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“Where does the square go?”: reinterpreting shapes when solving a tangram puzzle

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Abstract

Thirty-seven third graders and thirty-two first graders engaged in solving a tangram puzzle in the shape of a fox. They had five minutes to solve the puzzle, and after this time, they received guidance on the particular piece they had difficulties with. Through the lenses of navigating flexible abstraction, reinterpretation, combinations, and borrowing structure to expand upon the existing 2D shape composition and decomposition learning trajectory, we examined ways in which students' puzzle-solving processes and their challenges related to the fox puzzle's features. Students' initial shape placements suggest that parts of the fox puzzle primed the use of particular pieces, which reduced the abstraction of the puzzle. The most challenging part of the puzzle for students was navigating reinterpretation to place the square and two small triangles on the fox's head in nonstandard orientations. Even though students faced challenges at different steps, they overcame them similarly by trying new combinations and by borrowing structure. Some students did not complete the puzzle even though they used flips and turns (reinterpretation) strategically. The results suggest potential modifications of the current learning trajectory to account for differences between tangram and pattern block puzzles and differences due to tangram puzzles' features. Because the puzzle's features played a role in students' challenges, future work needs to focus on the interaction between students' puzzle-solving processes and puzzles' features for a variety of tangram puzzles.

Keywords Tangram puzzles · Puzzle features · Visual representation lenses · Shape composition · Geometry · Elementary

Children's geometric world emerges through play as they visualize the spatial environment (Clements & Sarama, 2000, 2009; Parks & Blom, 2013; van Hiele, 1999). “Spatial

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understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world" (National Council of Teachers of Mathematics (NCTM), Commission on Standards for School Mathematics, 1989, p. 48; see van den Heuvel-Panhuizen & Buys, 2008), and essential big ideas in geometric and spatial thinking include the composition and decomposition of shapes (e.g., Clements & Sarama, 2009; Clements et al., 2004; Parks & Blom, 2013; Riddell, 2016). In order for children to compose and decompose shapes, they need to recognize shapes' attributes (in various orientations) and develop relations among these attributes as classes of properties (Clements & Sarama, 2009; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; van den Heuvel-Panhuizen & Buys, 2008). For example, children might have difficulty making a rectangle from two congruent right triangles if they see only isosceles and equilateral triangles with horizontal bases (Clements & Sarama, 2000, 2009).

A key instructional strategy for helping students progress on the learning trajectory for 2D shape composition and decomposition (Composition LT) is to provide them with shape puzzles (i.e., pattern block and tangram puzzles) that gradually increase in difficulty (Clements & Sarama, 2017/2019). Generally, shape puzzles without internal lines, also known as silhouette puzzles (see the Shape Composer puzzle in Table 1), are considered hardest, and within these, puzzles in which the individual pieces are more identifiable are considered easiest (van den Heuvel-Panhuizen & Buys, 2008). Children are considered to have mastered shape puzzles, primarily focused on pattern block puzzles, when they use flips and turns strategically to place the shapes (Clements & Sarama, 2009); however, recent research with children on a 2D and 3D shape matching task and adults on a tangram task suggests that additional factors may influence puzzle-solving (Baran et al., 2007; Hallowell et al., 2015).

For example, turning may be more difficult for shapes that need to be placed in a non-standard orientation, such as a square with a corner pointing upward, i.e., ◆ (Baran et al., 2007; Clements & Sarama, 2009), or students could over-focus on certain properties of shapes, such as the corners of triangles (Hallowell et al., 2015). Baran et al. (2007) found that the degree to which tangram puzzles had distinct sections influenced how adults focused on the puzzle-solving, so paying attention to students' use of the puzzles' sections may also be important.

While the Composition LT details that students use turns and flips intentionally by the time that they are solving silhouette puzzles, there is little detail on students' puzzle-solving processes in relation to silhouette puzzles' features, especially for tangram puzzles. In fact, aside from the learning trajectory work, which focuses primarily on students solving pattern block puzzles, there is surprisingly little research investigating how children solve shape puzzles, let alone tangram puzzles. Instead, the literature focuses on ways pattern blocks and tangram puzzles are used in instruction (e.g., Lee et al., 2009), how students collaborate when solving puzzles together (e.g., Maheux & Proulx, 2015), or engagement levels with virtual versus physical puzzles (e.g., Schroth et al., 2019). Because tangram shapes differ in key ways from pattern block shapes and tangram puzzles use a standard set of pieces, students (and even adults!) may intentionally turn and flip but still have difficulty solving the puzzles. Attention to these and other factors in students' puzzle-solving process in relation to the puzzles' features could better indicate which types of puzzles to provide to help students make sense of silhouette puzzles' features, particularly tangram puzzles, and encourage strategic puzzle-solving.

Table 1 2D shape composition and decomposition learning trajectory levels[a]

Level	Piece Assembler	Picture Maker	Shape Composer	Substitution Composer
Puzzle-Solving Process	Uses trial and error.	Places shapes and uses turns and flips by trial and error. May use general side lengths or shape ideas to place shapes when suggested by the lines.	Uses shapes’ angles and side lengths and strategically uses turns and flips to place shapes. Can anticipate what will fit, e.g., identify that similar parts of a puzzle can be completed in the same way.	Uses combinations of shapes by trial and error to substitute for other shapes and fill a puzzle in multiple ways.
Types of Puzzles Solved: Puzzle Difficulty	Fills simple puzzles where each piece is outlined. Outlined pieces start out representing a separate part of the picture, and later puzzles may have multiple outlined pieces making up a part.	Fills simple puzzles where each piece is outlined, even if multiple pieces make up a part. Later puzzles may have fewer internal lines as guides.	Fills puzzles that are larger and have no internal lines.	Fills puzzles without internal lines in several ways.
Example Puzzle				

[a]Clements et al. (2004); Clements and Sarama (2009)

1 Learning trajectory for shape composition and decomposition

The Composition LT outlines a progression of students' developing strategies (e.g., flipping, turning) and types of shape puzzles to use at each level (e.g., with or without internal lines; Clements & Sarama, 2009) and was largely developed based on students' solving of puzzles made from pattern block shapes (see Fig. 1; Clements et al., 2004). The square is the only shape with a right angle, which may make them easier to distinguish in the puzzles.

Further, Table 1 provides a summary of the relevant levels in the Composition LT, detailing students' expected puzzle-solving processes and the type of puzzles students would solve.

Students classified at a shape composer level should be able to complete silhouette puzzles using rotation and flips intentionally (Clements & Sarama, 2009). Yet, restricting the number of pieces students can use to solve the puzzles, as with tangram puzzles (Fig. 2), might also influence and highlight new puzzle-solving processes and difficulties that may not be as prevalent with pattern block puzzles. Solving tangram puzzles, and composing and decomposing shapes in general, requires students to think creatively and focus on different aspects of shapes: length of sides in relation to other shapes, orientation, angles,

Fig. 1 Pattern block shapes (any number of these can be used to make a pattern block puzzle)

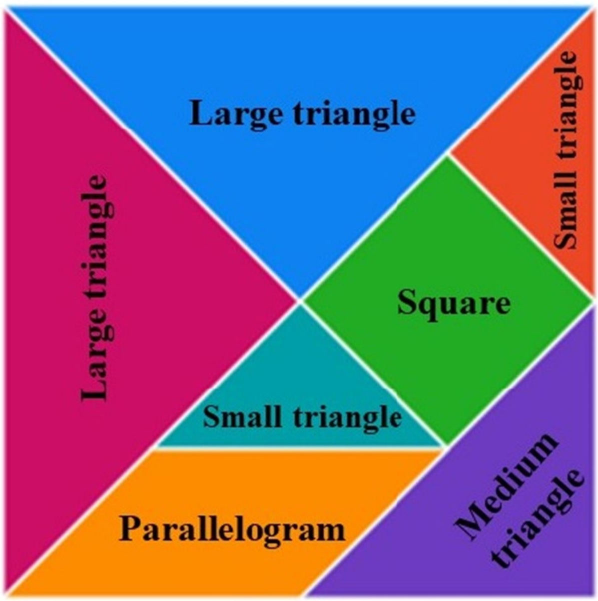


Fig. 2 Tangram shapes (traditionally, all are used to make each tangram puzzle)

size, etc. (Sarama & Clements, 2009; van den Heuvel-Panhuizen & Buys, 2008). Thus, to further explore how students' puzzle-solving processes and puzzle features might interact, we leverage a visual representation framework, originally proposed to describe how visualizations can support creative thinking (Martin & Schwartz, 2014), as an analytical tool for deepening our understanding of students' puzzle-solving processes.

2 Application of the visual representation framework to tangram puzzles

Martin and Schwartz's (2014) framework highlights cognitive mechanisms that support creativity through visualizations, which they organize in a two-dimensional space (see Fig. 3):

> The horizontal dimension depicts whether the creative move primarily involves decomposing and subtracting or whether it involves adding and combining. The vertical dimension represents the continuum of focusing on the elements within a visualisation or the structures they create. (pp. 80–81)

They argue that these mechanisms could increase the possibility of creative thought, which they define as purposeful novelty (Martin & Schwartz, 2014).

First, at the intersection of decomposing and elements, is the cognitive mechanism of *flexible abstraction*. Flexible abstraction involves removing details to see abstraction; for example, a map preserves location and distance while removing many details of the landscape. Second, at the intersection of integrating and elements is the cognitive mechanism of *combinations*. Combinations involve combining different parts and types of information (Martin & Schwartz, 2014). Third, at the intersection of decomposing and structures is the cognitive mechanism of *reinterpretation*, a form of intrinsic-dynamic spatial reasoning (Uttal et al., 2013). Reinterpretation involves letting go of visual prototypes. Finally, at the intersection of integrating and structure is the cognitive mechanism of *borrowing structure*. Borrowing structure involves using knowledge inherent in one visualization to support understanding in a different situation (Martin & Schwartz, 2014).

Currently, there are no empirical studies applying this framework to early geometry learning, spatial thinking, or puzzle-solving. However, the framework does provide insight into the cognitive mechanisms that could be involved in the creation of shape puzzles and connected to puzzles' features and students' potential puzzle-solving processes (see Bofferding & Foster, 2021). We outline these connections in relation to the tangram fox puzzle we used in Table 2.

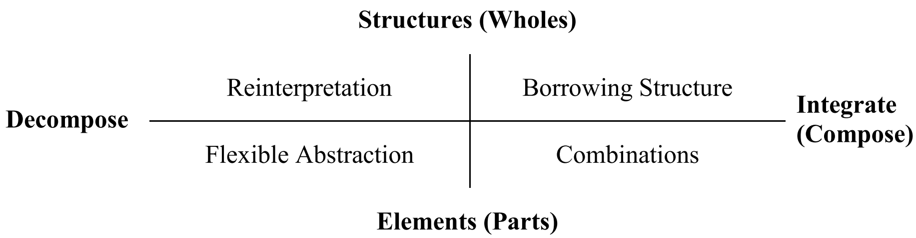
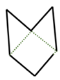


Fig. 3 Visual representation framework (Martin & Schwartz, 2014)

Table 2 How the visual representation framework relates to creating and solving a tangram puzzle

Cognitive Mechanism	How *Puzzle-Makers* Might Use the Mechanisms to Create Puzzles and Impact Their Difficulty	How *Puzzle-Solvers* Might Navigate the Mechanisms to Solve Puzzles	Relation to the Fox Puzzle We Used
Flexible Abstraction	Create a more difficult version of a puzzle by removing the puzzle's internal lines to create a silhouette, leaving a puzzle with a certain number of sides and angles visible and different sections.	Choose a section of the puzzle and a piece based on the puzzle's outline to reduce the abstraction and use other mechanisms to solve it.	The fox puzzle has three sections with no internal lines except the tail piece is completely outlined. Several visible acute angles in the fox's outline align with the angles on the triangles. 
Combinations	Create a puzzle with one or more solutions that supports one or more combinations of pieces in a section.	Fill a section of the puzzle by combining different pieces to solve the puzzle.	The tail can be composed with two small triangles or a parallelogram, the body can be composed in two ways (see Fig. 4), and the head can be composed in three ways (see Fig. 5). The correct solution involves composing the head and tail in one particular arrangement while the body pieces can be arranged in either of its two ways.
Reinterpretation	Create a puzzle where pieces are used with nonstandard perspectives or orientations.	Turn or flip pieces to fit them into empty spaces with the puzzle's outlines.	The small triangles and square in the head are oriented in nonstandard ways. Further, the parallelogram in the tail may need to be flipped. 
Borrowing Structure	Create a puzzle with common structure by organizing similar pieces in a reflected or translated way.	Look for common outlines or spaces and place similar pieces in a reflected or translated way to solve the puzzle.	The left ear borrows the structure of the right ear (or vice versa) using a reflection. Similarly, the left leg borrows the structure of the right leg (or vice versa) using a reflection. 

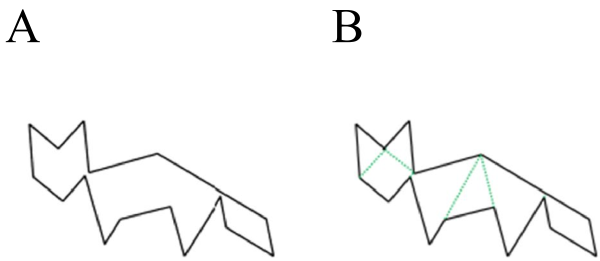


Fig. 4 Tangram puzzle of fox (**A**) and its two solutions (**B** and **C**)

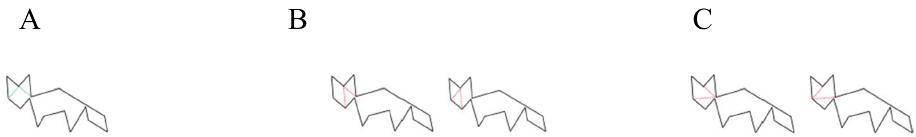


Fig. 5 Solutions to the fox's head that are correct (**A**) and incorrect (**B** and **C**)

3 Navigating cognitive mechanisms to solve a puzzle

Although navigating flexible abstraction involves undoing the abstraction created by the puzzle-maker, navigating the other three mechanisms involves using the mechanisms much in the same way as the puzzle-maker (e.g., putting together combinations of shapes to make the same combinations as the puzzle-maker).

3.1 Flexible abstraction

In general, silhouette tangram puzzles are an abstraction from puzzles with internal lines. Some tangram pieces might still be fully or almost fully outlined (e.g., the tail in our fox puzzle), while others are hidden. Students might flexibly navigate the abstraction by focusing on a particular section of the puzzle or starting with a particular piece to break up the space (Baran et al., 2007). In one study with graduate students, participants placed easy-to-distinguish, larger pieces first, leaving smaller pieces for later, a potential strategy for reimagining the abstract space (Baran et al., 2007). Clements et al. (2004) describe Kevin, a first-grader, as at the picture maker level because he did not appear to place shapes with intention. Part of their justification was that Kevin originally tried to fill a portion of the puzzle, removed the pieces to try again, and then placed a few pieces that created "two simple frames that did allow him to complete the puzzle" (Clements et al., 2004, p. 174). Through the *navigating flexible abstraction* lens, Kevin tried to simplify the space in one way before trying it another way, which helped him reduce the abstraction sufficiently and fill the less-defined inner space. What looks like trial-and-error with the learning trajectory lens could be considered productive attempts to break up the space using this navigating flexible abstraction lens.

3.2 Combinations

Early composition of figures, within the Composition LT, involves children using shapes to create their own pictures at the piece assembler level (see Table 1; Clements & Sarama, 2009; van Hiele, 1986, 1999). Completing shape puzzles beyond the piece assembler level is difficult because children have to figure out how to compose multiple shapes within sections, and shapes' orientations might not be aligned with their mental images of typical shapes (Clements & Sarama, 2009; Clements et al., 2004). Although spatial thinking interventions are not common (Newcombe & Shipley, 2015), they sometimes focus on combinations, particularly targeting triangles. In one study with tangrams, third graders learned that two small triangles could be put together to form a square (Siew et al., 2013). As part of an instructional unit on area, secondary students had to determine all the ways to use tangram pieces to make different sized squares; some students strategically used or made right triangles to make a square, while others moved pieces around randomly until they made a square (Tchoshanov, 2011). In another study, kindergarteners listened to narratives involving shape composition of isosceles triangles and participated in activities to help out the characters in the narrative. Girls particularly benefited from the narrative when solving similar composition tasks, but the effect did not translate to them solving broader tangram tasks (Casey et al., 2008). Because some tangram pieces can be composed in multiple ways from smaller pieces (e.g., the large triangle can be composed of two small triangles plus either the square or medium triangle), students may be able to solve tangram puzzles in multiple ways or may need to try different combinations of shapes within the same section of the puzzle to make the other pieces fit elsewhere.

3.3 Reinterpretation

Tangram puzzles consist of seven pieces that fit together to make a larger square (see Fig. 2). Interestingly, the parallelogram only has rotational symmetry, so in some puzzles, it must be flipped. As children progress on the Composition LT, they become more strategic in navigating reinterpretation (i.e., flips and turns), initially trying them by trial-and-error and moving toward anticipating how the shape will look after the transformation (Clements & Sarama, 2009). Turning and flipping shapes can help students reinterpret how the pieces might fit together. A key indicator that students are at the shape composer level is when they intentionally turn and flip (e.g., matching angles; Clements & Sarama, 2009; Clements et al., 2004, 2019).

Students' use of rotation and flips could be constrained by their familiarity and classification of shapes (Clements, 2004; Nurnberger-Haag, 2017; van Hiele, 1999). Unfortunately, children are overexposed to prototypical shapes in standard orientations only (e.g., ▲, ■), as in common shape books and shape sorters (Clements & Sarama, 2009; Nurnberger-Haag, 2017). Even older elementary children tend to say that a turned square is "not a square anymore, it's a diamond" (Clements & Sarama, 2009, p. 126). Not seeing shapes in other orientations may make students less inclined to turn and flip the shapes. For example, in an eye-tracking study, graduate students had difficulty placing the square when solving tangram puzzles if it needed to be rotated so that its corner pointed upward (Baran et al., 2007). In another study, kindergarteners had difficulty matching the angles of the shapes to the puzzle outline and had the hardest time placing two of the triangles that were oriented in non-prototypical ways (Chou et al., 2013).

Mary, a preschooler classified at the piece assembler level, turned a trapezoid so that the longest base was parallel to the right side of the page (a nonstandard orientation) to fit the piece correctly; likewise, Kevin at the picture maker level was also able to turn the trapezoid so that one side was parallel to the bottom of the page (Clements et al., 2004). Given some students’ resistance to accepting shapes as the same when they are turned (Clements & Sarama, 2009), these details seem noteworthy even if the students did not anticipate that turning the shape would work. Further, children may have difficulty explaining why shapes match and considering multiple aspects of shape, including paying attention to both orientation and how well the shapes fit in boundaries (Hallowell et al., 2015). Shape puzzles differ in priming students to think about common ways of seeing shapes (or not) or in using different combinations of shapes. This priming could interact with students’ attempts to manipulate shapes in ways that go against standard ways of seeing shapes (e.g., a square has horizontal sides versus slanted sides) or composing shapes (e.g., not having sides fully align).

3.4 Borrowing structure

When students solve tangram puzzles, borrowing structure could involve being able to recognize shapes within larger compositions, a characteristic of students at the substitution composer level (Clements & Sarama, 2009). In the study with graduate students, when the individual puzzle pieces were less distinct, making borrowing or noticing similar structure difficult, the participants focused more on the overall picture. The authors concluded that the complexity of the puzzles may interact with participants’ strategy formation (Baran et al., 2007). Within puzzles, noticing similarities in sections could help students find correct shape matches.

3.5 Current study

In our study, we explored students’ tangram puzzle-solving through lenses highlighted by the visual representation framework. The fox tangram puzzle was chosen because its space can be broken up in different ways (revealing how students navigate *flexible abstraction*); it requires shape placements in nonstandard orientations (casting insight into students’ navigating *reinterpretation* through flips and turns); the puzzle sections allow for multiple combinations (highlighting navigating *combinations*); and it has a structure where the correct placement of one piece can highlight a similar placement of a duplicate piece (indicating students’ noticing and *borrowing of structure*). Particularly, we examined the following research questions: How do first and third graders solve the fox tangram puzzle, when interpreted through the lenses of navigating abstraction, reinterpretation, combinations, and borrowing structure? What are their common difficulties in relation to the puzzle’s features, and how do they overcome them?

4 Methods

4.1 Participants and setting

The larger study, from which this data is drawn, took place at a public elementary school in a midwestern district in the USA (~46% qualified for free and reduced-price lunch and ~ 10% classified as English learners). After receiving approval from our Institutional Review Board and the school district, we sent home consent forms with all first and third graders at one school. Overall, we recruited students who returned signed forms: 32 first graders (14 female and 18 male) and 37 third graders (15 female and 22 male). We were interested in students' solving of silhouette puzzles, so we targeted students who, based on their ages and the Composition LT, would typically be near, at, or just beyond the shape composer level. Although we did not ask every student if they had previously worked with tangram puzzles, some did tell us that they had worked with them previously, while others indicated that they had not.

4.2 Design and materials

The tangram puzzle was shaped like a fox; in terms of *flexible abstraction*, the puzzle had three main sections: the head, body, and tail, which could be further broken down into ears, face, legs, body, and tail (see Fig. 4A and Table 2). Although the head could be *composed* in three different ways (see Fig. 5) and the tail could be composed in two different ways—with the parallelogram or two triangles—only one way in the head and tail sections would allow for a correct puzzle completion. The body, however, had two possible solutions (see Fig. 4B and C). To complete the puzzle, several pieces had to be *reinterpreted* from their standard orientations; the small triangles had to be positioned with the longest sides vertical, and the square had to be positioned with one corner pointing upward (see Table 2). Finally, because the ears of the fox are mirror images, students could *borrow structure* from one ear to complete the second ear; likewise, students could borrow the structure of one leg to complete the second leg (see Fig. 4B and Table 2).

We randomly scattered the tangram pieces in front of the students during individual interviews and asked them to use the shape pieces to make the fox while we video-recorded their puzzle-solving process. Similar to methods used by Evans et al. (2011), students had five minutes to explore this puzzle on their own, after which the researcher provided help with the piece or pieces that seemed to be the most challenging for them, which varied among students.

4.3 Data analysis

First, we transcribed students' explanations during their puzzle-solving process, recorded their steps (each time they attempted to place a piece), and determined whether each step was associated with a correct placement of the shape (students could remove a correctly placed shape, place it incorrectly, and replace it correctly again; this counted as two correct placements). Next, we analyzed students' puzzle-solving processes through the lenses of navigating abstraction, combinations, reinterpretations, and borrowing structure (Martin & Schwartz, 2014).

4.4 Navigating flexible abstraction

When students place a piece within the puzzle's outline, they change or reduce the puzzle's abstraction. As students place more pieces, the other mechanisms take on a larger role in puzzle-solving processes as students must continue reinterpreting pieces and combining them with other pieces. Therefore, as an indication of how students navigated the puzzle's abstract space before navigating other mechanisms, we identified which piece students first used (as this would break up the space in different ways) and which sections they first attempted to place it. We then looked for patterns in students' initial placements with their overall success in solving the puzzle and number of steps they used.

4.5 Navigating combinations and reinterpretation

Although students could turn and flip a shape on its own or make a combination without turning and flipping two shapes, typically these two mechanisms influenced each other. To preserve the richness of these interactions and because students' use of turns and flips were directly related to which shapes they combined and how they combined them, we analyzed these two mechanisms together. When identifying students' navigating combinations, we noted the extent to which students tried different combinations of shapes within the same section (i.e., head, body, tail) and whether students purposefully substituted pieces. Across students' solution steps, we identified if students reinterpreted shapes from their standard orientations by turning and flipping. When they did, we noted if they were trying to match angles or sides, as appropriate.

4.6 Navigating borrowing structure

For sequential steps, we noted instances where students first placed one shape and then directly placed a similar shape as evidence that they noticed the common structure. Students could borrow structure by noticing that one ear is the same as the other (just flipping a small triangle), that one leg is the same as the other (just flipping a large triangle), or that the medium triangle can be placed in the right leg in a similar orientation as a large triangle placed in the left leg.

Across these mechanism categories, we identified student cases to exemplify common difficulties and their ways of overcoming or avoiding these challenges.

5 Results

5.1 Navigating flexible abstraction

Interestingly, there was no noticeable difference in whether students correctly solved the puzzle based on their choice of initial piece or where they placed it first. Rather, differences emerged between the average number of steps in their puzzle-solving process and where they placed their initial piece.

5.1.1 Start in tail

Students were least likely to start by placing any shape in the tail section. When they did, they overwhelmingly tried the parallelogram. Half of the students who started with the parallelogram in the tail later removed it to try it on the head or body. Others who started with the parallelogram in the head or body left it in the tail once they placed it there (except in one case). Therefore, students who started with the parallelogram in the tail took more steps ($n=15$, $M=31$, $Mdn=30.5$) than those who started with the parallelogram in the head or body ($n=7$, $M=19$, $Mdn=16$).

5.1.2 Start in head

Overall, third graders were most likely to start by placing one of the shapes in the head section; however, they were split between whether they first tried the small triangle or parallelogram. First and third graders who started with the small triangle in the head used fewer steps ($n=11$, $M=18$, $Mdn=12$) than students who placed it elsewhere first ($n=4$, $M=25$, $Mdn=21$). For description of parallelogram steps, refer back to subsection *Start in tail.* Unlike with the parallelogram and small triangle, starting with the medium triangle in the head did not seem to help students finish the puzzle in fewer steps. In fact, of the five students who started with the medium triangle in the head, three placed it in the head again after trying it in the body or tail.

5.1.3 Start in body

Out of all pieces, first graders were most likely to start by placing the large triangle in the body section. Overall, 16 first and third graders started by placing a large triangle in one of the legs (especially the right leg) compared to four who started by placing it in the main body. However, students' general puzzle-solving processes did not obviously differ when they started the large triangle in the body versus the legs. Even so, students used fewer steps when starting the large triangle in a leg than in the main body ($n=16$, $M=17$, $Mdn=17$ versus $n=4$, $M=26$, $Mdn=25$, respectively). Four first graders who initially tried four of the five shapes in the body still ended up correctly solving the puzzle. One of these first graders, Halle, demonstrated an interesting process for navigating

Table 3 Pieces students first used and where they first placed them in the tangram fox

Placement	Parallelogram	Large triangle	Medium triangle	Small triangle	Square	Total
First grade ($n=31$[a])	29%	39%	10%	19%	3%	
Placed in head	6%	—	6%	19%	3%	35%
Placed in body	3%	35%	3%	—	—	42%
Placed in tail	19%	3%	—	—	—	23%
Third grade ($n=37$)	35%	30%	11%	24%	—	
Placed in head	11%	5%	8%	14%	—	38%
Placed in body	—	24%	—	5%	—	30%
Placed in tail	24%	—	3%	5%	—	32%
Overall ($n=68$[a])	32%	34%	10%	22%	1%	
Placed in head	9%	3%	7%	16%	1%	37%
Placed in body	1%	29%	1%	3%	—	35%
Placed in tail	22%	1%	1%	3%	—	28%

[a]We only had a partial video for one of the first graders, so totals for first graders are out of 31 instead of 32 participants.

the abstraction. Originally, Halle correctly placed the large triangles in the legs of the fox. However, she then tested placing the pieces on different parts of the body (by placing and quickly removing pieces in succession), effectively breaking up the sections to see the space in a new way. Following this testing, Halle rapidly completed the puzzle (see Fig. 4C). See Table 3 for a breakdown of where students placed their first piece.

5.2 Navigating combinations and reinterpretation

To combine pieces, students often had to reinterpret (i.e., turn and sometimes flip) them. Their combinations in one section often left students unsatisfied with other sections of the puzzle. As a result, sometimes students changed their correct placement of a piece by removing and replacing it to another spot, requiring further turns or flips and changes in combinations. In fact, 12 (32%) third graders and five (16%) first graders re-organized their correct placement of tangram pieces, moving them elsewhere and then replacing them correctly later, to solve the whole puzzle correctly. Their moving of pieces corresponded to students’ struggles to combine and reinterpret some of the tangram pieces in the fox puzzle or a section of the fox puzzle. We present these challenges by highlighting the interaction between navigating reinterpretation and combinations.

5.2.1 Fox tail—example of reinterpretation

On occasion, students tried to complete the fox’s tail with two small triangles and then replaced them with the parallelogram. However, the main difficulty with placing the parallelogram in the tail was related to students’ attempts to reinterpret it by flipping it. Reyna’s, a first grader’s, response to originally placing the parallelogram in the tail with a flipped orientation was typical of most students (see Fig. 6A). When it did not fit, she first rotated it 180° (reinterpretation). Still it did not fit, so she asked, “Is it backwards?” and flipped the parallelogram to correctly fit it into the tail (see Fig. 6B).

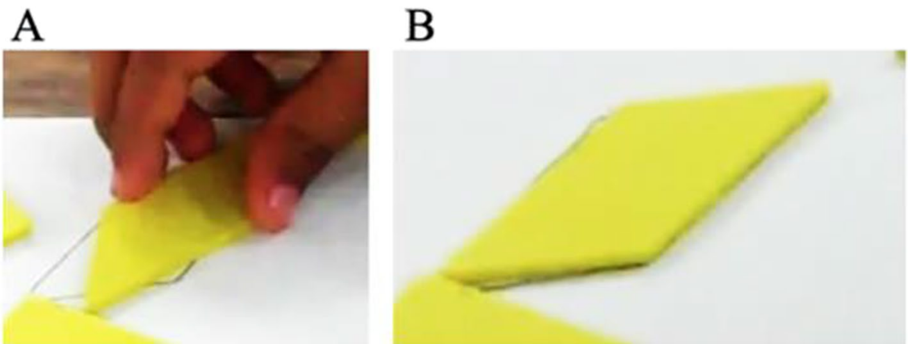


Fig. 6 Reyna’s (**A**) flipped and (**B**) correct placement of the parallelogram in the tail

5.2.2 Fox body—example of reinterpretation and combinations

Another prevalent difficulty students experienced was coordinating their reinterpretation of the two large triangles by turning them while also combining them with the medium triangle to fit them into the fox body. There were two ways students could successfully organize them (see Fig. 1B and C); however, some students either placed the two large triangles in the legs and had trouble reinterpreting the medium triangle's orientation or placed the medium triangle in one leg and had difficulty reinterpreting the orientations of the large triangles. One first grader, Carlos, exemplifies this latter group. Carlos started out by correctly placing the parallelogram on the tail and the square on the fox's head (see Fig. 7A). After briefly trying to place one of the small triangles on the left leg (see Fig. 7B), he spent some time trying to reinterpret how to place the two small triangles in the ears, removing them and the square, and trying them again while turning them until he combined them within the outline (see Fig. 7C-E). When he refocused on the body, his initial difficulties arose from trying to place the medium triangle on the left leg; with the two large triangles also placed on the body, this left a space similar to the small triangle (see Fig. 7F). After trying and rejecting the attempt to place a small triangle on the body (see Fig. 7G), he then tried the medium triangle on the right leg, but due to the orientation, when he placed a large triangle in the left leg, he was not able to fit the other large triangle on the body (see Fig. 7H).

5.2.3 Fox head—example of reinterpretation and combinations

A main challenge students had in solving the fox's head was reinterpreting the shapes' orientations: (1) correctly placing the square but not reinterpreting the orientations of one or both of the small triangles to a nonstandard orientation; (2) correctly placing the two small triangles but not reinterpreting the orientation of the square to a nonstandard orientation; (3) correctly placing one small triangle, incorrectly placing the other small triangle, and not being able to fit the square on the empty space; or (4) not being able to figure out any of the pieces in the fox's head. While some students started working on the head, others had partially correct bodies (e.g., the two legs were correct, but the tail and medium triangle were not used) or completely misplaced the tangram pieces when they decided to try out the fox's head. Therefore, students' initial challenges to solve the fox's head also involved trying to figure out which combination of shapes should go in that section, exacerbated by

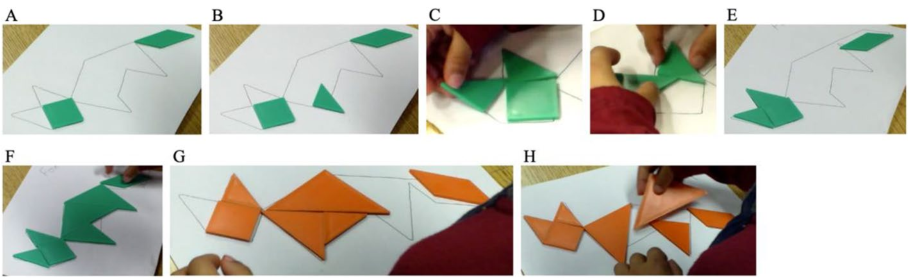


Fig. 7 Carlos's steps for attempting to complete the fox puzzle

the fact that students could fill the head outline with different combinations. Overall, 21 (57%) third- and 18 (55%) first-grade students struggled with solving the fox's head. To better understand students' challenges when solving the fox's head, we include examples of more vocal third graders to illustrate their puzzle-solving processes.

Some students tried combining *one medium and two small triangles to solve the head.* Frank, a third grader, first correctly solved the tail using the parallelogram. Then, he placed one of the large triangles on the fox's right leg. The fox's body was partially filled when he decided to try the head (see Fig. 8A). He took the medium and small triangles from the body and tried placing them on the head (see Fig. 8B-D) before he was able to turn (reinterpret) and combine them to fit (see Fig. 8E). Next, he laid the other large triangle on the fox's left leg correctly (see Fig. 8F). He was left with the square and muttered to himself, "Where does the square go?" He removed all the tangram pieces from the head, eventually took the medium triangle, and combined it with the large triangles on the body correctly (see Fig. 8G). From here, he tried to fit the two small triangles on the fox's ears, but their orientations were incorrect (see Fig. 8H). He tested the square by placing it on the head using a standard orientation (see Fig. 8I) and said, "But this [the medium triangle] should be here [on the body] and at the same time it needs to be here [on the head]." The researcher acknowledged his frustration, "But it can only be in one place." He moved around the square and other tangram pieces and said, "I am actually surprised that I don't know how to put these pieces together." He said he wanted to keep trying, so we did not give him a hint, but he did not solve the puzzle.

Some students tried combining *one parallelogram and two small triangles to solve the head.* Initially, Yara, a third grader, placed the two small triangles and the parallelogram on the fox's head stating, "This is his head" (see Fig. 9A-B). After briefly trying the square on the body (Fig. 9C), she correctly solved the body (see Fig. 9D) and only had the square left, puzzling, "I have this [square] and try to fill that [tail]." She explained, "I'm not doing anything wrong on the body or the head or am I?" The researcher gave her a playful questioning shrug and a little laugh. Yara removed the parallelogram from the head, questioning, "I want to see what this [parallelogram] does" and placed it on the tail correctly (see Fig. 9E). Then, she tried to solve the head using only two small triangles in a new way compared to Frank, placing one at the bottom of the face and one on an ear (see Fig. 9F). The researcher asked, "What are you going to do with the square?", and she responded, "I don't know…I know this goes somewhere." She played with the tangram pieces more and moved the two small triangles to the tail (see Fig. 9G) saying, "This is a very confusing

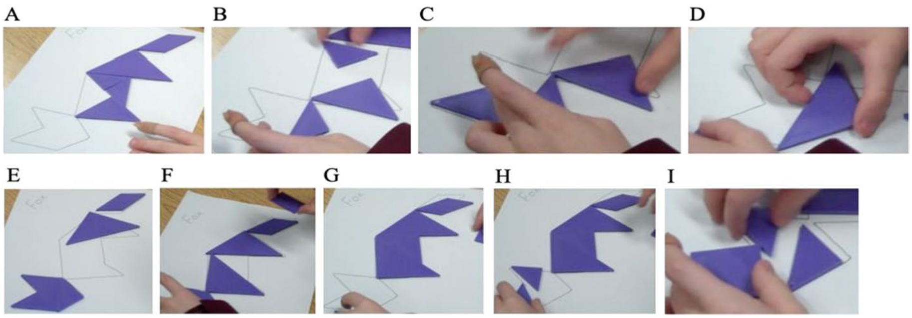


Fig. 8 Frank's steps for attempting to solve the fox puzzle

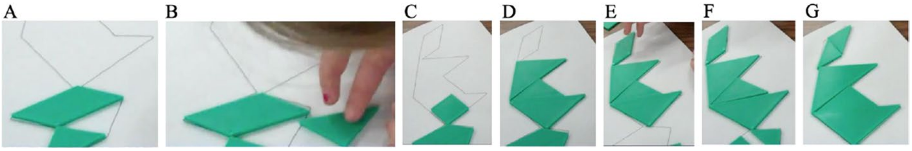


Fig. 9 Yara's steps for attempting to solve the fox puzzle

puzzle." She continued, "We technically know where all our pieces go except the square." The researcher responded, "Mmhmm, tricky square." After five minutes, she was able to solve the puzzle with guidance on how to place the square on the fox's head.

Some students tried combining *all possible ways to solve the head.* Yvette, a third grader, did not talk when solving the puzzle. At the beginning, she correctly solved the tail and then body (see Fig. 10A). When she started with the head, she compared the two small triangles and realized they were the same. Initially, she tried placing the triangle in each ear using a standard right triangle presentation (see Fig. 10B). Then she rotated them to correctly place them on the fox's head, but, like Frank, she tried the square with a standard orientation (see Fig. 10C). Afterward, she considered trying the medium triangle on the head (see Fig. 10D) but decided to start over. In her second exploration, she made the head with the parallelogram and two small triangles (see Fig. 10E), like Yara. She placed the two large triangles on the body as before (see Fig. 10F) and realized that the square did not fit in the empty space. Trying to solve the fox's body another way, she moved around, rotated, and tested all triangles on it while also correctly placing the parallelogram in the tail (see Fig. 10G-J). Then, she created the fox's head with the medium and two small triangles (see Fig. 10K-L). After placing the large triangles back on the body, she was left with the square that did not fit in the empty space (see Fig. 10M). Her next step was to move the medium triangle to the empty space on the body correctly (see Fig. 10N) and change the orientations of two small triangles on the head correctly. However, she was not able to fit the square on the head (see Fig. 10O). Unlike Frank and Yara, she removed the two triangles, finally rotated and placed the square correctly (successfully reinterpreting it), and then re-fit the two small triangles in the correct spot (see Fig. 10P).

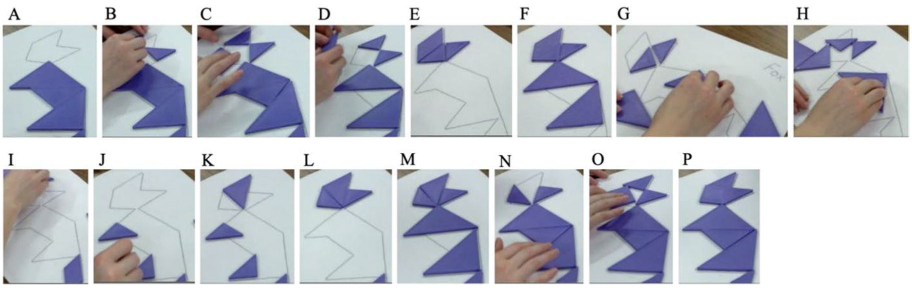


Fig. 10 Yvette's steps for completing the fox

5.3 Navigating borrowing structure

Students' noticing of common structures within sections of the puzzle played a role in their puzzle-solving process. Holly, a first grader, navigated borrowing structure on two occasions. Initially, she tried and removed several pieces (a large triangle in the right leg, the medium triangle in the head, a small triangle in the left leg). Then she borrowed structure by placing a large triangle in the left leg (see Fig. 11A) and immediately flipping the other large triangle to fit in a reflected fashion on the right leg (see Fig. 11B). She was able to correctly fit the medium triangle in the empty space on the body. Moving to the head, she briefly tried and removed both the square and a small triangle. Then she placed the square correctly, placed the small triangle on the right ear correctly, and immediately borrowed structure and reoriented the other small triangle to place it on the left ear correctly.

For the sake of this analysis, we considered students' actions as borrowing structure when they did so productively; however, there were also times when students copied a structure in an unproductive way. For example, Curt, a first grader, originally borrowed structure by correctly placing the two small triangles in the ears of the fox (see Fig. 12A). He then placed a medium triangle in the right leg (see Fig. 12B) and copied this structure to try placing a large triangle in the left leg (see Fig. 12C). Ultimately, he was successful at completing the puzzle, but others like him continued to struggle.

Overall, 20 (65%) first graders borrowed structure, and of those, 13 (65%) correctly solved the puzzle. Of the 11 (35%) first graders who did not borrow structure, only one (9%) of them completed the puzzle without a hint. Likewise, 29 (78%) third graders borrowed structure, and of those, 16 (55%) correctly solved the puzzle. Interestingly, of the eight (22%) third graders who did not borrow structure, four (50%) of them still completed the puzzle without a hint. Students at both grade levels were most likely to borrow structure when solving the ears (38 students); about one-third of these students noticed the common structure of the ears only when they were part of the last two or three pieces to place.

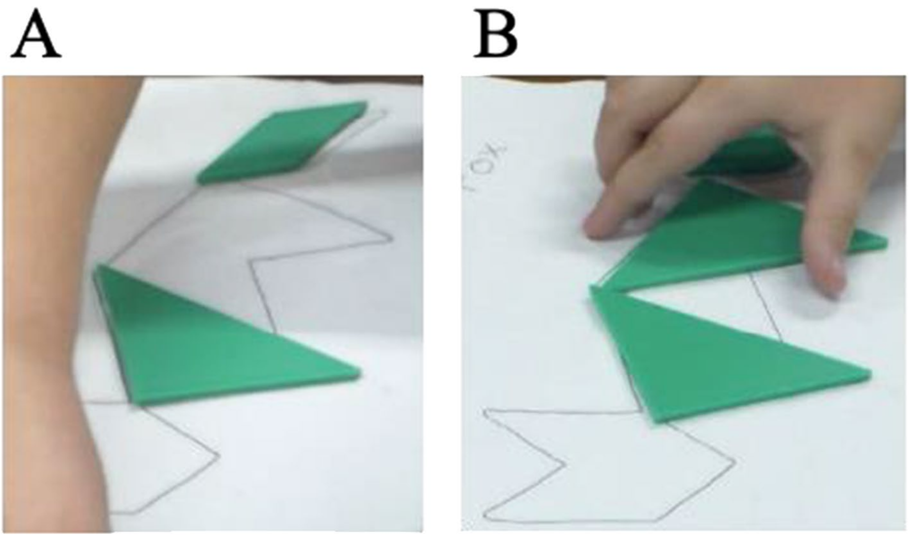


Fig. 11 Holly's use of borrowing structure with the large triangles

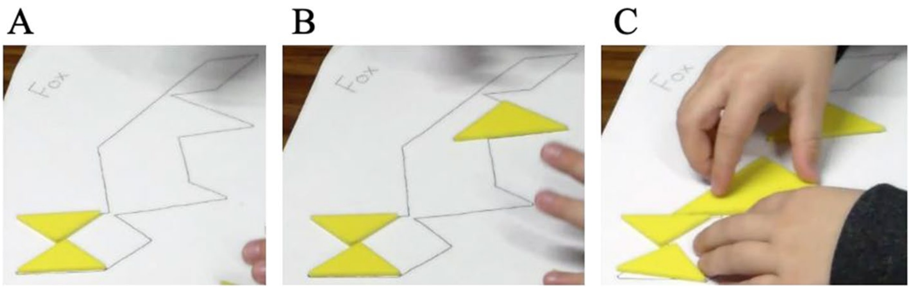


Fig. 12 Curt's use of borrowing structure for the ears and copying structure for the legs

Additionally, 26 students borrowed structure when using the large triangles to complete the legs, and only five students borrowed structure to place the medium and one large triangle in the legs.

6 Discussion

In this study, we explored first and third graders' tangram puzzle-solving processes and ways they overcame (or attempted to overcome) challenges. Students' challenges arose as their navigation of the cognitive mechanisms (Martin & Schwartz, 2014) interacted with the tangram puzzle's features. This interaction suggests a need to further tease apart the use of silhouette tangram puzzles at the shape composer level and the criteria for a student being a shape composer versus substitution composer (Clements & Sarama, 2009).

6.1 Navigating flexible abstraction

Overall, students' common initial placements of the tangram pieces suggest that certain puzzle features (i.e., outlines, size, and angles) reduced abstraction and guided students on where to place the pieces. The puzzle appeared to productively prime students' placement of the parallelogram in the tail, which is unsurprising given that the parallelogram was fully outlined, a feature for easier puzzles (Clements & Sarama, 2009; van den Heuvel-Panhuizen & Buys, 2008). Therefore, even students at a lower Composition LT level (i.e., the picture maker level) should be able to place the parallelogram in the tail. At the same time, students who first placed the parallelogram in the tail often then tried it in the head before putting it back in the tail. From a Composition LT perspective, these students might be classified as being unintentional or using trial and error. From a navigating flexible abstraction perspective, however, moving the same tangram piece to different spots could be productive, helping students see the space in a new way.

Although two small triangles also fit into the tail, students' favoritism toward the parallelogram suggests that they prioritize placing the largest shape they can for a space. Such an approach might explain why a subset of students initially placed the parallelogram in the fox's head and the large triangle in the body. In fact, students at both grade levels tried to break up the larger space of the body with the largest piece, the large triangle, similar to the strategy used by adults (Baran et al., 2007). Further, more students first tried the large triangle in the legs of the body, suggesting attention to the angles created by the partial outline, characteristic of students at the shape composer level (Clements & Sarama, 2009). Therefore, students' efforts to navigate flexible abstraction appear to result from an interaction among the size of the space, the outlines, and the particular pieces used.

Although students' piece placements displayed strong trends, one limitation of our study is that we did not ask students to talk aloud or explain their choices when solving the puzzle. Asking students to explain their puzzle-solving processes could lead toward frustration or them changing their piece placements. However, in future investigations, encouraging students to talk aloud or asking targeted questions (e.g., Why did you start with this piece? Why did you move this piece?) could clarify how they navigate the flexible abstraction.

6.2 Navigating combinations

Students could compose the fox head in three ways, but only one combination allowed for a correctly solved puzzle. Therefore, although some students changed their composition of the head section (i.e., substitution composer level; Clements & Sarama, 2009), they did not always complete the puzzle, an expectation of the earlier shape composer level. Even students who were not successful at solving the puzzle showed intentionality in placing the pieces. For example, Carlos placed triangles in the legs, matching the angles without any difficulty, even if they were the wrong sized triangles or facing the wrong direction. For the Composition LT to effectively translate to tangram puzzles, our analysis using the visual representation framework (Martin & Schwartz, 2014) suggests that additional sublevels of the Composition LT are needed to better represent the connections between puzzle-making and puzzle-solving. For this puzzle, the possibility of multiple sections having different combinations appeared to contribute to the difficulty of the puzzle and is an area we need to investigate further. Based on this work, continued investigations should also identify whether students focus more on *composing regions* of puzzles versus determining *placements of specific pieces*.

6.3 Navigating reinterpretation

According to the Composition LT (Clements & Sarama, 2009), all of the students in our study should have been able to complete the fox puzzle using flips and turns intentionally. However, when focusing on reinterpretation through the visual representation framework, we found that students were more intentional with some shapes than others. Even though there were right angles present in the puzzle outline, the head section did not prime the use of the square, likely because the outline of the head did not have any flat sides parallel to edges of the paper. Although students' struggles to rotate the square and orient it with non-horizontal sides is consistent with previous research (e.g., Baran et al., 2007; Clements & Sarama, 2009), some students struggled with reinterpreting the square even after correctly reinterpreting and placing the small triangles in the ears and completing *all other pieces* of the puzzle! The right angle of the square may not have been as salient to students as it might be in pattern block puzzles because all of the triangle tangram pieces also had a right angle. Future work should explore the extent to which separating the square in a nonstandard orientation (see Fig. 13) helps students reinterpret and correctly place it.

Compared to their use of the square, students may have been more likely to turn the triangles because of the number of places where the puzzle primed the use of triangles (e.g., two legs, tail, bottom of face, and two ears). The outline of the puzzle also likely contributed to students' use of triangles and focus on their vertices as found by Hallowell et al.

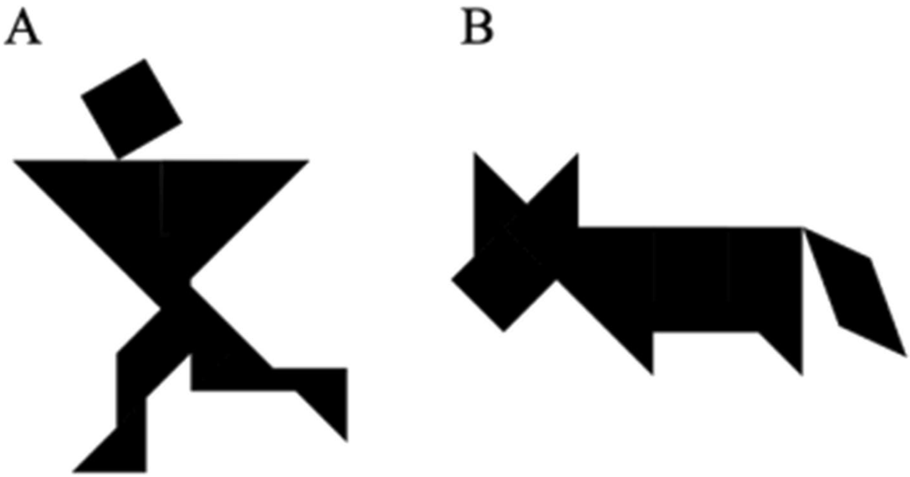


Fig. 13 Examples of tangram puzzles with the nonstandard square defined (**A**) and less defined (**B**)

(2015); the ears incorrectly primed the use of the small right triangles in a typical right triangle presentation (◣ or ◢), and several students oriented the small triangles this way. Additionally, the bottom of the face primed use of the triangle with the vertex opposite the hypotenuse facing downward (▼) (as opposed to upward), while the legs primed use of the triangles with the other vertices pointing downward (◤ or ◥). The visual representation framework (Martin & Schwartz, 2014) can help researchers and teachers focus on the interactions between students' use of reinterpretation and combinations to better understand what factors influence their placement of pieces. Based on results such as ours, curriculum developers could specifically design and choose puzzles to encourage students' use of reinterpretation. For example, they might keep certain combinations of tangram pieces constant and vary the orientation of others over a series of puzzles.

6.4 Navigating borrowing structure

One potential reason students may have been willing to turn pieces or use them strategically is their noticing of structure within the puzzle, a characteristic of students at the shape composer level (Clements & Sarama, 2014). Students who completed the ear last often intentionally flipped one of the triangles, likely noticing that the two ears were mirror images. Helping students notice this structure might encourage them to strategically turn and flip pieces in subsequent puzzles and avoid unproductive borrowing structure (e.g., Curt's placement of large triangles). Because we looked for instances where students placed the triangles in the legs or ears sequentially, we potentially underestimated their navigating borrowing structure. For example, students might have noticed structure even in instances where they correctly placed one small triangle in one ear, placed a different shape elsewhere, and then came back to the other ear to place the second small triangle. As mentioned previously, having students talk aloud while completing the puzzle might provide additional insight into their navigating borrowing structure.

Curriculum developers and teachers could promote and capitalize on borrowing structure by creating a series of puzzles: (1) promote *translation* by having students solve puzzles where two large and two small triangles are placed next to each other in the same orientation, (2) promote *reinterpretation* by having students solve puzzles where the triangles are placed symmetrically, and (3) present puzzles targeting *both translations and reinterpretation* mechanisms.

7 Conclusion

The use of the visual representation framework (Martin & Schwartz, 2014) highlights a need to reconsider the cognitive mechanisms involved in the Composition LT (Clements & Sarama, 2009) when shifting from using pattern blocks to using tangram puzzles, particularly in the case of silhouette puzzles. Trial and error in the Composition LT indicates less-advanced composition skills. However, trial and error followed by purposeful placing of shapes might indicate students' strategic attempts to navigate flexible abstraction and see the space in new ways using the visual representation framework (Martin & Schwartz, 2014). Further, because each tangram piece must only be used once in the puzzle, the interaction between the cognitive mechanisms and a puzzle's features takes on heightened importance. Focus on the mechanisms suggests that with tangram puzzles (or other puzzles where each piece must be used once), there might be an intermediary level where students substitute combinations of shapes using trial and error. One case to illustrate this is that of students who composed the head section in multiple ways (i.e., substitute composers, Clements & Sarama, 2009) but who did not successfully complete the puzzle (i.e., picture maker to shape composer levels). Further, rather than all silhouette puzzles being taken as equal at the shape composer level, silhouette puzzles (or puzzle sections) could be ranked based on their difficulty in relation to their features and potential to encourage students to use particular mechanisms strategically (e.g., easier puzzles have common structure to encourage navigating borrowing structure).

Author contribution Laura Bofferding designed the study and analyzed and wrote the manuscript; Mahtob Aqazade analyzed and wrote the manuscript.

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Data availability Not applicable.

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Declarations

Ethics approval This research was approved by Purdue University's Institutional Review Board.

Conflict of interest The authors declare no competing interests.

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