Four-field Hamiltonian fluid closure of the one-dimensional Vlasov–Poisson equation

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We consider a reduced dynamics for the first four fluid moments of the one-dimensional Vlasov-Poisson equation, namely, the fluid density, fluid velocity, pressure and heat flux. This dynamics depends on an equation of state to close the system. This equation of state (closure) connects the fifth order moment –related to the kurtosis in velocity of the Vlasov distribution– with the first four moments. By solving the Jacobi identity, we derive an equation of state which ensures that the resulting reduced fluid model is Hamiltonian. We show that this Hamiltonian closure ensures that symmetric homogeneous equilibria of the reduced fluid model are stable.

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I. INTRODUCTION

In order to simulate the dynamics of a plasma, there is a variety of models which are used according to the type of question and the level of detail in the description of the plasma. Most of these models can be categorized as kinetic or fluid, whether the dynamical field variables are functions of the phase space coordinates (x, v) of the particles or just configuration space coordinates x. Compared to kinetic models, fluid models have the significant advantage to be defined in a dimensionally reduced space, which makes them particularly desirable from a computational viewpoint. The central question is how to define these fluid models from a parent kinetic model. There are plethora of methods to do this, some better suited than others depending on the specific problem at hand. For instance, some reductions rely on an assumption on the shape of the distribution function,[1–5] or introduce suitably designed dissipative terms.[6–8] Here we follow a different route by requiring that the reduced fluid model preserves an important dynamical property of the parent model, namely its Hamiltonian structure.[9–11] Rather than being an additional constraint on the reduction, we will see that this requirement provides a way to perform the reduction, and precisely define the relevant closures leading to the definition of Hamiltonian fluid model(s). In order to illustrate this point we consider the one-dimensional Vlasov–Poisson equation. This equation describes the evolution of the distribution function $f(x, v, t)$ of charged particles (of charge e and mass m) in an electric field $E(x, t)$:

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} - \frac{e\widetilde{E}}{m}\frac{\partial f}{\partial v},$$

where $\widetilde{E}$ is the fluctuating part of the electric field E whose dynamics is given by

$$\frac{\partial E}{\partial t} = -4\pi\widetilde{j},$$

and $j = e\int vf\mathrm{d}v$ is the current density. We assume periodic boundary conditions in x with period $2L_x$, so that the fluctuating part is defined as

$$\widetilde{E} = E - \frac{1}{2L_x}\int_{-L_x}^{L_x} E\,\mathrm{d}x.$$

We consider a fluid description obtained by using the first four fluid moments of the distribution function, more precisely, the density $\rho(x, t)$, the fluid velocity $u(x, t)$, the pressure

$P(x,t)$ and the heat flux $q(x,t)$ defined by

$$\rho = \int f \mathrm{d}v,$$
$$u = \rho^{-1} \int v f \mathrm{d}v,$$
$$P = \int (v-u)^2 f \mathrm{d}v,$$
$$q = \frac{1}{2} \int (v-u)^3 f \mathrm{d}v.$$

From the Vlasov equation, we obtain the equations of motion for these moments:

$$\partial_t \rho = -\partial_x(\rho u), \tag{1a}$$
$$\partial_t u = -u\partial_x u - \frac{1}{\rho}\partial_x P + \frac{e\widetilde{E}}{m} \tag{1b}$$
$$\partial_t P = -u\partial_x P - 3P\partial_x u - 2\partial_x q, \tag{1c}$$
$$\partial_t q = -u\partial_x q - 4q\partial_x u + \frac{3P}{2\rho}\partial_x P - \frac{1}{2}\partial_x R, \tag{1d}$$
$$\partial_t E = -4\pi e\widetilde{\rho u}, \tag{1e}$$

where

$$R = \int (v-u)^4 f \mathrm{d}v,$$

which is related to the kurtosis (in velocity) of the distribution function f. Here and in what follows, ∂_x and ∂_v denote the partial derivatives of a function of x and v with respect to x and v, respectively. In order to close the set of equations of motion, we need an equation of state of the form

$$R = R(\rho, u, P, q).$$

An example of closure is obtained by assuming a Gaussian distribution for f (see Ref. 3),

$$f(x,v,t) = \frac{\rho}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-(v-u)^2/(2\sigma^2)},$$

which leads to $R = 3P^2/\rho$, independent of u and q. One of the main problems of the Gaussian closure is that the resulting model breaks the original Hamiltonian structure of the parent model, the Vlasov–Poisson equation.[12] As a consequence, this closure introduces unphysical dissipation.

Based on the preservation of the Hamiltonian structure, another closure based on dimensional analysis was proposed in Ref. 10, namely

$$R = \frac{P^2}{\rho} + \frac{4q^2}{P}. \tag{2}$$

We notice that this closure depends explicitly on the asymmetries of the distribution function, measured by q, and is still independent of the fluid velocity u. However this closure has a fundamental drawback which is that homogeneous equilibria are all unstable. In order to see this, we linearize the equations of motion around one of such equilibria with $q_0 = 0$, $u_0 = 0$ and $E_0 = 0$, i.e., $\rho = \rho_0 + \delta\rho$, $u = \delta u$, $P = P_0 + \delta P$, $q = \delta q$ and $E = \delta E$. The linearized equations of motion for $\delta\mathbf{X} = (\delta\rho, \delta u, \delta P, \delta q, \delta E)$ in Fourier space, i.e., for

$$\delta\mathbf{X} = \sum_{k=-\infty}^{\infty} \delta\mathbf{X}_k \, \mathrm{e}^{ikx},$$

reduce to

$$\delta\dot{\mathbf{X}}_k = A\,\delta\mathbf{X}_k, \tag{3}$$

where

$$A = \begin{pmatrix} 0 & -ik\rho_0 & 0 & 0 & 0 \\ 0 & 0 & -ik\rho_0^{-1} & 0 & e/m \\ 0 & -3ikP_0 & 0 & -2ik & 0 \\ ik\rho_0^{-2}P_0^2/2 & 0 & ik\rho_0^{-1}P_0/2 & 0 & 0 \\ 0 & -4\pi e\rho_0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix A does not have purely imaginary eigenvalues for

$$k^2 < k_c^2 = \frac{4\pi e^2\rho_0^2}{mP_0},$$

from which we conclude that all equilibria with $q_0 = 0$ are unstable.

Here we are looking for a closure which combines two important properties of the Vlasov–Poisson equation, namely, the stability of symmetric homogeneous equilibria, and its Hamiltonian structure.

We do not assume any particular form for the distribution function. Instead we solve the Jacobi identity in order to determine all possible $R(\rho, u, P, q)$ for which this identity is satisfied. As a result, we unveil a one-parameter family of Hamiltonian fluid closures. We show that for these closures, the associated Poisson bracket has two Casimir invariants of the entropy type, i.e., two observables C of the form $C = \int \mathrm{d}x \; \rho\, \Gamma(\rho, P, q)$. These Casimir invariants provide normal variables in which the closure in parametric form is found to be polynomial. We then examine numerically some properties of the resulting Hamiltonian model in two cases: plasma oscillations and the two-stream instability.

II. DERIVATION OF THE FOUR-FIELD HAMILTONIAN CLOSURE

The one-dimensional Vlasov–Poisson equation has a Hamiltonian structure[13] (see also Refs. 14 and 15 for a review), i.e., the equations of motion can be recast using a Hamiltonian and a Poisson bracket:

$$\partial_t f = \{f, H\}, \tag{4a}$$

$$\partial_t E = \{E, H\}, \tag{4b}$$

where

$$H[f, E] = \int \mathrm{d}x\mathrm{d}v f \frac{mv^2}{2} + \int \mathrm{d}x \frac{E^2}{8\pi}.$$

The Poisson bracket between two scalar functionals of f and E is given by

$$\{F, G\} = \frac{1}{m} \int f \left[\partial_x \frac{\delta F}{\delta f} \partial_v \frac{\delta G}{\delta f} - \partial_v \frac{\delta F}{\delta f} \partial_x \frac{\delta G}{\delta f} - 4\pi e \left(\widetilde{\frac{\delta F}{\delta E}} \partial_v \frac{\delta G}{\delta f} - \partial_v \frac{\delta F}{\delta f} \widetilde{\frac{\delta G}{\delta E}} \right) \right] \mathrm{d}x\mathrm{d}v, \tag{5}$$

where $\frac{\delta F}{\delta f}$ and $\frac{\delta F}{\delta E}$ denote the functional derivatives of F with respect to f and E respectively. In particular, this bracket satisfies the Jacobi identity, i.e.,

$$\{F, \{G, K\}\} + \{K, \{F, G\}\} + \{G, \{K, F\}\} = 0,$$

for all observables F, G and K. For simplicity of the notations and without loss of generality, we assume that $m = 1$.

Remark: Gauss's law is derived from a Casimir invariant of the bracket (5):

$$C[f, E] = \partial_x E - 4\pi e \int \mathrm{d}v f.$$

Here we consider a neutral plasma, i.e., such that the value of this Casimir invariant is $-4\pi e$ which expresses the presence of a neutralizing background.

Regardless of the truncation, Eq. (1) can be recast in the following form (see Ref. 10 for more details):

$$\partial_t \mathbf{X} = \{\mathbf{X}, H\},$$

where $\mathbf{X} = (\rho, u, S_2 = P/\rho^3, S_3 = 2q/\rho^4, E)$, and

$$H[\rho, u, S_2, S_3, E] = \frac{1}{2} \int \mathrm{d}x \left[\rho u^2 + \rho^3 S_2 + \frac{E^2}{4\pi} \right],$$

and

$$\{F,G\} = \int \mathrm{d}x \left[\frac{\delta G}{\delta u}\,\partial_x \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta u}\,\partial_x \frac{\delta G}{\delta \rho} - 4\pi e \left(\frac{\delta G}{\delta u}\,\widetilde{\frac{\delta F}{\delta E}} - \frac{\delta F}{\delta u}\,\widetilde{\frac{\delta G}{\delta E}} \right) \right.$$
$$\left. -\frac{1}{\rho} \left(\frac{\delta G}{\delta u}\,\frac{\delta F}{\delta S_i} - \frac{\delta F}{\delta u}\,\frac{\delta G}{\delta S_i} \right) \partial_x S_i + \alpha_{ij}\,\frac{1}{\rho^2}\,\frac{\delta F}{\delta S_i}\,\frac{\delta G}{\delta S_j} + \partial_x \left(\frac{1}{\rho}\,\frac{\delta F}{\delta S_i} \right) \beta_{ij}\,\frac{1}{\rho}\,\frac{\delta G}{\delta S_j} \right]. \quad (6)$$

The 2×2 matrices $\alpha = \partial_x \gamma$ and β are given by

$$\gamma = \begin{pmatrix} 2S_3 & 2S_4 - 3S_2^2 \\ 3S_4 - 6S_2^2 & 3S_5 - 12S_2S_3 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} 4S_3 & 5S_4 - 9S_2^2 \\ 5S_4 - 9S_2^2 & 6S_5 - 24S_2S_3 \end{pmatrix},$$

where $S_4 = R/\rho^5$ and S_5 is an arbitrary function of ρ, u, S_2 and S_3. As a consequence, since the bracket is antisymmetric, the models are all conserving energy regardless of the closure $S_4 = S_4(\rho, u, S_2, S_3)$ and $S_5 = S_5(\rho, u, S_2, S_3)$. We notice that $\beta = \gamma + \gamma^T$. This allows us to rewrite the Poisson bracket in a more antisymmetric way

$$\{F,G\} = \int \mathrm{d}x \left[\frac{\delta G}{\delta u}\,\partial_x \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta u}\,\partial_x \frac{\delta G}{\delta \rho} - 4\pi e \left(\frac{\delta G}{\delta u}\,\widetilde{\frac{\delta F}{\delta E}} - \frac{\delta F}{\delta u}\,\widetilde{\frac{\delta G}{\delta E}} \right) \right.$$
$$\left. -\frac{1}{\rho}\,\partial_x S_i \left(\frac{\delta G}{\delta u}\,\frac{\delta F}{\delta S_i} - \frac{\delta F}{\delta u}\,\frac{\delta G}{\delta S_i} \right) + \frac{1}{\rho}\,\frac{\delta G}{\delta S_i}\,\gamma_{ij}\,\partial_x \left(\frac{1}{\rho}\,\frac{\delta F}{\delta S_j} \right) - \frac{1}{\rho}\,\frac{\delta F}{\delta S_i}\,\gamma_{ij}\,\partial_x \left(\frac{1}{\rho}\,\frac{\delta G}{\delta S_j} \right) \right].$$

The Jacobi identity for the above bracket leads to the following constraints on the matrix γ:

$$(\gamma_{kn} + \gamma_{nk})\frac{\partial \gamma_{ij}}{\partial S_n} = (\gamma_{jn} + \gamma_{nj})\frac{\partial \gamma_{ik}}{\partial S_n}, \quad \text{(7a)}$$
$$\frac{\partial \gamma_{in}}{\partial S_m}\frac{\partial \gamma_{jk}}{\partial S_n} = \frac{\partial \gamma_{jn}}{\partial S_m}\frac{\partial \gamma_{ik}}{\partial S_n}, \quad \text{(7b)}$$

for all i, j, k, m (and repeated summation over n).

A. Explicit expression for the Hamiltonian closure

In Ref. 10, it was shown that in order for the bracket (6) to be Hamiltonian, the closure S_4 needs to be of the form $S_4 = S_4(S_2, S_3)$, i.e., it does not depend on ρ and u. The conditions (7) boil down to three constraints

$$
\begin{aligned}
6S_5 &= 12S_2S_3 + 4S_3\frac{\partial S_4}{\partial S_2} + (5S_4 - 9S_2^2)\frac{\partial S_4}{\partial S_3},\\
\frac{\partial S_5}{\partial S_2} &= 4S_3 + \frac{\partial S_4}{\partial S_3}\left(\frac{\partial S_4}{\partial S_2} - 3S_2\right),\\
\frac{\partial S_5}{\partial S_3} &= \frac{\partial S_4}{\partial S_2} + \left(\frac{\partial S_4}{\partial S_3}\right)^2.
\end{aligned}
$$

Equivalently, a necessary and sufficient condition is that the closure function S_4 satisfies the following two coupled nonlinear partial differential equations

$$
4S_3\frac{\partial^2 S_4}{\partial S_2^2} - (9S_2^2 - 5S_4)\frac{\partial^2 S_4}{\partial S_2 \partial S_3} - \frac{\partial S_4}{\partial S_2}\frac{\partial S_4}{\partial S_3} - 12S_3 = 0, \tag{8a}
$$

$$
4S_3\frac{\partial^2 S_4}{\partial S_2 \partial S_3} - (9S_2^2 - 5S_4)\frac{\partial^2 S_4}{\partial S_3^2} - \left(\frac{\partial S_4}{\partial S_3}\right)^2 - 2\frac{\partial S_4}{\partial S_2} + 12S_2 = 0. \tag{8b}
$$

From these equations, we readily check that the Gaussian closure $S_4 = 3S_2^2$ is not a solution of these equations, which means that the Gaussian closure is not Hamiltonian. In addition, we check that the solution given by Eq. (2), corresponding to the dimensional analysis of Ref. 10, i.e., $S_4 = S_2^2 + S_3^2/S_2$, is the simplest solution. However, this is not an adequate solution since all homogeneous equilibria are always found to be unstable, as pointed out above. To solve Eqs. (8), we start by looking for solutions close to symmetric distributions, i.e.,

$$
S_4(S_2, S_3) = f_0(S_2) + S_3^2 f_1(S_2) + O(S_3^4).
$$

We insert this expansion in Eqs. (8) and consider their leading behavior near $S_3 = 0$. This lead to a set of two coupled ordinary differential equations

$$
\begin{aligned}
&2f_0'' - (9S_2^2 - 5f_0)f_1' - f_1' f_0 - 6 = 0,\\
&-(9S_2^2 - 5f_0)f_1 - f_0' + 6S_2 = 0.
\end{aligned}
$$

By combining these two equations, we obtain one single ordinary differential equation

$$
f_0''(9S_2^2 - 5f_0) + 2f_0'^2 - 18S_2 f_0' + 20f_0 = 0.
$$

Near $S_2 = 0$, we look for solutions of the type

$$
f_0(S_2) = kS_2^\alpha.
$$

A possible solution is obviously the one obtained using the dimensional analysis[10], i.e., $f_0(S_2) = S_2^2$. In addition there is a less trivial family of solutions for $\alpha = 5/3$. More generally, we look at solutions which can be expanded in Puiseux series

$$f_0(S_2) = \sum_{n=5}^{\infty} a_n S_2^{n/3}.$$

We show that the only possible solutions are $f_0(S_2) = S_2^2$ and

$$f_0(S_2) = kS_2^{5/3},$$

for any value of k. For practical purposes, we define $\kappa = 5k/9$. We notice that contrary to the solution provided by dimensional analysis, the second solution comes as a family parameterized by κ. The interesting feature is that this family extends to a Hamiltonian closure for arbitrary large values of S_3. Indeed we are looking for a solution which can be expanded as

$$S_4(S_2, S_3) = \sum_{n=0}^{\infty} f_n(S_2) S_3^{2n}. \tag{9}$$

Inserting this ansatz in Eq. (8) leads to a recurrence relation for the coefficients $f_n(S_2)$:

$$f_0(S_2) = \frac{9\kappa}{5} S_2^{5/3}, \tag{10a}$$

$$f_1(S_2) = \frac{\kappa - 2S_2^{1/3}}{3S_2(\kappa - S_2^{1/3})}, \tag{10b}$$

$$f_{n+1} = -\frac{1}{9(n+1)(2n+1)S_2^{5/3}(\kappa - S_2^{1/3})} \Bigg[(4n-1) f_n' + \sum_{m=1}^{n} m(12m - 7 - 2n) f_m f_{n+1-m} \Bigg], \tag{10c}$$

and an addition constraint where f_n has to satisfy

$$S_2^{5/3}(\kappa - S_2^{1/3})(n+1) f_{n+1}' - (n+1)\frac{\kappa}{3} S_2^{2/3} f_{n+1} + \frac{2}{9} f_n'' + \frac{1}{9} \sum_{m=1}^{n} (6m - n - 1) f_m' f_{n+1-m} = 0, \tag{11}$$

for all $n \geq 1$. The first few terms are given by

$$f_2(S_2) = \frac{1}{3^4 S_2^{11/3}(\kappa - S_2^{1/3})},$$
$$f_3(S_2) = 2\frac{5\kappa - 3S_2^{1/3}}{3^8 S_2^{19/3}(\kappa - S_2^{1/3})^3},$$
$$f_4(S_2) = \frac{48\kappa^2 - 61\kappa S_2^{1/3} + 18 S_2^{2/3}}{3^{11} S_2^9(\kappa - S_2^{1/3})^5}.$$

The expression of other terms of the series expansion of S_4 can be obtained using a MATLAB[16] code available at Ref. 17. We are not able to prove directly that for all n, the f_ns obtained by the recursion relation (10) satisfy Eq. (11). However, we have checked that for n below 25, these conditions are satisfied using symbolic computations available from the MATLAB[16] code. Beyond this value of 25, the symbolic computations are too complex to allow simplifications in a reasonable amount of time. By truncating the series (9), i.e., by considering

$$S_4(S_2, S_3) = \sum_{n=0}^{n_{\max}} f_n(S_2) S_3^{2n},$$

we have found that the Jacobi identity is satisfied up to orders $S_3^{2n_{\max}}$ for the values of $n_{\max}$ we have tested. This led us to conjecture that the limit $n_{\max} \to \infty$ corresponds to a Hamiltonian closure. We notice that the closure is singular at

$$S_2^{(c)} = \kappa^3,$$

so this explicit closure $S_4 = S_4(S_2, S_3)$ is valid only in the range $S_2 \in [0, S_2^{(c)}[$.

Remark: Scaling. We notice that the functions f_n satisfy

$$f_n(\lambda^2 S_2; \lambda^{2/3}\kappa) = \lambda^{4-6n} f_n(S_2; \kappa),$$

for all $n \geq 0$. Therefore, we have a scaling relationship for S_4:

$$S_4(\lambda^2 S_2, \lambda^3 S_3; \lambda^{2/3}\kappa) = \lambda^4 S_4(S_2, S_3; \kappa).$$

A contour plot of S_4 in the plane (S_2, S_3) is represented in Fig. 1 for $\kappa = 1$. The equations of motion are given by Eqs. (1) with

$$R(\rho, P, q) = \rho^5 \sum_{n=0}^{\infty} f_n\left(\frac{P}{\rho^3}\right)\left(\frac{2q}{\rho^4}\right)^{2n}.$$

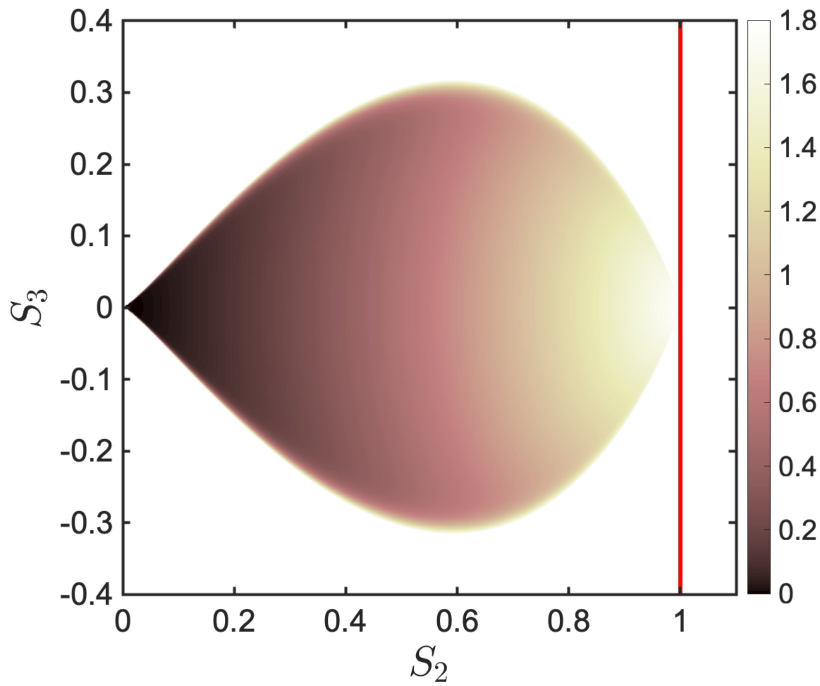

FIG. 1. Contourplot of S_4 given by Eq. (9) as a function of S_2 and S_3 for $\kappa = 1$. The summation is computed up to S_3^{30}. The vertical red line corresponds to the location of the singularity at $S_2 = S_2^{(c)}$. The MATLAB[16] code to compute symbolically the terms of the closure and obtain numerically this figure is available at Ref. 17.

In particular one interesting feature is that the first order of the closure does not depend on ρ, i.e.,

$$R(\rho, P, q = 0) = \frac{9\kappa}{5} P^{5/3}.$$

Remark: Relation between the kurtosis and the skewness. A scaling of kurtosis (related to S_4) with squared skewness (related to S_3) for plasma density fluctuations and sea-surface temperature fluctuations was found in Refs. 18–22. Using the Hamiltonian closure, this relation is found as the first two terms of the closure, i.e., $S_4 = b + aS_3^2 + O(S_3^4)$ where a and b are functions of ρ and P.

B. Casimir invariants

A very interesting property of the noncanonical Poisson bracket (6) is that it possesses a number of Casimir invariants, i.e., observables C such that $\{C, F\} = 0$ for any other observable F. First we are looking for Casimir invariants of the entropy type, i.e.,

$$C = \int \mathrm{d}x \rho \Gamma(S_2, S_3).$$

The function Γ satisfies the following conditions:

$$\beta_{ij} \frac{\partial^2 \Gamma}{\partial S_i \partial S_n} + \frac{\partial \gamma_{ij}}{\partial S_n} \frac{\partial \Gamma}{\partial S_i} = 0, \tag{12}$$

for all j, k and n in $(2,3)$ (and where we assumed implicit summation over the repeated index i). We assume that we have K solutions, denoted Γ_k for $k = 2, \ldots, K$. Using the property $\beta = \gamma + \gamma^\dagger$, we prove that the above-conditions are equivalent to

$$\frac{\partial}{\partial S_n} \left(\frac{\partial \Gamma_k}{\partial S_i} \beta_{ij} \frac{\partial \Gamma_l}{\partial S_j} \right) = 0, \tag{13}$$

for all n, k and l.

Using series expansions, we found two solutions to Eq. (13):

$$C_2 = \sum_{n=0}^{\infty} \int \mathrm{d}x \rho g_n(S_2) S_3^{2n},$$
$$C_3 = \sum_{n=0}^{\infty} \int \mathrm{d}x \rho h_n(S_2) S_3^{2n+1},$$

where the first elements in the series are:

$$g_0(S_2) = S_2^{1/3},$$
$$g_1(S_2) = -\frac{1}{3^3 S_2^{7/3} (\kappa - S_2^{1/3})},$$
$$g_2(S_2) = \frac{-4\kappa + 3 S_2^{1/3}}{3^6 S_2^5 (\kappa - S_2^{1/3})^3},$$
$$h_0(S_2) = \frac{1}{3 S_2 (\kappa - S_2^{1/3})},$$
$$h_1(S_2) = \frac{2}{3^4 S_2^{11/3} (\kappa - S_2^{1/3})^2},$$
$$h_2(S_2) = \frac{2(5\kappa - 4 S_2^{1/3})}{3^7 S_2^{19/3} (\kappa - S_2^{1/3})^4}.$$

The functions h_n and g_n for $n \geq 1$ are determined from the recurrence relations:

$$g_{n+1} = -\frac{1}{9(n+1)(2n+1)S_2^{5/3}(\kappa - S_2^{1/3})}\left[(4n+1)g_n' + \sum_{m=0}^{n} m(6n+4m+1)f_{n+1-m}g_m\right],$$

$$h_{n+1} = -\frac{1}{9(n+1)(2n+3)S_2^{5/3}(\kappa - S_2^{1/3})}\left[(4n+3)h_n' + \sum_{m=0}^{n}(2m+3n+3)(2m+1)f_{n+1-m}h_m\right],$$

which are both obtained from Eq. (12) with $j=3$ and $n=3$.

These Casimir invariants allow us to define particularly relevant variables, referred to as normal variables, in which the Hamiltonian system is greatly simplified. We perform a local change of variables: $(S_2, S_3) \mapsto (\Gamma_2, \Gamma_3)$, where

$$\Gamma_2 = \sum_{n=0}^{\infty} g_n(S_2)S_3^{2n},$$

$$\Gamma_3 = \sum_{n=0}^{\infty} h_n(S_2)S_3^{2n+1}.$$

The bracket (6) becomes

$$\{F,G\} = \int \mathrm{d}x \left[\frac{\delta G}{\delta u}\partial_x \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta u}\partial_x \frac{\delta G}{\delta \rho} - 4\pi e\left(\frac{\delta G}{\delta u}\widetilde{\frac{\delta F}{\delta E}} - \frac{\delta F}{\delta u}\widetilde{\frac{\delta G}{\delta E}}\right)\right.$$
$$\left. -\frac{\partial_x \Gamma_i}{\rho}\left(\frac{\delta G}{\delta u}\frac{\delta F}{\delta \Gamma_i} - \frac{\delta F}{\delta u}\frac{\delta G}{\delta \Gamma_i}\right) + \frac{1}{\rho}\partial_x\left(\frac{1}{\rho}\frac{\delta F}{\delta \Gamma_i}\right)\tilde{\beta}_{ij}\frac{\delta G}{\delta \Gamma_j}\right], \qquad (14)$$

where here F_i is the functional derivative of F with respect to Γ_i and $\tilde{\beta}$ is a symmetric matrix whose elements are

$$\tilde{\beta}_{kl} = \frac{\partial \Gamma_k}{\partial S_i}\beta_{ij}\frac{\partial \Gamma_l}{\partial S_j},$$

with an implicit summation over repeated indices. From Eq. (13), we deduce that the matrix $\tilde{\beta}$ is constant. As a consequence, the bracket (14) always satisfies the Jacobi identity. Therefore the existence of two Casimir invariants of the entropy type for the bracket (6) is sufficient to ensure that it is a Poisson bracket. Note that we use the terminology Casimir invariant also for a bracket which is a priori not of the Poisson type. Using the expressions for $S_3 = 0$, the matrix $\tilde{\beta}$ takes the very simple form

$$\tilde{\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In addition, the existence of two Casimir invariants of the entropy type ensures a third Casimir invariant:

$$C_1 = \int \mathrm{d}x \left(u - \frac{\rho}{2}\Gamma_k(\tilde{\beta}^{-1})_{kl}\Gamma_l\right),$$

which is equal to

$$C_1 = \int \mathrm{d}x\, (u - \rho \Gamma_2 \Gamma_3)\,.$$

Its expansion is given by

$$C_1 = \int \mathrm{d}x \left(u - \rho \sum_{n=0}^{\infty} k_n(S_2) S_3^{2n+1} \right),$$

where

$$k_n = \sum_{m=0}^{n} g_{n-m} h_m,$$

for $n \geq 0$, and the first elements of the series are given by

$$k_0(S_2) = \frac{1}{S_2^{2/3}(\kappa - S_2^{1/3})},$$
$$k_1(S_2) = \frac{1}{3^3 S_2^{10/3}(\kappa - S_2^{1/3})^2},$$
$$k_2(S_2) = \frac{4\kappa - 3S_2^{1/3}}{3^6 S_2^6 (\kappa - S_2^{1/3})^4}.$$

We notice that three Casimir invariants similar to C_1, C_2 and C_3 (but of course, different) have been found for the Hamiltonian closure obtained using the dimensional analysis (see Ref. 10).

The advantage of working in the variables Γ_i instead of the variables S_i is that the closure functions S_4 and S_5 are no longer present in the Poisson bracket. They are now in the Hamiltonian through the change of variables $(S_2, S_3) \mapsto (\Gamma_2, \Gamma_3)$. If we truncate the closure functions S_4 and S_5 –a natural step since these functions are given as series in S_3– the system remains Hamiltonian in the variables Γ_i whereas if these truncations are performed in the bracket in the variables S_i, the system would likely loose the Hamiltonian property.

C. Parametric expression for the Hamiltonian closure

There is another significant advantage to working with normal variables Γ_i: What is not fully satisfactory with the variables S_i is that the closure is given as a relatively complex expansion, and consequently we were not been able to check the Jacobi identity at all orders in the expansion. We will see below that the origin of this complication is due to the search for an explicit closure function $S_4(S_2, S_3)$, not to the search of a Hamiltonian closure per se. Here instead we are looking at a parametric expression of the closure, and we consider the normal variables as parameters of the closure. More precisely, we consider an arbitrary change of coordinates from some variables Γ_i to variables S_i:

$$
\begin{aligned}
S_2 &= \overline{S_2}(\Gamma_2, \Gamma_3),\\
S_3 &= \overline{S_3}(\Gamma_2, \Gamma_3),
\end{aligned}
$$

and the closure functions are given by

$$
\begin{aligned}
S_4 &= \overline{S_4}(\Gamma_2, \Gamma_3),\\
S_5 &= \overline{S_5}(\Gamma_2, \Gamma_3).
\end{aligned}
$$

We start with the bracket (14) which is a Poisson bracket since the matrix $\tilde{\beta}$ is constant. The question of finding Hamiltonian closures is reformulated as follows: What are the functions $\overline{S_i}$ for which the bracket (14) expressed in the variables S_i is the original bracket (6)? The answer is given by two sets of equations

$$
\frac{\partial \overline{S_i}}{\partial \Gamma_k} \widetilde{\beta}_{kl} \frac{\partial \overline{S_i}}{\partial \Gamma_l} = \beta_{ij}, \tag{15}
$$

$$
\frac{\partial^2 \overline{S_i}}{\partial \Gamma_n \partial \Gamma_k} \widetilde{\beta}_{kl} \frac{\partial \overline{S_i}}{\partial \Gamma_l} = \frac{\partial \gamma_{ij}}{\partial \Gamma_n}, \tag{16}
$$

for all i, j and n. The first set of equations (15) defines parametrically the functions $\overline{S_3}$, $\overline{S_4}$ and $\overline{S_5}$:

$$
\begin{aligned}
\overline{S_3} &= \frac{1}{2} \frac{\partial \overline{S_2}}{\partial \Gamma_2} \frac{\partial \overline{S_2}}{\partial \Gamma_3},\\
\overline{S_4} &= \frac{9}{5} \overline{S_2}^2 + \frac{1}{5} \frac{\partial \overline{S_2}}{\partial \Gamma_2} \frac{\partial \overline{S_3}}{\partial \Gamma_3} + \frac{1}{5} \frac{\partial \overline{S_2}}{\partial \Gamma_3} \frac{\partial \overline{S_3}}{\partial \Gamma_2},\\
\overline{S_5} &= 4 \overline{S_2}\ \overline{S_3} + \frac{1}{3} \frac{\partial \overline{S_3}}{\partial \Gamma_2} \frac{\partial \overline{S_3}}{\partial \Gamma_3}.
\end{aligned}
$$

Once the function $\overline{S_2}$ is specified, all of the other functions $\overline{S_i}$ are uniquely determined by the above equations. By inverting the equations $\Gamma_i = \Gamma_i(S_2, S_3)$ or by solving one of the constraints (16), we obtain the following expression for $\overline{S_2}(\Gamma_2, \Gamma_3)$:

$$
\overline{S_2}(\Gamma_2, \Gamma_3) = \Gamma_2^3 + \Gamma_2(\kappa - \Gamma_2)\Gamma_3^2. \tag{17}
$$

Inserting this expression in the parametric equations for S_3, S_4 and S_5 leads to the following expressions:

$$\overline{S_3}(\Gamma_2,\Gamma_3) = \Gamma_2\Gamma_3(\kappa-\Gamma_2)\left(3\Gamma_2^2+(\kappa-2\Gamma_2)\Gamma_3^2\right), \tag{18a}$$

$$\begin{aligned}\overline{S_4}(\Gamma_2,\Gamma_3) = {} & \frac{9\kappa}{5}\Gamma_2^5 + 6\Gamma_2^3(\kappa-\Gamma_2)^2\Gamma_3^2 \\ & + \Gamma_2(\kappa-\Gamma_2)\left(\kappa^2-3\Gamma_2(\kappa-\Gamma_2)\right)\Gamma_3^4,\end{aligned} \tag{18b}$$

$$\begin{aligned}\overline{S_5}(\Gamma_2,\Gamma_3) = {} & 9\kappa\Gamma_2^5(\kappa-\Gamma_2)\Gamma_3 + 10\Gamma_2^3(\kappa-\Gamma_2)^3\Gamma_3^3 \\ & + \Gamma_2(\kappa-\Gamma_2)(\kappa-2\Gamma_2)(\kappa^2-2\kappa\Gamma_2+2\Gamma_2^2)\Gamma_3^5.\end{aligned} \tag{18c}$$

We notice that the closure is no longer given as an infinite series. In particular, the functions $\overline{S_n}$ for $n = 2, 3, 4, 5$ are polynomials in the two variables Γ_2 and Γ_3, and the degree in Γ_3 is n and the degree in Γ_2 is $n+1$. Using Mathematica[23], we have checked that the constraints (16) are all satisfied. The code is available at Ref. 17. The series expansion of the explicit closure $S_4 = S_4(S_2, S_3)$ given in Eqs. (9) is obtained by inverting Eqs. (17) and (18a), and inserting them in Eq. (18b).

For S_2 to be positive, a necessary and sufficient condition is that $\kappa > \Gamma_2 > 0$ or if $\Gamma_2 > \kappa$, $\Gamma_3^2 < \Gamma_2^2/(\Gamma_2-\kappa)$. This means that S_2 can take arbitrarily large values, provided that S_3 is not too large. We notice that the point $(S_2 = \kappa, S_3 = 0)$ in Fig. 1 is obtained for $\Gamma_2 = \kappa$ regardless of the value of Γ_3.

In Fig. 2, we have represented the closure function S_4 given parametrically by Eqs. (18) for a selected range of parameters (Γ_2, Γ_3). The surface gets more complicated, with more branches, as the range of (Γ_2, Γ_3) is extended (see the Mathematica code available at Ref. 17). We notice that there is a central brighter patch where there is a single value of S_4 for a given (S_2, S_3). It corresponds to the explicit closure $S_4 = S_4(S_2, S_3)$ as depicted in Fig. 1.

D. Equations of motion

The Poisson bracket (14) becomes

$$\begin{aligned}\{F,G\} = \int \mathrm{d}x \Bigg[& \frac{\delta G}{\delta u}\partial_x\frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta u}\partial_x\frac{\delta G}{\delta \rho} - 4\pi e\left(\frac{\delta G}{\delta u}\widetilde{\frac{\delta F}{\delta E}} - \frac{\delta F}{\delta u}\widetilde{\frac{\delta G}{\delta E}}\right) \\ & - \frac{\partial_x\Gamma_i}{\rho}\left(\frac{\delta G}{\delta u}\frac{\delta F}{\delta \Gamma_i} - \frac{\delta F}{\delta u}\frac{\delta G}{\delta \Gamma_i}\right) + \frac{1}{\rho}\frac{\delta G}{\delta \Gamma_2}\partial_x\left(\frac{1}{\rho}\frac{\delta F}{\delta \Gamma_3}\right) - \frac{1}{\rho}\frac{\delta F}{\delta \Gamma_2}\partial_x\left(\frac{1}{\rho}\frac{\delta G}{\delta \Gamma_3}\right)\Bigg],\end{aligned}$$

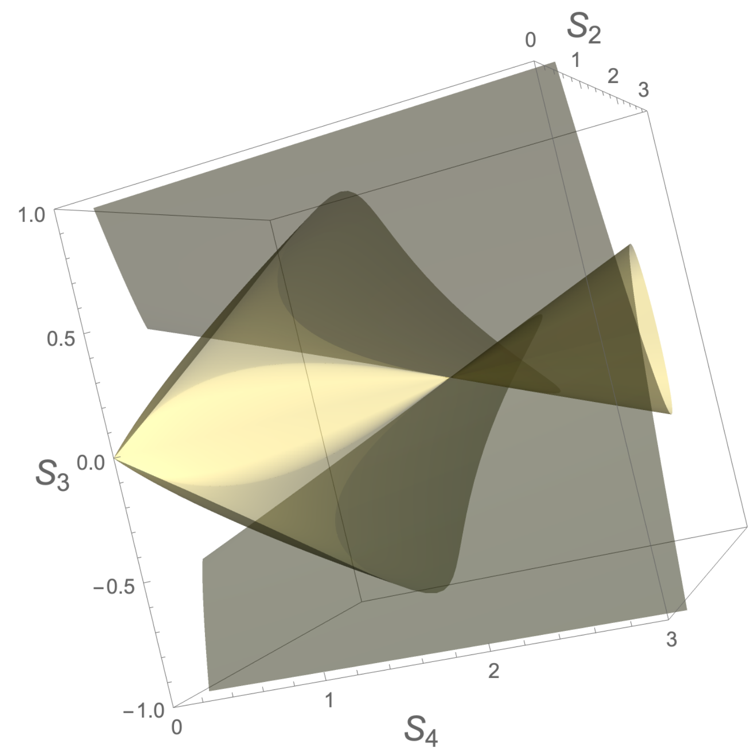

FIG. 2. Parametric representation of S_4 given by Eqs. (18) as a function of S_2 and S_3 for $\kappa = 1$. The Mathematica code is available at Ref. 17

and the Hamiltonian is

$$H[\rho, u, \Gamma_2, \Gamma_3, E] = \frac{1}{2}\int \mathrm{d}x \left[\rho u^2 + \rho^3 \overline{S_2}(\Gamma_2, \Gamma_3) + \frac{E^2}{4\pi}\right],$$

where $\overline{S_2}$ is given by Eq. (17). The equations of motion are given by $\dot{F} = \{F, H\}$:

$$\partial_t \rho = -\partial_x(\rho u), \tag{19a}$$

$$\partial_t u = -u\partial_x u - \frac{1}{\rho}\partial_x\left(\rho^3 \overline{S_2}\right) + e\widetilde{E} \tag{19b}$$

$$\partial_t \Gamma_2 = -u\partial_x \Gamma_2 - \frac{1}{2\rho}\partial_x\left(\rho^2 \frac{\partial \overline{S_2}}{\partial \Gamma_3}\right), \tag{19c}$$

$$\partial_t \Gamma_3 = -u\partial_x \Gamma_3 - \frac{1}{2\rho}\partial_x\left(\rho^2 \frac{\partial \overline{S_2}}{\partial \Gamma_2}\right), \tag{19d}$$

$$\partial_t E = -4\pi e\widetilde{\rho u}. \tag{19e}$$

Remark 1: In the case of an external time-dependent electric field $E_0(x,t)$, the closure is identical. First we need to autonomize the bracket. For the Vlasov–Poisson equation, the variables are the fields $f(x,v,t)$ and $E_1(x,t)$, together with t and K (K being the canonically conjugate variable to time t), such that the total electric field is $E = E_0 + E_1$. The Hamiltonian is

$$H[f, E_1, t, K] = \int \mathrm{d}x\mathrm{d}v f \frac{v^2}{2} + \int \mathrm{d}x \frac{E_1^2 + 2E_1E_0}{8\pi} + K,$$

and the Poisson bracket

$$\{F, G\} = \int f \left[\partial_x \frac{\delta F}{\delta f} \partial_v \frac{\delta G}{\delta f} - \partial_v \frac{\delta F}{\delta f} \partial_x \frac{\delta G}{\delta f} - 4\pi e \left(\widetilde{\frac{\delta F}{\delta E_1}} \partial_v \frac{\delta G}{\delta f} - \partial_v \frac{\delta F}{\delta f} \widetilde{\frac{\delta G}{\delta E_1}} \right) \right] \mathrm{d}x\mathrm{d}v$$
$$+ F_t G_K - F_K G_t.$$

For the reduced fluid equations, the Hamiltonian becomes

$$H[\rho, u, \Gamma_2, \Gamma_3, E_1, t, K] = \frac{1}{2} \int \mathrm{d}x \left[\rho u^2 + \rho^3 \overline{S_2} + \frac{E_1^2 + 2E_1E_0}{4\pi} \right] + K,$$

and the Poisson bracket

$$\{F, G\} = \int \mathrm{d}x \left[\frac{\delta G}{\delta u} \partial_x \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta u} \partial_x \frac{\delta G}{\delta \rho} - 4\pi e \left(\frac{\delta G}{\delta u} \widetilde{\frac{\delta F}{\delta E_1}} - \frac{\delta F}{\delta u} \widetilde{\frac{\delta G}{\delta E_1}} \right) \right.$$
$$\left. - \frac{\partial_x \Gamma_i}{\rho} \left(\frac{\delta G}{\delta u} \frac{\delta F}{\delta \Gamma_i} - \frac{\delta F}{\delta u} \frac{\delta G}{\delta \Gamma_i} \right) + \frac{1}{\rho} \frac{\delta G}{\delta \Gamma_2} \partial_x \left(\frac{1}{\rho} \frac{\delta F}{\delta \Gamma_3} \right) - \frac{1}{\rho} \frac{\delta F}{\delta \Gamma_2} \partial_x \left(\frac{1}{\rho} \frac{\delta G}{\delta \Gamma_3} \right) \right]$$
$$+ F_t G_K - F_K G_t.$$

The equations of motion consists in changing E by $E_0 + E_1$ in the Vlasov equation and in the momentum equation, and replacing E by E_1 in the Ampère equation.

Remark 2: By rescaling the parameters Γ_2 and Γ_3, and by rescaling the density ρ in the following way

$$\Gamma_2 = \kappa \, \Gamma_2^{(\mathrm{r})},$$
$$\Gamma_3 = \sqrt{\kappa} \, \Gamma_3^{(\mathrm{r})},$$
$$\rho = \kappa^{-3/2} \rho^{(\mathrm{r})},$$

the equations of motion (19a)-(19d) are not longer explicitly depending on κ. The parameter κ appears only in Ampère's equation or equivalently in Gauss' law. This means that the parameter of the closure κ can be viewed as the coupling parameter between the fluid part and the electrostatic part. The parameter κ can also be removed completely from the equations of motion by rescaling the charge and the electric field as

$$e = \kappa^{3/4} e^{(r)},$$
$$E = \kappa^{-3/4} E^{(r)}.$$

As a consequence, the one-parameter family of Hamiltonian closures can be seen as a unique Hamiltonian model, and the parameter κ is now in the initial condition.

E. Stability of the symmetric and homogeneous equilibria

We have found a one-parameter family of closures which fulfill the first requirement, namely, the resulting models are Hamiltonian. The second requirement is the stability of the equilibria $q_0 = 0$. The linearized equations of motion reduce to Eq. (3) with

$$A = \begin{pmatrix} 0 & -ik\rho_0 & 0 & 0 & 0 \\ 0 & 0 & -ik\rho_0^{-1} & 0 & e/m \\ 0 & -3ikP_0 & 0 & -2ik & 0 \\ 0 & 0 & -3ik(\kappa P_0^{2/3} - \rho_0^{-1}P_0)/2 & 0 & 0 \\ 0 & -4\pi e\rho_0 & 0 & 0 & 0 \end{pmatrix}.$$

From the dispersion relation, we define

$$\omega_0^2 = \omega_p^2 + 3\kappa P_0^{2/3} k^2,$$

where $\omega_p = \sqrt{4\pi e^2\rho_0/m}$ is the plasma frequency. The eigenvalues of A are all purely imaginary if

$$\omega_0^2 > \omega_{BG}^2,$$

where $\omega_{BG}(k)$ is the Bohm-Gross dispersion relation given by

$$\omega_{BG}^2 = \omega_p^2 + 3\frac{P_0}{\rho_0}k^2.$$

The non-zero eigenvalues of A are

$$i\omega = \pm\frac{i}{\sqrt{2}}\left(\omega_0^2 \pm \sqrt{\omega_0^4 - 4\omega_p^2(\omega_0^2 - \omega_{BG}^2)}\right)^{1/2}.$$

Therefore the homogeneous equilibria are stable for $\omega_0^2 > \omega_{BG}^2$, which is equivalent to requiring that $S_2 < S_2^{(c)}$ or $\Gamma_2 < \kappa$. In terms of the parameters of the equilibrium, this means that the pressure P_0 is such that $P_0^{1/3}/\rho_0 < \kappa$. A crucial factor is that the closure $R(\rho, P, q = 0)$ does not depend on ρ, and in this case, the necessary and sufficient condition for stability is

$$\omega_p^2 + k^2\frac{\partial R}{\partial P} > \omega_{BG}^2.$$

We recall that

$$R(\rho, P, 0) = \rho^5 S_4\left(\frac{P}{\rho^3}, 0\right).$$

The fractional exponent 5/3 in the closure comes from the requirement that R does not depend on ρ, ensuring the stability of the equilibria. More general cases for stability would be that at $q = 0$

$$\frac{\partial R}{\partial P} > \frac{3P}{\rho},$$
$$\frac{\partial R}{\partial \rho} \leq 0,$$

for all $\rho > 0$ and $P > 0$. However, these conditions do not ensure that the resulting model is Hamiltonian. As expected, the requirement that the model is Hamiltonian is more stringent than requiring that homogeneous equilibria are stable.

III. NUMERICAL APPLICATIONS

The objective of this section is not to offer a detailed comparison between the numerical implementation of the Hamiltonian fluid model and the one of the parent kinetic model. The objective is more modest since we limit ourselves to a couple of illustrations of the Hamiltonian fluid model, demonstrating the feasibility and practicality of the fluid model, which could trigger further questions of a more practical nature than the ones we consider in what follows. We consider two applications, one where the fluid model leads to stable plasma oscillations and the other one where it is unstable. In all the simulations, we consider a domain $x \in [-L_x, L_x]$ and $v \in [-L_v, L_v]$ with $L_v = 10$.

A. Plasma oscillations

We consider the following initial distribution function, built from a skew-normal distribution,

$$f(x, v, 0) = \frac{1}{\sqrt{2\pi}} (1 - A \cos k\, x) \left[1 + \mathrm{erf}\left(\frac{\alpha v}{\sqrt{2}}\right)\right] \mathrm{e}^{-v^2/2},$$

with $A = 10^{-4}$, $k = \lambda_D/2$ and $\alpha = 0.1$ (where λ_D is the Debye length). Here the velocities are in units of the thermal velocity $v_{\mathrm{th}} = \sqrt{k_B T}$. Given that the equilibrium has some initial fluid velocity, the Bohm-Gross dispersion relation is becomes

$$\omega_{BG} = \pm\omega_p \left[1 \pm u\frac{k}{\omega_p} + \frac{3}{2}\rho^2 S_2 \frac{k^2}{\omega_p^2} \pm 2\rho^3 S_3 \frac{k^3}{\omega_p^3} + O\left(\frac{k^4}{\omega_p^4}\right)\right].$$

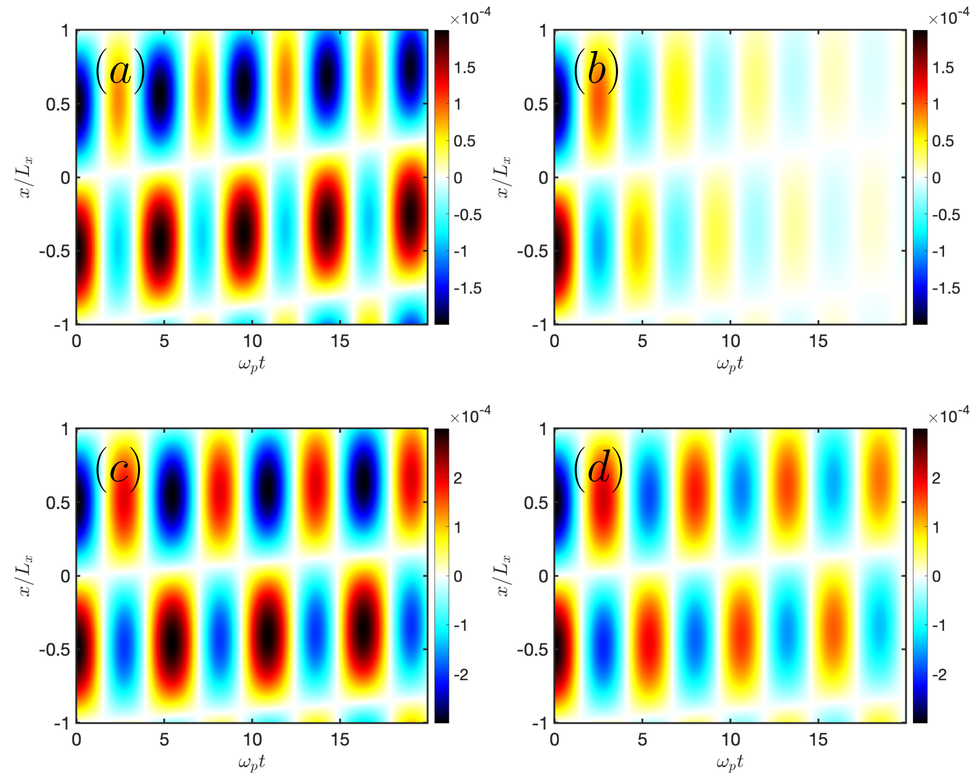

FIG. 3. Contour plot of $E(x,t)$: Panel (a): Hamiltonian fluid model with $\kappa = 1$ and $L_x/\lambda_D = 2\pi$. Panel (b): one-dimensional Vlasov-Poisson equation with $L_x/\lambda_D = 2\pi$. Panel (c): Hamiltonian fluid model with $\kappa = 1$ and $L_x/\lambda_D = 3\pi$. Panel (d): one-dimensional Vlasov–Poisson equation with $L_x/\lambda_D = 3\pi$.

This is the same dispersion relation given by the fluid and the kinetic models. For the skew-normal equilibrium,

$$\begin{aligned}
&\rho = 1,\\
&u = \alpha\sqrt{\frac{2}{\pi(1+\alpha^2)}},\\
&S_2 = 1 - \frac{2\alpha^2}{\pi(1+\alpha^2)},\\
&S_3 = \frac{\alpha^3}{(1+\alpha^2)^{3/2}}\sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi} - 1\right).
\end{aligned}$$

We consider the fluid model with $\kappa = 1$. Given the initial values of S_2 and S_3, we compute the initial values for Γ_2 and Γ_3. We represent the values of $E(x,t)$ in Fig. 3 obtained with the fluid and the kinetic model. We notice some qualitative similarities between the kinetic and the fluid model, such as plasma oscillations. However, as expected, the fluid model does not capture the damping of the field (clearly visible for $L_x/\lambda_D = 2\pi$), which is a purely kinetic effect. For larger values of L_x, i.e., $L_x/\lambda_D = 3\pi$ the damping is reduced as expected, and the agreement between the kinetic and the fluid simulations gets better.

B. Two-stream instability

Next, we consider the two-stream instability with the initial distribution

$$f(x,v,0) = (1 - A\cos kx)\frac{v^2 \mathrm{e}^{-v^2/2v_0^2}}{\sqrt{2\pi}\, v_0^3}.$$

For this distribution, $v_{\mathrm{th}} = \sqrt{3}\, v_0$. To simplify comparison with the existing literature, we take $\bar{\lambda} = \lambda_D/\sqrt{3}$ and v_0 to be our length and velocity scales, respectively. We set $A = 10^{-6}$ and $k\,\bar{\lambda} = 1/2$. From the previous section, we know that the Hamiltonian fluid model leads to an instability if $\kappa < 3^{1/3} \approx 1.44$ (since $\rho_0 = 1$ and $P_0 = 3$). Here we consider the fluid model with $\kappa = 1.30834$.

In Fig. 4, we compare the growth of the first four Fourier modes of the electric field, i.e., with $k\,\bar{\lambda} = 1/2$ (fundamental), $k\,\bar{\lambda} = 1$, $k\,\bar{\lambda} = 3/2$ and $k\bar{\lambda} = 2$ for $L_x/\lambda_D = 2\pi$. As expected, both models, fluid and kinetic, display the instability, i.e., the growth of the electric field with time. The numerical algorithm for the fluid model fails at $\omega_p t \approx 47$, at which time particle trapping becomes predominant in the kinetic model.

The parameter κ has been chosen such that the slope of the linear part of the first mode $k\,\bar{\lambda} = 0.5$ obtained with the fluid model matches the one obtained with the linear kinetic model, i.e., a growth rate of $0.25924553\,\omega_p$ (which has been corrected for the effects of the spatial grid). We notice that both models display some similar features, such as the oscillations at the beginning. Also, the slope of the higher-order modes corresponds rather well, despite the fact that these modes are higher in amplitude for the fluid model.

The main discrepancy between both models occur when the amplitude of the field saturates, which is when the kinetic effects are predominant, and these cannot be described by the fluid model. In addition, all wavenumbers are unstable in the Hamiltonian fluid model while only the fundamental mode is unstable in the kinetic model (the higher harmonics are driven by the fundamental mode through nonlinear couplings). For both models, the initial electric field has the same initial amplitude. Nonetheless, the amplitude of the fundamental

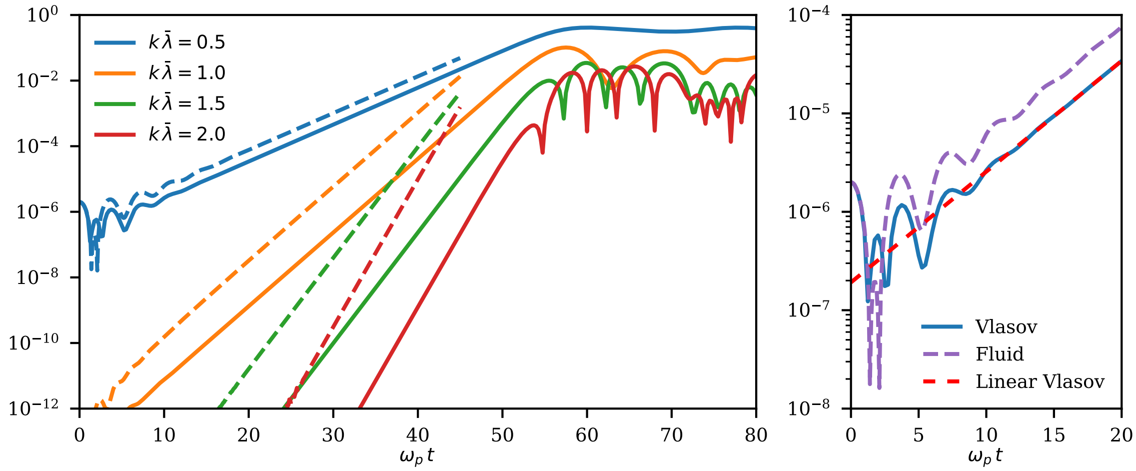

FIG. 4. The left panel shows magnitude of the Fourier modes $E_k(t)$ of the electric field $E(x,t)$ as functions of time for $k\,\bar{\lambda} = 1/2$ (blue curves), $k\,\bar{\lambda} = 1$ (orange curves), $k\,\bar{\lambda} = 3/2$ (green curves) and $k\,\bar{\lambda} = 2$ (read curves). The continuous curves are for the kinetic model, and the dashed curves are for the Hamiltonian fluid model with $\kappa = 1.30834$. The right panel shows the amplitudes for $k\,\bar{\lambda} = 1/2$ for Hamiltonian fluid model (dashed violet), kinetic (blue) and unstable mode from linear kinetic theory (red).

mode is slightly larger in the fluid model compared to the kinetic model (cf. the blue curves on the left panel of Fig. 4). This is due to differences in how the initial condition projects onto the system modes in the two models. In both cases, a linear anylsis produces mode amplitudes that are in excellent agreement with the numerical results.

CONCLUSIONS

We have exhibited a one-parameter family of Hamiltonian fluid models with the first four fluid moments – fluid density, fluid velocity, pressure and heat flux – as a result of the reduction of the one-dimensional Vlasov–Poisson equation. The closure involves an equation for the kurtosis in velocity of the distribution function. In the course of the reduction to a Hamiltonian fluid model, we have identified some normal variables in which the closure expressed parametrically is found to be polynomial in the normal variables. Each reduced Hamiltonian fluid model possesses three Casimir invariants, two of the entropy type and one generalized velocity. We have shown that some of these models ensures the stability of symmetric homogeneous equilibria, depending on the parameter of the closure and the initial conditions.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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