

The dualizing module and top-dimensional cohomology group of $\mathrm{GL}_n(\mathcal{O})$

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Abstract

For a number ring \mathcal{O} , Borel and Serre proved that $\mathrm{SL}_n(\mathcal{O})$ is a virtual duality group whose dualizing module is the Steinberg module. They also proved that $\mathrm{GL}_n(\mathcal{O})$ is a virtual duality group. In contrast to $\mathrm{SL}_n(\mathcal{O})$, we prove that the dualizing module of $\mathrm{GL}_n(\mathcal{O})$ is sometimes the Steinberg module, but sometimes instead is a variant that takes into account a sort of orientation. Using this, we obtain vanishing and nonvanishing theorems for the cohomology of $\mathrm{GL}_n(\mathcal{O})$ in its virtual cohomological dimension.

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1 Introduction

The following contrasting theorems are two of the main results of this paper. Let $\text{cl}(\mathcal{O})$ denote the class group of a number ring \mathcal{O} .

Theorem A (Vanishing). *Let \mathcal{O} be the ring of integers in a number field K and let vcd be the virtual cohomological dimension of $\text{GL}_n(\mathcal{O})$. Assume that n is even and that \mathcal{O}^\times contains an element of norm -1 . Also, letting r and $2s$ be the number of real and complex embeddings of K , assume that $r + s \geq n$. Then $H^{\text{vcd}}(\text{GL}_n(\mathcal{O}); \mathbb{Q}) = 0$.*

Theorem B (Nonvanishing). *Let \mathcal{O} be the ring of integers in a number field K and let vcd be the virtual cohomological dimension of $\text{GL}_n(\mathcal{O})$. Assume either that n is odd or that \mathcal{O}^\times does not contain an element of norm -1 . Then the dimension of $H^{\text{vcd}}(\text{GL}_n(\mathcal{O}); \mathbb{Q})$ is at least $(|\text{cl}(\mathcal{O})| - 1)^{n-1}$.*

In the rest of the introduction, we will explain the origin and motivation for these results. In particular, we will explain why the parity of n and the (non)existence of elements in \mathcal{O}^\times of norm -1 should have something to do with the cohomology of $\text{GL}_n(\mathcal{O})$ in its virtual cohomological dimension.

Remark 1.1. In light of the dichotomy suggested by Theorems A and B, it is natural to wonder which of their hypotheses are necessary. In particular, it is unclear whether the restrictive hypothesis $r + s \geq n$ is needed in Theorem A. We will discuss this at the end of the introduction.

Remark 1.2. Theorem B is closely connected to a recent theorem of Church–Farb–Putman [7] that says that if ν is the virtual cohomological dimension of $\text{SL}_n(\mathcal{O})$, then the dimension of $H^\nu(\text{SL}_n(\mathcal{O}); \mathbb{Q})$ is at least $(|\text{cl}(\mathcal{O})| - 1)^{n-1}$. Note that no assumption on n or \mathcal{O}^\times is necessary. The paper [7] also proves a vanishing theorem for $H^\nu(\text{SL}_n(\mathcal{O}); \mathbb{Q})$ that bears a superficial relationship to Theorem A, but in fact the mechanisms behind the results are completely different. We will discuss this more later in the introduction.

Duality. Let \mathcal{O} be the ring of integers in a number field K and let r and $2s$ be the numbers of real and complex embeddings of K . A fundamental result of Borel–Serre [4] says that the virtual cohomological dimension of $\text{GL}_n(\mathcal{O})$ is

$$\text{vcd} = r \binom{n+1}{2} + sn^2 - n.$$

Even better, they proved that $\text{GL}_n(\mathcal{O})$ is a virtual duality group of dimension vcd . By definition, this means that there is a $\mathbb{Z}[\text{GL}_n(\mathcal{O})]$ -module \mathfrak{D} called the *virtual dualizing module* such that the following holds. Let $G \subset \text{GL}_n(\mathcal{O})$ be a finite-index subgroup, including possibly $G = \text{GL}_n(\mathcal{O})$. Let R be a commutative ring such that for all finite subgroups $F < G$, the order $|F|$ is invertible in R . We thus can take $R = \mathbb{Z}$ if G is torsion-free and $R = \mathbb{Q}$ in all cases. Then for all $R[G]$ -modules M , we have

$$H^{\text{vcd}-i}(G; M) \cong H_i(G; M \otimes \mathfrak{D})$$

for all $i \geq 0$.

Remark 1.3. In most treatments of virtual duality groups, the duality relation is only discussed for torsion-free subgroups of finite index. It is well-known that the above holds for subgroups with torsion, but we do not know a source that gives a detailed proof of this. We will describe how this works for $\mathrm{GL}_n(\mathcal{O})$ in §2.1.

Specializing to $i = 0$ and $G = \mathrm{GL}_n(\mathcal{O})$ and $M = \mathbb{Q}$, this says that

$$\mathrm{H}^{\mathrm{vcd}}(\mathrm{GL}_n(\mathcal{O}); \mathbb{Q}) \cong \mathrm{H}_0(\mathrm{GL}_n(\mathcal{O}); \mathbb{Q} \otimes \mathfrak{D}) \cong (\mathbb{Q} \otimes \mathfrak{D})_{\mathrm{GL}_n(\mathcal{O})},$$

where the subscript indicates that we are taking coinvariants. Theorems A and B can thus be translated into results about the action of $\mathrm{GL}_n(\mathcal{O})$ on its virtual dualizing module \mathfrak{D} . The third main result of this paper identifies \mathfrak{D} .

Special linear group and the Steinberg module. To motivate this identification, we first explain the better-understood case of $\mathrm{SL}_n(\mathcal{O})$. Just like for $\mathrm{GL}_n(\mathcal{O})$, Borel–Serre proved that $\mathrm{SL}_n(\mathcal{O})$ is a virtual duality group of virtual cohomological dimension

$$\nu = r \binom{n+1}{2} + sn^2 - n - r - s + 1.$$

They also gave the following beautiful description of the virtual dualizing module for $\mathrm{SL}_n(\mathcal{O})$: it is the *Steinberg module* for $\mathrm{SL}_n(K)$, which we now describe. Let $\mathcal{T}_n(K)$ be the Tits building for $\mathrm{SL}_n(K)$, i.e. the geometric realization of the poset of K -parabolic subgroups of SL_n . The K -parabolic subgroups of SL_n are precisely the stabilizers of flags

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_r \subsetneq K^n, \tag{1.1}$$

and $\mathcal{T}_n(K)$ can alternately be described as the simplicial complex whose r -simplices are flags as in (1.1). The Solomon–Tits theorem [18, 6] says that $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of $(n-2)$ -spheres. The Steinberg module $\mathrm{St}_n(K)$ is $\tilde{\mathrm{H}}_{n-2}(\mathcal{T}_n(K))$. The action of $\mathrm{SL}_n(\mathcal{O})$ on $\mathrm{St}_n(K)$ is the restriction to $\mathrm{SL}_n(\mathcal{O})$ of the one induced by the action of $\mathrm{SL}_n(K)$ on $\mathcal{T}_n(K)$.

Borel–Serre proved their theorem by constructing a bordification of the symmetric space for $\mathrm{SL}_n(\mathcal{O})$. The boundary of this bordification has a stratification whose combinatorics are encoded by those of the K -parabolic subgroups of SL_n . As a result, the boundary is homotopy equivalent to $\mathcal{T}_n(K)$.

General linear group. To prove that $\mathrm{GL}_n(\mathcal{O})$ is a virtual duality group, Borel–Serre constructed a bordification of its associated symmetric space in terms of the K -parabolic subgroups of GL_n . Since the K -parabolic subgroups of GL_n are also the stabilizers of flags in K^n , it follows that the boundary of their bordification for $\mathrm{GL}_n(\mathcal{O})$ is homotopy equivalent to $\mathcal{T}_n(K)$. This might lead the reader to expect that the virtual dualizing module for $\mathrm{GL}_n(\mathcal{O})$ is also the Steinberg module $\mathrm{St}_n(K)$.

Unfortunately, this is false (see, e.g., [16, §3] and [9, §3.1]). Here is an easy example of this failure. We would like to thank Jeremy Miller and Peter Patzt for pointing it out to us.

Example 1.4. The virtual cohomological dimension of $\mathrm{GL}_2(\mathbb{Z})$ is 1. Let $\Gamma_2(2)$ denote the level-2 principal congruence subgroup of $\mathrm{GL}_2(\mathbb{Z})$, i.e. the kernel of the map $\mathrm{GL}_2(\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{F}_2)$

that reduces matrix entries modulo 2. Letting \mathfrak{D} be the virtual dualizing module for $\mathrm{GL}_2(\mathbb{Z})$ and thus also for its finite-index subgroup $\Gamma_2(2)$, we have

$$H^1(\Gamma_2(2); \mathbb{Q}) \cong H_0(\Gamma_2(2); \mathbb{Q} \otimes \mathfrak{D}) = (\mathbb{Q} \otimes \mathfrak{D})_{\Gamma_2(2)},$$

where the subscripts indicate that we are taking the coinvariants. As the following calculations show, $H^1(\Gamma_2(2); \mathbb{Q}) = 0$ and $(\mathbb{Q} \otimes \mathrm{St}_2(\mathbb{Q}))_{\Gamma_2(2)} \neq 0$, so $\mathrm{St}_2(\mathbb{Q}) \neq \mathfrak{D}$.

- The group $\Gamma_2(2)$ is generated by the matrices

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $c^2 = d^2 = 1$. Also, $cac^{-1} = a^{-1}$ and $cbc^{-1} = b^{-1}$. It follows that all the generators become torsion in the abelianization of $\Gamma_2(2)$, so $H^1(\Gamma_2(2); \mathbb{Q}) = 0$.

- The space $\mathcal{T}_2(\mathbb{Q})$ is the discrete set of lines in \mathbb{Q}^2 . Such lines are in bijection with rank-1 direct summands of \mathbb{Z}^2 , and thus can be reduced modulo 2 to give lines in \mathbb{F}_2^2 . This gives a surjection $\mathcal{T}_2(\mathbb{Q}) \twoheadrightarrow \mathcal{T}_2(\mathbb{F}_2)$ and hence a surjection $\pi: \mathrm{St}_2(\mathbb{Q}) \twoheadrightarrow \mathrm{St}_2(\mathbb{F}_2)$. Since π is $\Gamma_2(2)$ -invariant, it induces a surjection

$$(\mathbb{Q} \otimes \mathrm{St}_2(\mathbb{Q}))_{\Gamma_2(2)} \twoheadrightarrow \mathbb{Q} \otimes \mathrm{St}_2(\mathbb{F}_2) \neq 0. \quad \square$$

What is happening in the above example is that $\mathrm{GL}_2(\mathbb{Z})$ acts in an orientation-reversing way on its symmetric space. The identification of the Steinberg module for $\mathrm{SL}_n(\mathcal{O})$ passes through Poincaré–Lefschetz duality, so to do the same for $\mathrm{GL}_n(\mathcal{O})$ we must take into account orientations.

Dualizing module. If G is a group and A is an abelian group and $\chi: G \rightarrow \{\pm 1\}$ is a homomorphism, then let A_χ denote A endowed with the $\mathbb{Z}[G]$ -module structure arising from the action

$$g \cdot a = \chi(g) \cdot a \quad \text{for all } g \in G \text{ and } a \in A.$$

Our third main theorem is then the following. Recall that the group of units \mathcal{O}^\times is precisely the set of elements of \mathcal{O} whose norm is ± 1 .

Theorem C (Dualizing module). *Let \mathcal{O} be the ring of integers in a number field K and let \mathfrak{D} be the virtual dualizing module of $\mathrm{GL}_n(\mathcal{O})$. Letting $\chi: \mathrm{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ be the composition of the determinant homomorphism with the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$, we then have $\mathfrak{D} \cong \mathrm{St}_n(K) \otimes (\mathbb{Z}_\chi)^{\otimes(n-1)}$.*

The virtual dualizing module of $\mathrm{GL}_n(\mathcal{O})$ is thus different from $\mathrm{St}_n(K)$ if and only if n is even and \mathcal{O}^\times has an element of norm -1 . This latter condition forces \mathcal{O} to have a real embedding, so for instance never holds for rings of integers in imaginary quadratic fields. Beyond this, it is poorly understood which number rings have elements of norm -1 , even for rings of integers in real quadratic fields.

Remark 1.5. Theorem C seems to have been known to the experts, and results like it are mentioned in the literature in several places (see, e.g., [16, §3] and [9, §3.1]). However, no source we are aware of contains a proof of it in complete generality. Since we need Theorem C for Theorems A and B, we have taken this opportunity to fill this hole in the literature.

Cohomology in the vcd. Having identified the virtual dualizing module \mathfrak{D} for $\mathrm{GL}_n(\mathcal{O})$ in Theorem C, we now discuss Theorems A and B, which concern

$$H^{\mathrm{vcd}}(\mathrm{GL}_n(\mathcal{O}); \mathbb{Q}) \cong (\mathbb{Q} \otimes \mathfrak{D})_{\mathrm{GL}_n(\mathcal{O})}.$$

The restriction of the $\mathrm{GL}_n(\mathcal{O})$ -module \mathfrak{D} to $\mathrm{SL}_n(\mathcal{O})$ is simply the Steinberg module $\mathrm{St}_n(K)$. Letting ν be the virtual cohomological dimension of $\mathrm{SL}_n(\mathcal{O})$, we thus have

$$H^\nu(\mathrm{SL}_n(\mathcal{O}); \mathbb{Q}) \cong (\mathbb{Q} \otimes \mathrm{St}_n(K))_{\mathrm{SL}_n(\mathcal{O})} = (\mathbb{Q} \otimes \mathfrak{D})_{\mathrm{SL}_n(\mathcal{O})}.$$

In [7], Church–Farb–Putman proved two results about these $\mathrm{SL}_n(\mathcal{O})$ -coinvariants.

The first result of [7] generalizes a theorem of Lee–Szczerba [11, Theorem 4.1] that says that if \mathcal{O} is Euclidean, then $(\mathbb{Q} \otimes \mathrm{St}_n(K))_{\mathrm{SL}_n(\mathcal{O})} = 0$. The paper [7] says that this also holds if $\mathrm{cl}(\mathcal{O}) = 0$ and \mathcal{O} has a real embedding. Since $(\mathbb{Q} \otimes \mathfrak{D})_{\mathrm{GL}_n(\mathcal{O})}$ is a quotient of $(\mathbb{Q} \otimes \mathrm{St}_n(K))_{\mathrm{SL}_n(\mathcal{O})}$, this implies that under these assumptions we have

$$H^{\mathrm{vcd}}(\mathrm{GL}_n(\mathcal{O}); \mathbb{Q}) \cong (\mathbb{Q} \otimes \mathfrak{D})_{\mathrm{GL}_n(\mathcal{O})} = 0.$$

This vanishing result was already noted by Church–Farb–Putman; we will later comment on its relationship to Theorem A (see the “Trouble” paragraph below).

The second result of [7] says that the dimension of $(\mathbb{Q} \otimes \mathrm{St}_n(K))_{\mathrm{SL}_n(\mathcal{O})}$ is at least $(|\mathrm{cl}(\mathcal{O})| - 1)^{n-1}$. In fact, the proof in [7] actually proves that the dimension of $(\mathbb{Q} \otimes \mathrm{St}_n(K))_{\mathrm{GL}_n(\mathcal{O})}$ is at least $(|\mathrm{cl}(\mathcal{O})| - 1)^{n-1}$, which is a stronger result. The hypotheses of Theorem B are precisely those needed to ensure that $\mathfrak{D} = \mathrm{St}_n(K)$, so Theorem B immediately follows.

A tempting but wrong proof. As we discussed above, Theorem B follows from Theorem C together with the work of Church–Farb–Putman, so it only remains to discuss Theorem A. In light of Theorem C, Theorem A is equivalent to the assertion that under its assumptions, we have

$$(\mathrm{St}_n(K) \otimes \mathbb{Q}_\chi)_{\mathrm{GL}_n(\mathcal{O})} = 0,$$

where $\chi: \mathrm{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ is the composition of the determinant homomorphism and the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$. The Solomon–Tits theorem says that $\mathrm{St}_n(K)$ is generated by apartment classes (see below for the definition), and it is tempting to try to prove this by showing that the images of these apartment classes in $\mathrm{St}_n(K) \otimes \mathbb{Q}_\chi$ vanish in the $\mathrm{GL}_n(\mathcal{O})$ -coinvariants.

The apartment classes $[A_B]$ are indexed by expressions $B = (L_1, \dots, L_n)$ such that the L_i are 1-dimensional subspaces in K^n with $K^n = L_1 \oplus \dots \oplus L_n$. For such a B , let A_B denote the full subcomplex of $\mathcal{T}_n(K)$ spanned by the vertices $\langle L_i \mid i \in I \rangle$, where $I \subset \{1, \dots, n\}$ is a nonempty proper subset. The complex A_B is thus isomorphic to the barycentric subdivision of the boundary of an $(n-1)$ -simplex, and hence is homeomorphic to an $(n-2)$ -sphere. The apartment class is then the image

$$[A_B] \in \tilde{H}_{n-2}(\mathcal{T}_n(K)) = \mathrm{St}_n(K)$$

of the fundamental class of A_B .

The most straightforward way to show that $[A_B] \otimes 1 \in \text{St}_n(K) \otimes \mathbb{Q}_\chi$ vanishes in the $\text{GL}_n(\mathcal{O})$ -coinvariants would be to find some $g \in \text{GL}_n(\mathcal{O})$ such that $g([A_B]) = [A_B]$ but $\chi(g) = -1$; in the $\text{GL}_n(\mathcal{O})$ -coinvariants the elements $[A_B] \otimes 1$ and $g([A_B] \otimes 1) = -([A_B] \otimes 1)$ would then be equal. For a general B , this seems difficult.

However, it is easy to find such $g \in \text{GL}_n(\mathcal{O})$ for the *integral apartments*, i.e. the $B = (L_1, \dots, L_n)$ such that

$$\mathcal{O}^n = (\mathcal{O}^n \cap L_1) \oplus \dots \oplus (\mathcal{O}^n \cap L_n).$$

Indeed, for such a B we can use a $g \in \text{GL}_n(\mathcal{O})$ that scales L_1 by an element of \mathcal{O}^\times whose norm is -1 and fixes all the other L_i . To prove Theorem A, it would thus be enough to prove that $\text{St}_n(K)$ is generated by integral apartments.

Ash–Rudolph [1] proved that if \mathcal{O} is Euclidean, then $\text{St}_n(K)$ is generated by integral apartments. This was extended in [7] to also include \mathcal{O} with a real embedding and $\text{cl}(\mathcal{O}) = 0$. Using a variant of the argument described above that avoids use of the χ -factor, [7] used this to prove their aforementioned vanishing theorem.

Trouble. This leaves the cases of Theorem A that are not consequences of Church–Farb–Putman’s work, i.e. those where $\text{cl}(\mathcal{O}) \neq 0$. Unfortunately, [7] also proves that if $\text{cl}(\mathcal{O}) \neq 0$, then $\text{St}_n(K)$ is *not* generated by integral apartments. Finding a nice generating set for $\text{St}_n(K)$ when \mathcal{O} is not Euclidean or a PID with a real embedding seems like a difficult problem, so we cannot use one to prove Theorem A.

What we do. Our proof of Theorem A is thus by necessity entirely different from the above sketch. Recall that we are trying to prove that $H_0(\text{GL}_n(\mathcal{O}); \text{St}_n(K) \otimes \mathbb{Q}_\chi) = 0$. Our proof of this has two steps. The first is to carefully study the action of $\text{GL}_n(\mathcal{O})$ on the simplicial chain complex of the Tits building to translate our theorem into a sequence of results about the stable *untwisted* cohomology of $\text{GL}_n(\mathcal{O})$. The precise results we need are a bit technical, but the following special case of one of them gives the general flavor. Define $\text{CL}_n(\mathcal{O})$ to be the kernel of the homomorphism $\text{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ obtained by composing the determinant and norm maps.

- Let r and $2s$ be the numbers of real and complex embeddings of K . Then the action of $\text{GL}_n(\mathcal{O})$ on its normal subgroup $\text{CL}_n(\mathcal{O})$ induces the trivial action on $H_k(\text{CL}_n(\mathcal{O}); \mathbb{Q})$ for $0 \leq k \leq \min(n, r + s) - 1$.

See Proposition 4.4 for a more general statement. For some range of k (up to around $\frac{n}{2}$), this could be easily deduced from Borel’s [3] computation of the stable rational cohomology of $\text{SL}_n(\mathcal{O})$. However, we really need the whole range of values of k above – even getting a result that was off by 1 would cause everything to break!

There is a vast literature on homological stability results, and we use some of the technology developed there in a rather non-standard way to prove our theorem. It is a bit surprising that while an optimal homological stability theorem for $\text{SL}_n(\mathcal{O})$ is not known, the technology that has been developed is just barely strong enough to prove a result like the above that gives information well outside the known stable range.

Necessity of hypotheses. Theorem A has three hypotheses:

- n is even, and
- \mathcal{O}^\times contains an element of norm -1 , and
- $r + s \geq n$, where r and $2s$ are the numbers of real and complex embeddings of K .

The third of these is quite restrictive, and it is natural to wonder whether or not it is necessary. For \mathcal{O}^\times to contain an element of norm -1 , it is necessary for \mathcal{O} to have a real embedding. Lee and Szczarba [11, Theorem 4.1] also proved¹ that if \mathcal{O} is Euclidean, then $H^{\text{vcd}}(\text{GL}_n(\mathcal{O}); \mathbb{Q}) = 0$ for $n \geq 2$. The simplest number rings not covered by Lee–Szczarba’s theorem for which an element of norm -1 might exist are therefore real quadratic \mathcal{O} with positive class numbers. For these, the group $\text{GL}_2(\mathcal{O})$ is covered by Theorem A (which says that vanishing does not hold), while the group $\text{GL}_3(\mathcal{O})$ is covered by Theorem B. The smallest possible interesting examples not covered by these known results are thus $\text{GL}_4(\mathcal{O})$ for real quadratic number rings \mathcal{O} with positive class number and elements of norm -1 . Unfortunately, these are complicated enough that we are unaware of any computational data concerning them.

Outline. The two theorems above we must prove are Theorems C and A. We prove Theorem C in §2, and we start the proof of Theorem A in §3, which reduces it to results proved in subsequent sections.

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2 Identifying the virtual dualizing module

In this section, we prove Theorem C. There are two subsections. In §2.1, we use standard techniques to reduce ourselves to the existence of an action of $\text{GL}_n(\mathcal{O})$ on a space with appropriate properties. This space was constructed by Borel–Serre [4], but they did not verify one key property we need. In §2.2 we recall the construction of the space and verify the key property.

2.1 Reduction to a group action

In this section, we will show how Theorem C follows from the following proposition, which is essentially due to Borel–Serre [4]. However, they did not verify all the properties in it, in

¹The reference [11, Theorem 4.1] actually states that if \mathcal{O} is Euclidean, then $H_0(\text{SL}_n(\mathcal{O}); \text{St}_n(K)) = 0$ for $n \geq 2$. Bieri–Eckmann duality implies that this is equivalent to the vanishing of the rational cohomology of $\text{SL}_n(\mathcal{O})$ in its virtual cohomological dimension. Using the Hochschild–Serre spectral sequence associated to the short exact sequence $1 \rightarrow \text{SL}_n(\mathcal{O}) \rightarrow \text{GL}_n(\mathcal{O}) \rightarrow \mathcal{O}^\times \rightarrow 1$, this implies that $H^{\text{vcd}}(\text{GL}_n(\mathcal{O}); \mathbb{Q}) = 0$.

particular conclusion (iv).

Proposition 2.1. *Let \mathcal{O} be the ring of integers in a number field K , and let r and $2s$ be the numbers of real and complex embeddings of K . Let $\chi: \mathrm{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ be the composition of the determinant with the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$. Then there exists a smooth contractible manifold with corners \overline{X} such that the following hold.*

- (i) *The group $\mathrm{GL}_n(\mathcal{O})$ acts smoothly, properly discontinuously, and cocompactly on \overline{X} .*
- (ii) *The boundary $\partial\overline{X}$ is homotopy equivalent to the Tits building $\mathcal{T}_n(K)$, and the restriction of the $\mathrm{GL}_n(\mathcal{O})$ -action to $\partial\overline{X}$ corresponds to the usual action of $\mathrm{GL}_n(\mathcal{O})$ on $\mathcal{T}_n(K)$.*
- (iii) *The dimension of \overline{X} is $d = r\binom{n+1}{2} + sn^2 - 1$.*
- (iv) *For $g \in \mathrm{GL}_n(\mathcal{O})$, the action of g on \overline{X} reverses orientation if and only if n is even and $\chi(g) = -1$.*

We will explain how to extract Proposition 2.1 from Borel–Serre’s work in §2.2. Here we show how to use it to derive Theorem C. This derivation is mostly standard, but we spell it out since we do not know a source that carefully deals with orientations and non-free actions. Indeed, many sources talk about virtual duality groups, but ignore the fact that they are also \mathbb{Q} -duality groups if they have torsion, which is essential for our applications.

We need two lemmas. The first is a tiny generalization of a familiar fact about representations of groups over fields. Recall that if F is a group and $\chi: F \rightarrow \{\pm 1\}$ is a homomorphism and R is a commutative ring, then R_χ is the $R[F]$ -module whose underlying R -module is R and where $g \in F$ acts as multiplication by $\chi(g)$.

Lemma 2.2. *Let F be a finite group, $\chi: F \rightarrow \{\pm 1\}$ be a homomorphism, and R be a commutative ring such that $|F|$ is invertible in R . Then the $R[F]$ -module R_χ is projective.*

Proof. The surjection $\pi: R[F] \rightarrow R_\chi$ defined via the formula

$$\pi(g) = \chi(g) \in R \quad (g \in F)$$

splits via the homomorphism $\iota: R_\chi \rightarrow R[F]$ taking $1 \in R_\chi$ to

$$\frac{1}{|F|} \sum_{g \in F} \chi(g) \cdot g.$$

Thus R_χ is a direct summand of the free $R[F]$ -module $R[F]$, and is therefore projective. \square

Lemma 2.3. *Let G be a group and let Z be a contractible simplicial complex upon which G acts properly discontinuously and cocompactly. Let R be a commutative ring such that for all finite subgroups $F < G$, the order $|F|$ is invertible in R . Then the simplicial chain complex $C_\bullet(Z; R)$ is a resolution of R by finitely generated projective $R[G]$ -modules.*

Proof (compare to [8, Lemma 3.2]). The fact that Z is contractible implies that $C_\bullet(Z; R)$ is a resolution of R . We must prove that each $C_n(Z; R)$ is a finitely generated projective $R[G]$ -module. For an oriented n -simplex σ of Z , let M_σ be the $R[G]$ -submodule of $C_n(Z; R)$ generated by the basis element corresponding to σ . Since G acts cocompactly on Z , there are

finitely many orbits of the action of G on the set of n -simplices of Z . Let $\{\sigma(1), \dots, \sigma(m)\}$ be a set of orbit representatives for this action. Fixing an orientation on each $\sigma(i)$, we have

$$C_n(Z; R) = \bigoplus_{i=1}^m M_{\sigma(i)}.$$

It is thus enough to prove that each $M_{\sigma(i)}$ is a projective $R[G]$ -module. Let $G_{\sigma(i)}$ be the setwise stabilizer of σ . Since the action of G on Z is properly discontinuous, $G_{\sigma(i)}$ is a finite subgroup of G . The action of $G_{\sigma(i)}$ on Z might reverse the orientation of $\sigma(i)$. Let $\chi: G_{\sigma(i)} \rightarrow \{\pm 1\}$ be the homomorphism that records whether or not an element of $G_{\sigma(i)}$ reverses the orientation of $\sigma(i)$. We then have

$$M_{\sigma(i)} = \text{Ind}_{G_{\sigma(i)}}^G R_\chi.$$

Lemma 2.2 says that R_χ is a projective $R[G_{\sigma(i)}]$ -module, i.e. a direct summand of a free $R[G_{\sigma(i)}]$ -module. Since

$$\text{Ind}_{G_{\sigma(i)}}^G R[G_{\sigma(i)}] \cong R[G],$$

it follows that $M_{\sigma(i)}$ is also a direct summand of a free $R[G]$ -module, and is thus projective, as desired. \square

Proof of Theorem C, assuming Proposition 2.1. Let us first recall what must be proved. This requires introducing a large amount of notation:

- Let \mathcal{O} be the ring of integers in a number field K .
- Let $\chi: \text{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ be the composition of the determinant homomorphism and the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$.
- Let r and $2s$ be the numbers of real and complex embeddings of K .
- Let $\text{vcd} = r \binom{n+1}{2} + sn^2 - n$.
- Let G be a finite-index subgroup of $\text{GL}_n(\mathcal{O})$.
- Let R be a commutative ring such that for all finite subgroups $F < G$, the order $|F|$ is invertible in R .

We must prove that G is an R -duality group of dimension vcd with R -dualizing module $\text{St}_n(K) \otimes (R_\chi)^{\otimes(n-1)}$. Since this purported dualizing module is a free R -module, the standard theory of R -duality group (see, e.g. [2, §9]) says that this is equivalent to showing that

$$H^k(G; R[G]) \cong \begin{cases} \text{St}_n(K) \otimes (R_\chi)^{\otimes(n-1)} & \text{if } k = \text{vcd}, \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

for all $k \geq 0$.

Let X and \overline{X} be as in Proposition 2.1, so \overline{X} is a

$$d = r \binom{n+1}{2} + sn^2 - 1$$

dimensional manifold with boundary. Fix a $\text{GL}_n(\mathcal{O})$ -equivariant triangulation of \overline{X} . Lemma 2.3 implies that the simplicial chain complex $C_\bullet(\overline{X}; R)$ is a resolution of R by finitely-generated projective $R[G]$ -modules.

The proof of [5, Proposition VIII.7.5] now shows that

$$H^k(G; R[G]) \cong H_c^k(\overline{X}; R). \quad (2.2)$$

Let R_{or} be the orientation module for the action of $\text{GL}_n(\mathcal{O})$ on \overline{X} , so $R_{\text{or}} = R$ and elements of $\text{GL}_n(\mathcal{O})$ act on R_{or} by ± 1 depending on whether or not they reverse the orientation of \overline{X} . Conclusion (iv) of Proposition 2.1 implies that

$$R_{\text{or}} \cong (R_\chi)^{\otimes(n-1)}. \quad (2.3)$$

Applying Poincaré-Lefschetz duality, we see that as a G -module, we have

$$H_c^k(\overline{X}; R) \cong H_{d-k}(\overline{X}, \partial\overline{X}; R) \otimes R_{\text{or}}. \quad (2.4)$$

Using the fact that \overline{X} is contractible, the long exact sequence of a pair gives

$$H_{d-k}(\overline{X}, \partial\overline{X}; R) \otimes R_{\text{or}} \cong \tilde{H}_{d-k-1}(\partial\overline{X}; R) \otimes R_{\text{or}} \cong \tilde{H}_{d-k-1}(\mathcal{T}_n(K)) \otimes R_{\text{or}}. \quad (2.5)$$

Since the Tits building $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of $(n-2)$ -spheres and

$$d - \text{vcd} - 1 = \left(r \binom{n+1}{2} + sn^2 - 1 \right) - \left(r \binom{n+1}{2} + sn^2 - n \right) - 1 = n - 2,$$

we have

$$\tilde{H}_{d-k-1}(\mathcal{T}_n(K)) \cong \begin{cases} \text{St}_n(K) & \text{if } k = \text{vcd}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.6)$$

Combining (2.2)–(2.6), we obtain (2.1). \square

2.2 The Borel–Serre bordification

Let \mathcal{O} be the ring of integers in an algebraic number field K . A space \overline{X} satisfying the conclusions of Proposition 2.1 was constructed by Borel–Serre [4], who proved that it satisfies the first three conclusions of that proposition. In this section, we recall their construction and verify that it also satisfies the fourth conclusion.

Algebraic groups setup. Let $\mathbf{G} = R_{K/\mathbb{Q}}(\text{GL}_n)$ be the \mathbb{Q} -algebraic group obtained as the restriction of scalars of the K -algebraic group GL_n . We thus have $\mathbf{G}(\mathbb{Q}) \cong \text{GL}_n(K)$ and $\mathbf{G}(\mathbb{Z}) \cong \text{GL}_n(\mathcal{O})$. Let r and $2s$ be the numbers of real and complex embeddings of K and let

$$G = \mathbf{G}(\mathbb{R}) = \text{GL}_n(K \otimes_{\mathbb{Q}} \mathbb{R}) \cong \prod_{i=1}^r \text{GL}_n(\mathbb{R}) \times \prod_{j=1}^s \text{GL}_n(\mathbb{C}).$$

The group $\text{GL}_n(\mathcal{O})$ is thus a discrete subgroup of the real Lie group G . Let

$$\mathfrak{K} = \prod_{i=1}^r \text{O}(n) \times \prod_{j=1}^s \text{U}(n),$$

so \mathfrak{K} is a maximal compact subgroup of G .

Center. Recall that a *bordification* of a smooth manifold Y is a smooth manifold with corners \overline{Y} such that $\text{Int}(\overline{Y}) = Y$. The space \overline{X} constructed by Borel–Serre is a bordification of an appropriate symmetric space X . If we were working with a semisimple group like SL_n , then X would simply be G/\mathfrak{K} . To deal with a reductive group like GL_n , we will have to further quotient G/\mathfrak{K} by the following subgroup of the center. The center of the K -algebraic group GL_n is the multiplicative group \mathbb{G}_m , so $Z(\mathbf{G}) = R_{K/\mathbb{Q}}(\mathbb{G}_m)$. Let \mathbf{S} be the maximal \mathbb{Q} -split torus in $Z(\mathbf{G})$, so

$$\mathbf{S}(\mathbb{Q}) = \mathbb{Q}^\times < K^\times = Z(\mathbf{G})(\mathbb{Q}).$$

Set $S = \mathbf{S}(\mathbb{R}) < G$. Letting Id be the $n \times n$ identity matrix, we have

$$S = \{(a \text{Id}, \dots, a \text{Id}) \mid a \in \mathbb{R}^\times\} \leq Z(G).$$

Symmetric space. Let X be the smooth manifold $G/(\mathfrak{K} \cdot S)$. The space X is a symmetric space of noncompact type, and is thus contractible. This can be seen in an elementary way using the Gram–Schmidt orthogonalization process. Its dimension is

$$\begin{aligned} \dim(X) &= r(\dim(\text{GL}_n(\mathbb{R})) - \dim(\text{O}(n))) + s(\dim(\text{GL}_n(\mathbb{C})) - \dim(\text{U}(n))) - 1 \quad (2.7) \\ &= r(n^2 - \frac{n(n-1)}{2}) + s(2n^2 - n^2) - 1 = r\frac{n(n+1)}{2} + sn^2 - 1. \end{aligned}$$

Since $\text{GL}_n(\mathcal{O}) \cap \mathfrak{K} \cdot S$ is finite, the smooth and properly discontinuous action of $\text{GL}_n(\mathcal{O})$ on G by left multiplication descends to a smooth and properly discontinuous action of $\text{GL}_n(\mathcal{O})$ on X .

Borel–Serre bordification. Borel–Serre [4] prove that X has the following properties.

Theorem 2.4 (Borel–Serre, [4]). *Let the notation be as above. The manifold X has a bordification \overline{X} with the following properties.*

- (i) *The action of $\text{GL}_n(\mathcal{O})$ on X extends to a smooth, properly discontinuous, and cocompact action on \overline{X} .*
- (ii) *The boundary $\partial\overline{X}$ is homotopy equivalent to the Tits building $\mathcal{T}_n(K)$, and the restriction of the $\text{GL}_n(\mathcal{O})$ -action to $\partial\overline{X}$ corresponds to the usual action of $\text{GL}_n(\mathcal{O})$ on $\mathcal{T}_n(K)$.*

Proof. The space X is a “space of type $S - \mathbb{Q}$ for G ” in the language of Borel–Serre; see [4, 2.5(2)]. Borel–Serre construct a manifold with corners \overline{X} containing X as an open submanifold; see [4, 7.1]. Their construction satisfies (i) by [4, 9.3] and satisfies (ii) by [4, 8.4.2]. \square

What remains. Theorem 2.4 says that the bordification \overline{X} satisfies the first two conclusions of Proposition 2.1, and (2.7) shows that \overline{X} satisfies the third conclusion. To prove Proposition 2.1, we must therefore only verify the fourth conclusion, which identifies the elements of $\text{GL}_n(\mathcal{O})$ that preserve the orientation of \overline{X} . Since a diffeomorphism of a smooth manifold with corners is orientation-preserving if and only if its restriction to the interior is orientation-preserving, it is enough to determine which elements of $\text{GL}_n(\mathcal{O})$ preserve the orientation of X . The advantage of doing this is that the whole Lie group G acts on X , and in fact we will determine which elements of G preserve the orientation of X .

Define $\chi: G \rightarrow \mathbb{R}$ via the formula

$$\chi(g_1, \dots, g_r, g'_1, \dots, g'_s) = \det(g_1) \cdots \det(g_r) \cdot |\det(g'_1)| \cdots |\det(g'_s)|.$$

The restriction of χ to $\mathrm{GL}_n(K) \subset \mathrm{GL}_n(K)$ (and hence to $\mathrm{GL}_n(\mathcal{O})$) is the composition of the determinant homomorphism $\mathrm{GL}_n(K) \rightarrow K^\times$ with the norm map $K^\times \rightarrow \mathbb{Q}^\times$. From this, we see that the following lemma generalizes the fourth conclusion of Proposition 2.1.

Lemma 2.5. *Let the notation be as above. For $g \in G$, the action of g on X reverses orientation if and only if n is even and $\chi(g) < 0$.*

Once we prove Lemma 2.5, the proof of Proposition 2.1 will be complete. Before we do this, we must discuss two preliminary results.

Homogeneous spaces and orientations. The first is the following lemma. To interpret it, observe that if M is a connected orientable manifold, then the question of whether a homeomorphism of M preserves the orientation is independent of a choice of orientation.

Lemma 2.6. *Let H be a Lie group and let M be smooth connected orientable homogeneous space for H . Fix a basepoint $p \in M$. Then the action of H on M preserves the orientation of M if and only if the stabilizer H_p preserves the orientation of the tangent space $T_p M$.*

Proof. If the action of H on M preserves the orientation of M , then clearly H_p preserves the orientation of $T_p M$. We must prove the converse. Assume that H_p preserves the orientation of $T_p M$. Since M is connected, it is enough to construct an H -invariant orientation of M . For this, let ω be an orientation on $T_p M$. We can then define an orientation on M by letting the orientation on $T_q M$ for $q \in M$ be $h_*(\omega)$, where $h \in H$ satisfies $h(p) = q$. This is independent of the choice of h , and clearly gives a H -invariant orientation on M . \square

Ignoring the center. Our second lemma will allow us to ignore the difference between $X = G/(\mathfrak{K} \cdot S)$ and G/\mathfrak{K} :

Lemma 2.7. *Let the notation be as above. For $g \in G$, the action of g on X preserves orientation if and only if the action of g on G/\mathfrak{K} preserves orientation.*

Proof. Define $\Psi: G \rightarrow \mathbb{R}_{>0}$ via the formula

$$\Psi(g_1, \dots, g_{r+s}) = |\det g_1| \cdots |\det g_{r+s}|.$$

Via Ψ , the group G acts in an orientation-preserving way on $\mathbb{R}_{>0}$. To prove the lemma, it is thus enough to prove that there is a G -equivariant homeomorphism

$$\mathbb{R}_{>0} \times X \cong G/\mathfrak{K}.$$

To do this, it is enough to prove that $\mathbb{R}_{>0} \times X$ is the homogeneous G -space G/\mathfrak{K} .

Since $\Psi(S) = \mathbb{R}_{>0}$, the subgroup $S < G$ acts transitively on $\mathbb{R}_{>0}$. The subgroup S lies in the center of G , so S acts trivially on $X = G/(\mathfrak{K} \cdot S)$. Together, these facts imply that $\mathbb{R}_{>0} \times X$

is a homogeneous G -space. As a G -space, $\mathbb{R}_{>0}$ is isomorphic to $G/\ker(\Psi)$. We conclude that $\mathbb{R}_{>0} \times X$ is isomorphic as a G -space to G modulo

$$\ker(\Psi) \cap (\mathfrak{K} \cdot S) = \mathfrak{K} \cdot (\ker(\Psi) \cap S) = \mathfrak{K} \cdot (S \cap \mathfrak{K}) = \mathfrak{K}.$$

Here we are using the fact that $\mathfrak{K} \subset \ker(\Psi)$ and that S is central. The lemma follows. \square

Completing the proof. We finally prove Lemma 2.5, thus completing the proof of Proposition 2.1.

Proof of Lemma 2.5. By Lemma 2.7, it is enough to prove that the action of $g \in G$ on G/\mathfrak{K} reverses orientation if and only if n is even and $\chi(g) < 0$. By definition,

$$G = \left(\prod_{i=1}^r \mathrm{GL}_n(\mathbb{R}) \right) \times \left(\prod_{j=1}^s \mathrm{GL}_n(\mathbb{C}) \right)$$

and

$$G/\mathfrak{K} = \left(\prod_{i=1}^r \frac{\mathrm{GL}_n(\mathbb{R})}{\mathrm{O}(n)} \right) \times \left(\prod_{j=1}^s \frac{\mathrm{GL}_n(\mathbb{C})}{\mathrm{U}(n)} \right).$$

The action of G on G/\mathfrak{K} respects these product decompositions. It follows that for $g = (g_1, \dots, g_r, g'_1, \dots, g'_s) \in G$, the action of g on G/\mathfrak{K} reverses orientation if and only if

$$\begin{aligned} & \# \left\{ 1 \leq i \leq r \mid g_i \text{ reverses orientation of } \frac{\mathrm{GL}_n(\mathbb{R})}{\mathrm{O}(n)} \right\} \\ & + \# \left\{ 1 \leq j \leq s \mid g'_j \text{ reverses orientation of } \frac{\mathrm{GL}_n(\mathbb{C})}{\mathrm{U}(n)} \right\} \end{aligned}$$

is odd. Since $\mathrm{GL}_n(\mathbb{C})$ is connected, the action of g'_j will preserve orientation for all $1 \leq j \leq s$. What is more, since

$$\chi(g_1, \dots, g_r, g'_1, \dots, g'_s) = \det(g_1) \cdots \det(g_r) \cdot |\det(g'_1)| \cdots |\det(g'_s)|,$$

we see that $\chi(g) < 0$ if and only if

$$\# \{ 1 \leq i \leq r \mid \det(g_i) < 0 \}$$

is odd. We conclude that to prove the lemma, it is enough to prove the following claim.

Claim. *The subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of elements that fix the orientation of*

$$Y = \frac{\mathrm{GL}_n(\mathbb{R})}{\mathrm{O}(n)}$$

is $\mathrm{GL}_n^{>0}(\mathbb{R})$ if n is even and is $\mathrm{GL}_n(\mathbb{R})$ if n is odd.

Since $\mathrm{GL}_n(\mathbb{R})$ has two components, the subgroup in question is either $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n^{>0}(\mathbb{R})$, so it is enough to prove that $\mathrm{GL}_n(\mathbb{R})$ itself preserves the orientation on Y if and only if n is odd.

By Lemma 2.6, the group $\mathrm{GL}_n(\mathbb{R})$ preserves the orientation on Y if and only if $\mathrm{O}(n)$ preserves the orientation on the tangent space at the identity coset. We can identify this tangent space as the quotient of Lie algebras

$$V = \frac{\mathfrak{gl}_n(\mathbb{R})}{\mathfrak{o}(n)},$$

and the action of $\mathrm{O}(n)$ on it is the one induced by conjugation.

Since $\mathrm{O}(n)$ has only two components and the component of the identity clearly preserve the orientation on this tangent space, it suffices to check a single element of the non-identity component. We will use the matrix $e_{11}(-1)$ obtained from the identity matrix by replacing the entry at $(1, 1)$ with -1 .

For $1 \leq i, j \leq n$, let $a_{ij} \in \mathfrak{gl}_n(\mathbb{R})$ be the matrix with a 1 at position (i, j) and zeros elsewhere. The vector space V has a basis consisting of the cosets of $\{a_{ij} \mid 1 \leq i \leq j \leq n\}$. Conjugation by $e_{11}(-1)$ fixes a_{ii} for $1 \leq i \leq n$, and also fixes a_{ij} for $2 \leq i < j \leq n$. However, conjugation by $e_{11}(-1)$ takes a_{1j} with $j \leq 2 \leq n$ to $-a_{1j}$. This conjugation action thus negates precisely $(n-1)$ elements of our basis, so the determinant of its action on V is $(-1)^{n-1}$. We conclude that $e_{11}(-1)$ preserves the orientation if and only if n is odd, as desired. \square

3 Reduction I: the action on flag stabilizers is trivial

We now begin our proof of Theorem A. In this section, we reduce this theorem to proving that a certain action is trivial.

Setup. Let \mathcal{O} be the ring of integers in a number field K such that \mathcal{O} has an element of norm -1 . Let $\chi: \mathrm{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ be the composition of the determinant with the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$, and define $\mathrm{CL}_n(\mathcal{O}) = \ker(\chi)$. Let \mathfrak{F} be a length- q flag in K^n , i.e. an increasing sequence of subspaces

$$0 \subsetneq \mathfrak{F}_0 \subsetneq \mathfrak{F}_1 \subsetneq \cdots \subsetneq \mathfrak{F}_q \subsetneq K^n.$$

By convention, the degenerate case $q = -1$ simply means the empty flag. Define $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ (resp. $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$) to be the subgroup of $\mathrm{GL}_n(\mathcal{O})$ (resp. $\mathrm{CL}_n(\mathcal{O})$) that preserves \mathfrak{F} . If $q = -1$, then $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F}) = \mathrm{GL}_n(\mathcal{O})$ and $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}) = \mathrm{CL}_n(\mathcal{O})$. The group $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ of index at most 2. See Remark 4.2 below for a proof that it has index equal to 2.

The reduction. The proof of the following proposition begins in §4.

Proposition 3.1. *Let \mathcal{O} be the ring of integers in a number field K and let \mathfrak{F} be a flag in K^n . Assume that \mathcal{O}^\times has an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Then the action of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ on its normal subgroup $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ induces the trivial action on $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$ for $0 \leq k \leq \min(r + s, n) - 1$.*

Here we will assume the truth of Proposition 3.1 and use it to prove Theorem A.

Proof of Theorem A, assuming Proposition 3.1. We start by recalling what we must prove. Let \mathcal{O} be the ring of integers in a number field K and let vcd be the virtual cohomological dimension of $\text{GL}_n(\mathcal{O})$. Assume that the following hold.

- n is even.
- \mathcal{O}^\times contains an element of norm -1 .
- Letting r and $2s$ be the numbers of real and complex embeddings of K , we have $r + s \geq n$.

Our goal is then to prove that $H^{\text{vcd}}(\text{GL}_n(\mathcal{O}); \mathbb{Q}) = 0$. Let $\chi: \text{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ be the composition of the determinant with the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$. Applying Borel–Serre duality and Theorem C, we see that our goal is equivalent to showing that

$$H_0(\text{GL}_n(\mathcal{O}); \text{St}_n(K) \otimes \mathbb{Q}_\chi) = 0.$$

For this, we must study the action of $\text{GL}_n(\mathcal{O})$ on the chain complex for the building $\mathcal{T}_n(K)$.

Let \tilde{C}_\bullet be the usual augmented chain complex calculating the reduced simplicial homology of $\mathcal{T}_n(K)$, so $\tilde{C}_{-1} = \mathbb{Z}$ and

$$H_k(\tilde{C}_\bullet) = \begin{cases} \text{St}_n(K) & \text{if } k = n - 2, \\ 0 & \text{if } k \neq n - 2. \end{cases}$$

The chain complex \tilde{C}_\bullet can be regarded as a chain complex of $\text{GL}_n(K)$ -modules, but we will only consider it as a chain complex of $\text{GL}_n(\mathcal{O})$ -modules. Define $D_\bullet = \tilde{C}_\bullet \otimes \mathbb{Q}_\chi$, so

$$H_k(D_\bullet) = \begin{cases} \text{St}_n(K) \otimes \mathbb{Q}_\chi & \text{if } k = n - 2, \\ 0 & \text{if } k \neq n - 2 \end{cases}$$

as $\text{GL}_n(\mathcal{O})$ -modules.

We will examine the homology of $\text{GL}_n(\mathcal{O})$ with coefficients in the chain complex D_\bullet in the sense of [5, §VII.5]. Letting F_\bullet be a projective resolution of the trivial $\text{GL}_n(\mathcal{O})$ -module \mathbb{Z} , by definition $H_*(\text{GL}_n(\mathcal{O}); D_\bullet)$ is the homology of the double complex $F_\bullet \otimes D_\bullet$. Just like for any double complex, there are two spectral sequences converging to the homology of $F_\bullet \otimes D_\bullet$. The first spectral sequence has

$$E_{pq}^2 = H_q(\text{GL}_n(\mathcal{O}); H_p(D_\bullet)) = \begin{cases} H_q(\text{GL}_n(\mathcal{O}); \text{St}_n(K) \otimes \mathbb{Q}_\chi) & \text{if } p = n - 2, \\ 0 & \text{if } p \neq n - 2. \end{cases}$$

This spectral sequence thus degenerates to show that

$$H_k(\text{GL}_n(\mathcal{O}); D_\bullet) = H_{k-(n-2)}(\text{GL}_n(\mathcal{O}); \text{St}_n(K) \otimes \mathbb{Q}_\chi).$$

We deduce that our goal is equivalent to showing that $H_{n-2}(\text{GL}_n(\mathcal{O}); D_\bullet) = 0$.

The second spectral sequence converging to the homology of $F_\bullet \otimes D_\bullet$ has

$$(E')_{pq}^1 = H_p(\text{GL}_n(\mathcal{O}); D_q).$$

To prove that $H_{n-2}(\mathrm{GL}_n(\mathcal{O}); D_\bullet) = 0$, it is enough to prove that $(E')_{pq}^1 = H_p(\mathrm{GL}_n(\mathcal{O}); D_q) = 0$ for all $p \geq 0$ and $q \geq -1$ such that $p + q = n - 2$. To that end, fix such p and q .

Let \mathcal{F} be the set of length- q flags in K^n ; by convention, for $q = -1$ the set \mathcal{F} consists of the single empty flag. The vector space D_q thus consists of formal \mathbb{Q} -linear combinations of elements of \mathcal{F} , where $\mathrm{GL}_n(\mathcal{O})$ acts on \mathcal{F} via its obvious action and on the coefficients \mathbb{Q} via χ . Let I be a set of orbit representatives for the action of $\mathrm{GL}_n(\mathcal{O})$ on \mathcal{F} . For $\mathfrak{F} \in I$, recall that $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ is the $\mathrm{GL}_n(\mathcal{O})$ -stabilizer of \mathfrak{F} . We have

$$D_q = \bigoplus_{\mathfrak{F} \in I} \mathrm{Ind}_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})}^{\mathrm{GL}_n(\mathcal{O})} \mathbb{Q}_\chi,$$

so

$$H_p(\mathrm{GL}_n(\mathcal{O}); D_q) = \bigoplus_{\mathfrak{F} \in I} H_p(\mathrm{GL}_n(\mathcal{O}); \mathrm{Ind}_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})}^{\mathrm{GL}_n(\mathcal{O})} \mathbb{Q}_\chi) = \bigoplus_{\mathfrak{F} \in I} H_p(\mathrm{GL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}_\chi),$$

where the final isomorphism comes from Shapiro's Lemma. It is thus enough to prove that $H_p(\mathrm{GL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}_\chi) = 0$ for all $\mathfrak{F} \in I$.

Fix $\mathfrak{F} \in I$. Recall that $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ is the kernel of the restriction of $\chi: \mathrm{GL}_n(\mathcal{O}) \rightarrow \{\pm 1\}$ to $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$. Since $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ is a finite-index normal subgroup of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$, the existence of the transfer map shows that

$$H_p(\mathrm{GL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}_\chi) = (H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}_\chi))_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})},$$

where the subscript indicates that we are taking the $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ -coinvariants. See [5, Proposition III.10.4] for more details.

We thus must show that these coinvariants vanish. Since $p = n - 2 - q \leq n - 1$ (with equality precisely when $q = -1$), we can apply Proposition 3.1 to deduce that the action of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ on $H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$ is trivial. Using this along with the fact that $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ acts trivially on \mathbb{Q}_χ , we compute as follows:

$$\begin{aligned} (H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}_\chi))_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})} &= (H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}) \otimes \mathbb{Q}_\chi)_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})} \\ &= H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}) \otimes (\mathbb{Q}_\chi)_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})} \\ &= H_p(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q}) \otimes 0 = 0. \end{aligned}$$

Here we are using the fact that \mathcal{O} has an element of norm -1 , so the group $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ acts nontrivially on \mathbb{Q}_χ and $(\mathbb{Q}_\chi)_{\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})} = 0$. The theorem follows. \square

4 Reduction II: splitting a flag

In the previous section, we reduced Theorem A to Proposition 3.1. In this section, we reduce Proposition 3.1 to two further propositions that will be proven in subsequent sections.

4.1 Basic facts about flags

Before we can do this reduction, we must discuss some basic facts about flags for which [13] is a suitable reference. Let \mathcal{O} be the ring of integers in a number field K . Fix a finite-rank projective \mathcal{O} -module Q and let $n = \text{rk}(Q)$. We can then identify K^n with $Q \otimes K$.

Subspace stabilizers and projective modules. For a subspace V of $K^n = Q \otimes K$, the intersection $V \cap Q$ is a direct summand of Q . Here is a quick proof of this standard fact: $Q/V \cap Q$ is a finitely generated \mathcal{O} -submodule of K^n/V , and thus is torsion-free and hence projective, allowing us to split the short exact sequence

$$0 \longrightarrow V \cap Q \longrightarrow Q \longrightarrow Q/V \cap Q \longrightarrow 0.$$

This implies that $V \cap Q$ is itself a projective \mathcal{O} -module.

Splitting flag stabilizers. Now consider a flag \mathfrak{F} in $K^n = Q \otimes K$ of the form

$$0 \subsetneq \mathfrak{F}_0 \subsetneq \mathfrak{F}_1 \subsetneq \cdots \subsetneq \mathfrak{F}_q \subsetneq K^n.$$

Just like we did for $\text{GL}_n(\mathcal{O})$, we will write $\text{GL}(Q, \mathfrak{F})$ for the subgroup of $\text{GL}(Q)$ stabilizing \mathfrak{F} . Intersecting our flag with Q , we obtain a flag

$$0 \subsetneq \mathfrak{F}_0 \cap Q \subsetneq \mathfrak{F}_1 \cap Q \subsetneq \cdots \subsetneq \mathfrak{F}_q \cap Q \subsetneq Q$$

of direct summands of Q . Each term of this flag is a direct summand of the next one. Iteratively splitting each off from the next, we obtain a decomposition

$$Q = P_0 \oplus P_1 \oplus \cdots \oplus P_{q+1}$$

such that

$$\mathfrak{F}_i \cap Q = P_0 \oplus \cdots \oplus P_i \quad (0 \leq i \leq q).$$

The P_i are all projective \mathcal{O} -modules, and we will call the sequence $\mathfrak{P} = (P_0, \dots, P_{q+1})$ a *projective splitting* of Q with respect to the flag \mathfrak{F} . Define

$$\text{GL}(Q, \mathfrak{P}) = \text{GL}(P_0) \times \cdots \times \text{GL}(P_{q+1}) \subset \text{GL}(Q, \mathfrak{F}).$$

If $Q \cong \mathcal{O}^n$, we will often write $\text{GL}_n(\mathcal{O}, \mathfrak{P})$ instead of $\text{GL}(\mathcal{O}^n, \mathfrak{P})$.

Determinants of automorphisms of projective modules. For a finite-rank projective \mathcal{O} -module P , we have

$$\text{GL}(P) \subset \text{GL}(P \otimes K) \cong \text{GL}_{\text{rk}(P)}(K),$$

so there is a well-defined determinant map $\text{GL}(P) \rightarrow K^\times$. In fact, the image of this map lies in \mathcal{O}^\times :

Lemma 4.1. *Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. Then $\det(f) \in \mathcal{O}^\times$ for $f \in \text{GL}(P)$.*

Proof. Since P is a finite-rank projective \mathcal{O} -module, there exists another finite-rank projective \mathcal{O} -module P' such that $P \oplus P' \cong \mathcal{O}^m$ for some m . Extending automorphisms of P over P' by the identity, we get an embedding $\mathrm{GL}(P) \hookrightarrow \mathrm{GL}(\mathcal{O}^m)$ that fits into a commutative diagram

$$\begin{array}{ccccccc} \mathrm{GL}(P) & \longrightarrow & \mathrm{GL}(P \otimes K) & \xrightarrow{\cong} & \mathrm{GL}_{\mathrm{rk}(P)}(K) & \xrightarrow{\det} & K^\times \\ \downarrow & & \downarrow & & \downarrow & & \downarrow = \\ \mathrm{GL}(\mathcal{O}^m) & \longrightarrow & \mathrm{GL}(\mathcal{O}^m \otimes K) & \xrightarrow{\cong} & \mathrm{GL}_m(K) & \xrightarrow{\det} & K^\times. \end{array}$$

We get an equality on the rightmost vertical arrow since, with respect to an appropriate basis, the map $\mathrm{GL}_{\mathrm{rk}(P)}(K) \rightarrow \mathrm{GL}_m(K)$ is the standard one induced by the inclusion $K^{\mathrm{rk}(P)} \hookrightarrow K^m$. Since matrices in $\mathrm{GL}(\mathcal{O}^n) \cong \mathrm{GL}_n(\mathcal{O})$ have determinant in \mathcal{O}^\times , so do matrices in $\mathrm{GL}(P)$. \square

Assuming now that \mathcal{O}^\times has an element of norm -1 , we can define $\mathrm{CL}(P)$ to be the kernel of the map $\chi: \mathrm{GL}(P) \rightarrow \{\pm 1\}$ obtained by composing the determinant with the norm map $\mathcal{O}^\times \rightarrow \{\pm 1\}$.

Splitting flag stabilizers II. Continue to assume that \mathcal{O}^\times has an element of norm -1 . Recall that Q is a fixed rank- n projective \mathcal{O} -module. If \mathfrak{F} is a length- q flag in $K^n = Q \otimes K$ and $\mathfrak{P} = (P_0, \dots, P_{q+1})$ is a projective splitting of Q with respect to \mathfrak{F} , then using the above we can define

$$\mathrm{CL}(Q, \mathfrak{P}) = \mathrm{CL}(P_0) \times \cdots \times \mathrm{CL}(P_{q+1}) \subset \mathrm{GL}(Q, \mathfrak{P}).$$

The group $\mathrm{CL}(Q, \mathfrak{P})$ is a normal subgroup of $\mathrm{GL}(Q, \mathfrak{P})$ of index 2^{q+2} (see Remark 4.2 below if this is not clear). Just like above, for $Q = \mathcal{O}^n$ we will sometimes write $\mathrm{CL}_n(\mathcal{O}, \mathfrak{P})$ instead of $\mathrm{CL}(Q, \mathfrak{P})$.

Remark 4.2. If \mathfrak{F} is a flag in K^n (possibly the empty flag), then the determinant map $\mathrm{GL}(Q, \mathfrak{F}) \rightarrow \mathcal{O}^\times$ is surjective (and thus if \mathcal{O} has an element of norm -1 , then $\mathrm{CL}(Q, \mathfrak{F})$ is an index-2 subgroup of $\mathrm{GL}(Q, \mathfrak{F})$). Indeed, without loss of generality we can assume that \mathfrak{F} is a maximal flag since this just replaces $\mathrm{GL}(Q, \mathfrak{F})$ by a subgroup. Let $\mathfrak{P} = (P_0, \dots, P_{n-1})$ be a projective splitting of Q with respect to \mathfrak{F} , so we have

$$\mathrm{GL}(Q, \mathfrak{P}) = \mathrm{GL}(P_0) \times \cdots \times \mathrm{GL}(P_{n-1}) \subset \mathrm{GL}(Q, \mathfrak{F}).$$

For all $d \in \mathcal{O}^\times$, the element of $\mathrm{GL}(Q, \mathfrak{P})$ that scales P_0 by d and fixes P_1, \dots, P_{n-1} lies in $\mathrm{GL}(Q, \mathfrak{F})$ and has determinant d .

4.2 The reduction

We now turn to Proposition 3.1. Our goal is to reduce it to two propositions. The first is the following, which informally says in a range of degrees the homology groups of a flag-stabilizer are completely supported on a projective splitting:

Proposition 4.3. *Let \mathcal{O} be the ring of integers in a number field K , let Q be a rank- n projective \mathcal{O} -module, let \mathfrak{F} be a flag in $Q \otimes K$, and let \mathfrak{P} be a projective splitting of Q with respect to \mathfrak{F} . Assume that \mathcal{O}^\times has an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Then the map $\mathrm{H}_k(\mathrm{CL}(Q, \mathfrak{P}); \mathbb{Q}) \rightarrow \mathrm{H}_k(\mathrm{CL}(Q, \mathfrak{F}); \mathbb{Q})$ is a surjection for $0 \leq k \leq r + s - 1$.*

The second is the following, which is a generalization from \mathcal{O}^n to an arbitrary finite-rank projective module of the special case of Proposition 3.1 where the flag is trivial, and thus the conclusion of Proposition 3.1 is that $\mathrm{GL}_n(\mathcal{O})$ acts trivially on the rational homology of $\mathrm{CL}_n(\mathcal{O})$.

Proposition 4.4. *Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. Assume that \mathcal{O}^\times has an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Then the action of $\mathrm{GL}(P)$ on its normal subgroup $\mathrm{CL}(P)$ induces the trivial action on $H_k(\mathrm{CL}(P); \mathbb{Q})$ for $0 \leq k \leq \min(r+s, \mathrm{rk}(P)) - 1$.*

We will prove Propositions 4.3 and 4.4 in §5 and §6, respectively. Here we will assume their truth and derive Proposition 3.1.

Proof of Proposition 3.1, assuming Propositions 4.3 and 4.4. Let us recall the setup. Let \mathcal{O} be the ring of integers in a number field K and let \mathfrak{F} be a flag in K^n . Assume that \mathcal{O}^\times contains an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Consider some $0 \leq k \leq \min(r+s, n) - 1$. We must prove that the action of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$ on its normal subgroup $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ induces the trivial action on $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$.

Let $\mathfrak{P} = (P_0, \dots, P_m)$ be a projective splitting of \mathcal{O}^n with respect to \mathfrak{F} . By Proposition 4.3, the map

$$H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{P}); \mathbb{Q}) \rightarrow H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$$

is surjective. The Künneth formula says that

$$\begin{aligned} H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{P}); \mathbb{Q}) &= H_k(\mathrm{CL}(P_0) \times \cdots \times \mathrm{CL}(P_m); \mathbb{Q}) \\ &\cong \bigoplus_{i_0 + \cdots + i_m = k} H_{i_0}(\mathrm{CL}(P_0); \mathbb{Q}) \otimes \cdots \otimes H_{i_m}(\mathrm{CL}(P_m); \mathbb{Q}). \end{aligned} \quad (4.1)$$

It is thus enough to show that $\mathrm{GL}(\mathcal{O}, \mathfrak{F})$ acts trivially on the images of each of these factors in $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$.

Consider a summand

$$V = H_{i_0}(\mathrm{CL}(P_0); \mathbb{Q}) \otimes \cdots \otimes H_{i_m}(\mathrm{CL}(P_m); \mathbb{Q})$$

of (4.1). Since inner automorphisms always act trivially on homology and $\mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ is an index-2 subgroup of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F})$, it is enough to find a single element of $\mathrm{GL}_n(\mathcal{O}, \mathfrak{F}) \setminus \mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$ that acts trivially on the image of V in $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$. Since

$$i_0 + \cdots + i_m = k \leq \min(r+s, n-1) \quad \text{and} \quad \mathrm{rk}(P_0) + \cdots + \mathrm{rk}(P_m) = n,$$

there must exist some $0 \leq j \leq m$ such that $i_j \leq \min(r+s, \mathrm{rk}(P_j)) - 1$. We can thus apply Proposition 4.4 to see that $\mathrm{GL}(P_j)$ acts trivially on $H_{i_j}(\mathrm{CL}(P_j); \mathbb{Q})$. Pick $x_j \in \mathrm{GL}(P_j)$ such that $x_j \notin \mathrm{CL}(P_j)$ (and thus $\chi(x_j) = -1$). For $0 \leq j' \leq m$ with $j' \neq j$, set $x_{j'} = 1 \in \mathrm{GL}(P_{j'})$. Set

$$x = (x_0, \dots, x_m) \in \mathrm{GL}(P_0) \times \cdots \times \mathrm{GL}(P_m) = \mathrm{GL}_n(\mathcal{O}, \mathfrak{P}).$$

We thus have $x \notin \mathrm{CL}_n(\mathcal{O}, \mathfrak{F})$, and by construction x acts trivially on the image of V in $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{P}); \mathbb{Q})$ and hence also on the image of V in $H_k(\mathrm{CL}_n(\mathcal{O}, \mathfrak{F}); \mathbb{Q})$, as desired. \square

5 The homology carried on a split flag

In this section, we will prove Proposition 4.3. We start in §5.1 with a basic structural result about flag stabilizers, and then in §5.2 we reduce the proof to a simpler homological lemma whose proof occupies the remaining subsections of this section.

5.1 Decomposing stabilizers of flags

Let \mathcal{O} be the ring of integers in an algebraic number field K and let Q be a finite-rank projective \mathcal{O} -module. Proposition 4.3 concerns the homology of the $\mathrm{GL}(Q)$ -stabilizer of a flag. This section shows how to decompose this stabilizer as a semidirect product.

Motivating example. To understand the form this decomposition takes, we start with a familiar example. Let $\Gamma \subset \mathrm{GL}_{n+n'}(\mathbb{R})$ be the subgroup consisting of matrices with an $n' \times n$ block of zeros in their lower left hand corner:

$$\Gamma = \left\{ \left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array} \right) \mid A \in \mathrm{GL}_n(\mathbb{R}) \text{ and } B \in \mathrm{GL}_{n'}(\mathbb{R}) \right\}.$$

The group Γ contains the subgroups

$$\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R}) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \mid A \in \mathrm{GL}_n(\mathbb{R}) \text{ and } B \in \mathrm{GL}_{n'}(\mathbb{R}) \right\}$$

and

$$\mathrm{Mat}_{n,n'}(\mathbb{R}) = \left\{ \left(\begin{array}{c|c} 1 & U \\ \hline 0 & 1 \end{array} \right) \mid U \in \mathrm{Mat}_{n,n'}(\mathbb{R}) \right\}.$$

The additive subgroup $\mathrm{Mat}_{n,n'}(\mathbb{R})$ is normal, and

$$\Gamma = \mathrm{Mat}_{n,n'}(\mathbb{R}) \rtimes (\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R})). \quad (5.1)$$

The action of $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_{n'}(\mathbb{R})$ on $\mathrm{Mat}_{n,n'}(\mathbb{R})$ in (5.1) arises from the identification $\mathrm{Mat}_{n,n'}(\mathbb{R}) = \mathrm{Hom}(\mathbb{R}^{n'}, \mathbb{R}^n)$.

Our decomposition. Our analogue of (5.1) is as follows:

Lemma 5.1. *Let \mathcal{O} be the ring of integers in a number field K , let Q be a rank- n projective \mathcal{O} -module, let \mathfrak{F} be a flag in $Q \otimes K = K^n$, and let $\mathfrak{P} = (P_0, \dots, P_t)$ be a projective splitting of Q with respect to \mathfrak{F} . Set $Q' = P_0 \oplus \dots \oplus P_{t-1}$, so $Q = Q' \oplus P_t$, and let \mathfrak{F}' be the flag in $Q' \otimes K$ obtained by omitting the last term of \mathfrak{F} . Then $\mathrm{GL}(Q, \mathfrak{F}) = \mathrm{Hom}(P_t, Q') \rtimes (\mathrm{GL}(Q', \mathfrak{F}') \times \mathrm{GL}(P_t))$.*

Proof. Elements of $\mathrm{GL}(Q, \mathfrak{F})$ preserve Q' , and thus also act on Q/Q' , which we can identify with P_t . Combining the resulting homomorphisms $\mathrm{GL}(Q, \mathfrak{F}) \rightarrow \mathrm{GL}(Q', \mathfrak{F}')$ and $\mathrm{GL}(Q, \mathfrak{F}) \rightarrow \mathrm{GL}(P_t)$, we get a homomorphism $\phi: \mathrm{GL}(Q, \mathfrak{F}) \rightarrow \mathrm{GL}(Q', \mathfrak{F}') \times \mathrm{GL}(P_t)$. The homomorphism ϕ is a split surjection via the evident inclusion $\mathrm{GL}(Q', \mathfrak{F}') \times \mathrm{GL}(P_t) \hookrightarrow \mathrm{GL}(Q, \mathfrak{F})$. Letting $U = \ker(\phi)$, we have $U \cong \mathrm{Hom}(P_t, Q')$ via the identification that takes $f: P_t \rightarrow Q'$ to the automorphism of Q taking $(x, y) \in Q' \oplus P_t = Q$ to $(x + f(y), y)$. The lemma follows. \square

5.2 A reduction

In this section, we reduce Proposition 4.3 to the following lemma. For later use, we state the lemma in more generality than we need.

Lemma 5.2. *Let \mathcal{O} be the ring of integers in a number field K and let Q and P be finite-rank projective \mathcal{O} -modules. Assume that \mathcal{O} contains an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Let G be an arbitrary subgroup of $\mathrm{GL}(Q)$ and let $\Gamma = \mathrm{Hom}(P, Q) \rtimes (G \times \mathrm{CL}(P))$. Then the map $\mathrm{H}_k(G \times \mathrm{CL}(P); \mathbb{Q}) \rightarrow \mathrm{H}_k(\Gamma; \mathbb{Q})$ is an isomorphism for $0 \leq k \leq r + s - 1$.*

The restriction on k in the statement of Lemma 5.2 is the reason for the restriction on n in Theorem A. The proof of Lemma 5.2 occupies the remaining subsections of this section. Here we show how to derive Proposition 4.3 from it.

Proof of Proposition 4.3, assuming Lemma 5.2. We first recall the setup. Let \mathcal{O} be the ring of integers in a number field K , let Q be a rank- n projective \mathcal{O} -module, let \mathfrak{F} be a flag in $Q \otimes K = K^n$, and let \mathfrak{P} be a projective splitting of Q with respect to \mathfrak{F} . Assume that \mathcal{O}^\times has an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . We must prove that the map $\mathrm{H}_k(\mathrm{CL}(Q, \mathfrak{P}); \mathbb{Q}) \rightarrow \mathrm{H}_k(\mathrm{CL}(Q, \mathfrak{F}); \mathbb{Q})$ is a surjection for $0 \leq k \leq r + s - 1$.

Write $\mathfrak{P} = (P_0, \dots, P_t)$. The proof will be by induction on t . The base case $t = 0$ being trivial, assume that $t \geq 1$ and that the result is true for all smaller t . Let $Q' = P_0 \oplus \dots \oplus P_{t-1}$, so $Q = Q' \oplus P_t$. Let \mathfrak{F}' be the flag in $Q' \otimes K$ obtained by omitting the last term of \mathfrak{F} and let $\mathfrak{P}' = (P_0, \dots, P_{t-1})$, so \mathfrak{P}' is a projective splitting of Q' with respect to \mathfrak{F}' .

Lemma 5.1 says that

$$\mathrm{GL}(Q, \mathfrak{F}) = \mathrm{Hom}(P_t, Q') \rtimes (\mathrm{GL}(Q', \mathfrak{F}') \times \mathrm{GL}(P_t)). \quad (5.2)$$

We factor the map $\mathrm{CL}(Q, \mathfrak{P}) \rightarrow \mathrm{CL}(Q, \mathfrak{F})$ as follows:

$$\begin{aligned} \mathrm{CL}(Q, \mathfrak{P}) &= \mathrm{CL}(Q', \mathfrak{P}') \times \mathrm{CL}(P_t) \xrightarrow{\phi_1} \mathrm{CL}(Q', \mathfrak{F}') \times \mathrm{CL}(P_t) \\ &\xrightarrow{\phi_2} \mathrm{Hom}(P_t, Q') \rtimes (\mathrm{CL}(Q', \mathfrak{F}') \times \mathrm{CL}(P_t)) \xrightarrow{\phi_3} \mathrm{CL}(Q, \mathfrak{F}). \end{aligned}$$

The map ϕ_3 comes from identifying the indicated semidirect product with a subgroup of $\mathrm{CL}(Q, \mathfrak{F})$ via (5.2). It is enough to prove that each ϕ_i induces a surjection on $\mathrm{H}_k(-; \mathbb{Q})$ for $0 \leq k \leq r + s - 1$:

- For ϕ_1 , this comes from combining the Künneth formula with our inductive hypothesis, which implies that the map $\mathrm{H}_k(\mathrm{CL}(Q', \mathfrak{P}'); \mathbb{Q}) \rightarrow \mathrm{H}_k(\mathrm{CL}(Q', \mathfrak{F}'); \mathbb{Q})$ is a surjection for $0 \leq k \leq r + s - 1$.
- For ϕ_2 , this follows from Lemma 5.2.
- For ϕ_3 , this follows from the fact that ϕ_3 is the inclusion of a finite-index subgroup and thus induces a surjection on $\mathrm{H}_k(-; \mathbb{Q})$ for all k , which follows from the existence of the transfer map (see, e.g. [5, §III.9]). \square

5.3 Killing homology with a center

We will prove Lemma 5.2 by studying the Hochschild–Serre spectral sequence of the indicated semidirect product. This spectral sequence is composed of various twisted homology groups, and our goal will be to show that most of them vanish. The following lemma gives a simple criterion for showing this.

Lemma 5.3. *Let G be a group and let M be a finite-dimensional vector space over a field of characteristic 0 upon which G acts. Assume that there exists a central element c of G that fixes no nonzero element of M . Then $H_k(G; M) = 0$ for all k .*

Proof. Let C be the cyclic subgroup of G generated by c . Since c is central, the subgroup C is central and hence normal in G . Define $Q = G/C$. We thus have a short exact sequence

$$1 \longrightarrow C \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

The associated Hochschild–Serre spectral sequence is of the form

$$E_{pq}^2 = H_p(Q; H_q(C; M)) \Rightarrow H_{p+q}(G; M).$$

To prove that $H_k(G; M) = 0$ for all k , it is enough to prove that all terms of this spectral sequence vanish. In fact, we will prove that $H_q(C; M) = 0$ for all q .

Since c fixes no nonzero element of M , the linear map $M \rightarrow M$ taking $x \in M$ to $cx - x \in M$ has a trivial kernel. It is thus an isomorphism, which immediately implies that the C -coinvariants $H_0(C; M) = M_C$ vanish. If c has finite order, then C is a finite group. Since M is a vector space over a field of characteristic 0, this implies that $H_q(C; M) = 0$ for all $q \geq 1$, and we are done. Otherwise, $C \cong \mathbb{Z}$ and we also have to check that $H_1(C; M) = 0$. For this, we apply Poincaré duality to \mathbb{Z} (the fundamental group of a circle!) to see that $H_1(C; M) \cong H^0(C; M) = M^C$. These invariants vanish by assumption. \square

The following lemma will help us recognize when Lemma 5.3 applies.

Lemma 5.4. *Let C be a group and let M be a finite-dimensional vector space on which C acts. Let $\phi: C \rightarrow \mathrm{GL}(M)$ be the associated homomorphism and let $\overline{C} \subset \mathrm{GL}(M)$ be the Zariski closure of $\phi(C)$. Assume that \overline{C} contains an element that fixes no nonzero element of M . Then C does as well.*

Proof. The set of $x \in \mathrm{GL}(M)$ that fix a nonzero element of M is a Zariski-closed subspace; indeed, it is precisely the set of all x such that $\det(x - 1) = 0$. By assumption, \overline{C} is not contained in it, so $\phi(C)$ must not be as well. \square

5.4 The Zariski closure of units

To apply Lemma 5.3 to the Hochschild–Serre spectral sequence associated to the split short exact sequence

$$1 \longrightarrow \mathrm{Hom}(P, Q) \longrightarrow \Gamma \longrightarrow G \times \mathrm{CL}(P) \longrightarrow 1$$

discussed in Lemma 5.2, we need some interesting central elements of $G \times \mathrm{CL}(P)$. Let \mathcal{O}_1^\times be the set of norm-1 units in \mathcal{O} . The central elements we will use are in the subgroup \mathcal{O}_1^\times of $\mathrm{CL}(P)$, which acts on P as scalar multiplication.

We will want to apply Lemma 5.4 to this, which requires identifying the Zariski closure of \mathcal{O}_1^\times in an appropriate real algebraic group. To state the general result we will prove, let r and $2s$ be the numbers of real and complex embeddings of the algebraic number field K , so $\mathcal{O} \otimes \mathbb{R} \cong \mathbb{R}^r \oplus \mathbb{C}^s$, where \mathbb{C}^s is regarded as a $2s$ -dimensional \mathbb{R} -vector space. The group \mathcal{O}^\times acts on $\mathcal{O} \otimes \mathbb{R}$, providing us with a representation

$$\mathcal{O}^\times \longrightarrow \mathrm{GL}(\mathcal{O} \otimes \mathbb{R}) \cong \mathrm{GL}_{r+2s}(\mathbb{R}).$$

The following lemma identifies the Zariski closure of the image of \mathcal{O}_1^\times in $\mathrm{GL}(\mathcal{O} \otimes \mathbb{R})$ when \mathcal{O}^\times has an element of norm -1 , since any such K has a real embedding.

Lemma 5.5. *Let \mathcal{O} be the ring of integers in an algebraic number field K . Assume that K has a real embedding, and let r and $2s$ be the numbers of real and complex embeddings of K , so $\mathcal{O} \otimes \mathbb{R} \cong \mathbb{R}^r \oplus \mathbb{C}^s$. The Zariski closure of the image of \mathcal{O}_1^\times in $\mathrm{GL}(\mathcal{O} \otimes \mathbb{R}) \cong \mathrm{GL}_{r+2s}(\mathbb{R})$ is*

$$\left\{ (a_1, \dots, a_r, b_1, \dots, b_s) \in (\mathbb{R}^\times)^r \times (\mathbb{C}^\times)^s \mid \prod_{j=1}^r a_j \prod_{k=1}^s |b_k| = 1 \right\}.$$

Remark 5.6. Lemma 5.5 is not true for all algebraic number fields. For instance, the norm-1 units in $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$, which are not Zariski dense in $\{b \in \mathbb{C}^\times \mid |b| = 1\}$. It turns out that the conclusion of Lemma 5.5 holds if and only if K does not contain a CM subfield. We will not need this stronger result, so we prove only the above for the sake of brevity.

Lemma 5.5 could be deduced from general results about algebraic tori (see, e.g., [17, Appendix to Chapter 2]). To make this paper more self-contained, we include an elementary proof. We would like to thank Will Sawin for showing it to us.

Our proof will require a consequence of the standard proof of the Dirichlet Unit Theorem (see, e.g., [14, §I.7]). Continuing the above notation, let $f_1, \dots, f_r: K \rightarrow \mathbb{R}$ and $f_{r+1}, \bar{f}_{r+1}, \dots, f_{r+s}, \bar{f}_{r+s}: K \rightarrow \mathbb{C}$ be the real and complex embeddings of K . For $x \in K$, the norm of x equals

$$f_1(x) \cdots f_r(x) \cdot |f_{r+1}(x)|^2 \cdots |f_{r+s}(x)|^2. \quad (5.3)$$

For $x \in \mathcal{O}^\times$, this will be ± 1 . To convert the multiplication in \mathcal{O}^\times into addition and also to eliminate the distinction between ± 1 , we take absolute values and logarithms, and define $\Phi: K^\times \rightarrow \mathbb{R}^{r+s}$ via the formula

$$\Psi(x) = (\log |f_1(x)|, \dots, \log |f_r(x)|, 2 \log |f_{r+1}(x)|, \dots, 2 \log |f_{r+s}(x)|) \in \mathbb{R}^{r+s}.$$

For $x \in \mathcal{O}^\times$, the fact that (5.3) is ± 1 implies that

$$\log |f_1(x)| + \cdots + \log |f_r(x)| + 2 \log |f_{r+1}(x)| + \cdots + 2 \log |f_{r+s}(x)| = 0.$$

In other words, for $x \in \mathcal{O}^\times$ the image $\Psi(x) \in \mathbb{R}^{r+s}$ lies in the hyperplane

$$H = \{(x_1, \dots, x_{r+s}) \mid x_1 + \cdots + x_{r+s} = 0\}.$$

The key step in the proof of the Dirichlet Unit Theorem is showing that $\Psi(\mathcal{O}^\times)$ is a lattice in H ; see [14, Theorem I.7.3]. Since the norm-1 units \mathcal{O}_1^\times are an index-2 subgroup of \mathcal{O}^\times , it follows that $\Psi(\mathcal{O}_1^\times)$ also forms a lattice in H . As a consequence, we deduce the following.

Lemma 5.7. *Letting the notation be as above, consider $c_1, \dots, c_{r+s} \in \mathbb{R}$ such that*

$$c_1 \log |f_1(x)| + \dots + c_{r+s} \log |f_{r+s}(x)| = 0 \quad \text{for all } x \in \mathcal{O}_1^\times.$$

Then $2c_1 = \dots = 2c_r = c_{r+1} = \dots = c_{r+s}$.

Proof of Lemma 5.5. Let $f_1, \dots, f_{r+2s}: \mathcal{O}_1^\times \rightarrow \overline{\mathbb{Q}}^\times$ be the restrictions to \mathcal{O}_1^\times of the different embeddings of K into $\overline{\mathbb{Q}}$, ordered in an arbitrary way. The norm of an element of K is the product of its images under the different embeddings of K into $\overline{\mathbb{Q}}$, so since \mathcal{O}_1^\times consists of elements of norm 1 we have $f_1 \cdots f_{r+2s} = 1$. Let Λ be the \mathbb{R} -algebra of \mathbb{C} -valued functions on \mathcal{O}_1^\times . Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and let $\phi: \mathbb{R}[x_1^{\pm 1}, \dots, x_{r+2s}^{\pm 1}] \rightarrow \Lambda$ be the algebra map taking x_i to f_i . We have $x_1 \cdots x_{r+2s} - 1 \in \ker(\phi)$, and the lemma is equivalent to the assertion that the ideal I in $\mathbb{R}[x_1^{\pm 1}, \dots, x_{r+2s}^{\pm 1}]$ generated by $x_1 \cdots x_{r+2s} - 1$ equals $\ker(\phi)$. Note that this is independent of the order of the embeddings f_1, \dots, f_{r+2s} .

The starting point is the following special case.

Claim. *Let $m \in \mathbb{R}[x_1^{\pm 1}, \dots, x_{r+2s}^{\pm 1}]$ be a monomial such that $m - 1 \in \ker(\phi)$. Then $m - 1 \in I$.*

Proof of claim. Write $m = x_1^{d_1} \cdots x_{r+2s}^{d_{r+2s}}$ with each $d_i \in \mathbb{Z}$. To prove the claim, it is enough to prove that all the d_i are equal. Since $m - 1 \in \ker(\phi)$, the function

$$\phi(m) = f_1^{d_1} \cdots f_{r+2s}^{d_{r+2s}}: \mathcal{O}_1^\times \rightarrow \overline{\mathbb{Q}}$$

is the trivial character. Reordering the f_i if necessary, we can assume that $d_1 \geq d_i$ for all $1 \leq i \leq r+2s$. Since there is at least one real embedding of K , we can change our embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ by precomposing it with an appropriate element of the absolute Galois group and ensure that f_1 is a real embedding. We finally reorder f_2, \dots, f_{r+2s} such that f_1, \dots, f_r are the real embeddings, such that f_{r+1}, \dots, f_{r+2s} are the complex embeddings, and such that $\overline{f_{r+i}} = f_{r+i+s}$ for all $1 \leq i \leq s$.

For $u \in \mathcal{O}_1^\times$, we have $f_1^{d_1}(u) \cdots f_{r+2s}^{d_{r+2s}}(u) = 1$, so

$$\begin{aligned} 1 &= \left(f_1^{d_1}(u) \cdots f_{r+2s}^{d_{r+2s}}(u) \right) \overline{\left(f_1^{d_1}(u) \cdots f_{r+2s}^{d_{r+2s}}(u) \right)} \\ &= |f_1(u)|^{2d_1} \cdots |f_r(u)|^{2d_r} |f_{r+1}(u)|^{2d_{r+1}+2d_{r+1+s}} \cdots |f_{r+s}(u)|^{2d_{r+s}+2d_{r+2s}}. \end{aligned}$$

Taking logarithms and dividing by 2, we see that

$$\begin{aligned} 0 &= d_1 \log |f_1(u)| + \dots + d_r \log |f_r(u)| \\ &\quad + (d_{r+1} + d_{r+1+s}) \log |f_{r+1}(u)| + \dots + (d_{r+s} + d_{r+2s}) \log |f_{r+s}(u)| \end{aligned}$$

for all $u \in \mathcal{O}_1^\times$. Lemma 5.7 then implies that

$$2d_1 = \dots = 2d_r = d_{r+1} + d_{r+1+s} = \dots = d_{r+s} + d_{r+2s}.$$

Since $d_1 \geq d_{r+i}$ and $d_1 \geq d_{r+i+s}$, the only way that we can have $d_{r+i} + d_{r+i+s} = 2d_1$ is for $d_{r+i} = d_{r+i+s} = d_1$, so in fact

$$d_1 = d_2 = \cdots = d_{r+2s},$$

as desired. \square

We now turn to the general case. Consider a nonzero $\theta \in \ker(\phi)$. Write

$$\theta = \sum_{i=1}^k \lambda_i m_i \quad \text{with } m_i \in \mathbb{R}[x_1^{\pm 1}, \dots, x_{r+2s}^{\pm 1}] \text{ a monomial and } \lambda_i \in \mathbb{R}.$$

Collecting terms, we can assume that the m_i are all distinct and that $\lambda_i \neq 0$ for all i . For $1 \leq i \leq k$, the image $\phi(m_i): \mathcal{O}_1^\times \rightarrow \mathbb{C}$ is a character. Since distinct characters on an abelian group are linearly independent, it follows that there are distinct $1 \leq j, j' \leq k$ such that $\phi(m_j) = \phi(m_{j'})$. This implies that $\phi(m_j m_{j'}^{-1})$ is the trivial character, so $m_j m_{j'}^{-1} - 1 \in \ker(\phi)$. The above claim thus implies that $m_j m_{j'}^{-1} - 1 \in I$, so

$$m_j - m_{j'} = m_{j'}(m_j m_{j'}^{-1} - 1) \in I.$$

Subtracting $\lambda_j(m_j - m_{j'}) \in I$ from θ eliminates its $\lambda_j m_j$ term. Collecting terms in θ and repeating the above argument over and over again, we conclude that $\theta \in I$, as desired. \square

5.5 The proof of Lemma 5.2

We finally prove Lemma 5.2.

Proof of Lemma 5.2. We start by recalling what we want to prove. Let \mathcal{O} be the ring of integers in a number field K and let Q and P be finite-rank projective \mathcal{O} -modules. Assume that \mathcal{O} contains an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . Let G be an arbitrary subgroup of $\mathrm{GL}(Q)$ and let $\Gamma = \mathrm{Hom}(P, Q) \rtimes (G \times \mathrm{CL}(P))$. Our goal is to prove that the map $\mathrm{H}_k(G \times \mathrm{CL}(P); \mathbb{Q}) \rightarrow \mathrm{H}_k(\Gamma; \mathbb{Q})$ is an isomorphism for $0 \leq k \leq r + s - 1$. It is a little easier (but equivalent) to prove this with real coefficients.

The Hochschild–Serre spectral sequence for the split extension

$$1 \longrightarrow \mathrm{Hom}(P, Q) \longrightarrow \Gamma \longrightarrow G \times \mathrm{CL}(P) \longrightarrow 1$$

is of the form

$$E_{pq}^2 = \mathrm{H}_p(G \times \mathrm{CL}(P); \mathrm{H}_q(\mathrm{Hom}(P, Q); \mathbb{R})) \Rightarrow \mathrm{H}_{p+q}(\Gamma; \mathbb{R}).$$

We have

$$E_{k0}^2 = \mathrm{H}_k(G \times \mathrm{CL}(P); \mathbb{R}),$$

and to prove the lemma it is enough to prove that $E_{pq}^2 = 0$ for all p and all $1 \leq q \leq r + s - 1$. Fix some $1 \leq q \leq r + s - 1$. The group $\mathrm{CL}(P)$ contains the central subgroup \mathcal{O}_1^\times , which acts on P as scalar multiplication. Combining Lemmas 5.3 and 5.4, it is enough to prove that the

Zariski closure of the image of \mathcal{O}_1^\times in the group $\mathrm{GL}(\mathrm{H}_q(\mathrm{Hom}(P, Q); \mathbb{R}))$ contains an element that fixes no nonzero vector of $\mathrm{H}_q(\mathrm{Hom}(P, Q); \mathbb{R})$.

We have

$$\mathrm{H}_q(\mathrm{Hom}(P, Q); \mathbb{R}) \cong \wedge^q(\mathrm{Hom}(P, Q) \otimes \mathbb{R}).$$

We now identify $\mathrm{Hom}(P, Q) \otimes \mathbb{R}$:

Claim. *Let $n = \mathrm{rk}(P)$ and $m = \mathrm{rk}(Q)$. We then have*

$$\mathrm{Hom}(P, Q) \otimes \mathbb{R} \cong \mathrm{Mat}_{n,m}(\mathcal{O} \otimes \mathbb{R}).$$

Proof of claim. By the classification of finitely generated projective modules over Dedekind domains (see, e.g. [13, §1]), there exist nonzero ideals $I, J \subset \mathcal{O}$ such that $P = \mathcal{O}^{n-1} \oplus I$ and $Q = \mathcal{O}^{m-1} \oplus J$. Using this identification, we see $\mathrm{Hom}(P, Q)$ can be viewed as

$$\left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \mid A \in \mathrm{Mat}_{m-1,n-1}(\mathcal{O}), B \in \mathrm{Mat}_{m-1,1}(I^{-1}), C \in \mathrm{Mat}_{1,n-1}(J), D \in JI^{-1} \right\}.$$

Here $I^{-1} \subset K$ is the inverse of I using the usual multiplication of fractional ideals in a Dedekind domain. The claim now follows from the fact that

$$\mathcal{O} \otimes \mathbb{R} = J \otimes \mathbb{R} = I^{-1} \otimes \mathbb{R} = JI^{-1} \otimes \mathbb{R} = K \otimes \mathbb{R}. \quad \square$$

From this, we see that the action of \mathcal{O}_1^\times on $\wedge^q(\mathrm{Hom}(P, Q) \otimes \mathbb{R})$ can be identified with the action of \mathcal{O}_1^\times on

$$V := \wedge^q(\mathcal{O} \otimes \mathbb{R})^{nm} \cong \wedge^q(\mathbb{R}^r \oplus \mathbb{C}^s)^{nm}.$$

Identify $\mathrm{GL}(\mathbb{R}^r \oplus \mathbb{C}^s)$ as a Zariski-closed subgroup of $\mathrm{GL}(V)$ in the natural way. By Lemma 5.5, the Zariski closure of the image of \mathcal{O}_1^\times in $\mathrm{GL}(V)$ can be identified with

$$\left\{ (a_1, \dots, a_r, b_1, \dots, b_s) \in (\mathbb{R}^\times)^r \times (\mathbb{C}^\times)^s \mid \prod_{j=1}^r a_j \prod_{k=1}^s |b_k| = 1 \right\}, \quad (5.4)$$

which acts on $\mathbb{R}^r \oplus \mathbb{C}^s$ by scalar multiplication. We claim that the element

$$x = (a_1, \dots, a_r, b_1, \dots, b_s) = (2, \dots, 2, \frac{1}{2^{r+s-1}})$$

fixes no nonzero vector in V . Indeed, the eigenvalues for the action of x on V lie in the set of elements that can be expressed as the product of q elements of $\{2, \frac{1}{2^{r+s-1}}\}$, and $q \leq r + s - 1$, so 1 cannot be expressed in this form. \square

6 The action on automorphisms of projectives is trivial

In this section, we prove Proposition 4.4. The actual proof is in §6.3. This is preceded by two sections of preliminary results.

6.1 Equivariant homology

Our proof of Proposition 4.4 will use a bit of equivariant homology. In this section, we review some standard facts about this. See [5, §VII.7] for a textbook reference.

Semisimplicial sets. The natural setting for our proof is that of semisimplicial sets, which are a technical variant on simplicial complexes whose definition we briefly recall. For more details, see [10], which calls them Δ -sets. Let Δ be the category with objects the sets $[k] = \{0, \dots, k\}$ for $k \geq 0$ and whose morphisms $[k] \rightarrow [\ell]$ are the strictly increasing functions. A *semisimplicial set* is a contravariant functor X from Δ to the category of sets. The k -simplices of X are the image $X^{(k)}$ of $[k] \in \Delta$. The maps $X^{(\ell)} \rightarrow X^{(k)}$ corresponding to the Δ -morphisms $[k] \rightarrow [\ell]$ are called the *boundary maps*.

Geometric properties. A semisimplicial set X has a geometric realization $|X|$ obtained by taking geometric k -simplices for each element of $X^{(k)}$ and then gluing these simplices together using the boundary maps. Whenever we talk about topological properties of a semisimplicial set, we are referring to its geometric realization. An action of a group G on a semisimplicial set X consists of actions of G on each $X^{(k)}$ that commute with the boundary maps. This induces an action of G on $|X|$. The quotient X/G is naturally a semisimplicial set with k -simplices $X^{(k)}/G$.

Definition of equivariant homology. Let G be a group and let X be a semisimplicial set on which G acts. For a ring R , there are two equivalent definitions of the equivariant homology groups $H_*^G(X; R)$:

- Let EG be a contractible semisimplicial set on which G acts freely, so EG/G is a $K(G, 1)$. The group G then acts freely on $EG \times X$, and $H_*^G(X; R)$ is the homology with coefficients in R of the quotient space $(EG \times X)/G$.
- Let $F_\bullet \rightarrow \mathbb{Z}$ be a projective resolution of the trivial $\mathbb{Z}[G]$ -module \mathbb{Z} and let $C_\bullet(X; R)$ be the simplicial chain complex of X with coefficients in R . Then $H_*^G(X; R)$ is the homology of the double complex $F_\bullet \otimes C_\bullet(X; R)$.

Neither of these definitions depends on any choices.

Functoriality. Equivariant homology is functorial in the following sense. If G and G' are groups acting on semisimplicial sets X and X' , respectively, and if $f: G \rightarrow G'$ is a group homomorphism and $\phi: X \rightarrow X'$ is a map such that $\phi(gx) = f(g)\phi(x)$ for all $g \in G$ and $x \in X$, then we get an induced map $H_*^G(X; R) \rightarrow H_*^{G'}(X'; R)$.

Map to a point. If $\{p_0\}$ is a single point on which G acts trivially, then $H_*^G(\{p_0\}; R) = H_*(G; R)$. For an arbitrary semisimplicial set X on which G acts, the projection $X \rightarrow \{p_0\}$ thus induces a map $H_*^G(X; R) \rightarrow H_*(G; R)$. For this map, we have the following lemma.

Lemma 6.1. *Let X be an n -connected semisimplicial set on which a group G acts and let R be a ring. Then the natural map $H_k^G(X; R) \rightarrow H_k(G; R)$ is an isomorphism for $k \leq n$ and a surjection for $k = n + 1$.*

Proof. See [5, Proposition VII.7.3]. This reference assumes that X is contractible, but its proof gives the desired conclusion when X is assumed to be merely n -connected. \square

The spectral sequence. One of the main calculational tools for equivariant homology is as follows.

Lemma 6.2. *Let X be a semisimplicial set on which a group G acts and let R be a ring. For each simplex σ of X/G , let $\tilde{\sigma}$ be a lift of σ to X and let $G_{\tilde{\sigma}}$ be the stabilizer of $\tilde{\sigma}$. Then there is a spectral sequence*

$$E_{pq}^1 \cong \bigoplus_{\sigma \in (X/G)^{(q)}} H_p(G_{\tilde{\sigma}}; R) \Rightarrow H_{p+q}^G(X; R).$$

Proof. See [5, VII.(7.7)]. □

Group actions on equivariant homology. Now let Γ be a group acting on a semisimplicial set X and let G be a normal subgroup of Γ . For $\gamma \in \Gamma$, the maps $G \rightarrow G$ and $X \rightarrow X$ taking $g \in G$ to $\gamma g \gamma^{-1}$ and $x \in X$ to γx induce a map $H_*^G(X; R) \rightarrow H_*^G(X; R)$. This recipe gives an action of Γ on $H_*^G(X; R)$. The restriction of this action to G is trivial (this can be proved just like [5, Proposition III.8.1], which proves that inner automorphisms act trivially on ordinary group homology), so we get an induced action of Γ/G on $H_*^G(X; R)$. It is clear from its construction that the spectral sequence in Lemma 6.2 is a spectral sequence of $R[\Gamma/G]$ -modules.

6.2 The complex of lines

Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. Assume that \mathcal{O}^\times has an element of norm -1 , so we can talk about the group $\text{CL}(P)$. This group acts on the following space.

Definition 6.3. Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. A *line decomposition* of P is an ordered sequence (L_1, \dots, L_n) of rank-1 projective submodules of P such that $P = L_1 \oplus \dots \oplus L_n$. The *complex of lines* in P , denoted $\mathcal{L}(P)$, is the semisimplicial set whose $(k-1)$ -simplices are ordered sequences (L_1, \dots, L_k) of rank-1 projective submodules of P that can be extended to a line decomposition (L_1, \dots, L_n) . □

We thus have the equivariant homology groups $H_k^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$. Our main result about these equivariant homology groups is as follows.

Lemma 6.4. *Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. Assume that \mathcal{O}^\times has an element of norm -1 . Then the natural map $H_k^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q}) \rightarrow H_k(\text{CL}(P); \mathbb{Q})$ is a surjection for $0 \leq k \leq \text{rk}(P) - 1$.*

Proof. If $\mathcal{L}(P)$ were $(\text{rk}(P) - 2)$ -connected, then this would follow from Lemma 6.1. Unfortunately, this is not known and is likely to be false² – a slight strengthening of this would

²In [12, Proposition 2.12], it is proven that $\mathcal{L}(\mathcal{O}^2)$ is not connected when \mathcal{O} is quadratic imaginary but not a PID; however, such \mathcal{O} cannot contain an element of norm -1 , so this does not quite give a counterexample under our given hypotheses.

allow one to run the argument used to prove [7, Theorem A] and prove a result contradicting [7, Theorem B' in §5.3]. We will need an alternative approach.

Let $n = \text{rk}(P)$. By the classification of finitely generated projective modules over Dedekind domains (see, e.g. [13, §1]), there exist a nonzero ideal $I \subset \mathcal{O}$ such that $P = \mathcal{O}^{n-1} \oplus I$. Using this identification, we see that every element of $\text{GL}(P)$ is of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad \text{with } A \in \text{Mat}_{n-1, n-1}(\mathcal{O}), B \in \text{Mat}_{n-1, 1}(I^{-1}), C \in \text{Mat}_{1, n-1}(I), D \in \mathcal{O}.$$

Here $I^{-1} \subset K$ is the inverse of I using the usual multiplication of fractional ideals in a Dedekind domain. Define $\Gamma' = \text{GL}(P) \cap \text{CL}_n(\mathcal{O})$, so Γ' is finite-index in both $\text{CL}(P)$ and $\text{CL}_n(\mathcal{O})$. Modulo I , the last row of an element of Γ' is of the form $(0, \dots, 0, *)$. Define Γ to be the subgroup of matrices in Γ' whose last row equals $(0, \dots, 0, 1)$ modulo I . The group Γ is thus finite-index in Γ' .

We now construct a space for Γ to act on. Define $B_n(\mathcal{O}, I)$ to be the semisimplicial set whose $(m-1)$ -simplices are ordered sequences (v_1, \dots, v_m) of elements of \mathcal{O}^n that can be extended to a sequence (v_1, \dots, v_n) with the following properties:

- The v_i form a free \mathcal{O} -basis for \mathcal{O}^n .
- The last coordinate of each v_i equals either 0 or 1 modulo I .
- Precisely 1 of the v_i has a last coordinate equal to 1 modulo I .

The action of the group Γ on \mathcal{O}^n fixes the last coordinate modulo I . It follows that Γ acts on $B_n(\mathcal{O}, I)$. We will prove in Lemma 6.5 below that $B_n(\mathcal{O}, I)$ is $(n-2)$ -connected; in fact, this result was *almost* proved in [7], and we will show how to derive it from results in this paper. For now we will continue with the proof of Lemma 6.4 assuming that $B_n(\mathcal{O}, I)$ is $(n-2)$ -connected.

We now come to the key fact that relates the above to $\mathcal{L}(P)$ and $\text{CL}(P)$:

Claim. *There is a simplicial map $\Psi: B_n(\mathcal{O}, I) \rightarrow \mathcal{L}(P)$ taking a vertex v of $B_n(\mathcal{O}, I)$ to the \mathcal{O} -submodule $P \cap (\mathcal{O} \cdot v)$ of P .*

Proof of claim. It is enough to prove that if (v_1, \dots, v_n) is a top-dimensional simplex of $B_n(\mathcal{O}, I)$, then $(P \cap (\mathcal{O} \cdot v_1), \dots, P \cap (\mathcal{O} \cdot v_n))$ is a line decomposition of P . In other words, letting $L_i = P \cap (\mathcal{O} \cdot v_i)$ we must prove that $P = L_1 \oplus \dots \oplus L_n$.

Consider $x \in P$. We must prove that x can be uniquely expressed as $x = x_1 + \dots + x_n$ with $x_i \in L_i$ for $1 \leq i \leq n$. Since the v_i form a free \mathcal{O} -basis of \mathcal{O}^n and $P \subset \mathcal{O}^n$, there exists unique $\lambda_1, \dots, \lambda_n \in \mathcal{O}$ such that $x = \lambda_1 v_1 + \dots + \lambda_n v_n$. We have to show that $\lambda_i v_i \in P$ for all $1 \leq i \leq n$.

Let $1 \leq i_0 \leq n$ be the unique index such that the last coordinate of v_{i_0} equals 1 modulo I . For $1 \leq i \leq n$ with $i \neq i_0$, the last coordinate of v_i thus equals 0 modulo I , so $v_i \in P$ and thus $\lambda_i v_i \in P$. As for $\lambda_{i_0} v_{i_0}$, we have

$$\lambda_{i_0} v_{i_0} = x - \sum_{i \neq i_0} \lambda_i v_i.$$

Each term on the right hand side is an element of P , so $\lambda_{i_0} v_{i_0}$ is as well. □

The map Ψ along with the inclusion $\Gamma \hookrightarrow \mathrm{CL}(P)$ induces a map $H_k^\Gamma(B_n(\mathcal{O}, I); \mathbb{Q}) \rightarrow H_k^{\mathrm{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$. This map fits into a commutative diagram

$$\begin{array}{ccc} H_k^\Gamma(B_n(\mathcal{O}, I); \mathbb{Q}) & \longrightarrow & H_k^{\mathrm{CL}(P)}(\mathcal{L}(P); \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_k(\Gamma; \mathbb{Q}) & \longrightarrow & H_k(\mathrm{CL}(P); \mathbb{Q}). \end{array}$$

Since Γ is a finite-index subgroup of $\mathrm{CL}(P)$, the existence of the transfer map (see [5, Proposition III.10.4]) implies that the bottom row of this diagram is a surjection. Since $B_n(\mathcal{O}, I)$ is $(n-2)$ -connected by Lemma 6.5 below, Lemma 6.1 implies that the left column of this diagram is a surjection. We conclude that the right column of this diagram is a surjection, as desired. \square

It remains to prove the following result, which was promised during the above proof.

Lemma 6.5. *Let \mathcal{O} be the ring of integers in a number field K and let $I \subset \mathcal{O}$ be a nonzero ideal. Assume that \mathcal{O} has a real embedding (which hold, for instance, if \mathcal{O}^\times has an element of norm -1). Then the space $B_n(\mathcal{O}, I)$ defined in the proof of Lemma 6.4 above is $(n-2)$ -connected.*

Proof. We start by introducing an auxiliary space. Define $\widehat{B}_n(\mathcal{O}, I)$ to be the simplicial complex whose $(m-1)$ -simplices are unordered sets $\{v_1, \dots, v_m\}$ of elements of \mathcal{O}^n such that some (equivalently, any) ordering is an $(m-1)$ -simplex of $B_n(\mathcal{O}, I)$.

Recall that a simplicial complex X is said to be *weakly Cohen–Macaulay of dimension r* if it satisfies the following two properties:

- X is $(r-1)$ -connected.
- For all m -dimensional simplices σ of X , the link $\mathrm{lk}_X(\sigma)$ of σ in X is $(r-m-2)$ -connected.

These two conditions can be combined if you regard $\sigma = \emptyset$ as a -1 -simplex of X with $\mathrm{lk}_X(\sigma) = X$.

By definition, the only difference between $B_n(\mathcal{O}, I)$ and $\widehat{B}_n(\mathcal{O}, I)$ is that the vertices in a simplex of $B_n(\mathcal{O}, I)$ are ordered. In [15, Proposition 2.14], it is proved that in this situation, if $\widehat{B}_n(\mathcal{O}, I)$ is weakly Cohen–Macaulay of dimension $(n-1)$, then $B_n(\mathcal{O}, I)$ is $(n-2)$ -connected. To prove the lemma, therefore, it is enough to prove that $\widehat{B}_n(\mathcal{O}, I)$ is weakly Cohen–Macaulay of dimension $(n-1)$.

We now introduce yet another space. Define $\widehat{B}'_n(\mathcal{O}, I)$ to be the simplicial complex whose $(m-1)$ -simplices are unordered sets $\{v_1, \dots, v_m\}$ of elements of \mathcal{O}^n that can be extended to an unordered set $\{v_1, \dots, v_n\}$ with the following properties:

- The v_i form a free \mathcal{O} -basis for \mathcal{O}^n .
- The last coordinate of each v_i equals either 0 or 1 modulo I .

We thus have $\widehat{B}_n(\mathcal{O}, I) \subset \widehat{B}'_n(\mathcal{O}, I)$. In [7, Theorem E' from §2.3], Church–Farb–Putman proved that $\widehat{B}'_n(\mathcal{O}, I)$ is weakly Cohen–Macaulay of dimension $(n-1)$. This is where we use the assumption that \mathcal{O} has a real embedding.

We now show how to use the fact that $\widehat{B}'(\mathcal{O}, I)$ is weakly Cohen–Macaulay of dimension $(n - 1)$ to prove the same fact for $\widehat{B}(\mathcal{O}, I)$. Let σ be an m -simplex of $\widehat{B}_n(\mathcal{O}, I)$, where we allow $\sigma = \emptyset$ and $m = -1$. We then have

$$\mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma) \subset \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma).$$

Since $\widehat{B}'_n(\mathcal{O}, I)$ is weakly Cohen–Macaulay of dimension $(n - 1)$, the space $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ is $(n - m - 3)$ -connected. To prove the same for $\mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$, it is enough to construct a retraction $\rho: \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \rightarrow \mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$. There are two cases.

Case 1. *There exists a vertex w of σ whose last coordinate equals 1 modulo I .*

Proof of case. In this case, the complex $\mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$ is the full subcomplex of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ spanned by vertices whose last coordinates equal 0 modulo I . For all vertices v of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$, we define

$$\rho(v) = \begin{cases} v - w & \text{if the last coordinate of } v \text{ equals 1 modulo } I, \\ v & \text{otherwise.} \end{cases}$$

The last coordinate of $\rho(v)$ thus equals 0 modulo I . This extends to a simplicial retraction $\rho: \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \rightarrow \mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$ due to the following fact:

- If $\{v_1, \dots, v_n\}$ is a free \mathcal{O} -basis of \mathcal{O}^n and $c_2, \dots, c_n \in \mathcal{O}$, then $\{v_1, v_2 + c_2 v_1, \dots, v_n + c_n v_1\}$ is a free \mathcal{O} -basis of \mathcal{O}^n .

This completes the proof for this case. \square

Case 2. *The last coordinate of all vertices of σ equals 0 modulo I .*

Proof of case. In this case, the complex $\mathrm{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$ is the subcomplex of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ consisting of simplices that contain no edges between vertices both of whose last coordinates equal 1 modulo I . We remark that this is not a full subcomplex.

Let \mathcal{E} be the set of edges of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ joining vertices both of whose last coordinates equal 1 modulo I . The retraction we will construct will depend on two arbitrary choices:

- An enumeration $\mathcal{E} = \{e_1, e_2, \dots\}$.
- For each $i \geq 1$, an enumeration $e_i = \{w_i, w'_i\}$ of the two vertices of e_i . Since w_i and w'_i are distinct vertices of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ whose last coordinates are 1 modulo I , we have that $w_i - w'_i$ is a vertex of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ whose last coordinate is 0 modulo I .

For each $i \geq 1$, we define a map $\rho_i: \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \rightarrow \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ as follows. First, let S_i be the result of subdividing the edge e_i of $\mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ with a new vertex κ_i . We then define a simplicial map $\rho'_i: S_i \rightarrow \mathrm{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ via the formula

$$\rho'_i(v) = \begin{cases} w_i - w'_i & \text{if } v = \kappa_i, \\ v & \text{if } v \neq \kappa_i. \end{cases} \quad (v \text{ a vertex of } S_i).$$

This is a simplicial map for the following reason. It is clear that ρ'_i extends over the simplices of S_i that do not contain κ_i . The simplices of S_i that do contain κ_i are of the form

$$\{w_i, \kappa_i, v_3, \dots, v_m\} \quad \text{and} \quad \{\kappa_i, w'_i, v_3, \dots, v_m\}$$

for a simplex $\{w_i, w'_i, v_3, \dots, v_m\}$ of $\text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$. The images under ρ'_i of these two simplices are

$$\{w_i, w_i - w'_i, v_3, \dots, v_m\} \quad \text{and} \quad \{w_i - w'_i, w'_i, v_3, \dots, v_m\},$$

both of which are simplices of $\text{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$. The map ρ_i is then the composition of ρ'_i with the (nonsimplicial) subdivision map $\text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \xrightarrow{\cong} S_i$.

Now define $\rho: \text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \rightarrow \text{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$ to be the composition

$$\text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \xrightarrow{\rho_1} \text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \xrightarrow{\rho_2} \text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma) \xrightarrow{\rho_3} \dots$$

This infinite composition makes sense and is continuous since for each simplex σ of $\text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$, the sequence

$$\sigma, \rho_1(\sigma), \rho_2 \circ \rho_1(\sigma), \rho_3 \circ \rho_2 \circ \rho_1(\sigma), \dots$$

of subsets eventually stabilizes. These images are not simplices, but rather finite unions of simplices. From its construction, it is clear that ρ is a retraction from $\text{lk}_{\widehat{B}'(\mathcal{O}, I)}(\sigma)$ to $\text{lk}_{\widehat{B}(\mathcal{O}, I)}(\sigma)$. \square

This completes the proof of the lemma. \square

6.3 The proof of Proposition 4.4

We finally prove Proposition 4.4, which completes the proof Theorem A.

Proof of Proposition 4.4. We start by recalling the setup. Let \mathcal{O} be the ring of integers in a number field K and let P be a finite-rank projective \mathcal{O} -module. Assume that \mathcal{O}^\times has an element of norm -1 , and let r and $2s$ be the numbers of real and complex embeddings of K . We must prove that the action of $\text{GL}(P)$ on its normal subgroup $\text{CL}(P)$ induces the trivial action on $H_k(\text{CL}(P); \mathbb{Q})$ for $0 \leq k \leq \min(r + s, \text{rk}(P)) - 1$. This action factors through $\text{GL}(P)/\text{CL}(P) \cong \mathbb{Z}/2$.

The group $\text{GL}(P)$ acts on both $\text{CL}(P)$ and on the complex of lines $\mathcal{L}(P)$. We thus get an induced action of $\text{GL}(P)/\text{CL}(P)$ on $H_k^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$. The natural map $H_k^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q}) \rightarrow H_k(\text{CL}(P); \mathbb{Q})$ is $\text{GL}(P)/\text{CL}(P)$ -equivariant, and by Lemma 6.4 is also surjective for $0 \leq k \leq \text{rk}(P) - 1$. We deduce that to prove that the action of $\text{GL}(P)/\text{CL}(P)$ on $H_k(\text{CL}(P); \mathbb{Q})$ is trivial for $0 \leq k \leq \min(r + s, \text{rk}(P)) - 1$, it is enough to prove that the $\text{GL}(P)/\text{CL}(P)$ -action on $H_k^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$ is trivial for $0 \leq k \leq r + s - 1$.

By Lemma 6.2 (and the paragraph following that lemma), we have a spectral sequence of $\mathbb{Q}[\text{GL}(P)/\text{CL}(P)]$ -modules of the form

$$E_{pq}^1 = \bigoplus_{\sigma \in \mathcal{L}(P)^{(q)}/\text{CL}(P)} H_p((\text{CL}(P))_{\tilde{\sigma}}; \mathbb{Q}) \Rightarrow H_{p+q}^{\text{CL}(P)}(\mathcal{L}(P); \mathbb{Q}). \quad (6.1)$$

Here $\tilde{\sigma} \in \mathcal{L}(P)^{(k)}$ is an arbitrary lift of σ . The key to the proof is the following.

Claim. *The group $\mathrm{GL}(P)/\mathrm{CL}(P)$ acts trivially on E_{pq}^1 for $0 \leq p \leq r + s - 1$.*

Proof of claim. The group $\mathrm{GL}(P)/\mathrm{CL}(P)$ acts trivially on the set $\mathcal{L}(P)^{(q)}/\mathrm{CL}(P)$, so it does not permute the terms in (6.1). Let $\tilde{\sigma} = (L_1, \dots, L_{q-1})$ be a q -simplex of $\mathcal{L}(P)$ such that $H_p((\mathrm{CL}(P))_{\tilde{\sigma}}; \mathbb{Q})$ is one of the terms in (6.1). We must prove that the group $\mathrm{GL}(P)/\mathrm{CL}(P)$ acts trivially on $H_p((\mathrm{CL}(P))_{\tilde{\sigma}}; \mathbb{Q})$. Since $\mathrm{GL}(P)/\mathrm{CL}(P) \cong \mathbb{Z}/2$, it is enough to find a single element of $\mathrm{GL}(P) \setminus \mathrm{CL}(P)$ that acts trivially. Extend $\tilde{\sigma}$ to a line decomposition $(L_1, \dots, L_{\mathrm{rk}(P)})$ of P . Set $P' = L_1 \oplus \dots \oplus L_{q-1}$ and $P'' = L_q \oplus \dots \oplus L_{\mathrm{rk}(P)}$, so $P = P' \oplus P''$. Let x be an element of $\mathrm{GL}(L_1)$ that does not lie in $\mathrm{CL}(L_1)$ and let

$$X = (x, 1, \dots, 1) \in \mathrm{GL}(L_1) \times \mathrm{GL}(L_2) \times \dots \times \mathrm{GL}(L_{q-1}) \times \mathrm{GL}(P'') \subset \mathrm{GL}(P).$$

Since $\mathrm{GL}(L_i) \cong \mathcal{O}^\times$ is abelian, the element X commutes with the subgroup

$$\Lambda = \mathrm{CL}(L_1) \times \mathrm{CL}(L_2) \times \dots \times \mathrm{CL}(L_{q-1}) \times \mathrm{CL}(P'')$$

of $\mathrm{CL}(P)$. It follows that X acts trivially on the image of $H_p(\Lambda; \mathbb{Q})$ in $H_p((\mathrm{CL}(P))_{\tilde{\sigma}}; \mathbb{Q})$. It is enough, therefore, to prove that the map $H_p(\Lambda; \mathbb{Q}) \rightarrow H_p((\mathrm{CL}(P))_{\tilde{\sigma}}; \mathbb{Q})$ is surjective.

It follows from Lemma 5.1 that the $\mathrm{GL}(P)$ -stabilizer of $\tilde{\sigma}$ can be written as

$$\mathrm{Hom}(P'', P') \rtimes (\mathrm{GL}(L_1) \times \dots \times \mathrm{GL}(L_{q-1}) \times \mathrm{GL}(P'')).$$

From this, we see that

$$\Lambda' = \mathrm{Hom}(P'', P') \rtimes (\mathrm{CL}(L_1) \times \dots \times \mathrm{CL}(L_{q-1}) \times \mathrm{CL}(P''))$$

is a finite-index subgroup of $\mathrm{CL}(P)_{\tilde{\sigma}}$. The existence of the transfer map (see [5, Proposition III.10.4]) implies that the map $H_p(\Lambda'; \mathbb{Q}) \rightarrow H_p(\mathrm{CL}(P)_{\tilde{\sigma}}; \mathbb{Q})$ is surjective. Finally, Lemma 5.2 (with $G = \mathrm{CL}(L_1) \times \dots \times \mathrm{CL}(L_{q-1})$) implies that the map $H_p(\Lambda; \mathbb{Q}) \rightarrow H_p(\Lambda'; \mathbb{Q})$ is surjective (this is where we use the assumption that $0 \leq p \leq r + s - 1$). We conclude that the map $H_p(\Lambda; \mathbb{Q}) \rightarrow H_p((\mathrm{CL}(P))_{\tilde{\sigma}}; \mathbb{Q})$ is surjective, as desired. \square

Now, the spectral sequence (6.1) computes the associated graded of a filtration \mathcal{F}_\bullet of $\mathbb{Q}[\mathrm{GL}(P)/\mathrm{CL}(P)]$ -modules on $H_k^{\mathrm{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$ for each k . The above claim implies that $\mathrm{GL}(P)/\mathrm{CL}(P)$ acts trivially on E_{pq}^∞ for $0 \leq p \leq r + s - 1$, so for $0 \leq k \leq r + s - 1$ the $\mathrm{GL}(P)/\mathrm{CL}(P)$ -action on the associated graded terms for the filtration $\mathcal{F}_\bullet H_k^{\mathrm{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$ are trivial. Since $\mathrm{GL}(P)/\mathrm{CL}(P) = \mathbb{Z}/2$ is a finite group, Maschke's theorem implies that the category of $\mathbb{Q}[\mathrm{GL}(P)/\mathrm{CL}(P)]$ -modules is semisimple, so this implies that the $\mathrm{GL}(P)/\mathrm{CL}(P)$ -action on $H_k^{\mathrm{CL}(P)}(\mathcal{L}(P); \mathbb{Q})$ for $0 \leq k \leq r + s - 1$ is also trivial. The lemma follows. \square

References

- [1] A. Ash and L. Rudolph, The modular symbol and continued fractions in higher dimensions, *Invent. Math.* 55 (1979), no. 3, 241–250.
- [2] R. Bieri, *Homological dimension of discrete groups*, second edition, Queen Mary College Mathematical Notes, Queen Mary College, Department of Pure Mathematics, London, 1981.

- [3] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup. (4)* 7 (1974), 235–272 (1975).
- [4] A. Borel and J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436–491.
- [5] K. S. Brown, *Cohomology of groups*, corrected reprint of the 1982 original, Graduate Texts in Mathematics, 87, Springer-Verlag, New York, 1994.
- [6] K. S. Brown, *Buildings*, reprint of the 1989 original, Springer Monographs in Mathematics, Springer-Verlag, New York, 1998.
- [7] T. Church, B. Farb and A. Putman, Integrality in the Steinberg module and the top-dimensional cohomology of $SL_n \mathcal{O}_K$, *Amer. J. Math.* 141 (2019), no. 5, 1375–1419.
- [8] T. Church and A. Putman, The codimension-one cohomology of $SL_n(\mathbb{Z})$, *Geom. Topol.* 21 (2017), no. 2, 999–1032.
- [9] M. Dutour Sikirić, H. Gangl, P. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, On the cohomology of linear groups over imaginary quadratic fields, *J. Pure Appl. Algebra* 220 (2016), no. 7, 2564–2589.
- [10] G. Friedman, Survey article: An elementary illustrated introduction to simplicial sets, *Rocky Mountain J. Math.* 42 (2012), no. 2, 353–423.
- [11] R. Lee and R. H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* 33 (1976), no. 1, 15–53.
- [12] J. Miller, P. Patzt, J. Wilson, and D. Yasaki, Non-integrality of some Steinberg modules, *J. Topol.* 13 (2020), no. 2, 441–459.
- [13] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, NJ, 1971.
- [14] J. Neukirch, *Algebraic number theory*, translated from the 1992 German original and with a note by Norbert Schappacher, Grundlehren der Mathematischen Wissenschaften, 322, Springer-Verlag, Berlin, 1999.
- [15] O. Randal-Williams and N. Wahl, Homological stability for automorphism groups, *Adv. Math.* 318 (2017), 534–626.
- [16] M. Reeder, The Steinberg module and the cohomology of arithmetic groups, *J. Algebra* 141 (1991), no. 2, 287–315.
- [17] J.-P. Serre, *Abelian l -adic representations and elliptic curves*, revised reprint of the 1968 original, Research Notes in Mathematics, 7, A K Peters, Ltd., Wellesley, MA, 1998.
- [18] L. Solomon, The Steinberg character of a finite group with BN -pair, in *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, 213–221, Benjamin, New York.

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