A Lyapunov-based Shaking Function for a Class of Non-bilinear Quantum Systems

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***Abstract*— This paper introduces a Lyapunov-based feedback law for quantum systems that are controlled by shifting the phase of an optical beam. The objective is to drive the system to a target eigenstate of the nominal Hamiltonian (i.e. phase equal zero) by designing a shaking function such that the Hilbert-Schmidt distance is monotonically decreasing. After identifying the control law that maximises the descent rate, two possible tuning strategies are proposed to address bounded input requirements. Convergence of the proposed controller is shown using the Krasovskii-LaSalle principle under the same conditions commonly found for bilinear quantum systems. Numerical simulations showcase the effectiveness of the proposed shaking function.**

I. Introduction

Quantum control [1]–[3] has played an essential role in recent advancements in various areas such as quantum computing, quantum information and quantum sensing. One of the fundamental control objectives for quantum systems is to steer the wavefunction to the desired target eigenstate of the Hamiltonian. Lyapunov-based control offers an intuitive and systematic approach for designing suitable control inputs and was first introduced in [4] for a very specific class of bilinear quantum systems. The method was then extended in [5], [6], where different Lyapunov methods based on the state distance, the average value of an imaginary mechanical quantity, and the state error were proposed. Over the past decade, significant effort has been spent on improving the performance and convergence rate of Lyapunov-based methods for bilinear systems [7]–[11]. Concurrently, Lyapunov-based controllers have been successfully implemented on a variety of quantum information processing and quantum computing tasks such as quantum synchronization [12] and superconducting qubits [13], [14].

However, no effort has been made to move beyond the original bilinear model formulation, despite the fact that many quantum systems are not bilinear. A notable example is the case of trapped ultracold atoms, which are typically controlled by shifting the phase of (i.e. "shaking") the laser beam used for the optical trap [15]–[18]. In this context, the dependency on the control input is trigonometric, whereas the bilinear model can only be obtained by invoking the small angle approximation.

This paper addresses a generalized version of the Schrödinger equation featured in [15]–[18] and proposes a novel Lyapunov-based controller capable of steering the wavefunction to a target eigenstate of the Hamiltonian. Comparisons with the controller that would be obtained using the small angle approximation show that taking into account the trigonometric dependency on the control input yields significant benefits in terms of convergence.

The paper is organized as follows. Section II presents the non-bilinear model, states the control objective, and describes the steps to design an appropriate shaking function for a quantum system. Section III introduces a control-Lyapunov candidate function based on state distance, whereas Section IV shows how do assign a control law such that the candidate function becomes a valid Lyapunov function. Asymptotic stability of the system is discussed in Section V using Krasovskii-LaSalle principle, which yields structural limitations that are identical to the bilinear case. Section VI introduces two different tuning strategies for the control law. Finally, Section VII uses a simple two-level system to perform numerical comparisons between the proposed feedback laws and existing bilinear controllers.

II. Modeling & Problem Statement

Consider the quantum system

$$i\,|\dot{\psi}\rangle = (H_0 + H(u))\,|\psi\rangle\,, \tag{1}$$

where $|\psi\rangle \in \mathbb{C}^n$ is the wave function, $u \in \mathbb{R}$ is the control input, H_0 describes the free evolution of the system under a nominal optical field, and $H(u)$ features a trigonometric dependency on the control input in the form

$$H(u) = H_1 \sin(u) + H_2(1 - \cos(u)), \tag{2}$$

which captures the effects of dephasing the optical field with respect to a nominal configuration, as seen in [15]–[18]. Given $\langle\psi(0)|\psi(0)\rangle = 1$ and $u(t) \in \mathbb{R},\ \forall t \geq 0$, system (1)-(2) satisfies the property $\langle\psi(t)|\psi(t)\rangle = 1,\ \forall t \geq 0$, if and only if the matrices H_0, H_1, and H_2 are Hermitian.

The dynamic behavior of system (1)-(2) in its nominal phase configuration is obtained by setting $u(t) = 0,\ \forall t$, which entails

$$i\,|\dot{\psi}\rangle = H_0\,|\psi\rangle\,. \tag{3}$$

Let λ_n, $|n\rangle$ be an eigenvalue/eigenvector pair of H_0 such that

$$H_0\,|n\rangle = \lambda_n\,|n\rangle\,. \tag{4}$$

Given an initial condition $|\psi(0)\rangle = e^{i\theta}\,|n\rangle$, with $\theta \in [0, 2\pi)$, system (3) admits the solution

$$|\psi(t)\rangle = e^{i(\theta - \lambda_n t)}\,|n\rangle\,,$$

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which belongs to the periodic orbit

$$\mathcal{O}_n = \{|\psi\rangle = e^{i\theta}|n\rangle \mid \theta \in (-\pi, \pi]\}.$$

The objective of this paper is to design a control law $u(|\psi\rangle)$, also known as a *shaking function*, such that a specific periodic orbit $\mathcal{O}_n$ becomes an asymptotically stable limit cycle for the shaken system (1)-(2).

A. Proposed Strategy

The design of an appropriate shaking function for system (1)-(2) will follow the general steps outlined in [5]–[7], [10], [11]. First, we propose a Control-Lyapunov Candidate Function $V(|\psi\rangle)$ that describes a "distance" between the wavefunction $|\psi\rangle$ and the target orbit $\mathcal{O}_n$. Then, we identify a control law $u(|\psi\rangle)$ which makes the time derivative $\dot{V}(|\psi\rangle, u)$ negative semi-definite. Finally, we use the Krasovskii-LaSalle principle to identify potentially undesirable accumulation sets of the closed-loop system.

III. Control-Lyapunov Candidate Function

Consider the Hilbert–Schmidt distance between the wavefunction $|\psi\rangle$ and the eigenvector $|n\rangle$, i.e.

$$V(|\psi\rangle) = 1 - |\langle\psi|n\rangle|^2. \tag{5}$$

Since $|\psi\rangle$, $|n\rangle$ are both unit vectors, their inner product satisfies $|\langle\psi|n\rangle| \in [0,1]$. Moreover, it follows from the definition of $\mathcal{O}_n$ that $|\langle\psi|n\rangle| = 1$ if and only if $|\psi\rangle \in \mathcal{O}_n$. As a result, the Hilbert-Schmidt distance (5) satisfies the requirements of a Control-Lyapunov Candidate Function

- $V(|\psi\rangle) = 0$ if and only if $|\psi\rangle \in \mathcal{O}_n$;
- $V(|\psi\rangle) > 0$ for all unit vectors $|\psi\rangle \notin \mathcal{O}_n$.

The time derivative of (5) is

$$\dot{V} = -(\langle n|\dot{\psi}\rangle\langle\psi|n\rangle + \langle n|\psi\rangle\langle\dot{\psi}|n\rangle). \tag{6}$$

By replacing the system dynamics (1) and using the Hermitian property $H = H^\dagger \;\Rightarrow\; \langle -iH\psi| = \langle\psi|\, iH$, we obtain

$$\begin{aligned}\langle n|\dot{\psi}\rangle &= \langle n| - i(H_0 + H(u))\,|\psi\rangle \\ \langle\dot{\psi}|n\rangle &= \langle\psi| \quad i(H_0 + H(u))\,|n\rangle\,.\end{aligned}$$

Moreover, it follows from (4) that

$$\begin{aligned}-\langle n|\, i(H_0 + H(u))\,|\psi\rangle &= -i\lambda_n\langle n|\psi\rangle - i\,\langle n|\, H(u)\,|\psi\rangle\,,\\ \langle\psi|\, i(H_0 + H(u))\,|n\rangle &= \quad i\lambda_n\langle\psi|n\rangle + i\,\langle\psi|\, H(u)\,|n\rangle\,.\end{aligned}$$

Thus, can be rewritten (6) as

$$\begin{aligned}\dot{V} &= \quad (i\lambda_n\langle n|\psi\rangle + i\,\langle n|\,H(u)\,|\psi\rangle)\,\langle\psi|n\rangle \\ &\quad - \langle n|\psi\rangle\,(i\lambda_n\langle\psi|n\rangle + i\,\langle\psi|\,H(u)\,|n\rangle) \\ &= \quad i\lambda_n|\langle n|\psi\rangle|^2 + i\,\langle n|\,H(u)\,|\psi\rangle\langle\psi|n\rangle \\ &\quad - i\lambda_n|\langle n|\psi\rangle|^2 - i\,\langle n|\psi\rangle\langle\psi|\,H(u)\,|n\rangle \\ &= \; i(\langle n|\,H(u)\,|\psi\rangle\langle\psi|n\rangle - \langle n|\psi\rangle\langle\psi|\,H(u)\,|n\rangle),\end{aligned}$$

which, using the property $i(a - a^\dagger) = -2\,\mathrm{Im}(a)$, becomes

$$\dot{V} = -2\,\mathrm{Im}(\langle n|\,H(u)\,|\psi\rangle\langle\psi|n\rangle).$$

By replacing the control Hamiltonian (2), we finally obtain

$$\dot{V}(|\psi\rangle, u) = \alpha(|\psi\rangle)\sin(u) + \beta(|\psi\rangle)(1 - \cos(u)), \tag{7}$$

where

$$\alpha(|\psi\rangle) = -2\,\mathrm{Im}(\langle n|\,H_1\,|\psi\rangle\langle\psi|n\rangle), \tag{8}$$
$$\beta(|\psi\rangle) = -2\,\mathrm{Im}(\langle n|\,H_2\,|\psi\rangle\langle\psi|n\rangle). \tag{9}$$

The next section addresses how to design $u(|\psi\rangle)$ so that (7) is negative semi-definite. Before doing so, we briefly summarize how to design a controller using the small angle approximation.

A. Approximate Bilinear Solution

For the sake of comparison with existing bilinear control laws, consider the small angle approximation $\sin(u) \approx u$ and $\cos(u) \approx 1$. Then, equation (7) reduces to

$$\dot{V} = -2\,\mathrm{Im}(\langle n|\,H_1\,|\psi\rangle\langle\psi|n\rangle)u, \tag{10}$$

which can be made negative semi-definite by assigning

$$u_0 = \begin{cases}\kappa_0\,\mathrm{Im}\left(\langle n|\,H_1\,|\psi\rangle\,\frac{\langle\psi|n\rangle}{|\langle\psi|n\rangle|}\right), & \text{if } |\langle\psi|n\rangle| > 0,\\ \kappa_0\,\mathrm{Im}\left(\langle n|\,H_1\,|\psi\rangle\right), & \text{if } |\langle\psi|n\rangle| = 0,\end{cases} \tag{11}$$

where $\kappa_0 > 0$ is a tuning parameter and the division by $|\langle\psi|n\rangle|$ is motivated by the fact that $u(|\psi\rangle)$ would otherwise incur in the property $u \to 0$ for $|\langle\psi|n\rangle| \to 0$, which can significantly slow down convergence [6].

IV. Control Design

For ease of notation, this section will omit the dependency of α and β from the wavefunction $|\psi\rangle$. To design a control input u such that (7) is negative semi-definite, we first note that, when $u = 0$,

$$\dot{V}(|\psi\rangle, 0) = 0, \qquad \forall\,|\psi\rangle \in \mathbb{C}^n.$$

Given a specified $|\psi\rangle \in \mathbb{C}^n$, it then follows any minimizer u^* satisfying $\dot{V}(|\psi\rangle, u^*) \leq \dot{V}(|\psi\rangle, u)$, $\forall u \in \mathbb{R}$ will necessarily satisfy $\dot{V}(|\psi\rangle, u^*) \leq 0$.

To identify a minimizer for (7), we then compute

$$\frac{\partial\dot{V}}{\partial u} = \alpha\cos(u) + \beta\sin(u) = 0, \tag{12}$$

which admits the solutions

$$\underline{u} = 2k\pi - \mathrm{atan2}(\alpha, \beta), \quad \overline{u} = 2(k{+}1)\pi - \mathrm{atan2}(\alpha, \beta),$$

with $k \in \mathbb{Z}$. With no loss of generality, we assign $k = 0$ to ensure $\underline{u} \in (-\pi, \pi]$, $\overline{u} \in (\pi, 3\pi]$. To determine which of the two is the minimum, we further compute

$$\frac{\partial^2\dot{V}}{\partial u^2} = -\alpha\sin(u) + \beta\cos(u), \tag{13}$$

and note that

$$\left.\frac{\partial^2\dot{V}}{\partial u^2}\right|_{\underline{u}} = \frac{\alpha^2 + \beta^2}{\sqrt{\alpha^2 + \beta^2}} \geq 0, \quad \left.\frac{\partial^2\dot{V}}{\partial u^2}\right|_{\overline{u}} = -\frac{\alpha^2 + \beta^2}{\sqrt{\alpha^2 + \beta^2}} \leq 0.$$

Thus, the feedback law that minimizes $\dot{V}(|\psi\rangle, u)$ is

$$u^*(|\psi\rangle) = -\mathrm{atan2}(\alpha(|\psi\rangle), \beta(|\psi\rangle)). \tag{14}$$

V. Krasovskii-LaSalle Principle

Although (14) ensures $\dot{V}(|\psi\rangle, u^*) \leq 0$, there is no guarantee that (5) is a *strictly* time decreasing function. To prove asymptotic stability, we must therefore identify the largest invariant set contained in $\Omega = \{|\psi\rangle \mid \dot{V}(|\psi\rangle, u^*(|\psi\rangle)) = 0\}$. Following (7) and (14), we note that

$$\dot{V}(|\psi\rangle, u^*(|\psi\rangle)) = 0 \iff \alpha(|\psi\rangle) = \beta(|\psi\rangle) = 0. \quad (15)$$

Noting that the local geometric behavior of (1)-(2) is identical to a bilinear system with $u_1 = \sin(u)$ and $u_2 = (1-\cos(u))$, it follows from [19] that, if H_0 is not degenerate and (1)-(2) is controllable, the only sets that are potentially invariant are

- The target set: O_n
- The perpendicular set:

$$\mathcal{P} = \{|\psi\rangle \mid \langle\psi|n\rangle = 0\} \quad (16)$$

- The decoupled set:

$$\mathcal{D} = \{|\psi\rangle \notin \mathcal{O}_n \mid \langle n| H_1 |\psi\rangle = \langle n| H_2 |\psi\rangle = 0\}. \quad (17)$$

To achieve global asymptotic stability, we would ideally like to ensure that the target set O_n is the only invariant set of the closed-loop system.

A. Perpendicular Set

The condition $\langle\psi(\tau)|n\rangle = 0$ identifies the subspace $\mathcal{P}$ of all wavefunctions that are perpendicular to $|n\rangle$. Fortunately, this set has the peculiar property of being non-attractive under the proposed controller. Indeed, $V = 1 - |\langle\psi|n\rangle|^2$ and $\dot{V} \leq 0$ ensure that $\langle\psi(\tau)|n\rangle \neq 0$ implies $\langle\psi(t)|n\rangle \neq 0,\ \forall t \geq \tau$. As a result, the set $\mathcal{P}$ can be easily destabilized by replacing the arguments of (14) with

$$\hat{\alpha}(|\psi\rangle) = \begin{cases} -\operatorname{Im}\left(\langle n| H_1 |\psi\rangle \langle\psi|n\rangle\right), & \text{if } |\langle\psi|n\rangle| > 0, \\ -\operatorname{Im}\left(\langle n| H_1 |\psi\rangle\right), & \text{if } |\langle\psi|n\rangle| = 0, \end{cases} \quad (18)$$

and

$$\hat{\beta}(|\psi\rangle) = \begin{cases} -\operatorname{Im}\left(\langle n| H_2 |\psi\rangle \langle\psi|n\rangle\right), & \text{if } |\langle\psi|n\rangle| > 0, \\ -\operatorname{Im}\left(\langle n| H_2 |\psi\rangle\right), & \text{if } |\langle\psi|n\rangle| = 0. \end{cases} \quad (19)$$

Indeed, given $\hat{\alpha} \neq 0$ or $\hat{\beta} \neq 0$, (14) implies $u^* \neq 0$. As detailed in [19], the controllabilty of the system then ensures the existence of a finite time τ such that $\langle\psi(\tau)|n\rangle \neq 0$.

Interestingly enough, unlike the bilinear controller (11), equations (18)-(19) do not require a division by $|\langle\psi|n\rangle|$ since $\text{atan2}(\hat{\alpha}, \hat{\beta})$ already ensures $u \not\to 0$ for $|\langle\psi|n\rangle| \to 0$.

B. Decoupled Set

The condition $\langle n| H_1 |\psi\rangle = \langle n| H_2 |\psi\rangle = 0$ identifies a subspace $\mathcal{D}$ that is not directly coupled to the target set $\mathcal{O}_n$ via the control matrices H_1, H_2. Unlike the previous case, this set may be attractive under the proposed controller and is therefore difficult to address. It then follows from the Krasovskii-LaSalle method that the system trajectories will asymptotically tend to $\mathcal{O}_n \cup \mathcal{D}$ as opposed to just $\mathcal{O}_n$. This issue is identical to the bilinear system case, and is a well-known structural limitation that follows from the geometric properties of H_1, H_2.

Conditions under which $\mathcal{D} = \emptyset$ are detailed in [7], [10]. Insight on how to reduce the region of attraction of undesirable set $\mathcal{D}$ versus that of the target set $\mathcal{O}_n$ can be found in [6].

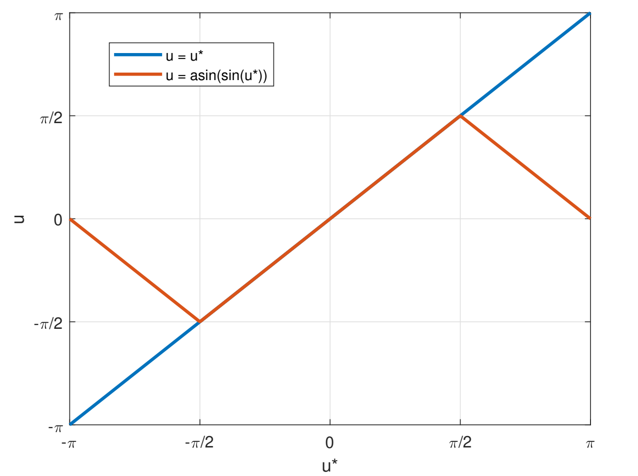

Fig. 1. Comparison between (21) and (22) given $\kappa_1 = \kappa_2 = 1$. The former takes full advantage of the range of u^*, but is discontinuous on the domain boundaries. The latter is continuous, but less aggressive for $|u^*| \in (\pi/2, \pi]$.

VI. Tuning Strategies

Although (14) yields the shaking function that maximizes the local decay rate of the Lyapunov function, it may be beneficial in some contexts (e.g. in the presence of control input bounds) to introduce a tuning parameter for the controller. Fortunately, the continuous function (7) is such that

$$\dot{V}(|\psi\rangle, u) \leq 0, \qquad \forall u \in [0, u^*(|\psi\rangle)] \quad (20)$$

Thus, a possible option for tuning the controller is to assign

$$u_1(|\psi\rangle) = \kappa_1 u^*(|\psi\rangle), \quad (21)$$

with $\kappa_1 \in (0, 1]$. This tuning strategy guarantees a bounded control input $u_1(|\psi\rangle) \in (-\kappa_1\pi, \kappa_1\pi],\ \forall |\psi\rangle \in \mathbb{C}^n$. Unfortunately, it also features a strong discontinuity when u^* jumps from $\pm\pi$ to $\mp\pi$, which might make it difficult to implement in an experimental setting.

As illustrated in Figure 1, another way to ensure $u \in [0, u^*]$ for $u^* \in (-\pi, \pi]$ is to assign

$$u_2(|\psi\rangle) = \text{asin}(\kappa_2 \sin(u^*(|\psi\rangle))), \quad (22)$$

with $\kappa_2 \in (0, 1]$. This tuning strategy guarantees a bounded control input $u_2(|\psi\rangle) \in (-\text{asin}(\kappa_2), \text{asin}(\kappa_2)],\ \forall |\psi\rangle \in \mathbb{C}^n$. Moreover, there is no longer a discontinuity when u^* jumps from $\pm\pi$ to $\mp\pi$, since both cases yield $u_2 = 0$. The advantages and disadvantages of these two options will be addressed in the next section.

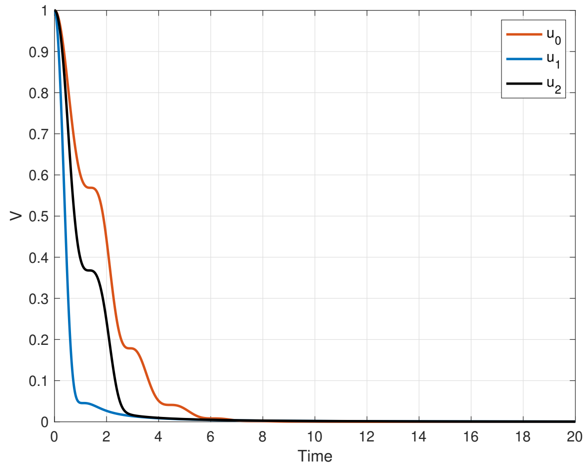

Fig. 2. Behavior of the Hilbert-Schmidt distance obtained using the bilinear controller u_0 ($\kappa_0 = 1$) and the two non-bilinear controllers u_1 ($\kappa_1 = 1$) and u_2 ($\kappa_2 = 1$). The control law u_1 follows the steepest descent and is the fastest to converge, while the bilinear controller u_0 converges the slowest.

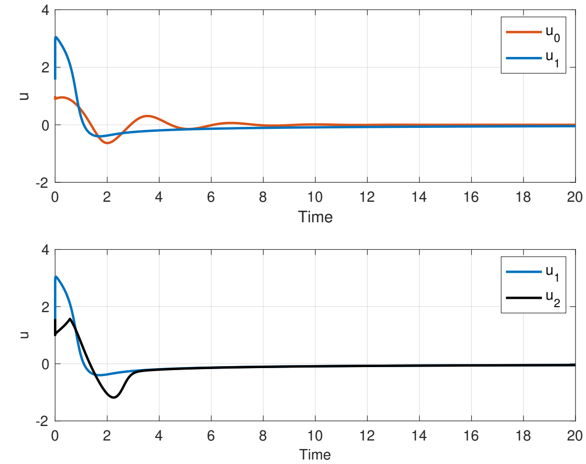

Fig. 3. **Top**: Comparison of the bilinear controller u_0 ($\kappa_0 = 1$) and the non-bilinear controller u_1 ($\kappa_1 = 1$); **Bottom**: Comparison of the two non-bilinear controllers u_1 ($\kappa_1 = 1$) and u_2 ($\kappa_2 = 1$).
The control effort associated to u_1 is nearly double the other two strategies.

VII. Numerical Example

Consider the two-level system

$$i\,|\dot{\psi}\rangle = (H_0 + H_1 \sin(u) + H_2(1 - \cos(u)))\,|\psi\rangle\,,$$

where $H_0 = \sigma_z$, $H_1 = \sigma_y$, and $H_2 = \sigma_x$ are the Pauli matrices

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{23}$$

Given the initial condition $|\psi(0)\rangle = |0\rangle = [1, 0]^T$, the control objective is to steer the system to the periodic orbit $\mathcal{O}_1$, where $|1\rangle = [0, 1]^T$ is an eigenstate of σ_z.

To achieve this objective, we first need to show that (17) is an empty set. To do so, we note that $\langle 1|\,\sigma_y = [i, 0]$ and $\langle 1|\,\sigma_x = [1, 0]$. Thus, $\langle 1|\,\sigma_y\,|\psi\rangle = 0$ and $\langle 1|\,\sigma_x\,|\psi\rangle = 0$ hold if and only if

$$|\psi\rangle = e^{i\theta} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \forall \theta \in (-\pi, \pi]. \tag{24}$$

Since (24) is nothing more than the periodic orbit $\mathcal{O}_1$, we have $\mathcal{D} = \emptyset$. As a result, the proposed controller ensures that $\mathcal{O}_1$ is a globally asymptotically stable limit cycle.

The following subsections compare the closed-loop behavior of the three controllers $u_0(|\psi\rangle)$, $u_1(|\psi\rangle)$, and $u_2(|\psi\rangle)$ given in (11), (21), and (22), respectively. The system trajectories will be plotted using the Bloch Sphere representation, which correlate the wavefunction $|\psi\rangle$ to the x, y, z coordinates

$$x = \langle\psi|\,\sigma_x\,|\psi\rangle\,, \quad y = \langle\psi|\,\sigma_y\,|\psi\rangle\,, \quad z = \langle\psi|\,\sigma_z\,|\psi\rangle\,. \tag{25}$$

This change of coordinates is such that the periodic orbit $\mathcal{O}_0$ reduces to the single point $[0, 0, 1]$ (i.e. the north pole), whereas the periodic orbit $\mathcal{O}_1$ reduces to the single point $[0, 0, -1]$ (i.e. the south pole).

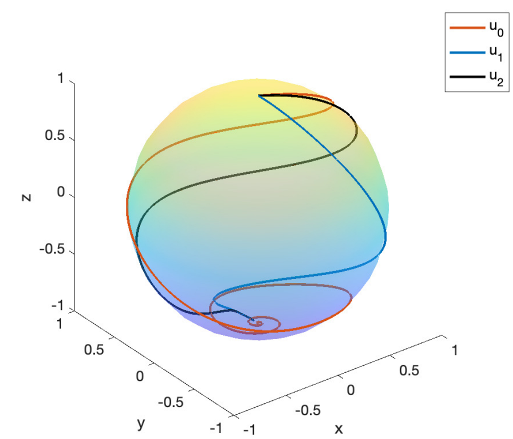

Fig. 4. Bloch Sphere trajectories obtained using the bilinear controller u_0 ($\kappa_0 = 1$) and the non-bilinear controllers u_1 ($\kappa_1 = 1$) and u_2 ($\kappa_2 = 1$). All trajectories succesfully travel from the north pole to the south pole. However, two non-bilinear control laws u_1 and u_2 hit the south pole directly, whereas the bilinear control law u_0 spirals around its target.

A. Unitary Input Gains

Figures 2–4 compare the behavior for $\kappa_0 = \kappa_1 = \kappa_2 = 1$. In this context, we note that $\kappa_1 = 1$ entails $u_1(|\psi\rangle) = u^*(|\psi\rangle)$, where u^* in (14) is the minimizer of $\dot{V}(|\psi\rangle\,, u)$. As such, it is not surprising that the control law $u_1(|\psi\rangle)$ is the fastest to converge (Figure 2), although this added performance also requires a higher control effort (Figure 3).

The advantage of keeping the non-bilinear formulation is clearly evidenced (Figure 2) by the fact that both $u_1(|\psi\rangle)$ and $u_2(|\psi\rangle)$ converge faster than the bilinear controller $u_0(|\psi\rangle)$. A justification for this faster convergence can be gleaned from the significantly different behavior of the Bloch Spere trajectories (Figure 4).

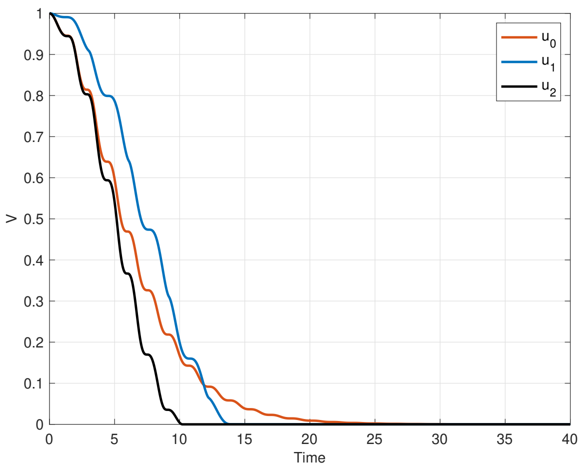

Fig. 5. Behavior of the Hilbert-Schmidt distance obtained using the bilinear controller u_0 ($\kappa_0 = 0.3$) and the non-bilinear controllers u_1 ($\kappa_1 = 0.3/\pi$) and u_2 ($\kappa_2 = \sin(0.3)$). This time, the fastest response is obtained using the control law u_2. The bilinear controller u_0 starts out equally aggressive, but slows down over time and is eventually overtaken by u_1.

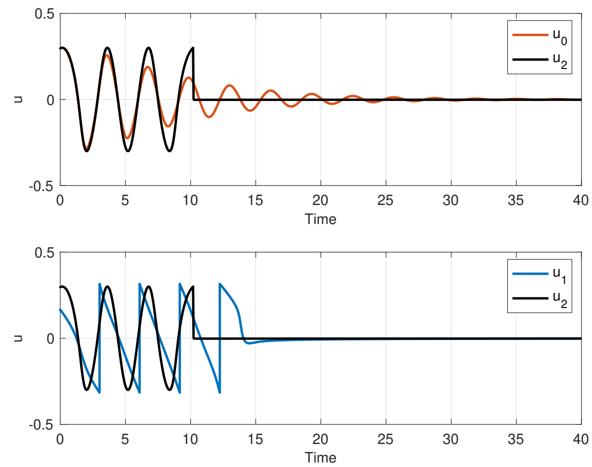

Fig. 6. **Top**: Comparison of the bilinear controller u_0 ($\kappa_0 = 0.3$) and the non-bilinear controller u_2 ($\kappa_2 = \sin(0.3)$); **Bottom**: Comparison of the two non-bilinear controllers u_1 ($\kappa_1 = 0.3/\pi$) and u_2 ($\kappa_2 = \sin(0.3)$). The control inputs u_1 and u_2 take full advantage of the bounds, whereas u_0 tapers off. Moreover, u_2 is smooth in the interval $[0, 10]$, whereas u_1 has multiple discontinuities.

B. Homogeneous Input Bounds

Figures 5–7 compare the behavior for $\kappa_0 = 0.3$, $\kappa_1 = 0.3/\pi$ and $\kappa_2 = \sin(0.3)$, which were chosen to ensure that all control inputs belongs to the interval $[-0.3, 0.3]$. In this case, it is interesting to note that the control $u_2(|\psi\rangle)$ achieves the fastest convergence (Figure 5), while also displaying a smooth behavior (Figure 6).

Once again, the advantage of the non-bilinear formulation is clearly evidenced (Figure 5) by the fact that both $u_1(|\psi\rangle)$ and $u_2(|\psi\rangle)$ converge faster than the bilinear controller $u_0(|\psi\rangle)$. The Bloch Spere trajectories (Figure 7) confirm this behavior by showing that the non-bilinear controllers are able to leverage the trigonometric dependency on the control input to cut straigh to the target, as opposed to slowly spiraling inwards.

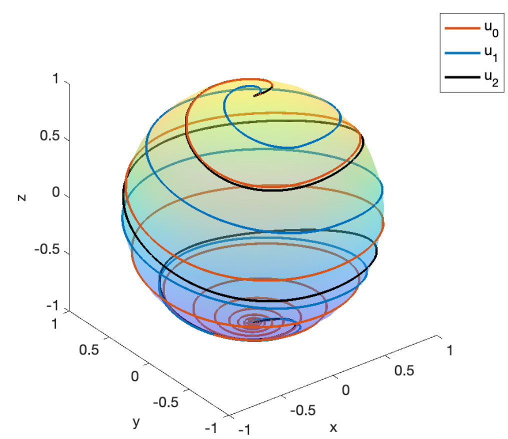

Fig. 7. Bloch Sphere trajectories obtained using the bilinear controller u_0 ($\kappa_0 = 0.3$) and the two non-bilinear controllers u_1 ($\kappa_1 = 0.3/\pi$) and u_2 ($\kappa_2 = \sin(0.3)$). Once again, all three strategies successfully steer the system from the nort to the south pole. However, u_1 and u_2 do so directly, whereas u_0 begins to spiral when it is close to the target.

VIII. Conclusion

This paper proposes a Lyapunov-based shaking function for quantum systems that feature a trigonometric dependency on the control input. Using the Hilbert-Schmidt distance as a control Lyapunov function, we identify the feedback law that maximises the descent rate and propose effective tuning strategies to handle the case of bounded input requirements. Asymptotic stability of the closed-loop system is proven under the same geometric limitations that affect Lyapunov-based controllers for bilinear systems. Comparisons with the bilinear controller obtained using the small angle approximation show that the proposed controller displays significantly better convergence properties, thereby justifying the potential impact of non-bilinear control stategies for quantum systems.

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