HIGHER-GENUS WALL-CROSSING IN THE GAUGED LINEAR SIGMA MODEL

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WITH AN APPENDIX BY YANG ZHOU

ABSTRACT. We introduce a technique for proving all-genus wall-crossing formulas in the gauged linear sigma model as the stability parameter varies, without assuming factorization properties of the virtual class. Implementing this technique to the gauged linear sigma model associated to a complete intersection in weighted projective space, we obtain a uniform proof of the wall-crossing formula in both the geometric and the Landau–Ginzburg phase.

1. INTRODUCTION

The gauged linear sigma model (GLSM) was introduced in the physics literature by Witten [45] in the early 1990's as an example of a two-dimensional quantum field theory whose target is a complete intersection in a GIT quotient. Since then, it has been extensively studied in physics (see [33, 34] for references) and has become a powerful tool in understanding many aspects of both physics and geometry.

The mathematical development of the GLSM as a curve-counting theory was carried out by Fan, Jarvis, and the third author in [25], where the foundations were laid for the theory in the "compact-type" subspace. The resulting theory is divided into chambers (or "phases") by variation of GIT on the target geometry. In the geometric phase, the GLSM recovers the quasimap theory developed by Ciocan-Fontanine, Kim, and Maulik [19], while in the Landau–Ginzburg phase, it recovers Fan–Jarvis–Ruan–Witten (FJRW) theory [24] when the target is a hypersurface in weighted projective space.

Our interest in the GLSM is motivated by the celebrated Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence. Originally framed mathematically as a connection between the Gromov–Witten theory of a hypersurface and the FJRW theory of its defining polynomial, the correspondence can be re-cast in this new language as a wall-crossing (or "phase transition") between different chambers of the GLSM. Two types of walls are relevant here: between phases there are chamber walls, and within each phase, there are walls as one varies the choice of a stability parameter ϵ. So far, very little is understood about the transition across the chamber walls, though a general conjecture posits that it should involve analytic continuation of generating functions. The ϵ-wall-crossing, on the other hand, has been made remarkably explicit by Ciocan-Fontanine and Kim, as we explain below; this is the subject of the current work. Our ultimate goal is to combine the two types of wall-crossings to yield a proof of the LG/CY correspondence in all genus. In genus zero, this program has been carried out when the target is a hypersurface (through the combined work of Ciocan-Fontanine–Kim [15], Ross and the third author [41], and Chiodo and the third author [11]), and in genus one, it has been carried out for the quintic hypersurface without marked points (through the work of Kim–Lho [37] and Guo–Ross [31, 32]).

1.1. **Statement of results.** For a complete intersection Y in projective space, the definition of quasimaps introduced by Ciocan-Fontanine, Kim, and Maulik generalizes the notion of stable maps to Y via the additional datum of a positive rational number ϵ. When $\epsilon \to \infty$, quasimaps coincide with the usual stable maps, so quasimap theory recovers Gromov–Witten theory. When $\epsilon \to 0$, on the other hand, quasimaps become stable quotients [40], and the resulting theory is thought to correspond to the mirror B-model of Y [12, 15, 16].

Quasimap theory changes only at certain discrete values of ϵ, so there is a wall-and-chamber structure on the space of stability parameters. Ciocan-Fontanine and Kim proved a wall-crossing formula in [15] exhibiting how the genus-zero theory varies with ϵ, and in [18, 17], they formulated the following conjecture in all genus:

Conjecture 1.1 (See [17]). *Let Y be a complete intersection in projective space, and fix $g, n \geq 0$. Then one has the following equality in $H_*(\overline{\mathcal{M}}_{g,n}^{\epsilon}(Y, \beta))[[q]]$:*

$$\sum_{\beta} q^{\beta} [\overline{\mathcal{M}}_{g,n}^{\epsilon}(Y, \beta)]^{\mathrm{vir}} = \sum_{\beta_0, \beta_1, \ldots, \beta_k} \frac{q^{\beta_0}}{k!} b_{\vec{\beta}*} c_* \left(\prod_{i=1}^{k} q^{\beta_i} \mathrm{ev}_{n+i}^* (\mu_{\beta_i}^{\epsilon}(-\psi_{n+i})) \cap [\overline{\mathcal{M}}_{g,n+k}^{\infty}(Y, \beta_0)]^{\mathrm{vir}} \right),$$

where $\mu_{\beta}^{\epsilon}(z)$ are certain coefficients of the I-function of Y, $b_{\vec{\beta}}$ is a morphism that converts marked points to basepoints, and c is the natural contraction morphism from ∞-stable to ϵ-stable quasimaps.

Ciocan-Fontanine and Kim proved this conjecture in [17], using virtual push-forward techniques and MacPherson's graph construction. A different proof was given in [21] under the assumption that $n \geq 1$, using localization on a "twisted graph space", which has the advantage that it can be adapted to the more general context of the GLSM.

To explain this generalized setting, we recall that the GLSM depends on the choice of (1) a GIT quotient

$$X_{\theta} = [V /\!\!/_{\theta} G],$$

in which V is a complex vector space, $G \subset \mathrm{GL}(V)$, and θ is a character of G; (2) a polynomial function $W \colon X_{\theta} \to \mathbb{C}$ known as the "superpotential"; and (3) an action of $\mathbb{C}^*$ on V known as the "R-charge". Associated to any such choice of input data and any stability parameter ϵ, there is a state space $\mathcal{H}_{\theta}$ and a moduli space $Z_{g,n,\beta}^{\epsilon,\theta}$ parameterizing ϵ-stable Landau–Ginzburg quasimaps to the critical locus of W (see Section 2 below). Under the technical requirement that the theory admits a "good lift", $Z_{g,n,\beta}^{\epsilon,\theta}$ is compact and admits a virtual fundamental class, which can be paired with elements of the compact-type subspace $\mathcal{H}_{\theta}^{\mathrm{ct}} \subset \mathcal{H}_{\theta}$ to yield invariants.

If Y is a nonsingular complete intersection in weighted projective space $\mathbb{P}(w_1, \ldots, w_M)$ defined by the vanishing of polynomials $F_1, \ldots, F_N$ of degrees $d_1, \ldots, d_N$, then one obtains a GIT quotient by letting $G = \mathbb{C}^*$ act on $V = \mathbb{C}^{M+N}$ with weights

$$(w_1, \ldots, w_M, -d_1, \ldots, -d_N).$$

Choosing $\theta \in \mathrm{Hom}(G, \mathbb{C}^*) \cong \mathbb{Z}$ to be any positive character, the resulting GIT quotient is

$$X_+ := \bigoplus_{j=1}^{N} \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_M)}(-d_j).$$

Denote the coordinates of X_+ by $(x_1, \ldots, x_M, p_1, \ldots, p_N)$, and define a superpotential by

$$W(x_1, \ldots, x_M, p_1, \ldots, p_N) := \sum_{j=1}^{N} p_j F_j(x_1, \ldots, x_M).$$

Finally, define an R-charge by letting $\mathbb{C}^*$ act with weight 1 on the p-coordinates. Then the critical locus of W is simply the complete intersection Y, the existence of a good lift is automatic, and the moduli space of the GLSM reduces to the moduli space of ordinary quasimaps to Y:

$$Z_{g,n,\beta}^{\epsilon,+} = \overline{\mathcal{M}}_{g,n}^{\epsilon}(Y, \beta).$$

The state space $\mathcal{H}_+$ is isomorphic to the cohomology of Y, and the compact-type subspace consists of classes pulled back from the ambient $\mathbb{P}(w_1, \ldots, w_M)$. Thus, we obtain a theory that agrees with the quasimap theory of Y with ambient insertions. [1]

On the other hand, when the character θ is negative, the resulting GIT quotient is

$$X_- := \bigoplus_{i=1}^{M} \mathcal{O}_{\mathbb{P}(d_1, \ldots, d_N)}(-w_i)$$

and the critical locus of the above W is the zero section. With the R-charge as above, the moduli space $Z_{g,n,\beta}^{\epsilon,-}$ parameterizes marked curves equipped with a line bundle L and a section

$$\vec{p} \in \Gamma \left(\bigoplus_{j=1}^{N} (L^{\otimes -d_j} \otimes \omega_{\log}) \right)$$

satisfying certain stability conditions depending on the parameter ϵ. In particular, when $N = 1$, one has a section $p \in \Gamma(L^{\otimes -d} \otimes \omega_{\log})$, and for $\epsilon \gg 0$, the stability condition enforces that p is nowhere vanishing; thus, p trivializes $L^{\otimes -d} \otimes \omega_{\log}$, and the resulting moduli space of d-spin structures is precisely the moduli space of FJRW theory.

The assumption that the theory admits a good lift, explained in detail in [25], amounts in this setting to the requirement that $d_1 = \cdots = d_N$, in which case the theory is known in the physics literature as a "hybrid model" and was developed mathematically in the narrow sectors (a subset of the state space properly contained in $\mathcal{H}_-^{ct}$) by the first author [20].

Conjecture 1.1 can be directly generalized to this setting, and our main theorem is a verification that the conjecture holds:

Theorem 1.2 (See Theorem 3.1). *Let $Y \subset \mathbb{P}(w_1, \ldots, w_M)$ be a nonsingular complete intersection defined by the vanishing of a collection of polynomials of degrees $d_1, \ldots, d_N$, where $w_i | d_j$ for all i and j. Fix $\theta \in \{+, -\}$ and suppose that the associated GLSM admits a good lift. Then, for any g and n and a tuple of insertions $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_\theta^{ct}$, one has the following*

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[1] In order to know that the two theories indeed coincide, we need to match the construction of the virtual cycle in the GLSM, defined via Kiem–Li's cosection technique, with the construction of the virtual cycle in quasimap theory by Ciocan-Fontanine–Kim–Maulik. This has been carried out for the quintic hypersurface (on the level of correlators) in the seminal work of Chang–Li [3], and has since then been generalized to the case of complete intersections in $\mathbb{P}(w_1, \ldots, w_M)$ [8, 38, 7]. See Remark 2.12 below for further discussion.

equality in $H_*(Z_{g,n,\beta}^{\epsilon,\theta})[[q]]$:

$$\sum_{\beta\geq 0} q^\beta \left(\prod_{a=1}^n \mathrm{ev}_a^*(\alpha_a) \cap [Z_{g,n,\beta}^{\epsilon,\theta}]^{\mathrm{vir}}\right) =$$

$$\sum_{k=0}^\infty \sum_{\beta_0,\beta_1,\ldots,\beta_k\geq 0} \frac{q^{\beta_0}}{k!} b_{\vec{\beta}*} c_* \left(\prod_{a=1}^n \mathrm{ev}_a^*(\alpha_a) \prod_{i=1}^k q^{\beta_i} \mathrm{ev}_{n+i}^*(\mu_{\beta_i}^\epsilon(-\psi_{n+i})) \cap [Z_{g,n+k,\beta_0}^{\infty,\theta}]^{\mathrm{vir}}\right).$$

Remark 1.3. In particular, taking $\theta = +$ and $w_1 = \cdots = w_M = 1$, the proof of Theorem 1.2 reproduces the results of [21] for ambient insertions, assuming the equivalence of quasimap theory with the positive phase of the GLSM mentioned above. For this reason, we focus in what follows on the case where $\theta = -$, remarking where necessary on the appropriate modifications for the positive (that is, the "geometric") phase.

The basic idea of the proof of Theorem 1.2 is the same as the authors' technique in the geometric phase, as presented in [21]. Namely, we construct a larger moduli space with a $\mathbb{C}^*$-action in which the theories at $\epsilon = \infty$ and arbitrary ϵ arise as fixed loci. (This larger moduli space is closely related to the space of mixed-spin p-fields considered by Chang–Li–Li–Liu [5, 6].) A direct application of the proof in [21], however, would require a factorization property of the virtual class along strata corresponding to nodal curves, and the verification of such a property is an unsolved issue dating back to the early days of FJRW theory; the problem is that, while the theory is only defined for compact-type insertions, decomposition at nodes may involve insertions that are not of compact type. In ongoing research, Ciocan-Fontanine–Favero–Guéré–Kim–Shoemaker [13] are working toward a resolution of this problem via the theory of matrix factorizations, and an analytic approach has also been proposed by Tian–Xu [42, 43] and Fan–Jarvis–Ruan [26].

Rather than awaiting the completion of the general theory in the non-compact-type case, though (since the compact-type theory is sufficient for the applications in which we are interested), we present in this work a technique that avoids the necessity of factorization along nodes. The idea is to introduce a new moduli space for each boundary stratum in $Z_{g,n,\beta}^{\epsilon,\theta}$ and to carefully choose the discrete data in the twisted graph space so that only very particular degenerations occur in the contributing fixed loci. We hope that this strategy may be useful in other situations where the factorization of the GLSM virtual class has been an obstacle.

As in the geometric phase, the proof of Theorem 1.2 initially requires that $n \geq 1$, since it relies crucially on the fact that a certain localization expression changes in a nontrivial way when an insertion is varied. However, in the appendix (written by Yang Zhou), the result for $n = 0$ is deduced from the result for $n \geq 1$ by leveraging a wall-crossing between "heavy points" and "light points." These ideas were developed in [46], in which Yang Zhou gives an alternative proof of Theorem 1.2 in the case where Y is a hypersurface.

1.2. **Plan of the paper.** In Section 2, we review the necessary theory of the GLSM in the setting of interest here. We state our main theorem in precise form in Section 3, and we state the progressive refinements of the theorem that are necessary for the proof. We introduce the twisted graph space in Section 4, where we explicitly calculate the fixed-locus contributions to the localization formula. We also review the setting where the target is a point, in which case the twisted graph space specializes to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \delta)$ and was studied in detail in [21]. Finally, in Section 5, we present the proof of Theorem 1.2. The structure of the proof is

to compute, via localization, the pushforward from the twisted graph space to $Z_{g,n,\beta}^{\epsilon,\theta}$ of a certain difference of cohomology classes. The contributions from degree-zero components are closely related to the calculations for a point, and using these calculations, we show that the class in question changes by an irrational function of the equivariant parameter when an insertion is varied. This implies that the class vanishes, and from here, we deduce the wall-crossing theorem when $n \geq 1$. In the appendix, the notion of "light" marked points is introduced, and the $n = 0$ case of the wall-crossing is deduced from the $n \geq 1$ case by proving an auxiliary wall-crossing that relates heavy to light markings and then applying the dilaton and divisor equations in the light-marking setting.

2. Definitions and setup

We review the definition of the gauged linear sigma model (GLSM) for the Landau–Ginzburg phase in the setting of Theorem 1.2. This is a very special case of the general construction of the GLSM by Fan, Jarvis, and the third author. When θ is negative and ϵ is sufficiently large, it recovers the hybrid model defined in [20].

2.1. **Gauged linear sigma model.** Throughout what follows, we fix a nonsingular complete intersection

$$Y := \{F_1 = \cdots = F_N = 0\} \subset \mathbb{P}(w_1, \ldots, w_M),$$

where $F_1, \ldots, F_N$ are polynomials of degrees $d_1, \ldots, d_N$ and $w_i | d_j$ for all i and j.

The GLSM, in general, depends on three pieces of input data: a GIT quotient $X_\theta = [V /\!\!/_\theta G]$, a polynomial function $W \colon X \to \mathbb{C}$ known as the superpotential, and an action of $\mathbb{C}^*$ on V known as the R-charge. In our case,

$$X_\theta := [(\mathbb{C}^M \times \mathbb{C}^N) /\!\!/_\theta G],$$

where

$$G := \{(g^{w_1}, \ldots, g^{w_M}, g^{-d_1}, \ldots, g^{-d_N}) \mid g \in \mathbb{C}^*\} \cong \mathbb{C}^*$$

acts diagonally on $V := \mathbb{C}^M \times \mathbb{C}^N$, and there are two phases depending on whether the character $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ is positive or negative. Denoting the coordinates on V by $(x_1, \ldots, x_M, p_1, \ldots, p_N)$, the unstable locus when θ is positive is $\{x_1 = \cdots = x_M = 0\}$, so we have

$$X_+ = \frac{(\mathbb{C}^M \setminus \{0\}) \times \mathbb{C}^N}{\mathbb{C}^*} \cong \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_M)}(-d_j).$$

Similarly, when θ is negative, the unstable locus of θ is $\{p_1 = \cdots = p_N = 0\}$, so

$$X_- = \frac{\mathbb{C}^M \times (\mathbb{C}^N \setminus \{0\})}{\mathbb{C}^*} \cong \bigoplus_{i=1}^M \mathcal{O}_{\mathbb{P}(d_1, \ldots, d_N)}(-w_i).$$

The superpotential in either phase is

$$W(x_1, \ldots, x_M, p_1, \ldots, p_N) := \sum_{i=1}^{N} p_j F_j(x_1, \ldots, x_M),$$

and the R-charge acts by multiplication on the p-coordinates. Using that Y is nonsingular, one can check that the critical locus Z_+ of W when θ is positive is the complete intersection Y inside the zero section of X_+, and the critical locus Z_- when θ is negative is the entire zero section in X_-.

From here forward, we restrict to the negative phase of the GLSM unless otherwise specified, and we assume that all of the degrees d_j are equal. To ease notation, we make use of the abbreviations

$$d := d_1 = \cdots = d_N,$$

$$X := X_- = \bigoplus_{i=1}^{M} \mathcal{O}_{\mathbb{P}(d,\ldots,d)}(-w_i),$$

and

$$Z := Z_- = \mathbb{P}(d, \ldots, d) \subset X.$$

2.2. **State space.** The state space of the GLSM is defined as the relative Chen–Ruan cohomology group

(1) $$\mathcal{H} := H^*_{\mathrm{CR}}(X, W^{+\infty}; \mathbb{C}),$$

in which $W^{+\infty}$ is a Milnor fiber of W—that is, $W^{+\infty} = W^{-1}(A)$ for a sufficiently large real number A. Thus, the state space decomposes into summands indexed by the components of the inertia stack $\mathcal{I}X$, which are labeled by elements $g \in G$ with nontrivial fixed locus $\mathrm{Fix}(g) \subset X$.

More concretely, the only $(g^{w_1}, \ldots, g^{w_M}, g^{-d}, \ldots, g^{-d}) \in G$ with nontrivial fixed locus are those for which $g^d = 1$. An element $(\vec{x}, \vec{p}) \in \mathrm{Fix}(g)$ must have $x_i = 0$ whenever $g^{w_i} \neq 1$, but there is no constraint on the x_i for which $g^{w_i} = 1$. Thus, denoting $\mathbb{P}(\vec{d}) := \mathbb{P}(d, \ldots, d)$ and

$$F(g) := \{i \mid g^{w_i} = 1\} \subset \{1, \ldots, M\},$$

we set

$$X_g := \bigoplus_{i \in F(g)} \mathcal{O}_{\mathbb{P}(\vec{d})}(-w_i) \subset X$$

and $W_g := W|_{X_g}$. Then

$$\mathcal{H} = \bigoplus_{g \in \mathbb{Z}_d} H^*(X_g, W_g^{+\infty}; \mathbb{C}).$$

An element $g \in \mathbb{Z}_d$ (or its corresponding component of $\mathcal{H}$) is referred to as *narrow* if $\mathrm{Fix}(g)$ is compact, which amounts to requiring that $F(g) = \emptyset$. In other words, if we set

(2) $$\mathrm{nar} := \left\{ m \in \left\{0, \frac{1}{d}, \cdots, \frac{d-1}{d}\right\} \,\middle|\, \nexists\, i \text{ such that } mw_i \in \mathbb{Z} \right\},$$

then the narrow sectors are indexed by $g = e^{2\pi im}$ with $m \in \mathrm{nar}$. Narrow sectors have $W|_{\mathrm{Fix}(g)} \equiv 0$, so their contribution to $\mathcal{H}$ is simply $H^*(\mathbb{P}(\vec{d})) \cong H^*(\mathbb{P}^{N-1})$.

For any $g \in \mathbb{Z}_d$, there is a natural map

$$\eta_g \colon H^{k-2|F(g)|}(X_g) \to H^k(X_g, W_g^{+\infty}).$$

To define η_g, we set $X_g^{\mathrm{ct}} \cong \mathbb{P}(\vec{d})$ to be the zero section inside X_g. The map η_g is defined as the composition

$$\text{(3)} \qquad H^{k-2|F(g)|}(X_g) \xrightarrow{\sim} H^{k-2|F(g)|}(X_g^{\mathrm{ct}}) \xrightarrow{\sim} H^k(X_g, X_g \setminus X_g^{\mathrm{ct}}) \to H^k(X_g, W_g^{+\infty}),$$

where the second map is the Thom isomorphism and the remaining maps are induced by the inclusions. Together, the η_g define a homomorphism

$$\text{(4)} \qquad \eta \colon H_{\mathrm{CR}}^*(X) \to \mathcal{H},$$

and we define the *compact-type state space* as

$$\mathcal{H}^{\mathrm{ct}} := \mathrm{image}(\eta) \subset \mathcal{H}.$$

Note that η_g is an isomorphism when g is narrow, so the compact-type state space contains all of the narrow sectors.

Instead of working with $\mathcal{H}^{\mathrm{ct}}$, in what follows we take insertions from

$$\widetilde{\mathcal{H}} := H_{\mathrm{CR}}^*(X),$$

which surjectively maps to $\mathcal{H}^{\mathrm{ct}}$ via η. If one wishes to take insertions from $\mathcal{H}^{\mathrm{ct}}$ instead, then one must first lift them to $H_{\mathrm{CR}}^*(X)$. We conjecture that the resulting correlators are independent of the choice of lift; see Lemma 2.15 below.

Remark 2.1. In the geometric chamber, there is an isomorphism

$$\mathcal{H}_+ := H_{\mathrm{CR}}^*(X_+, W^{+\infty}; \mathbb{C}) \cong H_{\mathrm{CR}}^*(Y)$$

(see [10, Proposition 3.4]). This has a twisted sector

$$X_{g,+} = \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}_g}(-d)$$

for any g such that $g^{w_i} = 1$ for some i, in which $\mathbb{P}_g \subset \mathbb{P}(w_1, \ldots, w_M)$ is the sub-projective space spanned by the coordinates x_i for which $g^{w_i} = 1$. The zero section is $X_{g,+}^{\mathrm{ct}} \cong \mathbb{P}_g$ in this case, and if $Y_g := \mathbb{P}_g \cap Y$, then η_g can be identified with the restriction $H^*(\mathbb{P}_g) \to H^*(Y_g)$. Thus, the compact-type state space $\mathcal{H}_+$ consists of the ambient classes in each sector of $H_{\mathrm{CR}}^*(Y)$ on the geometric side, and inserting classes from $\widetilde{\mathcal{H}}_+ := H_{\mathrm{CR}}^*(X_+)$ simply means lifting an ambient class $\iota^*\alpha$ (in which $\iota \colon Y_g \to \mathbb{P}_g$ is the inclusion) to an insertion of α.

2.3. **Moduli space.** The definition of the GLSM moduli space, which depends on the choice of a stability parameter ϵ, is based on the notion of quasimaps introduced by Ciocan-Fontanine, Kim, and Maulik [19] and studied extensively by Ciocan-Fontanine and Kim [14, 15, 16, 17, 18].

Fix a genus g, a degree $\beta \in \mathbb{Z}$, a nonnegative integer n, and a positive rational number ϵ.

Definition 2.2. A *prestable Landau–Ginzburg quasimap* to Z consists of an n-pointed prestable orbifold curve $(C; q_1, \ldots, q_n)$ of genus g with nontrivial isotropy only at special points, an orbifold line bundle L on C, and a section

$$\vec{p} = (p_1, \ldots, p_N) \in \Gamma((L^{\otimes -d} \otimes \omega_{\log})^{\oplus N})$$

(where

$$\omega_{\log} := \omega_C \otimes \mathcal{O}_C([q_1] + \cdots + [q_n]),$$

is the logarithmic dualizing sheaf) such that the zero set of $\vec{p}$ (that is, the set of points $q \in C$ such that $p_1(q) = \cdots = p_N(q) = 0$) is finite. An *$\epsilon$-stable Landau–Ginzburg quasimap* to Z is a prestable Landau–Ginzburg quasimap to Z, satisfying the following further conditions:

- *Representability*: For every $q \in C$ with isotropy group G_q, the homomorphism $G_q \to \mathbb{C}^*$ giving the action of the isotropy group on the bundle L is injective.
- *Nondegeneracy*: The zero set of $\vec{p}$ is disjoint from the marked points and nodes of C, and for each zero q of $\vec{p}$, the order of the zero (that is, the common order of vanishing of $p_1, \ldots, p_N$) satisfies

(5)
$$\mathrm{ord}_q(\vec{p}) \leq \frac{1}{\epsilon}.$$

  (Zeroes of $\vec{p}$ are referred to as *basepoints* of the quasimap.)
- *Stability*: The $\mathbb{Q}$-line bundle

(6)
$$(L^{\otimes -d} \otimes \omega_{\log})^{\otimes \epsilon} \otimes \omega_{\log}$$

  is ample.

The *degree* of a quasimap is defined as

(7)
$$\beta := \deg(L^{\otimes -d} \otimes \omega_{\log}).$$

By the nondegeneracy condition, the degree β is a non-negative integer.

Notice that when $N = 1$ and $\epsilon > 2$, condition (5) implies that the section p_1 is nowhere-vanishing, so it gives a trivialization $L^{\otimes -d} \otimes \omega_{\log} \cong \mathcal{O}_C$. In this case, the degree must be zero, and condition (6) amounts to the requirement that $(C; q_1, \ldots, q_n)$ be a stable orbifold curve. The definition of an ϵ-stable quasimap, then, recovers the notion of a d-spin curve. On the other hand, when $\epsilon \leq \frac{1}{\beta}$, condition (5) puts no restriction on the orders of the basepoints, and (6) is equivalent to imposing the analogous requirement for all $\epsilon > 0$. The resulting moduli space is analogous to the moduli space of stable quotients [40].

Remark 2.3. An alternative way to view the choice of stability parameter ϵ is to replace, in the definition of the GLSM, the character $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, \mathbb{C}^*)$ by the rational character $\theta \cdot (d\epsilon) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$; see [16, Section 2]. We return to this perspective later when defining the twisted graph space.

Proposition 2.4. *There is a proper Deligne–Mumford stack $Z^\epsilon_{g,n,\beta}$ parameterizing genus-g, n-pointed, ϵ-stable Landau–Ginzburg quasimaps of degree β to Z up to isomorphism.*

We will prove Proposition 2.4 by relating the above definition of ϵ-stable Landau–Ginzburg quasimap to Z [25, Definition 4.2.11] in the setting of [25, Example 7.2.2]. For this, and for the construction of virtual cycles, we introduce the following notion:

Definition 2.5. An *ϵ-stable Landau–Ginzburg quasimap* to X is a tuple $(C; q_1, \ldots, q_n; L; (\vec{x}, \vec{p}))$, in which $(C; q_1, \ldots, q_n; L; \vec{p})$ is an ϵ-stable Landau–Ginzburg quasimap to Z, and

$$\vec{x} \in \Gamma\left(\bigoplus_{i=1}^{M} L^{\otimes w_i} \right).$$

Proof of Proposition 2.4. We first note that the notion of ϵ-stable Landau–Ginzburg quasimap to X agrees with the notion of ϵ-stable Landau–Ginzburg quasimap of [25, Definition 4.2.11] in the setting of [25, Example 7.2.2] for a suitably chosen good lift. Using the notation of [25], the groups G and Γ are $\mathbb{C}^* = \mathbb{C}^* \times \{1\}$ and $\mathbb{C}^* \times \mathbb{C}^*_R$ respectively, the character $\theta \colon G \to \mathbb{C}^*$

is given by $\lambda \mapsto \lambda^{-d}$, and the good lift $\vartheta \colon G \to \mathbb{C}^*$ is $(g, \mu) \mapsto g^{-d}\mu$. Tracing through the definitions, we match the two notions.

We remark here that the trivial lift $\vartheta_0 \colon G \to \mathbb{C}^*$ defined by $(g, \mu) \mapsto g^{-d}$ is another good lift (as remarked in [25, Example 7.2.2]). Using this lift would lead to a slightly different looking but closely related stability condition.

By [25, Theorem 5.4.1], we conclude that the substack $Z_{g,n,\beta}^{\epsilon}$ of Landau–Ginzburg quasimaps to the critical locus Z (see the discussion of [25, Example 7.2.2]) is proper. $\qquad\square$

Remark 2.6. The perspective of viewing an ϵ-stable Landau–Ginzburg quasimap to Z as a section

$$(\vec{x}, \vec{p}) \in \Gamma \left(\bigoplus_{i=1}^{M} L^{\otimes w_i} \oplus \bigoplus_{j=1}^{N} (L^{\otimes -d_j} \otimes \omega_{\log}) \right)$$

whose image lies in the affine cone over the critical locus Z and whose order of contact with the unstable locus is bounded, is important for defining the virtual cycle, and it also points to how one must modify Definition 2.2 for the positive phase. Namely, Landau–Ginzburg quasimaps to Z_+ consist of sections $(\vec{x}, \vec{p})$ as above whose image lies in the affine cone over the critical locus of Z_+, which precisely recovers the definition of quasimaps to the complete intersection Y as in [19]. Replacing $\vec{p}$ by $\vec{x}$ in (5) and $L^{\otimes -d} \otimes \omega_{\log}$ by L in both (6) and (7) yields the definition of ϵ-stability for quasimaps. Thus, $Z_{g,n,\beta}^{\epsilon,+}$ is the moduli space $\overline{\mathcal{M}}_{g,n}^{\epsilon}(Y, \beta)$ of (ordinary) quasimaps defined in [19].

2.4. **Multiplicities and evaluation maps.** Recall that the *multiplicity* of an orbifold line bundle L at a point $q \in C$ with isotropy group $\mathbb{Z}_r$ is defined as the number $m \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ such that the canonical generator of $\mathbb{Z}_r$ acts on the total space of L in local coordinates around q by

$$(x, v) \mapsto \left(e^{2\pi i \frac{1}{r}} x, e^{2\pi i m} v \right).$$

In our case, r is a divisor of d, so that we can take the multiplicities to lie in the set $\{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\}$, and for a tuple $\vec{m} = (m_1, \ldots, m_n)$ with $m_i \in \{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\}$, we denote by

$$Z_{g,\vec{m},\beta}^{\epsilon} \subset Z_{g,n,\beta}^{\epsilon}$$

the open and closed substack consisting of quasimaps for which the multiplicity of L at q_i is m_i. We occasionally wish to leave some multiplicities undetermined, so we denote by

$$Z_{g,\vec{m}+k,\beta}^{\epsilon} \subset Z_{g,n+k,\beta}^{\epsilon}$$

the substack on which the multiplicity of L at q_i is m_i for $1 \le i \le n$, while the last k marked points are allowed any multiplicity.

A crucial feature of the multiplicities is that they determine the relationship between L and its pushforward $|L|$ to the coarse underlying curve. Specifically, suppose that $C' \subset C$ is an irreducible component of C with special points $\{q_k\}$ at which the multiplicities of L are $\{m_k\}$. Then, if $\rho \colon C' \to |C'|$ is the natural map to the coarse underlying curve, we have

$$L|_{C'} = \rho^* |L|_{C'} | \otimes \mathcal{O}_{C'} \left(\sum_k m_k [\overline{q_k}] \right),$$

for the degree-1 divisors $\overline{q_k}$ pulled back from the coarse underlying curve $|C|$. Applying this equation in the case where C is smooth yields a compatibility condition on the multiplicities,

since the degree of $\rho^*|L|$ is an integer. Namely, we have:

$$
(8) \qquad \frac{-\beta + 2g - 2 + n}{d} - \sum_{i=1}^{n} m_i \in \mathbb{Z}.
$$

By a component-wise argument, the same equality (8) still holds when C is not smooth.

Note that equation (8) is independent of m_i if and only if $m_i = \frac{1}{d}$, so this is the only case in which there is a forgetful map on $Z^{\epsilon}_{g,\vec{m},\beta}$ forgetting q_i and its orbifold structure. Thus, the role of the unit in the $\epsilon \gg 0$ hybrid theory—in particular, in the string and dilaton equation—is played by the fundamental class in the narrow sector $H^*(\mathbb{P}^{N-1}) \subset \widetilde{\mathcal{H}}$ indexed by $e^{2\pi i \frac{1}{d}} \in \mathbb{Z}_d$. For this reason, we denote this element of $\widetilde{\mathcal{H}}$ by $\mathbf{1}$ in what follows.

Remark 2.7. The same string and dilaton equations do *not* hold in the theory of ϵ-stable Landau–Ginzburg quasimaps for more general ϵ. One can deduce the appropriate modification from Theorem 1.2; see the discussion in [18, Section 3.4].

Remark 2.8. In the geometric chamber, our convention is that $\beta = \deg(L)$, so the analogue of condition (8) is $\beta - \sum_{i=1}^{n} m_i \in \mathbb{Z}$. In particular, there is a forgetful map on $Z^{\epsilon,+}_{g,\vec{m},\beta} = \overline{\mathcal{M}}^{\epsilon}_{g,\vec{m}}(Y,\beta)$ forgetting q_i only if $m_i = 0$, and the role of the unit in this chamber is played by the usual fundamental class $\mathbf{1} \in H^*_{\mathrm{CR}}(Y) \cong \mathcal{H}_+$.

For each $i = 1, \ldots, n$, there is an evaluation map

$$
\mathrm{ev}_i \colon Z^{\epsilon}_{g,n,\beta} \to \overline{\mathcal{I}}X.
$$

Here, $\overline{\mathcal{I}}X$ is the rigidified inertia stack of X, which in this case is simply

$$
(9) \qquad \overline{\mathcal{I}}X := \bigsqcup_{a \in \mathbb{Z}_d} \bigoplus_{i \in F(a)} \mathcal{O}_{\mathbb{P}(\vec{d}/\gcd(a,d))}(-w_i/\gcd(a,d)).
$$

(In fact, the evaluation maps land in $\overline{\mathcal{I}}Z = \bigsqcup_{a \in \mathbb{Z}_d} \mathbb{P}(\vec{d}/\gcd(a,d))$, but given that our insertions are elements of $\widetilde{\mathcal{H}} := H^*_{\mathrm{CR}}(X)$, it is more natural and more consistent with the geometric phase if the target of evaluation is understood as $\overline{\mathcal{I}}X$.) To define the evaluation maps, let $\pi \colon \mathcal{C} \to Z^{\epsilon}_{g,n,\beta}$ be the universal curve and $\mathcal{L}$ the universal line bundle, and let

$$
\sigma \in \Gamma\left(\bigoplus_{i=1}^{M} \mathcal{L}^{\otimes w_i} \oplus \bigoplus_{j=1}^{N} (\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}) \right)
$$

be the universal section, whose first M coordinates are zero. If $\Delta_i \subset \mathcal{C}$ denotes the stacky divisor corresponding to the ith marked point, then

$$
\sigma\big|_{\Delta_i} \in \Gamma\left(\bigoplus_{i=1}^{M} \mathcal{L}^{\otimes w_i} \oplus \bigoplus_{j=1}^{N} \mathcal{L}^{\otimes -d} \bigg|_{\Delta_i} \right),
$$

using the fact that $\omega_{\pi,\log}\big|_{\Delta_i}$ is trivial. Thus, evaluating $\sigma\big|_{\Delta_i}$ at the fiber over a point $(C; q_1, \ldots, q_n; L; \vec{p})$ in the moduli space yields an element of $\mathbb{P}(\vec{d}/\gcd(a,d))$, and by definition, ev_i sends $Z^{\epsilon}_{g,\vec{m},\beta}$ to the zero section $\mathbb{P}(\vec{d}/\gcd(a,d))$ in the component of (9) indexed by $a := e^{2\pi i m_i} \in \mathbb{Z}_d$.

Remark 2.9. In the geometric chamber, these recover the composition of the usual evaluation map with the inclusion,

$$\overline{\mathcal{M}}^{\epsilon}_{g,n}(Y,\beta) \to \overline{\mathcal{I}}Y \to \overline{\mathcal{I}}X_+,$$

so in particular, they factor through the inclusion $\overline{\mathcal{I}}Y \to \overline{\mathcal{I}}\mathbb{P}(w_1,\ldots,w_M)$.

2.5. **Virtual cycle and correlators.** By the results of [25, Section 5.1], there exists a virtual cycle

$$[Z^{\epsilon}_{g,n,\beta}]^{\mathrm{vir}} \in A_*(Z^{\epsilon}_{g,n,\beta}).$$

It is constructed using the cosection technique of Kiem–Li [37], following the application of the technique by Chang–Li [3] to the Gromov–Witten theory of the quintic threefold and Chang–Li–Li [4] to spin theory. We outline the basic idea, referring the reader to [25] for details.

The key point is that $Z^{\epsilon}_{g,n,\beta}$ sits inside of the noncompact moduli space $X^{\epsilon}_{g,n,\beta}$ of stable Landau–Ginzburg quasimaps to X as defined in Definition 2.5. By [3, Proposition 2.5], $X^{\epsilon}_{g,n,\beta}$ admits a relative perfect obstruction theory

$$(10) \qquad \mathbb{E}^{\bullet} := \left(R\pi_* \left(\bigoplus_{i=1}^{M} \mathcal{L}^{\otimes w_i} \oplus \bigoplus_{j=1}^{N} (\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}) \right) \right)^{\vee} \to \mathbb{L}^{\bullet}_{X^{\epsilon}_{g,n,\beta}/D_{g,n,\beta}}$$

relative to the moduli space $D_{g,n,\beta}$ parameterizing only $(C; q_1, \ldots, q_n; L)$, in which $\pi \colon \mathcal{C} \to X^{\epsilon}_{g,n,\beta}$ is the universal curve and $\mathcal{L}$ the universal line bundle on $\mathcal{C}$. One hopes to define a homomorphism

$$\mathcal{O}b_{X^{\epsilon}_{g,n,\beta}/D_{g,n,\beta}} \to \mathcal{O}_{X^{\epsilon}_{g,n,\beta}}$$

via the derivatives of the superpotential that descends to a cosection $\sigma \colon \mathcal{O}b_{X^{\epsilon}_{g,n,\beta}} \to \mathcal{O}_{X^{\epsilon}_{g,n,\beta}}$ whose fiber is zero exactly over $Z^{\epsilon}_{g,n,\beta} \subset X^{\epsilon}_{g,n,\beta}$. On the components $X^{\epsilon}_{g,\vec{m},\beta}$ for which each m_i is narrow, this procedure works, and the cosection technique outputs a virtual cycle $[Z^{\epsilon}_{g,\vec{m},\beta}]^{\mathrm{vir}}$ supported on $Z^{\epsilon}_{g,\vec{m},\beta}$.

On the components of $X^{\epsilon}_{g,n,\beta}$ where not all multiplicities are narrow, on the other hand, an additional step is necessary. Let $X^{\epsilon,\mathrm{ct}}_{g,n,\beta} \subset X^{\epsilon}_{g,n,\beta}$ denote the subspace where $x_1(q_k) = \cdots = x_M(q_k) = 0$ for each marked point q_k. (We have $X^{\epsilon,\mathrm{ct}}_{g,\vec{m},\beta} = X^{\epsilon}_{g,\vec{m},\beta}$ if all multiplicities are narrow, since this condition implies that $L^{\otimes w_i}$ has nonzero multiplicity at q_k for each i and k, and hence all sections of $L^{\otimes w_i}$ must vanish at q_k.)

The relative perfect obstruction theory $\mathbb{E}^{\bullet}$ can be modified to yield a relative perfect obstruction theory for $X^{\epsilon,\mathrm{ct}}_{g,n,\beta}$. Namely, we set
(11)

$$\mathbb{E}^{\bullet}_{\mathrm{ct}} := \left(R\pi_* \left(\bigoplus_{i=1}^{M} \left(\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}\left(-\sum_{k=1}^{n} \Delta_k\right) \right) \oplus \bigoplus_{i=1}^{N} (\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}) \right) \right)^{\vee} \to \mathbb{L}^{\bullet}_{X^{\epsilon,\mathrm{ct}}_{g,n,\beta}/D_{g,n,\beta}},$$

where Δ_k again denotes the stacky divisor in the universal curve corresponding to the kth marked point with its orbifold structure. Again, using [3, Proposition 2.5], one checks that (11) is indeed a relative perfect obstruction theory. (In the narrow case, the bundles $L^{\otimes w_i}$ and $L^{\otimes w_i} \otimes \mathcal{O}(-\sum [q_k])$ have the same coarse underlying bundle and hence the same cohomology, so $\mathbb{E}^{\bullet}_{\mathrm{ct}} = \mathbb{E}^{\bullet}$.) The derivatives of W now define a homomorphism

$$\mathcal{O}b_{X^{\epsilon,\mathrm{ct}}_{g,n,\beta}/D_{g,n,\beta}} \to \mathcal{O}_{X^{\epsilon,\mathrm{ct}}_{g,n,\beta}}.$$

Namely, the fiber over a point $(C; q_1, \ldots, q_n; (\vec{x}, \vec{p})) \in X_{g,n,\beta}^{\epsilon,\mathrm{ct}}$ is the homomorphism

$$\bigoplus_{i=1}^{M} H^1\left(L^{\otimes w_i} \otimes \mathcal{O}\left(-\sum_{k=1}^{n}[q_k]\right)\right) \oplus \bigoplus_{j=1}^{N} H^1(L^{\otimes -d} \otimes \omega_{\log}) \to \mathbb{C}$$

$$(\dot{x}_1, \ldots, \dot{x}_M, \dot{p}_1, \ldots, \dot{p}_N) \mapsto \sum_{i=1}^{M} \frac{\partial W}{\partial x_i}(\vec{x}, \vec{p}) \cdot \dot{x}_i + \sum_{j=1}^{N} \frac{\partial W}{\partial p_j}(\vec{x}, \vec{p}) \cdot \dot{p}_j.$$

A straightforward application of Serre duality checks that this indeed lands in $\mathbb{C}$. Furthermore, one can check that it descends to a cosection—that is, a homomorphism

$$\sigma \colon \mathcal{O}b_{X_{g,n,\beta}^{\epsilon,\mathrm{ct}}} \to \mathcal{O}_{X_{g,n,\beta}^{\epsilon,\mathrm{ct}}}$$

out of the absolute obstruction sheaf. The degeneracy locus of σ (the locus of points in $X_{g,n\beta}^{\epsilon,\mathrm{ct}}$ over which the fiber of σ is the zero homomorphism, see [3, Definition 3.3]) is precisely the closed subset $Z_{g,n,\beta}^{\epsilon} \subset X_{g,n,\beta}^{\epsilon,\mathrm{ct}}$, and the cosection technique outputs a virtual cycle supported on this locus.

From this discussion, it is straightforward to compute the virtual dimension of each component $Z_{g,\vec{m},\beta}^{\epsilon}$. In particular, we have

(12)

$$\mathrm{vdim}(Z_{g,\vec{m},\beta}^{\epsilon}) = \mathrm{vdim}(X_{g,\vec{m},\beta}^{\epsilon,\mathrm{ct}})$$

$$= \dim(D_{g,\vec{m},\beta}) + (h^0 - h^1)\left(\bigoplus_{i=1}^{M}\left(L^{\otimes w_i} \otimes \mathcal{O}\left(-\sum_{k=1}^{n}[q_k]\right)\right) \oplus \bigoplus_{j=1}^{N}(L^{\otimes -d} \otimes \omega_{\log})\right)$$

$$= \mathrm{vdim}(\overline{\mathcal{M}}_{g,\vec{m}}^{\epsilon}(Z,\beta)) + \sum_{i=1}^{M} \chi\left(L^{\otimes w_i} \otimes \mathcal{O}\left(-\sum_{k=1}^{n}[q_k]\right)\right).$$

Remark 2.10. In genus zero, the definition of the virtual cycle simplifies substantially. Indeed, the condition that $w_i | d$ implies that $\deg(|L^{\otimes w_i} \otimes \mathcal{O}(-\sum[q_k])|) < 0$ for each i (see [20, Section 4.2.9]). Thus, the cosection is identically zero, so the cosection-localized virtual class is the usual virtual class of $Z_{0,n,\beta}^{\epsilon} = X_{0,n,\beta}^{\epsilon}$ defined by way of the perfect obstruction theory $\mathbb{E}^{\bullet}$. Furthermore, $\mathbb{E}^{\bullet}$ is quasi-isomorphic to a vector bundle over a smooth space, so we have

$$[Z_{0,n,\beta}^{\epsilon}]^{\mathrm{vir}} = e\left(\bigoplus_{i=1}^{M} R^1\pi_*(\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}\left(-\sum_{k=1}^{n}\Delta_k\right))\right) \cap [Z_{0,n,\beta}^{\epsilon}].$$

We are now ready to define correlators in the GLSM, following [25]. For this, we recall that the psi classes are defined by

$$\psi_i = c_1(\mathbb{L}_i) \in A^*(Z_{g,n,\beta}^{\epsilon}), \quad i = 1, \ldots, n,$$

where $\mathbb{L}_i$ is the line bundle whose fiber over a moduli point is the cotangent line to the coarse curve $|C|$ at q_i.

Definition 2.11. Given

$$\phi_1, \ldots, \phi_n \in \widetilde{\mathcal{H}}$$

and nonnegative integers $a_1, \ldots, a_n$, the associated *genus-g, degree-β, ϵ-stable GLSM corre-lator* is defined as

$$\langle \phi_1 \psi^{a_1} \cdots \phi_n \psi^{a_n} \rangle_{g,n,\beta}^\epsilon := \int_{[Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_1) \psi_1^{a_1} \cdots \mathrm{ev}_n^*(\phi_n) \psi_n^{a_n}.$$

Remark 2.12. The cosection construction outlined above can also be used to define a virtual cycle $[\overline{\mathcal{M}}_{g,n}^\epsilon(Y, \beta)]^{\mathrm{vir}}$ in the geometric chamber. To do so, we set $X_{g,n,\beta}^{\epsilon,+}$ to be the moduli space parameterizing tuples $(\vec{x}, \vec{p})$ as above, satisfying representability and the geometric-phase nondegeneracy and stability conditions that

$$\mathrm{ord}_q(\vec{x}) \leq \frac{1}{\epsilon}$$

and $L^\epsilon \otimes \omega_{\log}$ is ample. This admits a relative perfect obstruction theory $\mathbb{E}_+^\bullet$ exactly as in (10). Inside $X_{g,n,\beta}^{\epsilon,+}$, we define $X_{g,n,\beta}^{\epsilon,\mathrm{ct},+}$ as the subspace in which $p_1(q_k) = \cdots = p_N(q_k) = 0$ for each marked point q_k, whose relative perfect obstruction theory $\mathbb{E}_{\mathrm{ct},+}^\bullet$ is obtained from $\mathbb{E}_+^\bullet$ by replacing $\omega_{\pi,\log}$ by ω_π. The derivatives of W define a homomorphism out of the obstruction sheaf of $\mathbb{E}_{\mathrm{ct},+}^\bullet$ that descends to a cosection, yielding a virtual cycle supported on $Z_{g,n,\beta}^{\epsilon,+} = \overline{\mathcal{M}}_{g,n}^\epsilon(Y, \beta)$.

It is a highly nontrivial statement that this construction agrees with the virtual cycle defined by Ciocan-Fontanine–Kim–Maulik in [19], which was what appeared in our previous work [21]. In fact, in the case where X_+ is the quintic threefold and $\epsilon \gg 0$, Chang–Li proved in [3] that the two definitions of the virtual cycle yield the same correlators up to a sign. Since then their work has been generalized to a statement on the level of virtual cycles, to quasi-maps, and to more general complete intersections [8, 38, 7]. Thus, while we work with the cosection-localized virtual class in what follows, the results of [21] for ambient insertions can be deduced as a special case.

We note, furthermore, that the analogue of Remark 2.10 in the geometric chamber (known as the orbifold quantum Lefschetz hyperplane principle [44]) requires that $w_i | d_j$ for all i and j, since this condition is equivalent to the requirement that the bundle $\bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}(w_1, \ldots, w_M)}(-d_j)$ be pulled back from the coarse underlying space of $\mathbb{P}(w_1, \ldots, w_M)$.

Recall that in replacing the compact-type state space $\mathcal{H}^{\mathrm{ct}}$ by $\widetilde{\mathcal{H}}$, it was necessary to choose lifts of elements of $\mathcal{H}^{\mathrm{ct}}$ under the map $\eta \colon \widetilde{\mathcal{H}} \to \mathcal{H}^{\mathrm{ct}}$. Now that we have defined correlators, we can more precisely address the issue of their independence of the choice of lift. The statement relies on the following conjecture:

Conjecture 2.13 (Broad vanishing). *Let $X^{ct} \subset X$ denote the zero section,[2] and let $\phi \in H_{\mathrm{CR}}^*(X)$ be such that*

$$e(T_{\overline{\mathcal{I}}X/\overline{\mathcal{I}}X^{ct}})\phi = 0,$$

where $T_{\overline{\mathcal{I}}X/\overline{\mathcal{I}}X^{ct}}$ denotes the relative tangent bundle of the projection $\overline{\mathcal{I}}X \to \overline{\mathcal{I}}X^{ct}$. Then, for any $i \in \{1, \ldots, n\}$, we have

$$\mathrm{ev}_i^*(\phi) \cap [Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}} = 0.$$

For example, Conjecture 2.13 holds in r-spin theory, and the analogue in the geometric phase also holds (see Remark 2.16 below).

---

[2]Although $X^{\mathrm{ct}} = Z$, we use different notation to clarify the parallel in the geometric phase.

Remark 2.14. The name "broad vanishing" refers to the fact that, if the theory satisfies the further condition that $|F(g)| \geq N$ for any $g \in \mathbb{Z}_d$ such that $F(g) \neq \emptyset$ (c.f. [22, Condition **(A2)**]), then Conjecture 2.13 implies $\mathrm{ev}_i^*(\phi) \cap [Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}} = 0$ whenever ϕ is not narrow. Indeed, under this assumption, by dimension considerations, $e(T_{\overline{\mathcal{I}X}/\overline{\mathcal{I}X}^{\mathrm{ct}}})$ vanishes on all noncompact twisted sectors, so any non-narrow element of $H_{\mathrm{CR}}^*(X)$ satisfies $e(T_{\overline{\mathcal{I}X}/\overline{\mathcal{I}X}^{\mathrm{ct}}})\phi = 0$.

From here, the independence of the correlators of the choice of lift under η is nearly immediate:

Lemma 2.15. *Assume Conjecture 2.13, and let* $\phi \in \ker(\eta)$*, where* η *is as in* (4)*. Then*

$$\mathrm{ev}_i^*(\phi) \cap [Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}} = 0$$

for any $i \in \{1, \ldots, n\}$*.*

Proof. By the definition of the Thom isomorphism, the composition of η with the natural restriction map $H_{\mathrm{CR}}^*(X, W^{+\infty}) \to H_{\mathrm{CR}}^*(X)$ is given by

$$\rho \colon H_{\mathrm{CR}}^*(X) \to H_{\mathrm{CR}}^*(X), \qquad \rho(\phi) = e(T_{\overline{\mathcal{I}X}/\overline{\mathcal{I}X}^{\mathrm{ct}}})\phi.$$

Since $\ker(\eta) \subset \ker(\rho)$, the lemma follows from Conjecture 2.13. $\qquad\square$

Remark 2.16. In the geometric phase, Conjecture 2.13 does hold, so the proof of Lemma 2.15 goes through in that setting with no additional assumptions. Indeed, by Remark 2.9, the evaluation maps factor through the inclusion

$$\iota \colon \overline{\mathcal{I}Y} \to \overline{\mathcal{I}}\mathbb{P}(w_1, \ldots, w_M),$$

and by Remark 2.1, we have $\eta = \iota^*$. Thus, in the geometric phase one actually has the stronger statement that $\mathrm{ev}_i^*(\phi) = 0$ whenever $\phi \in \ker(\eta)$.

2.6. **The J-function.** The small J-function for ϵ-stable quasimap theory was defined by Ciocan-Fontanine and Kim in [15] using localization on a graph space, generalizing the original definition in Gromov–Witten theory due to Givental [27, 28, 29] and the interpretation in terms of contraction maps due to Bertram [1]. The appropriate modifications for Landau–Ginzburg theory, which we recall below, were carried out by Ross and the third author in [41].

Let $\mathcal{G}Z_{0,1,\beta}^\epsilon$ denote a "graph space" version of the moduli space $Z_{0,1,\beta}^\epsilon$, which parameterizes the same objects as $Z_{0,1,\beta}^\epsilon$ together with the additional datum of a degree-1 map $C \to \mathbb{P}^1$, or equivalently, a parameterization of one component of C, and the ampleness condition (6) is not required on the parameterized component (see [19, Definition 7.2.1 (4)]). Note that the compatibility condition (8) implies that the multiplicity at the single marked point must be

$$(13) \qquad\qquad m_1 = \left\langle \frac{-\beta - 1}{d} \right\rangle,$$

in which the symbol $\langle a \rangle$ for a rational number a is defined by the requirement that $0 \leq \langle a \rangle < 1$ and $\langle a \rangle \equiv a \mod \mathbb{Z}$.

There is an action of $\mathbb{C}^*$ on $\mathcal{G}Z_{0,1,\beta}^\epsilon$ given by multiplication on the parameterized component $C_0 \cong \mathbb{P}^1$. The fixed loci consist of quasimaps for which the marked point and all of the degree β lies either over 0 or over ∞ on C_0. We denote by $F_\beta^\epsilon \subset \mathcal{G}Z_{0,1,\beta}^\epsilon$ the fixed locus on which the marked point lies at ∞ and all of the degree lies over 0. More precisely, when $\beta > 1/\epsilon$, an

element of F_β^ϵ consists of an ϵ-stable Landau–Ginzburg quasimap to Z attached at a single marked point to C_0, so
$$F_\beta^\epsilon \cong Z_{0,1,\beta}^\epsilon.$$
When $\beta \leq 1/\epsilon$, on the other hand, such a quasimap would not be stable; instead, C_0 is the entire source curve, and the quasimap has a single basepoint of order β at 0. In either case, there is an evaluation map
$$\mathrm{ev}_\bullet \colon F_\beta^\epsilon \to \overline{\mathcal{I}}X,$$
defined by evaluation at the single marked point $\infty \in C_0$.

The evaluation map we require must be slightly modified from the above, in order to account for the orbifold structure of X (see [9, Section 3.1]). Let
$$\mathbf{r}_\beta : H^*(\overline{\mathcal{I}}X) \to H^*(\overline{\mathcal{I}}X)$$
be multiplication by the number d_{m_1}, in which $d_{m_1} := d/\gcd(d \cdot m_1, d)$, and m_1 is as defined by (13), and let
$$\iota : \overline{\mathcal{I}}X \to \overline{\mathcal{I}}X$$
be the involution coming from inversion on the indexing set $\mathbb{Z}_d$ of the twisted sectors, as well as inverting the banding. Then we define
$$(\widetilde{\mathrm{ev}_\bullet})_* = \iota_* \circ \mathbf{r}_\beta \circ (\mathrm{ev}_\bullet)_* \colon H^*_{\mathbb{C}^*}(F_\beta^\epsilon) \to H^*_{\mathbb{C}^*}(\overline{\mathcal{I}}X).$$

Analogously to $Z_{0,1,\beta}^\epsilon$, the graph space admits a virtual cycle defined by cosection localization. From here, the J-function is defined as follows:

Definition 2.17. Let z denote the equivariant parameter for the action of $\mathbb{C}^*$ on $\mathcal{G}Z_{0,1,\beta}^\epsilon$, and let q be a formal Novikov variable. The *small ϵ-stable J-function* is
$$J^\epsilon(q,z) := -z^2 \sum_{\beta \geq 0} q^\beta (\widetilde{\mathrm{ev}_\bullet})_* \left(\frac{[F_\beta^\epsilon]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_{F_\beta^\epsilon/\mathcal{G}Z_{0,1,\beta}^\epsilon}^{\mathrm{vir}})} \right) \in \widetilde{\mathcal{H}}[[q,z,z^{-1}]].$$

Remark 2.18. In the geometric phase, this coincides up to the prefactor of $-z^2$ with the image under the pushforward
$$H^*_{\mathrm{CR}}(Y) \to H^*_{\mathrm{CR}}(X_+)$$
of the small J-function of [15]. The discrepancy in the prefactor is due to two differences between our set-up and theirs. First, we use a one-pointed graph space instead of a zero-pointed graph space (a necessary modification, since the marked point in general carries orbifold structure), which changes the localization contributions by a factor of $-z$ to cancel the contribution of automorphisms moving the unmarked point at ∞. Second, our conventions differ by an overall factor of z; for example, in the case where Y is semi-positive, the J-function of [15] is of the form $I_0(q) + O(z^{-1})$, whereas ours is of the form $I_0(q)z + O(z^0)$.

Using Remark 2.10 and the discussion above, we can make the J-function more explicit. In particular, when $\beta > 1/\epsilon$, we have
$$[F_\beta^\epsilon]^{\mathrm{vir}} = [Z_{0,1,\beta}^\epsilon]^{\mathrm{vir}}$$
and
$$e_{\mathbb{C}^*}(N_{F_\beta^\epsilon/\mathcal{G}Z_{0,1,\beta}^\epsilon}^{\mathrm{vir}}) = z(z - \psi_1),$$
so the terms of J^ϵ with $\beta > 1/\epsilon$ can be expressed as GLSM correlators. The terms of J^ϵ with $\beta \leq 1/\epsilon$ are referred to as *unstable terms*, and can be computed explicitly. For the details of both types of terms, we refer the reader to [23, Section 2.7].

Taking $\epsilon \to 0+$ (that is, requiring the stability condition (6) for all $\epsilon > 0$), every term of the J-function becomes unstable, so one obtains a generating function that can be computed exactly. The result is known as the I-function $I(q, z) := J^{0+}(q, z)$, and can be calculated explicitly, as was carried out for hypersurfaces in [41]. Truncating $I(q, z)$ to powers of q less than or equal to $1/\beta$, more generally, yields an explicit expression for the unstable part of J^ϵ for any ϵ.

We denote by

$$[J^\epsilon]_+(q, z) \in \widetilde{\mathcal{H}}[[q, z]]$$

the part of the J-function with non-negative powers of z, which has contributions only from the unstable terms, and we let $\mu_\beta^\epsilon(z)$ denote the q^β-coefficient in $-z\mathbf{1} + [J]_+^\epsilon(q, z)$:

$$\sum_\beta q^\beta \mu_\beta^\epsilon(z) = -z\mathbf{1} + [J^\epsilon]_+(q, z).$$

This series, which is sometimes called the "mirror transformation", plays a particularly important role in the wall-crossing formula.

We occasionally require a generalization of $J^\epsilon(q, z)$ in which descendent insertions are allowed. This is defined, for $\mathbf{t} = \mathbf{t}(z) \in \widetilde{\mathcal{H}}[\![z]\!]$, by

$$(14) \qquad J^\epsilon(q, \mathbf{t}, z) := -z^2 \sum_{\beta, k \geq 0} \frac{q^\beta}{k!} (\widetilde{\mathrm{ev}}_\bullet)_* \left(\prod_{i=1}^k \mathrm{ev}_i^*(\mathbf{t}(\psi_i)) \cap \frac{[F_{k,\beta}^\epsilon]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N_{F_{k,\beta}^\epsilon/\mathcal{G}Z_{0,k+1,\beta}^\epsilon}^{\mathrm{vir}})} \right),$$

where $F_{k,\beta}^\epsilon \subset \mathcal{G}Z_{0,k+1,\beta}^\epsilon$ is defined as the fixed locus in which all but the last marked point and all of the degree are concentrated over $0 \in C_0$, while the last point marked lies at $\infty \in C_0$. The small J-function is recovered by setting $\mathbf{t} = 0$.

3. STATEMENT OF RESULTS

Ciocan-Fontanine and Kim conjectured wall-crossing formulas for stable quasimap invariants in [15, 18], and they have proven these conjectures in many cases; in [17], they prove a wall-crossing theorem in all genus for any projective complete intersection. In this section, we review Ciocan-Fontanine–Kim's conjecture in the Landau–Ginzburg context, and we state our main theorem in precise form.

3.1. **Wall-crossing for J-functions.**
In genus zero, the wall-crossing conjecture states that the function $J^\epsilon(q, z)$ lies on the Lagrangian cone defined by the ∞-stable hybrid theory— that is, there exists $\mathbf{t} \in \mathcal{H}[[z]]$ such that $J^\epsilon(q, z) = J^\infty(q, \mathbf{t}, z)$. The $\mathbf{t}$ in this equation can be determined explicitly from the fact that $J^\infty(q, \mathbf{t}, z) = z\mathbf{1} + \mathbf{t}(-z) + O(z^{-1})$, so we must have

$$(15) \qquad J^\epsilon(q, z) = J^\infty(q, z\mathbf{1} + [J^\epsilon]_+(q, -z), z) = J^\infty \left(q, \sum_\beta q^\beta \mu_\beta^\epsilon(-z), z \right).$$

This statement was proved in the hypersurface case by Ross and the third author [41], and in the hybrid case by Ross and the first author [23]. In the geometric phase, (15) was proved by Cheong–Ciocan-Fontanine–Kim [15, 9].

3.2. **Wall-crossing for virtual cycles.** More generally, Ciocan-Fontanine and Kim lift their conjecture in any genus to the level of virtual cycles. To state the analogue in Landau–Ginzburg theory, which is the statement of our main theorem in the negative phase, we require some further notation. The relevant ideas are based on [15, Section 3.2].

Let $\vec{\beta} = (\beta_1, \ldots, \beta_k)$ be a tuple of nonnegative integers with $\beta_i \leq 1/\epsilon$ for each i. Let $m_i := \left\langle \frac{\beta_i + 1}{d} \right\rangle$ for all i, and let $\vec{m} = (m_1, \ldots, m_k)$. Then there is a morphism

$$b_{\vec{\beta}} \colon Z^{\epsilon}_{g, n + \vec{m}, \beta - \sum_{i=1}^{k} \beta_i} \to Z^{\epsilon}_{g, n, \beta},$$

defined as follows. For $(C; q_1, \ldots, q_{n+k}; L; \vec{p}) \in Z^{\epsilon}_{g, n + \vec{m}, \beta - \sum_{i=1}^{k} \beta_i}$, let $\widetilde{C}$ be the partial coarsening of C obtained by forgetting the last k marked points and their orbifold structure. The bundle

$$L \otimes \mathcal{O} \left(\sum_{i=1}^{k} \frac{-\beta_i - 1}{d} [\bar{q}_{n+i}] \right)$$

has multiplicity zero at each of the last k marked points, and hence it is pulled back from a bundle $\widetilde{L}$ on $\widetilde{C}$. For $j = 1, \ldots, N$, let

$$\widetilde{p}_j \in \Gamma \left(\widetilde{L}^{\otimes -d} \otimes \omega^{\widetilde{C}}_{\log} \right)$$
$$= \Gamma \left(\left(\left(\widetilde{L}^{\otimes -d} \left(\sum_{i=1}^{k} (-\beta_i - 1) [\bar{q}_{n+i}] \right) \otimes \omega^{\widetilde{C}}_{\log} \left(\sum_{i=1}^{k} [\bar{q}_{n+i}] \right) \right) \otimes \mathcal{O} \left(\sum_{i=1}^{k} \beta_i [\bar{q}_{n+i}] \right) \right)$$

be the section on $\widetilde{C}$ obtained from $p_j \in \Gamma(L^{\otimes -d} \otimes \omega^{C}_{\log})$ by the natural inclusion $\mathcal{O} \subset \mathcal{O}(\sum_i \beta_i [\bar{q}_{n+i}])$. Then, setting $\widetilde{\vec{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_N)$, we define

$$b_{\vec{\beta}}(C; q_1, \ldots, q_{n+k}; L; \vec{p}) := (\widetilde{C}; q_1, \ldots, q_n; \widetilde{L}; \widetilde{\vec{p}}),$$

assuming the latter is stable. It is possible, however, that $\widetilde{C}$ contains rational tails—that is, genus-zero components with a single special point—on which

$$(16) \qquad \deg \left(\widetilde{L}^{\otimes -d} \otimes \omega^{\widetilde{C}}_{\log} \right) \leq \frac{1}{\epsilon},$$

and which thus violate the ampleness condition (6). To define $b_{\vec{\beta}}$, we contract such a component and replace it with a basepoint of order equal to the left-hand side of (16) at the point where the component was attached, as formalized below. This may create a new rational tail, so we repeat the process inductively until stability is achieved.

The definition of the morphism

$$c \colon Z^{\infty}_{g, n, \beta} \to Z^{\epsilon}_{g, n, \beta},$$

which contracts unstable rational tails and replaces them with basepoints, is similar. Namely, suppose that $(C; q_1, \ldots, q_n; L; \vec{p}) \in Z^{\infty}_{g, n, \beta}$, where C has a rational tail C_0 and

$$\beta_0 := \deg(L^{\otimes -d} \otimes \omega^{C}_{\log}|_{C_0}) \leq 1/\epsilon.$$

Let $\widetilde{C}$ be obtained from $\overline{C \setminus C_0}$ by forgetting the orbifold structure at the point q where C_0 meets the rest of C. Then the line bundle

$$L|_{\overline{C \setminus C_0}} \otimes \mathcal{O} \left(\frac{-\beta_0 - 1}{d} [\bar{q}] \right)$$

has multiplicity zero at q, so as above, it is pulled back from a bundle $\widetilde{L}$ on $\widetilde{C}$. For $j = 1, \ldots, N$, we let

$$\widetilde{p}_j \in \Gamma\left(\widetilde{L}^{\otimes -d} \otimes \omega_{\log}^{\widetilde{C}}\right)$$

$$= \Gamma\left(\left(\widetilde{L}^{\otimes -d}((-\beta_0 - 1)[\overline{q}]) \otimes \omega^{\widetilde{C}_{\log}}([\overline{q}])\right) \otimes \mathcal{O}\left(\beta_0[\overline{q}]\right)\right)$$

be obtained from the restriction of p_j to $\overline{C \setminus C_0}$. We then set $\widetilde{\vec{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_N)$ and let

$$c(C; q_1, \ldots, q_n; L; \vec{p}) := (\widetilde{C}; q_1, \ldots, q_n; \widetilde{L}; \widetilde{\vec{p}}),$$

assuming the latter is stable. If not, we iterate the procedure until we reach an ϵ-stable Landau–Ginzburg quasimap.

By a slight abuse of notation, we extend ev_i^* to $\widetilde{\mathcal{H}}[\psi_i]$ by linearity in ψ_i, and similarly, we allow $b_{\vec{\beta}*}$ and c_* to operate linearly in q. Equipped with these definitions, we can give a precise statement of Theorem 1.2 in the Landau–Ginzburg phase:

Theorem 3.1. *Fix $g \geq 0$, $n \geq 1$, and a tuple of insertions $\alpha_1, \ldots, \alpha_n \in \widetilde{\mathcal{H}}$. Then*

$$\sum_{\beta \geq 0} q^\beta \left(\prod_{a=1}^n \mathrm{ev}_a^*(\alpha_a) \cap [Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}}\right) =$$

$$\sum_{k=0}^\infty \sum_{\beta_0 \geq 0} \sum_{0 \leq \beta_1, \ldots, \beta_k \leq 1/\epsilon} \frac{q^{\beta_0}}{k!} b_{\vec{\beta}*} c_* \left(\prod_{a=1}^n \mathrm{ev}_a^*(\alpha_a) \prod_{i=1}^k q^{\beta_i} \mathrm{ev}_{n+i}^*(\mu_{\beta_i}^\epsilon(-\psi_{n+i})) \cap [Z_{g,n+k,\beta_0}^\infty]^{\mathrm{vir}}\right).$$

(Note that, by the definition of $\mu_\beta^\epsilon(z)$, the expression inside $c_*(\cdots)$ is supported on the substack of $Z_{g,n+k,\beta_0}^\infty$ on which the multiplicity at the marked point q_{n+i} is $\langle\frac{\beta_i+1}{d}\rangle$, so the morphism $b_{\vec{\beta}}$ is well-defined.)

Remark 3.2. Note that, in contrast to the statement of Theorem 1.2, we now assume that $n \geq 1$. The $n = 0$ case is handled separately in the appendix.

Remark 3.3. The analogue of Theorem 3.1 in the geometric phase is Theorem 2.6 of [21], except that in that case (1) the insertions are not required to be ambient, and (2) the weights w_i are all equal to one. Thus, assuming the equivalence of Ciocan-Fontanine–Kim–Maulik's virtual cycle and the cosection-localized virtual cycle (Remark 2.12), the proof discussed below reproduces, in a fundamentally similar but more complicated way, the ambient case of [21].

3.3. **Twisted theory.** In order to prove Theorem 3.1, we must introduce a twist of the virtual class by a certain equivariant Euler class.

Denoting the universal curve over $Z_{g,n,\beta}^\epsilon$ again by $\pi \colon \mathcal{C} \to Z_{g,n,\beta}^\epsilon$ and the universal line bundle on $\mathcal{C}$ by $\mathcal{L}$, we let $\mathbb{C}^*$ act trivially on $Z_{g,n,\beta}^\epsilon$. We denote the equivariant parameter by λ, and to distinguish this action from the one on the graph space considered above, we write $\mathbb{C}^*$ as $\mathbb{C}_\lambda^*$. Furthermore, we denote by $\mathbb{C}_{(\lambda)}$ a nonequivariantly trivial line bundle with a $\mathbb{C}^*$-action of weight 1.

Define

$$[Z_{g,n,\beta}^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}} := \frac{[Z_{g,n,\beta}^\epsilon]^{\mathrm{vir}}}{e_{\mathbb{C}_\lambda^*}\left(R\pi_*\left(\mathcal{P}^\vee \otimes \mathbb{C}_{(\lambda)}\right)\right)},$$

where
$$\mathcal{P} := \mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}.$$
We can also define a twisted J-function by

$$J^{\epsilon}_{\text{tw}}(q,z) := -z^2 e_{\mathbb{C}^*_\lambda}(\mathcal{O}_X(-1) \otimes \mathbb{C}_{(\lambda)}) \sum_{\beta \geq 0} q^{\beta}(\widetilde{\text{ev}_\bullet})_* \left(\frac{[F^{\epsilon}_{\beta}]^{\text{vir}} \cap e_{\mathbb{C}^*_\lambda \times \mathbb{C}^*_z}(-R\pi_*(\mathcal{P}^\vee \otimes \mathbb{C}_{(\lambda)}))}{e_{\mathbb{C}^*_z}(N^{\text{vir}}_{F^{\epsilon}_{\beta}/\mathcal{G}Z^{\epsilon}_{0,1,\beta}})} \right),$$

where $\mathbb{C}^*_z$ denotes the $\mathbb{C}^*$-action on the graph space, and where $\mathcal{O}(-1)$ is the tautological line bundle, which descends to $\overline{\mathcal{I}}X$. From here, we define a twisted mirror transformation by

$$\sum_{\beta} q^{\beta} \mu^{\epsilon,\text{tw}}_{\beta}(z) = -\mathbf{1}z + [J^{\epsilon}_{\text{tw}}(z)]_+ \in \widetilde{H}(\lambda)[\![q,z]\!].$$

The twisted version of Theorem 3.1 is the following:

Theorem 3.4. *Fix $g \geq 0$, $n \geq 1$, and a tuple of insertions $\alpha_1, \ldots, \alpha_n \in \widetilde{\mathcal{H}}$. Then*

$$\sum_{\beta \geq 0} q^{\beta} \left(\prod_{a=1}^{n} \text{ev}^*_a(\alpha_a) \cap [Z^{\epsilon}_{g,n,\beta}]^{\text{vir}}_{\text{tw}} \right) =$$

$$\sum_{k=0}^{\infty} \sum_{\beta_0,\beta_1,\ldots,\beta_k \geq 0} \frac{q^{\beta_0}}{k!} b_{\vec{\beta}*} c_* \left(\prod_{a=1}^{n} \text{ev}^*_a(\alpha_a) \prod_{i=1}^{k} q^{\beta_i} \text{ev}^*_{n+i}(\mu^{\epsilon,\text{tw}}_{\beta_i}(-\psi_{n+i})) \cap [Z^{\infty}_{g,n+k,\beta_0}]^{\text{vir}}_{\text{tw}} \right).$$

In fact, the twisted wall-crossing theorem implies the untwisted one:

Lemma 3.5. *Theorem 3.4 implies Theorem 3.1.*

Proof. The proof is identical to Lemma 2.10 of [21]: one recovers the untwisted theorem from the twisted one by taking the top power of λ on both sides. $\square$

Remark 3.6. In the geometric phase, the definitions of $[\overline{\mathcal{M}}^{\epsilon}_{g,n}(Y,\beta)]^{\text{vir}}_{\text{tw}}$, J^{ϵ}_{tw}, and $\mu^{\epsilon,\text{tw}}_{\beta}$ are exactly parallel to the above, except that $\mathcal{P}$ is replaced by $\mathcal{L}$; see [21, Section 2.4].

3.4. **Wall-crossing for dual graphs.** The quasimap wall-crossing of [21] uses a factorization property of the virtual cycle along nodes that is not in general known to hold for the GLSM. Thus, the proof of Theorem 3.4 requires a new idea: an induction on topological types, for which we must introduce one further refinement of the wall-crossing statement.

Let Γ be a prestable dual graph—that is, a set of vertices, edges, and numbered legs—for which each vertex v is decorated with a genus $g(v) \geq 0$ and a degree $\beta(v) \geq 0$, and each half-edge h (including the legs) is decorated with a multiplicity $m(h) \in \{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\}$. We assume that for each edge e, the multiplicities at the two half-edges h and h' satisfy

$$(17) \qquad\qquad m(h) + m(h') \in \mathbb{Z}$$

and for each vertex v, as a consequence of (8), the multiplicities at the set $H(v)$ of incident half-edges satisfy

$$(18) \qquad\qquad \frac{-\beta(v) + 2g(v) - 2 + |H(v)|}{d} - \sum_{h \in H(v)} m(h) \in \mathbb{Z}.$$

Denote by $\vec{m}(v)$ the tuple $\{m(h)\}_{h \in H(v)}$.

In addition, we equip Γ with a number of further decorations. Namely, let

$$v_\bullet \in V(\Gamma)$$

be any vertex such that $\beta(v_\bullet) > 0$, and let

$$n' \colon V(\Gamma) \to \mathbb{Z}^{\geq 0}$$

be any function with $n'(v_\bullet) = 0$. We denote by $\Gamma_{n'}$ the graph obtained from Γ by adding $n'(v)$ additional legs to each vertex v and assigning multiplicity $\frac{1}{d}$ to each of them, and we say that $(\Gamma, v_\bullet, n')$ is *stable* if $\Gamma_{n'}$ is the dual graph of an element of $Z_{g,n+n',\beta}^\epsilon$.

Similarly to the recursive structure of the boundary of the moduli space of curves, for any stable $(\Gamma, v_\bullet, n')$ and any stability parameter ϵ, we construct an explicit fiber product Z_Γ^ϵ and a morphism

$$\iota_\Gamma \colon Z_\Gamma^\epsilon \to Z_{g,n+n',\beta}^\epsilon$$

that is a finite cover of the closure of the locus of ϵ-stable Landau–Ginzburg quasimaps with decorated dual graph $\Gamma_{n'}$. Here, denoting by $V(\Gamma)$, $E(\Gamma)$, and $L(\Gamma)$ the vertex, edge and leg sets, respectively, we have

(19)
$$g = h^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v),$$
$$\beta = \sum_{v \in V(\Gamma)} \beta(v),$$
$$n = |L(\Gamma)|,$$
$$n' = \sum_{v \in V(\Gamma)} n'(v).$$

Explicitly, for each vertex $v \in V(\Gamma)$, denote by $\vec{m}'(v)$ the tuple of multiplicities at half-edges incident to v together with $n'(v)$ additional multiplicities of $\frac{1}{d}$. Then Z_Γ^ϵ is the fiber product of the moduli spaces $Z_{g(v),\vec{m}'(v),\beta(v)}^\epsilon$ for each $v \in V(\Gamma)$, where the fiber product is over $(\overline{\mathcal{I}Z})^{|E(V)|}$ via the evaluation maps at the half-edges and for exactly one of the half-edges of each edge, we compose the evaluation map with the inversion of band automorphism. (The reason for this inversion is related to (17).) The degree of ι_Γ onto its image is given by

(20)
$$\frac{|\operatorname{Aut}(\Gamma_{n'})|}{\prod_h d_{m(h)}},$$

where the product is over all half-edges. Note that Z_Γ^ϵ depends on the choice of n', although we suppress it from the notation.

There is also another description of Z_Γ^ϵ useful for the definition its virtual class. Let D_Γ be the fiber product of the stacks $D_{g(v),n(v)+n'(v),\beta(v)}$ of curves with a line bundle (see Section 2.5), which is defined analogously to Z_Γ^ϵ. This stack comes with a local complete intersection morphism $j_\Gamma \colon D_\Gamma \to D_{g,n+n',\beta}$ (see [4, Lemma 4.4]). Using the cartesian diagram

$$
\begin{array}{ccc}
Z_\Gamma^\epsilon & \xrightarrow{\iota_\Gamma} & Z_{g,n+n',\beta}^\epsilon \\
\downarrow & & \downarrow \\
D_\Gamma & \xrightarrow{j_\Gamma} & D_{g,n+n',\beta},
\end{array}
$$

(21)

we can define a virtual cycle for Z_Γ^ϵ: namely, defining a fiber product $X_\Gamma^{\epsilon,\mathrm{ct}}$ in analogy to Section 2.5, we set $[Z_\Gamma^\epsilon]^{\mathrm{vir}} := [X_\Gamma^{\epsilon,\mathrm{ct}}]_\sigma^{\mathrm{vir}}$ by virtual pullback along ι_Γ. Alternatively, we can define a relative perfect obstruction theory similarly to (11), except that we now utilize the

universal curve over $X_\Gamma^{\epsilon,\mathrm{ct}}$ and the base of the obstruction theory is D_Γ. Then $[Z_\Gamma^\epsilon]^{\mathrm{vir}}$ is a cosection localized virtual class of $X_\Gamma^{\epsilon,\mathrm{ct}}$. Let us point out that there is an additional subtlety in the defininition of the fiber product: in addition to the inversion of band automorphism mentioned in the previous paragraph, we need to compose one of the evaluation maps at each edge with another automorphism, as explained on [24, Page 28].

Similarly to Section 3.3, there is also a twisted virtual cycle for Z_Γ^ϵ, but this requires the additional choice of a coloring of the vertices

$$V(\Gamma) = V_0 \sqcup V_\infty$$

such that $v_\bullet \in V_0$. For any such coloring, we define

$$(22) \qquad [Z_\Gamma^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}} = \frac{[Z_\Gamma^\epsilon]^{\mathrm{vir}}}{\prod_{v \in V_0} e_{\mathbb{C}_\lambda^*}(R\pi_{v,*}(\mathcal{P}^\vee) \otimes \mathbb{C}_{(\lambda)}) \prod_{v \in V_\infty} e_{\mathbb{C}_\lambda^*}(R\pi_{v,*}(\mathcal{P}) \otimes \mathbb{C}_{(-\lambda)})},$$

in which $\pi_v : \mathcal{C}_v \to Z_\Gamma^\epsilon$ denotes the projection from the component $\mathcal{C}_v$ of the universal curve corresponding to the vertex v—that is, $\mathcal{C}_v$ is the pullback of the universal curve over $Z_{g(v),\vec{m}'(v),\beta(v)}^\epsilon$ under the projection $Z_\Gamma^\epsilon \to Z_{g(v),\vec{m}'(v),\beta(v)}^\epsilon$.

Our goal, now, is to state a wall-crossing theorem for each choice of decorated dual graph. We require a bit of notation. First, for each $l \geq 0$, let $\Gamma + l$ be the graph obtained from Γ by adding l additional legs of multiplicity $\frac{1}{d}$ to the vertex $v_\bullet$, and let

$$\pi_l \colon Z_{\Gamma+l}^\epsilon \to Z_\Gamma^\epsilon$$

be the morphism that forgets these additional legs.

The insertions at these l additional legs in the wall-crossing theorem are expressed in terms of a universal series

$$\epsilon^{\lambda_0}(z) \in R[[z]],$$

where the ground ring is $R := \widetilde{\mathcal{H}}(\lambda)[[y]]$ for a formal parameter y that is given geometric meaning in what follows. The precise definition of $\epsilon^{\lambda_0}(z)$ is given in Sections 4.4 and 5 below, in terms of the contribution to a virtual localization from unmarked trees of rational curves; for now, we require just one property:

$$(23) \qquad \epsilon^{\lambda_0}(z) \in y \cdot R[[z]],$$

which is immediate from the definition in Section 4.4.

For each $i \in \{1, \ldots, l\}$, let

$$\mathrm{ev}_{v_\bullet,i} \colon Z_{\Gamma+l}^\epsilon \to X_{(1/d)} \subset \overline{\mathcal{I}}X$$

be the evaluation map at the ith of the l additional markings on $v_\bullet$, which by construction maps to the multiplicity-$\frac{1}{d}$ sector $X_{(1/d)}$ (the summand $a = 1$ in (9)) in $\overline{\mathcal{I}}X$.

Theorem 3.7. *Fix $g \geq 0$, $\beta \geq 0$, and $n \geq 1$, and let Γ be a prestable dual graph with these discrete data. Fix decorations $v_\bullet$ and $n' \colon V(\Gamma) \to \mathbb{Z}^{\geq 0}$ such that $(\Gamma, v_\bullet, n')$ is stable. Then,*

for any coloring $V(\Gamma) = V_0 \cup V_\infty$ as above and any tuple of insertions $\alpha_1, \ldots, \alpha_n \in \widetilde{\mathcal{H}}$,

(24)

$$
\sum_{l=0}^\infty \frac{1}{l!} \pi_{l*} \left(\prod_{a \in L'} \mathrm{ev}_{v_\bullet, a}^*(\epsilon^{\lambda_0}(\psi)) \prod_{\substack{v \in V(\Gamma) \\ a \in L(v)}} \mathrm{ev}_{v,a}^*(\alpha_a) \cap [Z_{\Gamma+l}^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}} \right) = \sum_{l=0}^\infty \frac{1}{l!} \pi_{l*} \left(\sum_{K=\{k_v\}_{v \in V(\Gamma)}} \frac{1}{\prod_v k_v!} \right.
$$

$$
\left. \sum_{\substack{\vec{\beta} = \{\beta_i^v\}_{v \in V(\Gamma), 0 \le i \le k_v} \\ \beta_0^v + \beta_1^v + \cdots + \beta_{k_v}^v = \beta(v)}} b_{\vec{\beta}*} c_* \left(\prod_{a \in L'} \mathrm{ev}_{v_\bullet, a}^*(\epsilon^{\lambda_0}(\psi)) \prod_{\substack{v \in V(\Gamma) \\ a \in L(v)}} \mathrm{ev}_{v,a}^*(\alpha_a) \prod_{\substack{v \in V(\Gamma) \\ j \in [k_v]}} \mathrm{ev}_{v,j}^*(\mu_{\beta_j^v}^{\epsilon, \mathrm{tw}}(-\psi)) \cap [Z_{\Gamma_{K,\vec{\beta}}+l}^\infty]_{\mathrm{tw}}^{\mathrm{vir}} \right) \right),
$$

where $L(v)$ denotes the leg set of v (excluding the extra legs specified by n'), L' denotes the set of ℓ extra legs on $v_\bullet$, and $\mathrm{ev}_{v,a}$ denote the evaluation maps at the corresponding marked points. The morphisms $b_{\vec{\beta}}$ and c are defined analogously to Section 3.2, and $\Gamma_{K,\vec{\beta}}$ is the graph obtained from Γ by modifying each vertex v to have degree β_0^v and k_v additional legs.

Remark 3.8. Although we leave the multiplicities at the k_v additional legs of each vertex unspecified, the expression on the right-hand side is in fact supported on the locus where the multiplicities are $\left\langle \frac{\beta_1^v + 1}{d} \right\rangle, \ldots, \left\langle \frac{\beta_{k_v}^v + 1}{d} \right\rangle$.

This statement implies the twisted, and hence the untwisted, wall-crossing of the previous subsection:

Lemma 3.9. *Theorem 3.7 implies Theorem 3.4.*

Proof. Take Γ to be a graph with a single vertex v, and the additional decorations to be $v = v_\bullet \in V_0$ and $n'(v) = 0$. Then $Z_\Gamma^\epsilon = Z_{g,n,\beta}^\epsilon$ and $Z_{\Gamma_{K,\vec{\beta}}}^\infty = Z_{g,n+k,\beta_0}^\infty$. The resulting equality depends on the parameter y, but setting $y = 0$ and applying (23), we see that only the $l = 0$ term contributes. This term is precisely the statement of Theorem 3.4. $\square$

Remark 3.10. The definition of Z_Γ^ϵ and its virtual cycle via (21) work equally well in the geometric phase, after replacing the condition (18) by

$$
\beta(v) - \sum_{h \in H(v)} m(h) \in \mathbb{Z}.
$$

Replacing $\mathcal{P}$ by $\mathcal{L}$ in (22) gives the definition of the twisted virtual cycle. The additional legs in $\Gamma + l$ should have multiplicity zero in the geometric phase (see Remark 2.8), but after this modification, Theorem 3.7 and Lemma 3.9 generalize immediately.

4. TWISTED GRAPH SPACE

The proof of Theorem 3.7 is by $\mathbb{C}^*$-localization on a "twisted graph space." This space, which is closely related to the space of mixed-spin p-fields considered by Chang–Li–Li–Liu [5, 6], was introduced in our previous work [21] in the geometric phase. We adapt the definition to the Landau–Ginzburg phase in this section and derive the properties we require for the proof of Theorem 3.7.

4.1. **Definition of the twisted graph space.** In the language of the GLSM, the twisted graph space $\mathbb{P}X^\epsilon_{g,n,\beta,\delta}$ consists of Landau–Ginzburg quasimaps to the GIT quotient

$$\mathbb{P}X := ((\mathbb{C}^M \times \mathbb{C}^N) \times \mathbb{C}^2) /\!\!/_{\widetilde{\theta}} (\mathbb{C}^*)^2,$$

in which the action of $(\mathbb{C}^*)^2$ is

$$(g,t) \cdot (\vec{x}, \vec{p}, z_1, z_2) = (g^{w_1}x_1, \ldots, g^{w_M}x_M, g^{-d}p_1, \ldots, g^{-d}p_N, g^d t z_1, t z_2)$$

and the rational character $\widetilde{\theta} \colon (\mathbb{C}^*)^2 \to \mathbb{C}^*$ (c.f. Remark 2.3) is

$$\widetilde{\theta}(g,t) = \theta(g)^\epsilon t^{3+\epsilon}$$

for the negative character θ used to define X. (As we will see shortly, this choice of $\widetilde{\theta}$ is chosen carefully to avoid basepoints with $z_1 = z_2 = 0$, and to have the loci of the moduli space where $z_1 \equiv 0$ or $z_2 \equiv 0$ equal to moduli spaces of Landau–Ginzburg quasimaps to X for different stability conditions.) The superpotential is extended trivially to the new factors, and the R-charge acts with weight 1 on the p-coordinates, weight -1 on z_1, and weight 0 on all other coordinates.

More concretely,

$$\mathbb{P}X^\epsilon_{g,n,\beta,\delta} = \{(C; q_1, \ldots, q_n; L_1, L_2; x_1, \ldots, x_M, p_1, \ldots, p_N, z_1, z_2)\},$$

where $(C; q_1, \ldots, q_n)$ is an n-pointed prestable orbifold curve of genus g with nontrivial isotropy only at special points, L_1 and L_2 are orbifold line bundles with $P_1 := L_1^{\otimes -d} \otimes \omega_{\log}$,

(25) $$\beta = \deg(P_1), \quad \delta = \deg(L_2),$$

and

$$(\vec{x}, \vec{p}, \vec{z}) := (x_1, \ldots, x_M, p_1, \ldots, p_N, z_1, z_2) \in \Gamma \left(\bigoplus_{i=1}^M L_1^{\otimes w_i} \oplus \bigoplus_{j=1}^N P_1 \oplus (P_1^\vee \otimes L_2) \oplus L_2 \right).$$

We require this data to satisfy the following conditions:

- *Representability*: For every $q \in C$ with isotropy group G_q, the homomorphism $G_q \to \mathbb{C}^*$ giving the action of the isotropy group on the bundle L_1 is injective.
- *Nondegeneracy*: The zero set of $\vec{p}$ is disjoint from the marked points and nodes of C, and the sections z_1 and z_2 never simultaneously vanish. Furthermore, for each point q of C at which $z_2(q) \neq 0$, we have

(26) $$\operatorname{ord}_q(\vec{p}) \leq \frac{1}{\epsilon},$$

  and for each point q of C at which $z_2(q) = 0$, we have

(27) $$\operatorname{ord}_q(\vec{p}) = 0,$$

  or in other words, $\vec{p}(q) \neq 0$. (We note that this can be phrased, as in [16, Section 2.1], as a length condition bounding the order of contact of $(\vec{x}, \vec{p}, \vec{z})$ with the unstable locus of $\mathbb{P}X$.)
- *Stability*: The $\mathbb{Q}$-line bundle

(28) $$(P_1^\epsilon \otimes L_2^{\otimes 3+\epsilon}) \otimes \omega_{\log}$$

  is ample.

This definition is chosen to coincide with the stable Landau–Ginzburg quasimaps to $\mathbb{P}X$ defined in [25, Definition 4.2.11] with stability parameter $\epsilon' = 1$. In our case (see the proof of Proposition 2.4), we have $G \cong (\mathbb{C}^*)^2$ and $\Gamma \cong G \times \mathbb{C}_R^*$, and we have chosen the good lift $\widetilde{\vartheta}$ of the rational character $\widetilde{\theta}$ with $\widetilde{\vartheta}(g, t, \mu) = \vartheta(g, \mu) \cdot t^{3+\epsilon}$.

We decompose $\mathbb{P}X^\epsilon_{g,n,\beta,\delta}$ by multiplicities, denoting by

$$\mathbb{P}X^\epsilon_{g,\vec{m},\beta,\delta} \subset \mathbb{P}X^\epsilon_{g,n,\beta,\delta}$$

the open and closed substack on which the multiplicity of L_1 at the marked point q_i is m_i. (The multiplicity of L_2 is always zero, since otherwise the sections z_1 and z_2 would both vanish.)

Note that the nondegeneracy condition implies that z_1 and z_2 define a map

$$f \colon C \to \mathbb{P}(P_1^\vee \oplus \mathcal{O}_C).$$

Analogously to what was stated in Section 2.5, we let

$$\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta} \subset \mathbb{P}X^\epsilon_{g,n,\beta,\delta}$$

be the locus on which $x_1(q_k) = \cdots = x_M(q_k) = 0$ for each marked point q_k. Then there is a relative perfect obstruction theory[3]

$$\widetilde{\mathbb{E}}^\bullet := \left(R\pi_* \left(\bigoplus_{i=1}^M \left(\mathcal{L}_1^{\otimes w_i}\left(-\sum_{k=1}^n \Delta_k\right)\right) \oplus \bigoplus_{j=1}^N \mathcal{P}_1 \right) \oplus R\pi_* f^* T_{\mathbb{P}(\mathcal{P}_1^\vee \oplus \mathcal{O})/\mathcal{C}} \right)^\vee \to \mathbb{L}^\bullet_{\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}/D_{g,n,\beta}}.$$

The derivatives of the superpotential once again define a homomorphism out of the relative obstruction sheaf, which descends to a cosection

$$\widetilde{\sigma} \colon \mathcal{O}b_{\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}} \to \mathcal{O}_{\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}}.$$

The degeneracy locus of $\widetilde{\sigma}$ is the substack

$$\mathbb{P}Z^\epsilon_{g,n,\beta,\delta} \subset \mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}$$

parameterizing Landau–Ginzburg quasimaps that land in the critical locus $\mathbb{P}Z \subset \mathbb{P}X$ of the extended superpotential $\widetilde{W} := \sum_{j=1}^N p_j F_j(\vec{x})$. Explicitly, $\mathbb{P}Z$ is a $\mathbb{P}^1$-bundle over $Z = \mathbb{P}(\vec{d})$ and $\mathbb{P}Z^\epsilon_{g,\vec{m},\beta,\delta}$ is the locus in $\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}$ on which the sections $x_1, \ldots, x_M$ are all identically zero. This locus is proper, by the existence of the good lift ϑ described above. The cosection construction thus yields a virtual cycle

$$[\mathbb{P}Z^\epsilon_{g,n,\beta,\delta}]^{\mathrm{vir}} \in A_*(\mathbb{P}Z^\epsilon_{g,n,\beta,\delta}),$$

and a similar computation to (12) shows that

$$\mathrm{vdim}(\mathbb{P}Z^\epsilon_{g,n,\beta,\delta}) = \mathrm{vdim}(Z^\epsilon_{g,n,\beta}) + 2\delta - \beta + 1 - g.$$

To define evaluation maps, let

$$\varsigma \in \Gamma(\mathcal{P}_1^{\oplus N} \oplus (\mathcal{P}_1^\vee \otimes \mathcal{L}_2) \oplus \mathcal{L}_2)$$

---

[3]This is not exactly the standard definition of the perfect obstruction theory of a GLSM moduli space, which in this case would be relative to a moduli space parameterizing $(C; q_1, \ldots, q_n; L_1, L_2)$. However, by the Euler sequence, the resulting (cosection-localized) virtual cycles agree.

denote the universal section over the universal curve $\pi\colon \mathcal{C} \to \mathbb{P}Z^\epsilon_{g,n,\beta,\delta}$. If $\Delta_i \subset \mathcal{C}$ denotes the divisor corresponding to the ith marked point, then

$$\varsigma|_{\Delta_i} \in \Gamma\left(\bigoplus_{j=1}^N \left(\mathcal{L}_1^{\otimes -d} \otimes \mathcal{O}\left(-\sum_{k=1}^n \Delta_k\right)\right) \oplus \left(\mathcal{L}_1^{\otimes d} \otimes \mathcal{L}_2\right) \oplus \mathcal{L}_2\Big|_{\Delta_i}\right),$$

since $\omega_{\pi,\log}|_{\Delta_i}$ is trivial. Thus, restricting $\varsigma|_{\Delta_i}$ to a fiber gives an element of $\overline{\mathcal{I}}\mathbb{P}Z \subset \overline{\mathcal{I}}\mathbb{P}X$, and we obtain an evaluation map

$$\mathrm{ev}_i \colon \mathbb{P}Z^\epsilon_{g,n,\beta,\delta} \to \overline{\mathcal{I}}\mathbb{P}X$$

that sends $\mathbb{P}Z^\epsilon_{g,\vec{m},\beta,\delta}$ to the twisted sector indexed by $e^{2\pi i m_i} \in \mathbb{Z}_d$. Analogously to the definition of the compact-type state space in Section 2.2, the insertions for the twisted graph space are drawn from $H^*_{\mathrm{CR}}(\mathbb{P}X)$, which, as a vector space, is isomorphic to $\widetilde{\mathcal{H}} \otimes H^*(\mathbb{P}^1)$.

As in Section 3.4, one can define substacks of $\mathbb{P}Z^\epsilon_{g,n,\beta,\delta}$ and finite covers thereof associated to decorated dual graphs. Namely, let $(\Gamma, v_\bullet, n')$ be as in Section 3.4, and define $\widetilde{\Gamma}$ by adding an additional decoration

$$D\colon V(\Gamma) \to \mathbb{Z}^{\geq 0}$$

to each vertex of Γ. We say that $(\widetilde{\Gamma}, v_\bullet, n')$ is *stable* if, after adding $n'(v)$ additional legs to each vertex v, the graph $\widetilde{\Gamma}$ becomes the dual graph of an element of $\mathbb{P}Z^\epsilon_{g,\vec{m}+n',\beta,\delta}$.

For any stable decorated graph $(\widetilde{\Gamma}, v_\bullet, n')$ as above, and any stability parameter ϵ, we denote by $\mathbb{P}Z^\epsilon_{\widetilde{\Gamma}}$ the fiber product of the moduli spaces $\mathbb{P}Z^\epsilon_{g(v),\vec{m}'(v),\beta(v),D(v)}$ over $\prod_{e \in E(\widetilde{\Gamma})} \overline{\mathcal{I}}\mathbb{P}Z$. Here, as above, $\vec{m}'(v)$ records the multiplicities at half-edges incident to v as well as $n'(v)$ additional multiplicities of $\frac{1}{d}$. This fiber product imposes the agreement, for each edge $e \in E(\widetilde{\Gamma})$ with half-edges h_1 and h_2 incident to vertices v_1 and v_2, of the evaluation maps corresponding to h_1 and h_2. (There are two subtleties in the fiber product, discussed on [24, page 28], but they are not relevant to what follows.) As before, there is a virtual local complete intersection map

$$\iota_{\widetilde{\Gamma}} \colon \mathbb{P}Z^\epsilon_{\widetilde{\Gamma}} \to \mathbb{P}Z^\epsilon_{g,n+n',\beta,\delta},$$

which is a finite cover of the closure of the locus in the twisted graph space whose elements have decorated dual graph $\widetilde{\Gamma}_{n'}$, where $\widetilde{\Gamma}_{n'}$ is obtained from $\widetilde{\Gamma}$ by adding $n'(v)$ additional legs of multiplicity $\frac{1}{d}$ to each vertex v.

By an analogous construction, we obtain a moduli space $\mathbb{P}X^{\epsilon,\mathrm{ct}}_{\widetilde{\Gamma}}$ equipped with an obstruction theory and cosection. By cosection localization, we obtain a virtual cycle $[\mathbb{P}Z^\epsilon_{\widetilde{\Gamma}}]^{\mathrm{vir}}$ supported on the degeneracy locus $\mathbb{P}Z^\epsilon_{\widetilde{\Gamma}}$.

Remark 4.1. The definition of $\mathbb{P}X^{\epsilon,+}_{g,n,\beta,\delta}$ in the geometric phase is given by replacing $\vec{p}$ by $\vec{x}$ in (26) and (27) and replacing P_1 by L_1 in (25) and (28). Setting $\mathbb{P}X^{\epsilon,\mathrm{ct},+}_{g,n,\beta,\delta}$ to be the locus in which $p_1(q_k) = \cdots = p_N(q_k) = 0$ for each marked point q_k, the above yields a virtual cycle supported on the $\mathbb{P}Z^{\epsilon,+}_{g,n,\beta,\delta}$. Analogously, one obtains moduli spaces spaces $\mathbb{P}Z^{\epsilon,+}_{\widetilde{\Gamma}}$ with virtual cycles for any decorated graph.

4.2. $\mathbb{C}^*$-action and fixed loci.
There is an action of $\mathbb{C}^*$ on $\mathbb{P}X^\epsilon_{g,n,\beta,\delta}$, given by acting on the z_1-coordinate with weight -1. Let λ be the equivariant parameter, which is defined as the dual of the weight-one representation of $\mathbb{C}^*$.

The fixed loci of this action are straightforward to compute using similar techniques to [30, 39, 41]. First, recall that (z_1, z_2) define a section $f \colon C \to \mathbb{P}(P_1^\vee \oplus \mathcal{O}_C)$, and from this perspective, the $\mathbb{C}^*$ action can be viewed as scaling on the first homogeneous coordinate of the projective bundle. The computation of the fixed loci then mimics the computation for stable maps to $\mathbb{P}^1$. In particular, suppose that C' is an irreducible component of C with special points $\vec{q}'$, and set $L' = L|_{C'}$ and $\vec{p}' = \vec{p}|_{C'}$. If $(C'; \vec{q}'; L; \vec{p})$ has finitely many automorphisms, then f must map C' into one of the fixed sets of $\mathbb{P}(P_1^\vee \oplus \mathcal{O}_C)$, which are the loci where $z_1 = 0$ or $z_2 = 0$. On the other hand, if $(C'; \vec{q}'; L; \vec{p})$ has infinitely many automorphisms, the component C' must be of genus zero, the map $f|_{C'}$ must be a multiple cover of a fiber of $f \colon C \to \mathbb{P}(P_1^\vee \oplus \mathcal{O}_C)$, and all special points and basepoints of C' must have either z_1 or z_2 vanishing.

These fixed loci can be indexed by decorated graphs. We denote such a graph by Λ (to avoid confusion with dual graphs Γ) and decorate it as follows:

- Each vertex v has an index $j(v) \in \{0, \infty\}$, a genus $g(v)$, and a degree $\beta(v) \in \mathbb{Z}^{\geq 0}$.
- Each edge e has a degree $\delta(e) \in \mathbb{Z}^{\geq 1}$.
- Each half-edge h (including the legs) has a multiplicity $m(h) \in \left\{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\right\}$.
- The legs are labeled with the numbers $\{1, \ldots, n\}$.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges, including legs.

The fixed locus in $\mathbb{P}X_{g,n,\beta,\delta}^\epsilon$ indexed by Λ parameterizes Landau–Ginzburg quasimaps for which:

- Each edge e corresponds to a genus-zero component C_e on which
$$\deg(L_2|_{C_e}) = \delta(e),$$
  and which has two distinguished "ramification points" q_1, q_2. The section z_1 (on C_e) vanishes nowhere except possibly at q_1, while z_2 vanishes only at q_2, and all of the marked points, all of the nodes, and all of the degree of $P_1|_{C_e}$ is concentrated at q_1 and q_2. That is,
$$\deg(P_1|_{C_e}) = \mathrm{ord}_{q_1}(\vec{p}) + \mathrm{ord}_{q_2}(\vec{p}),$$
  so if both ramification points are special points (at which $\vec{p}$ is necessarily nonvanishing), it follows that the degree of $P_1|_{C_e}$ is zero.
- Each vertex v for which $j(v) = 0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C over which $z_1 \equiv 0$, and each vertex v for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve over which $z_2 \equiv 0$. The labels $g(v)$ and $\beta(v)$ denote the genus of C_v and the degree of $P_1|_{C_v}$, respectively, and the legs incident to v indicate the marked points on C_v.
- A vertex v is *unstable* if stable sub-curves of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case, v corresponds to a single point of the component C_e for each adjacent edge e, which may be a node at which C_e meets $C_{e'}$, a marked point of C_e, or a basepoint on C_e of order $\beta(v)$ (with the special case $\beta(v) = 0$, in which v stands for an unmarked point of C_e).
- The index $m(l)$ on a leg l indicates the multiplicity of L_1 at the corresponding marked point.

- A half-edge h incident to a stable vertex v corresponds to a node at which components C_e and C_v meet, and $m(h)$ indicates the multiplicity of L_1 at the branch of the node on C_v. If v is unstable and hence h corresponds to a single point on a component C_e, then $m(h)$ is the *negative* (additive inverse) in $\mathbb{Q}/\mathbb{Z}$ of the multiplicity of L_1 at this point.

In particular, we note that the decorations at each stable vertex v yield a tuple

$$\vec{m}(v) \in \left\{0, \frac{1}{d}, \ldots, \frac{d-1}{d}\right\}^{\mathrm{val}(v)}$$

recording the multiplicities of L_1 at every special point of C_v.

The crucial observation, now, is the following. For a stable vertex v such that $j(v) = 0$, we have $z_1|_{C_v} \equiv 0$, so the stability condition (26) implies that $\mathrm{ord}_q(\vec{p}) \leq 1/\epsilon$ for each $q \in C_v$. That is, the restriction of $(C; q_1, \ldots, q_n; L_1; \vec{p})$ to C_v defines an element of $X^{\epsilon}_{g(v), \vec{m}(v), \beta(v)}$. On the other hand, for a stable vertex v such that $j(v) = \infty$, we have $z_2|_{C_v} \equiv 0$, so the stability condition (27) implies that $\mathrm{ord}_q(\vec{p}) = 0$ for each $q \in C_v$. Thus, the restriction of $(C; q_1, \ldots, q_n; L_1; \vec{p})$ to C_v defines an element of $X^{\infty}_{g(v), \vec{m}(v), \beta(v)}$. Finally, for each edge e, the restriction of $(\vec{x}, \vec{p})$ to C_e defines a constant map to X (possibly with an additional basepoint at the ramification point q_1).

Remark 4.2. It is important in what follows to observe that, if C_e is an edge component containing a basepoint of some order $\beta(e)$, then one must have $\delta(e) > \beta(e)$. Indeed, if this is not the case, then $z_1 \equiv 0$. Given that z_2 must vanish somewhere, this is impossible without violating the nondegeneracy condition in the definition of $\mathbb{P}X^{\epsilon}_{g,n,\beta,\delta}$.

Denote

$$(29) \qquad FX_\Lambda := \prod_{\substack{v \text{ stable} \\ j(v)=0}} X^{\epsilon}_{g(v), \vec{m}(v), \beta(v)} \times_{\overline{\mathcal{I}}X} \prod_{\text{edges } e} \overline{\mathcal{I}}X^{\frac{1}{\delta(e)}} \times_{\overline{\mathcal{I}}X} \prod_{\substack{v \text{ stable} \\ j(v)=\infty}} X^{\infty}_{g(v), \vec{m}(v), \beta(v)},$$

where $\overline{\mathcal{I}}X^{\frac{1}{\delta(e)}}$ is a $\delta(e)$th root gerbe over $\overline{\mathcal{I}}X$; the explicit root gerbe, and the reason for its appearance, are explained in Section 4.3.2 below. Here, the fiber products are taken over the evaluation map at the half-node on the vertex moduli space and the composition of the evaluation map at the half-node on the edge moduli space with the involution $\iota : \overline{\mathcal{I}}X \to \overline{\mathcal{I}}X$ coming from inversion (to ensure that all nodes are balanced).

The preceding discussion implies that there is a canonical family of $\mathbb{C}^*$-fixed elements of $\mathbb{P}X^{\epsilon}_{g,n,\beta,\delta}$ over FX_Λ, yielding a morphism

$$FX_\Lambda \to \mathbb{P}X^{\epsilon}_{g,n,\beta,\delta}.$$

This is not exactly the inclusion of the associated fixed locus, because elements of $\mathbb{P}X^{\epsilon}_{g,n,\beta,\delta}$ have additional automorphisms from permuting the components via an automorphism of Λ, and scaling the fibers of $L_1|_{C_e}$ by dth roots of unity.[4] In particular, ι_Γ decomposes into a finite (étale) map of degree

$$(30) \qquad \frac{|\mathrm{Aut}(\Lambda)|}{\prod_{h \in \widetilde{H}(\Lambda)} d_{m(h)}}$$

---

[4]The scalings of L_1 on adjacent components do not yield independent automorphisms, however, when the components meet at a node with nontrivial isotropy; see [36, Proposition 1.18].

and a closed embedding. Here, $\widetilde{H}(\Lambda)$ is a set of half-edges containing one half-edge for each node of a generic curve in FX_Λ. This means that $\widetilde{H}(\Lambda)$ contains all non-leg half-edges at stable vertices and exactly one half-edge at each non-legged valence-two vertex with genus zero and β-degree zero. Furthermore, we define $d_{m(h)} = d/\gcd(d \cdot m(h), d)$, which does not depend on which half-edge at an unstable vertex is chosen.

The $\mathbb{C}^*$-action preserves the locus $\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta} \subset \mathbb{P}X^{\epsilon}_{g,n,\beta,\delta}$, and we can similarly define $FX^{\mathrm{ct}}_\Lambda \subset FX_\Lambda$ by the requirement that $x_1(q_k) = \cdots = x_M(q_k) = 0$ whenever q_k corresponds to a leg of Λ. The perfect obstruction theory on $\mathbb{P}X^{\epsilon,\mathrm{ct}}_{g,n,\beta,\delta}$ and the cosection $\widetilde{\sigma}$ are both $\mathbb{C}^*$-equivariant, so by [2], each fixed locus admits a cosection-localized virtual class, and the virtual localization formula expresses $[\mathbb{P}Z^{\epsilon}_{g,n,\beta,\delta}]^{\mathrm{vir}}$ in terms of these contributions. We set

$$(31) \qquad FZ_\Lambda := \prod_{\substack{v \text{ stable} \\ j(v)=0}} Z^{\epsilon}_{g(v),\vec{m}(v),\beta(v)} \times_{\overline{\mathcal{I}}X^{\mathrm{ct}}} \prod_{\text{edges } e} \overline{\mathcal{I}}X^{\mathrm{ct}, \frac{1}{\delta(e)}} \times_{\overline{\mathcal{I}}X^{\mathrm{ct}}} \prod_{\substack{v \text{ stable} \\ j(v)=\infty}} Z^{\infty}_{g(v),\vec{m}(v),\beta(v)},$$

where, again, the fiber product is taken over evaluation on the vertex side and evaluation followed by inversion on the edge side. We denote by $[FZ_\Lambda]^{\mathrm{vir}}$ the pullback of the cosection-localized virtual class on the fixed locus associated to Λ under the étale map.

More generally, one can do all of this on each of the stacks $\mathbb{P}X^{\epsilon}_{\widetilde{\Gamma}}$ associated to a decorated dual graph $(\widetilde{\Gamma}, v_\bullet, n')$. That is, let $\widetilde{\Gamma}$ be a stable decorated dual graph as in Section 4.1, and let Λ be any graph obtained from $\widetilde{\Gamma}$ by replacing each vertex v by a localization graph for $\mathbb{P}X^{\epsilon}_{g(v),\vec{m}'(v),\beta(v),D(v)}$. For each such Λ, there is a fixed locus in $\mathbb{P}X^{\epsilon,\mathrm{ct}}_{\widetilde{\Gamma}}$, and for the same fiber product FX^{ct}_Λ defined above, we have a morphism

$$\iota_{\widetilde{\Gamma},\Lambda} \colon FX^{\mathrm{ct}}_\Lambda \to \mathbb{P}X^{\epsilon,\mathrm{ct}}_{\widetilde{\Gamma}},$$

whose image is the fixed locus associated to Λ.

Remark 4.3. The fixed loci have an analogous structure in the geometric phase; see [21, Section 3.2].

4.3. Localization contributions.

The virtual localization formula, first proved by Graber–Pandharipande [30] and adapted to the setting of cosection-localized virtual classes by Chang–Kiem–Li [2], expresses $[\mathbb{P}Z^{\epsilon}_{g,n,\beta,\delta}]^{\mathrm{vir}}$ in terms of contributions from each localization graph:

$$(32) \qquad [\mathbb{P}Z^{\epsilon}_{g,n,\beta,\delta}]^{\mathrm{vir}} = \sum_\Lambda \frac{\prod_{h \in \widetilde{H}(\Lambda)} d_{m(h)}}{|\mathrm{Aut}(\Lambda)|} \iota_{\Lambda,*} \left(\frac{[FZ_\Lambda]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_\Lambda)} \right).$$

Here, $[FZ_\Lambda]^{\mathrm{vir}}$ and N^{vir}_Λ are the virtual cycle of the fixed locus, and virtual normal bundle defined via restricting the obstruction theory to the fixed locus. By abuse of notation, we have pulled them back under the étale map from FZ_Λ. There is also a technical assumption necessary in order to be able to apply this formula: the restriction of the perfect obstruction theory to each fixed locus has to admit a global resolution by a perfect complex. In our case, this is clear from the analysis of the perfect obstruction theory in the following sections.

More generally, all of this can be done for each dual graph $\widetilde{\Gamma}$. Note that if Λ is a localization graph contributing to $[\mathbb{P}Z^{\epsilon}_{\widetilde{\Gamma}}]^{\mathrm{vir}}$, then the space FZ_Λ appears in the localization both for $[\mathbb{P}Z^{\epsilon}_{g,n,\beta,\delta}]^{\mathrm{vir}}$ and for $[\mathbb{P}Z^{\epsilon}_{\widetilde{\Gamma}}]^{\mathrm{vir}}$. However, its contribution is *a priori* different: in the localization for $[\mathbb{P}Z^{\epsilon}_{g,n,\beta,\delta}]^{\mathrm{vir}}$, we have a virtual cycle $[FZ_\Lambda]^{\mathrm{vir}}$ as defined above via the obstruction theory

on $\mathbb{P}X_{g,n,\beta,\delta}^{\epsilon,\mathrm{ct}}$, whereas in the localization for $[\mathbb{P}Z_{\widetilde{\Gamma}}^{\epsilon}]^{\mathrm{vir}}$, we have a virtual cycle that we denote $[FZ_{\Lambda}]_{\widetilde{\Gamma}}^{\mathrm{vir}}$, defined via the obstruction theory on $\mathbb{P}X_{\widetilde{\Gamma}}^{\epsilon,\mathrm{ct}}$. Similarly, we have virtual normal bundles $N_{\Lambda}^{\mathrm{vir}}$ for the fixed locus associated to Λ inside $\mathbb{P}X_{g,n,\beta,\delta}^{\epsilon,\mathrm{ct}}$, and $N_{\Lambda,\widetilde{\Gamma}}^{\mathrm{vir}}$ for the fixed locus inside $\mathbb{P}X_{\widetilde{\Gamma}}^{\epsilon,\mathrm{ct}}$.

In fact, however, these two contributions are closely related:

Lemma 4.4. *With notation as in* (32), *we have*

$$[FZ_{\Lambda}]^{\mathrm{vir}} = [FZ_{\Lambda}]_{\widetilde{\Gamma}}^{\mathrm{vir}}$$

and

$$e(N_{\Lambda}^{\mathrm{vir}}) = e(N_{\Lambda,\widetilde{\Gamma}}^{\mathrm{vir}}) \prod_{e \in E(\widetilde{\Gamma})} c_1(T_{e,1} \otimes T_{e,2}),$$

where, for each edge $e \in E(\widetilde{\Gamma})$, $T_{e,1}$ *and* $T_{e,2}$ *denote the relative tangent line bundles at the corresponding node of the* orbifold *curve.*

Proof. We have the following commutative diagram, whose right square is cartesian:

$$
\begin{array}{ccccc}
FX_{\Lambda}^{\mathrm{ct}} & \xrightarrow{\iota_{\widetilde{\Gamma},\Lambda}} & \mathbb{P}X_{\widetilde{\Gamma}}^{\epsilon,\mathrm{ct}} & \xrightarrow{\iota_{\widetilde{\Gamma}}} & \mathbb{P}X_{g,n,\beta,\delta}^{\epsilon,\mathrm{ct}} \\
& & \downarrow{\pi} & & \downarrow{\pi} \\
& & D_{\widetilde{\Gamma}} & \xrightarrow{j_{\widetilde{\Gamma}}} & D_{g,n,\beta}.
\end{array}
$$

Here, we define $D_{\widetilde{\Gamma}} = D_{\Gamma}$, and $j_{\widetilde{\Gamma}} = j_{\Gamma}$ as appearing in (21). By definition, the perfect obstruction theory on $\mathbb{P}X_{\widetilde{\Gamma}}^{\epsilon,\mathrm{ct}}$ relative to $D_{\widetilde{\Gamma}}$ is $j_{\widetilde{\Gamma}}^{*}\widetilde{\mathbb{E}}^{\bullet}$. By [4, Lemma 4.4], the map $j_{\widetilde{\Gamma}}$ is a finite and local complete intersection morphism, so its cotangent complex $\mathbb{L}_{j_{\widetilde{\Gamma}}}^{\bullet}$ is concentrated in degree -1, and it is quasi-isomorphic to the complex

$$\left(\bigoplus_{e \in E(\widetilde{\Gamma})} T_{e,1} \otimes T_{e,2} \right)^{\vee} \to 0$$

of vector bundles. Since the corresponding π-relative obstruction theories agree, thus, the absolute perfect obstruction theories $\mathbb{E}_{\widetilde{\Gamma},\mathrm{abs}}^{\bullet}$ and $\widetilde{\mathbb{E}}_{\mathrm{abs}}^{\bullet}$ fit into a distinguished triangle

$$\pi^{*}(\mathbb{L}_{j_{\widetilde{\Gamma}}}^{\bullet})[-1] \xrightarrow{+1} \widetilde{\mathbb{E}}_{\mathrm{abs}}^{\bullet} \to \mathbb{E}_{\widetilde{\Gamma},\mathrm{abs}}^{\bullet} \to \pi^{*}(\mathbb{L}_{j_{\widetilde{\Gamma}}}^{\bullet}).$$

The virtual classes $[FZ_{\Lambda}]^{\mathrm{vir}}$ and $[FZ_{\Lambda}]_{\widetilde{\Gamma}}^{\mathrm{vir}}$ are defined by the $\mathbb{C}^{*}$-fixed part of $\iota_{\widetilde{\Gamma},\Lambda}^{*}\left(\widetilde{\mathbb{E}}_{\mathrm{abs}}^{\bullet} \right)$ and $\iota_{\widetilde{\Gamma},\Lambda}^{*}\left(\mathbb{E}_{\widetilde{\Gamma},\mathrm{abs}}^{\bullet} \right)$, respectively, and the virtual normal bundles are defined by the corresponding moving parts. The lemma therefore follows from the fact that, as in the calculation in Section 4.3.3 below, the line bundles $T_{e,1} \otimes T_{e,2}$ are moving. $\square$

The goal of the remainder of this subsection is to compute the contributions to (32) of each graph Λ explicitly. (All of this can also be done for each $[\mathbb{P}Z_{\widetilde{\Gamma}}^{\epsilon}]^{\mathrm{vir}}$, but by Lemma 4.4, the contributions of each localization graph Λ in that case are straightforward to deduce from the corresponding contributions to $[\mathbb{P}Z_{g,n,\beta,\delta}^{\epsilon}]^{\mathrm{vir}}$.) We first concentrate on the virtual normal bundle, identifying its equivariant Euler class with an explicit cohomology class pulled back

from FZ_Λ. To do so, we apply the normalization exact sequence to express the moving part of the perfect obstruction theory in terms of vertex, edge, and node factors. Then, in Section 4.3.4, we study the fixed part of the perfect obstruction theory, decomposing the cosection-localized virtual class along certain nodes. Finally, in Section 4.3.5, we push forward $[\mathbb{P}Z^\epsilon_{g,n,\beta,\delta}]^{\mathrm{vir}}$ to $Z^\epsilon_{g,n,\beta}$ by forgetting L_2 and $\vec{z}$, and we use the above calculations to express the contribution of each graph Λ to the resulting pushforward in terms of a genus-g contribution and contributions from certain trees of rational components. Crucially, we observe that the genus-0 contributions depend in only a very controlled way on the particular GIT quotient X at hand, which sets the stage for a more explicit study of those contributions in the following subsection.

4.3.1. *Vertex contributions.* For a vertex v of Λ, the sections $\vec{x}$ and $\vec{p}$ are $\mathbb{C}^*$-fixed, so the deformations of these sections, together with the deformations of the curve C_v and the line bundle $L_1|_{C_v}$, are part of the $\mathbb{C}^*$-fixed part of the perfect obstruction theory. The map f is moving, and its contribution to the inverse Euler class of the virtual normal bundle is

$$\frac{1}{e^{\mathbb{C}^*}(R\pi_* f^* T_{\mathbb{P}_v/\mathcal{C}_v})} = \frac{1}{e^{\mathbb{C}^*}(R\pi_* N_{j(v)})},$$

where

$$N_0 = \mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}, \qquad N_\infty = \mathcal{P}_1 \otimes \mathbb{C}_{(-\lambda)}$$

are the normal bundles to the zero and infinity sections, respectively, in the projectivized bundle $\mathbb{P}_v := \mathbb{P}((\mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}) \oplus \mathcal{O}_{\mathcal{C}_v})$ on the universal curve $\mathcal{C}_v$.

4.3.2. *Edge contributions.* For an edge e, the corresponding factor of FZ_Λ is isomorphic to the $\delta(e)$th root stack $\widetilde{X}^{\mathrm{ct}}_e := \overline{\mathcal{I}}X^{\mathrm{ct}, \frac{1}{\delta(e)}}$ of a component X^{ct}_e of the rigidified inertia stack $\overline{\mathcal{I}}X^{\mathrm{ct}}$ with respect to the line bundle $\mathcal{O}_{\mathbb{P}^{N-1}}(-1)|_{X^{\mathrm{ct}}_e} \otimes \mathbb{C}_{(\lambda)}$. By virtue of its universal property, on $\widetilde{X}^{\mathrm{ct}}_e$, there is a universal $\delta(e)$th root $\mathcal{R}$ of $\mathcal{O}_{\mathbb{P}^{N-1}}(-1)|_{\widetilde{X}^{\mathrm{ct}}_e} \otimes \mathbb{C}_{(\lambda)}$. With this root, the universal family over X^{ct}_e takes the form

$$\mathcal{C}_e := \mathbb{P}(\mathcal{R} \oplus \mathcal{O}_{\widetilde{X}^{\mathrm{ct}}_e}) \xrightarrow{\ f\ } \mathbb{P}((\mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}) \oplus \mathcal{O}_{\widetilde{X}^{\mathrm{ct}}_e}) =: \mathbb{P}_e$$

$$\pi \downarrow$$

$$\widetilde{X}^{\mathrm{ct}}_e,$$

at least in the case that the multiplicity of the edge is zero. In general, the projective bundle defining $\mathcal{C}_e$ is defined over the universal cyclic gerbe over $\widetilde{X}^{\mathrm{ct}}_e$, hence $\mathcal{C}_e \to \widetilde{X}^{\mathrm{ct}}_e$ is a family of twisted curves with orbifold structure at the zero and infinity section.

Let us make the universal sections and universal map explicit. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\widetilde{X}^{\mathrm{ct}}_e$, there is a line bundle $\mathcal{O}_{\mathcal{C}_e}(1)$ together with universal sections

$$(x, y) \in H^0((\mathcal{O}_{\mathcal{C}_e}(1) \otimes \pi^*\mathcal{R}) \oplus \mathcal{O}_{\mathcal{C}_e}(1))$$

such that x (respectively, y) vanishes precisely at the zero section (respectively, at the infinity section) of $\mathcal{C}_e$ with order one. The universal line bundle $\mathcal{P}_1$ is given by

$$\mathcal{P}_1 = \pi^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)|_{X^{\mathrm{ct}}_e} \otimes \mathcal{O}_{\mathcal{C}_e}(\beta(e)[0]) = \pi^*\mathcal{R}^{-\delta(e)} \otimes \mathbb{C}_{(\lambda)} \otimes \mathcal{O}_{\mathcal{C}_e}(\beta(e)[0]),$$

where $[0]$ is the zero section in the projectivized bundle.

$$\beta(e) := \begin{cases} \beta(v) & \text{if there is a genus zero, valence one vertex } v \text{ at } e, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the universal section x gives an isomorphism

$$\mathcal{O}_{\mathcal{C}_e}([0]) \cong \mathcal{O}_{\mathcal{C}_e}(1) \otimes \pi^*\mathcal{R}.$$

The universal map f is given by the sections

$$(\zeta_1, \zeta_2) = (x^{\delta(e)-\beta(e)}, y^{\delta(e)}) \in H^0((\mathcal{L}_2 \otimes \mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}) \oplus \mathcal{L}_2),$$

where

$$\mathcal{L}_2 = \mathcal{O}_{\mathcal{C}_e}(\delta(e)).$$

Note that

$$f^*T_{\mathbb{P}_e/\mathcal{C}_e} = \mathcal{O}_{\mathcal{C}_e}((\delta(e) - \beta(e))[0] + \delta(e)[\infty]),$$

where $[\infty]$ denotes the infinity section of $\mathcal{C}_e$.

It is useful for what follows to introduce the notation

$$\lambda_j = \begin{cases} \lambda - H & \text{if } j = 0 \\ -\lambda + H & \text{if } j = \infty \end{cases}$$

for the relative tangent line classes at the zero and infinity section of $\mathbb{P}_e$. We interpret λ_j as an equivariant cohomology class on $\widetilde{X}_e^{\text{ct}}$, where H denotes the hyperplane class pulled back from $\mathbb{P}^{N-1}$.

Equipped with this notation, we turn to the contributions of the edge. First of all, there are contributions from the sections $\vec{x}$ and $\vec{p}$. For these, it is useful to consider both the moving and fixed parts of the obstruction theory, since the total contribution to the localization from these sections is essentially identical to the corresponding contribution to the graph-space localization used to define the J-function. There are two differences between this setting and the J-function, however. First, we must divide the J-function by a factor of z to cancel the contribution $-z^{-1}$ in the localization on the ordinary graph space and the prefactor of $-z^2$; see Remark 2.18. Second, the tangent space at the zero section of the universal curve of the ordinary graph space has first Chern class z, while in our situation the corresponding Chern class is $\lambda_0/\delta(e)$. Thus, one sees that z must be replaced in the J-function by $\lambda_0/\delta(e)$. In all, then, we thus far have a contribution of

$$(33) \qquad \frac{1}{d_{m(h)}} \left[z^{-1} J^\epsilon(q, z) \big|_{z=\lambda_0/\delta(e)} \right]_{q^{\beta(e)}}.$$

Furthermore, there are contributions from deformations of the map f. The vector bundle

$$R^0\pi_*\big(f^*T_{\mathbb{P}_e/\mathcal{C}_e}\big) = R^0\pi_*\bigg(\mathcal{O}_{\mathcal{C}_e}\big((\delta(e) - \beta(e))[0] + \delta(e)[\infty]\big)\bigg)$$

has a trivial, $\mathbb{C}^*$-fixed factor $R^0\pi_*\mathcal{O}$. The complement of this factor is moving, and the inverse of its Euler class can be written as

$$\frac{1}{\prod_{b=1}^{\delta(e)-\beta(e)} \frac{b\lambda_0}{\delta(e)} \prod_{b=1}^{\delta(e)} \frac{b\lambda_\infty}{\delta(e)}} = \frac{\prod_{b=\delta(e)-\beta(e)+1}^{\delta(e)} \frac{b\lambda_0}{\delta(e)}}{\prod_{b=1}^{\delta(e)} \frac{b\lambda_0}{\delta(e)} \prod_{b=1}^{\delta(e)} \frac{b\lambda_\infty}{\delta(e)}}.$$

In the case where there is no basepoint, so $\beta(e) = 0$, only the denominator of the above appears. The numerator, on the other hand, can be rewritten as

$$\prod_{b=\delta(e)-\beta(e)+1}^{\delta(e)} \frac{b\lambda_0}{\delta(e)} = \prod_{b=0}^{\beta(e)-1} \left(\lambda_0 - \frac{b\lambda_0}{\delta(e)}\right).$$

Using

$$e^{\mathbb{C}^*}(-R\pi_*(\mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)})) = e^{\mathbb{C}^*}((R^0\pi_*(\omega_\pi \otimes \mathcal{P}_1))^\vee \otimes \mathbb{C}_{(\lambda)})$$

$$= e^{\mathbb{C}^*}\left(\left(R^0\pi_*\mathcal{O}((\beta(e)-1)[0] - [\infty])\right)^\vee \otimes \mathcal{O}_{\mathbb{P}^{N-1}}(-1) \otimes \mathbb{C}_{(\lambda)}\right) = \lambda_0^{-1} \prod_{b=0}^{\beta(e)-1} \left(\lambda_0 - \frac{b\lambda_0}{\delta(e)}\right),$$

we can write

$$(34) \qquad \frac{1}{e^{\mathbb{C}^*}(R^0\pi_*(f^*T_{\mathbb{P}_e/\mathcal{C}_e}))} = \frac{\lambda_0 e^{\mathbb{C}^*}(-R\pi_*(\mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}))}{\prod_{b=1}^{\delta(e)} \frac{b\lambda_0}{\delta(e)} \prod_{b=1}^{\delta(e)} \frac{b\lambda_\infty}{\delta(e)}},$$

which is the contribution to the inverse Euler class of the virtual normal bundle coming from the sections $\vec{z}$.

Together, (33) and (34) yield a contribution of

$$\frac{\delta(e)}{\lambda_0 d_{m(h)}} \left[J^{\epsilon,\text{tw}}(q, z)\big|_{z=\lambda_0/\delta(e)} \right]_{q^{\beta(e)}} \cdot \frac{1}{\prod_{b=1}^{\delta(e)} \frac{b\lambda_0}{\delta(e)} \prod_{b=1}^{\delta(e)} \frac{b\lambda_\infty}{\delta(e)}}$$

to the localization.

Finally, there are additional contributions from automorphisms of C_e. These have nontrivial torus weight only if there is an unmarked ramification point, so the edge e must be incident to an unstable vertex v of valence 1. The contribution from such automorphisms to the inverse Euler class of the virtual normal bundle is the tangent line class at the point corresponding to v, which is

$$\frac{\lambda_{j(v)}}{\delta(e)}.$$

4.3.3. *Node contributions.* In the normalization exact sequence for $R\pi_* f^* T_{\mathbb{P}((\mathcal{P}_1^\vee \otimes \mathbb{C}_{(\lambda)}) \oplus \mathcal{O})/\mathcal{C}}$, a node corresponding to a vertex v contributes

$$e^{\mathbb{C}^*}(N_{j(v)}) = \lambda_{j(v)}$$

to the inverse Euler class of the virtual normal bundle. The deformations in $D_{g,n,\beta}$ smoothing the node also give a contribution of

$$\frac{1}{e^{\mathbb{C}^*}(T_1) + e^{\mathbb{C}^*}(T_2)} = \frac{d_{m(h)}}{e^{\mathbb{C}^*}(\overline{T}_1) + e^{\mathbb{C}^*}(\overline{T}_2)},$$

where T_1 and T_2 denote the tangent lines of the orbifold curve at the two branches of the node, $\overline{T}_1$ and $\overline{T}_2$ denote the corresponding tangent lines pulled back from the underlying coarse curve, and where h is any of the half-edges of e. This inverse makes sense since at least one of the T_i (say T_1) corresponds to an edge e, so we have

$$e^{\mathbb{C}^*}(T_1) = \frac{\lambda_{j(v)}}{\delta(e)}.$$

4.3.4. *Virtual cycle.* We now look at the fixed part of the perfect obstruction theory in more detail, decomposing it along certain nodes.

We first introduce some notation and terminology. Let Λ be a localization graph, and let $\widetilde{\Gamma}$ be the prestable dual graph of a generic element of the associated fixed locus in $\mathbb{P}X^\epsilon_{g,n,\beta,\delta}$. There is a forgetful map

$$p : \mathbb{P}X^\epsilon_{g,n,\beta,\delta} \to X^\epsilon_{g,n,\beta}$$

that forgets L_2, z_1, and z_2 (and stabilizes as necessary), and a vertex v of $\widetilde{\Gamma}$ whose corresponding irreducible component is not contracted by this stabilization is called *very stable*. Let Γ be the dual graph obtained from $\widetilde{\Gamma}$ according to p, with an extra leg recording where any tree of rational components in $\widetilde{\Gamma}$ was attached. See Figure 1.

The virtual cycle $[FZ_\Lambda]^{\mathrm{vir}}$ is not pulled back under this forgetful map, but the source of the discrepancy lies only in the vertex components of Λ that are contracted, since edge components to not contribute to the virtual cycle. Furthermore, components corresponding to vertices v with $\beta(v) = 0$ also do not contribute to the virtual cycle, since the corresponding factor of (31) is a smooth moduli space (and as a result the map $\widetilde{p}$ defined below is smooth). Thus, the vertices relevant for expressing the discrepancy between $[FZ_\Lambda]^{\mathrm{vir}}$ and a cycle pulled back under p are

$$T := \{v \in V(\Gamma) \mid v \text{ not very stable}, v \notin V_{E(\Lambda)}(\widetilde{\Gamma}), \beta(v) > 0\}.$$

For each $v \in T$, define $X^{\epsilon,\mathrm{ct}-1}_{0,n(v),\beta(v)}$ as the locus inside $X^\epsilon_{0,n(v),\beta(v)}$ where $x_1(q_k) = \cdots = x_M(q_k) = 0$ for all $k \in \{1,\ldots,n(v)\}$ except for the unique leg that is closest to one of the very stable vertices.[5] Define FZ'_Λ analogously to FZ_Λ, but replacing the factor of $Z^\epsilon_{0,n(v),\beta(v)}$ by $X^{\epsilon,\mathrm{ct}-1}_{0,n(v),\beta(v)}$ for each $v \in T$. Let

$$\iota : FZ_\Lambda \hookrightarrow FZ'_\Lambda$$

be the proper inclusion map.

If the smooth map

$$(35) \qquad \widetilde{p} \colon FZ'_\Lambda \to X^{\epsilon,\mathrm{ct}}_\Gamma \times_{(\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|}} \prod_{v \in T} X^{\epsilon,\mathrm{ct}-1}_{0,n(v),\beta(v)}$$

is defined by contracting vertices that are neither very stable nor in T (see Figure 1), and

$$\widetilde{\Delta} \colon (\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|} \to (\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|} \times (\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|}$$

is the diagonal map, then the aim of this section is to prove the following formula:

Lemma 4.5.

$$\iota_*[FZ_\Lambda]^{\mathrm{vir}} = \widetilde{p}^* \widetilde{\Delta}^! \left([Z^\epsilon_{\Gamma'}]^{\mathrm{vir}} \times \prod_{v \in T} [X^{\epsilon,\mathrm{ct}-1}_{0,n(v),\beta(v)}]^{\mathrm{vir}} \right)$$

The first step toward proving Lemma 4.5 is to re-write $[FZ_\Lambda]^{\mathrm{vir}}$ via a more convenient perfect obstruction theory. Thus far, $[FZ_\Lambda]^{\mathrm{vir}}$ has been defined using the fixed part of (the

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[5]In fact, $X^{\epsilon,\mathrm{ct}-1}_{0,n(v),\beta(v)} = Z^\epsilon_{0,n(v),\beta(v)}$ in the Landau–Ginzburg phase, but we use different notation to clarify the parallel argument in the geometric phase.
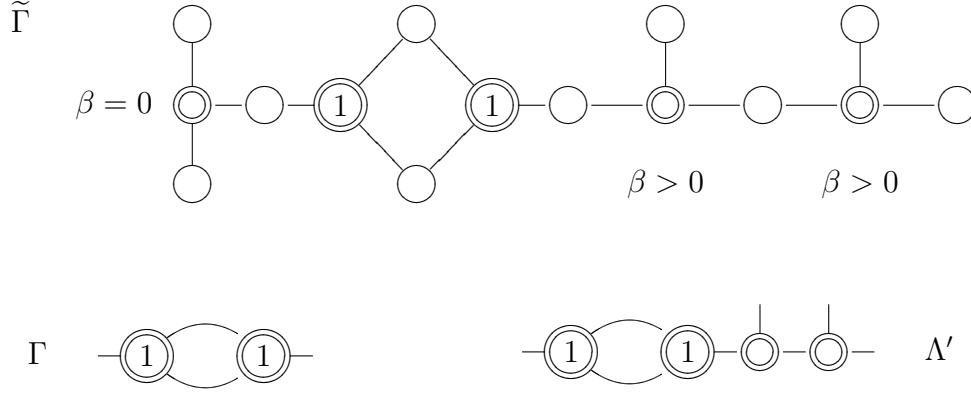
FIGURE 1. On the first line, the dual graph $\widetilde{\Gamma}$ associated to a localization graph Λ, where the number within a vertex indicates its genus and empty vertices are understood as genus zero. Vertices with a double circle correspond to vertex components of Λ, vertices with a single circle correspond to edge components of Λ, and the genus-zero vertex components are labeled according to whether their β-degree equals zero. The second line shows the graph Γ and the graph Λ' represented by the fiber product (35). Note that the two vertices of Λ' that are not in Γ are precisely the elements of T.

absolute perfect obstruction theory induced by) $\widetilde{\mathbb{E}}^\bullet$, which is a perfect obstruction theory for FX_Λ^{ct} relative to $D_{g,n,\beta}$; we denote this fixed part by $\mathbb{E}_{\mathrm{old}}^\bullet$. On the other hand, if

$$\mathcal{E} := \bigoplus_{i=1}^M (\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}(-\textstyle\sum_{k=1}^n \Delta_k)) \oplus \bigoplus_{j=1}^N (\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}),$$

in which $\mathcal{C}$ is the universal curve on FX_Λ^{ct} and $\mathcal{L}$ the universal line bundle, then the complex $(R\pi_*\mathcal{E})^\vee$ is a perfect obstruction theory for FX_Λ^{ct} relative to the smooth Artin stack D_Λ defined from $D_{\widetilde{\Gamma}}$ by taking the substack where the gerbes at the opposite sides of each edge component are identified (with inverted band), and by rigidifying along the automorphisms of each edge component that move the map f of Section 4.3.2. Let $\mathbb{E}_{\mathrm{new}}^\bullet$ be the fixed part of the corresponding absolute perfect obstruction theory.[6]

Lemma 4.6. $\mathbb{E}_{\mathrm{old}}^\bullet$ and $\mathbb{E}_{\mathrm{new}}^\bullet$ are quasi-isomorphic.

Proof. This follows from the analysis of the virtual normal bundle in the previous sections. As we have seen, the curve deformation factors corresponding to smoothing the nodes are moving. Therefore, for $\mathbb{E}_{\mathrm{old}}^\bullet$, we can work relative to the Artin stack $D_{\widetilde{\Gamma}}$ instead of $D_{g,n,\beta}$. We have also seen that, if $\mathbb{P} := \mathbb{P}(\mathcal{P}_1^\vee \oplus \mathcal{O})$, then the factor $R\pi_* f^* \mathbb{L}_{\mathbb{P}/\mathcal{C}}$ in $\widetilde{\mathbb{E}}^\bullet$ is moving except for one trivial factor for each edge in Λ. This factor is identified with the corresponding edge automorphism factor in the cotangent complex of D_Λ. Finally, the remaining edge automorphisms are moving. Therefore, working relative to D_Λ and removing the factor $R\pi_* f^* \mathbb{L}_{\mathbb{P}/\mathcal{C}}$ from $\widetilde{\mathbb{E}}^\bullet$ gives $\mathbb{E}_{\mathrm{old}}^\bullet$, which is the same as the description of $\mathbb{E}_{\mathrm{new}}^\bullet$. □

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[6]$R\pi_*\mathcal{E}$ has moving factors corresponding to edges with basepoints.

To continue, we separate the dual graph $\widetilde{\Gamma}$ corresponding to Λ into two parts $\widetilde{\Gamma}_g$ and $\widetilde{\Gamma}_0$, where $\widetilde{\Gamma}_g$ is formed by all very stable vertices together with all chains of rational components connecting them, and $\widetilde{\Gamma}_0$ consists of the remaining vertices of $\widetilde{\Gamma}$, which form a disjoint union of trees of rational components. Correspondingly, we can decompose the universal curve

$$\mathcal{C} = \mathcal{C}_g \cup \mathcal{C}_0.$$

Let $\Delta = \mathcal{C}_g \cap \mathcal{C}_0$, and define the following subsheaf:

$$\mathcal{E}' := \bigoplus_{i=1}^{M}(\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}(-\Delta - \sum_{k=1}^{n}\Delta_k))|_{\mathcal{C}_g} \oplus \bigoplus_{j=1}^{N}(\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}) \subset \mathcal{E}.$$

The cokernel of the inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$ is given by

$$\mathcal{E}'' := \bigoplus_{i=1}^{M}(\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}(-\sum_{k=1}^{n}\Delta_k))|_{\mathcal{C}_0}.$$

Note that by Remark 2.10, we have $\pi_*\mathcal{E}'' = 0$, so that $\pi_*\mathcal{E}' \cong \pi_*\mathcal{E}$. Therefore, $(R\pi_*\mathcal{E}')^\vee$ provides an alternative perfect obstruction theory for FX_Λ^{ct} relative to D_Λ; moreover, the cosection σ restricts to a cosection σ' on this alternative perfect obstruction theory. The degeneracy locus of σ' is precisely the locus FZ_Λ' defined above, so we denote by $[FZ_\Lambda']^{\mathrm{vir}}$ the induced cosection-localized virtual class. Cosection-localized pullback then implies the following:

Lemma 4.7.

$$\iota_*[FZ_\Lambda]^{\mathrm{vir}} = e(R^1\pi_*\mathcal{E}'')^{\mathrm{fix}} \cap [FZ_\Lambda']^{\mathrm{vir}}$$

The vector bundle $R^1\pi_*\mathcal{E}''$ can be easily decomposed. First, $R^1\pi_*\mathcal{E}''$ has a summand for each connected component of $\mathcal{C}_0$. Then, using short exact sequences similar to $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we can inductively split $e(R^1\pi_*\mathcal{E}'')^{\mathrm{fix}}$ into a factor for each irreducible component of $\mathcal{C}_0$. Each time we split at a node, we make the choice to twist down by Δ on the side closest to the trunk of the tree. Now, we can check that the only fixed factors of $e(R^1\pi_*\mathcal{E}'')$ are on the vertices of $\widetilde{\Gamma}$ in the set T. Therefore, we can write

$$e(R^1\pi_*\mathcal{E}'')^{\mathrm{fix}} = \prod_{v \in T} e(R^1\pi_*\mathcal{E}_v''),$$

where $\mathcal{E}_v''$ denotes the restriction of $\mathcal{E}''$ to the component of $\mathcal{C}_0$ corresponding to v and twisted down at all but the node closest to $\mathcal{C}_g$.

The factor of $[FZ_\Lambda']^{\mathrm{vir}}$, on the other hand, is pulled back under $\widetilde{p}$. More specifically, let Λ' denote the lower-right graph in Figure 1, and let $FX_{\Lambda'}^{\mathrm{ct}}$ be the fiber product defined in (35) as the target of $\widetilde{p}$. The virtual cycle $[FZ_\Lambda']^{\mathrm{vir}}$ is defined via cosection localization from the larger space FX_Λ^{ct}, to which $\widetilde{p}$ extends. Denoting this extension also by $\widetilde{p}$, we have a cartesian diagram

$$
\begin{array}{ccc}
FX_\Lambda^{\mathrm{ct}} & \xrightarrow{\widetilde{p}} & FX_{\Lambda'}^{\mathrm{ct}} \\
\downarrow & & \downarrow \\
D_\Lambda & \longrightarrow & D_{\Lambda'},
\end{array}
$$

where $D_{\Lambda'}$ denotes the Artin stack of curves with a line bundle corresponding to the dual graph associated with Λ'. Note that the lower horizontal map is smooth.

Since the map $p\colon \mathcal{C}_g \to \mathcal{C}'_g$ contracting chains of rational components of $\mathcal{C}_g$ is log étale (in the sense that $p^*\omega_{\mathcal{C}'_g,\log} = \omega_{\mathcal{C}_g,\log}$ and therefore also $p^*\mathcal{L} = \mathcal{L}$), the perfect obstruction theory $(R\pi_*\mathcal{E}')^\vee$ of FX_Λ^{ct} relative to D_Λ is pulled back from the analogously-defined perfect obstruction theory of $FX_{\Lambda'}^{\mathrm{ct}}$ relative to the stack $D_{\Lambda'}$. The cosection σ' is also pulled back from the analogously-defined cosection σ' of $FX_{\Lambda'}^{\mathrm{ct}}$. Therefore,

$$(36) \qquad \iota_*[FZ_\Lambda]^{\mathrm{vir}} = \prod_{v\in T} e(R^1\pi_*\mathcal{E}''_v) \cap \widetilde{p}^*[FX_{\Lambda'}^{\mathrm{ct}}]_{\sigma'}^{\mathrm{vir}}.$$

Finally, we split $[FX_{\Lambda'}^{\mathrm{ct}}]_{\sigma'}^{\mathrm{vir}}$ along nodes joining the T-components to the rest of the curve. For this, first note that the universal curve $\mathcal{C}'$ over $FX_{\Lambda'}^{\mathrm{ct}}$ is of the form

$$\mathcal{C}' = \mathcal{C}'_g \cup \bigcup_{v\in T} \mathcal{C}_v,$$

where $\mathcal{C}_v$ is the rational component corresponding to v. The curves intersect in a set Δ' of nodes, with $|\Delta'| = |T|$. In particular, given that Λ' consists precisely of Γ together with the vertices of T, we can form a cartesian diagram

$$\begin{array}{ccc}
FX_{\Lambda'}^{\mathrm{ct}} & \longrightarrow & FX_{\Lambda'}^{\mathrm{ct}'} \\
\downarrow & & \downarrow \\
(\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|} & \xrightarrow{\ \widetilde{\Delta}\ } & (\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|} \times (\overline{\mathcal{I}}X^{\mathrm{ct}})^{|T|},
\end{array}$$

where

$$FX_{\Lambda'}^{\mathrm{ct}'} := X_\Gamma^{\epsilon,\mathrm{ct}} \times \prod_{v\in T} X_{0,n(v),\beta(v)}^{\epsilon,\mathrm{ct}-1}.$$

Note that the perfect obstruction theory of $FX_{\Lambda'}^{\mathrm{ct}}$ relative to $D_{\Lambda'}$ is defined by $(R\pi_*\mathcal{E}')^\vee$, for

$$\mathcal{E}' := \mathcal{E}'_x \oplus \mathcal{E}'_p := \bigoplus_{i=1}^M (\mathcal{L}^{\otimes w_i} \otimes \mathcal{O}(-\Delta - \textstyle\sum_{k=1}^n \Delta_k))|_{\mathcal{C}'_g} \oplus \bigoplus_{j=1}^N (\mathcal{L}^{\otimes -d} \otimes \omega_{\pi,\log}),$$

where $\Delta \subset \Delta'$ are the nodes at $\mathcal{C}'_g$. By the normalization exact sequence, we may write $R\pi_*\mathcal{E}'$ as the cone of

$$R\pi_*\mathcal{E}'_x|_{\mathcal{C}'_g} \oplus R\pi_*\mathcal{E}'_p|_{\mathcal{C}'_g} \oplus \bigoplus_{v\in T} R\pi_*\mathcal{E}'_p|_{\mathcal{C}'_v} \to \mathcal{E}'_p|_{\Delta'}.$$

Define $D_{\Lambda''}$ in the same way as $D_{\Lambda'}$, but instead of one line bundle L on all of C, take one line bundle L_g on $\mathcal{C}'_g$ and one line bundle L_v on each $\mathcal{C}_v$ for $v\in T$. We can then use the cone of

$$R\pi_*\mathcal{E}'_x|_{\mathcal{C}'_g} \oplus R\pi_*\mathcal{E}'_p|_{\mathcal{C}'_g} \oplus \bigoplus_{v\in T} R\pi_*\mathcal{E}'_p|_{\mathcal{C}'_v} \to \bigoplus_{v\in T} T_{\overline{\mathcal{I}}X^{\mathrm{ct}}},$$

which is induced by the map $\mathcal{E}'_p|_q \to T_{\overline{\mathcal{I}}X^{\mathrm{ct}}}$ coming from the Euler sequence at each node $q\in\Delta'$, to define a perfect obstruction theory of $FX_{\Lambda'}^{\mathrm{ct}}$ relative to $D_{\Lambda''}$. Note that the two relative perfect obstruction theories of $FX_{\Lambda'}^{\mathrm{ct}}$ yield the same absolute perfect obstruction theory.

For $FX_{\Lambda'}^{\mathrm{ct}'}$, we can define a perfect obstruction theory relative to $D_{\Lambda''}$ via

$$\bigoplus_{i=1}^M (\mathcal{L}_g^{\otimes w_i} \otimes \mathcal{O}(-\Delta - \textstyle\sum_{k=1}^n \Delta_k)) \oplus \bigoplus_{j=1}^N \left((\mathcal{L}_g^{\otimes -d} \otimes \omega_{\pi,\log}) \oplus \bigoplus_{v\in T} (\mathcal{L}_v^{\otimes -d} \otimes \omega_{\pi,\log}) \right),$$

where $\mathcal{L}_g$ and $\mathcal{L}_v$ are the universal line bundles on $\mathcal{C}_g'$ and $\mathcal{C}_v$. Then, the relative perfect obstruction theories of $FX_{\Lambda'}^{\mathrm{ct}}$ and $FX_{\Lambda'}^{\mathrm{ct}'}$ relative to $D_{\Lambda''}$ are compatible with respect to the morphism $\widetilde{\Delta}$. On the direct product $FX_{\Lambda'}^{\mathrm{ct}'}$, we may define a cosection σ'' as in Section 2.5, but working only on the component $\mathcal{C}_g$. This cosection pulls back to the cosection σ' of $FX_{\Lambda'}^{\mathrm{ct}}$. Therefore, by cosection-localized pullback, we have

$$(37) \qquad [FX_{\Lambda'}^{\mathrm{ct}}]_{\sigma'}^{\mathrm{vir}} = \widetilde{\Delta}^! \left([FX_{\Lambda'}^{\mathrm{ct}'}]_{\sigma''}^{\mathrm{vir}} \right).$$

Combining (36) and (37), we have shown that

$$\iota_* [FZ_\Lambda]^{\mathrm{vir}} = \widetilde{p}^* \widetilde{\Delta}^! \left([FX_{\Lambda'}^{\mathrm{ct}}]_{\sigma''}^{\mathrm{vir}} \right) \cap \prod_{v \in T} e(R^1 \pi_* \mathcal{E}_v'').$$

Since $FX_{\Lambda'}^{\mathrm{ct}}$ is defined as a product, $[FX_{\Lambda'}^{\mathrm{ct}'}]_{\sigma''}^{\mathrm{vir}}$ clearly splits as a product of virtual cycles for each factor. The factor of $X_\Gamma^{\epsilon,\mathrm{ct}}$ yields a virtual cycle $[Z_\Gamma^\epsilon]^{\mathrm{vir}}$, while the factor corresponding to each $v \in T$, after combining with $e(R^1 \pi_* \mathcal{E}_v'')$, yields $[X_{0,n(v),\beta(v)}^{\epsilon,\mathrm{ct}-1}]^{\mathrm{vir}}$. Thus, we recover the statement of Lemma 4.5.

4.3.5. *Comparison.* We are finally ready to express the total contribution of each localization graph Λ in the form needed for what follows. Let the dual graph Γ' associated to Λ be defined as in Subsection 4.3.4, and let

$$p \colon FX_\Lambda^{\mathrm{ct}} \to X_{\Gamma'}^{\epsilon,\mathrm{ct}}$$

be the map that forgets L_2 and $\vec{z}$ and contracts unstable components (while preserving markings at the stable vertices to keep track of where trees of rational components were attached). Then, by Lemma 4.5 and the virtual normal bundle computations of the previous subsections, we can write

$$p_* \left(\frac{[FZ_\Lambda]^{\mathrm{vir}}}{e(N_\Lambda^{\mathrm{vir}})} \right) = \deg(p) \cdot [Z_{\Gamma'}^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}} \cdot \prod_{h \in H(\Gamma')} \mathrm{ev}_h^*(F_h(\psi_h)),$$

where $H(\Gamma')$ is the set of half-edges of Γ'. The factors $F_h(z)$ are valued in $\mathcal{H}(\lambda)[z]$ and, in the case that h is a leg, are obtained by collecting localization factors and performing integrals over moduli spaces $Z_{0,n(v),\beta(v)}^\epsilon$ corresponding to h. If h belongs to an edge $e = \{h, h'\}$, we choose any way of distributing localization factors corresponding to e among F_h and $F_{h'}$. In the case where $\beta(v) > 0$, the integrals over $Z_{0,n(v),\beta(v)}^\epsilon$ are against the virtual cycle $[X_{0,n(v),\beta(v)}^{\epsilon,\mathrm{ct}-1}]_\sigma^{\mathrm{vir}}$, where $X_{0,n(v),\beta(v)}^{\epsilon,\mathrm{ct}-1}$ is defined as in the previous subsection. The degree $\deg(p)$ consists of root stack automorphism factors $(\delta(e))^{-1}$ for each edge of Λ, as well as up to two factors of $d_{m(h)}^{-1}$ for each edge of Λ due to the fact that the edge moduli space uses the ordinary inertia stack while the fiber products are relative to the rigidified inertia stack. More precisely, all edges in Λ that have a basepoint give only one factor of $d_{m(h)}^{-1}$ while any other edge gives $d_{m(h)}^{-2}$.

Note that only a few factors appearing in the localization formula depend on multiplicities:

- the prefactors of (32), which give a factor of $d_{m(h)}$ for each edge in the dual graph Γ
- the factors $d_{m(h)}$ from Section 4.3.3 for each edge of Γ
- the factors $d_{m(h)}^{-1}$ from Section 4.3.2 for each edge of Λ with a base point

The second and third factors are included in the F_h. Let us define a modified series $F_h'(\psi_h)$ is obtained from $F_h(\psi_h)$ by removing the second and third factors, while adding in the factors $(\delta(e))^{-1}$ from $\deg(p)$. Then, we can write

$$\prod_{h \in \widetilde{H}(\Lambda)} d_{m(h)} \cdot p_* \left(\frac{[FZ_\Lambda]^{\mathrm{vir}}}{e(N_\Lambda^{\mathrm{vir}})} \right) = [Z_{\Gamma'}^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}} \prod_{e \in E(\Gamma')} d_{m(e)}^2 \prod_{h \in H(\Gamma')} \mathrm{ev}_h^*(F_h'(\psi_h)).$$

The key point in what follows is that the insertion F_h' corresponding to a tree of unstable rational components with β-degree zero does not depend on the GIT quotient X if one regards the hyperplane class H as a formal variable. Therefore, F_h' equals the corresponding factor in the case where the GIT quotient X is a single point, under the substitution $\lambda \to \lambda_0$. We consider this setting further in the next subsection.

Remark 4.8. Our calculations of the vertex, edge, and node contributions to the virtual normal bundle carry over directly to the geometric chamber and can be found in [21]. The decomposition of the virtual cycle carried out in Lemma 4.5 also holds in the geometric chamber; this can be obtained by the same proof as above, or, if one assumes the equivalence of the cosection-localized virtual cycle with Ciocan-Fontanine–Kim–Maulik's virtual cycle on $\overline{\mathcal{M}}_{g,n}^\epsilon(Y, \beta)$, it follows from the known decomposition along nodes. Thus, as we saw in [21], it is also the case that the insertion corresponding to a tree of rational components with β-degree zero in the geometric chamber equals the corresponding insertion for $X = \{\text{point}\}$ under the substitution $\lambda \to \lambda_0$, which is the crucial fact needed for what follows.

4.4. **Equivariant orbifold projective line.**

If the GIT quotient X is replaced by a single point (so that there are no sections $\vec{x}, \vec{p}$), then the twisted graph space reduces to the usual moduli space of stable maps to $\mathbb{P}^1$. In this section, we summarize explicit computations from [21, Section 4] of related generating series that play a role in the twisted graph space localization to follow.

As in Section 4.2, the $\mathbb{C}^*$-fixed loci in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \delta)$ can be indexed by n-legged graphs Γ, where each vertex v is decorated by an index $j(v) \in \{0, \infty\}$ and a genus $g(v)$, and each edge e is decorated by a degree $\delta(e) \in \mathbb{Z}^{\geq 1}$. Each vertex v corresponds to a maximal sub-curve of genus $g(v)$ contracted to the single point $j(v) \in \mathbb{P}^1$, or, in the unstable case where the vertex has genus zero and valence one or two, to a single point in the source curve. Each edge e corresponds to a noncontracted component, which is necessarily of genus zero, and on which the map to $\mathbb{P}^1$ is of the form $[x : y] \mapsto [x^{\delta(e)} : y^{\delta(e)}]$ in coordinates.

Fix insertions $\alpha_1, \ldots, \alpha_n \in H^*_{\mathbb{C}^*}(\mathbb{P}^1)$, and let $p \colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \delta) \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then the localization formula expresses the class

$$p_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \delta)]^{\mathrm{vir}} \right)$$

as a sum over contributions from each fixed-point graph Λ. These expressions can be stated more efficiently by considering the generating series

$$(38) \qquad \sum_{\delta=0}^\infty y^\delta p_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \delta)]^{\mathrm{vir}} \right)$$

for a Novikov variable y.

Let Φ denote the sum of all contributions to (38) from graphs Λ on which there is a vertex v with $g(v) = g$ and $j(v) = 0$, and such that, after stabilization, the generic curve in the

moduli space corresponding to Λ is smooth; the second condition means that there is no tree emanating from v that contains more than one marking. Therefore, emanating from the vertex v on such a graph, there are n (possibly empty) trees on which at least one marking lies and l trees with no marking, for some integer l. It follows that

$$(39) \qquad \Phi = \sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\sum_{i=0}^{g} c_i(\mathbb{E}) \lambda^{g-1-i} \prod_{k=1}^{n} S(\alpha_k, \psi_k) \prod_{k=n+1}^{n+l} \epsilon(\psi_k) \right),$$

in which $\pi_l \colon \overline{\mathcal{M}}_{g,n+l} \to \overline{\mathcal{M}}_{g,n}$ denotes the forgetful map, and $\mathbb{E}$ is the Hodge bundle. The series $S(\alpha, z)$ in (39) is the universal generating series of localization contributions of a tree emanating from a vertex v with $j(v) = 0$ that contains exactly one of the markings and has an insertion of $\alpha \in H^*_{\mathbb{C}^*}(\mathbb{P}^1)$. The series $\epsilon(z)$, similarly, is the generating series of localization contributions of a tree containing none of the markings. Clearly, such a tree needs to have degree at least one, so $\epsilon(z)$ is a multiple of y.

Let $\overline{\psi}_k$ be the pullback under π_l of the class ψ_k on $\overline{\mathcal{M}}_{g,n}$. It is well-known that $\overline{\psi}_k$ differs from ψ_k exactly on the boundary divisors of $\overline{\mathcal{M}}_{g,n}$ where the kth marking and some of the last l markings lie on a rational tail. By rewriting the classes $\psi_1, \ldots, \psi_n$ in terms of $\overline{\psi}_1, \ldots, \overline{\psi}_n$ and boundary divisors, and for each summand integrating along the fibers of the map forgetting all markings of the involved boundary divisors, we can rewrite Φ in the form

$$\Phi = \sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\sum_{i=0}^{g} c_i(\mathbb{E}) \lambda^{g-1-i} \prod_{k=1}^{n} \widetilde{S}(\alpha_k, \overline{\psi}_k) \prod_{k=n+1}^{n+l} \epsilon(\psi_k) \right)$$

for modified universal series $\widetilde{S}(\alpha, z)$. Surprisingly, the series $\widetilde{S}(\alpha, z)$ is easier to compute than $S(\alpha, z)$. In fact, it is closely related to the R-matrix for the equivariant Gromov–Witten theory of $\mathbb{P}^1$.

For the proof of Theorem 3.1, what is relevant from this formula is $\widetilde{S}(\alpha, 0)$, and we use the following result from [21]. (Note that $\widetilde{S}$ is called $\widetilde{S}_0$ in [21].)

Lemma 4.9. *We have the identities*

$$\widetilde{S}(\mathbf{1}_{\mathbb{P}^1}, 0) = \phi^{-1/4},$$

$$\widetilde{S}(H, 0) = \phi^{-1/4} \left(\frac{\lambda}{2} + \frac{\lambda}{2} \sqrt{\phi} \right),$$

in which

$$\phi := 1 + \frac{4y}{\lambda^2}.$$

5. Proof of Theorem 3.1

We now turn to the proof of Theorem 3.7, which, as explained above, implies Theorem 3.1.

First, we introduce a bracket notation in order to write Theorem 3.7 in a more economical way:

$$
\left(\prod_{h \in H} \mathrm{ev}_h^*(\phi_h \psi^{a_h})\right)_\Gamma^{\mathrm{WC}} := \prod_{h \in H} \mathrm{ev}_h^*(\phi_h \psi^{a_h}) \cap [Z_\Gamma^\epsilon]_{\mathrm{tw}}^{\mathrm{vir}}
$$

$$
- \sum_{K=\{k_v\}_{v \in V(\Gamma)}} \frac{1}{\prod_v k_v!} \sum_{\substack{\vec{\beta}=\{\beta_i^v\}_{v \in V(\Gamma), 0 \le i \le k_v} \\ \beta_0^v + \beta_1^v + \cdots + \beta_{k_v}^v = \beta(v)}} b_{\vec{\beta}*} c_* \left(\prod_{h \in H} \mathrm{ev}_h^*(\phi_h \psi^{a_h}) \prod_{v \in V(\Gamma)} \prod_{j=1}^{k_v} \mathrm{ev}_{v,j}^*(\mu_{\beta_j^v}^{\epsilon,\mathrm{tw}}(-\psi)) \cap [Z_{\Gamma_{K,\vec{\beta}}}^\infty]_{\mathrm{tw}}^{\mathrm{vir}}\right).
$$

Here, H is any subset of the half-edges of $\Gamma_{n'}$, and otherwise the notation is as in Theorem 3.7. One can extend the definition of $(\cdots)_\Gamma^{\mathrm{WC}}$ by linearity to allow any insertions in $\mathcal{H}[[\psi]]$.

With this notation, the statement of Theorem 3.7 can be written as

$$
(40) \qquad \sum_{l=0}^\infty \frac{1}{l!} \pi_{l*} \left(\prod_{h' \in L'} \mathrm{ev}_{h'}^*(\epsilon^{\lambda_0}(\psi)) \prod_{h \in L(\Gamma)} \mathrm{ev}_h^*(\alpha_h)\right)_{\Gamma+l}^{\mathrm{WC}} = 0,
$$

where $L(\Gamma)$ is the leg set of Γ and L' stands for the set of the l additional legs. It suffices to prove the theorem without any insertions α_h, with the understanding that the only state-space elements that can be pulled back under our evaluation maps and paired with the virtual class are those of compact type.

Remark 5.1. The proof is essentially formal once one assumes wall-crossing results in genus zero and the localization calculations of Sections 4.3 and 4.4. In particular, by [15] and Remark 4.8, the following proof applies without modification to the geometric chamber, yielding the other half of Theorem 1.2.

Proof of Theorem 3.7. First, apply induction on the degree β. When $\beta = 0$, the result is trivially true, since ϵ-stable and ∞-stable quasimaps agree in degree zero.

Assume, then, that $\beta > 0$ and that the theorem holds for all $\beta' < \beta$, and apply induction on the decorated graph $(\Gamma, v_\bullet, n')$. The partial order on the set of such data used to organize the induction is given by setting

$$
(\Gamma', v_\bullet', n'') \le (\Gamma, v_\bullet, n')
$$

if Γ is obtained from Γ' by replacing a subgraph $\Gamma'' \subset \Gamma'$ containing $v_\bullet'$ by the single vertex $v_\bullet$, and $n''(v) = n'(v)$ for all $v \in \Gamma \setminus \{v_\bullet\}$. This partial order is illustrated in Figure 2. In order to be able to apply induction, it is important to note that there is no infinite chain with respect to this partial order.

Now, suppose that the theorem holds for all $(\Gamma', v_\bullet', n'') < (\Gamma, v_\bullet, n')$. In order to prove that it holds for $(\Gamma, v_\bullet, n')$, the strategy is to lift to the twisted graph space and apply localization. There are many dual graphs for the twisted graph space that correspond (under the map p that forgets L_2, z_1, and z_2) to Γ, but we describe a very special set of such graphs that are constructed to control the localization graphs that appear. The idea is that, for each vertex v of Γ other than $v_\bullet$, we replace v with a vertex of δ-degree zero in the twisted graph space, so that it is forced to be a vertex component in any contributing localization graph. We replace each edge e of Γ with a vertex of δ-degree 1, so that it becomes an edge component in the localization graphs. All the remaining δ-degree is concentrated at $v_\bullet$, which means
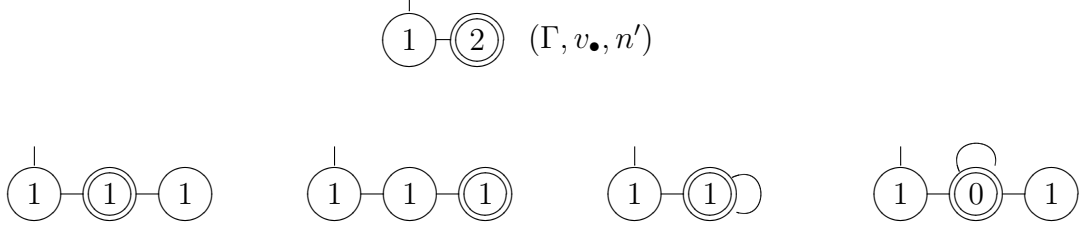
FIGURE 2. An example of a graph $(\Gamma, v_\bullet, n')$ and some of the graphs below it in the partial order. The vertex $v_\bullet$ (or, in the second line, $v'_\bullet$) is marked with a double circle, and the numbers in each circle indicate the genus. The data of n' (or, in the second line, n'') is not shown, but can be added arbitrarily as long as the left-most vertex agrees between the first and second lines.

that $v_\bullet$ may decompose into an entire subgraph in the contributing localization graphs—but, away from $v_\bullet$, these graphs are identical to Γ.

To carry this out precisely, first choose a decomposition of the half-edges of Γ at $v_\bullet$ into

$$H(v_\bullet) = N_0^\bullet \cup N_\infty^\bullet.$$

Let N_0' be the set of all half-edges of Γ at vertices in $V_0 \setminus \{v_\bullet\}$, and similarly, let N_∞' be the set of all half-edges of Γ at vertices in V_∞. Let

$$N_0 = N_0^\bullet \cup N_0', \qquad N_\infty = N_\infty^\bullet \cup N_\infty'.$$

Finally, choose an integer $\delta \geq 0$. Define a dual graph $\widetilde{\Gamma}_\delta$ for the twisted graph space by taking any edge e of Γ consisting of a half-edge in N_0 and a half-edge in N_∞, and subdividing e by adding a single new vertex v with $g(v) = \beta(v) = 0$. (This creates new half-edges, but the multiplicities on these new half-edges are determined.) Extend the coloring on vertices of Γ to vertices of $\widetilde{\Gamma}_\delta$ by setting

$$V(\widetilde{\Gamma}_\delta) = V_0 \cup V_\infty \cup V_{\text{new}},$$

where V_{new} denotes the vertices of $\widetilde{\Gamma}_\delta$ not coming from vertices of Γ. If we define

$$D \colon V(\widetilde{\Gamma}_\delta) \to \mathbb{Z}^{\geq 0}$$

by

$$D(v) = \begin{cases} \delta & \text{if } v = v_\bullet \\ 1 & \text{if } v \in V_{\text{new}} \\ 0 & \text{otherwise} \end{cases}$$

and define $\delta' = \delta + |V_{\text{new}}|$, then $\widetilde{\Gamma}_\delta$ defines a moduli space $\mathbb{P}Z_{\widetilde{\Gamma}_\delta}^\epsilon$ with a finite map to a substack of $\mathbb{P}Z_{g,n+n',\beta,\delta'}^\epsilon$. Note that we have precisely constructed $\widetilde{\Gamma}_\delta$ so that we have

$$p \colon \mathbb{P}Z_{\widetilde{\Gamma}_\delta}^\epsilon \to Z_\Gamma^\epsilon,$$

where p is the morphism that forgets L_2, z_1, and z_2 (and stabilizes as necessary).

The proof of the theorem now proceeds by computing, via localization, the difference between the expressions

(41) $$\sum_{\delta=0}^\infty y^\delta p_* \left(\prod_{n \in N_0} \text{ev}_n^*(\mathbf{1} \otimes \varphi_0) \prod_{n \in N_\infty} \text{ev}_n^*(\mathbf{1} \otimes \varphi_\infty) \cap [\mathbb{P}Z_{\widetilde{\Gamma}_\delta}^\epsilon]^{\text{vir}} \right)$$

and

$$(42) \quad \sum_{\delta=0}^{\infty} \sum_{K=\{k_v\}_{v\in V}} \frac{1}{\prod_v k_v!} \sum_{\substack{\vec{\beta}_v=\{\beta_j^v\}_{j\in[k_v]} \\ \beta_1^v+\cdots+\beta_{k_v}^v=\beta(v)}} \sum_{\substack{\vec{\delta}_v=\{\delta_j^v\}_{j\in[k_v]} \\ \delta_j^v\leq\beta_j^v \\ \delta_j^v=0 \text{ if } v\neq v_\bullet}} y^{\delta+\delta_1^{v_\bullet}+\cdots+\delta_{k_{v_\bullet}}^{v_\bullet}}$$

$$p_*\widetilde{b}_{\vec{\beta}_v*}\widetilde{c}_{v*}\left(\prod_{n\in N_0} \mathrm{ev}_n^*(\mathbf{1}\otimes\varphi_0) \prod_{n\in N_\infty} \mathrm{ev}_n^*(\mathbf{1}\otimes\varphi_\infty) \prod_{v\in V}\prod_{j=1}^{k_v} \mathrm{ev}_{v,j}^*(\widetilde{\mu}_{\beta_j^v,\delta_j^v}^\epsilon(-\psi)) \cap [\mathbb{P}Z_{\widetilde{\Gamma}_{\delta,K}}^\infty]^{\mathrm{vir}} \right),$$

which, crucially, are both rational functions (in fact, Laurent polynomials) in λ. Here,

$$(43) \qquad \varphi_0 = \frac{[0]}{\lambda} = \frac{H}{\lambda}, \qquad \varphi_\infty = -\frac{[\infty]}{\lambda} = \mathbf{1} - \frac{H}{\lambda}.$$

The morphisms $\widetilde{b}_{\vec{\beta}_v}$ and $\widetilde{c}_v$ are lifts of $b_{\vec{\beta}_v}$ and c_v to the twisted graph space, and $\widetilde{\Gamma}_{\delta,K}$ is obtained from $\widetilde{\Gamma}_\delta$ by setting the β-degree of all vertices to zero and adding k_v additional legs of multiplicity $\frac{1}{d}$ to each vertex v. The mirror transformation $\widetilde{\mu}^\epsilon$ of the twisted graph space is defined by

$$\sum_{\delta=0}^{\infty} y^\delta \widetilde{\mu}_{\beta,\delta}^\epsilon(z) = \mu_\beta^{\epsilon,\mathrm{tw}}(z)\otimes\varphi_0 - \sum_{\delta=1}^{\beta} y^\delta \frac{\left[J_{\mathrm{tw}}^\epsilon\left(q,\frac{\lambda_0}{\delta}\right)\right]_{q^\beta}\lambda_\infty}{\left(\frac{\lambda_\infty}{\delta}+z\right)\prod_{i=1}^{\delta}\prod_{j\in\{0,\infty\}}\frac{i\lambda_j}{\delta}}\otimes\varphi_\infty.$$

Finally, we note that in (41) and (42), insertions at $n\in N_0$ and $n\in N_\infty$ can occur at both marked points and half-nodes; the latter is well-defined by pushforward from the fiber product defining $\mathbb{P}Z_{\widetilde{\Gamma}_\delta}^\epsilon$ or $\mathbb{P}Z_{\widetilde{\Gamma}_{\delta,K}}^\infty$.

The definition of $\widetilde{\Gamma}_\delta$ is specifically chosen so that the only localization graphs contributing to (41) are those obtained from $\widetilde{\Gamma}_\delta$ by

- replacing each $v\in V_{\mathrm{new}}$ by an edge e with $\delta(e)=1$;
- keeping each vertex $v\in(V_0\cup V_\infty)\setminus\{v_\bullet\}$ the same;
- replacing $v_\bullet$ by a localization graph $\Lambda_\delta^\bullet$ of genus $g(v_\bullet)$, total β-degree $\beta(v_\bullet)$, and total edge degree δ.

The description of the possible localization graphs for (42) is similar, except that in that case $\Lambda_\delta^\bullet$ has β-degree zero and additional markings depending on the choice of K.

We can view both (41) and (42) as sums over the same set of graphs in the following way. For each tuple K and each choice of a contributing localization graph Λ_K to (42), and each choice of $\vec{\beta}_{v_\bullet}$ and $\vec{\delta}_{v_\bullet}$, we define Λ by applying the following operations to Λ_K:

- Remove all extra legs at vertices v with $j(v)=0$ and add their β-degree to the incident vertex.
- Replace each extra leg at a vertex v by $j(v)=\infty$ with a new edge connected to a new genus-zero, valence-one vertex. The degree of the new edge is prescribed by $\vec{\delta}_{v_\bullet}$, and the β-degree of the former extra leg is put onto the new vertex.

The resulting Λ is a localization graph contributing to (41), so long as it satisfies $\delta(e)>\beta(v)$ for each genus-zero, valence-one vertex v with unique incident edge e (see Remark 4.2). If Λ does not satisfy this degree condition, but it satisfies all other conditions of a dual graph contributing to (41), then we refer to it as a "fake localization graph." Conversely, given a

fake localization graph Λ, we refer to the sum of all contributions of localization graphs Λ_K corresponding to Λ as the "contribution of Λ to (42)."

There are several summands in the contribution of Λ to (42). For each vertex v of genus zero and valence one with unique incident edge e such that $\delta(e) > \beta(v)$, the local contribution to the localization formula is given by

$$\delta_{\beta(v),0}\frac{\lambda_0}{\delta(e)} + \mu_{\beta(v)}^{\epsilon,\mathrm{tw}}\left(\frac{\lambda_0}{\delta(e))}\right)$$

$$+ \sum_{\substack{k\geq 1, \beta_0+\cdots+\beta_k=\beta(v) \\ k+\beta_0>1}} \mathrm{ev}_{1,*}\left(\frac{\prod_{i=1}^k \mathrm{ev}_{i+1}^*\left(\mu_{\beta_i}^{\epsilon,\mathrm{tw}}(-\psi_{i+1})\right)}{\left(\frac{\lambda_0}{\delta(e)}-\psi_1\right)k!} \cap [Z_{0,1+k,\beta_0}^\infty]_{\mathrm{tw}}^{\mathrm{vir}}\right)$$

$$= \left[J^{\infty,\mathrm{tw}}\left(q, \sum_\beta q^\beta \mu_\beta^{\epsilon,\mathrm{tw}}\left(-\frac{\lambda_0}{\delta(e)}\right), \frac{\lambda_0}{\delta(e)}\right)\right]_{q^{\beta(v)}},$$

where the first two summands correspond to the cases where v is unstable and the last summand corresponds to the case where v is stable. By the J-function wall-crossing (15), this contribution is equal to

$$\left[J^{\epsilon,\mathrm{tw}}\left(q, \frac{\lambda_0}{\delta(e)}\right)\right]_{q^{\beta(v)}}.$$

The definition of $\widetilde{\mu}^\epsilon$ is exactly chosen such that the local contribution to the localization formula of a vertex v of genus zero and valence one with unique incident edge vanishes if $\delta(e) \leq \beta(v)$. This means that the contribution to (42) of any fake localization graph Λ that is not an actual localization graph contributing to (41) vanishes. Because of that, from now on we only consider the contribution of actual localization graphs to the difference of (41) and (42).

Fix a localization graph Λ contributing to (41), and let Γ_Λ be the prestable dual graph of the image under p of a generic element of the fixed locus associated to Λ (in other words, Γ_Λ is the graph Γ of Section 4.3.4). If Λ has a genus-zero component of positive β-degree that is contracted under the forgetful map, then the degree of the vertex of Γ_Λ to which this component is contracted is strictly less than β. It follows that the contribution of this vertex to (41) equals its contribution to (42), by the inductive hypothesis together with an application of the genus-zero wall-crossing theorem [23] and the above application of the J-function wall-crossing. Thus, for the study of the difference of (41) and (42), we can from now on assume that the localization graphs have no genus-zero, valence-one vertex of positive β-degree.

Notice that Γ_Λ is obtained from the original dual graph Γ by (possibly) degenerating the vertex $v_\bullet$ but leaving all other vertices the same, and Λ is obtained from Γ_Λ by attaching trees of the following types, which we refer to as type-1, type-2, and type-3 trees, respectively:

(1) those that connect two vertices of Γ_Λ that come from the degeneration of $v_\bullet$;
(2) those that are attached to a single vertex of Γ_Λ and contain exactly one leg of $v_\bullet$;
(3) those that are attached to a single vertex of Γ_Λ and contain no leg of $v_\bullet$.

The local contribution of any such tree to the localization formulas for (41) or (42) are evidently identical.

Notice, furthermore, that many localization graphs Λ yield the same dual graph Γ_Λ; in light of this, we rearrange the localization graphs according to the dual graph they produce. Specifically, let Γ' be any dual graph arising as Γ_Λ for some Λ. We define the decorations on Γ' by first choosing an arbitrary new vertex $v'_\bullet$ of Γ' such that $j(v'_\bullet) = 0$ for all Λ such that $\Gamma_\Lambda = \Gamma'$. Then, for each such Λ, define $n''_\Lambda : V(\Lambda) \to \mathbb{Z}^{\geq 0}$ by

$$n''_\Lambda(v) = \begin{cases} n'(v), & \text{if } v \notin \Lambda^\bullet_\delta, \\ 0, & \text{if } v = v'_\bullet, \\ \#\{\text{type-3 trees at } v\} & \text{otherwise}, \end{cases}$$

Finally, define the coloring $V(\Gamma') = V'_0 \cup V'_\infty$ of the vertices of Γ' via the labels $j(v)$ on the stable vertices of any Λ such that $\Gamma_\Lambda = \Lambda$.

Fix a decorated dual graph $(\Gamma', v'_\bullet, n'')$, and consider the set of localization graphs

$$\mathrm{Loc}(\Gamma') = \{\Lambda \mid \Gamma_\Lambda = \Gamma', \ n'_\Lambda = n''\}.$$

Furthermore, within $\mathrm{Loc}(\Gamma')$, fix a set of Λ that agree with each other except for type-3 trees at $v'_\bullet$. Let $S^{\lambda_0}(\varphi_j, z)$ be the generating series keeping track of the contribution to the localization formula for (41) (or (42)) of all type-2 trees attached to $v'_\bullet$ and with insertion φ_j at the leg, and let $\epsilon^{\lambda_0}(z)$ be the similar generating series corresponding to type-3 trees attached to $v'_\bullet$. We note that $S^{\lambda_0}(\varphi_j, z)$ agrees with $S(\varphi_j, z)$ from Section 4.4 except for the substitution $\lambda \to \lambda_0$, and the relation between ϵ and ϵ^{λ_0} is the same.

The contribution of our fixed set of localization graphs to the difference of (41) and (42) is essentially

$$\sum_{l=0}^\infty \frac{1}{l!} \pi_{l*} \left(\prod_{h \in H} \mathrm{ev}_h^*(U_h(\psi)) \prod_{h \in L'} \mathrm{ev}_h^*(\epsilon^{\lambda_0}(\psi)) \right)^{\mathrm{WC}}_{\Gamma'+l},$$

where H and L' denote the set of half-edges at $v'_\bullet$ or the l additional half-edges, respectively, and $U_h(\psi)$ stands for the localization contribution of the (possibly empty) tree at h. The above formula differs from the correct formula by an invertible factor from the localization contribution of trees that are not attached to $v'_\bullet$.

We claim that Theorem 3.7 would imply that this contribution is zero. Indeed, rewriting the insertions in terms of the ψ-classes $\overline{\psi}$ pulled back under π_l and using a slightly generalized bracket notation, it becomes

$$(44) \qquad \sum_{l=0}^\infty \frac{1}{l!} \pi_{l*} \left(\prod_{h \in H} \mathrm{ev}_h^*(\widetilde{U}_h(\overline{\psi})) \prod_{h \in L'} \mathrm{ev}_h^*(\epsilon^{\lambda_0}(\psi)) \right)^{\mathrm{WC}}_{\Gamma'+l},$$

where the relation between U_h and $\widetilde{U}_h$ is the same as the one between S and $\widetilde{S}$ discussed in Section 4.4. Adding insertions has the effect of multiplying (40) by a cycle, so if Theorem 3.7 holds for $(\Gamma', v'_\bullet, n'')$, then the contribution of $(\Gamma', v'_\bullet, n'')$ to the localization must vanish. In particular, if $v_\bullet$ degenerates at all in the passage from Γ to Γ', then the contribution of $(\Gamma', v'_\bullet, n'')$ to the difference of (41) and (42) vanishes by the induction hypothesis.

Thus, all that remains is the contribution to the localization formula from Λ for which the decorated stabilized graph is the same as $(\Gamma, v_\bullet, n')$ except for possible additional legs at $v_\bullet$. The contribution from these remaining graphs to the difference of (41) and (42) can be

written as

$$\sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\prod_{h \in N_0^{\bullet}} \mathrm{ev}_h^*(\widetilde{S}^{\lambda_0}(\varphi_0, \overline{\psi})) \prod_{h \in N_{\infty}^{\bullet}} \mathrm{ev}_h^*(\widetilde{S}^{\lambda_0}(\varphi_{\infty}, \overline{\psi})) \prod_{h \in L'} \mathrm{ev}_h^*(\epsilon^{\lambda_0}(\psi)) \right)_{\Gamma+l}^{\mathrm{WC}}.$$

We filter these remaining localization contributions (as well as the statement of Theorem 3.7) according to cohomological degree. This can be made explicit by separating the "insertion" $\exp(\epsilon^{\lambda_0})$ into its components of degree i:

$$\sum_{l=0}^{\infty} \frac{1}{l!} \epsilon^{\lambda_0}(z_1) \otimes \cdots \otimes \epsilon^{\lambda_0}(z_l) = \sum_{i=-\infty}^{\infty} \epsilon_i,$$

where the degree of a homogeneous element

$$\phi_1 \otimes \cdots \otimes \phi_l \, \lambda^i y^j z_1^{a_1} \cdots z_l^{a_l} \in \widetilde{\mathcal{H}}^{\otimes l}(\lambda)[\![y, z_1, \ldots, z_l]\!]$$

is defined by

$$\mathrm{codim}(\phi_1) + \cdots + \mathrm{codim}(\phi_l) + 0 \cdot (i+j) + (a_1 + \cdots + a_l) - l.$$

We can further decompose

$$\epsilon_i = \sum_{l=0}^{\infty} \epsilon_i^l,$$

such that $\epsilon_i^l \in \mathcal{H}^{\otimes l}(\lambda)[\![y, z_1, \ldots, z_l]\!]$.

We now apply a further induction on i to prove that the ϵ_i-part of the localization contribution, namely

$$(45) \qquad \sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\prod_{h \in N_0^{\bullet}} \mathrm{ev}_h^*(\widetilde{S}^{\lambda_0}(\varphi_0, \overline{\psi})) \prod_{h \in N_{\infty}^{\bullet}} \mathrm{ev}_h^*(\widetilde{S}^{\lambda_0}(\varphi_{\infty}, \overline{\psi})) \cdot \mathrm{ev}_{H_l}^*(\epsilon_i^l(\psi_{H_l})) \right)_{\Gamma+l}^{\mathrm{WC}},$$

and the ϵ_i-part of Theorem 3.7, that is

$$(46) \qquad \sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\mathrm{ev}_{H_l}^*(\epsilon_i^l(\psi_{H_l})) \right)_{\Gamma+l}^{\mathrm{WC}},$$

both vanish. Here, abusing notation somewhat, we denote by ev_{H_l} the evaluation map at all of the l additional markings and by ψ_{H_l} the corresponding collection of ψ-classes.

To start the induction, notice that for $i \ll 0$, the ϵ_i-part of the localization contribution of Γ has dimension greater than the dimension of each component of Z_{Γ}^{ϵ}, so this contribution vanishes. Suppose, then, that (45) vanishes whenever $i < i_0$. Consider the part of the contribution of Γ in dimension

$$(47) \qquad \mathrm{vdim}(Z_{\Gamma}^{\epsilon}) - i_0.$$

The part in this dimension only involves ϵ_i for $i \leq i_0$. Moreover, all of its terms ϵ_i for $i < i_0$ are obtained from (46) by multiplication by a cohomology class, and hence they vanish by the inductive hypothesis. Thus, the part of the localization contribution of dimension (47) equals

$$(\widetilde{S}(\varphi_0, 0))^{|N_0^{\bullet}|} (\widetilde{S}(\varphi_{\infty}, 0))^{|N_{\infty}^{\bullet}|} \sum_{l=0}^{\infty} \frac{1}{l!} \pi_{l*} \left(\mathrm{ev}_{H_l}^*(\epsilon_i^l(\psi_{H_l})) \right)_{\Gamma+l}^{\mathrm{WC}}.$$

The key observation, then, is that by Lemma 4.9 and (43), this contribution changes by a factor of

$$
(48) \qquad\qquad \frac{1 - \sqrt{\phi}}{1 + \sqrt{\phi}}
$$

if we change our choice of $N_\infty^\bullet$ such that it contains one additional element, and so that $N_0^\bullet$ contains one element less. This is possible because either Γ is the trivial graph and $v_\bullet$ has all of the legs (of which there are at least one), or Γ is a non-trivial graph and there is therefore at least one edge at $v_\bullet$.

The change (48) in the contribution, though, is not a rational function in λ, despite the fact that, before and after the change, the contribution is a Laurent polynomial in λ. This is only possible if the contributions at hand are identically zero, which completes the proof that the ϵ_{i_0}-part of the contribution of Γ to the difference between (41) and (42) vanishes. Dividing by

$$
(\widetilde{S}(\varphi_0, 0))^{|N_0^\bullet|} (\widetilde{S}(\varphi_\infty, 0))^{|N_\infty^\bullet|}
$$

proves the ϵ_{i_0}-part of Theorem 3.7.   □

APPENDIX A. WALL-CROSSING WHEN $n = 0$ (BY YANG ZHOU)

In this appendix, we extend the wall-crossing in the Landau–Ginzburg phase to the case $n = 0$, thus completing the proof of Theorem 1.2. Thanks to Theorem 3.1, in order to express $[Z_{g,0,\beta}^\epsilon]^{\mathrm{vir}}$ in terms of ∞-theory virtual cycles, it suffices to express it in terms of $[Z_{g,(1/d),\beta}^\epsilon]^{\mathrm{vir}}$. This can be done via an analogue of the dilaton equation when $g > 1$, and via an analogue of the divisor equation when $N > 1$.

The direct generalizations of the dilaton and divisor equations fail in this setting, due to the incompatibility of the virtual cycles under the forgetful map

$$
Z_{g,(1/d),\beta}^\epsilon \to Z_{g,0,\beta}^\epsilon.
$$

However, this incompatibility can be remedied by replacing the marking with a "light" marking, which can indeed be forgotten without affecting the virtual cycle. Applying the techniques developed in the previous work [46], the virtual cycle with a light marking can then be related to $[Z_{g,(1/d),\beta}^\epsilon]^{\mathrm{vir}}$ via another wall-crossing formula.

A.1. **The light marking.** The first task is to define ϵ-stable Landau–Ginzburg quasimaps with a light marking of multiplicity $1/d$. We assume that ϵ is not on a wall—i.e., $k\epsilon \neq 1$ for any integer k—and that $2g - 2 + n + \epsilon\beta > 0$. We then pick a positive rational number δ small enough so that

$$
(*) \qquad\qquad k\epsilon - 1 \quad \text{and} \quad k\epsilon - 1 + \delta \text{ have the same sign for any integer } k.
$$

Definition A.1. An ϵ-stable Landau–Ginzburg quasimap to Z with a light marking of multiplicity $1/d$ and n heavy markings consists of an $(n+1)$-pointed prestable Landau–Ginzburg quasimap of genus g to Z

$$
(C; q_1, \ldots, q_{n+1}; L; \vec{p})
$$

satisfying the following conditions:

- *Light marking*: The isotropy group at q_1 is cyclic of order d, and the line bundle L has multiplicity $1/d$ at q_1.

- *Nondegeneracy*: The zero set of $\vec{p}$ is disjoint from the markings $q_2, \ldots, q_{n+1}$ and the nodes of C, and for each zero q of $\vec{p}$, the order of the zero satisfies

(49)
$$\mathrm{ord}_q(\vec{p}) \leq \frac{1}{\epsilon}.$$

  Here, when $q = q_1$, we define the vanishing order of p_i to be the multiplicity of the divisor (p_i) on the coarse curve, or in other words, the integer k such that

$$(p_i) = kd[q_1] + [\text{other points}].$$

  In particular, by (*), condition (49) for $q = q_1$ is equivalent to $\delta + \epsilon \mathrm{ord}_{q_1}(\vec{p}) < 1$.
- *Stability*: The $\mathbb{Q}$-line bundle

$$(L^{\otimes -d} \otimes \omega_{\mathrm{log}})^{\otimes \epsilon} \otimes \omega_{\mathrm{log}}(d(\delta - 1) \cdot [q_1])$$

  is ample. (Note, here, that the line bundle $\omega_{\mathrm{log}}(d(\delta - 1) \cdot [q_1])$ is the pullback of $\omega_{|C|}(\delta \overline{q_1} + \overline{q_2} + \cdots + \overline{q_{n+1}})$, where $\overline{q_i}$ is the image of q_i in the coarse curve $|C|$.)

The stability condition above is equivalent to q_1 being a weight-δ ("light") marking and $q_2, \ldots, q_{n+1}$ being weight-1 ("heavy") markings, in the sense of [16]. It is clear that the definition is independent of the choice of δ.

The theory of quasimaps with a light marking is parallel to the case with only heavy markings. In particular, there is a proper Deligne–Mumford stack $Z^{\epsilon,\delta}_{g,(1/d)+n,\beta}$ parameterizing genus-g, ϵ-stable Landau–Ginzburg quasimaps of degree β to Z, with a light marking of multiplicity $1/d$ and n heavy markings, up to isomorphism. The compact-type virtual cycle $[Z^{\epsilon,\delta}_{g,(1/d)+n,\beta}]^{\mathrm{vir}}$ and evaluation map ev_1 are defined in the same way as before, with the only difference being that ev_1 maps to the rigidified inertia stack $\overline{\mathcal{I}}[\mathbb{C}^N/\mathbb{C}^*]$ for the action of $\mathbb{C}^*$ on $\mathbb{C}^N$ with weights $(d, \ldots, d)$. Indeed, it lands in the component indexed by $\bar{1} \in \mathbb{Z}_d$, which is isomorphic to the quotient $[\mathbb{C}^N/\mathbb{C}^*]$ by the weight-1 action.

There is a forgetful morphism

$$\tau : Z^{\epsilon,\delta}_{g,(1/d)+n,\beta} \to Z^{\epsilon}_{g,n,\beta}$$

forgetting the light marking, which only contracts components on which the degree of L is zero. Thus, $Z^{\epsilon,\delta}_{g,(1/d)+n,\beta}$ is identified with the universal curve of $Z^{\epsilon}_{g,n,\beta}$, and in particular, τ is flat. Moreover, the universal line bundles of the two moduli spaces are compatible under τ, and hence, so are the relative perfect obstruction theories. Thus, we obtain:

Lemma A.2. $\tau^*[Z^{\epsilon}_{g,n,\beta}]^{\mathrm{vir}} = [Z^{\epsilon,\delta}_{g,(1/d)+n,\beta}]^{\mathrm{vir}}$.

A.2. **The master space.** Our next goal is to compare the virtual cycle for the moduli space $Z^{\epsilon,\delta}_{g,(1/d),\beta}$ with one light marking (and no heavy markings) to the virtual cycle for the moduli space $Z^{\epsilon}_{g,(1/d),\beta}$ with one heavy marking (and no light markings). The technique for doing so is localization on a larger moduli space that we refer to as the "master space".

Assume that $2g - 2 + n + 1 + \epsilon\beta > 0$ and $2g - 2 + n \geq -1$, and let S be any scheme. Then the S-points of the master space are as follows:

Definition A.3. An S-family of ϵ-stable Landau–Ginzburg quasimaps to Z *with a mixed marking* consists of

$$(\pi : C \to S; q_1, q_2, \ldots, q_{n+1}; L, N; \vec{p}, v_1, v_2),$$

where

(1) $(\pi : C \to S; \vec{q}; L; \vec{p})$ is an S-family of $(n + 1)$-pointed prestable Landau–Ginzburg quasimaps of genus g to Z;

(2) N is a line bundle on S;

(3) $v_1 \in H^0(S, T_{q_1} \otimes N)$ and $v_2 \in H^0(S, N)$ are sections without common zeros, where T_{q_1} is the line bundle on S formed by the relative tangent spaces to the *coarse* curves at q_1.

We require that this data satisfy the following conditions on each geometric fiber:

- *Mixed marking:* The isotropy group at q_1 is cyclic of order d, and the line bundle L has multiplicity $1/d$ at q_1.
- *Nondegeneracy:* The zero set of $\vec{p}$ is disjoint from markings $q_2, \ldots, q_{n+1}$ and the nodes of C, and for each zero q of $\vec{p}$, the order of the zero satisfies

$$\operatorname{ord}_q(\vec{p}) \leq \frac{1}{\epsilon}.$$

- *Generic Stability:* The $\mathbb{Q}$-line bundle

$$(L^{\otimes -d} \otimes \omega_{\log})^{\otimes \epsilon} \otimes \omega_{\log}$$

  is ample.
- When $v_1 = 0$, $\vec{p}$ does not vanish at q_1.
- When $v_2 = 0$, the $\mathbb{Q}$-line bundle

$$(L^{\otimes -d} \otimes \omega_{\log})^{\otimes \epsilon} \otimes \omega_{\log}(d(\delta - 1) \cdot [q_1])$$

  is ample.

Thus, a family over a point consists of a prestable Landau–Ginzburg quasimap to Z together with $v_1/v_2 \in T_{q_1} C \cup \{\infty\}$, where q_1 behaves as a heavy marking when $v_1 = 0$ and as a light marking when $v_2 = 0$. We call q_1 a *mixed marking* in what follows, whereas $q_2, \ldots, q_{n+1}$ are heavy markings.

Theorem A.4. *There is a proper Deligne–Mumford stack $\widetilde{Z}^{\epsilon,\delta}_{g,(1/d)+n,\beta}$ (the "master space") parameterizing genus-g, ϵ-stable Landau–Ginzburg quasimaps of degree β to Z, with a mixed marking of multiplicity $1/d$ and n heavy markings, up to isomorphism.*

Proof. The same argument as in the proof of [46, Theorem 4] shows that the moduli problem is represented by a Deligne–Mumford stack of finite type over $\mathbb{C}$. We now use the valuative criterion to prove its properness. The proof is similar to that of [46, Theorem 5].

Let R be a Henselian discrete valuation ring with residue field $\mathbb{C}$. Let $B = \operatorname{Spec} R$, with closed point $b \in B$ and generic point $B^\circ = B \backslash \{b\}$. For a stable family

$$\xi^\circ = (\pi^\circ : C^\circ \to B^\circ; \vec{q^\circ}; L^\circ, N^\circ; \vec{p^\circ}, v_1^\circ, v_2^\circ)$$

over B°, we must show that (possibly after a finite base change) we can extend ξ° to a B-family

$$\xi = (\pi : C \to B; \vec{q}; L, N; \vec{p}, v_1, v_2),$$

and the extension is unique up to unique isomorphism.

Let $|C^\circ|$ be the coarse moduli of C°. The line bundle

$$(L^\circ)^{\otimes -d} \otimes \omega_{\pi^\circ, \log}$$

descends to a line bundle M° on $|C^\circ|$, and the section $\vec{p^\circ}$ descend to a section $\vec{f^\circ}$ of $(M^\circ)^{\oplus N}$. Let $\overline{q_i^\circ}$ be the image of q_i° on $|C^\circ|$, and we set

$$\eta^\circ := (|\pi^\circ| : |C^\circ| \to B^\circ; \overline{q_1^\circ}, \dots, \overline{q_{n+1}^\circ}; M^\circ; \vec{f^\circ}),$$

which is a family of prestable quasimaps to $\mathbb{P}^{N-1}$.

We claim that, for any B-family of prestable quasimaps

(50) $$\eta = (|\pi| : |C| \to B; \overline{q_1}, \dots, \overline{q_{n+1}}; M; \vec{f})$$

extending η°, there is unique prestable extension ξ of ξ° whose underlying family of quasimaps is η. Indeed, given η, by a standard argument (see, for example, [35, Theorem 1.5.1] or [20, Thoerem 4.1.7]), there is a unique way (possibly after a finite base change) to add stack structure at nodes and markings and extend L° to a representable d-th root L with

$$L^{\otimes d} \cong \omega_{\pi,\log} \otimes M^\vee.$$

Then, as in [46, Theorem 5], there is a unique extension of $(N^\circ, v_1^\circ, v_2^\circ)$ to (N, v_1, v_2) such that v_1 and v_2 have no common zeros.

The stability condition for ξ can be reformulated in terms of η, v_1, and v_2 as follows: ξ is stable if and only if $(|\pi| : |C| \to B; \overline{q_1}, \dots, \overline{q_{n+1}})$ is a family of prestable $(n+1)$-pointed curves and for each geometric fiber,

- $\vec{f}$ does not vanish at the nodes;
- for each zero q of $\vec{f}$, we have

$$\mathrm{ord}_q(\vec{f}) < \frac{1}{\epsilon};$$

- the $\mathbb{Q}$-line bundle

$$M^{\otimes \epsilon} \otimes \omega_{|C|/B}([\overline{q_1}] + \cdots [\overline{q_{n+1}}])$$

 is ample;
- when $v_1 = 0$, $\vec{f}$ does not vanish at q_1;
- when $v_2 = 0$, the $\mathbb{Q}$-line bundle

$$M^{\otimes \epsilon} \otimes \omega_{|C|/B}(\delta[\overline{q_1}] + [\overline{q_2}] + \cdots [\overline{q_{n+1}}])$$

 is ample.

We will show that, up isomorphism, there is a unique η satisfying the above conditions.

As in the proof of [46], we may assume that π° is smooth and both v_1° and v_2° are non-zero. We first consider the case

(51) $$2g - 2 + n + \epsilon \deg M + \delta > 0.$$

When viewing $\overline{q_1}$ as a light marking, the generic fiber is ϵ-stable. Possibly after finite base change, we extend the quasimap η° to an ϵ-stable B-family with light marking $\overline{q_1}$, and we get a unique extension (N, v_1, v_2) such that v_1 and v_2 have no common zeros. The only situation that violates the master-space stability condition is if

(52) $$v_1(b) = 0 \quad \text{and} \quad \overline{q_1} \text{ is a zero of } \vec{f} \text{ in the special fiber.}$$

If this happens, we blow up the total space of the family at the marking $\overline{q_1}$ of the special fiber. It is easy to see that $\vec{f^\circ}$ uniquely extends to a B-family of prestable quasimaps from the new family of curves to $\mathbb{P}^{N-1}$.

We repeat this procedure until (52) does not hold. As in [46] , the vanishing order of v_1 at b drops by one after each step, so the procedure eventually terminates. Finally, we contract the exceptional divisors on which M is trivial. Note that M has nontrivial degree on the exceptional divisor of the last blowup, which contains $\overline{q_1}$ and is not contracted. Thus, it is easy to see that after contractions, (52) still does not hold, so we do get a stable family in the case of (51).

Suppose, now, that

$$(53) \qquad\qquad 2g - 2 + n + \epsilon \deg M + \delta \leq 0.$$

Since we have assumed that $2g - 2 + n \geq -1$ and δ is sufficiently small, (53) only holds when $g = 0$, $n = 1$, and $\epsilon \deg M < 1$. In this case, we can find a B°-isomorphism between $|C^\circ|$ and $\mathbb{P}^1 \times B^\circ$, identifying $\overline{q_1^\circ}$ with $\{0\} \times B^\circ$, $\overline{q_2^\circ}$ with $\{\infty\} \times B^\circ$, and v_1° / v_2° with the standard tangent vector $\partial/\partial z$, where z is the coordinate on $\mathbb{P}^1$. We first take the constant family as the extension of the curves, and the markings and the prestable quasimaps extend uniquely. The only situation that violates the master-space stability condition is when

$$(54) \qquad\qquad \overline{q_2} \text{ is a zero of } \vec{f} \text{ in the special fiber.}$$

If this happens, we modify the underlying curves by repeated blowups at q_2 of the special fiber until (54) no longer holds. Finally, we blow down the unstable components. This produces a stable reduction. Note that the irreducible component of the special fiber containing $\overline{q_1}$ needs to be blown down precisely when its M-degree is zero, which takes v_1 to 0 and $\overline{q_1}$ to some point where $\vec{f}$ does not vanish. This completes the construction in the case of (53).

The uniqueness part is standard, so we omit the details. The key is that, if C' is a smooth rational subcurve containing $\overline{q_1}$ and only one other special point, then C' cannot be contracted unless $\deg M|_{C'} = 0$, since contracting C' takes v_1 to 0. $\qquad\square$

A.3. **Virtual cycle for the master space.** Let $\widetilde{X}^{\epsilon,\delta}_{g,(1/d),\beta}$ be the master-space analogue of $X^\epsilon_{g,(1/d),\beta}$. More precisely, an S-point of $\widetilde{X}^{\epsilon,\delta}_{g,(1/d),\beta}$ is

$$(\pi : C \to S; q_1; L, N; \vec{p}, v_1, v_2) \in \widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}(S)$$

together with a section

$$\vec{x} = (x_1, \cdots, x_M) \in \Gamma(C, \bigoplus_{i=1}^M L^{\otimes w_i}).$$

As before, $\widetilde{X}^{\epsilon,\delta}_{g,(1/d),\beta}$ admits a perfect obstruction theory relative to the smooth Artin stack $\widetilde{D}_{g,(1/d),\beta}$ parametrizing only $(C; q_1; L, N; v_1, v_2)$. The relative perfect obstruction theory is defined by the same formula as (11), and the marking is narrow. The same formula as before defines a cosection whose degeneracy locus is $\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}$, and we get a cosection-localized virtual class

$$[\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}} \in A_*(\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}).$$

A.4. **Localization on the master space.** From now on, we assume that $g \geq 1$.

Define a $\mathbb{C}^*$-action on $\widetilde{X}^{\epsilon,\delta}_{g,(1/d),\beta}$ by[7]

$$t \cdot (\pi : C \to S; q_1; L, N; \vec{p}, \vec{x}, v_1, v_2) = (\pi : C \to S; q_1; L, N; \vec{p}, \vec{x}, t^{-1}v_1, v_2), \quad t \in \mathbb{C}^*.$$

---

[7]We note that this action is opposite to the one in [46].

The perfect obstruction theory is equivariant and the cosection is invariant, and the restriction of the perfect obstruction theory to each fixed locus has a global resolution. The degeneracy locus $\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}$ of the cosection has three types of fixed loci:

(1) $F_0\widetilde{Z}$ is the vanishing locus of v_1.
(2) $F_\infty\widetilde{Z}$ is the vanishing locus of v_2.
(3) For each $0 < \beta' < 1/\epsilon$, $F_{\beta'}\widetilde{Z}$ is the locus where
   - $C = C_g \cup C_0$, where C_0 is a smooth rational subcurve and $\deg((L^{\otimes -d} \otimes \omega_{\log})|_{C_0}) = \beta'$;
   - $q_1 \in C_0$ and $C_g \cap C_0$ are the only two special points of C_0;
   - neither v_1 nor v_2 is zero;
   - $\vec{p}$ has vanishing order β' at q_1.

Set-theoretically, these are the only fixed loci. Indeed, if $v_1, v_2 \neq 0$, then the only way that an automorphism can re-scale $t^{-1}v_1$ back to v_1 is by acting nontrivially on the component C_0 of C on which q_1 lies. This changes the moduli point unless C_0 has just one other special point and all of $\deg(L^{\otimes -d} \otimes \omega_{C,\log}|_{C_0})$ is concentrated in a single basepoint at q_1, which forces C_0 to have genus zero, as claimed.

We now describe the stack structure of the fixed loci and compute their contributions to the localization formula. The first two cases are the same as in [46, Lemma 7.15], so we only state the results. The key to the proof is that $F_0\widetilde{Z}$ and $F_\infty\widetilde{Z}$ are effective Cartier divisors defined by the vanishing of v_1 and v_2, respectively.

Lemma A.5. *The substack $F_0\widetilde{Z}$ is isomorphic to $Z^\epsilon_{g,(1/d),\beta}$, where q_1 is viewed as a heavy marking. Its localization contribution is*

$$\frac{[F_0\widetilde{Z}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_0\widetilde{Z}})} = \frac{[Z^\epsilon_{g,(1/d),\beta}]^{\mathrm{vir}}}{-z - \psi_1}.$$

The substack $F_\infty\widetilde{Z}$ is isomorphic to $Z^{\epsilon,\delta}_{g,(1/d),\beta}$, where q_1 is viewed as a light marking. Its localization contribution is

$$\frac{[F_\infty\widetilde{Z}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\infty\widetilde{Z}})} = \frac{[Z^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}}}{z + \psi_1}.$$

Here, ψ_1 is the ψ-class of the coarse curves at q_1.

We now come to $F_{\beta'}\widetilde{Z}$. Recall that the multiplicity of L at q_1 is $1/d$, so the compatibility condition (8) applied to C_g implies that the multiplicity of $L|_{C_g}$ at $C_g \cap C_0$ is

$$m_{\beta'} := \left\langle \frac{\beta' + 1}{d} \right\rangle.$$

Thus, the automorphism group of the node $C_g \cap C_0$ is cyclic or order $d_m := d/\gcd(d \cdot m_{\beta'}, d)$. Since the node is balanced, the multiplicity of $L|_{C_0}$ at $C_g \cap C_0$ is $m_\bullet := \langle -m_{\beta'} \rangle$.

We claim, in fact, that $F_{\beta'}\widetilde{Z}$ is isomorphic to a fiber product of the moduli space $Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}$ with the locus $F^\epsilon_{\beta'}$ in the graph space $\mathcal{G}Z^\epsilon_{0,1,\beta'}$. Recall from the definition of the J-function that $F^\epsilon_{\beta'} \subset \mathcal{G}Z^\epsilon_{0,1,\beta'}$ is defined as the fixed locus of the graph space where the only marking lies at ∞ and all of the degree lies over 0. The multiplicity at the marking must be $m_\bullet$. In the definition of the J-function, the virtual normal bundle $N^{\mathrm{vir}}_{F^\epsilon_{\beta'}/\mathcal{G}Z^\epsilon_{0,1,\beta'}}$ has two parts:

(1) a rank-1 part coming from deforming the marking away from $\infty \in \mathbb{P}^1$, whose Euler class is $(-z)$;

(2) the moving part of the relative obstruction theory (10), which we denote by $N^{\mathrm{vir,rel}}_{F^\epsilon_{\beta'}/\mathcal{G}Z^\epsilon_{0,1,\beta'}}$.

We now form a morphism

(55) $$\iota_{\beta'} : Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'} \times_{\overline{\mathcal{I}}Z} F^\epsilon_{\beta'} \to F_{\beta'}\widetilde{Z}.$$

Here, let us denote the marking, ψ-class, and evaluation map of $Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}$ by

$$q'_1, \psi'_1, \text{ and } \mathrm{ev}'_1,$$

and denote by

$$\widehat{\mathrm{ev}}_\bullet : F^\epsilon_{\beta'} \to \overline{\mathcal{I}}Z$$

the evaluation map of $F^\epsilon_{\beta'}$ at the unique marking ∞ with the inverted banding. The fiber product in (55) is defined via ev'_1 and $\widehat{\mathrm{ev}}_\bullet$.

To define $\iota_{\beta'}$, take any S-point of $Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'} \times_{\overline{\mathcal{I}}Z} F^\epsilon_{\beta'}$, which consists of

(56)
$$\eta_g = (\pi_g : C_g \to S, q'_1; L_g, \vec{p}_g) \in Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}(S),$$
$$\eta_0 = (\pi_0 : C_0 \to S, q_\bullet; L_0; \vec{p}_0) \in F^\epsilon_{\beta'}(S), \text{ and}$$
$$\text{a 2-morphism } \theta : \mathrm{ev}'_1 \overset{\cong}{\Rightarrow} \widehat{\mathrm{ev}}_\bullet.$$

By the definition of $F^\epsilon_{\beta'}$, C_0 is $\mathbb{P}^1 \times S$ with additional stack structure along the marking $q_\bullet = \{\infty\} \times S$. The idea is to glue q'_1 with $q_\bullet$ and place the light marking q_1 at $0 \in \mathbb{P}^1$. Let

(57) $$\rho : C'_0 \to C_0$$

be the d-th root stack of the divisor $\{0\} \times S$ with universal root $q_1 \subset C'_0$. We also view $q_\bullet$ as a marking on C'_0. Note that, as relative Cartier divisors, $d[q_1] = \rho^*(\{0\} \times S)$.

Let $L'_0 = \rho^* L_0 \otimes \mathcal{O}_E([q_1])$. Then

(58) $$\rho_*(L'_0) = L_0 \quad \text{and} \quad \rho_*\left(L'_0{}^{\otimes-d} \otimes \omega_{C'_0/S,\log}\right) = L_0{}^{\otimes-d} \otimes \omega_{C_0/S,\log}.$$

Via the second isomorphism, $\vec{p}_0$ induces a section $\vec{p'_0} \in \Gamma((L'_0{}^{\otimes-d} \otimes \omega_{C'_0/S,\log})^{\oplus N})$. Thus, we get a family of genus-0 Landau–Ginzburg quasimaps

$$\eta'_0 = (\pi_0 \circ \rho : C'_0 \to S; q_1, q_\bullet; L'_0; \vec{p'_0}).$$

Note that the evaluation map of η'_0 at $q_\bullet$ is also $\mathrm{ev}_\bullet$. The map $\iota_{\beta'}$ is defined by gluing the Landau–Ginzburg quasimaps η_g and η'_0 along the markings q'_1 and $q_\bullet$ and setting $N = \mathcal{O}_S$, $v_2 \equiv 1$, and $v_1 \equiv \partial/\partial z$, where z is the coordinate on $\mathbb{P}^1$. This defines a morphism to $\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}$ that maps each closed point to $F_{\beta'}\widetilde{Z}$. Moreover, the resulting family is $\mathbb{C}^*$-invariant, so it factors through $F_{\beta'}\widetilde{Z}$.

Lemma A.6. *The morphism $\iota_{\beta'}$ is an isomorphism onto $F_{\beta'}\widetilde{Z}$.*

Proof. We construct a quasi-inverse. Given any family

(59) $$(\pi : C \to S; q_1; L, N; \vec{p}, v_1, v_2) \in \widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}(S)$$

in $F_{\beta'}\widetilde{Z}$, we must recover the gluing data (56). By the definition of $F_{\beta'}\widetilde{Z}$, over any closed point $s \in S$, (59) comes from gluing the data (56), and the key is to show that we can split the node in families over any (possibly non-reduced) base S.

Since the family is $\mathbb{C}^*$-fixed, $\mathbb{C}^*$ acts on C and the morphism $\pi : C \to S$ is invariant. On the fiber $C_s = C_{g,s} \cup C_{0,s}$ over each closed point $s \in S$, $\mathbb{C}^*$ acts on $C_{0,s}$ fixing q_1 and the node $q_n := C_{g,s} \cap C_{0,s}$, and $t \in \mathbb{C}^*$ maps the tangent vector $t^{-1} v_1/v_2$ back to v_1/v_2. Hence, $\mathbb{C}^*$ acts nontrivially on the tangent space to $C_{0,s}$ at q_n. It is obvious that $\mathbb{C}^*$ acts trivially on $C_{g,s}$. Hence it acts nontrivially on the first order deformation smoothing the node q_n. Since the family is fixed by $\mathbb{C}^*$, the S-family of curves C can be decomposed as C_g and C_0 glued at a pair of markings.

Now it is clear how to recover (56): restricting the Landau–Ginzburg quasimaps to C_g and C_0 separately, we recover η_0' and η_g. It is easy to see that η_0' is equivalent to η_0, and this defines a quasi-inverse to $\iota_{\beta'}$. $\qquad\square$

Let $\mathrm{pr}_1 : F_{\beta'}\widetilde{Z} \to Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}$ and $\mathrm{pr}_2 : F_{\beta'}\widetilde{Z} \to F^\epsilon_{\beta'}$ be the projections.

Lemma A.7. *We have*
$$[F_{\beta'}\widetilde{Z}]^{\mathrm{vir}} = \mathrm{pr}_1^*[Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}]^{\mathrm{vir}},$$

and the inverse of the equivariant Euler class of the virtual normal bundle restricted to $F_{\beta'}\widetilde{Z}$ is

$$\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\beta'}\widetilde{Z}})} = \mathrm{pr}_1^*\left(\frac{d_m}{-\psi_1' - z}\right) \cdot \mathrm{pr}_2^*\left(\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir,rel}}_{F^\epsilon_{\beta'}/\mathcal{G}Z^\epsilon_{0,1,\beta'}})}\right).$$

Proof. The proof is similar to that of Lemma 4.5.

Let $\mathfrak{Z} \subset \widetilde{D}_{g,(1/d),\beta}$ be the reduced, locally-closed substack where q_1 is on a rational tail of degree β' and v_1 and v_2 are both nonzero. The normal bundle of $\mathfrak{Z}$ is moving. As in the proof of Lemma 4.5, $[F_{\beta'}\widetilde{Z}]^{\mathrm{vir}}$ can be defined by the fixed part of the absolute obstruction theory induced by

$$(60) \qquad \left(R\pi_*\left(\bigoplus_{i=1}^M \left(\mathcal{L}^{\otimes w_i}\left(-\sum_{k=1}^n \Delta_k\right)\right) \oplus \bigoplus_{j=1}^N \mathcal{P}\right)\right)^\vee,$$

which is a relative perfect obstruction theory relative to $\mathfrak{Z}$.

The universal curve decomposes as $\mathcal{C}_g \cup \mathcal{C}_0$, where $\mathcal{C}_0$ is the rational tail containing q_1. Let $\Delta_0 \subset \mathcal{C}_0$ and $\Delta_g \subset \mathcal{C}_g$ be the node $\mathcal{C}_g \cap \mathcal{C}_0$. The contributions from the subsheaf $\bigoplus_{j=1}^N \mathcal{P}|_{\mathcal{C}_0}(-\Delta_0)$ and the quotient sheaf $\bigoplus_{i=1}^M \left(\mathcal{L}^{\otimes w_i}\left(-\sum_{k=1}^n \Delta_k\right)\right)|_{\mathcal{C}_0}$ to (60) are both moving, and this moving part is identified with $\mathrm{pr}_2^*(N^{\mathrm{vir,rel}}_{F^\epsilon_{\beta'}/\mathcal{G}Z^\epsilon_{0,1,\beta'}})$ thanks to the relation (58). The remaining part is the fixed part, given by

$$\left(R\pi_*\left(\bigoplus_{i=1}^M \left(\mathcal{L}^{\otimes w_i}\left(-\Delta_g - \sum_{k=1}^n \Delta_k\right)|_{\mathcal{C}_g}\right) \oplus \bigoplus_{j=1}^N \mathcal{P}|_{\mathcal{C}_g}\right)\right)^\vee.$$

This is exactly the pullback of the compact-type perfect obstruction theory of $Z^\epsilon_{g,(m_{\beta'}),\beta-\beta'}$ relative to $D_{g,(m_{\beta'}),\beta-\beta'}$. Moreover, the cosections are compatible and the forgetful morphism $\mathfrak{Z} \to D_{g,(m_{\beta'}),\beta-\beta'}$ is étale. Thus, the identity of virtual cycles follows from cosection-localized pullback. The moving part of the tangent complex of $\widetilde{D}_{g,(1/d),\beta}$ along $\mathfrak{Z}$ is the normal bundle of $\mathfrak{Z}$, whose Euler class is $\frac{1}{d_m}(-\psi_1' - z)$. $\qquad\square$

Having established the contributions from each fixed locus, we can collect all of them to get the localization formula on the master space. In particular, there is a morphism

$$\rho : \widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta} \to Z^{\epsilon,\delta}_{g,(1/d),\beta}$$

that forgets N, v_1, and v_2 and stabilizes as necessary, and we calculate the coefficient of z^{-2} in the class $\rho_*(\psi_1 \cap [\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}})$ by $\mathbb{C}^*$-localization. This coefficient is zero, but expressing it in terms of contributions from each fixed locus yields a nontrivial relation. To calculate these contributions, we first note that, under the isomorphism $F_0\widetilde{Z} \cong Z^{\epsilon}_{g,(1/d),\beta}$ from Lemma A.5, the restriction of ρ to $F_0\widetilde{Z}$ is the morphism

$$c : Z^{\epsilon}_{g,(1/d),\beta} \to Z^{\epsilon,\delta}_{g,(1/d),\beta}$$

that replaces the heavy marking with a light marking and then stabilizes. Under the isomorphism $F_\infty\widetilde{Z} \cong Z^{\epsilon,\delta}_{g,(1/d),\beta}$, the restriction of ρ to $F_\infty\widetilde{Z}$ is the identity. Finally, under the isomorphism $F_{\beta'}\widetilde{Z}$ from Lemma A.6, the restriction of ρ to $F_{\beta'}\widetilde{Z}$ is the composition of pr_1 with the morphism

$$b : Z^{\epsilon}_{g,(m_{\beta'}),\beta-\beta'} \to Z^{\epsilon,\delta}_{g,(1/d),\beta}$$

that replaces the last marking with a light marking that is also a basepoint of order β'.

Using Lemma A.5, Lemma A.6, and Lemma A.7 together with the projection formula, the relation that results from localization is as follows:

(61)
$$\psi_1 \cap [Z^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}} = c_* \left(\psi_1 \cap [Z^{\epsilon}_{g,(1/d),\beta}]^{\mathrm{vir}} \right) - \sum_{\beta'} b_* \left((\mathrm{ev}'_1)^* \left(\mu^{\epsilon}_{\beta'}(-\psi'_1) \right) \cap [Z^{\epsilon}_{g,(m_{\beta'}),\beta-\beta'}]^{\mathrm{vir}} \right).$$

Another relation is obtained by calculating $\rho_*(\mathrm{ev}_1^*(H) \cap [\widetilde{Z}^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}})$, where H is the "hyperplane class" on $[\mathbb{C}^N/\mathbb{C}^*] \subset \overline{\mathcal{I}Z}$. Concretely, $\mathrm{ev}_1^*(H)$ is the Euler class of the line bundle $L^{\otimes -d} \otimes \omega_{C,\log}|_{q_1}$, and the restriction of $\mathrm{ev}_1^*(H)$ to $F_{\beta'}\widetilde{Z}$ is equal to $(\mathrm{ev}'_1)^*(H) + \beta' z$. The coefficient of z^{-1} in this pushforward is zero, but expressing it in terms of contributions from each fixed locus and applying the same reasoning as above yields:

(62)
$$\begin{aligned}
\mathrm{ev}_1^*(H) \cap [Z^{\epsilon,\delta}_{g,(1/d),\beta}]^{\mathrm{vir}} =&c_* \left(\mathrm{ev}_1^*(H) \cap [Z^{\epsilon}_{g,(1/d),\beta}]^{\mathrm{vir}} \right) \\
&+ \beta' \sum_{\beta'} b_* \left((\mathrm{ev}'_1)^* \left(\mu^{\epsilon}_{\beta'}(-\psi'_1) \right) \cap [Z^{\epsilon}_{g,(m_{\beta'}),\beta-\beta'}]^{\mathrm{vir}} \right) \\
&- \sum_{\beta'} b_* \left((\mathrm{ev}'_1)^* \left(\frac{H}{\psi'_1} \mu^{\epsilon}_{\beta'}(-\psi'_1) \right) \cap [Z^{\epsilon}_{g,(m_{\beta'}),\beta-\beta'}]^{\mathrm{vir}} \right).
\end{aligned}$$

Here, in the expression $\frac{H}{\psi'_1} \mu^{\epsilon}_{\beta'}(-\psi'_1)$, we simply discard the term of $\mu^{\epsilon}_{\beta'}(-\psi'_1)$ that does not involve ψ'_1.

A.5. Wall-crossing without a heavy marking.

Equations (61) and (62) can be viewed as wall-crossing formulas for converting a light marking to a heavy marking. Equipped with them, we are prepared to prove the $n = 0$ case of the main wall-crossing theorem.

From now on, we compute on $Z^{\epsilon}_{g,0,\beta}$ and suppress all the pushforward notations. In particular, the class that was denoted ψ'_1 in the previous subsection will now be denoted simply ψ_1, and similarly for q'_1 and ev'_1. The meaning of $\mathrm{ev}_i^*(H)$ and ψ_i are determined by the virtual cycle that they are capped with.

First, suppose $N > 1$. We then have $\beta[Z_{g,0,\beta}^{\epsilon}]^{\mathrm{vir}} = \mathrm{ev}_1^*(H) \cap [Z_{g,(1/d),\beta}^{\epsilon,\delta}]^{\mathrm{vir}}$ by Lemma A.2. Combining this with Theorem 3.1 and (62), we get

$$
(63) \quad \begin{aligned}
\beta[Z_{g,0,\beta}^{\epsilon}]^{\mathrm{vir}} =& \sum_{\beta_0+\cdots+\beta_k=\beta} \frac{1}{k!} \mathrm{ev}_1^*(H) \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}} \\
&+ \sum_{\beta_0+\cdots+\beta_k+\beta'=\beta} \frac{\beta'}{k!} \mathrm{ev}_1^* \left(\mu_{\beta'}^{\epsilon}(-\psi_1)\right) \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(m_{\beta'})+k,\beta_0}^{\infty}]^{\mathrm{vir}} \\
&- \sum_{\beta_0+\cdots+\beta_k+\beta'=\beta} \frac{1}{k!} \mathrm{ev}_1^* \left(\frac{H}{\psi_1} \mu_{\beta'}^{\epsilon}(-\psi_1)\right) \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(m_{\beta'})+k,\beta_0}^{\infty}]^{\mathrm{vir}}.
\end{aligned}
$$

Applying the divisor equation

$$
(64) \quad \begin{aligned}
\mathrm{ev}_1^*(H) \cdot \prod_{i=1}^k \psi_{1+i}^{a_i} \cap [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}} =& \beta_0 \prod_{i=1}^k \psi_i^{a_i} \cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}} \\
&+ \sum_{j=1}^k \left(\psi_j^{a_j-1} \prod_{i\neq j} \psi_i^{a_i} \right) \cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}}
\end{aligned}
$$

for ∞-theory, we then obtain

$$
(65) \quad \begin{aligned}
\mathrm{ev}_1^*(H) \cdot \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}} =& \beta_0 \prod_{i=1}^k \mathrm{ev}_i^*(\mu_{\beta_i}^{\epsilon}(-\psi_i)) \cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}} \\
+ \sum_{j=1}^k \left(\mathrm{ev}_j^*(\frac{H}{\psi_j} \mu_{\beta_i}^{\epsilon}(-\psi_j)) \prod_{i\neq j} \mathrm{ev}_i^*(\mu_{\beta_i}^{\epsilon}(-\psi_i)) \right) &\cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}}.
\end{aligned}
$$

Substituting (65) into (63) and rearranging the summation using the symmetry of the marked points, we get

$$
(66) \quad \beta[Z_{g,0,\beta}^{\epsilon}]^{\mathrm{vir}} = \beta \sum_{\beta_0+\cdots+\beta_k=\beta} \frac{1}{k!} \prod_{i=1}^k \mathrm{ev}_i^*(\mu_{\beta_i}^{\epsilon}(-\psi_i)) \cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}},
$$

which is exactly the statement of the wall-crossing theorem in this case.

Now, suppose that $g > 1$. In this case, we have $(2g-2)[Z_{g,0,\beta}^{\epsilon}]^{\mathrm{vir}} = \psi_1 \cap [Z_{g,(1/d),\beta}^{\epsilon,\delta}]^{\mathrm{vir}}$. Combining this with Theorem 3.1 and (61), we get

$$
(67) \quad \begin{aligned}
(2g-2)[Z_{g,0,\beta}^{\epsilon}]^{\mathrm{vir}} =& \sum_{\beta_0+\cdots+\beta_k=\beta} \frac{\psi_1}{k!} \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}} \\
&- \sum_{\beta_0+\cdots+\beta_k+\beta'=\beta} \frac{1}{k!} \mathrm{ev}_1^* \left(\mu_{\beta'}^{\epsilon}(-\psi_1)\right) \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}}.
\end{aligned}
$$

Using the dilaton equation for ∞-theory, we get

$$
(68) \quad \frac{\psi_1}{k!} \prod_{i=1}^k \mathrm{ev}_{1+i}^*(\mu_{\beta_i}^{\epsilon}(-\psi_{1+i})) \cap [Z_{g,(1/d)+k,\beta_0}^{\infty}]^{\mathrm{vir}} = \frac{2g-2+k}{k!} \prod_{i=1}^k \mathrm{ev}_i^*(\mu_{\beta_i}^{\epsilon}(-\psi_i)) \cap [Z_{g,k,\beta_0}^{\infty}]^{\mathrm{vir}}.
$$

Substituting this into (67), we again obtain the desired wall-crossing formula.

The only remaining case is when $g = N = 1$, but this can be handled exactly analogously to the work of Guo–Ross [31] for the quintic threefold.

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