Some recent results on renormalization-group properties of quantum field theories

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Abstract

We discuss some higher-loop studies of renormalization-group flows and fixed points in various quantum field theories

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1 Introduction

A fundamental question in quantum field theory (QFT) concerns how the running coupling of a theory changes as a function of the reference Euclidean energy/momentum scale μ where it is measured. The variation of this coupling with μ is described by the renormalization group (RG) beta function of the theory. Here we will discuss some results that we have obtained in this area. Much of this work was with T. A. Ryttov. We will focus mainly on vectorial asymptotically free nonabelian gauge theories in $d = 4$ dimensions, but also discuss some other asymptotically free theories, namely the 2D finite-N Gross-Neveu model and 6D ϕ^3 theories, as well as some infrared-free theories, including U(1) gauge theory, O(N) ϕ^4 theory, and chiral gauge theories..

2 Asymptotically Free Nonabelian Gauge Theories

Let us consider an asymptotically free (AF) vectorial nonabelian gauge theory (in $d = 4$ dimensions) with gauge group G and N_f massless fermions ψ_j, $j = 1, ..., N_f$, transforming according to a representation R of G. We denote the running gauge coupling as $g(\mu)$ and define $\alpha(\mu) \equiv g(\mu)^2/(4\pi)$ and $a(\mu) \equiv g(\mu)^2/(16\pi^2)$. The dependence of $\alpha(\mu)$ on μ is described by the RG beta function, $\beta = d\alpha(\mu)/dt$, where $dt = d\ln\mu$. This has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell \ , \tag{1}$$

where b_ℓ is the ℓ-loop coefficient. For a general operator $\mathcal{O}$, we denote the full scaling dimension as $D_{\mathcal{O}}$ and its free-field value as $D_{\mathcal{O},free}$. The anomalous dimension of this operator, denoted $\gamma_{\mathcal{O}}$, is defined via $D_{\mathcal{O}} = D_{\mathcal{O},free} - \gamma_{\mathcal{O}}$. The coefficients b_1 and b_2 are independent of the scheme used for regularization and renormalization and are $b_1 = (1/3)[11C_A - 4T_f N_f]$ [1,2] and $b_2 = (1/3)[34C_A^2 - 4(5C_A + 3C_f)N_f T_f]$ [3,4], where $C_2(R)$ is the quadratic Casimir invariant, and $T(R)$ is the trace invariant, for the representation R, and we use the notation $C_2(adj) \equiv C_A$, $T(R) \equiv T_f$, and $C_2(R) \equiv C_f$. The AF condition means that $b_1 > 0$, i.e., $N_f < N_u$, where $N_u = 11C_A/(4T_f)$. Since $\alpha(\mu)$ is small at large μ, one can self-consistently calculate β as a power series in $\alpha(\mu)$. As μ decreases from large values in the ultraviolet (UV) to small values in the infrared (IR), $\alpha(\mu)$ increases.

A situation of special interest occurs if β has a zero at a nonzero (physical) value $\alpha = \alpha_{IR}$. In the asymptotically free regime, this happens if the condition $N_u > N_f > 17C_A^2/[2(5C_A + 3C_f)T_f]$ holds, so that $b_2 < 0$. At the two-loop (2ℓ) level, the zero in β occurs at $\alpha_{IR,2\ell} = -4\pi b_1/b_2$. If N_f is close enough to N_u that this IR zero of β occurs at small enough coupling so that the gauge interaction does not produce any spontaneous chiral symmetry breaking (SχSB), then it is an exact IR fixed point (IRFP) of the RG. The theory at this IRFP exhibits scale invariance and is inferred to exhibit conformal invariance, whence the term "conformal window" for this regime. In this IR limit, the theory is in a chirally symmetric, deconfined, nonabelian Coulomb phase (NACP). If, on the other hand, as μ decreases and $\alpha(\mu)$ increases toward α_{IR}, there is a scale $\mu = \Lambda$ at which $\alpha(\mu)$ exceeds a critical value, α_{cr}, for the formation of a fermion condensate $\langle\bar\psi\psi\rangle$ with associated SχSB, then the fermions gain dynamical masses of order Λ. These fermions are then integrated out of the low-energy effective field theory operative for $\mu < \Lambda$. In this case, α_{IR} is only an approximate IRFP. We define $N_{f,cr}$ to be the critical value of N_f such that as N_f decreases below $N_{f,cr}$, there is SχSB. If N_f is only slightly less than N_u, so that α_{IR} is small, then the theory at the IRFP is weakly coupled and is amenable to perturbative analysis [5]. A case of interest for studies of physics beyond the Standard Model (BSM) is N_f slightly less than $N_{f,cr}$. In this case, there is slow-running, quasi-conformal behavior of $\alpha(\mu)$ over an extended interval of μ. The dynamical breaking of the approximate scale (dilatation) symmetry then leads to a light pseudo-Nambu-Goldstone boson, the dilaton. In a BSM application, with the Higgs boson being at least partially a dilaton, this might help to solve the fine-tuning problem of why the Higgs mass is protected against large radiative corrections.

It is of interest to investigate the properties of IRFPs in these vectorial AF gauge theories. Among these properies are the anomalous dimensions of (gauge-invariant) operators, such as $\bar\psi\psi = \sum_{i=1}^{N_f} \bar\psi_i\psi_i$, denoted $\gamma_{\bar\psi\psi,IR}$. In general, one can express the anomalous dimension $\gamma_{\bar\psi/\psi}$ as the series expansion

$$\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell\, a^\ell \ , \tag{2}$$

where c_ℓ is the ℓ-loop coefficient. Evaluating this with α set equal to the IRFP value, calculated to a given n-loop ($n\ell$) order then yields $\gamma_{\bar\psi\psi,IR}$ to this order, denoted as $\gamma_{\bar\psi\psi,IR,n\ell}$. Another operator of interest is $\mathrm{Tr}(F_{\lambda\rho}F^{\lambda\rho})$, where $F^b_{\lambda\rho}$ is the field-strength tensor (with b a group index). The anomalous dimension $\gamma_{F^2,IR}$ of this operator at the IRFP satisfies $\gamma_{F^2} = -\beta'_{IR}$, where $\beta' = d\beta/d\alpha$.

As N_f decreases through the conformal regime, α_{IR} increases, motivating higher-loop calculations of anomalous dimensions. We have carried out this program of calculating the UV to IR renormalization-group evolution and anomalous dimensions at an IRFP to higher-loop order in a series of papers, many with T. A. Ryttov, including [6]- [23]. Our first calculations were at the 4-loop level [6], and subsequently, we have extended these to the 5-loop level, with inputs (in the $\overline{\mathrm{MS}}$ scheme) up to the 5-loop level from [24,25]. (At the 4-loop level, see also [26]). Our calculations to higher-loop order enable us to describe the IR properties of the theory throughout a larger portion of the conformal window than would be possible with the lowest-order (two-loop) results. As N_f decreases below $N_{f,cr}$, the properties of the IR theory change qualitatively, and the perturbative calculations do not apply. A unitarity upper bound in the conformal regime is $\gamma_{\bar{\psi}\psi,IR} < 2$ (reviewed in [27]), and studies of Schwinger-Dyson equations [28] suggest that the onset of SχSB occurs if $\gamma_{\bar{\psi}\psi,IR} > 1$. Thus, for a given G and R, our higher-loop calculations of $\gamma_{\bar{\psi}\psi,IR}$ yield estimates for $N_{f,cr}$; in turn, this information is relevant for the above-mentioned BSM theories.

There is an intensive ongoing program of research in the lattice gauge theory community to study this physics. Much work has been done for $G = \mathrm{SU}(3)$ with R equal to the fundamental representation. For this theory, $N_u = 16.5$ (where a formal continuation from physical integer N_f to real N_f is understood). There is not yet a consensus among lattice groups concerning the value of $N_{f,cr}$ (i.e., the lower end of the conformal window as a function of N_f) for this theory. As an example, we consider the case $N_f = 12$. Several lattice groups [29–34] have found that this theory is IR-conformal, while Ref. [35] has argued that it is chirally broken and hence not IR-conformal. For our 5-loop analysis, we have made use of Padé resummation methods in addition to direct analysis of series expansions. As above, we denote our n-loop value of $\gamma_{\bar{\psi}\psi,IR}$ as $\gamma_{\bar{\psi}\psi,IR,n\ell}$. We calculate $\gamma_{\bar{\psi}\psi,IR,2\ell} = 0.773$, $\gamma_{\bar{\psi}\psi,IR,3\ell} = 0.312$, $\gamma_{\bar{\psi}\psi,IR,4\ell} = 0.253$, and $\gamma_{\bar{\psi}\psi,IR,5\ell} = 0.255$. These results show reasonable convergence at the 4-loop and 5-loop levels, and our values of $\gamma_{\bar{\psi}\psi,IR,4\ell}$ and $\gamma_{\bar{\psi}\psi,IR,5\ell}$ are in very good agreement with the values $\gamma_{\bar{\psi}\psi,IR} = 0.23(6)$ [33] (in accord with [31,32]) and $\gamma_{\bar{\psi}\psi,IR} = 0.235(46)$ [34] measured in lattice simulations. Our values are also in agreement with the range of effective values reported in [35]. For β'_{IR} in this $N_f = 12$ theory, as calculated via power series in the IR coupling, we find $\beta'_{IR,2\ell} = 0.360$, $\beta'_{IR,3\ell} = 0.295$, and $\beta'_{IR,4\ell} = 0.282$. Again, these values show good convergence, and the 4-loop value is in very good agreement with the value $\beta'_{IR} = 0.26(2)$ obtained from lattice measurements [32]. In our papers we have discussed corresponding comparisons with lattice results for other gauge groups G, fermion representations R, and flavor numbers N_f. We have also studied theories with fermions in multiple different representations [23].

Since the b_ℓ for $\ell \geq 3$ and the c_ℓ for $\ell \geq 2$ depend on the scheme used for regularization and renormalization, it is important to assess the effects of this scheme dependence. We have done this in [10–14]. This scheme dependence is a generic feature of higher-order perturbative calculations, e.g., in QCD. A scheme transformation can be expressed as a mapping between α and α', or equivalently, a and a', which we write as $a = a' f(a')$, where $f(a')$ is the scheme transformation function. We can write $f(a')$ as a series expansion

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s \ , \tag{3}$$

where s_{max} may be finite or infinite. In the new scheme, the beta function is $\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell$. We have calculated the b'_ℓ in terms of the b_ℓ and k_s. In addition to the results $b'_1 = b_1$ and $b'_2 = b_2$, we find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ , \tag{4}$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1 \ , \tag{5}$$

and so forth for higher ℓ. We have specified a set of conditions that a physically acceptable scheme transformation must satisfy and have shown that although these can easily be satisfied in the vicinity of zero coupling, they are not automatic, and can be quite restrictive, at a nonzero coupling, as is relevant for an IRFP in an UV-free (AF) theory, or a UVFP in an IR-free theory. As part of this work, we have constructed scheme transformations that can map to a scheme with vanishing coefficients at loop level $\ell \geq 3$ in the vicinity of the origin, but we have also shown that it is more difficult to try to do this at a zero of β away from the origin. We have applied these results to assess the degree of scheme dependence in our higher-loop calculations of anomalous dimensions at IRFPs in AF gauge theories and have shown that this dependence is small. This is similar to the experience in QCD, where calculations performed to higher order exhibited reduced scheme dependence (e.g. [36] and references therein).

The anomalous dimensions of gauge-invariant operators at the IRFP are physical and hence cannot depend on the scheme used for regularization and renormalization. However, this property is not maintained by finite-order perturbative series expansions beyond the lowest orders. It is therefore useful to calculate these anomalous dimensions in a scheme-independent (SI) manner [5,37,38]. To do this, one utilizes the fact that $\alpha_{IR} \to 0$ as $N_f \to N_u$. Hence, one can reexpress the anomalous dimensions as series expansions in the manifestly scheme-independent variable $\Delta_f = N_u - N_f$, rather than as power series in the IR coupling:

$$\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j \, \Delta_f^j \tag{6}$$

and

$$\beta'_{IR} = \sum_{j=1}^{\infty} d_j \, \Delta_f^j \ , \tag{7}$$

where $d_1 = 0$. In general, the calculation of the coefficient κ_j in Eq. (6) requires, as inputs, the values of the b_ℓ for $1 \leq \ell \leq j+1$ and the c_ℓ for $1 \leq \ell \leq j$. The calculation of the coefficient d_j in Eq. (7) requires, as inputs, the values of the b_ℓ for $1 \leq \ell \leq j$. We denote the truncation of these series to maximal power $j = p$ as $\gamma_{\bar{\psi}\psi,IR,\Delta_f^p}$ and β'_{IR,Δ_f^p}, respectively. With Ryttov we have calculated (i) the κ_j up to $j = 4$, and thus the series expansion for $\gamma_{\bar{\psi}\psi,IR}$ to $O(\Delta_f^4)$, and (ii) the d_j up to $j = 5$ and hence β'_{IR} to $O(\Delta_f^5)$ for general G and R. We have studied a number of specific theories in detail, including the gauge groups SU(N_c) with R equal to the fundamental, adjoint, and rank-2 symmetric and antisymmetric tensor representations, and similarly for SO(N_c) and Sp(N_c) for various N_c. For the illustrative theory discussed above, namely SU(3) with $N_f = 12$ fermions in the fundamental representation, our calculations of $\gamma_{\bar{\psi}\psi,IR}$ via Eq. (6) yield slightly larger values than our calculations via Eq. (2), and our computations of β'_{IR} yield slightly smaller values than those that we obtained via series expansions in the IR coupling.

An interesting feature of our scheme-independent results is that κ_1 and κ_2 are manifestly positive, and this positivity also holds for κ_3 and κ_4 for a general G and all of the representations R that we have studied. This leads to two monotonicity properties in the conformal regime: (i) for a fixed p with $1 \leq p \leq 4$, the anomalous dimension $\gamma_{\bar{\psi}\psi,IR,\Delta_f^p}$ is a monotonically increasing function of Δ_f, i.e., increases monotonically with decreasing N_f; (ii) for a fixed N_f, $\gamma_{\bar{\psi}\psi,IR,\Delta_f^p}$ is a monotonically increasing function of p in the range $1 \leq p \leq 4$. From our analyis of a $\mathcal{N} = 1$ supersymmetric SU(N_c) gauge theory with N_f conjugate pairs of chiral superfields [19], we have found that this positivity property of the κ_j is true for all j

3 RG Studies of Other Theories

We have also performed higher-loop studies of RG flows and possible zeros of beta functions for other theories, including (i) the 2D finite-N Gross-Neveu model [39], (ii) various ϕ^3 theories in 6D [40,41], (iii) 4D U(1) gauge theory [42], (iv) 4D nonabelian gauge theories with $N_f > N_u$ [42], and (v) 4D O(N) $\lambda|\vec{\phi}|^4$ theory [43–45]. The theories (i) and (ii) are UV-free (i.e., AF), while the theories (iii)-(v) are IR-free. In these studies, we combined direct analyses of higher-loop beta functions with Padé approximants and scheme transformations to derive results.

3.1 Finite-N Gross-Neveu Model

The Gross-Neveu (GN) model [46] is a 2D QFT with an N-component massless fermion, ψ_j, $j = 1, ..., N$ and a four-fermion interaction. This model has been of interest because it exhibits some properties similar to QCD, namely asymptotic freedom and formation of massive bound states of fermions. The model was solved exactly in the $N \to \infty$ limit in [46]. In this limit, the beta function has no IR zero. This leaves open the question of whether the beta function has an IR zero for finite N. We investigated this in [39], using the beta function up to the 4-loop level from [47]. We found that, where the perturbative calculation of the beta function is reliable, it does not exhibit robust evidence for an IR zero.

3.2 6D ϕ^3 Theories

ϕ^3 theories in $d = 6$ dimensions are asymptotically free, and it is of interest to investigate whether they exhibit IRFPs. We have done this in [40] with Gracey and Ryttov, using beta functions calculated up to the 4-loop order. As before, without loss of generality, we take the matter field to be massless, since a ϕ field with nonzero mass m_ϕ would be integrated out of the low-energy effective theory for momentum scales $\mu < m_\phi$ and hence is not relevant for the IR limit $\mu \to 0$. We have studied ϕ^3 theories with a real 1-component ϕ field and also with an N-component field ϕ_i transforming according to the fundamental representation of a global SU(N) symmetry, with a self-interaction $\propto d_{ijk}\phi^i\phi^j\phi^k + h.c.$. For both of these theories, we find evidence against an IRFP. An interesting study of ϕ^3 theory in a 6D spacetime with two compact dimensions by Kisselev and Petrov is [48]. In [41], we show that if a beta function is not identically zero but has a vanishing one-loop term, then it is not, in general, possible to use scheme transformations to eliminate ℓ-loop terms with $\ell \geq 3$ in the beta function, even in the vicinity of the origin in coupling constant space.

3.3 Studies of IR-free Theories, Including 4D U(1) and O(N) $\lambda|\vec{\phi}|^4$

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as the reference scale μ decreases, the coupling decreases toward 0. As μ increases from the IR, the coupling increases, and a basic question is whether the beta function has a UV zero (in the perturbatively calculable range), which would be a UV fixed point (UVFP) of the RG.

An explicit example of a UVFP in an IR-free theory occurs in the O(N) nonlinear σ model in $d = 2 + \epsilon$ dimensions. From a solution of this model in the $N \to \infty$ limit, one finds, for small ϵ [49, 50],

$$\beta(\xi) = \epsilon\xi\Big(1 - \frac{\xi}{\xi_c}\Big), \tag{8}$$

where ξ is the effective coupling and $\xi_c = 2\pi\epsilon$. Hence, assuming that ξ is small for small μ, it follows that $\lim_{\mu\to\infty} \xi(\mu) = \xi_c$, so the theory has a UVFP at ξ_c.

Let us consider a 4D U(1) gauge theory with N_f fermions with a charge q. This theory is IR-free, and the 1-loop and 2-loop coefficients in β have the same sign, so there is no UV zero in β at the maximal scheme-independent order. In [42] we investigated a possible UVFP at higher-loop order. One part of our work in [42] was an analysis of the beta function using the 5-loop coefficient [51, 52]. Another part made use of exact closed-form results for $N_f \to \infty$ [53]. In [42] we also performed a corresponding investigation of possible UVFP for a nonabelian gauge theory with $N_f > N_u$. In both the U(1) and nonabelian case, we found evidence against a UVFP. Of course, in neither case does this imply that the theory has a Landau pole, because the running gauge coupling gets too large for perturbative calculations to be reliable before one actually reaches this would-be pole.

In [43–45] we investigated the RG behavior of 4D O(N) $\lambda|\vec{\phi}|^4$ theory to six-loop order, using b_5 from [54] and b_6 from [55] (in the $\overline{\text{MS}}$ scheme). Again, for values of the interaction coupling where the perturbative (and Padé resummation) methods were applicable, we did not find robust evidence for a UVFP.

4 Asymptotically Free Chiral Gauge Theories

The analysis of asymptotically free chiral gauge theories is also of considerable interest. The (massless) fermion content is chosen so as to avoid any gauge anomalies, mixed gauge-gravitational anomalies, and global anomaly. As the theory flows from the UV to the IR and the coupling grows, several possible types of behavior can occur, including (i) an exact IRFP in a conformal phase; (ii) bilinear fermion condensate formation with dynamical breaking of gauge and global symmetries; or (iii) confinement with formation of massless composite fermions. These theories have been of interest for BSM physics (e.g, [56]). Our works in this area include [57]- [62], which contain references to the extensive literature.

5 Conclusion

Studies of RG flows and possible RG fixed points in quantum field theories continue to be of great interest, both from the point of view of formal theory and for applications to BSM physics. Here we have briefly discussed some of our results on higher-loop perturbative calculations with inputs up to the five-loop level for anomalous dimensions at IR fixed points in asymptotically free nonabelian gauge theories and comparisons of these results with lattice measurements. We have also discussed our results on RG flows and investigation of possible RG fixed points for several other UV-free theories and for several IR-free theories. There are many opportunities for further work in this area.

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