal{F}(\sigma)$ such that

$$\left(\prod_{i \in I_\tau^c} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right) \left(\prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \right) \geq \frac{r^{2r} \cdot \tilde{f}(r)}{2^\ell \ell!} \geq \tilde{f}(r).$$

Let the cycle decomposition of $\sigma = \{\pi_1, \pi_2, \dots, \pi_k\}$ with $\pi_i = \{j_1, j_2, \dots, j_{x_i}\}$. For each vector u_i , we define symbols p_i indicating whether u_i is present as a perpendicular vector in τ , i.e., $p_i = \parallel$ if $i \in I_\tau^c$ and $p_i = \perp$ otherwise.

Then each cyclic permutation π_i corresponds to a cycle C_i in $\tilde{G}(S)$ given by

$$C_i = (u_{j_1}^{p_{j_1}} \rightarrow v_{j_2} \rightarrow u_{j_2}^{p_{j_2}} \rightarrow v_{j_3} \dots u_{j_{x_i}}^{p_{j_{x_i}}} \rightarrow v_{j_1} \rightarrow u_{j_1}^{p_{j_1}}),$$

and for every $i \in [k]$, $\left(\prod_{j \in I_\tau^c \cap \pi_i} e^{-\tilde{w}_0(u_j^\parallel, v_{\sigma(j)})} \right) \cdot \left(\prod_{j \in I_\tau^0 \cap \pi_i} e^{-\tilde{w}_0(u_j^\perp, v_{\sigma(j)})} \right) = e^{-\tilde{w}_0(C_i)}$ and therefore,

$$\left(\prod_{i \in I_\tau^c} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right) \cdot \left(\prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \right) = \prod_{i=1}^k e^{-\tilde{w}_0(C_i)} \geq \tilde{f}(r). \quad (21)$$

At least one of the C_i 's must be an \tilde{f} -violating cycle. If not, then $e^{-\tilde{w}_0(C_i)} \leq \tilde{f}(|C_i|/2)$ and

$$\left(\prod_{i \in I_\tau^c} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right) \left(\prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \right) = \prod_{i=1}^k e^{-\tilde{w}_0(C_i)} \leq \prod_{i=1}^k \tilde{f}(|C_i|/2) \leq \tilde{f}(r), \quad (22)$$

where the last inequality follows from $\sum_{i=1}^k |C_i| = 2\ell \leq 2r$. Equation (22) contradicts (21), so there exist i such that C_i is an \tilde{f} -violating cycle in $\tilde{G}(S)$. \square

B.2 Analysis of Stage 2

In this section, we prove that if the algorithm finds an \tilde{f} -violating cycle in Stage 2, then exchanging on this cycle increases the volume by a constant factor. However, since the algorithm fails to find a cycle in Stage 1, a cycle in Stage 2 must contain a perpendicular vertex. We first bound the number of vertices a minimal \tilde{f} -violating cycle can contain, and use this fact to prove that exchanging on such a cycle increases the volume.

Lemma 13 *If C is a minimal \tilde{f} -violating cycle in $\tilde{G}(S)$ and $\tilde{C} = \{u \in [n] \setminus S : u^\perp \text{ or } u^\parallel \in C\} \cup \{v \in S : v \in C\}$, then $S\Delta\tilde{C}$ is independent in \mathcal{M} .*

Proof Observation 2 and Lemma 9 imply that $S\Delta\tilde{C}$ is independent in $\tilde{\mathcal{M}}(S)$, and the second part of Observatio 2 then implies that $S\Delta\tilde{C}$ is independent in \mathcal{M} . \square

To make our later calculations possible, we first prove that any minimal \tilde{f} -violating cycle in $\tilde{G}(S)$ contains exactly one perpendicular vector. The following lemma is slightly more general than we need now, but will be useful later.

Lemma 14 *Let C , $|C| = 2y$ be a cycle in $\tilde{G}(S)$ such that there are no \tilde{f} -violating cycles with less than $2y$ hops in $\tilde{G}(S)$. Let C contain $k \geq 2$ perpendicular vertices $u_1^\perp, u_2^\perp, \dots, u_k^\perp$ such that the section of C from u_i^\perp to $u_{(i \bmod k)+1}^\perp$ has $2x_i$ hops for $i \in [k]$. Then*

$$e^{-\tilde{w}_0(C)} \leq \prod_{i=1}^k \tilde{f}(x_i).$$

Proof The proof is by induction on $k > 1$. Let the perpendicular vertices $u_1^\perp, u_2^\perp, \dots, u_k^\perp$ appear in order along around C . Let C' and C'' be the two cycles created by replacing the edges $u_1^\perp \rightarrow v_1$ and $u_2^\perp \rightarrow v_2$ with the new edges $u_1^\perp \rightarrow v_2$ and $u_2^\perp \rightarrow v_1$, so that C' is the cycle containing u_1^\perp . Note that C' has $2y - 2x_1 < 2y$ hops and C'' has $2x_1 < 2y$ hops. So by the hypothesis of the Lemma, both C' and C'' are not \tilde{f} -violating.

Additionally,

$$\tilde{w}_0(u_1^\perp \rightarrow v_2) + \tilde{w}_0(u_2^\perp \rightarrow v_1) = -\log \left(\frac{\|u_1^\perp\| \|u_2^\perp\|}{\|v_1\| \|v_2\|} \right) = \tilde{w}_0(u_1^\perp \rightarrow v_1) + \tilde{w}_0(u_2^\perp \rightarrow v_2),$$

and therefore

$$e^{-\tilde{w}_0(C)} = e^{-\tilde{w}_0(C') - \tilde{w}_0(C'')}. \quad (23)$$

When $k = 2$, both cycles C' and C'' contain exactly one perpendicular vertex, and since they are not \tilde{f} -violating, $e^{-\tilde{w}_0(C')} \leq \tilde{f}(2y - 2x_1)$ and $e^{-\tilde{w}_0(C'')} \leq \tilde{f}(2x_1)$.

Using equation (23), we conclude that $e^{-\tilde{w}_0(C)} = e^{-\tilde{w}_0(C') - \tilde{w}_0(C'')} \leq \tilde{f}(2y - 2x_1) \cdot \tilde{f}(x_1)$, as desired.

When $k > 2$ the cycle C' has $k - 1 \geq 2$ perpendicular vertices since it no longer contains u_1^\perp . Thus, the induction hypothesis implies that $e^{-\tilde{w}_0(C')} \leq \prod_{i=2}^k \tilde{f}(x_i)$. Since C'' is not \tilde{f} -violating, it satisfies $e^{-\tilde{w}_0(C'')} \leq \tilde{f}(2x_1)$.

Again using equation (23), we conclude that $e^{-\tilde{w}_0(C)} = e^{-\tilde{w}_0(C') - \tilde{w}_0(C'')} \leq \prod_{i=1}^k \tilde{f}(x_i)$, as desired. \square

When $y = \sum_{i=1}^k x_i$, we know that $\tilde{f}(y) \geq \prod_{i=1}^k \tilde{f}(x_i)$. Thus, we obtain the following corollary:

Corollary 1 *If C is a minimal \tilde{f} -violating cycle in $\tilde{G}(S)$, then C contains exactly one perpendicular vector.*

Now we will prove that for a minimal \tilde{f} -violating cycle C , the volume of $S\Delta C$ is strictly larger than $\text{vol}(S)$.

Lemma 15 *If C is a minimal \tilde{f} -violating cycle in $\tilde{G}(S)$, then $\text{vol}(S\Delta C) \geq 2 \cdot \text{vol}(S)$.*

Proof If $|C| = 2$, then let $C = (u^\perp \rightarrow v)$. Since C is an \tilde{f} -violating cycle, $e^{-\tilde{w}_0(C)} = \frac{\|u^\perp\|}{\|v^\perp\|} > \tilde{f}(1) \geq 2$. By Lemma 10,

$$\frac{\text{vol}(S - v + u)}{\text{vol}(S)} = \sqrt{a_{uv}^2 + \frac{\|u^\perp\|^2}{\|v^\perp\|^2}} \geq e^{-\tilde{w}_0(C)} \geq 2.$$

This concludes the proof when C has 2 arcs.

Now consider the case when C has at least 4 arcs. By Corollary 1, we can assume that C contains exactly one perpendicular vertex. Let $C = (u_1^\perp \rightarrow v_1 \rightarrow u_2^\perp \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\perp \rightarrow v_\ell \rightarrow u_1^\perp)$, where $\ell \geq 2$, $v_i \in S$, and $v_i \rightarrow u_{i+1}$ is an arc in $G(S)$.

Define $T := S\Delta C$, $X := C \setminus S = \{u_1, u_2, \dots, u_\ell\}$, and $Y := S \cap C = \{v_1, v_2, \dots, v_\ell\}$, and let $S \setminus Y = \{v_{\ell+1}, \dots, v_r\}$. Let \mathbf{P} denote the projection matrix orthogonal to the span of $S \setminus Y$.

Then

$$\frac{\text{vol}(T)}{\text{vol}(S)} = \frac{\text{vol}(\{S \setminus Y\} \cup X)}{\text{vol}(\{S \setminus Y\} \cup Y)} = \frac{\text{vol}(u_1, u_2, \dots, u_\ell, v_{\ell+1}, \dots, v_r)}{\text{vol}(v_1, v_2, \dots, v_\ell, v_{\ell+1}, \dots, v_r)}.$$

Note that for any set of vectors $\{x_1, \dots, x_\ell\}$,

$$\text{vol}(x_1, x_2, \dots, x_\ell, v_{\ell+1}, \dots, v_r) = \text{vol}(\mathbf{P}x_1, \mathbf{P}x_2, \dots, \mathbf{P}x_\ell) \cdot \text{vol}(v_{\ell+1}, \dots, v_r).$$

Applying this to the sets X and Y , we get

$$\frac{\text{vol}(T)}{\text{vol}(S)} = \frac{\text{vol}(\mathbf{P}u_1, \mathbf{P}u_2, \dots, \mathbf{P}u_\ell)}{\text{vol}(\mathbf{P}v_1, \mathbf{P}v_2, \dots, \mathbf{P}v_\ell)} = \frac{\text{vol}(\mathbf{P}X)}{\text{vol}(\mathbf{P}Y)}.$$

By definition, $u_i = \sum_{j=1}^r a_{i,j}v_j + u_i^\perp$. Taking the projection of u_i orthogonal to $\text{Span}(S \setminus Y)$, we get $\mathbf{P}u_i = \sum_{j=1}^\ell a_{i,j}\mathbf{P}v_j + u_i^\perp$, since $Pv_j = 0$ for all $j > \ell$. So we can decompose each vector $\mathbf{P}u_i$ into a sum of $\ell + 1$ vectors, i.e., $\mathbf{P}u_i = \sum_{j=0}^\ell u_i^{(j)}$ where

- $u_i^{(0)} := u_i^\perp$ for all i , and
- $u_i^{(j)} := a_{i,j}\mathbf{P}v_j$.

Note that for any $i_1, i_2 \in [\ell]$, the vectors $u_{i_1}^{(j)}$ and $u_{i_2}^{(j)}$ are linearly dependent for any $j > 0$.

Let I_C denote the tuple $(0, 2, 3, \dots, \ell)$. Using Observation 3, we can lower bound $\text{vol}(\mathbf{P}X)$ as

$$\text{vol}(\mathbf{P}X) \geq \text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)}) - \sum_{\substack{i_1, i_2, \dots, i_\ell \in \{0, 1, \dots, \ell\} \\ (i_1, i_2, \dots, i_\ell) \neq I_C}} \text{vol}(u_1^{(i_1)}, u_2^{(i_2)}, \dots, u_\ell^{(i_\ell)}). \quad (24)$$

For any permutation $\sigma \in \mathcal{S}_\ell$, define $\mathcal{F}(\sigma) := \{(i_1, i_2, \dots, i_\ell) : i_j \in \{\sigma(j), 0\}\}$. Since $u_{i_1}^{(j)}$ and $u_{i_2}^{(j)}$ are linearly dependent, we can rewrite (24) as

$$\begin{aligned} \text{vol}(\mathbf{P}X) &\geq \text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)}) - \sum_{\tau \in \cup_{\sigma \in \mathcal{S}_\ell} \mathcal{F}(\sigma) \setminus I_C} \text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)}) \\ &\geq \text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)}) - \sum_{\sigma \in \mathcal{S}_\ell} \sum_{\tau \in \mathcal{F}(\sigma) \setminus I_C} \text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)}). \end{aligned} \quad (25)$$

For a fixed $\sigma \neq id_\ell$ and some $\tau \in \mathcal{F}(\sigma)$, let $I_\tau^0 := \{j \in [\ell] : \tau(j) = 0\}$ and $I_\tau^\sigma := [\ell] \setminus I_\tau^0$. Also, let $X_\tau = \{v_{\sigma(i)} : i \in I_\tau^\sigma\}$. Then

$$\begin{aligned} \frac{\text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)})}{\text{vol}(\mathbf{P}Y)} &= \frac{\text{vol}\left(\bigcup_{i \in I_\tau^0} \{u_i^\perp\}\right) \cdot \text{vol}\left(\bigcup_{i \in I_\tau^\sigma} \{a_{i, \sigma(i)} \mathbf{P}v_{\sigma(i)}\}\right)}{\text{vol}(\mathbf{P}Y)} \\ &\leq \prod_{i \in I_\tau^0} \|u_i^\perp\| \cdot \prod_{i \in I_\tau^\sigma} |a_{i, \sigma(i)}| \cdot \frac{\text{vol}\left(\bigcup_{i \in I_\tau^\sigma} \{\mathbf{P}v_{\sigma(i)}\}\right)}{\text{vol}(\mathbf{P}Y)} \\ &\leq \prod_{i \in I_\tau^0} \|u_i^\perp\| \cdot \prod_{i \in I_\tau^\sigma} |a_{i, \sigma(i)}| \cdot \frac{1}{\prod_{i \in I_\tau^0} \|v_{\sigma(i)}^\perp\|} \quad (\text{from Observation 4}) \\ &= \prod_{i \in I_\tau^0} \frac{\|u_i\|}{\|v_{\sigma(i)}^\perp\|} \cdot \prod_{i \in I_\tau^\sigma} |a_{i, \sigma(i)}| = \prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \cdot \prod_{i \in I_\tau^\sigma} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})}. \end{aligned}$$

Summing over all tuples $\tau \in \mathcal{F}(\sigma)$, we get

$$\begin{aligned} \sum_{\tau \in \mathcal{F}(\sigma)} \frac{\text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)})}{\text{vol}(\mathbf{P}Y)} &\leq \prod_{i \in I_\tau^0} e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} \cdot \prod_{i \in I_\tau^\sigma} e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \\ &= \prod_{i=1}^\ell \left(e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} + e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})} \right). \end{aligned} \quad (26)$$

Let $W_C \in \mathbb{R}^{\ell \times \ell}$ be a matrix with $[W_C]_{i,j} = e^{-\tilde{w}_0(u_i^\perp, v_{\sigma(i)})} + e^{-\tilde{w}_0(u_i^\parallel, v_{\sigma(i)})}$. Then we can rewrite (26) as

$$\sum_{\tau \in \mathcal{F}(\sigma)} \frac{\text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)})}{\text{vol}(\mathbf{P}Y)} \leq \prod_{i=1}^\ell [W_C]_{i, \sigma(i)}. \quad (27)$$

Similarly, the sum of tuples for the identity permutation id_ℓ gives

$$\frac{\text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{P}Y)} + \sum_{\tau \in \mathcal{F}(id_\ell) \setminus I_C} \frac{\text{vol}(u_1^{(\tau_1)}, u_2^{(\tau_2)}, \dots, u_\ell^{(\tau_\ell)})}{\text{vol}(\mathbf{P}X)} \leq \prod_{i=1}^\ell [W_C]_{i,i}. \quad (28)$$

Plugging equation (27) and equation (28) in (25), we get

$$\begin{aligned}
\frac{\text{vol}(\mathbf{PX})}{\text{vol}(\mathbf{PY})} &\geq \frac{\text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{PY})} - \sum_{\sigma \neq \text{id}} \prod_{i=1}^{\ell} [W_C]_{i, \sigma(i)} - \left(\prod_{i=1}^{\ell} [W_C]_{i,i} - \frac{\text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{PY})} \right) \\
&= \frac{2 \text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{PY})} - \sum_{\sigma \neq \text{id}} \prod_{i=1}^{\ell} [W_C]_{i, \sigma(i)} - \prod_{i=1}^{\ell} [W_C]_{i,i} \\
&= \frac{2 \text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{PY})} - \text{perm}(W_C). \tag{29}
\end{aligned}$$

From Lemma 16, we have

$$\text{perm}(W_C) \leq 1.56 \cdot \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}|. \tag{30}$$

Note that

$$\frac{\text{vol}(u_1^{(0)}, u_2^{(2)}, \dots, u_\ell^{(\ell)})}{\text{vol}(\mathbf{PY})} = \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}|. \tag{31}$$

Inserting the bounds from (30) and (31) in (29) gives

$$\frac{\text{vol}(\mathbf{PX})}{\text{vol}(\mathbf{PY})} \geq \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}| \cdot (2 - 1.56) = 0.44 \cdot \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}|.$$

Since C is an \tilde{f} -Violating cycle, $e^{-\tilde{w}_0(C)} = \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}| \geq \tilde{f}(\ell)$. Therefore,

$$\frac{\text{vol}(S\Delta C)}{\text{vol}(S)} = \frac{\text{vol}(\mathbf{PX})}{\text{vol}(\mathbf{PY})} \geq 0.44 \cdot \tilde{f}(\ell) \geq 2.$$

□

B.3 Miscellaneous Lemmas

Lemma 16 *Let $C = (v_0 \rightarrow u_1^\perp \rightarrow v_1 \rightarrow u_2^\parallel \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\parallel \rightarrow v_0)$ be a minimal \tilde{f} -violating cycle in $\tilde{G}(S)$ with $\ell \geq 2$ and let W_C be a matrix with $[W_C]_{i,j} = |a_{i,j}| + \frac{\|u_i^\perp\|}{\|v_j^\perp\|}$, then*

$$\text{perm}(W_C) \leq 1.56 \cdot \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}|.$$

Proof Define $z_1 := \frac{\|u_1^\perp\|}{\|v_1^\perp\|}$, $p_1 = \perp$, and $z_i := |a_{i,i}|$, $p_i = \parallel$ for all $i \geq 2$.

We show upper bounds on the absolute value of each entry of W_C as a function of z_i 's. Consider the i, j -th entry of W_C . For $i > j$, let $C_{i,j}^\perp := (u_i^\perp \rightarrow v_j \rightarrow u_{j+1}^{p_{j+1}} \rightarrow v_{j+1} \rightarrow \dots \rightarrow v_{i-1} \rightarrow u_i^\perp)$. $C_{i,j}^\perp$ is a cycle of with $2(i-j)$ hops and $V(C_{i,j}^\perp) \subset V(C)$. C being a minimal \tilde{f} -violating cycle implies that

$C_{i,j}^\perp$ is not an \tilde{f} -violating cycle. Therefore, $e^{-\tilde{w}_0(C_{i,j}^\perp)} = \frac{\|u_i^\perp\|}{\|v_j^\perp\|} \cdot \prod_{s=j+1}^{i-1} z_s < \tilde{f}(i-j)$. This implies

$$\frac{\|u_i^\perp\|}{\|v_j^\perp\|} < \frac{\tilde{f}(i-j)}{\prod_{s=j+1}^{i-1} z_s}. \quad (32)$$

Similarly for $j = \ell$, we have $\frac{\|u_i^\perp\|}{\|v_0^\perp\|} < \frac{\tilde{f}(i)}{\prod_{s=1}^{i-1} z_s}$.

Similarly, let $C_{i,j}^\parallel := (u_i^\parallel \rightarrow v_j \rightarrow u_{j+1}^{p_{j+1}} \rightarrow v_{j+1} \rightarrow \dots \rightarrow v_{i-1} \rightarrow u_i^\parallel)$. Again, using the fact that $C_{i,j}^\parallel$ is not \tilde{f} -Violating we get

$$|a_{i,j}| < \frac{\tilde{f}(i-j)}{\prod_{s=j+1}^{i-1} z_s}, \quad (33)$$

for any $i > j$ and for $j = \ell$, we have $a_{i,0} < \frac{\tilde{f}(i)}{\prod_{s=1}^{i-1} z_s}$.

For $i < j < \ell$, let $C_{i,j}^\perp := (v_0 \rightarrow u_1^{p_1} \rightarrow v_1 \rightarrow \dots \rightarrow u_i^\perp \rightarrow v_j \rightarrow \dots \rightarrow u_\ell^{p_\ell} \rightarrow v_0)$. Again, $C_{i,j}^\perp$ is a cycle with $2(\ell - j + i)$ hops and which is not \tilde{f} -violating. Therefore,

$$\frac{\|u_i^\perp\|}{\|v_j^\perp\|} \cdot \prod_{s=1}^{i-1} z_s \cdot \prod_{s=j+1}^{\ell} z_s < \tilde{f}(\ell - j + i). \quad (34)$$

Since C is an \tilde{f} -violating cycle, we also have

$$e^{-\tilde{w}_0(C)} = \prod_{s=1}^{\ell} z_s > \tilde{f}(\ell). \quad (35)$$

Combining (34) and (35) gives

$$\frac{\|u_i^\perp\|}{\|v_j^\perp\|} < \frac{\tilde{f}(\ell - j + i)}{\tilde{f}(\ell)} \cdot \prod_{s=i}^j z_s.$$

Similarly, for $i < j < \ell$,

$$|a_{i,j}| < \frac{\tilde{f}(\ell - j + i)}{\tilde{f}(\ell)} \cdot \prod_{s=i}^j z_s.$$

Define $x_1 := (|a_{1,1}| + \frac{\|u_1^\perp\|}{\|v_1^\perp\|}) / \frac{\|u_1^\perp\|}{\|v_1^\perp\|}$ and $x_i := (|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|}) / |a_{i,i}|$ for $i \geq 2$. Then $x_i \geq 1$ and the i -th diagonal entry of W_C is $x_i \cdot z_i$.

Let B_ℓ be the matrix obtained by applying the following operations to W_C

- Multiply the last column by z_1 and for $j > 1$, divide the j -th column by $\prod_{s=2}^j z_s$
- Divide the first row by z_1 and for $i > 1$, multiply the i -th row by $\prod_{s=2}^{i-1} z_s$
- Divide the last column by $\prod_{i=1}^{\ell} z_i$.

- Divide the j -th column by x_j .

Then $|\text{perm}(W_C)| = (\prod_{i=1}^{\ell} x_i z_i) \cdot |\text{perm}(B_\ell)|$, and B_ℓ satisfies the following properties:

- $b_{i,i} = 1$ for all $i \in [\ell]$,
- $|b_{i,j}| \leq 2 \cdot \tilde{f}(i-j)/x_j \leq 2 \cdot \tilde{f}(i-j)$ for all $j < i \leq \ell$, and
- $|b_{i,j}| \leq 2 \cdot \tilde{f}(\ell-j+i)/(\tilde{f}(\ell)x_j) \leq 2 \cdot \tilde{f}(\ell-j+i)/\tilde{f}(\ell)$ for all $i < j \leq \ell$.

If $\ell = 1$, $\text{perm}(B_\ell) = 1$ and for $\ell \geq 2$, Lemma 18 gives $\text{perm}(B_\ell) \leq 1.3$. Therefore, we have

$$\text{perm}(W_C) \leq 1.3 \cdot \prod_{i=1}^{\ell} x_i z_i = 1.3 \cdot \prod_{i=1}^{\ell} \left(|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|} \right) \leq 1.3 \cdot 1.2 \cdot \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}| = 1.56 \cdot e^{-\tilde{w}_0(C)},$$

where the last inequality follows from Lemma 17. \square

Lemma 17 *If $C = (v_0 \rightarrow u_1^\perp \rightarrow v_1 \rightarrow u_2^\parallel \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\parallel \rightarrow v_0)$ is a minimal \tilde{f} -violating cycle with $\ell \geq 2$, then*

$$\prod_{i=1}^{\ell} \left(|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|} \right) \leq 1.2 \cdot e^{-\tilde{w}_0(C)}. \quad (36)$$

Proof Since C is an \tilde{f} -violating cycle,

$$e^{-\tilde{w}_0(C)} = \frac{\|u_1^\perp\|}{\|v_1^\perp\|} \cdot \prod_{i=2}^{\ell} |a_{i,i}| \geq \tilde{f}(\ell). \quad (37)$$

Define $x_{i,0} := \frac{\|u_i^\perp\|}{\|v_i^\perp\|}$ and $x_{i,1} := |a_{i,i}|$ for all $i \in [\ell]$. Then we can decompose the LHS of (36) as follows:

$$\prod_{i=1}^{\ell} \left(|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|} \right) = \sum_{Z \subseteq [\ell]} \prod_{i \in Z} x_{i,0} \cdot \prod_{i \notin Z} x_{i,1} =: \sum_{Z \subseteq [\ell]} w(Z),$$

We now upper bound $w(Z)$ for all $Z \neq \{1\}$ based on the cardinality of Z . For the empty set, i.e., $|Z| = 0$, since Algorithm 4 did not return a cycle in Stage 1, $w(Z) = \prod_{i=1}^{\ell} |a_{i,i}| = e^{-w_0(C)} \leq f(\ell)$.

Consider a subset Z with $|Z| = k$ where $k \geq 2$. We define a cycle C_Z such that C_Z contains the same arcs as C but a vertex $u_i \in C \setminus S$ is present as a perpendicular vertex in C_Z if and only if $u \in Z$. Then $w(Z) = e^{-\tilde{w}_0(C_Z)}$. Let the distance between successive perpendicular vertices in C_Z be b_1, b_2, \dots, b_k . Then $\sum_{i=1}^k b_i = \ell$ and by Lemma 14, $w(Z) = e^{-\tilde{w}_0(C_Z)} \leq \prod_{i=1}^k \tilde{f}(b_i)$.

Since C_Z has the same structure as C , we can completely characterize C_Z by the first vertex among u_1, u_2, \dots, u_ℓ which is perpendicular, and the distances between successive perpendicular vertices in C_Z , i.e., b_1, b_2, \dots, b_k . There are only ℓ options for the first perpendicular vertex. Therefore,

$$\sum_{Z: |Z|=k} w(Z) \leq \ell \cdot \sum_{\substack{\sum_{j=1}^k b_j = \ell \\ b_1, b_2, \dots, b_k \geq 1}} \prod_{i=1}^k \tilde{f}(b_i) \leq \ell \cdot \binom{\ell-1}{k-1} \cdot \tilde{f}(\ell-k+1) \leq \frac{\tilde{f}(\ell)}{\ell^8}. \quad (38)$$

For a subset with one element, i.e., $Z = \{x\}$, $x \neq 1$, $-\log w(Z)$ is the same as the weight of the cycle

$$C_x := (v_0 \rightarrow u_1^\perp \rightarrow v_1 \rightarrow \dots \rightarrow u_x^\perp \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\perp \rightarrow v_0).$$

Combining C_x with C , we get two cycles

$$\begin{aligned} C_x^{(1)} &= (v_0 \rightarrow u_1^\perp \rightarrow v_1 \rightarrow \dots \rightarrow u_x^\perp \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\perp \rightarrow v_0) \\ C_x^{(2)} &= (v_0 \rightarrow u_1^\parallel \rightarrow v_1 \rightarrow \dots \rightarrow u_x^\parallel \rightarrow v_2 \rightarrow \dots \rightarrow u_\ell^\parallel \rightarrow v_0), \end{aligned}$$

such that $\tilde{w}_0(C) \cdot \tilde{w}_0(C_x) = \tilde{w}_0(C_x^{(1)}) \cdot \tilde{w}_0(C_x^{(2)})$. By Lemma 14, we know that $e^{-\tilde{w}_0(C_x^{(1)})} \leq \tilde{f}(x-1)\tilde{f}(\ell-x+1)$. Also, since $C_x^{(2)}$ only contains parallel vectors, $\tilde{w}_0(C_x^{(2)}) = w_0(C_x^{(2)})$; and since $C_x^{(2)}$ is not f -violating, $e^{-\tilde{w}_0(C_x^{(2)})} \leq f(\ell)$. Therefore, $e^{-\tilde{w}_0(C) - \tilde{w}_0(C_x)} \leq \tilde{f}(x-1)\tilde{f}(\ell-x+1) \cdot f(\ell)$. C being \tilde{f} -violating implies $e^{-\tilde{w}_0(C)} \geq \tilde{f}(\ell)$, and therefore $e^{-\tilde{w}_0(C)} \leq \tilde{f}(x-1)\tilde{f}(\ell-x+1)f(\ell)/\tilde{f}(\ell)$.

Summing over all choices of x , we get

$$\sum_{Z, |Z|=1} w(Z) = \sum_{x=2}^{\ell} e^{-\tilde{w}_0(C_x)} \leq \sum_{x=2}^{\ell} \frac{\tilde{f}(x-1)\tilde{f}(\ell-x+1)f(\ell)}{\tilde{f}(\ell)} \leq \frac{4f(\ell)}{\ell}. \quad (39)$$

Combining (37), (38), and (39), we get

$$\begin{aligned} \prod_{i=1}^{\ell} \left(|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|} \right) &= \sum_{Z \subseteq [\ell]} w(Z) \leq w(\emptyset) + \sum_{Z: |Z|=1} w(Z) + \sum_{Z: |Z| \geq 2} w(Z) \\ &\leq f(\ell) + e^{-\tilde{w}_0(C)} + \frac{4f(\ell)}{\ell} + \frac{\tilde{f}(\ell)}{\ell^8}. \end{aligned}$$

Since $e^{-\tilde{w}_0(C)} \geq \tilde{f}(\ell) \geq (\ell!)^4 \cdot f(\ell)$,

$$\prod_{i=1}^{\ell} \left(|a_{i,i}| + \frac{\|u_i^\perp\|}{\|v_i^\perp\|} \right) \leq e^{-\tilde{w}_0(C)} \left(1 + \frac{3}{(\ell!)^4} + \frac{1}{\ell^8} \right) \leq 1.2 \cdot e^{-\tilde{w}_0(C)}.$$

□

Lemma 18 Let $B_\ell \in \mathbb{R}^{\ell \times \ell}$ satisfy the following properties:

- $b_{i,i} = 1$ for all $i \in [\ell]$,
- $0 \leq b_{i,j} \leq 2 \cdot \tilde{f}(i-j)$ for all $j < i \leq \ell$, and
- $0 \leq b_{i,j} \leq 2 \cdot \tilde{f}(\ell-j+i)/\tilde{f}(\ell)$ for all $i < j \leq \ell$.

Then the permanent of B_ℓ is at most 1.13.

Proof For $\ell = 2$, $\text{perm}(B_2) = 1 + b_{1,2} \cdot b_{2,1} \leq 1 + \frac{4\tilde{f}(1)^2}{\tilde{f}(2)} < 1.1$.

Now, consider the case when $\ell \geq 3$. Let id_ℓ denote the identity permutation on ℓ elements. Expanding the permanent of B_ℓ gives

$$\text{perm}(B_\ell) = \sum_{\sigma \in \mathcal{S}_\ell} \prod_{i=1}^{\ell} b_{i,\sigma(i)} = 1 + \sum_{\sigma \in \mathcal{S}_\ell \setminus \{id_\ell\}} \prod_{i=1}^{\ell} b_{i,\sigma(i)}. \quad (40)$$

We categorize all permutations in $\mathcal{S}_\ell \setminus \{id_\ell\}$ based on the number of fixed points and the number of exceedances. The set of fixed points of a permutation $\sigma \in \mathcal{S}_\ell$ is defined as $\{i \in [\ell] : \sigma(i) = i\}$ and the exceedance of σ is defined to be the number of indices i such that $\sigma(i) > i$ (for more details, see Remark 1 and Lemma 19). Let $\mathcal{S}_\ell(n, k)$ denote the subset of \mathcal{S}_ℓ with $\ell - n$ fixed points and k exceedances. Then

$$|\mathcal{S}_\ell(n, k)| = \binom{\ell}{n} \cdot T(n, k),$$

where $T(n, k)$ is the derangement number defined in Remark 1.

Since all permutations in $\mathcal{S}_\ell \setminus \{id_\ell\}$ have at most $\ell - 2$ fixed points and at least 1 exceedance, we can further expand (40) as

$$\text{perm}(B_\ell) = 1 + \sum_{n=2}^{\ell} \sum_{k=1}^{n-1} \sum_{\sigma \in \mathcal{S}_\ell(n, k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)}. \quad (41)$$

For a permutation $\sigma \in \mathcal{S}_\ell(n, k)$,

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} = \prod_{i>\sigma(i)} b_{i,\sigma(i)} \cdot \prod_{i<\sigma(i)} b_{i,\sigma(i)} \leq \prod_{i>\sigma(i)} 2\tilde{f}(i - \sigma(i)) \cdot \prod_{i<\sigma(i)} \frac{2\tilde{f}(\ell - \sigma(i) + i)}{\tilde{f}(\ell)}.$$

where the last inequality follows from the hypothesis of the Lemma.

Since $\sum_{i=1}^{\ell} i - \sigma(i) = 0$ for any permutation σ , $\sum_{i>\sigma(i)} i - \sigma(i) + \sum_{i<\sigma(i)} \ell - \sigma(i) + i = \ell \cdot |\{i : \sigma(i) > i\}| = \ell \cdot k$ for any $\sigma \in \mathcal{S}_\ell(n, k)$. Therefore, if $\sigma \in \mathcal{S}_\ell(n, k)$, then there exist integers $1 \leq x_1, x_2, \dots, x_n \leq n - 1$ with $\sum_{i=1}^n x_i = \ell \cdot k$, such that

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq 2^n \cdot \frac{\prod_{i=1}^n \tilde{f}(x_i)}{\tilde{f}(\ell)^k}.$$

Since \tilde{f} satisfies $\tilde{f}(a + b) \geq \tilde{f}(a) \cdot \tilde{f}(b)$, under the constraints on x_i 's, for any $k \geq n/2$,

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq 2^n \cdot \frac{\tilde{f}(\ell - 1)^k \cdot \tilde{f}(2k - n + 1) \cdot \tilde{f}(1)^{n-k-1}}{\tilde{f}(\ell)^k},$$

and for $k < n/2$,

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq 2^n \cdot \frac{\tilde{f}(\ell - 1)^{k-1} \cdot \tilde{f}(\ell - n + 2k - 1) \cdot \tilde{f}(1)^{n-k}}{\tilde{f}(\ell)^k}.$$

Using the definition of \tilde{f} , for any $k \geq n/2$,

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq 2^{2n-k-1} \cdot \frac{((2k-n+1)!)^{11}}{\ell^{11k}}, \quad (42)$$

and for $k < n/2$,

$$\prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq 2^{2n-k} \cdot \frac{1}{\ell^{11(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^{11}}. \quad (43)$$

For some k and n with $k < n/2$, summing over all permutations in $\mathcal{S}_{\ell}(n, k)$ gives

$$\sum_{\sigma \in \mathcal{S}_{\ell}(n, k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \binom{\ell}{n} \cdot T(n, k) \cdot 2^{2n-k} \cdot \frac{1}{\ell^{11(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^{11}}. \quad (44)$$

We will bound the three terms of equation (44), namely $\binom{\ell}{n}$, $T(n, k)$, and 2^{2n-k} separately.

Expanding the first term, we get

$$\binom{\ell}{n} \cdot \frac{1}{\ell^{(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))} = \frac{\ell \cdot (\ell-1) \dots (\ell-n+1)}{n! \cdot \ell^{(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))} < \frac{1}{n! \cdot \ell^{k-1}}. \quad (45)$$

For the third term, note that $k < n/2$ implies that $2^{2n-k} < 2^{3n-3k}$, and since $k \geq 1$, $\ell - n + 2k > 2$. Using these two facts, we get

$$2^{2n-k} \cdot \frac{1}{\ell^{3(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^3} < \frac{2^{3(n-k)}}{(\ell-n+2k)^{3(n-k)}} < 1. \quad (46)$$

Plugging (45) and (46) in (44), we get

$$\sum_{\sigma \in \mathcal{S}_{\ell}(n, k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{n! \cdot \ell^{k-1}} \cdot T(n, k) \cdot \frac{1}{\ell^{7(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^7}. \quad (47)$$

For $k = 1$, $T(n, k) = 1$ and therefore

$$\sum_{\sigma \in \mathcal{S}_{\ell}(n, k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{n!} \cdot \frac{1}{(\ell \cdot (\ell-1) \dots (\ell-n+2))^7}. \quad (48)$$

For any $2 \leq k < n/2$, using Lemma 19, we have $T(n, k) \leq (2k+3)^n$. Since $k \geq 2$, $2k+3 \leq 2 \cdot 2k$, and therefore $T(n, k) \leq (2 \cdot 2k)^{n+2}$. Plugging this in (47), we get

$$\sum_{\sigma \in \mathcal{S}_{\ell}(n, k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{n! \cdot \ell^{k-1}} \cdot (2k)^n \cdot 2^n \cdot \frac{1}{\ell^{7(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^7}. \quad (49)$$

Moreover $k < n/2$ implies that $n - k > n/2$, and as a result $n \leq 2(n - k)$ and $2^n \cdot (2k)^n \leq$

$2^{2(n-k)} \cdot (2k)^{2(n-k)}$. Therefore,

$$\begin{aligned}
& 2^n \cdot (2k)^n \cdot \frac{1}{\ell^{4(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^4} \\
& < 2^{2(n-k)} \cdot (2k)^{2(n-k)} \cdot \frac{1}{\ell^{4(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^4} \\
& < \frac{(2)^{2(n-k)}}{(\ell-n+2k)^{2(n-k)}} \cdot \frac{(2k)^{2(n-k)}}{(\ell-n+2k)^{2(n-k)}} < 1,
\end{aligned} \tag{50}$$

where the last inequality follows from $2k \leq \ell - n + 2k$ and $2 \leq \ell - n + 2k$.

Combining (49) and (50), we have for any $2 \leq k < n/2$,

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{n! \cdot \ell^{k-1}} \cdot \frac{1}{\ell^{3(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^3}. \tag{51}$$

For a fixed k and n with $k \geq n/2$, summing over all permutations in $\mathcal{S}_\ell(n,k)$,

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \binom{\ell}{n} \cdot T(n,k) \cdot 2^{2n-k-1} \cdot \frac{((2k-n+1)!)^{11}}{\ell^{11k}}. \tag{52}$$

We will again bound the three terms, namely $\binom{\ell}{n}$, $T(n,k)$, and 2^{2n-k-1} separately.

For the first term, since $n/2 \leq k$, $2n-k-1 \leq 3k-1$, and therefore

$$2^{2n-k-1} \cdot \frac{((2k-n+1)!)^3}{\ell^{3k}} \leq 2^{3k-1} \cdot \frac{((2k-n+1)!)^3}{\ell^{3k}}.$$

Since $2k-n+1 \leq k$, and $k+1 \leq \ell$,

$$2^{3k-1} \cdot \frac{((2k-n+1)!)^3}{\ell^{3k}} \leq \frac{1}{2} \cdot \frac{(2^k k!)^3}{(k+1)^{3k}}.$$

For $k = 1, 2, 3, 4, 5$, $\frac{(2^k k!)^3}{(k+1)^{3k}} \leq 1$. For $k \geq 6$, $2^k k! \leq k^k$. Therefore,

$$2^{2n-k-1} \cdot \frac{((2k-n+1)!)^3}{\ell^{3k}} \leq \frac{1}{2}. \tag{53}$$

Expanding the third term,

$$\begin{aligned}
\binom{\ell}{n} \cdot \frac{((2k-n+1)!)^2}{\ell^{2k}} &= \frac{\ell \cdot (\ell-1) \dots (\ell-n+1)}{n!} \cdot \frac{((2k-n+1)!)^2}{\ell^{2k}} \\
&\leq \frac{1}{n!} \cdot \frac{((2k-n+1)!)^2}{\ell^{2k-n}} = \frac{1}{\ell(n-1)!} \cdot \frac{((2k-n+1)!)^2}{n\ell^{2k-n-1}} \leq \frac{(2k-n+1)!}{\ell(n-1)!}.
\end{aligned}$$

Since $k+1 \leq n$, we have $2k-n+1 \leq n-1$, and therefore

$$\binom{\ell}{n} \cdot \frac{((2k-n+1)!)^2}{\ell^{2k}} \leq \frac{(2k-n+1)!}{\ell(n-1)!} \leq \frac{1}{\ell}. \tag{54}$$

Plugging in (53) and (54) in (52),

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{2\ell} \cdot T(n,k) \cdot \frac{((2k-n+1)!)^6}{\ell^{6k}}. \quad (55)$$

Since $T(n, n-1) = 1$, for $k = n-1$, we have

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{2\ell} \cdot \frac{((n-1)!)^6}{\ell^{6(n-1)}}. \quad (56)$$

By Lemma 19, for $k < n-1$, $T(n,k) \leq (2n-2k+5)^n$ for $k \geq n/2$, and

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{2\ell} \cdot (2n-2k+5)^n \cdot \frac{((2k-n+1)!)^6}{\ell^{6k}}.$$

Let $n = 2k - z$, then

$$\begin{aligned} (2n-2k+5)^n \cdot \frac{((2k-n+1)!)^6}{\ell^{6k}} &= (2k-2z+5)^{2k-z} \cdot \frac{((z+1)!)^6}{\ell^{6k}} \\ &\leq (2k-z+5)^{2k-z} \cdot \frac{((z+1)!)^6}{\ell^{6k}}. \end{aligned}$$

Taking derivative of $\frac{(2k-z+5)^{2k-z+2}}{\ell^{6k}}$ with respect to k ,

$$\frac{(2k-z+5)^{2k-z}}{\ell^{6k}} \left(2 \cdot \log(2k-z+5) + 2 \cdot \frac{2k-z}{2k-z+5} - 6 \log(\ell) \right) \leq 0.$$

Therefore $\frac{(2k-z+5)^{2k-z+2}}{\ell^{6k}}$ is a non-increasing function of k . Since $n = 2k - z \geq 1$, k satisfies $2k \geq z + 1$.

So $\frac{(2k-z+5)^{2k-z}}{\ell^{6k}}$ is maximized when $2k = z + 1$. Therefore,

$$(2k-z+5)^{2k-z} \cdot \frac{((z+1)!)^6}{\ell^{6k}} \leq 6 \cdot \frac{((z+1)!)^6}{\ell^{6k}} = 6 \cdot \frac{((2k-n+1)!)^6}{\ell^{6k}} \leq 6 \cdot \frac{1}{\ell^{6(n-k-1)}},$$

where the last inequality follows from $2k - n + 1 \leq \ell$. Plugging this bound in (55) gives

$$\sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \leq \frac{1}{2\ell} \cdot \frac{6}{\ell^{6(n-k-1)}}. \quad (57)$$

Plugging in (48), (51), (56), and (57) into (41),

$$\begin{aligned} \text{perm}(B_\ell) &= 1 + \sum_{n=2}^{\ell} \sum_{k=1}^{n-1} \sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \\ &= 1 + \sum_{n=2}^{\ell} \sum_{\sigma \in \mathcal{S}_\ell(n,1)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} + \sum_{n=5}^{\ell} \sum_{2 \leq k < n/2} \sum_{\sigma \in \mathcal{S}_\ell(n,k)} \prod_{i=1}^{\ell} b_{i,\sigma(i)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=2}^{\ell} \sum_{\sigma \in \mathcal{S}_{\ell}(n, n-1)} \prod_{i=1}^{\ell} b_{i, \sigma(i)} + \sum_{n=2}^{\ell} \sum_{k=\lceil n/2 \rceil}^{n-2} \sum_{\sigma \in \mathcal{S}_{\ell}(n, k)} \prod_{i=1}^{\ell} b_{i, \sigma(i)} \\
\leq & 1 + \sum_{n=2}^{\ell} \frac{1}{n!} \cdot \frac{1}{(\ell \cdot (\ell-1) \dots (\ell-n+2))^7} \\
& + \sum_{n=5}^{\ell} \sum_{2 \leq k < n/2} \frac{1}{n! \cdot \ell^{4(k-1)} \cdot (\ell \cdot (\ell-1) \dots (\ell-n+2k))^3} \\
& + \sum_{n=2}^{\ell} \frac{1}{2\ell} \cdot \frac{((n-1)!)^6}{\ell^{6(n-1)}} + \sum_{n=2}^{\ell} \sum_{k \geq n/2} \frac{1}{2\ell} \cdot \frac{6}{\ell^{6(n-k-1)}} \\
\leq & 1 + \frac{1}{\ell^5} + \frac{1}{\ell^2} + \frac{1}{2\ell^6} + \sum_{n=2}^{\ell} \frac{3}{\ell} \cdot \frac{1}{\ell^6 - 1} \\
\leq & 1 + \frac{1}{\ell^5} + \frac{1}{\ell^2} + \frac{1}{2\ell^6} + \frac{3}{\ell^6 - 1} \leq 1.13,
\end{aligned}$$

for $\ell \geq 3$. □

Remark 1 (Exceedances and the Eulerian Number) For a permutation $\sigma \in \mathcal{S}_n$,

- The exceedance of σ is defined as $|\{i \in [n-1] : \sigma(i) > i\}|$.
- The Eulerian number $E(n, k)$ is defined to be the number of permutations in \mathcal{S}_n with $k-1$ exceedances.
- The derangement number $T(n, k)$ is defined to be the number of derangements in \mathcal{S}_n with k exceedances.

The explicit formula for $E(n, k)$ is $E(n, k) = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} \cdot (k+1-j)^n$ (page 273, [Com74]). The exponential generating function of $E(n, k)$ is given by (page 273, [Com74])

$$\sum_{n=0}^{\infty} \sum_{k=0}^n E(n, k) t^k \frac{x^n}{n!} = \frac{t-1}{t - e^{(t-1)x}}. \quad (58)$$

The exponential generating function of $T(n, k)$ is given by (Proposition 5, [Bre90])

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n-1} T(n, k) t^k \frac{x^n}{n!} = e^{-t} \cdot \frac{t-1}{t - e^{(t-1)x}}. \quad (59)$$

Comparing (58) and (59), we infer

$$T(n, k) = \sum_{j=0}^n (-1)^{(n-j)} \cdot \binom{n}{j} \cdot E(j, k). \quad (60)$$

Lemma 19 For any positive integer n ,

- $T(n, 1) = T(n, n-1) = 1$,

- $T(n, k) = T(n, n + 1 - k) < (2k + 3)^n$ for any $k \in \{2, \dots, n - 2\}$.

Proof Since $E(n, m) = \sum_{k=0}^{m+1} (-1)^k \binom{n+1}{k} \cdot (m+1-k)^n$, taking absolute values, we get

$$\begin{aligned} E(n, m) &< \binom{n+1}{0} \cdot (m+1)^n + \binom{n+1}{2} \cdot (m+1-2)^n + \dots \\ &< (m+1)^n \cdot \left(\sum_{i=0}^n \binom{n+1}{2i} \right) \leq (m+1)^n \cdot 2^n. \end{aligned}$$

Using equation 60 and taking absolute values, we get

$$T(n, k) < \sum_{j=0}^n \binom{n}{j} \cdot E(j, k) < \sum_{j=0}^n \binom{n}{j} \cdot 2^j \cdot (k+1)^j = (1 + 2(k+1))^n.$$

□

The following lemma is an extension to lemma 6.

Lemma 20 *Let S be a subset of $r < d$ vectors in \mathbb{R}^d . Let C be a cycle in $G(S)$, and let $X = C \setminus S$ and $Y = S \setminus C$ such that $|X| = |Y| = \ell$. Let $X = YA_C + X_\perp$, where X_\perp denotes the projection of vectors in X orthogonal to $\text{Span}(Y)$. Then the change in objective value is given by*

$$\frac{\text{vol}(S\Delta C)^2}{\text{vol}(S)^2} \geq \det(A_C^\top A_C).$$

In particular, if C' is the projection of the vectors in C onto $\text{Span}(S)$, then $\text{vol}(S\Delta C)^2 \geq \text{vol}(S\Delta C')^2$.

Proof Let $W = S \setminus Y$. Let us abuse notation to denote by X the matrix whose columns are the vectors in set X and similarly for other sets defined above. Then let $X = YA_C + WA' + Z$, where Z is the component of X which is orthogonal to $\text{Span}(S)$. Let $Y = Y_\perp + Y_W$, where Y_\perp is the component of Y orthogonal to $\text{Span}(W)$.

Let M denote the matrix whose columns are the elements of $S\Delta C$. Concretely, $M = [X \ W]$ following the notation above. Note that

$$\begin{aligned} \text{vol}(S\Delta C)^2 &= \det(M^\top M) = \det \left(\begin{bmatrix} X^\top \\ W^\top \end{bmatrix} [X \ W] \right) = \det \left(\begin{bmatrix} X^\top X & X^\top W \\ W^\top X & W^\top W \end{bmatrix} \right) \\ &= \det(W^\top W)^{1/2} \det(X^\top X - X^\top W(W^\top W)^{-1}W^\top X), \end{aligned} \quad (61)$$

where the last equation follows from the fact that $M^\top M$ is positive definite.

Now, since Z is orthogonal to both Y and W , we get

$$X^\top X = (YA_C)^\top (YA_C) + (WA')^\top (WA') + Z^\top Z + (YA_C)^\top (WA') + (WA')^\top (YA_C). \quad (62)$$

Additionally, $W(W^\top W)^{-1}W^\top$ is the projection matrix on the column space of W , so

$$W(W^\top W)^{-1}W^\top X = Y_W A_C + WA'. \quad (63)$$

Multiplying equation (63) with X gives

$$X^\top W(W^\top W)^{-1}W^\top X = (YA_C)^\top(Y_W A_C) + (YA_C)^\top(WA') + (WA')^\top(Y_W A_C) + (WA')^\top(WA'). \quad (64)$$

Subtracting equation (64) from equation (62), we see that

$$\begin{aligned} X^\top X - X^\top W(W^\top W)^{-1}W^\top X &= (YA_C)^\top(Y - Y_W)A_C + (WA')^\top(Y - Y_W)A_C + Z^\top Z \\ &= (Y_\perp A_C)^\top(Y_\perp A_C) + Z^\top Z. \end{aligned}$$

Substituting the value of $X^\top X - X^\top W(W^\top W)^{-1}W^\top X$ from the above equation in equation (61), we conclude that

$$\text{vol}(S\Delta C)^2 = \det(W^\top W) \det\left((Y_\perp A_C)^\top(Y_\perp A_C) + Z^\top Z\right).$$

Similarly,

$$\begin{aligned} \text{vol}(S)^2 &= \det\left(\begin{bmatrix} Y^\top \\ W^\top \end{bmatrix} \begin{bmatrix} Y & W \end{bmatrix}\right) = \det\left(\begin{bmatrix} Y^\top Y & Y^\top W \\ W^\top Y & W^\top W \end{bmatrix}\right) \\ &= \det(W^\top W)^{1/2} \det\left(Y^\top Y - Y^\top W(W^\top W)^{-1}W^\top Y\right) \\ &= \det(W^\top W)^{1/2} \det\left(Y^\top Y - Y^\top Y_W\right) \end{aligned} \quad (65)$$

$$= \det(W^\top W)^{1/2} \det\left(Y_\perp^\top Y_\perp\right). \quad (66)$$

Finally, dividing equation (64) by equation (13) gives

$$\frac{\text{vol}(S\Delta C)^2}{\text{vol}(S)^2} = \frac{\det\left((Y_\perp A_C)^\top(Y_\perp A_C) + Z^\top Z\right)}{\det(Y_\perp^\top Y_\perp)}.$$

Since $Z^\top Z$ is positive semidefinite, we conclude that

$$\begin{aligned} \frac{\text{vol}(S\Delta C)^2}{\text{vol}(S)^2} &= \frac{\det\left((Y_\perp A_C)^\top(Y_\perp A_C) + Z^\top Z\right)}{\det(Y_\perp^\top Y_\perp)} \\ &\geq \frac{\det\left((Y_\perp A_C)^\top(Y_\perp A_C)\right)}{\det(Y_\perp^\top Y_\perp)} = \det(A_C^\top A_C). \end{aligned}$$

If C' is the projection of C onto $\text{Span}(S)$, then Lemma 6 implies that $\text{vol}(S\Delta C')^2 = \text{vol}(S)^2 \cdot \det(A_C^\top A_C) \leq \text{vol}(S\Delta C)^2$. \square