Exact results for average cluster numbers in bond percolation on infinite-length lattice strips

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We calculate exact analytic expressions for the average cluster numbers $\langle k\rangle_{\Lambda_s}$ on infinite-length strips Λ_s, with various widths, of several different lattices, as functions of the bond occupation probability p. It is proved that these expressions are rational functions of p. As special cases of our results, we obtain exact values of $\langle k\rangle_{\Lambda_s}$ and derivatives of $\langle k\rangle_{\Lambda_s}$ with respect to p, evaluated at the critical percolation probabilities $p_{c,\Lambda}$ for the corresponding infinite two-dimensional lattices Λ. We compare these exact results with an analytic finite-size correction formula and find excellent agreement. We also analyze how unphysical poles in $\langle k\rangle_{\Lambda_s}$ determine the radii of convergence of series expansions for small p and for p near to unity. Our calculations are performed for infinite-length strips of the square, triangular, and honeycomb lattices with several types of transverse boundary conditions.

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I. INTRODUCTION

The study of percolation on lattice graphs elucidates the effect of vacant sites and/or bonds on the connectedness properties of the system. Here we consider bond percolation, in which the bonds of the lattice are randomly present with probability p and thus absent with probability $1-p$. Percolation is relevant for the analysis of such phenomena as the flow of liquids through porous rock, electrical conduction through composite materials, and the magnetic properties of materials with lattice defects and impurities. On an infinite lattice Λ, as p decreases from 1 to 0, the probability $P(p)$ for a site to be part of an infinite connected cluster decreases and vanishes at a critical value $p_{c,\Lambda}$, remaining identically zero for $0 \leqslant p < p_{c,\Lambda}$. Other quantities also behave nonanalytically at $p = p_{c,\Lambda}$. For example, as p increases toward $p_{c,\Lambda}$ from below, the average cluster size $S(p)$ diverges. Thus, the percolation transition is a geometrical transition from a region $0 \leqslant p < p_{c,\Lambda}$, in which only finite connected clusters exist, to a region $p_{c,\Lambda} \leqslant p \leqslant 1$, in which there is a percolating cluster containing an infinite number of sites and bonds. The singularities in various quantities such as $P(p)$ and $S(p)$ are described by a set of critical exponents depending only on the dimensionality d of Λ but independent of the specific type of lattice and type (site or bond) of percolation (some reviews include [1–4]).

One of the interesting quantities in percolation is the average number of (connected) clusters per site on a lattice graph G, in particular, the limit as the number of sites $n \to \infty$,

$$\langle k\rangle_{\{G\}} = \lim_{n\to\infty} n^{-1}\langle k\rangle_G, \tag{1.1}$$

where $\{G\}$ denotes the given $n \to \infty$ limit of the family of n-vertex graphs G. Here, as in mathematical graph theory [3], a cluster is defined as a connecting subgraph of G, including single sites. Since as $p \to 0$ there are no bonds and each site is a cluster, it follows that $\lim_{p\to 0}\langle k\rangle_{\{G\}} = 1$. On the other hand, as $p \to 1$ there is just one cluster, namely, Λ, so $\langle k\rangle_{\{G\}} = 0$. This function $\langle k\rangle_{\{G\}}$ is a monotonically decreasing function of p for $0 \leqslant p \leqslant 1$; it is continuous but nonanalytic at $p = p_{c,\Lambda}$, with a finite singularity of the form $(\langle k\rangle_{\{G\}})_{\rm sing} \propto |p - p_{c,\Lambda}|^{2-\alpha}$. There is no exact solution for $\langle k\rangle_\Lambda$ as a general function of p for (site or bond) percolation on a regular lattice of dimension $d \geqslant 2$, although a solution has been calculated for the Bethe lattice [5]. Much has been learned from series expansions [1,6–8] and Monte Carlo simulations [4,9].

Although the critical exponents describing singularities in quantities such as $P(p)$ and $S(p)$ are universal, the critical (threshold) values of p depend on the type (site or bond) of percolation and on the type of lattice Λ. For bond percolation on the two-dimensional lattices considered here, exact expressions are known for these critical percolation threshold values $p_{c,\Lambda}$ [10]. The exact values of $\langle k\rangle_\Lambda$ on each of these lattices Λ, evaluated at the respective critical values $p = p_{c,\Lambda}$, have also been determined [11] (see also [12–14]), as have the finite-size corrections [15,16].

In [17,18] we gave exact analytic calculations of average cluster numbers $\langle k\rangle_{\Lambda_s}$ as functions of p in bond percolation for infinite-length strips, with various widths, of a variety of lattices with certain transverse boundary conditions. We also gave numerical values of $\langle k\rangle_{\Lambda_s}$ evaluated at $p = p_{c,\Lambda}$ to five-digit accuracy.

In the present paper we report a far-reaching extension of this earlier work, which has enabled us to substantially increase the number of lattice strips for which we are able to calculate exact analytical expressions for the average cluster numbers as functions of p. This is based on a method of calculation that we have devised, which is much more powerful than the method that we used in [17], as we explain in

Sec. III. In addition to making possible exact calculations of $\langle k\rangle_{\Lambda_s}$ on considerably wider infinite-length strips, this method has enabled us to prove an important theorem that $\langle k\rangle_{\Lambda_s}$ is a rational function of p (see Sec. IV A).

By convention, the longitudinal (horizontal) direction along a given lattice strip is taken to be the x direction and the transverse (vertical) direction to be the y direction. We denote a given infinite-length strip of a lattice Λ with width L_y sites and with free or periodic transverse boundary conditions (FBC_y or PBC_y, respectively) by $[\Lambda, (L_y)_F]$ or $[\Lambda, (L_y)_P]$. For the case of the square lattice, we have also calculated $\langle k\rangle$ for the case of infinite-length strips with self-dual (sd) transverse boundary conditions (i.e., such that a finite strip graph is invariant under a planar duality transformation that maps the vertices and faces of a given graph to the faces and vertices of the dual graph, respectively) and we denote these strips by $[\mathrm{sq}, (L_y)_{\mathrm{sd}}]$. Here we will often use the compact notation

$$\Lambda_s \equiv [\Lambda, (L_y)_{\mathrm{BC}_y}] \tag{1.2}$$

for infinite-length lattice strips (where the subscript s stands for strip).

Our results include the following.

(1) We prove a theorem showing that for an infinite-length lattice strip $[\Lambda, (L_y)_{\mathrm{BC}_y}]$, the average number of clusters per site $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is a rational function of the bond occupation probability p.

(2) We calculate exact expressions for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ as functions of p, for a variety of infinite-length lattice strips with width L_y and certain transverse boundary conditions. The lattices are square, triangular, and honeycomb.

(3) We calculate the exact values of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ evaluated at the critical value of p for the corresponding infinite two-dimensional lattice Λ, $p_{c,\Lambda}$, which we denote by

$$\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\Lambda}}. \tag{1.3}$$

The numerical values of these exact analytic expressions agree with the numerical values that we presented in [17] for the respective infinite-length strips.

(4) We quantitatively study how these values approach the critical value $\langle k\rangle_{c,\Lambda}$ for the infinite two-dimensional lattice Λ as the strip width L_y increases and in particular compare with the exact results from Refs. [15,16] for the leading finite-size correction term in the case of periodic transverse boundary conditions

$$\langle k\rangle_{[\Lambda,(L_y)_P]}|_{p=p_{c,\Lambda}} = \langle k\rangle|_{c,\Lambda} + \frac{c_\Lambda \tilde{b}}{L_y^2} + \cdots, \tag{1.4}$$

where we use the notation

$$\langle k\rangle_{c,\Lambda} \equiv \langle k\rangle_\Lambda|_{p=p_{c,\Lambda}} \tag{1.5}$$

for the average cluster number, per site, on the infinite two-dimensional lattice Λ evaluated at $p = p_{c,\Lambda}$ and the ellipsis denotes higher-order terms in $1/L_y$. The coefficient $\tilde{b}$ is [15]

$$\tilde{b} = \frac{5\sqrt{3}}{24} = 0.360\,844 \tag{1.6}$$

and c_Λ is a mathematical constant that takes account of the geometry of the lattice [see Eqs. (4.9) and (4.10) below]. With this geometric relation incorporated, the coefficient $\tilde{b}$ is universal. Our exact results are in very good agreement with this formula (1.4), including (i) the $(L_y)^{-2}$ dependence of the leading correction term, (ii) the value of $\tilde{b}$, and (iii) the universality with respect to lattice type. The universality of $\tilde{b}$ was previously demonstrated from a comparative analysis of the square and triangular lattices in [16]. In this context, we recall that our results for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$, and hence for the values $\langle k\rangle_{[\Lambda,(L_y)_P]}|_{p=p_{c,\Lambda}}$, are independent of the longitudinal boundary conditions imposed on the lattice strips. For the comparison with $\langle k\rangle_\Lambda|_{p=p_{c,\Lambda}}$ on the corresponding infinite two-dimensional lattice Λ, we define the ratio

$$R_{[\Lambda,(L_y)_{\mathrm{BC}_y}],c} = \frac{\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\Lambda}}}{\langle k\rangle_{c,\Lambda}}. \tag{1.7}$$

(5) As a corollary of our theorem, we prove that for infinite-length square-lattice strips, the critical values $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\Lambda}}$ are rational numbers and for infinite-length strips of the triangular and honeycomb lattices they are rational functions of the quantity $\sin(\pi/18)$ that appears in $p_{c,\mathrm{tri}}$ and $p_{c,\mathrm{hc}}$.

(6) We calculate $d^j\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}/(dp)^j$ with $j = 1, 2, 3$, evaluated at $p = p_{c,\Lambda}$, for infinite-length lattice strips Λ_s with a resultant determination of coefficients in the expansion of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ about this value $p_{c,\Lambda}$.

(7) We study the poles in $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ involving the determination of the pole or the complex-conjugate pair of poles closest to the origin in the complex-p plane, which thus set the radius of convergence of the small-p series. A corresponding analysis is given of the poles of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in the complex-r plane, where

$$r \equiv 1 - p. \tag{1.8}$$

Some of our results are summarized in Tables I–VI . Our results in Ref. [17] included analytic calculations of $\langle k\rangle_{\Lambda_s}$ for $\Lambda_s = [\mathrm{sq}, 2_F]$, $[\mathrm{sq}, 2_P]$, $[\mathrm{sq}, 1_{\mathrm{sd}}]$, $[\mathrm{tri}, 2_F]$, $[\mathrm{tri}, 2_P]$, and $[\mathrm{hc}, 2_P]$. We also presented numerical calculations of $\langle k\rangle_{\Lambda_s}$ and $\langle k\rangle_{\Lambda_s}|_{p=p_{c,\Lambda}}$ for several other infinite-length strips. These included plots of $\langle k\rangle_{\Lambda_s}$ as functions of p for $p \in [0, 1]$ for the square, triangular, and honeycomb lattices with the various transverse boundary conditions for widths up to $L_y = 5$. We refer the reader to Ref. [17] for these results.

There have been a number of developments in percolation theory since our Ref. [17] that serve as motivation for the present work, e.g., [4,19,20]. These include studies of the behavior of $\langle k\rangle_\Lambda$ at $p = p_{c,\Lambda}$ [19] for two-dimensional lattices Λ, relevant to our results mentioned in item 6 above, and a recent calculation of $\langle k\rangle_{\mathrm{sq,diag}}|_{p=p_{c,sq}}$ on infinite-length diagonal strips of the square lattice [20]. The work in Ref. [20] is complementary to ours, since [20] does not calculate $\langle k\rangle_{\mathrm{sq,diag}}$ as a general function of p, but instead calculates the evaluation of $\langle k\rangle_{\mathrm{sq,diag}}$ at the special point $p = p_{c,\mathrm{sq}}$ for general width (and also uses strips oriented in a diagonal direction rather than along the lattice axes, as we do). However, Ref. [19] shows the insights into percolation that can be gained by exact calculations on infinite-length lattice strips, just as our previous studies in [17,18] did. We discuss this further in Sec. VIII.

In statistical mechanics, it had been very valuable to use high-temperature series expansions of thermodynamic

TABLE I. Structural features of exact expressions for average cluster numbers $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ on infinite-length strips of various lattices Λ with width L_y and specified transverse boundary conditions $(\mathrm{BC})_y$, expressed as functions of bond occupation probability p and bond vacancy probability $r = 1 - p$. For each such lattice strip we list the degrees $\deg(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})$ and $\deg(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})$ of the numerator and denominator of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ (as polynomials in p or r) and the degree $\deg[\mathrm{PF}(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})]$. To save space, in the table we write $\deg(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}) \equiv \deg(N)$, $\deg(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}) \equiv \deg(D)$, and $\deg[\mathrm{PF}(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})] \equiv \deg[\mathrm{PF}(N)]$.

Λ	$(L_y)_{\mathrm{BC}_y}$	deg(N)	deg(D)	deg[PF(N)]
sq	1_F	1	0	1
sq	2_F	4	3	2
sq	3_F	13	12	3
sq	4_F	46	45	3
sq	2_P	5	4	2
sq	3_P	11	10	3
sq	4_P	31	30	4
sq	5_P	63	62	4
sq	1_{sd}	3	2	3
sq	2_{sd}	13	12	3
sq	3_{sd}	53	52	3
tri	2_F	3	2	3
tri	3_F	16	15	4
tri	4_F	42	41	4
tri	2_P	10	6	4
tri	3_P	23	17	6
tri	4_P	55	47	6
hc	2_F	6	5	2
hc	3_F	13	12	2
hc	4_F	72	71	2
hc	2_P	4	3	2
hc	4_P	32	31	3

quantities to determine the critical temperature. This was determined via the estimate of the radius of convergence of these series. However, the application of this procedure in studies of percolation encountered a complication, namely, that the radii of convergence of these series expansions were typically determined not by the actual critical values of $p_{c,\Lambda}$ or $r_{c,\Lambda}$, but instead by unphysical singularities in the respective complex-p plane and r plane that lie closer to the origin than $p_{c,\Lambda}$ or $r_{c,\Lambda}$ [7,8]. Although this complication was circumvented, e.g., by the use of Padé approximants, to get accurate determinations of critical behavior at the percolation transition, it raises an intriguing question, namely, whether one would encounter the presence of similar unphysical singularities in analyses of exact expressions for average cluster numbers on infinite-length lattice strips. Our work in [17] provided some initial insight into this question. Our present results go substantially further in answering this question, since we have now succeeded in calculating exact expressions for $\langle k\rangle_{\Lambda_s}$ for considerably greater strip widths. Indeed, one of the interesting results of our study of the poles in the exact expressions for $\langle k\rangle_{\Lambda_s}$ on various infinite-length lattice strips Λ_s is that we find that, as the strip width L_y increases, it is generic that there is a pole on the negative real axis or a complex-conjugate pair of poles in the complex-p plane closer to the origin than the value $p_{c,\Lambda}$ for the infinite lattice, and similarly for the pole(s) in the r plane.

TABLE II. Values of average cluster numbers $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ on infinite-length strips of various lattices with specified transverse boundary conditions, evaluated at the critical bond occupation probabilities $p = p_{c,\Lambda}$ for the corresponding infinite two-dimensional lattices. These values are given analytically and numerically, to the indicated floating-point accuracy. The entries in the rightmost column of the table are the values of the ratio $R_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in Eq. (1.7).

Λ	$(L_y)_{\mathrm{BC}_y}$	$\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$	$\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}],\mathrm{num}}$	$R_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$
sq	1_F	1/2	0.5	5.098076
sq	2_F	2/7	0.285714	2.913186
sq	3_F	147/670	0.219403	2.237066
sq	4_F	27229/145196	0.187533	1.912112
sq	2_P	1/5	0.2	2.039230
sq	3_P	11/78	0.141026	1.437919
sq	4_P	677/5572	0.121500	1.238836
sq	5_P	85013/753370	0.112844	1.150571
sq	1_{sd}	1/6	0.166667	1.699359
sq	2_{sd}	17/118	0.144068	1.468937
sq	3_{sd}	2051/15474	0.132545	1.351448
sq	∞	Eq. (2.12)	0.0980762	1
tri	2_F	Eq. (6.3)	0.359575	3.214963
tri	3_F	Eq. (6.2)	0.271487	2.427362
tri	4_F	Eq. (6.8)	0.229460	2.051605
tri	2_P	Eq. (6.14)	0.190910	1.706929
tri	3_P	Eq. (6.13)	0.146651	1.311205
tri	4_P	Eq. (6.19)	0.131378	1.174651
tri	∞	Eq. (2.13)	0.111844	1
hc	2_F	Eq. (7.3)	0.204751	2.663717
hc	3_F	Eq. (7.2)	0.160002	2.081555
hc	4_F	Eq. (7.9)	0.138341	1.799749
hc	2_P	Eq. (7.15)	0.127450	1.658066
hc	4_P	Eq. (7.17)	0.0898337	1.168696
hc	∞	Eq. (2.14)	0.076867	1

TABLE III. Values of $a_{i,[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}$ for $i = 1, 2$ in Eq. (2.6) for infinite-length, finite-width square-lattice strips with various transverse boundary conditions.

$(L_y)_{\mathrm{BC}_y}$	$a_{1,[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}$	$a_{2,[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}$
1_F	−1	0
2_F	−1.204082	0.921283
3_F	−1.214450	1.464119
4_F	−1.200912	1.833688
2_P	−1	1.493333
3_P	−1	2.201755
4_P	−1	2.617979
5_P	−1	2.898863
1_{sd}	−1	1.777778
2_{sd}	−1	2.186394
3_{sd}	−1	2.4668475

TABLE IV. Small-p and small-r expansions of the average cluster number $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ for the infinite-length strip of the lattice Λ with width L_y and transverse boundary conditions BC_y.

Λ	$(L_y)_{\mathrm{BC}_y}$	Small-p series	Small-r series
sq	1_F	$1-p$ (exact)	r (exact)
sq	2_F	$1-\frac{3}{2}p+\frac{1}{2}p^4+\frac{1}{2}p^6+O(p^7)$	$\frac{1}{2}r^2+2r^3-\frac{7}{2}r^5-\frac{3}{2}r^6+O(r^7)$
sq	3_F	$1-\frac{5}{3}p+\frac{2}{3}p^4+p^6+O(p^7)$	$r^3+\frac{7}{3}r^4+2r^5-\frac{11}{3}r^6+O(r^7)$
sq	4_F	$1-\frac{7}{4}p+\frac{3}{4}p^4+\frac{5}{4}p^6+O(p^7)$	$\frac{1}{2}r^3+\frac{5}{4}r^4+2r^5+\frac{19}{4}r^6+O(r^7)$
sq	2_P	$1-2p+\frac{1}{2}p^2+2p^4-2p^5+\frac{5}{2}p^6+O(p^7)$	$\frac{1}{2}r^2+2r^4-2r^5+\frac{5}{2}r^6-6r^7+O(r^8)$
sq	3_P	$1-2p+\frac{1}{3}p^3+p^4+2p^5+O(p^7)$	$\frac{1}{3}r^3+r^4+2r^5-2r^7+O(r^8)$
sq	4_P	$1-2p+\frac{5}{4}p^4+5p^6+O(p^7)$	$\frac{5}{4}r^4+5r^6-4r^7+O(r^8)$
sq	5_P	$1-2p+p^4+\frac{1}{5}p^5+2p^6+O(p^7)$	$r^4+\frac{1}{5}r^5+2r^6+2r^7+O(r^8)$
sq	1_{sd}	$1-2p+p^3+p^4-p^6-p^7+O(p^9)$	$r^3+r^4-r^6-r^7+O(r^9)$
sq	2_{sd}	$1-2p+\frac{1}{2}p^3+p^4+\frac{1}{2}p^5+p^6+O(p^7)$	$\frac{1}{2}r^3+r^4+\frac{1}{2}r^5+r^6+r^7+O(r^8)$
sq	3_{sd}	$1-2p+\frac{1}{3}p^3+p^4+\frac{1}{3}p^5+\frac{4}{3}p^6+O(p^7)$	$\frac{1}{3}r^3+r^4+\frac{1}{3}r^5+\frac{4}{3}r^6+\frac{1}{3}r^7+O(r^8)$
sq	∞	$1-2p+p^4+2p^6+O(p^7)$	$r^4+2r^6-2r^7+O(r^8)$
tri	2_F	$1-2p+p^3+p^4-p^6+O(p^7)$	$r^3+r^4-r^6-r^7+r^9+r^{10}+O(r^{12})$
tri	3_F	$1-\frac{7}{3}p+\frac{4}{3}p^3+\frac{5}{3}p^4+p^5-\frac{1}{3}p^6+O(p^7)$	$\frac{2}{3}r^4+\frac{4}{3}r^5+r^6+\frac{2}{3}r^7-\frac{1}{3}r^8-\frac{17}{3}r^9-\frac{5}{3}r^{10}+O(r^{11})$
tri	4_F	$1-\frac{5}{2}p+\frac{3}{2}p^3+2p^4+\frac{3}{2}p^5+\frac{1}{2}p^6+O(p^7)$	$\frac{1}{2}r^4+r^6+\frac{3}{2}r^7+2r^8+2r^9-\frac{1}{2}r^{10}+O(r^{11})$
tri	2_P	$1-3p+\frac{1}{2}p^2+4p^3+\frac{9}{2}p^4-10p^5-10p^6+O(p^7)$	$\frac{1}{2}r^4+2r^6-2r^8+O(r^{10})$
tri	3_P	$1-3p+\frac{7}{3}p^3+6p^4+11p^5-\frac{17}{3}p^6+O(p^7)$	$\frac{4}{3}r^6+2r^8+2r^{10}+O(r^{11})$
tri	4_P	$1-3p+2p^3+\frac{13}{4}p^4+7p^5+22p^6+O(p^7)$	$r^6+\frac{1}{4}r^8+6r^{10}+O(r^{11})$
tri	∞	$1-3p+2p^3+3p^4+3p^5+3p^6+O(p^7)$	$r^6+3r^{10}+3r^{11}+O(r^{12})$
hc	2_F	$1-\frac{5}{4}p+\frac{1}{4}p^6+\frac{1}{4}p^{10}+O(p^{11})$	$\frac{3}{2}r^2+3r^3-\frac{31}{4}r^4-7r^5+35r^6+O(r^7)$
hc	3_F	$1-\frac{4}{3}p+\frac{1}{3}p^6+\frac{2}{3}p^{10}+O(p^{11})$	$\frac{1}{3}r^2+3r^3+\frac{17}{3}r^4-\frac{22}{3}r^5-53r^6+O(r^7)$
hc	4_F	$1-\frac{11}{8}p+\frac{3}{8}p^6+\frac{7}{8}p^{10}+O(p^{11})$	$\frac{1}{4}r^2+\frac{3}{2}r^3+\frac{29}{8}r^4+\frac{93}{8}r^5+\frac{35}{8}r^6+O(r^7)$
hc	2_P	$1-\frac{3}{2}p+\frac{1}{2}p^4+\frac{1}{2}p^6-\frac{1}{2}p^7+\frac{1}{2}p^8-p^9+p^{10}+O(p^{11})$	$\frac{1}{2}r^2+2r^3-\frac{7}{5}r^5-\frac{3}{2}r^6+O(r^7)$
hc	4_P	$1-\frac{3}{2}p+\frac{1}{2}p^6+\frac{3}{4}p^8+\frac{5}{2}p^{10}+O(p^{11})$	$r^3+\frac{9}{4}r^4+\frac{11}{2}r^5+7r^6+O(r^7)$
hc	∞	$1-\frac{3}{2}p+\frac{1}{2}p^6+\frac{3}{2}p^{10}+O(p^{11})$	$r^3+\frac{3}{2}r^4+\frac{3}{5}r^6+O(r^7)$

II. BACKGROUND

In this section we review some relevant background. We begin with an important connection between percolation and the Potts model. For the sake of generality, let us consider the q-state Potts model on a connected graph $G=(V,E)$ defined by its set of sites (vertices) V and its set of bonds (called edges in mathematical graph terminology) E. In graph theory, a percolation cluster is a connected subgraph of G. The partition function of the q-state Potts model on G is [21]

$$Z(G,q,v)=\sum_{G'\subseteq G} q^{k(G')}v^{e(G')}, \tag{2.1}$$

where $G'=(V,E')$ is a spanning subgraph of G, i.e., a subgraph containing all of the sites in G and a subset $E'\subseteq E$ of the bonds of G, $e(G')$ is the number of bonds in G', and, as above, $k(G')$ is the number of connected components in G'. In the thermal context, $v=e^K-1$ is a temperature-dependent Boltzmann variable, with $K=J/(k_BT)$, where J is the spin-spin coupling in the Potts Hamiltonian $\mathcal{H}=-J\sum_{e_{ij}}\delta_{\sigma_i,\sigma_j}$, where e_{ij} is the bond connecting sites i and j in G. The dimensionless free energy is then defined as

$$f(\{G\},q,v)=\lim_{n\to\infty}\frac{1}{n}\ln[Z(G,q,v)], \tag{2.2}$$

where, as above, n denotes the number of sites in G and $\{G\}$ denotes the $N\to\infty$ limit of G. Now in $f(\{G\},q,v)$, set $v=v_p$, where

$$v_p=\frac{p}{1-p}. \tag{2.3}$$

Then the average number of clusters per site is

$$\langle k\rangle_{\{G\}}=\frac{\partial f(\{G\},q,v_p)}{\partial q}\bigg|_{q=1}, \tag{2.4}$$

which shows the correspondence with the q-state Potts model in the limit $q\to 1$.[1] Now we specialize to the case where G is a lattice graph. The relation (2.4) leads to the inference that the percolation transition on these lattices is in the universality class of the q-state Potts model in the limit where $q\to 1$, with critical exponents $\alpha=-2/3$, $\beta=5/36$, $\gamma=43/18$, etc.

[1]By correspondence with the Potts model in the limit $q\to 1$, one means that the thermodynamic limit is taken first for $q\neq 1$, after which one then takes the limit $q\to 1$. On a finite graph with q set equal to 1, the partition function is trivially given by Eq. (3.4), and, in particular, on a Δ-regular graph, in the limit $n(G)\to\infty$, $f=(\Delta/2)\ln(v+1)=(\Delta/2)K$, which is an analytic function of K and has no nonanalytic critical behavior.

TABLE V. For each infinite-length strip of the lattice Λ with width L_y and transverse boundary conditions BC_y, denoted by $[\Lambda, (L_y)_{\mathrm{BC}_y}]$, this table lists information about the pole or complex-conjugate pair of poles located nearest the origin in the complex-p or -r plane, in the exact expression for the average cluster number $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$. The columns are 1, Λ; 2, $(L_y)_{\mathrm{BC}_y}$; 3, $p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}$; 4, $|p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$; 5, whether $|p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$ is larger or smaller than the critical bond occupation probability $p_{c,\Lambda}$ on the corresponding infinite two-dimensional lattice; 6, $r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}$; 7, $|r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$; and 8, whether $|r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$ is larger or smaller than the critical bond occupation probability $r_{c,\Lambda}$ on the corresponding infinite two-dimensional lattice. The values of $p_{c,\Lambda}$ are given in Eqs. (2.8)–(2.10) and $r_{c,\Lambda} = 1 - p_{c,\Lambda}$. For brevity of notation, column 5 is labeled with the symbol rpc, standing for $|p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$ "relative to $p_{c,\Lambda}$," and similarly, column 8 is labeled with the symbol rrc, standing for $|[r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}|$ "relative to $r_{c,\Lambda}$." No entry indicates that an entry is not applicable.

Λ	$(L_y)_{\mathrm{BC}_y}$	$p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}$	$\lvert p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}\rvert$	rpc	$r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}$	$\lvert r_{[\Lambda,(L_y)_{\mathrm{BC}_y}],np}\rvert$	rrc
sq	1_F	none			none		
sq	2_F	-0.754878	0.754878	$>p_{c,\mathrm{sq}}$	$0.122561\pm0.744862i$	0.754878	$>r_{c,\mathrm{sq}}$
sq	3_F	$-0.400758\pm0.399068i$	0.565564	$>p_{c,\mathrm{sq}}$	-0.411578	0.411578	$<r_{c,\mathrm{sq}}$
sq	4_F	-0.492588	0.492588	$<p_{c,\mathrm{sq}}$	$-0.317578\pm0.244625i$	0.400871	$<r_{c,\mathrm{sq}}$
sq	2_P	-0.618034	0.618034	$>p_{c,\mathrm{sq}}$	-0.618034	0.618034	$>r_{c,\mathrm{sq}}$
sq	3_P	$-0.354731\pm0.319907i$	0.477676	$<p_{c,\mathrm{sq}}$	$-0.354731\pm0.319907i$	0.477676	$<r_{c,\mathrm{sq}}$
sq	4_P	-0.424294	0.424294	$<p_{c,\mathrm{sq}}$	-0.424294	0.424294	$<r_{c,\mathrm{sq}}$
sq	5_P	$-0.371844\pm0.169863i$	0.408805	$<p_{c,\mathrm{sq}}$	$-0.371844\pm0.169863i$	0.408805	$<r_{c,\mathrm{sq}}$
sq	1_{sd}	$e^{\pm i\pi/3}$	1	$>p_{c,\mathrm{sq}}$	$e^{\pm i\pi/3}$	1	$>r_{c,\mathrm{sq}}$
sq	2_{sd}	-0.483657	0.483657	$<p_{c,\mathrm{sq}}$	-0.483657	0.483657	$<r_{c,\mathrm{sq}}$
sq	3_{sd}	$-0.341129\pm0.289364i$	0.447326	$<p_{c,\mathrm{sq}}$	$-0.341129\pm0.289364i$	0.447326	$<p_{c,\mathrm{sq}}$
tri	2_F	$e^{\pm i\pi/3}$	1	$>p_{c,\mathrm{tri}}$	$e^{\pm i\pi/3}$	1	$>r_{c,\mathrm{tri}}$
tri	3_F	$-0.300743\pm0.259341i$	0.397120	$>p_{c,\mathrm{tri}}$	-0.599392	0.599392	$<r_{c,\mathrm{tri}}$
tri	4_F	-0.335309	0.335309	$<p_{c,\mathrm{tri}}$	$-0.419061\pm0.379572i$	0.565408	$<r_{c,\mathrm{tri}}$
tri	2_P	-0.374357	0.374357	$>p_{c,\mathrm{tri}}$	-0.6538705	0.6538705	$>r_{c,\mathrm{tri}}$
tri	3_P	$-0.2277805\pm0.175218i$	0.287376	$<p_{c,\mathrm{tri}}$	-0.594760	0.594760	$<r_{c,\mathrm{tri}}$
tri	4_P	-0.260779	0.260779	$<p_{c,\mathrm{tri}}$	-0.570571	0.570571	$<r_{c,\mathrm{tri}}$
hc	2_F	-0.856675	0.856675	$>p_{c,\mathrm{hc}}$	$-0.0783889\pm0.496940i$	0.503084	$>r_{c,\mathrm{hc}}$
hc	3_F	$-0.492595\pm0.542272i$	0.732604	$>p_{c,\mathrm{hc}}$	$0.123348\pm0.377252i$	0.396906	$>r_{c,\mathrm{hc}}$
hc	4_F	$-0.552838\pm0.373251i$	0.667042	$>p_{c,\mathrm{hc}}$	$-0.212449\pm0.136692i$	0.252625	$<r_{c,\mathrm{hc}}$
hc	2_P	-0.754878	0.754878	$>p_{c,\mathrm{hc}}$	$0.122561\pm0.744862i$	0.754878	$>r_{c,\mathrm{hc}}$
hc	4_P	-0.585767	0.585767	$<p_{c,\mathrm{hc}}$	-0.270891	0.270891	$<r_{c,\mathrm{hc}}$

[22,23] and an associated conformal field theory having a Virasoro algebra with central charge $c = 0$ [24] for the case of dimensionality $d = 2$ relevant here. In [17,18] we used the relation (2.4) together with our earlier exact calculations of f on strips of lattices with arbitrarily great length and fixed width, with various transverse boundary conditions (BC_y) to obtain new analytic expressions and numerical values for $\langle k\rangle_{\Lambda_s}$ for infinite-length strips of these types.

TABLE VI. Values of $\tilde{b}_{[\Lambda,(L_y)_P]}$ in Eq. (1.6) for infinite-length, finite-width lattice strips with periodic transverse boundary conditions, including a comparison with the value $\tilde{b} = 5\sqrt{3}/24 = 0.360\,844$ in Eq. (1.4) from Ref. [15] (see also [16]).

Λ	$(L_y)_P$	$\tilde{b}_{[\Lambda,(L_y)_P]}$	$\frac{\tilde{b}_{[\Lambda,(L_y)_P]}}{\tilde{b}_\Lambda}$
sq	2_P	0.407695	1.129838
sq	3_P	0.386545	1.071225
sq	4_P	0.374786	1.038638
sq	5_P	0.369185	1.023116
tri	2_P	0.365190	1.012044
tri	3_P	0.361720	1.002428
tri	4_P	0.360890	1.0001279
hc	2_P	0.350452	0.971201
hc	4_P	0.359354	0.995871

For a lattice Λ, in the thermodynamic limit, the average cluster number per site has the following expansion in the local neighborhood of $p_{c,\Lambda}$:

$$\langle k\rangle_\Lambda = \langle k\rangle_{c,\Lambda} + a_{1,\Lambda_s}(p - p_{c,\lambda}) + a_{2,\Lambda_s}(p - p_{c,\lambda})^2 + \mathcal{A}_{\Lambda,\pm}|p - p_{c,\lambda}|^{2-\alpha}. \qquad (2.5)$$

Here $\alpha = -2/3$ for $d = 2$, as noted above, and the amplitudes $\mathcal{A}_{\Lambda,\pm}$ refer to the limits $p - p_{c,\Lambda} \to 0^\pm$, respectively. Thus, $\langle k\rangle_\Lambda$ has a finite branch-point singularity at $p = p_{c,\Lambda}$. A recent discussion of the coefficients in this expansion is [19] (where $\langle k\rangle_\Lambda$ is defined per bond rather than per site).

A theorem that we present below shows that on an infinite-length strip of a lattice Λ with width L_y and some prescribed transverse boundary conditions BC_y, $\langle k\rangle_{\Lambda_s}$, evaluated at $p = p_{c,\Lambda}$, is a rational function of p. Although it is therefore meromorphic, none of its poles occur in the physical interval $p \in [0, 1]$. Hence, for p in this interval, it has a Taylor series expansion, and if one evaluates this at the value of p equal to the critical value for the infinite lattice $p_{c,\Lambda}$, then one obtains

$$\langle k\rangle_{\Lambda_s}|_{p=p_{c,\Lambda}} = \langle k\rangle_{\Lambda_s,c} + \sum_{j=1}^{\infty} a_{j,\Lambda_s}(p - p_{c,\Lambda})^j, \qquad (2.6)$$

where

$$a_{j,\Lambda_s} = \frac{1}{j!}\frac{d^j\langle k\rangle_{\Lambda_s}}{(dp)^j}\bigg|_{p=p_{c,\Lambda}}. \tag{2.7}$$

As our results in [17] showed, and our current results further demonstrate, for a given infinite-length strip $[\Lambda, (L_y)_{\mathrm{BC}_y}]$, as L_y increases, $\langle k\rangle_{\Lambda_s}|_{p=p_{c,\Lambda}}$ approaches the critical value $\langle k\rangle_{c,\Lambda}$ for the infinite two-dimensional lattice.

The known values of critical bond occupation probabilities for the square (sq), triangular (tri), and honeycomb (hc) lattices are [10] (see also [12,14])

$$p_{c,\mathrm{sq}} = \frac{1}{2}, \tag{2.8}$$

$$p_{c,\mathrm{tri}} = 2\sin\left(\frac{\pi}{18}\right) = 0.347\,296, \tag{2.9}$$

and

$$p_{c,\mathrm{hc}} = 1 - p_{c,\mathrm{tri}} = 1 - 2\sin\left(\frac{\pi}{18}\right) = 0.652\,704. \tag{2.10}$$

Here and below, floating-point values are given to the indicated number of significant figures. It will be convenient to introduce the shorthand symbol

$$s \equiv \sin\left(\frac{\pi}{18}\right). \tag{2.11}$$

Exact analytic expressions for $\langle k\rangle_{c,\Lambda}$ were presented in [11] (see also related results in [12,14]):

$$\langle k\rangle_{c,\mathrm{sq}} = \frac{3\sqrt{3}-5}{2} = 0.098\,076\,2, \tag{2.12}$$

$$\langle k\rangle_{c,\mathrm{tri}} = \frac{35}{4} - \frac{3}{p_{c,\mathrm{tri}}} = \frac{-6+35s}{4s} = 0.111\,844, \tag{2.13}$$

and

$$\langle k\rangle_{c,hc} = \frac{1}{2}\big(\langle k\rangle_{c,\mathrm{tri}} + p_{c,\mathrm{tri}}^3\big) = \frac{-6+31s+24s^2}{8s} = 0.076\,866\,7. \tag{2.14}$$

On a lattice Λ with coordination number Δ_Λ, the small-p series expansion for $\langle k\rangle_\Lambda$ has the generic form $\langle k\rangle_\Lambda = 1-(\Delta_\Lambda/2)p+\cdots$, where the ellipsis indicates higher-order terms. For the square, triangular, and honeycomb lattices, the small-p series expansions are [7]

$$\langle k\rangle_{\mathrm{sq}} = 1-2p+p^4+2p^6-2p^7+7p^8+O(p^9), \tag{2.15}$$

$$\langle k\rangle_{\mathrm{tri}} = 1-3p+2p^3+3p^4+3p^5+3p^6+6p^7+O(p^9), \tag{2.16}$$

and

$$\langle k\rangle_{hc} = 1-\tfrac{3}{2}p+\tfrac{1}{2}p^6+\tfrac{3}{2}p^{10}+O(p^{11}). \tag{2.17}$$

These have been calculated to higher order than shown here, but we will only need the expansions to these respective orders for comparison with the small-p expansions of our exact expressions for average cluster numbers on infinite-length strips of various lattices with specified transverse boundary conditions.

It has also been valuable to calculate Taylor series expansions of average cluster numbers in terms of the expansion variable r for small r. On (the thermodynamic limit of) a lattice Λ with coordination number Δ_Λ, the small-r series expansion for $\langle k\rangle_\Lambda$ has the generic form $\langle k\rangle_\Lambda = r^{\Delta_\Lambda}+\cdots$, where the ellipsis indicates higher-order terms. For the square, triangular, and honeycomb lattices, the small-r series expansions are [7]

$$\langle k\rangle_{\mathrm{sq}} = r^4+2r^6-2r^7+7r^8+O(r^9), \tag{2.18}$$

$$\langle k\rangle_{\mathrm{tri}} = r^6+3r^{10}-3r^{11}+2r^{12}+O(r^{14}), \tag{2.19}$$

and

$$\langle k\rangle_{\mathrm{hc}} = r^3+\tfrac{3}{2}r^4+\tfrac{3}{2}r^6+O(r^7). \tag{2.20}$$

III. CALCULATIONAL METHODS

We consider strip graphs of a lattice Λ of finite width L_y and arbitrarily great length $m = L_x$, with a given set of longitudinal and transverse boundary conditions. For these strip graphs, the Potts model partition function Z has the form of a finite sum of mth powers

$$Z([\Lambda, L_x, L_y, \mathrm{BC}_x, \mathrm{BC}_y], q, v) = \sum_j c_j(\lambda_j)^m, \tag{3.1}$$

where c_j are coefficients and λ_j are certain functions that depend the type of strip, but are independent of the length m. The λ_j functions are eigenvalues of a transfer matrix and also determine the form of a recursion relation satisfied by the Potts model partition function or equivalent Tutte-Whitney polynomial for the given strip graph [25,26]. In the limit of infinite length $m\to\infty$, this sum is dominated by the λ of largest magnitude, so the reduced dimensionless free energy is

$$f\big([\Lambda, (L_y)_{\mathrm{BC}_y}], q, v\big) = \frac{1}{L_y}\ln\big(\lambda_{\mathrm{dom},[\Lambda,(L_y)_{\mathrm{BC}_y}]}\big). \tag{3.2}$$

In previous work, we have determined the λ functions and in particular λ_{dom} for a number of lattice strips Λ_s (e.g., [27–36]). As was shown in this earlier work, the above-mentioned dominant λ function, and hence the resultant reduced free energy f, is independent of the type of longitudinal boundary conditions used for the finite-m lattice strips. We will make use of a general property of $Z(G, q, v)$, which holds for any graph G, namely,

$$Z(G, q=1, v) = (v+1)^{e(G)}, \tag{3.3}$$

where, as above, $e(G)$ denotes the number of edges (bonds) on G. This follows because if $q = 1$, then the Potts model Hamiltonian $\mathcal{H}$ reduces simply to $\mathcal{H} = -Je(G)$, so

$$Z(G, q=1, v) = e^{Ke(G)} = (v+1)^{e(G)}. \tag{3.4}$$

For a Δ-regular graph G, $e(G) = (\Delta/2)n(G)$. More generally, for a graph which is not Δ regular, one can define an effective vertex degree Δ_{eff} (e.g., [37]). as

$$\Delta_{\mathrm{eff}} = \lim_{n(G)\to\infty}\frac{2e(G)}{n(G)}. \tag{3.5}$$

Hence, for a family of Δ-regular lattice strip graphs Λ_s, Eq. (3.2) applies for $q = 1$ with

$$\lambda_{\mathrm{dom},\Lambda_s}|_{q=1} = (v+1)^{(\Delta/2)L_y}, \tag{3.6}$$

and similarly for non-Δ-regular graphs, with Δ replaced by Δ_{eff}. In particular, for the application to percolation, setting $v = v_p = p/(1-p)$, we have

$$\lambda_{\mathrm{dom},\Lambda_s}|_{q=1,v=v_p} = \left(\frac{1}{1-p}\right)^{(\Delta/2)L_y}. \tag{3.7}$$

Each of the λ functions appearing in Eq. (3.1), and in particular the dominant λ, is a solution to an algebraic equation

$$\sum_{j=0}^{j_{\max}} \kappa_{\Lambda_s,j} \left(\lambda_{\Lambda_s}\right)^j = 0, \tag{3.8}$$

where the coefficients $\kappa_{\Lambda_s,j}$ are polynomials in q and v. The property that the $\kappa_{\Lambda_s,j}$ are polynomials in q and v follows from a combination of the properties that (i) $Z(G,q,v)$ is a polynomial in q and v, as is evident from Eq. (3.1); (ii) the sums of mth powers of λ_j that enter in Eq. (3.1) determining the dominant λ_j arise as traces of the mth power of a transfer matrix and hence are symmetric polynomials in the roots of the characteristic equation for the transfer matrix; and (iii) a theorem embodied in Newton's identities that states that a symmetric polynomial in the roots of an algebraic equation is expressible as a polynomial in the coefficients entering in the equation [38,39].

For many strip graphs, $j_{\max}$ in the equation of the form (3.8) for the dominant λ is $j_{\max} \geqslant 5$, so one cannot solve for $\lambda_{\mathrm{dom},\Lambda_s}$ in terms of radicals. Fortunately, however, one does not need to do this; all that one needs to do is to calculate $\lambda_{\mathrm{dom},\Lambda_s}$ and $d\lambda_{\mathrm{dom},\Lambda_s}/dq$, both evaluated at $q = 1$, for insertion into Eq. (2.4). We can do this as follows. Differentiating Eq. (3.8) with respect to q and solving for $d\lambda_{\mathrm{dom},\Lambda_s}/dq$, we have

$$\frac{d\lambda_{\mathrm{dom},\Lambda_s}}{dq} = -\frac{\sum_{j=0}^{j_{\max}} \left(\lambda_{\mathrm{dom},\Lambda_s}\right)^j \frac{d\kappa_{\Lambda_s,j}}{dq}}{\sum_{j=1}^{j_{\max}} j\kappa_{\Lambda_s,j}\left(\lambda_{\mathrm{dom},\Lambda_s}\right)^{j-1}}. \tag{3.9}$$

Evaluating this equation at $q = 1$ and $v = v_p$, we have

$$\left.\frac{d\lambda_{\mathrm{dom},\Lambda_s}}{dq}\right|_{q=1,v=v_p} = -\frac{\sum_{j=0}^{j_{\max}}(1-p)^{-j}\left[\frac{d\kappa_{\Lambda_s,j}}{dq}\right]|_{q=1,v=v_p}}{\sum_{j=1}^{j_{\max}} j\left[\kappa_{\Lambda_s,j}|_{q=1,v=v_p}\right](1-p)^{1-j}}. \tag{3.10}$$

This is a powerful result, because it means that in calculating $\langle k\rangle_{\Lambda_s}$, one does not have to actually solve for the dominant root $\lambda_{\mathrm{dom},\Lambda_s}$ but instead only use its derivative evaluated at $v = v_p$ and $q = 1$, which can be expressed as a rational function of p. As discussed in Refs. [27,28], our method of calculating $Z(G,q,v)$, and hence, in particular, Eq. (3.8) for a given lattice strip, is an iterative use of the deletion-contraction relation $Z(G,q,v) = Z(G-e,q,v) + vZ(G/e,q,v)$, where $G-e$ denotes the graph obtained from G by deleting the edge (i.e., the bond) e and G/e denotes the graph obtained from G by deleting the bond e and identifying the two vertices that it connected, i.e., contracting on this bond. This is equivalent to a transfer matrix method, which we have also used [36,40]. The $\kappa_{\Lambda_s,j}$ are the coefficients in the indicial equation for the dominant eigenvalue $\lambda_{\mathrm{dom},\Lambda_s}$ of this transfer matrix for the given strip Λ_s and are determined from the entries in the transfer matrix. The iterative use of the deletion-contraction method for this calculation is a generalization of its previous use in calculating generating functions for chromatic polynomials of lattice strip graphs [41]. With the requisite Eq. (3.8) for a given lattice strip Λ_s, we then proceed to calculate Eq. (3.10), which determines $\langle k\rangle_{\Lambda_s}$.

Having explained our method of calculation, we next discuss the analytic structure of the results and their pertinence to series expansions. Because the Potts model is a discrete spin model, the series expansions for $\langle k\rangle_{\Lambda_s}$ for small p or for small r are Taylor series expansions, with finite radii of convergence. Owing to the fact that $v_p = p/(1-p)$, a small-p expansion for a (bond or site) percolation problem is formally analogous to a high-temperature expansion of the corresponding Potts model. Normally, a high-temperature expansion in a Potts model has a radius of convergence equal to the critical point. However, the radii of convergence of Taylor series expansions around both $p = 0$ and $p = 1$ were typically set by unphysical singularities, and these radii of convergence were less than the distance from the expansion point to the physical singularity $p_{c,\Lambda}$ for the small-p expansions and $r_{c\Lambda} = 1 - p_{c,\Lambda}$ for small-r expansions [7,8]. We showed in [17], using the exact expressions that we calculated for $\langle k\rangle_{\Lambda_s}$ on infinite-length, finite-width lattice strips, that these expressions also exhibited poles nearer to the origin in the complex-p plane than the respective value of $p_{c,\Lambda}$ on the infinite two-dimensional lattice. Similarly, we showed that these expressions, as functions of r, exhibited poles closer to the origin in the complex-r plane than $r_{c,\Lambda} = 1 - p_{c,\Lambda}$ for the corresponding infinite two-dimensional lattices. Thus, the calculations of $\langle k\rangle_{\Lambda_s}$ on infinite-length lattice strips Λ_s in [17] provided insight into the influence of unphysical poles in the small-p and small-r series expansions on infinite two-dimensional lattices. Our present results provide further insight into this phenomenon.

Our results on radii of convergence and pole structure are based on a general property that we have proved above, that $\langle k\rangle_{\Lambda_s}$ is a rational function of p and hence also of $r = 1 - p$. For a given infinite-length strip Λ_s of the lattice Λ of finite width L_y and specified transverse boundary conditions BC_y, let us denote the set of poles in the complex-p plane by $p_{\Lambda_s,i}$ with the index i enumerating the number of poles. For each infinite-length lattice strip Λ_s, we determine the pole or complex-conjugate pair of poles closest to the origin, which thus determines the radius of convergence of the small-p series. In a similar way, our exact expressions $\langle k\rangle_{\Lambda_s}$ as functions of r provide insight into this, since we can determine the poles in each of them and in particular the pole or complex-conjugate pair of poles closest to the origin in the complex-r plane, which thus set the radius of convergence of the respective small-r series expansions of $\langle k\rangle_{\Lambda_s}$. It should be noted that it is not the case that there is a simple relation between the pole(s) nearest to the origin in the p plane and the pole(s) nearest to the origin in the complex r plane. To illustrate this, let us consider a hypothetical example in which, for an infinite-length lattice strip Λ_s, the exact expression for the average cluster number $\langle k\rangle_{\Lambda_s}$ has poles at $p = -0.4$ and

$p = 0.7$. The pole nearest to the origin in the p plane is at $p = -0.4$, so the radius of convergence of the small-p series expansion of $\langle k\rangle_{\Lambda_s}$ is 0.4. In this hypothetical example, the poles in $\langle k\rangle_{\Lambda_s}$, expressed as a function of r, are at $r = 0.3$ and $r = 1.4$, so the radius of convergence of the small-r series is 0.3. Thus, although there is a one-to-one correspondence between the full set of poles of $\langle k\rangle$ in the complex-p and r planes, it is not in general true that the nearest pole to the origin in the complex-r plane is equal to 1 minus the value of the nearest pole to the origin in the complex-r plane.

A word is in order concerning how the longitudinal and transverse directions of our lattice strips relate to the lattice vectors. For the square-lattice strips, we take these longitudinal and transverse directions to be the lattice axes. The strips of the triangular lattice are constructed by starting with a square-lattice strip with the same boundary conditions and adding diagonal bonds to each square, say, from the lower left site to the upper right site of each square. A picture of several illustrative finite-length sections of these triangular-lattice strips was included as Fig. 1 in Ref. [29]. Pictures of finite-length sections of strips of the honeycomb (brick) lattice were given as Figs. 16 and 18 in Ref. [31]. In Ref. [32] we presented results for square-lattice strip graphs with several types of self-dual transverse boundary conditions (see also [42]). To construct a strip of the square lattice with one type of self-dual boundary condition, one starts with a square-lattice strip of length L_x and width L_y vertices and periodic longitudinal boundary conditions. One then adds bonds connecting each site on the upper side of the strip to a single external vertex. For a picture of a finite-length section of this self-dual square-lattice strip graph, we refer the reader to Fig. 1 of Ref. [32]. These all yield the same expression for $\langle k\rangle_{\mathrm{sq},(L_y)_{\mathrm{sd}}}$.

The expressions for the effective coordination numbers Δ_{eff}, as defined in Eq. (3.5), for the infinite-length strips that we consider here are listed below:

$$\Delta_{[\mathrm{sq},(L_y)_F],\mathrm{eff}} = 4 - \frac{2}{L_y}, \tag{3.11}$$

$$\Delta_{[\mathrm{tri},(L_y)_F],\mathrm{eff}} = 6 - \frac{4}{L_y}, \tag{3.12}$$

and

$$\Delta_{[\mathrm{hc},(L_y)_F],\mathrm{eff}} = 3 - \frac{1}{L_y}. \tag{3.13}$$

For the infinite-length limit of the first type of self-dual square-lattice strip, we have

$$\Delta_{[\mathrm{sq},(L_y)_{\mathrm{sd}}],\mathrm{eff}} = 4. \tag{3.14}$$

IV. SOME GENERAL PROPERTIES

In this section we prove several general theorems and discuss some general structural features of our exact calculations of average cluster numbers $\langle k\rangle_{\Lambda,\mathrm{BC}_y}$ for infinite-length strips of lattices Λ with finite width L_y and various transverse boundary conditions BC_y. (As noted before, all results are independent of the longitudinal boundary conditions used for a given lattice strip.)

A. $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is a rational function of p

We first prove an important theorem stating that for an infinite-length strip graph $\Lambda_s = [\Lambda, (L_y)_{\mathrm{BC}_y}]$, the average cluster number per site $\langle k\rangle_{\Lambda_s}$ is a rational function of p and hence also of r, that is,

$$\langle k\rangle_{\Lambda_s} = \frac{N_{\Lambda_s}}{D_{\Lambda_s}}, \tag{4.1}$$

where N and D denote numerator and denominator polynomials in p. In factorized form,

$$\langle k\rangle_{\Lambda_s} = \frac{\prod_{i=1}^{\deg_p(N_{\Lambda_s})}(1 - p/a_i)}{\prod_{j=1}^{\deg_p(D_{\Lambda_s})}(1 - p/b_j)}. \tag{4.2}$$

This applies to an arbitrary two-dimensional lattice and is not limited to the specific types of lattices (square, triangular, and honeycomb) for which we calculate $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ here. To prove this theorem, we note that, from Eq. (2.4),

$$\langle k\rangle_{\Lambda_s} = \frac{1}{L_y}\frac{\left(\frac{d\lambda_{\mathrm{dom},\Lambda_s}}{dq}\right)\Big|_{q=1,v=v_p}}{\lambda_{\mathrm{dom},\Lambda_s}|_{q=1,v=v_p}}. \tag{4.3}$$

From Eqs. (3.6) and (3.10), it follows that this is a rational function of p.

This is a very interesting and useful result, because naively, if one were to make direct use of $\langle k\rangle_{\Lambda_s}$ via Eq. (2.4) as the derivative of $f = \ln(\lambda_{\mathrm{dom},\Lambda_s})$ with respect to q, evaluated at $q = 1$, one might naturally think that it would be necessary first to calculate $\lambda_{\mathrm{dom},\Lambda_s}$. With strips for which this is possible, the algebraic equation that yields $\lambda_{\mathrm{dom},\Lambda_s}$ is of degree 2 to 4, so $\lambda_{\mathrm{dom},\Lambda_s}$ would be an algebraic, but not rational, function of q, and for wider strips, the algebraic equation that yields $\lambda_{\mathrm{dom},\Lambda_s}$ is of degree 5 or higher, so one would not be able to solve for $\lambda_{\mathrm{dom},\Lambda_s}$ analytically at all. As our method of calculation presented in Sec. III shows, one can avoid this problem by making use of Eq. (3.10), which does not require solving for λ_{dom} itself as a general function of q, but only the evaluation at $v = v_p$ and $q = 1$.

From our theorem in Eq. (4.1) above, it follows that $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is a meromorphic function of p, with poles at

$$p = p_{[\Lambda,(L_y)_{\mathrm{BC}_y}],j} = b_j, \quad j = 1, \dots, \deg_p\big(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}\big). \tag{4.4}$$

Clearly, when expressed as a function of r, $\langle k\rangle_{\Lambda,(L_y)_{\mathrm{BC}_y}}$ is again a rational function

$$\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]} = \frac{N_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}}{D_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}}, \tag{4.5}$$

where $N_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}$ and $D_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}$ are polynomials in r of degree $\deg_r(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r})$ and $\deg_r(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r})$, respectively, with

$$\deg_p\big(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}\big) = \deg_r\big(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}\big) \tag{4.6}$$

and

$$\deg_p\big(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}\big) = \deg_r\big(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}],r}\big). \tag{4.7}$$

Furthermore, there is a one-to-one correspondence between the poles of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in the p plane and in the r plane.

B. $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\mathrm{sq}}}$ is a rational number

An important corollary of our theorem in Eq. (4.1) is that in the case of square-lattice strips, when one evaluates $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}$ at $p = p_{c,\mathrm{sq}} = 1/2$, the result, namely, $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\mathrm{sq}}}$, is a rational number. Although this property does not hold for strips of other lattices such as triangle or honeycomb, one has an analogous result, namely, that because $p_{c,\mathrm{tri}}$ is a polynomial of the quantity $s \equiv \sin(\pi/18)$ defined in Eq. (2.11) and $p_{c,\mathrm{hc}}$ is a polynomial function of s, $\langle k\rangle_{[\mathrm{tri},(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\mathrm{tri}}}$ and $\langle k\rangle_{[\mathrm{hc},(L_y)_{\mathrm{BC}_y}]}|_{p=p_{c,\mathrm{hc}}}$ are rational functions of s.

C. Agreement with universal finite-size scaling formula

As noted in the Introduction, our exact results for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ evaluated at $p = p_{c,\Lambda}$ enable us to make several comparisons, to check agreement with (a) the values $\langle k\rangle_{c,\Lambda}$ and (b) the formula (1.4) from [15,16] for the finite-size correction term, involving three individual checks: (i) the $(L_y)^{-2}$ dependence on strip width of the leading finite-size correction, (ii) the coefficient $\tilde{b}$ in Eq. (1.6), and (iii) the universality with respect to lattice type. For the comparison (b), we define a constant

$$\tilde{b}_{[\Lambda,(L_y)_{\mathrm{BC}_y}]} = c_\Lambda^{-1} L_y^2 \big[\langle k\rangle_{[\Lambda,(L_y)_P]} - \langle k\rangle_{c,\Lambda}\big]. \tag{4.8}$$

The c_Λ^{-1} in Eq. (1.4) is a geometrical factor connected with the relation between the area A_ℓ of a regular ℓ-sided polygon and the length a of a side (equal to the lattice spacing in our case), $A_\ell = \ell a^2/[4\tan(\pi/\ell)]$. The role of c_{tri} in the universality of $\tilde{b}$ for the square and triangular lattices was shown in [16]. For the lattices that we consider, $c_{\mathrm{sq}} = 1$ and, with our notational conventions in [29–31],

$$c_{\mathrm{tri}} = \frac{\sqrt{3}}{2} \tag{4.9}$$

and

$$c_{\mathrm{hc}} = \frac{1}{\sqrt{3}}, \tag{4.10}$$

so that $c_{\mathrm{tri}}\tilde{b} = 5/16$ and $c_{\mathrm{hc}}\tilde{b} = 5/24$. Agreement with the formula (1.4) requires that, as the width L_y of the infinite-length strip increases, the quantity $\tilde{b}_{[\Lambda,(L_y)_P]}$ should approach the value $\tilde{b} = 5\sqrt{3}/24$, independent of lattice type. We find excellent agreement with both (a) and all three parts (i)–(iii) of property (b). Our results are listed in Table VI and show excellent concordance, in particular, with part (iii) of condition (b), for all of the types of lattice that we consider, namely, square, triangular, and honeycomb. Quantitatively, as is evident in Table VI, the ratios $\tilde{b}_{[\mathrm{sq},5_P]}/\tilde{b}$, $\tilde{b}_{[\mathrm{hc},4_P]}/\tilde{b}$, and $\tilde{b}_{[\mathrm{tri},4_P]}/\tilde{b}$ differ from unity by the respective amounts $1\times10^{-2}, 4\times10^{-3}$, and 1×10^{-4}. These ratios are thus quite close to unity even for these modest-width strips.

D. Property of poles for square-lattice strips with periodic and self-dual transverse boundary conditions

We find an interesting special property of the expressions for $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$ and $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$, i.e., of the average cluster numbers for the infinite-length strips of the square lattice with width L_y and either periodic or self-dual (sd) transverse boundary conditions. For each such strip, we find that the denominator of $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$ (or $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$), expressed as a function of p, is the same as this denominator expressed as a function of r, with the interchange $r \leftrightarrow p$. That is, for the strips with periodic transverse boundary conditions, given

$$\langle k\rangle_{[\mathrm{sq},(L_y)_P]} = \frac{N_{[\mathrm{sq},(L_y)_P]}}{D_{[\mathrm{sq},(L_y)_P]}}, \tag{4.11}$$

with

$$N_{[\mathrm{sq},(L_y)_P]} = (1-p)^m \left(L_y + \sum_\ell c_{[\mathrm{sq},(L_y)_P],\ell}\, p^\ell\right), \tag{4.12}$$

where m is a certain power depending on L_y, and

$$D_{[\mathrm{sq},(L_y)_P]} = L_y \left(1 + \sum_\ell d_{[\mathrm{sq},(L_y)_P],\ell}\, p^\ell\right), \tag{4.13}$$

the denominator polynomial has the form

$$D_{[\mathrm{sq},(L_y)_P]} = L_y \left(1 + \sum_\ell d_{[\mathrm{sq},(L_y)_P],\ell}\, r^\ell\right). \tag{4.14}$$

The same property expressed in Eqs. (4.11)–(4.14) also holds for the square-lattice strips with self-dual boundary conditions. Hence, the set of poles of $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$ in the p plane has the same values as the set of poles in the r plane, and similarly for the set of poles of $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$. Note that this equality of coefficients for p^j and r^j terms in Eqs. (4.13) and (4.14) is not implied by the fact that that the denominator of a given strip, expressed in terms of p, is equal to this denominator, written in terms of $r = 1-p$. Indeed, the special coefficient equality embodied in Eqs. (4.13) and (4.14) is not true for the other infinite-length, finite-width strips for which we have obtained exact calculations of the average cluster number.

E. Some properties of the derivatives $\frac{d^j\langle k\rangle_{\Lambda_s}}{(dp)^j}$

We have found several properties of the jth derivatives $d^j\langle k\rangle_{\Lambda_s}/(dp)^j$ for general infinite-length lattice strips. First, as a corollary of our theorem (4.1) that $\langle k\rangle_{\Lambda_s}$ is a rational function of p, it follows that the jth derivative $d^j\langle k\rangle_{\Lambda_s}/(dp)^j$ is also a rational function of p and that any evaluation of this function for rational p is a rational number.

Second, for an infinite-length strip graph Λ_s which is Δ regular,

$$\left.\frac{d\langle k\rangle_{\Lambda_s}}{dp}\right|_{p=0} = -\frac{\Delta}{2}. \tag{4.15}$$

If the infinite-length strip graph Λ_s is not Δ regular, then this relation holds with Δ replaced by Δ_{eff} on the right-hand side.

Third, for all infinite-length lattice strips Λ_s with $\Delta \geqslant 3$ (in the Δ-regular case) or, more generally, $\Delta_{\mathrm{eff}} \geqslant 3$,

$$\left.\frac{d\langle k\rangle_{\Lambda_s}}{dp}\right|_{p=1} = 0. \tag{4.16}$$

This property (4.16) holds for all of the lattice strips considered here, given our condition on the vertex degree. [This condition excludes the one-dimensional strip, for which $\Delta = 2$ and $\langle k\rangle_{1D} = 1-p$, so $d\langle k\rangle_{1D}/dp = -1$ independent of p.]

F. Structural properties of $\frac{d^j \langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}}{(dp)^j}$ and $a_{j,[\mathrm{sq},(L_y)_{\mathrm{BC}_y}]}$

For infinite-length strips of the square lattice with width L_y and either periodic or self-dual boundary transverse conditions, we find several general results concerning $\frac{d^j \langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}}{(dp)^j}$ and $a_{j,\mathrm{sq},(L_y)_{\mathrm{BC}_y}}$ for $1 \leqslant j \leqslant 3$. For compact notation, we will denote infinite-length square-lattice strips with either of these two types of transverse boundary conditions by $[\mathrm{sq}, (L_y)_{P,\mathrm{sd}}]$. First, $d^3\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}/(dp)^3$ has the symmetry property that under a replacement of $p \to 1-p$, this third derivative reverses in sign,

$$\frac{d^3\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}}{(dp)^3}(p) = -\frac{d^3\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}}{(dp)^3}(1-p), \qquad (4.17)$$

where the (p) and $(1-p)$ indicate the arguments of the respective functions. Consistent with this symmetry property, we find that $\frac{d^3\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}}{(dp)^3}$ contains the factor $(1-2p)$.

Concerning evaluations of $\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}$ at the critical value of p for the infinite lattice, namely, $p_{c,\mathrm{sq}} = 1/2$, which yield the coefficients $a_{1,[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}$, we find that

$$a_{1,[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]} = -1. \qquad (4.18)$$

This agrees with Ref. [19], when one takes account of the fact that we define $\langle k\rangle$ per site here, while Ref. [19] defines $\langle k\rangle$ per bond. Our calculations of $a_{1,[\mathrm{sq},(L_y)_F]}$ for the strips with free transverse boundary conditions are consistent with the inference that these coefficients approach the value -1 in the $L_y \to \infty$ limit. The fact that the value is already reached for finite L_y on the square-lattice strips with periodic or self-dual transverse boundary conditions shows the advantage in the use of these latter boundary conditions, since they remove boundary effects and render the strip graphs 4-regular. Finally, given that $d^3\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}/(dp)^3$ contains the factor $(2p-1)$, it follows that

$$a_{3,[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]} = 0. \qquad (4.19)$$

G. Relation between small-p and small-r series expansions of $\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]}$

From our calculations of $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$ and $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$, we find that in all cases, the small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$, and separately the small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$, are closely related and are of the form

$$\langle k\rangle_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}]} = 1 - 2p + \sum_{\ell=L_y}^{\infty} h_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}],\ell}\, p^\ell \qquad (4.20)$$

and

$$\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]} = \sum_{\ell=L_y}^{\infty} h_{[\mathrm{sq},(L_y)_{P,\mathrm{sd}}],\ell}\, r^\ell, \qquad (4.21)$$

where, as before, the subscript P, sd means that the equality holds separately for the square-lattice strips with periodic or self-dual transverse boundary conditions. Thus, except for the first two terms in the small-p series, all of the coefficients in both of these series, from the respective $O(p^{L_y})$ and $O(r^{L_y})$ orders to infinity, are the same. Since the radii of convergence of these series are determined by the behavior of the small-p and small-r series as the order goes to infinity (e.g., by the ratio test), this equality of the coefficients is in accord with the property discussed in the preceding subsection, that the poles are at the same positions in the p plane and in the r plane for each of these strips, so that the pole (or complex-conjugate pair of poles) that is closest to the origin is the same in the p and r planes, and hence the small-p and small-r series expansions have the same radius of convergence. In contrast, for other infinite-length, finite-width strips of various lattices, the radius of convergence of the small-p expansion is not in general equal to the radius of convergence of the small-r expansion.

H. Some general properties of the numerator and denominator polynomials in $\langle k\rangle_{\Lambda_s}$

For many of the infinite-length, finite-width lattice strips Λ_s for which we have calculated the exact expressions $\langle k\rangle_{\Lambda_s}$, we find that the degree of the numerator, as a polynomial in p or r, is greater, by one unit, than the degree of the denominator, i.e.,

$$\deg_p\big(N_{\Lambda_s}\big) = \deg_p\big(D_{\Lambda_s}\big) + 1 \qquad (4.22)$$

for these strips. These include the $[\mathrm{sq}, (L_y)_{\mathrm{BC}_y}]$ strips with $\mathrm{BC}_y = F, P$, sd; the $[\mathrm{hc}, (L_y)_{\mathrm{BC}_y}]$ strips with $\mathrm{BC}_y = F, P$; and the $[\mathrm{tri}, (L_y)_F]$ strips. This is not the case with the $[\mathrm{tri}, (L_y)_P]$ strips. For the $[\mathrm{tri}, (L_y)_P]$ strips for which we have obtained $\langle k\rangle_{[\mathrm{tri},(L_y)_P]}$, namely, those with widths $L_y = 2, 3, 4$, we find that

$$\deg_p\big(N_{[\mathrm{tri},(L_y)_P]}\big) = \deg_p\big(D_{[\mathrm{tri},(L_y)_P]}\big) + 2L_y. \qquad (4.23)$$

Calculations of $\langle k\rangle_{\Lambda_s}$ for larger values of L_y would be necessary to determine if these patterns persist for wider strips.

We find that the numerator $N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ always contains a prefactor (abbreviated PF) equal to $(1-p) = r$ raised to a certain power depending on $[\Lambda, (L_y)_{\mathrm{BC}_y}]$, which we denote by $\deg[\mathrm{PF}(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})]$. This power is equal to the minimum power of r in the small-r expansion of $N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$. In Table I we list the values of $\deg(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})$, $\deg[\mathrm{PF}(N_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})]$, and $\deg(D_{[\Lambda,(L_y)_{\mathrm{BC}_y}]})$ for the strips for which we have calculated the average cluster numbers $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$.

V. STRIPS OF THE SQUARE LATTICE

A. 3_F square-lattice strip

As noted above, in [17] we calculated $\langle k\rangle_{[\mathrm{sq},2_F]}$. Here we make use of our more powerful calculational method described in Sec. III to obtain exact results on infinite-length lattice strips with substantially greater widths. As the first of our results for explicit expressions of $\langle k\rangle_{\Lambda_s}$ on infinite-length lattice strips, we present our calculation of this average cluster number for the $[\mathrm{sq}, 3_F]$ strip:

$$\langle k\rangle_{[\mathrm{sq},3_F]} = \frac{(1-p)^3(3+4p-3p^2-8p^3+9p^4+12p^5-26p^6+9p^7+11p^8-11p^9+3p^{10})}{3(1+p-p^2)(1-p+p^2)(1-p^2-2p^3+6p^4-2p^5-3p^6+3p^7-p^8)}$$
$$= \frac{r^3(3+4r-7r^2+5r^3+8r^4-31r^5-61r^7+47r^8-19r^9+3r^{10})}{3(1+r-r^2)(1-r+r^2)(1-r-r^2+9r^3-14r^4+13r^5-10r^6+5r^7-r^8)}. \quad (5.1)$$

As indicated, it is useful to express this and other average cluster numbers $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ as functions of p and also, equivalently, as functions of r. As noted above, in Table I we list the degrees of the numerator and denominator of $\langle k\rangle_{[\mathrm{sq},3_F]}$ as polynomials in p or equivalently in r, together with the degree of the prefactor $(1-p)$.

When evaluated at $p = p_{c,\mathrm{sq}}$, $\langle k\rangle_{[\mathrm{sq},3_F]}$ has the value

$$\langle k\rangle_{[\mathrm{sq},3_F]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{147}{670} = 0.219\,403. \quad (5.2)$$

In Table II we list this critical value. It is of interest to compare the critical value (5.2) with $\langle k\rangle_{c,\mathrm{sq}}$ on the infinite square lattice. For this purpose, we list the values of the ratio (1.7) for the present lattice strips and others in Table II. Tables I and II also list the corresponding results for the other infinite-length, finite-width lattice strips with various widths and transverse boundary conditions denoted by BC_y for which we have calculated $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$.

It is instructive to study derivatives of $\langle k\rangle_{[\mathrm{sq},3_F]}$ and to apply these to calculate the coefficients $a_{[\mathrm{sq},3_F],j}$ in Eq. (2.6) for the first several values of j. Doing this, we obtain the results

$$a_{1,[\mathrm{sq},3_F]} = -\frac{16\,355}{3\times(67)^2} = -1.214\,450, \quad (5.3)$$

$$a_{2,[\mathrm{sq},3_F]} = \frac{297\,238\,112}{3^3\times 5^2\times(67)^3} = 1.464\,119, \quad (5.4)$$

and

$$a_{3,[\mathrm{sq},3_F]} = \frac{1\,004\,115\,424}{3^3\times(67)^4} = 1.845\,528. \quad (5.5)$$

We list these values in Table III, which also lists the analogous values of these coefficients for other infinite-length lattice strips. In Eqs. (5.3)–(5.5) we have indicated the factorizations of the denominators. In general, the numerators of these expressions do not have similarly simple factorizations; for example, the numerators of $a_{j,[\mathrm{sq},3_F]}$ for $j = 1, 2$ have the respective factorizations 5×3271, and $2^5\times 9\,288\,691$. For an infinite two-dimensional lattices Λ, the leading singularity in $\langle k\rangle_\Lambda$ occurs in the $|p-p_{c,\Lambda}|^{2-\alpha} = |p-p_{c,\Lambda}|^{8/3}$ term in Eq. (2.5), but, as a consequence of our theorem (4.1), it follows that $\langle k\rangle_{\Lambda_s}$ does not have any branch-point singularities such as $|p-p_{c,\Lambda}|^{8/3}$.

The Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},3_F]}$ for small p and r are

$$\langle k\rangle_{[\mathrm{sq},3_F]} = 1-\tfrac{5}{3}p+\tfrac{2}{3}p^4+p^6-p^7+\tfrac{7}{3}p^8-4p^9+O(p^{10}) \quad (5.6)$$

and

$$\langle k\rangle_{[\mathrm{sq},3_F]} = r^3+\tfrac{7}{3}r^4+2r^5-\tfrac{11}{3}r^6-9r^7-\tfrac{49}{3}r^8 + \tfrac{86}{3}r^9+O(r^{10}). \quad (5.7)$$

For comparison with results for other strips, we list the first few terms of these series in Table IV.

Of the 12 poles of $\langle k\rangle_{[\mathrm{sq},3_F]}$ in the complex-p plane, the ones nearest to the origin are a complex-conjugate pair at

$$p_{[\mathrm{sq},3_F],np} = -0.400\,758 \pm 0.399\,068i, \quad (5.8)$$

of magnitude

$$|p_{[\mathrm{sq},3_F],np}| = 0.565\,564, \quad (5.9)$$

which is therefore the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{sq},3_F]}$, as indicated in Table V. In the complex-r plane, the pole of $\langle k\rangle_{[\mathrm{sq},3_F]}$ nearest to the origin occurs at the value

$$r_{[\mathrm{sq},3_F],np} = -0.411\,578, \quad (5.10)$$

of magnitude $|r_{[\mathrm{sq},3_F],np}| = 0.411\,578$, which is thus the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{sq},3_F]}$. Note that for this infinite-length strip, we have the generic behavior that $1-|r_{[\mathrm{sq},3_F],np}|$ is not equal to $|p_{[\mathrm{sq},3_F],np}|$.

B. 4_F square-lattice strip

We calculate

$$\langle k\rangle_{[\mathrm{sq},4_F]} = \frac{N_{[\mathrm{sq},4_F]}}{D_{[\mathrm{sq},4_F]}}, \quad (5.11)$$

where the numerator and denominator polynomials $N_{[\mathrm{sq},4_F]}$ and $D_{[\mathrm{sq},4_F]}$ are given in Eqs. (A1) and (A2) in the Appendix. At $p = p_{c,\mathrm{sq}}$, $\langle k\rangle_{[\mathrm{sq},4_F]}$ has the value

$$\langle k\rangle_{[\mathrm{sq},4_F]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{27\,229}{145\,196} = 0.187\,533. \quad (5.12)$$

This and the other exact values of $\langle k\rangle_{\Lambda,(L_y)_{\mathrm{BC}_y}}$ evaluated at $p = p_{c,\Lambda}$ for moderately wide strips of the square lattice with various transverse boundary conditions do not have particularly simple factorizations. For example, the factorizations of the numerator and denominator of Eq. (5.12) are $27\,229 = 73\times 373$ and $145\,196 = 2^2\times 36\,299$.

For the coefficient $a_{1,[\mathrm{sq},4_F]}$ we calculate

$$a_{1,[\mathrm{sq},4_F]} = -\tfrac{14\,241\,087\,916}{11\,858\,556\,609} = -1.200\,912. \quad (5.13)$$

The analytic expressions for the coefficients $a_{2,[\mathrm{sq},4_F]}$ and $a_{3,[\mathrm{sq},4_F]}$ are sufficiently lengthy that we only give the floating-point values:

$$a_{2,[\mathrm{sq},4_F]} = 1.833\,688 \quad (5.14)$$

and

$$a_{3,[\mathrm{sq},4_F]} = 2.277\,750\,5. \quad (5.15)$$

The Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},4_F]}$ for small p and r are listed in Table IV. Of the 45 poles of $\langle k\rangle_{[\mathrm{sq},4_F]}$ in the complex-p plane, the pole nearest to the origin is

$$p_{[\mathrm{sq},4_F],np} = -0.492\,588, \quad (5.16)$$

which thus sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{sq},4_F]}$ as $|p_{[\mathrm{sq},3_F],np}| = 0.492\,588$. Note that this is smaller than the physical singularity of $\langle k\rangle_{\mathrm{sq}}$ on the infinite square lattice, at $p = p_{c,\mathrm{sq}} = 1/2$, as indicated in Table V. In the complex-r plane, the poles of $\langle k\rangle_{\mathrm{sq},4_F}$ nearest to the origin are the complex-conjugate pair

$$r_{[\mathrm{sq},4_F],np} = -0.317\,578 \pm 0.244\,625i, \quad (5.17)$$

of magnitude $|r_{[\mathrm{sq},4_F],np}| = 0.400\,871$, which is thus the radius of convergence of the small-r series expansion of $\langle k\rangle_{[\mathrm{sq},4_F]}$. Note that for this strip, we again have the generic behavior that $1 - |r_{[\mathrm{sq},4_F],np}|$ is not equal to $|p_{[\mathrm{sq},4_F],np}|$.

C. 3_P square-lattice strip

Results for infinite-length strips with periodic transverse boundary conditions have the advantage, relative to those with free transverse boundary conditions, that they are free of boundary effects, although of course they still reflect the finite transverse size of the strips. In [17] we calculated $\langle k\rangle_{[\mathrm{sq},2_P]}$. We present here our calculation of the average cluster number for the infinite-length strip of the square lattice with width $L_y = 3$ and periodic transverse boundary conditions:

$$\langle k\rangle_{[\mathrm{sq},3_P]} = \frac{(1-p)^3(3+3p-3p^2-14p^3+18p^4-p^5-13p^6+11p^7-3p^8)}{3(1-p^2-2p^3+11p^4-11p^5-p^6+10p^7-10p^8+5p^9-p^{10})}$$
$$= \frac{r^3(1+3r+5r^2-5r^3-7r^4+16r^5-20r^6+13r^7-3r^8)}{3(1-r^2-2r^3+11r^4-11r^5-r^6+10r^7-10r^8+5r^9-r^{10})}. \quad (5.18)$$

At $p = p_{c,\mathrm{sq}} = 1/2$, this has the value

$$\langle k\rangle_{[\mathrm{sq},3_P]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{11}{78} = 0.141\,025\,6. \quad (5.19)$$

As is evident from Eq. (5.18) and is also true for all of the other $(L_y)_P$ strips of the square lattice for which we have calculated $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}$, the poles in the complex-p and -r planes have the same values. The coefficients $a_{1,[\mathrm{sq},3_P]}$ and $a_{3,[\mathrm{sq},3_P]}$ are given by our general results (4.18) and (4.19). For $a_{2,[\mathrm{sq},3_P]}$ we calculate

$$a_{2,[\mathrm{sq},3_P]} = \tfrac{77\,024}{34\,983} = 2.201\,755. \quad (5.20)$$

The first few terms of the small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},3_P]}$ are given in Table IV. Since the poles of $\langle k\rangle_{[\mathrm{sq},3_P]}$ are the same when expressed in the variables p and r, it follows that the poles nearest to the origin in the complex-p and -r planes have the same value. This is

$$p_{[\mathrm{sq},3_P],np} = r_{[\mathrm{sq},3_P],np} = -0.354\,731 \pm 0.319\,907i, \quad (5.21)$$

with magnitude

$$|p_{[\mathrm{sq},3_P],np}| = |r_{[\mathrm{sq},3_P],np}| = 0.477\,676. \quad (5.22)$$

These poles thus determine the radii of convergence of the respective small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},3_P]}$ as 0.477 676. As indicated in Table V, this radius of convergence is smaller than $p_{c,\mathrm{sq}} = r_{c,\mathrm{sq}} = 1/2$.

D. 4_P square-lattice strip

For this strip we calculate

$$\langle k\rangle_{[\mathrm{sq},4_P]} = \frac{N_{[\mathrm{sq},4_P]}}{D_{[\mathrm{sq},4_P]}}, \quad (5.23)$$

where the numerator and denominator polynomials $N_{[\mathrm{sq},4_P]}$ and $D_{[\mathrm{sq},4_P]}$, which are rather lengthy, are given in Eqs. (A3) and (A4) in the Appendix. At $p = p_{c,\mathrm{sq}} = 1/2$, $\langle k\rangle_{[\mathrm{sq},4_P]}$ has the value

$$\langle k\rangle_{[\mathrm{sq},4_P]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{677}{5572} = 0.121\,500. \quad (5.24)$$

For $a_{2,[\mathrm{sq},4_P]}$ we calculate

$$a_{2,[\mathrm{sq},4_P]} = \tfrac{3\,398\,556\,656}{1\,298\,160\,381} = 2.617\,979. \quad (5.25)$$

In $\langle k\rangle_{[\mathrm{sq},4_P]}$, the nearest poles to the origin in both the complex-p plane and the complex-r plane are at

$$p_{[\mathrm{sq},4_P],np} = r_{[\mathrm{sq},4_P],np} = -0.424\,294, \quad (5.26)$$

which determine the radii of convergence of the respective small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},4_P]}$. This radius of convergence is again smaller than $p_{c,\mathrm{sq}} = r_{c,\mathrm{sq}} = 1/2$.

E. 5_P square-lattice strip

We calculate

$$\langle k\rangle_{[\mathrm{sq},5_P]} = \frac{N_{[\mathrm{sq},5_P]}}{D_{[\mathrm{sq},5_P]}}, \quad (5.27)$$

where $N_{[\mathrm{sq},5_P]}$ and $D_{[\mathrm{sq},5_P]}$ are given in Eqs. (A5) and (A6) in the Appendix. At $p = p_{c,\mathrm{sq}} = 1/2$, this has the value

$$\langle k\rangle_{[\mathrm{sq},5_P]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{85\,013}{753\,370} = 0.112\,844. \quad (5.28)$$

This is only 15% larger than the value for the infinite square lattice

$$R_{[\mathrm{sq},5_P],c} = 1.150\,571, \quad (5.29)$$

where $R_{[\Lambda,(L_y)_{\mathrm{BC}_y}],c}$ was defined in Eq. (1.7).

For $a_{2,[\mathrm{sq},5_P]}$ we calculate

$$a_{2,[\mathrm{sq},5_P]} = \tfrac{1\,275\,302\,677\,055\,206\,848}{439\,932\,074\,289\,972\,983} = 2.898\,863. \quad (5.30)$$

The small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},5_P]}$ are given in Table IV. Of the 62 poles of $\langle k\rangle_{[\mathrm{sq},5_P]}$ when expressed as a function of p, which are the same when expressed as a function of r, the nearest poles to the origin in both the complex-p and -r planes are the complex-conjugate pair

$$p_{[\mathrm{sq},5_P],np} = r_{[\mathrm{sq},5_P],np} = -0.371\,844 \pm 0.169\,863i, \quad (5.31)$$

with magnitude

$$|p_{[\mathrm{sq},5_P],np}| = |r_{[\mathrm{sq},5_P],np}| = 0.408\,805. \tag{5.32}$$

These poles thus determine the radii of convergence of the respective small-p and small-r Taylor series expansions of $\langle k\rangle_{[\mathrm{sq},5_P]}$ as 0.408 805. As was the case with the 3_P and 4_P strips of the square lattice, this radius of convergence is smaller than $p_{c,\mathrm{sq}} = r_{c,\mathrm{sq}} = 1/2$.

F. Square-lattice strips with self-dual transverse boundary conditions and $L_y = 2$

Since the square lattice is self-dual, it is also useful to employ boundary conditions for strip graphs of the square lattice that obey this property even for finite L_x and L_y [32,42]. We denote the average cluster number for the strip of the square lattice with width L_y and self-dual transverse boundary conditions by $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}$. In [17] we calculated $\langle k\rangle_{[\mathrm{sq},1_{\mathrm{sd}}]}$. Here, for the 2_{sd} strip of the square lattice, we calculate

$$\begin{aligned}\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]} &= \frac{(1-p)^3(2-2p-4p^2+15p^3-17p^4+p^5+24p^6-34p^7+24p^8-10p^9+2p^{10})}{2(1-2p+9p^3-18p^4+16p^5+5p^6-32p^7+44p^8-35p^9+18p^{10}-6p^{11}+p^{12})}\\ &= \frac{r^3(1-3r^2+9r^3-2r^4-19r^5+38r^6-38r^7+24r^8-10r^9+2r^{10})}{2(1-2r+9r^3-18r^4+16r^5+5r^6-32r^7+44r^8-35r^9+18r^{10}-6r^{11}+r^{12})}.\end{aligned} \tag{5.33}$$

At $p = p_{c,\mathrm{sq}} = 1/2$, $\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]}$ has the value

$$\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{17}{118} = 0.144\,068. \tag{5.34}$$

The comparison of this with the value of $\langle k\rangle_{\mathrm{sq},c}$ for the infinite square lattice is indicated by the ratio

$$R_{[\mathrm{sq},2_{\mathrm{sd}}],c} = 1.468\,937\,2. \tag{5.35}$$

For $a_{2,[\mathrm{sq},2_{\mathrm{sd}}]}$ we calculate

$$a_{2,[\mathrm{sq},2_{\mathrm{sd}}]} = \tfrac{235\,936}{107\,911} = 2.186\,394. \tag{5.36}$$

The first few terms of the small-p and small-r series expansions of $\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]}$ are given in Table IV. In $\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]}$, the nearest pole to the origin in the complex-p and -r planes is

$$p_{[\mathrm{sq},2_{\mathrm{sd}}],np} = r_{[\mathrm{sq},2_{\mathrm{sd}}],np} = -0.483\,656\,7, \tag{5.37}$$

which sets the radius of convergence of the small-p and small-r series expansions for $\langle k\rangle_{[\mathrm{sq},2_{\mathrm{sd}}]}$.

G. 3_{sd} square-lattice strip

For the 3_{sd} strip of the square lattice, we calculate

$$\langle k\rangle_{[\mathrm{sq},3_{\mathrm{sd}}]} = \frac{N_{[\mathrm{sq},3_{\mathrm{sd}}]}}{D_{[\mathrm{sq},3_{\mathrm{sd}}]}}, \tag{5.38}$$

where $N_{[\mathrm{sq},3_{\mathrm{sd}}]}$ and $D_{[\mathrm{sq},3_{\mathrm{sd}}]}$ are given in Eqs. (A7) and (A8) in the Appendix. At $p = p_{c,\mathrm{sq}} = 1/2$, this has the value

$$\langle k\rangle_{[\mathrm{sq},3_{\mathrm{sd}}]}|_{p=p_{c,\mathrm{sq}}} = \tfrac{2051}{15\,474} = 0.132\,545. \tag{5.39}$$

For $a_{2,[\mathrm{sq},3_{\mathrm{sd}}]}$ we calculate

$$a_{2,[\mathrm{sq},3_{\mathrm{sd}}]} = \tfrac{4\,105\,669\,781\,114\,576}{1\,664\,338\,698\,530\,559} = 2.466\,847\,5. \tag{5.40}$$

The poles in $\langle k\rangle_{[\mathrm{sq},3_{\mathrm{sd}}]}$, nearest to origin in the complex-p and -r planes are the complex-conjugate pair

$$p_{[\mathrm{sq},3_{\mathrm{sd}}],np} = r_{[\mathrm{sq},3_{\mathrm{sd}}],np} = -0.341\,129 \pm 0.289\,364i, \tag{5.41}$$

with magnitude

$$|p_{[\mathrm{sq},3_{\mathrm{sd}}],np}| = |r_{[\mathrm{sq},3_{\mathrm{sd}}],np}| = 0.447\,326, \tag{5.42}$$

which is thus the radius of convergence of the small-p and small-r series expansions of $\langle k\rangle_{[\mathrm{sq},3_{\mathrm{sd}}]}$.

Thus, the square-lattice strips with periodic transverse boundary conditions and the square-lattice strips with self-dual transverse boundary conditions most closely replicate the properties of the infinite square lattice, namely, absence of boundary effects and self-duality. For this reason, one expects that for a given width L_y, the values of $\langle k\rangle$ and its critical value at $p = p_{c,\mathrm{sq}}$, $\langle k\rangle_{[\mathrm{sq},(L_y)_P]}|_{p=p_{\mathrm{sq},c}}$ or $\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}|_{p=p_{\mathrm{sq},c}}$, will be closer to the values on the infinite square lattice than is the case for free transverse boundary conditions, and our exact results confirm this general expectation.

VI. TRIANGULAR-LATTICE STRIPS

A. 3_F triangular-lattice strips

In [17] we presented calculations of $\langle k\rangle_{[\mathrm{tri},2_F]}$ and $\langle k\rangle_{[\mathrm{tri},2_P]}$. Here, again making use of our more powerful calculational methods, we calculate

$$\begin{aligned}\langle k\rangle_{[\mathrm{tri},3_F]} &= \frac{(1-p)^4(3+2p-3p^2-14p^3+48p^4-62p^5+7p^6+90p^7-144p^8+123p^9-66p^{10}+21p^{11}-3p^{12})}{3(1-p-2p^3+22p^4-56p^5+72p^6-29p^7-76p^8+179p^9-210p^{10}+166p^{11}-94p^{12}+37p^{13}-9p^{14}+p^{15})}\\ &= \frac{r^4(2+2r+r^2-r^3-4r^4+2r^5+7r^6-27r^8+42r^9-33r^{10}+15r^{11}-3r^{12}}{3(1-r+r^2-2r^3+2r^4+7r^5-17r^6+22r^7-28r^8+29r^9-12r^{10}-13r^{11}+23r^{12}-16r^{13}+6r^{14}-r^{15})}.\end{aligned} \tag{6.1}$$

At $p = p_{c,\mathrm{tri}}$,

$$\langle k\rangle_{[\mathrm{tri},3_F]}|_{p=p_{c,tri}} = \frac{306\,241-2\,163\,343s+2\,302\,182s^2}{3(25\,781-182\,124s+193\,812s^2)} = 0.271\,486\,6. \tag{6.2}$$

Here and below we use the symbol $s = \sin(\pi/18)$, as defined in Eq. (2.11). In obtaining this and analytic evaluations of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}|_{p=p_{c,\Lambda}}$ for other strips of the triangular lattice and for strips of the honeycomb lattice, we have used the trigonometric identity $\sin^3(\pi/18) = (1/8)[6\sin(\pi/18) - 1]$, which enables us to reduce any (finite-degree) polynomial in s to a polynomial of degree 2. We also note an analytic result for the [tri, 2_F] strip that was not given before:

$$\langle k\rangle_{[\mathrm{tri},2_F]}|_{p=p_{c,\mathrm{tri}}} = \frac{2(1-6s+6s^2)}{1-2s+4s^2} = 0.359\,575. \quad (6.3)$$

In Table IV we list the first few terms in the small-p and small-r series expansions of $\langle k\rangle_{[\mathrm{tri},3_F]}$. The poles in $\langle k\rangle_{[\mathrm{tri},3_F]}$ nearest to the origin in the complex-p plane are the complex-conjugate pair

$$p_{[\mathrm{tri},3_F],np} = -0.300\,743 \pm 0.259\,341i, \quad (6.4)$$

with magnitude

$$|p_{[\mathrm{tri},3_F],np}| = 0.397\,120, \quad (6.5)$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{tri},3_F]}$. The pole in $\langle k\rangle_{[\mathrm{tri},3_F]}$ nearest to the origin in the complex-r plane occurs at

$$r_{[\mathrm{tri},3_F],np} = -0.599\,392, \quad (6.6)$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{tri},3_F]}$ as 0.599 392. These values are listed in Table V.

B. 4_F triangular-lattice strips

For the 4_F strip of the triangular lattice, we calculate

$$\langle k\rangle_{[\mathrm{tri},4_F]} = \frac{N_{[\mathrm{tri},4_F]}}{D_{[\mathrm{tri},4_F]}}, \quad (6.7)$$

where the numerator and denominator polynomials are given in Eqs. (A9) and (A10) in the Appendix. At $p = p_{c,\mathrm{tri}}$,

$$\langle k\rangle_{[\mathrm{tri},4_F]}|_{p=p_{c,\mathrm{tri}}} = \frac{7\,325\,865\,108\,433\,807 - 51\,751\,213\,463\,154\,938s + 55\,072\,491\,066\,145\,656s^2}{8(225\,167\,815\,542\,115 - 1\,590\,625\,477\,629\,565s + 1\,692\,708\,277\,627\,262s^2)}.$$
$$= 0.229\,460. \quad (6.8)$$

The pole in $\langle k\rangle_{[\mathrm{tri},4_F]}$ nearest to the origin in the complex-p plane occurs at

$$p_{[\mathrm{tri},4_F],np} = -0.335\,309, \quad (6.9)$$

with magnitude $|p_{[\mathrm{tri},4_F],np}| = 0.335\,309$, which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{tri},4_F]}$. The poles in $\langle k\rangle_{[\mathrm{tri},4_F]}$ nearest to the origin in the complex-r plane are the complex-conjugate pair

$$r_{[\mathrm{tri},4_F],np} = -0.419\,061 \pm 0.379\,572i, \quad (6.10)$$

with magnitude

$$|r_{[\mathrm{tri},4_F],np}| = 0.565\,408, \quad (6.11)$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{tri},4_F]}$. In contrast to the situation with the 2_F and 3_F strips of the triangular lattice, $|p_{[\mathrm{tri},4_F],np}| < p_{\mathrm{tri},c}$ and $|r_{[\mathrm{tri},4_F],np}| < r_{\mathrm{tri},c}$. Thus, for this strip, the radii of convergence of the small-p and small-r series are not set by the respective physical critical values $p_{c,\mathrm{tri}}$ and $r_{c,\mathrm{tri}}$, on the infinite triangular lattice, but instead by unphysical singularities.

C. 3_P triangular-lattice strip

We denote the average cluster number for the infinite-length strip of the triangular lattice with width L_y and periodic transverse boundary conditions by $\langle k\rangle_{[\mathrm{tri},(L_y)_P]}$. For the 3_P strip of the triangular lattice, we calculate

$$\langle k\rangle_{[\mathrm{tri},3_P]} = \frac{N_{[\mathrm{tri},3_P]}}{D_{[\mathrm{tri},3_P]}}, \quad (6.12)$$

where the numerator and denominator polynomials are given in Eqs. (A11) and (A12) in the Appendix. At $p = p_{c,\mathrm{tri}}$,

$$\langle k\rangle_{[\mathrm{tri},3_P]}|_{p=p_{c,\mathrm{tri}}} = \frac{2(74\,704\,191 - 527\,723\,687s + 561\,591\,818s^2)}{9(939\,965 - 6\,640\,082s + 7\,066\,228s^2)} = 0.146\,651. \quad (6.13)$$

We note an analytic result that was not given in [17], namely,

$$\langle k\rangle_{[\mathrm{tri},2_P]}|_{p=p_{c,\mathrm{tri}}} = \frac{3(251 - 1774s + 1888s^2)}{2(33 - 240s + 256s^2)} = 0.190\,910. \quad (6.14)$$

The poles in $\langle k\rangle_{[\mathrm{tri},3_P]}$ nearest to the origin in the complex-p plane are the complex-conjugate pair

$$p_{[\mathrm{tri},3_P],np} = -0.227\,780\,5 \pm 0.175\,218i, \quad (6.15)$$

with magnitude

$$|p_{[\mathrm{tri},3_P],np}| = 0.287\,376, \quad (6.16)$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{tri},3_P]}$. The pole in $\langle k\rangle_{[\mathrm{tri},3_P]}$ nearest to the origin in the complex-r plane is

$$r_{[\mathrm{tri},3_P],np} = -0.594\,760, \tag{6.17}$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{tri},3_P]}$ as 0.594 760. These radii of convergence are both smaller than the respective critical values $p_{c,\mathrm{tri}}$ and $r_{c,\mathrm{tri}}$.

D. 4_P triangular-lattice strip

For the 4_P triangular lattice strip, we calculate

$$\langle k\rangle_{[\mathrm{tri},4_P]} = \frac{N_{[\mathrm{tri},4_P]}}{D_{[\mathrm{tri},4_P]}}, \tag{6.18}$$

where $N_{[\mathrm{tri},4_P]}$ and $D_{[\mathrm{tri},4_P]}$ are given in Eqs. (A13) and (A14) in the Appendix. At $p = p_{c,\mathrm{tri}}$,

$$\langle k\rangle_{[\mathrm{tri},4_P]}|_{p=p_{c,\mathrm{tri}}} = \frac{574\,004\,215\,646\,387\,707\,017 - 4\,054\,867\,821\,476\,682\,227\,104s + 4\,315\,100\,205\,943\,310\,268\,010s^2}{2(8\,584\,252\,854\,733\,404\,261 - 60\,640\,688\,209\,720\,609\,514s + 64\,532\,472\,500\,426\,786\,720s^2)}$$
$$= 0.131\,378. \tag{6.19}$$

Relative to the critical value $\langle k\rangle_{c,\mathrm{tri}}$ for the infinite triangular lattice,

$$R_{[\mathrm{tri},4_P,c]} = 1.174\,651. \tag{6.20}$$

Interestingly, this ratio is approaching reasonably close to unity already when the strip width has the modest value of $L_y = 4$, if one uses periodic transverse boundary conditions. The approach to the infinite-width limit is slower if one uses free transverse boundary conditions. This is similar to the behavior that we found for the square lattice and, as in that case, one can understand it as a consequence of the absence of any boundaries for PBC_y.

The pole in $\langle k\rangle_{[\mathrm{tri},4_P]}$ nearest to the origin in the complex-p plane occurs at

$$p_{[\mathrm{tri},4_P],np} = -0.260\,779, \tag{6.21}$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{tri},4_P]}$ as 0.260 779. The pole in $\langle k\rangle_{[\mathrm{tri},4_P]}$ nearest to the origin in the complex-r plane occurs at

$$r_{[\mathrm{tri},4_P],np} = -0.570\,571, \tag{6.22}$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{tri},4_P]}$ as 0.570 571. Both of these radii of convergence are smaller than the respective critical values $p_{c,\mathrm{tri}}$ and $r_{c,\mathrm{tri}}$.

VII. HONEYCOMB-LATTICE STRIPS

A. 3_F honeycomb-lattice strip

We calculated the average cluster number for the infinite-length 2_F strip of the honeycomb lattice in Ref. [17]. Here we calculate

$$\langle k\rangle_{[\mathrm{hc},3_F]} = \frac{(1-p)^2(3+2p-2p^2-5p^3-p^4+5p^5+p^6+4p^7-10p^8+7p^{10}-3p^{11})}{3(1-p^2+p^3)(1-2p^3+p^4+2p^6-2p^8+p^9)}$$
$$= \frac{r^2(1+5r-4r^2+14r^3-41r^4+87r^5-167r^6+226r^7-190r^8+95r^9-26r^{10}+3r^{11})}{3(1-r+2r^2-r^3)(1-3r+10r^2-14r^3+17r^4-26r^5+30r^6-20r^7+7r^8-r^9)}. \tag{7.1}$$

At $p = p_{c,\mathrm{hc}}$, this has the value

$$\langle k\rangle_{[\mathrm{hc},3_F]}|_{p=p_{c,\mathrm{hc}}} = \frac{-12\,803 + 90\,443s - 96\,244s^2}{12(772 - 5453s + 5803s^2)} = 0.160\,002. \tag{7.2}$$

We note a related analytic result

$$\langle k\rangle_{[\mathrm{hc},2_F]}|_{p=p_{c,\mathrm{hc}}} = \frac{-55 + 392s - 408s^2}{8(5 - 32s + 34s^2)} = 0.204\,751. \tag{7.3}$$

The poles in $\langle k\rangle_{[\mathrm{hc},3_F]}$ nearest to the origin in the complex-p plane are the complex-conjugate pair

$$p_{[\mathrm{hc},3_F],np} = -0.492\,595 \pm 0.542\,272i, \tag{7.4}$$

with magnitude

$$|p_{[\mathrm{hc},3_F],np}| = 0.732\,604, \tag{7.5}$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{hc},3_F]}$. The poles in $\langle k\rangle_{[\mathrm{hc},3_F]}$ nearest to the origin in the complex-r plane are the complex-conjugate pair

$$r_{[\mathrm{hc},3_F],np} = 0.123\,348 \pm 0.377\,252i, \tag{7.6}$$

with magnitude

$$|r_{[\mathrm{hc},3_F],np}| = 0.396\,906, \tag{7.7}$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{hc},3_F]}$.

B. 4_F honeycomb-lattice strip

For the 4_F strip of the honeycomb lattice, we calculate

$$\langle k\rangle_{[\mathrm{hc},4_F]} = \frac{N_{[\mathrm{hc},4_F]}}{D_{[\mathrm{hc},4_F]}}, \tag{7.8}$$

where $N_{[\mathrm{hc},4_F]}$ is a polynomial of degree 72 in p containing a factor of $(1-p)^2$ and $D_{[\mathrm{hc},4_F]}$ is a polynomial of degree 71 in p that we have calculated. At $p = p_{c,\mathrm{hc}}$,

$$\begin{aligned}\langle k\rangle_{[\mathrm{hc},4_F]}|_{p=p_{c,\mathrm{hc}}} \\ = \frac{-113\,592\,578\,275\,136\,635\,723\,243\,683 + 8\,024\,381\,665\,694\,504\,094\,459\,981\,670s - 8\,539\,368\,606\,495\,326\,857\,081\,040\,364s^2}{16(53\,721\,138\,617\,890\,198\,824\,050\,135 - 379\,495\,673\,336\,597\,286\,155\,883\,324s + 403\,850\,860\,315\,586\,504\,368\,203\,856s^2)} \\ = 0.138\,340\,7.\end{aligned} \tag{7.9}$$

The poles in $\langle k\rangle_{[\mathrm{hc},4_F]}$ nearest to the origin in the complex-p plane are the complex-conjugate pair

$$p_{[\mathrm{hc},4_F],np} = -0.552\,838 \pm 0.373\,251i, \tag{7.10}$$

with magnitude

$$|p_{[\mathrm{hc},4_F],np}| = 0.667\,042, \tag{7.11}$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{hc},4_F]}$. The poles in $\langle k\rangle_{[\mathrm{hc},4_F]}$ nearest to the origin in the complex-r plane are the complex-conjugate pair

$$r_{[\mathrm{hc},4_F],np} = -0.212\,449 \pm 0.136\,692i, \tag{7.12}$$

with magnitude

$$|r_{[\mathrm{tri},4_F],np}| = 0.252\,625, \tag{7.13}$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{hc},4_F]}$.

C. 2_P honeycomb-lattice strip

Strips of the honeycomb lattice require that L_y be even. For the 2_P strip of the honeycomb lattice we calculate

$$\langle k\rangle_{[\mathrm{hc},2_P]} = \langle k\rangle_{[\mathrm{sq},2_F]}. \tag{7.14}$$

For the value evaluated at $p = p_{c,\mathrm{hc}}$, we find

$$\langle k\rangle_{[\mathrm{hc},2_P]} = \frac{-3 + 22s - 20s^2}{4(1-2s)^2} = 0.127\,450. \tag{7.15}$$

D. 4_P honeycomb-lattice strip

For the 4_P honeycomb strip, we calculate

$$\langle k\rangle_{[\mathrm{hc},4_P]} = \frac{N_{[\mathrm{hc},4_P]}}{D_{[\mathrm{hc},4_P]}}, \tag{7.16}$$

where $N_{[\mathrm{hc},4_P]}$ and $D_{[\mathrm{hc},4_P]}$ are given in Eqs. (A15) and (A16) in the Appendix. At $p = p_{c,\mathrm{hc}}$, this has the value

$$\langle k\rangle_{[\mathrm{hc},4_P]}|_{p=p_{c,\mathrm{hc}}} = \frac{736\,538\,075\,855 - 5\,203\,035\,904\,036s + 5\,536\,955\,158\,472s^2}{32(-19\,547\,696\,983 + 138\,088\,406\,531s - 146\,950\,612\,867s^2)} = 0.089\,833\,7. \tag{7.17}$$

The pole in $\langle k\rangle_{[\mathrm{hc},4_P]}$ nearest to the origin in the complex-p plane is

$$p_{[\mathrm{hc},4_P],np} = -0.585\,767, \tag{7.18}$$

which sets the radius of convergence of the small-p series for $\langle k\rangle_{[\mathrm{hc},4_P]}$ as 0.585 767. The pole in $\langle k\rangle_{[\mathrm{hc},4_P]}$ nearest to the origin in the complex-r plane is

$$r_{[\mathrm{hc},4_P],np} = -0.270\,891, \tag{7.19}$$

which sets the radius of convergence of the small-r series for $\langle k\rangle_{[\mathrm{hc},4_P]}$ as 0.270 891.

VIII. COMPARATIVE DISCUSSION

As noted in the Introduction, our main results here include (i) the theorem (4.1), showing that the average cluster number per site on infinite-length lattice strips with width L_y and specified transverse boundary conditions BC_y, $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$, is a rational function of the bond occupation probability p; (ii) the calculation of the exact expressions for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ as a function of p; (iii) exact values of these average cluster numbers at $p = p_{c,\Lambda}$, the critical bond occupation probability for the corresponding infinite-length lattices; (iv) a study of the L_y dependence of these values (discussed further below); (v) calculations of $d^j\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}/(dp)^j$ with $j = 1, 2, 3$, evaluated at $p = p_{c,\Lambda}$, for infinite-length lattice strips Λ_s with a resultant determination of coefficients in the expansion of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in Eq. (2.5); and (vi) a study of the poles in $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ and the insight that these yield concerning the role of unphysical singularities setting the radii of convergence in small-p and small-r series expansions of various quantities in percolation on infinite two-dimensional lattices. That is, one encounters this phenomenon even for finite-width strips of modest widths, before the limit $L_y \to \infty$ is taken to obtain $\langle k\rangle_\Lambda$.

Here we give some further comparative discussion of these results. First, our exact results strengthen and extend two monotonicity relations that we found in our previous study [17]. We find that for fixed $p \in (0, 1)$, $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ is a monotonically decreasing function of the strip width L_y for all of the lattices considered here. (At the end points of the physical interval in p, the values are fixed, as $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]} = 1$ at $p = 0$ and $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]} = 0$ at $p = 1$, independent of L_y.) Second, for fixed L_y, $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is a monotonically decreasing function of p in the physical interval $0 \leqslant p \leqslant 1$.

Furthermore, with our present exact analytic results, we have strengthened the finding from our previous study in [17] that for a given lattice type and set of transverse boundary conditions, over the range of strip widths L_y that we have studied, the behavior of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is consistent with the inference that, for a fixed $p \in (0, 1)$, the average cluster number on the infinite-length strip, $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$, approaches $\langle k\rangle_\Lambda$ as $L_y \to \infty$. (This is automatic for the two end points $p = 0$ and $p = 1$, where $\langle k\rangle_{\Lambda_s} = 1$ and $\langle k\rangle_{\Lambda_s} = 0$.)

In particular, for each type of infinite-length, finite-width lattice strip Λ_s for which we have calculated exact expressions for $\langle k\rangle_{\Lambda_s}$, as L_y increases, the evaluation with p set equal to the critical bond occupation probability for the corresponding infinite two-dimensional lattice Λ, $p = p_{c,\Lambda}$, approaches the known critical value for the infinite lattice $\langle k\rangle_\Lambda|_{p=p_{c,\Lambda}}$. As expected, for a given width L_y, the deviation from this critical value for the infinite two-dimensional lattice is smallest for the infinite-length strips with periodic transverse boundary conditions, since these remove boundary effects, as contrasted with the strips with free transverse boundary conditions:

$$|\langle k\rangle_{[\Lambda,(L_y)_P]}|_{p=p_{c,\Lambda}} - \langle k\rangle_\Lambda|_{p=p_{c,\Lambda}}| < |\langle k\rangle_{[\Lambda,(L_y)_F]}|_{p=p_{c,\Lambda}} - \langle k\rangle_\Lambda|_{p=p_{c,\Lambda}}|. \tag{8.1}$$

Thus, with periodic transverse boundary conditions, the only finite-size effect that remains on the infinite-length lattices is the fact that L_y is finite, i.e., there is a finite-length path crossing the lattice strip in a transverse direction. For a given infinite-length square-lattice strip of width L_y, the deviation of the average cluster number at $p = p_{c,sq}$ from its value on the infinite square lattice is also smaller with self-dual boundary conditions, as contrasted with free transverse boundary conditions:

$$[\langle k\rangle_{[\mathrm{sq},(L_y)_{\mathrm{sd}}]}|_{p=p_{c,\mathrm{sq}}} - \langle k\rangle_{\mathrm{sq}}|_{p=p_{c,\mathrm{sq}}}] < [\langle k\rangle_{[\mathrm{sq},(L_y)_F]}|_{p=p_{c,\mathrm{sq}}} - \langle k\rangle_{\mathrm{sq}}|_{p=p_{c,\mathrm{sq}}}]. \tag{8.2}$$

Our work here is complementary to the calculation in Ref. [20] of $\langle k\rangle_{\mathrm{sq,diag}}|_{p=p_{c,\mathrm{sq}}}$ on infinite-length diagonal strips of arbitrary widths (with toroidal boundary conditions) on the square lattice, since we calculate $\langle k\rangle_{\Lambda,(L_y)_{\mathrm{BC}_y}}$ as a function of p, not just for the single value $p = p_{c,\Lambda}$, while Ref. [20] calculates the values only at $p = p_{c,\mathrm{sq}}$. (Another difference is that we have also calculated exact values of $\langle k\rangle_{\Lambda,(L_y)_{\mathrm{BC}_y}}$ for triangular and honeycomb lattices and in [17] for the kagome lattice.) As the strip width increases, the approach to the value $\langle k\rangle_{c,\mathrm{sq}}$ in Eq. (2.12) is comparably rapid. For example, for the index $N = 3$ (corresponding to a width across the diagonal of $3\sqrt{2} = 4.243$), Ref. [20] obtains $\langle k\rangle_{\mathrm{sq,diag}} = 79/672 = 0.117\,560$, which lies between our values $\langle k\rangle_{\mathrm{sq},4_P}|_{p=p_{c,\mathrm{sq}}} = 677/5572 = 0.121\,500$ in Eq. (5.24) and $\langle k\rangle_{\mathrm{sq},5_P}|_{p=p_{c,\mathrm{sq}}} = 85\,013/753\,370 = 0.112\,844$ in Eq. (5.28). This is in accord with one's expectation, since the width $3\sqrt{2}$ is intermediate between the widths $L_y = 4$ and $L_y = 5$.

An important result of our calculations is the comparison with the formula for the finite-size correction to $\langle k\rangle_{c,\Lambda}$ derived in [15,16], given above in Eq. (1.4), both concerning the constant $5\sqrt{3}/24$ in the $O(1/L_y^2)$ term and concerning the universality of this finite-size correction as regards the type of lattice, with the geometrical factors (4.9) and (4.10) incorporated. For this comparison, we list in Table VI the values of $\tilde{b}_{\Lambda,L_y}$ that we extract from our fit to Eq. (4.8) for the infinite-length strips of the square, triangular, and honeycomb lattices. As is evident from this table, as L_y increases, our results approach the value $\tilde{b} = 5\sqrt{3}/24$ in [15] (see also [16]) and furthermore are consistent with being equal for all three of these types of lattices, in agreement with the universality property of this finite-size correction. Indeed, with rather modest strip widths, we find excellent agreement with the value of $\tilde{b}$ in Eq. (1.4). This is a valuable universality check using exact results for different types of lattice strips, including square, triangular, and honeycomb lattices.

Another interesting application of our calculations of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ on these infinite-length lattice strips is to investigate how the small-p and small-r Taylor series expansions compare with those for the corresponding infinite two-dimensional lattices. The entries in Table IV are useful for this purpose. As is evident from this table, for the infinite-length $[\Lambda, (L_y)_P]$ strips, which are Δ regular, we find that the small-p expansions have the general form $\langle k\rangle_{\Lambda_s} = 1 - (\Delta/2)p + \cdots$, where the ellipsis denotes higher-order terms, in accord with Eq. (4.15). This form for the first two terms is the same as with the infinite two-dimension lattices. For the infinite-length strips that are not Δ regular, such as those with free transverse boundary conditions, we find that the small-p expansion has the form $\langle k\rangle_{\Lambda_s} = 1 - (\Delta_{\rm eff}/2)p + \cdots$, where $\Delta_{\rm eff}$ was defined in Eq. (3.5).

Concerning the rest of the small-p series, by inspecting the series for $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ on infinite-length lattice strips of a given lattice Λ with some specified transverse boundary conditions, one can see how, as a function of increasing strip width L_y, coefficients of certain terms for these strips approach the values that they have in the corresponding small-p or small-r expansion of $\langle k\rangle_\Lambda$ on the infinite two-dimensional lattice Λ. For example, consider the $(L_y)_F$ strips of the square, triangular, and honeycomb lattices. One sees that the coefficient of the respective linear terms in the small-p series expansions increase monotonically toward the respective values 4, 6, and 3, in agreement with the discussion above.

The next higher-order term in the small-p expansion of $\langle k\rangle_{\rm sq}$ for the infinite square lattice is p^4, and one can see from Table IV how, as the width L_y of the $(L_y)_F$ square-lattice strips increases from 2 to 4, the coefficient of the p^4 term in the series expansion of $\langle k\rangle_{[{\rm sq},(L_y)_F]}$ increases toward 1, taking on the respective values 1/2, 2/3, and 3/4. Similarly, the next term higher than linear in the small-p expansion of $\langle k\rangle_{\rm tri}$ on the infinite triangular lattice is $2p^3$, and the coefficients of the p^3 terms in $\langle k\rangle_{[{\rm tri},(L_y)_F]}$ increase toward this value, as 1, 4/3, and 3/2 with $L_y = 2$, 3, and 4, respectively. Finally, the next term higher than linear in the small-p expansion of $\langle k\rangle_{\rm hc}$ on the infinite honeycomb lattice is $(1/2)p^6$, and the coefficients of the p^6 terms in $\langle k\rangle_{[{\rm hc},(L_y)_F]}$ increase toward 1/2, taking on the values 1/4, 1/3, and 3/8 as L_y increases from 2 to 4.

For the infinite-length strips with periodic transverse boundary conditions, the linear terms in p are equal to their values for the corresponding infinite two-dimensional lattices, and again the rest of the small-p series become more similar to the series for the two-dimensional lattices as the width increases. As an example, consider the $[{\rm sq}, (L_y)_P]$ strips. The small-p series for $\langle k\rangle_{[{\rm sq},2_P]}$ has a nonzero p^2 term, but it is absent in the series expansion of $\langle k\rangle_{[{\rm sq},3_P]}$ on the next wider strip of this type. In turn, the small-p series for $\langle k\rangle_{[{\rm sq},3_P]}$ contains a nonzero p^3 term, but it is absent in the series expansion of $\langle k\rangle_{[{\rm sq},4_P]}$ on the next wider strip of this type. The small-p series expansion of $\langle k\rangle_{[{\rm sq},5_P]}$ matches not just the linear term, but also the p^4 term of $\langle k\rangle_{\rm sq}$ exactly. Corresponding comments apply for the $(L_y)_P$ strips of the triangular and honeycomb lattices. One might anticipate some special properties of the small-p series expansions of $\langle k\rangle_{[{\rm sq},(L_y)_{\rm sd}]}$ owing to the inclusion of the self-duality property. Interestingly, one sees that with all three widths for which we have calculated $\langle k\rangle_{[{\rm sq},(L_y)_{\rm sd}]}$, namely, $L_y = 1$, 2, and 3, the small-p expansions match not just the linear term, but also the p^4 term in $\langle k\rangle_{\rm sq}$ exactly. Over this range of L_y values, one observes that the coefficient of the p^3 term decreases monotonically, consistent with its vanishing as $L_y \to \infty$. Analogous comments apply for the small-r series expansions of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$.

Finally, we have used our exact calculations of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ for these lattice strips to answer an intriguing question concerning the presence of unphysical singularities that were found, in analyses of small-p and small-r series calculations of average cluster numbers on two-dimensional lattices [7,8], to be closer to the respective origins in these planes than the physical $p_{c,\Lambda}$ and $r_{c,\Lambda} = 1 - p_{c,\Lambda}$ for these lattices. The question is whether such unphysical singularities (which are manifested as poles in Padé approximants of series) would also be encountered in the exact expressions for $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$. Our earlier analytic results in [17] showed the presence of poles, but were limited to rather narrow strip widths. With our present calculations of $\langle k\rangle_{\Lambda_s}$ for considerably greater strip widths, we have answered this question, in the affirmative. This is evident in Table V. Furthermore, we find that with all of the strips for which we have performed exact calculations, for a given type of lattice strip Λ and specified transverse boundary conditions ${\rm BC}_y$, the magnitude of the pole(s) of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ nearest to the origin in the complex p plane decreases monotonically with increasing L_y, and similarly, the magnitude of the pole(s) of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ nearest to the origin in the r plane decreases monotonically with increasing L_y. Thus, the corresponding radii of convergence of the small-p and small-r series also decrease with increasing L_y. One knows rigorously that the small-p and small-r series expansions of $\langle k\rangle_\Lambda$ for infinite-length strips of arbitrarily large width, and also for the infinite lattices Λ, are Taylor series with finite radii of convergence, given the connection via (2.4) with the Potts model. This follows because $v_p = p/(1-p)$, so that the small-p and small-r expansions in this bond percolation problem correspond, respectively, to high-temperature and low-temperature expansions in the Potts model. In general, the high- and low-temperature expansions of a discrete spin model such as the Potts model are Taylor series expansions with finite radii of convergence. Our results are thus consistent with the inference that, as $L_y \to \infty$, the magnitude of the pole(s) nearest to the origin in the complex-p plane and the resultant radius of convergence of the small-p series expansions of $\langle k\rangle_{[\Lambda,(L_y)_{BC_y}]}$ will approach the value obtained from analyses of small-p series expansions of $\langle k\rangle_\Lambda$ on the corresponding infinite two-dimensional lattices. A similar comment applies to the poles in the r plane. For example, regarding the poles in the p plane, from analyses in Ref. [8] of small-p series expansions for the average cluster number on the square lattice $\langle k\rangle_{\rm sq}$, evidence was reported for an unphysical singularity at $p = -0.41 \pm 0.02$ (see also [7]). Our results, as listed in Table V, show a decrease in the magnitude of the unphysical pole(s) nearest to the origin in the complex-p plane consistent with the inference that with increasing strip width L_y, this magnitude approaches this value $\simeq 0.41$ [8] obtained from series analyses for the infinite square lattice. Indeed, the magnitude of the complex-conjugate pair of poles nearest to the origin in the $[{\rm sq}, 5_P]$ strip is already equal to 0.41 to two significant figures. Our exact results on these lattice strips thus give additional insight into this phenomenon of unphysical singularities closer to the origin than $p_{c,\Lambda}$ that were noticed in early series analyses [7,8].

IX. CONCLUSION

In this paper we have presented a number of new exact results for average cluster numbers $\langle k\rangle_{\Lambda,(L_y)_{\mathrm{BC}_y}}$ in the bond percolation problem on infinite-length lattice strips of the square, triangular, and honeycomb lattices with various transverse boundary conditions. We have proved a theorem that $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ is a rational function of the bond occupation probability p. We have evaluated our expressions for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ with p set equal to the critical values $p=p_{c,\Lambda}$ for the corresponding infinite two-dimensional lattices. We have also calculated coefficients of $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ in an expansion around $p=p_{c,\Lambda}$. Using our calculations on infinite-length strips of several different widths and lattices types, we have checked and found excellent agreement with the functional form and coefficient describing the finite-size correction to the infinite-width limit. Finally, we have carried out a study of the poles in the expressions for $\langle k\rangle_{[\Lambda,(L_y)_{\mathrm{BC}_y}]}$ and how these determine the radii of convergence of the small-p and small-r Taylor series expansions of these quantities. In turn, this has given insight into the appearance of unphysical singularities that were found in early series expansions of $\langle k\rangle_\Lambda$ on two-dimensional lattices Λ.

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APPENDIX: SOME DETAILED RESULTS OF CALCULATIONS

We list here numerator and denominator polynomials in Eq. (4.1) for various infinite-length strips that are too lengthy to give in the main text:

$$\begin{aligned}N_{[\mathrm{sq},4_F]} &= (1-p)^3(4+5p-13p^2-22p^3+13p^4+120p^5-35p^6-342p^7+67p^8+800p^9-42p^{10}-2243p^{11}\\&+2042p^{12}+867p^{13}-1632p^{14}+2066p^{15}-8992p^{16}+14\,900p^{17}-3933p^{18}-15\,767p^{19}+19\,105p^{20}\\&-10\,149p^{21}+17\,236p^{22}-37\,363p^{23}+39\,047p^{24}-19\,238p^{25}-6431p^{26}+58\,942p^{27}-158\,184p^{28}\\&+235\,049p^{29}-176\,732p^{30}-19\,602p^{31}+213\,240p^{32}-267\,764p^{33}+182\,599p^{34}-59\,067p^{35}-17\,833p^{36}\\&+35\,509p^{37}-24\,007p^{38}+10\,257p^{39}-2997p^{40}+589p^{41}-71p^{42}+4p^{43}),\end{aligned}\tag{A1}$$

$$\begin{aligned}D_{[\mathrm{sq},4_F]} &= 4(1-4p^2+8p^4+21p^5-45p^6-50p^7+125p^8+106p^9-262p^{10}-388p^{11}+1257p^{12}-911p^{13}-353p^{14}\\&+1392p^{15}-3441p^{16}+7214p^{17}-7659p^{18}-33p^{19}+10\,102p^{20}-13\,234p^{21}+12\,476p^{22}-17\,624p^{23}+25\,847p^{24}\\&-24\,760p^{25}+10\,265p^{26}+17\,864p^{27}-67\,400p^{28}+131\,039p^{29}-160\,372p^{30}+101\,976p^{31}+31\,616p^{32}\\&-155\,851p^{33}+192\,656p^{34}-139\,509p^{35}+55\,077p^{36}+4708p^{37}-24\,705p^{38}+20\,289p^{39}-10\,358p^{40}+3729p^{41}\\&-961p^{42}+171p^{43}-19p^{44}+p^{45}),\end{aligned}\tag{A2}$$

$$\begin{aligned}N_{[\mathrm{sq},4_P]} &= (1-p)^4(4+8p-16p^3-39p^4+112p^5-20p^6-208p^7+315p^8-223p^9+248p^{10}-647p^{11}\\&+1106p^{12}-1318p^{13}+1453p^{14}-766p^{15}-2735p^{16}+8742p^{17}-12\,662p^{18}+10\,502p^{19}-4091p^{20}\\&-1358p^{21}+3122p^{22}-2307p^{23}+1033p^{24}-297p^{25}+51p^{26}-4p^{27}),\end{aligned}\tag{A3}$$

$$\begin{aligned}D_{[\mathrm{sq},4_P]} &= 4(1-p+p^2)(1+p-2p^2-3p^3-3p^4+41p^5-36p^6-62p^7+140p^8-131p^9+120p^{10}-226p^{11}\\&+460p^{12}-649p^{13}+688p^{14}-480p^{15}-654p^{16}+3216p^{17}-5785p^{18}+5926p^{19}-3292p^{20}+99p^{21}\\&+1578p^{22}-1584p^{23}+912p^{24}-351p^{25}+90p^{26}-14p^{27}+p^{28}),\end{aligned}\tag{A4}$$

$$\begin{aligned}N_{[\mathrm{sq},5_P]} &= (1-p)^4(5+15p+10p^2-40p^3-115p^4-29p^5+660p^6+132p^7-1709p^8-877p^9+3950p^{10}\\&+2877p^{11}-7215p^{12}-8662p^{13}+7196p^{14}+40\,393p^{15}-53\,232p^{16}+13\,204p^{17}-51\,313p^{18}+19\,634p^{19}\\&+377\,380p^{20}-503\,109p^{21}-570\,329p^{22}+1\,553\,036p^{23}-65\,274p^{24}-2\,873\,234p^{25}+4\,621\,549p^{26}\\&-7\,720\,349p^{27}+15\,352\,272p^{28}-16\,433\,567p^{29}-12\,262\,362p^{30}+78\,782\,168p^{31}-158\,447\,809p^{32}\\&+214\,186\,307p^{33}-230\,019\,014p^{34}+216\,228\,871p^{35}-186\,980\,567p^{36}+142\,532\,407p^{37}-68\,762\,291p^{38}\\&-55\,618\,898p^{39}+243\,770\,621p^{40}-473\,742\,752p^{41}+674\,493\,935p^{42}-757\,917\,965p^{43}+682\,330\,188p^{44}\\&-487\,268\,491p^{45}+263\,968\,633p^{46}-92\,180\,540p^{47}+443\,325p^{48}+27\,880\,668p^{49}-24\,713\,816p^{50}+14\,007\,915p^{51}\\&-5\,985\,845p^{52}+2\,011\,895p^{53}-535\,627p^{54}+111\,561p^{55}-17\,625p^{56}+1993p^{57}-144p^{58}+5p^{59}),\end{aligned}\tag{A5}$$

$$D_{[\mathrm{sq},5_P]} = 5(1 + p - 2p^2 - 6p^3 - 3p^4 + 26p^5 + 103p^6 - 244p^7 - 142p^8 + 516p^9 + 420p^{10} - 1159p^{11} - 928p^{12} + 1992p^{13} + 1578p^{14} + 2395p^{15} - 23\,040p^{16} + 39\,567p^{17} - 38\,811p^{18} + 26\,672p^{19} + 64\,051p^{20} - 272\,943p^{21} + 288\,026p^{22} + 249\,844p^{23} - 779\,755p^{24} + 77\,897p^{25} + 2\,020\,147p^{26} - 4\,713\,372p^{27} + 8\,356\,354p^{28} - 12\,526\,447p^{29} + 9\,141\,812p^{30} + 17\,671\,571p^{31} - 77\,282\,022p^{32} + 157\,595\,007p^{33} - 229\,642\,624p^{34} + 269\,077\,829p^{35} - 270\,785\,628p^{36} + 242\,608\,396p^{37} - 188\,685\,316p^{38} + 99\,894\,346p^{39} + 41\,909\,295p^{40} - 247\,536\,567p^{41} + 498\,168\,300p^{42} - 733\,592\,566p^{43} + 871\,224\,361p^{44} - 853\,561\,993p^{45} + 688\,825\,282p^{46} - 448\,046\,653p^{47} + 220\,169\,597p^{48} - 63\,265\,190p^{49} - 13\,220\,314p^{50} + 33\,003\,789p^{51} - 26\,681\,313p^{52} + 15\,080\,448p^{53} - 6\,672\,527p^{54} + 2\,389\,811p^{55} - 697\,964p^{56} + 165\,068p^{57} - 31\,003p^{58} + 4465p^{59} - 464p^{60} + 31p^{61} - p^{62}), \tag{A6}$$

$$N_{[\mathrm{sq},3_{\mathrm{sd}}]} = (1-p)^3(3 - 3p - 21p^2 + 37p^3 + 97p^4 - 265p^5 - 275p^6 + 1559p^7 - 735p^8 - 4454p^9 + 6397p^{10} + 7719p^{11} - 25\,594p^{12} + 461p^{13} + 76\,993p^{14} - 100\,105p^{15} - 48\,081p^{16} + 240\,589p^{17} - 133\,404p^{18} - 299\,125p^{19} + 397\,672p^{20} + 468\,568p^{21} - 1\,660\,402p^{22} + 1\,467\,662p^{23} + 705\,502p^{24} - 2\,859\,795p^{25} + 2\,447\,284p^{26} - 148\,761p^{27} + 71\,758p^{28} - 4\,717\,102p^{29} + 10\,333\,853p^{30} - 9\,242\,363p^{31} - 2\,195\,761p^{32} + 18\,554\,630p^{33} - 29\,140\,317p^{34} + 26\,914\,438p^{35} - 13\,774\,889p^{36} - 2\,046\,623p^{37} + 12\,789\,267p^{38} - 15\,764\,460p^{39} + 13\,053\,019p^{40} - 8\,354\,879p^{41} + 4\,320\,752p^{42} - 1\,833\,211p^{43} + 638\,949p^{44} - 181\,329p^{45} + 41\,088p^{46} - 7182p^{47} + 912p^{48} - 75p^{49} + 3p^{50}), \tag{A7}$$

$$D_{[\mathrm{sq},3_{\mathrm{sd}}]} = 3(1 - 2p - 5p^2 + 19p^3 + 13p^4 - 112p^5 + 32p^6 + 542p^7 - 883p^8 - 788p^9 + 3568p^{10} - 1056p^{11} - 9489p^{12} + 11\,669p^{13} + 18\,234p^{14} - 61\,546p^{15} + 42\,562p^{16} + 71\,008p^{17} - 151\,651p^{18} + 24\,638p^{19} + 201\,958p^{20} - 77\,630p^{21} - 616\,216p^{22} + 1\,248\,416p^{23} - 776\,173p^{24} - 859\,257p^{25} + 2\,199\,256p^{26} - 1\,891\,602p^{27} + 827\,535p^{28} - 1\,633\,704p^{29} + 5\,205\,016p^{30} - 8\,217\,812p^{31} + 5\,395\,624p^{32} + 4\,995\,137p^{33} - 18\,179\,344p^{34} + 25\,979\,096p^{35} - 23\,240\,423p^{36} + 11\,597\,999p^{37} + 2\,383\,494p^{38} - 12\,270\,154p^{39} + 15\,475\,815p^{40} - 13\,388\,744p^{41} + 9\,080\,298p^{42} - 5\,043\,522p^{43} + 2\,332\,946p^{44} - 902\,428p^{45} + 290\,686p^{46} - 77\,040p^{47} + 16\,440p^{48} - 2725p^{49} + 330p^{50} - 26p^{51} + p^{52}), \tag{A8}$$

$$N_{[\mathrm{tri},4_F]} = (1-p)^4(2 - 5p - 4p^2 + 41p^3 - 52p^4 - 80p^5 + 164p^6 + 838p^7 - 4165p^8 + 8517p^9 - 8197p^{10} - 1589p^{11} + 13\,355p^{12} - 8786p^{13} - 4606p^{14} - 61\,665p^{15} + 374\,163p^{16} - 1\,043\,384p^{17} + 1\,905\,928p^{18} - 2\,421\,614p^{19} + 1\,878\,238p^{20} + 140\,422p^{21} - 3\,349\,440p^{22} + 6\,775\,564p^{23} - 9\,239\,709p^{24} + 9\,983\,576p^{25} - 9\,009\,248p^{26} + 6\,948\,032p^{27} - 4\,628\,988p^{28} + 2\,674\,993p^{29} - 1\,339\,759p^{30} + 578\,446p^{31} - 213\,006p^{32} + 65\,734p^{33} - 16\,546p^{34} + 3259p^{35} - 470p^{36} + 44p^{37} - 2p^{38}), \tag{A9}$$

$$D_{[\mathrm{tri},4_F]} = 2(1 - 4p + 4p^2 + 18p^3 - 60p^4 + 43p^5 + 80p^6 + 225p^7 - 2534p^8 + 8252p^9 - 15\,122p^{10} + 15\,527p^{11} - 3457p^{12} - 12\,950p^{13} + 17\,747p^{14} - 40\,206p^{15} + 232\,540p^{16} - 868\,823p^{17} + 2\,151\,018p^{18} - 3\,857\,167p^{19} + 5\,099\,429p^{20} - 4\,529\,584p^{21} + 1\,058\,496p^{22} + 5\,305\,640p^{23} - 13\,128\,301p^{24} + 20\,075\,271p^{25} - 24\,025\,016p^{26} + 24\,060\,724p^{27} - 20\,745\,034p^{28} + 15\,614\,712p^{29} - 10\,330\,842p^{30} + 6\,022\,412p^{31} - 3\,090\,155p^{32} + 1\,389\,384p^{33} - 542\,808p^{34} + 181\,832p^{35} - 51\,206p^{36} + 11\,780p^{37} - 2122p^{38} + 280p^{39} - 24p^{40} + p^{41}), \tag{A10}$$

$$N_{[\mathrm{tri},3_P]} = (1-p)^6(3 + 9p - 50p^3 + 84p^4 - 24p^5 - 192p^6 + 554p^7 - 844p^8 + 812p^9 - 516p^{10} + 246p^{11} - 151p^{12} + 143p^{13} - 112p^{14} + 56p^{15} - 16p^{16} + 2p^{17}), \tag{A11}$$

$$D_{[\mathrm{tri},3_P]} = 3(1 - 3p^2 - 3p^3 + 68p^4 - 187p^5 + 190p^6 + 162p^7 - 1035p^8 + 2404p^9 - 3822p^{10} + 4494p^{11} - 3954p^{12} + 2580p^{13} - 1215p^{14} + 391p^{15} - 77p^{16} + 7p^{17}), \tag{A12}$$

$$N_{[\mathrm{tri},4_P]} = (1-p)^6(4 + 12p - 4p^2 - 100p^3 - 83p^4 + 1290p^5 - 2067p^6 - 2512p^7 + 11\,219p^8 + 3776p^9 - 91\,473p^{10} + 237\,866p^{11} - 238\,434p^{12} - 355\,578p^{13} + 2\,194\,759p^{14} - 5\,879\,228p^{15} + 10\,734\,693p^{16} - 11\,817\,298p^{17}$$

$$- 3\,000\,450p^{18} + 49\,716\,006p^{19} - 133\,234\,513p^{20} + 226\,293\,288p^{21} - 260\,526\,672p^{22} + 145\,333\,622p^{23} + 188\,864\,004p^{24} - 743\,143\,968p^{25} + 1\,422\,696\,984p^{26} - 2\,051\,609\,680p^{27} + 2\,439\,158\,465p^{28} - 2\,472\,507\,822p^{29} + 2\,176\,639\,966p^{30} - 1\,694\,462\,238p^{31} + 1\,200\,557\,665p^{32} - 813\,001\,894p^{33} + 559\,873\,482p^{34} - 405\,661\,962p^{35} + 300\,606\,573p^{36} - 214\,020\,448p^{37} + 138\,935\,000p^{38} - 79\,728\,612p^{39} + 39\,753\,500p^{40} - 17\,021\,640p^{41} + 6\,188\,754p^{42} - 1\,884\,492p^{43} + 471\,692p^{44} - 94\,508p^{45} + 14\,570p^{46} - 1622p^{47} + 116p^{48} - 4p^{49}), \tag{A13}$$

$$D_{[\mathrm{tri},4_P]} = 4(1 - 4p^2 - 8p^3 + 42p^4 + 258p^5 - 1514p^6 + 2760p^7 + 81p^8 - 7196p^9 - 8065p^{10} + 116\,560p^{11} - 367\,969p^{12} + 562\,624p^{13} + 89\,861p^{14} - 3\,170\,072p^{15} + 10\,610\,941p^{16} - 22\,666\,338p^{17} + 32\,951\,037p^{18} - 21\,009\,590p^{19} - 50\,850\,559p^{20} + 222\,682\,606p^{21} - 494\,752\,835p^{22} + 772\,870\,308p^{23} - 843\,281\,180p^{24} + 417\,584\,024p^{25} + 751\,628\,022p^{26} - 2\,729\,780\,780p^{27} + 5\,298\,392\,040p^{28} - 7\,950\,694\,944p^{29} + 10\,021\,670\,376p^{30} - 10\,934\,130\,274p^{31} + 10\,454\,310\,676p^{32} - 8\,802\,418\,934p^{33} + 6\,535\,447\,502p^{34} - 4\,275\,633\,432p^{35} + 2\,459\,294\,308p^{36} - 1\,239\,099\,924p^{37} + 543\,948\,012p^{38} - 206\,498\,264p^{39} + 67\,102\,916p^{40} - 18\,406\,832p^{41} + 4\,181\,204p^{42} - 765\,524p^{43} + 108\,540p^{44} - 11\,180p^{45} + 744p^{46} - 24p^{47}), \tag{A14}$$

$$N_{[\mathrm{hc},4_P]} = (1-p)^3(4 + 6p + 2p^2 - 6p^3 - 20p^4 + 12p^5 + 12p^6 - 2p^7 - 11p^8 + 23p^9 - 9p^{10} - 46p^{11} + 118p^{12} - 207p^{13} + 257p^{14} - 159p^{15} - 63p^{16} + 194p^{17} - 126p^{18} + 24p^{19} - 71p^{20} + 309p^{21} - 623p^{22} + 705p^{23} - 391p^{24} - 20p^{25} + 178p^{26} - 116p^{27} + 34p^{28} - 4p^{29}), \tag{A15}$$

$$D_{[\mathrm{hc},4_P]} = 4(1 - p^2 - p^3 - 2p^4 + 10p^5 - 5p^6 - 3p^7 + 10p^9 - 13p^{10} - 6p^{11} + 47p^{12} - 105p^{13} + 167p^{14} - 182p^{15} + 99p^{16} + 39p^{17} - 118p^{18} + 95p^{19} - 55p^{20} + 110p^{21} - 286p^{22} + 481p^{23} - 515p^{24} + 317p^{25} - 43p^{26} - 104p^{27} + 98p^{28} - 43p^{29} + 10p^{30} - p^{31}). \tag{A16}$$

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