

OPENNESS OF K-SEMISTABILITY FOR FANO VARIETIES

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ABSTRACT. In this paper, we prove the openness of K-semistability in families of log Fano pairs by showing that the stability threshold is a constructible function on the fibers. We also prove that any special test configuration arises from a log canonical place of a bounded complement and establish properties of any minimizer of the stability threshold.

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1. INTRODUCTION

Throughout the paper, we work over an algebraically closed characteristic zero field.

K-stability was invented as an algebraic condition to characterize when a Fano variety admits a Kähler-Einstein metric (see [Tia97, Don02]). In recent years, the question of whether one can construct the moduli space parametrizing K-polystable \mathbb{Q} -Fano varieties with fixed numerical invariants, as well as establish nice properties for it, has attracted significant interest. Previously, the construction of the moduli space relied on the properties, especially the existence, of Kähler-Einstein metrics (see e.g. [LWX19]). Nevertheless, using the valuative criterion developed in [Fuj19a, Li17], a purely algebro-geometric approach has been dramatically advanced. See [BX19, ABHLX20] for more background.

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This paper aims to settle one of the main steps of the construction. Namely, we prove that in a \mathbb{Q} -Gorenstein family $(X, \Delta) \rightarrow B$ of log Fano pairs over a normal base, the locus $B^\circ \subset B$ parametrizing K-semistable fibers is a Zariski open set. This is the last ingredient needed to conclude that the moduli space of K-polystable \mathbb{Q} -Fano varieties exists. See Theorem 1.3 for a more precise statement.

1.1. Main theorems. Before proving the openness result, we first establish the following property of the stability threshold as well as Tian's α -invariant.

Theorem 1.1. *If $(X, \Delta) \rightarrow B$ is a \mathbb{Q} -Gorenstein family of log Fano pairs over a normal base B , then the functions*

$$B \ni b \mapsto \min\{\alpha(X_{\bar{b}}, \Delta_{\bar{b}}), 1\} \quad \text{and} \quad B \ni b \mapsto \min\{\delta(X_{\bar{b}}, \Delta_{\bar{b}}), 1\}$$

are constructible and lower semicontinuous.

Recall that the α -invariant of a log Fano pair (X, Δ) was introduced in [Tia87] and the stability threshold (also known as the δ -invariant) in [FO18]. It was shown in [FO18][BJ20] that $\delta(X, \Delta) \geq 1$ if and only if (X, Δ) is K-semistable, based on the valuative criteria for K-semistability proved in [Fuj19a][Li17]. Therefore, we have the following immediate corollary of Theorem 1.1

Corollary 1.2. *If $(X, \Delta) \rightarrow B$ is a \mathbb{Q} -Gorenstein family of log Fano pairs over a normal base B , then*

$$B^\circ := \{b \in B \mid (X_{\bar{b}}, \Delta_{\bar{b}}) \text{ is K-semistable}\}$$

is a Zariski open subset of B .

Together with the main theorems in [Jia20][BX19][ABHLX20], we deduce

Theorem 1.3. *The moduli functor $\mathfrak{X}_{V,n}^{\text{Kss}}$ of K-semistable \mathbb{Q} -Fano varieties of dimension n and volume V is an Artin stack of finite type over k and admits a separated good moduli space $\mathfrak{X}_{V,n}^{\text{Kss}} \rightarrow X_{V,n}^{\text{Kps}}$, whose k -points parameterize K-polystable \mathbb{Q} -Fano varieties of dimension n and volume V .*

In fact, the boundedness of $\mathfrak{X}_{V,n}^{\text{Kss}}$ was settled in [Jia20], which heavily relied on results in [Bir19]. Corollary 1.2 then implies that the moduli functor $\mathfrak{X}_{V,n}^{\text{Kss}}$ is an Artin stack of finite type over k . With the latter step completed, it follows from the main theorems in [BX19][ABHLX20] that $\mathfrak{X}_{V,n}^{\text{Kss}}$ admits the separated good moduli space $\mathfrak{X}_{V,n}^{\text{Kss}} \rightarrow X_{V,n}^{\text{Kps}}$.

As a consequence of Corollary 1.2 and Theorem 1.3 we show that K-stability (resp. K-polystability) is an open (resp. constructible) condition for \mathbb{Q} -Gorenstein families of log Fano pairs; see Theorem 4.5.

Remark 1.4. An analogue of Theorem 1.1 in a local setting was proved in [Xu20] for the normalized volume function defined in [Li18]. Corollary 1.2 and Theorem 1.3 can also be obtained independently as a consequence of the local result via the cone construction.

1.2. Outline of the proof. Our strategy of proving Theorem 1.1 is approximating the infimum

$$\delta(X_{\bar{b}}, \Delta_{\bar{b}}) = \inf_E \frac{A_{X_{\bar{b}}, \Delta_{\bar{b}}}(E)}{S(E)} \quad \text{for all divisors } E \text{ over } X_{\bar{b}}$$

by the values on lc places E of bounded complements. We then deduce constructibility by using a theorem on invariance of log plurigenera established in [HMX13, Theorem 1.8].

More precisely, the proof of Theorem 1.1 relies on combining two techniques. The first one is the special degeneration theory initiated in [LX14] and later developed in [LX20, Fuj19a, Fuj19b, BX19] etc. Roughly speaking, to compute $\delta(X, \Delta)$ for a log Fano pair (X, Δ) , instead of testing general divisorial valuations, we can focus on a special class of valuations, which are those that arise from a special degeneration. It was known previously that it suffices to consider such degenerations for studying K-stability. Our new strategy, which is the second ingredient in this paper, is to use global complements to study them.

The concept of a *complement* was introduced in [Sho92]. Since then it has been a particularly effective tool in birational geometry for understanding Fano varieties. In particular, a profound theorem on the existence of bounded global complements for log Fano pairs was proved by [Bir19]. By using the techniques from [Fuj19b, LX20], one can show that the valuation computing $\min\{\delta, 1\}$ can be approximated by special divisors (see [BLZ19, ZZ19]). By applying Birkar's result, we deduce that all these special divisors are lc places of a bounded family of complements.

The above discussion can be easily extended to a \mathbb{Q} -Gorenstein family of log Fano pairs $(X, \Delta) \rightarrow B$, and finally we can use [HMX13] to conclude that

$$b \mapsto \frac{A_{X_b, \Delta_b}(E_b)}{S(E_b)}$$

is a constant function if the special divisor E_b over (X_b, Δ_b) varies in a family giving fiberwise log resolutions.

The arguments in Section 1-4 are of a global nature. In the appendix, we will develop this strategy further using local techniques.

1.3. Appendix. In Appendix A, we will use complements to further study the K-stability of a log Fano pair (X, Δ) . The results proved in Appendix A are not needed elsewhere in this paper. However, we expect it will be useful for future research.

We first prove the following theorem which gives a characterization of a valuation v computing $\delta(X, \Delta)$ when $\delta(X, \Delta) \leq 1$.

Theorem 1.5 (=Theorem A.7). *Let n be a positive integer and $I \subset \mathbb{Q}$ a finite set. Then there exists a positive integer $N = N(n, I)$ satisfying the following:*

Let (X, Δ) be an n -dimensional log Fano pair such that coefficients of Δ belong to I . If $\delta(X, \Delta) \leq 1$, and v is a valuation computing $\delta(X, \Delta)$, then v is quasi-monomial and an lc place of an N -complement.

While part of Theorem 1.5 can be proved by a global method similar to our proof of Theorem 1.1 (see Proposition 3.8 and Theorem 4.6), the statement in the full generality has to be established in a somewhat different way. For this we have to invoke the cone construction and use some arguments from Xu20. The technique of using the cone construction to study the K-stability of a log Fano pair was initiated in Li17 and played a key role in proving results in LX20, LWX21, BX19.

We also show the following theorem which gives a characterization of weakly special test configurations. It is obtained by combining arguments in LWX21 and Xu20, which uses the existence of bounded local complements.

Theorem 1.6 (=Theorem A.2). *Let n be a positive integer and $I \subset \mathbb{Q}$ a finite set. Then there exists a positive integer $N = N(n, I)$ satisfying the following:*

If (X, Δ) is an n -dimensional log Fano pair such that coefficients of Δ belong to I , then a finite set of \mathbb{Z} -valued divisorial valuations $\{v_1, \dots, v_d\} \subset \text{Val}_X$ is a weakly special collection (see Definition A.1) if and only if there exists an N -complement Δ^+ of (X, Δ) such that each v_i is an lc place of (X, Δ^+) .

By LX14, to study K-(semi,poly)stability, we can concentrate on the class of weakly special test configurations. Theorem 1.6 says that this class of test configurations comes from a somewhat ‘bounded’ amount of information.

Postscript remarks. Since the first version of this article appeared on the arXiv, there has been works generalizing and strengthening our results. We list a few related works below.

- (1) In LXZ21, Theorem 1.1], it is shown that any valuation computing $\delta(X, \Delta) < \frac{n+1}{n}$ for an n -dimensional log Fano pair (X, Δ) has a finitely generated associated graded ring. Theorem 1.5 is a crucial step in proving this result. This result together with BHLLX21, XZ20 implies that the K-moduli space $X_{V,n}^{\text{Kps}}$ is a projective scheme.
- (2) In LXZ21 Corollary 3.7], it is shown in the setting of Theorem 1.1 that $B \ni b \mapsto \min\{\delta(X_b, \Delta_b), \frac{n+1}{n}\}$ is constructible and lower semicontinuous, where n is the relative dimension of X/B .
- (3) Zhuang found a characterization of special prime divisors over a log Fano pair, that are, prime divisors induced by special test configurations in Xu21 Theorem 4.12] as a strengthening of Theorem 1.6

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2. PRELIMINARIES

2.1. Conventions. We will follow standard terminologies in KM98, Kol13. A (normal) pair (X, Δ) is composed of a normal variety X and an effective \mathbb{Q} -divisor Δ on

X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. See [KM98, 2.34] for the definitions of *klt*, *plt*, and *lc* pairs.

A pair (X, Δ) is *log Fano* if X is projective, (X, Δ) is klt, and $-K_X - \Delta$ is ample. A variety X is *\mathbb{Q} -Fano* if $(X, 0)$ is log Fano. More generally, a variety X is of *Fano type* if it is projective and there exists a \mathbb{Q} -divisor Δ such that (X, Δ) is klt and $-K_X - \Delta$ is big and nef.

For a \mathbb{Q} -divisor L , we write $|L|_{\mathbb{Q}}$ for the set effective \mathbb{Q} -divisors which are \mathbb{Q} -linearly equivalent to L . For a subset $I \subseteq [0, 1]$, we set

$$I_+ = \{0\} \cup \left\{ j \in [0, 1] \mid j = \sum_{p=1}^l i_p \text{ for some } i_1, \dots, i_l \in I \right\}$$

and

$$D(I) = \left\{ \frac{m-1+a}{m} \mid a \in I_+ \text{ and } m \in \mathbb{N} \right\}.$$

2.2. Families of pairs.

Definition 2.1. A \mathbb{Q} -Gorenstein family of (normal) pairs $f : (X, \Delta) \rightarrow B$ over a normal base is the data of a flat surjective morphism of varieties $f : X \rightarrow B$ and a \mathbb{Q} -divisor Δ on X satisfying

- (1) B is normal and f has normal, connected fibers (hence, X is normal as well),
- (2) $\text{Supp}(\Delta)$ does not contain a fiber, and
- (3) $K_{X/B} + \Delta$ is \mathbb{Q} -Cartier.

We say $(X, \Delta) \rightarrow B$ is a \mathbb{Q} -Gorenstein family of log Fano pairs if in addition (X_b, Δ_b) is log Fano for all $b \in B$. Here, Δ_b is the cycle pull-back of Δ to the fiber X_b . See [Kol20, Section 4] for more background.

In birational geometry, we should usually allow the fibers to be slc pairs. However, in this note, we are only interested in families whose fibers are of Fano type. Thus, we can assume all fibers are normal.

Definition 2.2. Let $f : (X, \Delta) \rightarrow B$ be a \mathbb{Q} -Gorenstein family of pairs with B smooth. A morphism $g : Y \rightarrow X$ is a *fiberwise log resolution* of $(X, \Delta) \rightarrow B$ if Y is smooth over B , $E := \sum_{i \in I} E_i = \text{Exc}(g) + \text{Supp}(g_*^{-1}D)$ is an snc divisor, and each stratum of E is smooth with irreducible fibers over B . (Here, the *strata* of E are the irreducible components of $E_J = \cap_{j \in J} E_j$, for some subset $J \subseteq I$.)

If $(X, \Delta) \rightarrow B$ is a \mathbb{Q} -Gorenstein family of pairs, then we can always find a nonempty open set $U \subset B$ and a finite étale map $U' \rightarrow U$ such that $(X_{U'}, \Delta_{U'}) \rightarrow U'$ admits a fiberwise log resolution.

2.3. Valuations. Let X be a variety. A valuation on X will mean a valuation $v : K(X)^\times \rightarrow \mathbb{R}$ that is trivial on k and has center on X . Recall, v has center on X if there exists a point $\xi \in X$ such that $v \geq 0$ on $\mathcal{O}_{X, \xi}$ and > 0 on $\mathfrak{m}_\xi \subset \mathcal{O}_{X, \xi}$. Since X is assumed to be separated, such a point ξ is unique, and we say v has *center* $c_X(v) := \xi$.

If X is proper, then the valuative criterion for properness implies such a center always exists uniquely. By convention, we set $v(0) = +\infty$.

Following [JM12, BdFFU15], we write Val_X for the set of valuations on X and Val_X^* for the set of non-trivial ones. We endow Val_X with the topology of pointwise convergence.

To any valuation $v \in \text{Val}_X$ and $p \in \mathbb{N}$, there is an associated *valuation ideal* $\mathfrak{a}_p(v)$. For an affine open subset $U \subseteq X$, $\mathfrak{a}_p(v)(U) = \{f \in \mathcal{O}_X(U) \mid v(f) \geq p\}$ if $c_X(v) \in U$ and $\mathfrak{a}_p(v)(U) = \mathcal{O}_X(U)$ otherwise.

For an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ and $v \in \text{Val}_X$, we set

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X, c_X(v)}\} \in [0, +\infty].$$

We can also make sense of $v(s)$ when \mathcal{L} is a line bundle and $s \in H^0(X, \mathcal{L})$. After trivializing \mathcal{L} at $c_X(v)$, we set $v(s)$ equal to the value of the local function corresponding to s under this trivialization; this is independent of the choice of trivialization.

Similarly, if D is a Cartier divisor, we set $v(D) := v(f)$, where f is a local equation for D at $c_X(v)$. If D is only \mathbb{Q} -Cartier, we set $v(D) := m^{-1}v(mD)$, where m is a positive integer so that mD is Cartier.

2.3.1. Divisorial valuations. Let $\mu : Y \rightarrow X$ be a proper birational morphism of varieties with Y normal. A prime divisor $E \subset Y$ (called a *prime divisor over X*) induces a valuation $\text{ord}_E : K(X)^\times \rightarrow \mathbb{Z}$ given by order of vanishing along E . A valuation of the form $c \cdot \text{ord}_E$, where $c \in \mathbb{Q}_{>0}$, is called *divisorial*. We write $\text{DivVal}_X \subset \text{Val}_X$ for the set of such valuations.

2.3.2. Quasi-monomial valuations. Let $\mu : Y \rightarrow X$ be a proper birational morphism with Y regular. Fix a not necessarily closed point $\eta \in Y$ and y_1, \dots, y_r a regular system of parameters for $\mathcal{O}_{Y, \eta}$. Given $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r$, we define a valuation v_α as follows: For $f \in \mathcal{O}_{Y, \eta}$, we can write f in $\widehat{\mathcal{O}_{Y, \eta}} \simeq k(\eta)[[y_1, \dots, y_r]]$ as $\sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta$, where $c_\beta \in k(\eta)$ and set

$$v_\alpha(f) := \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}. \quad (2.1)$$

Note that v_α is determined by the Newton polygon of $\sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta$.

A valuation of the form v_α is called *quasi-monomial*. If $\alpha \in \mathbb{Q}_{\geq 0}^r$, then v_α is a divisorial valuation. Indeed, after a sequence of smooth blowups $Y' \rightarrow Y$ that are toroidal with respect to the coordinates y_1, \dots, y_r , we may find a prime divisor $F \subset Y'$ and $c \in \mathbb{Q}_{>0}$ so that $v_\alpha = c \text{ord}_F$.

Let $E = E_1 + \dots + E_d$ be a reduced snc divisor on Y . Fix a subset $J \subseteq \{1, \dots, d\}$ and an irreducible component $Z \subseteq \cap_{i \in J} E_i$. Write $\eta \in Y$ for the generic point of Z and choose a regular system of parameters $(y_i)_{i \in J}$ at η such that each y_i locally defines E_i at η . We write $\text{QM}_\eta(Y, E) \subseteq \text{Val}_X$ for the set of quasi-monomial valuations that can be described at η with respect to $(y_i)_{i \in J}$ and note that $\text{QM}_\eta(Y, E) \simeq \mathbb{R}_{\geq 0}^r$. We set $\text{QM}(Y, E) := \cup_\eta \text{QM}_\eta(Y, E)$, which has the structure of a simplicial cone complex, and $\text{QM}(Y, E)^*$ for the non-trivial valuations in $\text{QM}(Y, E)$.

2.3.3. *Log discrepancy.* For a pair (X, Δ) , we write

$$A_{X,\Delta} : \text{Val}_X^* \rightarrow \mathbb{R} \cup \{+\infty\}$$

for the log discrepancy function with respect to (X, Δ) as in [JM12][BdFFU15] (see [Blu18] for the case when $\Delta \neq 0$). The function $A_{X,\Delta}$ is homogeneous of degree 1 and lower semicontinuous.

A pair (X, Δ) is klt (resp., lc) if and only if $A_{X,\Delta}(v) > 0$ (resp., ≥ 0) for all $v \in \text{Val}_X^*$. If D is an effective \mathbb{Q} -Cartier divisor, then $A_{X,\Delta+D}(v) = A_{X,\Delta}(v) - v(D)$ for all $v \in \text{Val}_X$.

When $\mu : Y \rightarrow X$ is a proper birational morphism with Y normal and $E \subset Y$ a prime divisor,

$$A_{X,\Delta}(\text{ord}_E) = 1 + \text{coeff}_E(K_Y - \mu^*(K_X + \Delta))$$

and we will often write $A_{X,\Delta}(E)$ for this value. If $\mu : Y \rightarrow X$ is a log resolution of (X, Δ) and $E := \text{Exc}(\mu) + \text{Supp}(\mu_*^{-1}\Delta)$, then $A_{X,\Delta}$ is linear on the cones in $\text{QM}(Y, E)$. Additionally, if we write Δ_Y for the \mathbb{Q} -divisor satisfying $K_Y + \Delta_Y = \mu^*(K_X + \Delta)$, then $A_{X,\Delta} = A_{Y,\Delta_Y}$.

The following result is well known.

Lemma 2.3. *Keep the above notation. If (X, Δ) is lc, then*

$$\text{QM}(Y, \Delta_Y^{-1}) = \{v \in \text{Val}_X \mid A_{X,\Delta}(v) = 0\},$$

where Δ_Y^{-1} is the sum of the prime divisors in Δ_Y with coefficient one. In particular, the set does not depend on Y .

Proof. Set $E := \text{Exc}(\mu) + \text{Supp}(\mu_*^{-1}\Delta)$ and observe that $A_{X,\Delta} = A_{Y,\Delta_Y}$ is zero on an extremal ray of a cone in $\text{QM}(Y, E)$ if and only if the corresponding prime divisor on Y has coefficient 1 in Δ_Y . Since $A_{X,\Delta}$ is linear on the cones in $\text{QM}(Y, E)$, this implies $\text{QM}(Y, \Delta_Y^{-1})$ is the locus of $\text{QM}(Y, E)$ where $A_{X,\Delta}$ is zero. By [Blu18] Prop 3.2.5, it is also the locus of Val_X where $A_{X,\Delta}$ is zero. \square

2.4. Invariants associated to log Fano pairs. Let (X, Δ) be a log Fano pair and r a positive integer such that $L := -r(K_X + \Delta)$ is a Cartier divisor. The section ring of L is given by

$$R(X, L) := R = \bigoplus_{m \in \mathbb{N}} R_m = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mL)).$$

2.4.1. *Filtrations induced by valuations and associated invariants.* For $v \in \text{Val}_X$ and $\lambda \in \mathbb{R}_{>0}$, we set

$$\mathcal{F}_v^\lambda R_m := \{s \in R_m \mid v(s) \geq \lambda\}.$$

If $v = \text{ord}_E$, where E is a divisor over X arising on a proper birational model $\mu : Y \rightarrow X$, then

$$\mathcal{F}_v^\lambda R_m \simeq H^0(Y, \mathcal{O}_Y(m\mu^*L - \lceil \lambda E \rceil)).$$

We consider the following invariants

$$T(v) := \sup_{m \in \mathbb{Z}_{>0}} T_{mr}(v), \quad \text{where} \quad T_{mr}(v) := \frac{1}{mr} \sup\{\lambda \mid \mathcal{F}^{\lambda m} R_m \neq 0\}$$

and

$$S(v) := \lim_{m \rightarrow \infty} S_{mr}(v), \quad \text{where} \quad S_{mr}(v) := \int_0^\infty \frac{\dim(\mathcal{F}^{\lambda m} R_m)}{mr \dim R_m} d\lambda.$$

When the choice of the log Fano pair (X, Δ) is not clear from context, we write $T_{X,\Delta}(v)$ and $S_{X,\Delta}(v)$ for these values.

Both invariants can be written in terms of the vanishing of v along classes of anti-canonical divisors. Specifically, for m divisible by r ,

$$T_m = \max\{v(\frac{1}{m}D) \mid D \in |-m(K_X + \Delta)|\}.$$

and

$$S_m(v) = \max\{v(D) \mid D \in |-K_X - \Delta|_{\mathbb{Q}} \text{ is } m\text{-basis type}\}.$$

Here, following [FO18, Def. 0.1], a \mathbb{Q} -divisor $D \in |-K_X - \Delta|_{\mathbb{Q}}$ is called *m-basis type* if there exists a basis $\{s_1, \dots, s_{N_m}\}$ of $H^0(X, \mathcal{O}_X(-m(K_X + \Delta)))$ such that

$$D = \frac{1}{mN_m} (\{s_1 = 0\} + \dots + \{s_{N_m} = 0\}).$$

The functions S and T are lower semicontinuous on Val_X [BJ20, Prop 3.13] and homogeneous of degree 1 [BJ20, §3.2]. When E is a divisor over X arising on a proper birational model $\mu : Y \rightarrow X$, then

$$T(\text{ord}_E) = \sup\{t \in \mathbb{R}_{>0} \mid -\mu^*(K_X + \Delta) - tE \text{ is pseudoeffective}\}$$

and

$$S(\text{ord}_E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \text{vol}(-\mu^*(K_X + \Delta) - tE) dt$$

We will often write $T(E)$ and $S(E)$ for these values. See [BJ20, Sect. 3] for further details.

2.4.2. Behaviour of S and T on a simplicial cone. Let $\mu : Y \rightarrow X$ be a proper birational morphism with Y regular and $E := \sum_{i=1}^d E_i$ a reduced snc divisor.

Proposition 2.4. *The functions S and T are continuous on $\text{QM}(Y, E)$.*

When X is smooth, the result is a special case of [BJ18, Prop 5.6]. We provide a proof that works for all log Fano pairs.

Proof. We will only prove the continuity statement for S , since the proof for T is similar. To proceed, we first show that for each positive integer m divisible by r , the function S_m is continuous on $\text{QM}(Y, E)$.

For a \mathbb{Q} -Cartier divisor D on X , write $\varphi_D : \text{QM}(Y, E) \rightarrow \mathbb{R}$ for the continuous function sending $v \mapsto v(D)$. With this notation, we have $S_m = \sup_D \varphi_D$, where the sup runs through all m -basis type divisors. Since any m -basis type divisor D lies in $\frac{1}{mN_m} |-mN_m(K_X + \Delta)|$, Lemma 2.5 implies the set of functions $\{\varphi_D \mid D \text{ is } m\text{-basis type}\}$

is finite. Therefore, $S_m : \text{QM}(Y, E) \rightarrow \mathbb{R}$ is the maximum of finitely many continuous functions and itself continuous.

We proceed to show S is continuous on $\text{QM}(Y, E)$. Since S is lower semicontinuous on Val_X [BJ20, Prop. 3.13], it suffices to show the upper semicontinuity. Pick any $t \in \mathbb{R}_{>0}$. We have to show $U := \{v \in \text{QM}(Y, E) \mid S(v) < t\}$ is open.

Pick any $w \in U$. We may choose $\varepsilon > 0$ so that $S(w) + \varepsilon A_{X,\Delta}(w) < t$. By the fact that S_m converges pointwise to S and [BL18, Thm. 5.13], which is a partial uniform convergence result, we may choose m divisible by r so that $S_m(w) + \varepsilon A_{X,\Delta}(w) < t$ and $S \leq S_m + \varepsilon A_{X,\Delta}$ on $\text{QM}(Y, E)$. Since S_m and $A_{X,\Delta}$ are continuous on $\text{QM}(Y, E)$, there exists an open neighborhood $W \subset \text{QM}(Y, E)$ so that $S_m + \varepsilon A_{X,\Delta} < t$ on W . Then $W \subseteq U$, which completes the proof. \square

We must prove the following lemma used in the above proposition. For a \mathbb{Q} -Cartier divisor D on X , write $\varphi_D : \text{QM}(Y, E) \rightarrow \mathbb{R}$ for the function sending $v \mapsto v(D)$.

Lemma 2.5. *If H is a Cartier divisor on X , then the set of functions $\{\varphi_D \mid D \in |H|\}$ is finite.*

Proof. It suffices to prove the statement for the restriction of φ_D to a fixed simplicial cone in $\text{QM}(Y, E)$. Choose any irreducible component $Z \subseteq \cap_{i \in J} E_i$. Write $\eta \in Y$ for the generic point of Z , set $r := |J|$, and fix a regular system of parameters $(y_i)_{i \in J}$ at $\eta \in Y$ such that y_i locally defines E_i .

Set $B := \mathbb{P}(H^0(X, \mathcal{O}_X(H))^*)$ and write \mathcal{H} for the universal divisor on $X \times B$ parameterizing elements of $|H|$. To prove the lemma, we will write $B = \cup B_i$ as a finite union of constructible subsets so that the restriction of $\varphi_{\mathcal{H}_b}$ to $\text{QM}_\eta(Y, E)$ is independent of $b \in B_i$.

Choose a nonempty affine subset $U \subseteq B$ and a function $f \in \mathcal{O}_{Y,\eta} \otimes_k \mathcal{O}(U)$ that defines the Cartier divisor $\mathcal{H}|_{Y \times B}$ in a neighborhood of $\eta \times U$. We can write the image of f in $\widehat{\mathcal{O}_{Y,\eta}} \otimes \mathcal{O}(U)$ as $\sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta$, where each $c_\beta \in k(\eta) \otimes \mathcal{O}(U)$ and consider the associated Newton polygon $N := \text{conv}\{\beta + \mathbb{R}_{\geq 0}^r \mid c_\beta \neq 0\}$. Note that N is determined by a finite collection of non-zero coefficients $c_{\beta(1)}, \dots, c_{\beta(m)}$. Hence, if we let $B_1 \subset U$ denote the open set where $c_\beta^{(i)} \neq 0$ for all $i = 1, \dots, m$, then the Newton polygon of the image of f in $\widehat{\mathcal{O}_{Y,\eta}} \otimes k(b)$ agrees with N for all $b \in B_1$. Hence, $\varphi_{\mathcal{H}_b}$ is independent of $b \in B_1$. Repeating this argument on the complement eventually yields such a decomposition. \square

2.4.3. K -stability. The definition of K -stability was originally defined in terms of degenerations [Tia97, Don02]. For this paper, we use the valuative characterization of K -stability invented in [Fuj19a, Li17], which suits our techniques better.

Let E be a prime divisor over a log Fano pair (X, Δ) arising on a proper normal model $\mu : Y \rightarrow X$. Following [Fuj19a], we set

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E)(-K_X - \Delta)^n - \int_0^\infty \text{vol}(-\mu^*(K_X - \Delta) - tE) dt.$$

Definition–Theorem 2.6. A log Fano pair (X, Δ) is *K-semistable* (resp., *K-stable*) if and only if $\beta_{X,\Delta}(E) \geq 0$ (resp., > 0) for all divisors E over X .

The equivalence of this definition with the definition in [Tia97][Don02][LX14] is addressed in [Fuj19a][Li17] (and [BX19] for part of the K-stable case).

2.4.4. Thresholds. Let (X, Δ) be a log Fano pair and r a positive integer so that $r(K_X + \Delta)$ is Cartier. We will consider two thresholds that measure the singularities of anticanonical divisors.

First is an invariant defined in [FO18]. For a positive integer m divisible by r , we set

$$\delta_m(X, \Delta) := \min\{\text{lct}(X, \Delta; D) \mid D \in |-K_X - \Delta|_{\mathbb{Q}} \text{ is } m\text{-basis type}\}.$$

The *stability threshold* of (X, Δ) is defined by $\delta(X, \Delta) := \limsup_{m \rightarrow \infty} \delta_{mr}(X, \Delta)$. As shown in [BJ20], the limsup in the definition of the stability threshold is in fact a limit and

$$\delta(X, \Delta) = \inf_E \frac{A_{X,\Delta}(E)}{S(E)} = \inf_v \frac{A_{X,\Delta}(v)}{S(v)}, \quad (2.2)$$

where the first infimum runs through all prime divisors E over X and the second through $v \in \text{Val}_X^*$ with $A_{X,\Delta}(v) < +\infty$. Therefore, the valuative criterion in Definition–Theorem 2.6 implies (X, Δ) is K-semistable if and only if $\delta(X, \Delta) \geq 1$.

Next is Tian’s α -invariant (also known as the *global log canonical threshold*) defined by

$$\alpha(X, \Delta) := \inf\{\text{lct}(X, \Delta; D) \mid D \in |-K_X - \Delta|_{\mathbb{Q}}\}.$$

Similar to the stability threshold, the invariant may be expressed in terms of valuations

$$\alpha(X, \Delta) := \inf_E \frac{A_{X,\Delta}(E)}{T(E)} = \inf_v \frac{A_{X,\Delta}(v)}{T(v)}; \quad (2.3)$$

see [Amb16][BJ20].

We say that a valuation computes the stability threshold (resp., global log canonical threshold) if it achieves the infimum in (2.2) (resp., (2.3)).

2.5. Complements. The theory of complements was introduced by Shokurov in his work on threefold log flips [Sho92]. The boundedness of complements proved in [Bir19] (also see its generalization in [HLS19]) plays a key role in this paper.

Definition 2.7 (Global complements). Let (X, Δ) be a projective lc pair. A \mathbb{Q} -complement of (X, Δ) is a \mathbb{Q} -divisor Δ^+ on X such that $\Delta^+ \geq \Delta$, (X, Δ^+) is lc, and $K_X + \Delta^+ \sim_{\mathbb{Q}} 0$. An N -complement of (X, Δ) is a \mathbb{Q} -complement Δ^+ satisfying $N(K_X + \Delta^+) \sim 0$.

The latter definition differs from the terminology in [Bir19], which is weaker. An N -complement Δ^+ of (X, Δ) as defined above agrees with the definition in *loc. cit.* of an N -complement Δ^+ of (X, Δ) satisfying $\Delta^+ \geq \Delta$, which is sometimes called a monotonic N -complement in the literature.

Clearly, if Δ^+ is an N -complement, then $\Delta^+ - \Delta \in |-K_X - \Delta|_{\mathbb{Q}}$. Additionally, if r is a positive integer so that $r(K_X + \Delta)$ is Cartier, then $rN(\Delta^+ - \Delta) \in |-rN(K_X + \Delta)|$.

One crucial input in Theorem 1.1 is the following statement, which follows from the deep result of [Bir19, Theorem 1.7].

Theorem 2.8 ([Bir19, Theorem 1.7]). *Let n be a natural number and $I \subset \mathbb{Q} \cap [0, 1]$ a finite set. There is a positive number $N := N(n, I)$ depending only on n and I satisfying the following:*

Assume (X, Δ) is an n -dimensional lc pair such that X is of Fano type and the coefficients of Δ belong to $D(I)$. If (X, Δ) admits a \mathbb{Q} -complement, then it admits an N -complement.

Proof. Since $-K_X - \Delta$ is not nef, we cannot directly apply [Bir19, Thm 1.7]. But, there is an easy reduction step (see e.g. [Bir19, (6.1)]).

Since X is Fano type and $-K_X - \Delta$ is linearly equivalent to the effective divisor $\Delta^+ - \Delta$, we can run an MMP for $-K_X - \Delta$ to get a birational model $h: X \dashrightarrow X'$ such that $-K_{X'} - h_*\Delta$ is nef. Since (X, Δ) has a \mathbb{Q} -complement, so does $(X', h_*\Delta)$. Now, $(X', h_*\Delta)$ has an N -complement by [Bir19, Theorem 1.7]. Therefore, [Bir19, (6.1)] implies (X, Δ) admits an N -complement as well. \square

We note that one can find more general statements on the existence of bounded complements in [HLS19, Thm 1.13].

3. APPROXIMATION AND BOUNDEDNESS

The idea of approximating a valuation by a sequence of divisors coming from a special type of birational morphisms was developed in [LX20, Fuj19b], modeled on the arguments in [LX14]. One key observation in this paper is that we can combine the boundedness of complements with the latter approximation process.

3.1. Approximation of thresholds and \mathbb{Q} -complements. We will proceed to discuss an important class of valuations over a log Fano pair and then describe their relation to the stability and global log canonical thresholds.

3.1.1. Lc places of \mathbb{Q} -complements.

Definition 3.1. Let (X, Δ) be a log Fano pair. We say $v \in \text{Val}_X$ is an *lc place of a \mathbb{Q} -complement* (resp., *N -complement*) if there exists a \mathbb{Q} -complement (resp., N -complement) Δ^+ of (X, Δ) such that $A_{X, \Delta^+}(v) = 0$. When $v = \text{ord}_E$ for some divisor E over X , we simply say E is an lc place of a \mathbb{Q} -complement (resp., N -complement).

In Appendix A we will see that lc places of a \mathbb{Q} -complements are closely related to weakly special test configurations with irreducible central fiber.

We state the following elementary lemma concerning divisorial valuations that are the lc place of a \mathbb{Q} -complement.

Lemma 3.2. *Let (X, Δ) be a log Fano pair and E a prime divisor over X . If E is an lc place of a \mathbb{Q} -complement, then there exists a proper birational morphism of normal varieties $\mu : Y \rightarrow X$ satisfying:*

- (1) E appears as a divisor on Y with $\text{Exc}(\mu) \subseteq E$,
- (2) $(Y, \mu_*^{-1}\Delta + (1-a)E)$ is lc and admits a \mathbb{Q} -complement, where $a := \text{coeff}_E(\Delta)$ if E is a prime divisor on X and zero otherwise, and
- (3) Y is Fano type.

Proof. Choose a \mathbb{Q} -complement Δ^+ of (X, Δ) such that $A_{X, \Delta^+}(E) = 0$ and set

$$D := \Delta^+ - \Delta \sim_{\mathbb{Q}} -(K_X + \Delta).$$

Fix $0 < c < 1$ so that $0 < A_{X, \Delta + cD}(E) < 1$.

By [BCHM10], there exists a proper birational morphism of normal varieties $\mu : Y \rightarrow X$ such that E appears as a divisor on Y with $\text{Exc}(\mu) \subseteq E$ such that $-E$ is \mathbb{Q} -Cartier and μ -ample. Write Γ for the \mathbb{Q} -divisor on Y so that

$$K_Y + \Gamma = \mu^*(K_X + \Delta^+).$$

Since (Y, Γ) is lc and $\Gamma \geq \mu_*^{-1}\Delta + (1-a)E$, (2) holds. Next, set $\Gamma' = \Gamma - (1-c)\mu^*(D)$. Note that (Y, Γ') is klt, since $(X, \Delta + cD)$ is klt. Additionally,

$$-K_Y - \Gamma' \sim_{\mathbb{Q}} -(1-c)\mu^*(K_X + \Delta)$$

is big and nef. Therefore, Y is of Fano type. \square

3.1.2. Stability threshold. We state the following characterization of the stability threshold.

Proposition 3.3. *Let (X, Δ) be a log Fano pair. If $\delta(X, \Delta) \leq 1$, then*

$$\delta(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{S(E)}$$

where the infimum runs through prime divisors E over X such that E is an lc place of a \mathbb{Q} -complement.

The result is proved in [BLZ19] in the case when $\delta(X, \Delta) < 1$. Using an argument from [ZZ19], the $\delta(X, \Delta) = 1$ case can be deduced from the < 1 case.

Proof. We first treat the case when $\delta(X, \Delta) < 1$ which is embedded in the proof of [BLZ19, Theorem 4.1]. For the convenience of the reader, we recall the argument in *loc. cit.*

By Equation [2.2] the inequality $\delta(X, \Delta) \leq \inf_E \frac{A_{X, \Delta}(E)}{S(E)}$ holds. For the reverse inequality, pick any $\varepsilon > 0$. By the fact that δ is a limit and [BJ20, Cor. 3.6], we may choose m so that

$$\delta_m(X, \Delta) < \min\{1, (1 + \varepsilon)\delta(X, \Delta)\}$$

and $S_m(v) \leq (1 + \varepsilon)S(v)$ for all $v \in \text{Val}_X^*$ with finite log discrepancy. Now, fix an m -basis type divisor B such that $\delta_m(X, \Delta) = \text{lc}(X, \Delta; B)$ and a divisor E over X

computing the lct (i.e. $\frac{A_{X,\Delta}(E)}{\text{ord}_E(B)} = \text{lct}(X, \Delta; B)$). Note that $\text{ord}_E(B) \leq S_m(E) \leq (1 + \varepsilon)S(E)$. Therefore,

$$\frac{A_{X,\Delta}(E)}{S(E)} \leq (1 + \varepsilon) \frac{A_{X,\Delta}(E)}{\text{ord}_E(B)} = (1 + \varepsilon)\delta_m(X, \Delta) \leq (1 + \varepsilon)^2\delta(X, \Delta).$$

We will show E is the lc place of a \mathbb{Q} -complement. To proceed, note that $(X, \Delta + \delta_m B)$ is lc and $A_{X,\Delta+\delta_m B}(E) = 0$. By [KM98, Lem. 5.17.2], we can choose a divisor $H \in |-K_X - \Delta|_{\mathbb{Q}}$ so that $(X, \Delta + \delta_m B + (1 - \delta_m)H)$ remains lc. Hence, $\Delta^+ := \Delta + \delta_m B + (1 - \delta_m)H$ is a \mathbb{Q} -complement of (X, Δ) and $A_{X,\Delta^+}(E) = 0$. Therefore, sending $\varepsilon \rightarrow 0$ shows that the reverse inequality holds.

We now assume $\delta(X, \Delta) = 1$.

Claim: For any $\varepsilon \in (0, \alpha(X, \Delta))$, there exists $D \in |-K_X - \Delta|_{\mathbb{Q}}$ such that $(X, \Delta + \varepsilon D)$ is klt and $\delta(X, \Delta + \varepsilon D) < 1$.

The claim follows immediately from [ZZ19, Theorem 1.2]. For the convenience of the reader, we recall the argument in *loc. cit.* Pick any $\varepsilon \in (0, \alpha(X, \Delta))$, then $(X, \Delta + \varepsilon D)$ is klt for any $D \in |-K_X - \Delta|_{\mathbb{Q}}$. Since $\delta(X, \Delta) = 1$, we may choose a prime divisor E over X such that $\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)} < 1 + \frac{\varepsilon}{3n}$, where $n := \dim(X)$. Using the inequality $T_{X,\Delta}(E) \geq (1 + \frac{1}{n})S_{X,\Delta}(E)$ [Fuj19a, Prop. 2.1], we may choose $D \in |-K_X - \Delta|_{\mathbb{Q}}$ such that $\text{ord}_E(D) \geq (1 + \frac{1}{2n})S_{X,\Delta}(E)$. Observe that

$$A_{X,\Delta+\varepsilon D}(E) = A_{X,\Delta}(E) - \varepsilon \cdot \text{ord}_E(D) \quad \text{and} \quad S_{X,\Delta+\varepsilon D}(E) = (1 - \varepsilon)S_{X,\Delta}(E),$$

where the second equation is [BJ20, Lem. 3.7.i]. Therefore,

$$\delta(X, \Delta + \varepsilon D) \leq \frac{A_{X,\Delta+\varepsilon D}(E)}{S_{X,\Delta+\varepsilon D}(E)} = \frac{1}{1 - \varepsilon} \left(\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)} - \varepsilon \cdot \frac{\text{ord}_E(D)}{S_{X,\Delta}(E)} \right).$$

Since $\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)} - \varepsilon \cdot \frac{\text{ord}_E(D)}{S_{X,\Delta}(E)} \leq 1 + \frac{\varepsilon}{3n} - \varepsilon \left(1 + \frac{1}{2n}\right) < 1 - \varepsilon$, we can conclude $\delta(X, \Delta + \varepsilon D) < 1$.

We now return to the proof of the proposition. By Equation [2.2] the inequality $\delta(X, \Delta) \leq \inf_E \frac{A_{X,\Delta}(E)}{S(E)}$ holds. For the reverse inequality, fix a rational number $\varepsilon \in (0, \alpha(X, \Delta))$. By the above claim, there exists $D \in |-K_X - \Delta|_{\mathbb{Q}}$ such that $\delta(X, \Delta + \varepsilon D) < 1$. Using the $\delta < 1$ case, we may find a divisor E over X such that $\frac{A_{X,\Delta+\varepsilon D}(E)}{S_{X,\Delta+\varepsilon D}(E)} < 1$ and E is the lc place of a \mathbb{Q} -complement Δ^+ of $(X, \Delta + \varepsilon D)$. Note that Δ^+ is also a \mathbb{Q} -complement of (X, Δ) , since by definition $\Delta^+ \geq \Delta + \varepsilon D \geq \Delta$.

We now estimate $\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}$. Using that $\alpha(X, \Delta) \leq \frac{A_{X,\Delta}(E)}{T_{X,\Delta}(E)}$ by [2.3], we see $\text{ord}_E(D) \leq T_{X,\Delta}(E) \leq \frac{A_{X,\Delta}(E)}{\alpha(X, \Delta)}$. Therefore,

$$1 > \frac{A_{X,\Delta+\varepsilon D}(E)}{S_{X,\Delta+\varepsilon D}(E)} = \frac{A_{X,\Delta}(E) - \varepsilon \text{ord}_E(D)}{(1 - \varepsilon)S_{X,\Delta}(E)} \geq \left(\frac{1 - \varepsilon/\alpha(X, \Delta)}{1 - \varepsilon} \right) \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}.$$

Sending $\varepsilon \rightarrow 0$ completes the proof. \square

3.1.3. Global log canonical threshold. We now prove an analog of Proposition 3.3 for the global log canonical threshold. The statement follows almost immediately from definitions.

Proposition 3.4. *Let (X, Δ) be a log Fano pair. If $\alpha(X, \Delta) < 1$, then*

$$\alpha(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{T(E)}$$

where the infimum runs through divisors E over X such that E is an lc place of \mathbb{Q} -complement.

By applying a deeper result [Bir21, Thm. 1.5], it follows that the above infimum is a minimum. Though, Proposition 3.4 will be sufficient for proving that $\min\{1, \alpha\}$ is constructible.

Proof. By Equation 2.3, $\alpha(X, \Delta) \leq \inf_E \frac{A_{X, \Delta}(E)}{T(E)}$ where the infimum runs through lc places of \mathbb{Q} -complements. For the reverse inequality, pick any $\varepsilon \in (0, 1 - \alpha(X, \Delta))$. We may choose $D \in |-K_X - \Delta|_{\mathbb{Q}}$ such that

$$c := \text{lct}(X, \Delta; D) \leq \alpha(X, \Delta) + \varepsilon,$$

and a divisor E over X computing $\text{lct}(X, \Delta; D)$. Note that $c < 1$ by our choices of ε .

Observe that $(X, \Delta + cD)$ is lc and $A_{X, \Delta + cD}(E) = A_{X, \Delta}(E) - c \cdot \text{ord}_E(D) = 0$. Since $-K_X - \Delta$ is ample, we may find $H \in |-K_X - \Delta|_{\mathbb{Q}}$ so that $(X, \Delta + cD + (1 - c)H)$ remains lc [KM98, Lem. 5.17.2]. Hence, $\Delta^+ := \Delta + cD + (1 - c)H$ is a \mathbb{Q} -complement of (X, Δ) with $A_{X, \Delta^+}(E) = 0$.

Now, observe that $\frac{A_{X, \Delta}(E)}{T(E)} \leq \text{lct}(X, \Delta; D) \leq \alpha(X, \Delta) + \varepsilon$, since $\text{ord}_E(D) \leq T(E)$. Therefore, sending $\varepsilon \rightarrow 0$ completes the proof. \square

3.2. Boundedness. Using the boundedness of complements, we will show that lc places of \mathbb{Q} -complements are in fact lc places of N -complements.

Theorem 3.5. *Let n be a natural number and $I \subseteq \mathbb{Q}$ a finite set. There is a positive integer $N := N(n, I)$ satisfying the following:*

Assume (X, Δ) is an n -dimensional log Fano pair such that the coefficients of Δ belong to $D(I)$. If E is a divisor over X that is the lc place of a \mathbb{Q} -complement, then E is an lc place of an N -complement.

The statement is a consequence of the boundedness of complements in [Bir19].

Proof. Let E be a divisor over a log Fano pair (X, Δ) such that E is an lc place of a \mathbb{Q} -complement. Applying Lemma 3.2 gives a proper birational morphism $\mu : Y \rightarrow X$ satisfying conditions (1)-(3) of the lemma.

Since the latter conditions are satisfied, we may apply Theorem 2.8 to find an integer $N := N(n, I)$, depending only on n and I , so that $(Y, \mu_*^{-1}\Delta + (1 - a)E)$ admits a N -complement Γ_Y .

Hence, $N(K_Y + \Gamma_Y) \sim 0$, (Y, Γ_Y) is lc, and Γ_Y has coefficient 1 along E . If we set $\Gamma := \mu_*(\Gamma_Y)$, then

$$K_Y + \Gamma_Y = \mu^*(K_X + \Gamma).$$

holds. This implies Γ is a N -complement of (X, Δ) and $A_{X, \Gamma}(E) = 0$. \square

The next two statements follow immediately from combining Theorem 3.5 with Propositions 3.3 and 3.4.

Corollary 3.6. *Let n be a natural number and $I \subseteq \mathbb{Q}$ a finite set. There is a positive integer $N := N(n, I)$ satisfying the following:*

If (X, Δ) is an n -dimensional log Fano pair such that the coefficients of Δ belong to $D(I)$ and $\delta(X, \Delta) \leq 1$, then

$$\delta(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{S(E)},$$

where the infimum runs through divisors over X that are lc places of an N -complement.

Corollary 3.7. *Let n be a natural number and $I \subseteq \mathbb{Q}$ a finite set. There is a positive integer $N := N(n, I)$ satisfying the following:*

If (X, Δ) is an n -dimensional log Fano pair such that the coefficients of Δ belong to $D(I)$ and $\alpha(X, \Delta) < 1$, then

$$\alpha(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{T(E)},$$

where the infimum runs through divisors over X that are lc places of an N -complement.

3.3. Approximating valuations computing the stability threshold. In this section, we show that if a valuation computes $\delta < 1$, then it is a limit of divisorial valuations that are lc places of bounded complements. The result will not be used in the proof of Theorem 1.1

We will obtain stronger results via passing to the cone to also cover the case when $\delta(X, \Delta) = 1$. Nevertheless, Proposition 3.8 is proved only using global arguments.

Proposition 3.8. *Let n be a natural number and $I \subseteq \mathbb{Q}$ a finite set. There is a positive integer $N := N(n, I)$ satisfying the following:*

Assume (X, Δ) is an n -dimensional log Fano pair such that the coefficients of Δ belong to $D(I)$ and $\delta(X, \Delta) < 1$. If $v^ \in \text{Val}_X^{\leq 1}$ computes $\delta(X, \Delta)$, then there exists a sequence of divisorial valuations $(v_k)_k$ in $\text{Val}_X^{\leq 1}$ converging to v^* such that each v_k is the lc place of a N -complement and $\lim_{k \rightarrow \infty} \frac{A_{X, \Delta}(v_k)}{S(v_k)} = \delta(X, \Delta)$.*

Here, $\text{Val}_X^{\leq 1}$ denotes the set $\{v \in \text{Val}_X \mid A_{X, \Delta}(v) = 1\}$. Before proving the proposition, we need the following lemma, which may be viewed as a global analogue of [LX20, Lemma 3.8].

Lemma 3.9. *Let (X, Δ) be a log Fano pair with $\delta(X, \Delta) < 1$. Assume $v^* \in \text{Val}_X^{\leq 1}$ computes $\delta(X, \Delta)$. For any ideal \mathfrak{b} on X and $\varepsilon > 0$, there exists a divisorial valuation $w \in \text{Val}_X^{\leq 1}$ such that*

- (1) w is the lc place of a \mathbb{Q} -complement and
- (2) $w(\mathbf{b}) \geq v^*(\mathbf{b})(1 - \varepsilon)$.

Proof. Let $\mu : Y \rightarrow X$ be a log resolution of (X, Δ, \mathbf{b}) . Write E for the Cartier divisor on Y such that $\mathbf{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ and Δ_Y for the \mathbb{Q} -divisor on Y such that

$$K_Y + \Delta_Y = \mu^*(K_X + \Delta).$$

Since $-K_X - \Delta$ is ample and $-E$ μ -semiample, there exists a rational number $0 < t \ll 1$ so that $-\mu^*(K_X + \Delta) - tE$ is semiample.

Claim: For any $\varepsilon' > 0$, there exists $D \in |-K_X - \Delta|_{\mathbb{Q}}$ such that

$$tw(\mathbf{b}) \leq w(D) \leq tw(\mathbf{b}) + \varepsilon' A_{X,\Delta}(w) \quad (3.1)$$

for all $w \in \text{Val}_X$ and $\text{lct}(X, \Delta; D) > \text{lct}(X, \Delta; \mathbf{b})/2$.

Fix $m \in \mathbb{Z}_{>0}$ sufficiently divisible so that

$$|m(-\mu^*(K_X + \Delta) - tE)|$$

is base point free. By Bertini's Theorem, we may choose a divisor H in the above linear system so that $\text{Supp}(\Delta_Y) + H$ is snc. Set $D := \mu_*(m^{-1}H + tE)$, which is an element of $|-K_X - \Delta|_{\mathbb{Q}}$. We will show D satisfies the claim if $m \gg 0$.

Note that $\mu^*D = m^{-1}H + tE$. Hence, for $w \in \text{Val}_X$,

$$w(D) = m^{-1}w(H) + tw(E) = m^{-1}w(H) + tw(\mathbf{b}).$$

Since $H + \text{Supp}(\Delta_Y)$ is snc, $\text{lct}(Y, \Delta_Y; H) = 1$. Therefore, $A_{X,\Delta}(w) = A_{Y,\Delta_Y}(w) \geq w(H)$, which implies (3.1) holds if $m \geq 1/\varepsilon'$.

To finish the claim, we compute

$$\begin{aligned} \text{lct}(X, \Delta; D)^{-1} &= \text{lct}(Y, \Delta_Y; m^{-1}H + tE)^{-1} \\ &\leq \text{lct}(Y, \Delta_Y; m^{-1}H)^{-1} + \text{lct}(Y, \Delta_Y; tE)^{-1} \\ &\leq 1/m + t \cdot \text{lct}(X, \Delta; \mathbf{b})^{-1}, \end{aligned}$$

where the first inequality follows from the log concavity of the log canonical threshold (e.g. see [JM12, Lemma 1.7.iv]). Since $t < 1$, we conclude $\text{lct}(X, \Delta; D) > \text{lct}(X, \Delta; \mathbf{b})/2$ when $m \gg 0$.

Returning to the proof of the lemma, set $\delta := \delta(X, \Delta)$, $c := \text{lct}(X, \Delta; \mathbf{b})$, and $\beta := \min\{1 - \delta, c/2\}$. Choose a divisor D satisfying the above claim with $\varepsilon' := \varepsilon t v^*(\mathbf{b})/2$. If we set $\Delta' := \Delta + \beta D$, then (X, Δ') is log Fano. Indeed, the pair is klt, since $\text{lct}(X, \Delta; D) > c/2 \geq \beta$. Additionally, $-K_X - \Delta' \sim_{\mathbb{Q}} -(1 - \beta)(K_X + \Delta)$ is ample.

Observe that for $w \in \text{Val}_X$,

$$A_{X,\Delta'}(w) = A_{X,\Delta}(w) - \beta w(D) \quad \text{and} \quad S_{X,\Delta'}(w) = (1 - \beta)S_{X,\Delta}(w),$$

where the second equality is by [BJ20, Lem. 3.7.i]. Therefore, (3.1) gives

$$A_{X,\Delta}(w)(1 - \beta\varepsilon') - \beta tw(\mathbf{b}) \leq A_{X,\Delta'}(w) \leq A_{X,\Delta}(w) - \beta tw(\mathbf{b}). \quad (3.2)$$

If we set $p := v^*(\mathbf{b})$, we see

$$\delta' := \delta(X, \Delta') \leq \frac{A_{X, \Delta'}(v^*)}{S_{X, \Delta'}(v^*)} \leq \frac{A_{X, \Delta}(v^*) - \beta t p}{(1 - \beta) S_{X, \Delta}(v^*)} = \frac{(1 - \beta t p) \delta}{1 - \beta}.$$

Since $1 - \beta \geq \delta$, it follows that $\delta' < 1$.

Applying Proposition 3.3 to (X, Δ') , we may find a divisorial valuation $w \in \text{Val}_X^=$ that is an lc place of a \mathbb{Q} -complement of (X, Δ') so that

$$\frac{A_{X, \Delta'}(w)}{S_{X, \Delta'}(w)} \leq \frac{(1 - \beta t p + \beta \varepsilon') \delta}{1 - \beta}. \quad (3.3)$$

Since $\Delta' \geq \Delta$, w is also an lc place of a \mathbb{Q} -complement of (X, Δ) . By 3.2 and the inequality $\delta \leq \frac{A_{X, \Delta}(w)}{S_{X, \Delta}(w)} = \frac{1}{S_{X, \Delta}(w)}$, we know

$$\frac{(1 - \beta \varepsilon' - \beta t w(\mathbf{b})) \delta}{1 - \beta} \leq \frac{A_{X, \Delta'}(w)}{S_{X, \Delta'}(w)}. \quad (3.4)$$

Analyzing 3.3 and 3.4, we see $p \leq w(\mathbf{b}) + \frac{2\varepsilon'}{t}$. Since $\frac{2\varepsilon'}{t} = \varepsilon v^*(\mathbf{b})$, the proof is complete. \square

Proof of Proposition 3.8. By Lemma 3.9, there exists a sequence of divisorial valuations $(v_k)_k$ in $\text{Val}_X^=$ that are lc places of \mathbb{Q} -complements satisfying

$$\frac{v_k(\mathbf{a}_k(v^*))}{k} \geq 1 - \frac{1}{k} \quad \text{and} \quad A_{X, \Delta}(v_k) = 1.$$

By Theorem 3.5 each v_k is also the lc place of a N -complement. We will show that a subsequence converges to v^* in the valuation space.

Let $\xi \in X$ denote the center of v^* . Write $\mathbf{m}_\xi \subset \mathcal{O}_X$ for the ideal of functions vanishing along ξ and set $r := v^*(\mathbf{m}_\xi)$, which is > 0 . Since $v^*(\mathbf{m}_\xi^{\lceil k/r \rceil}) = \lceil k/r \rceil r \geq k$, $\mathbf{m}_\xi^{\lceil k/r \rceil} \subseteq \mathbf{a}_k(v^*)$. Therefore,

$$\lceil k/r \rceil v_k(\mathbf{m}_\xi) \geq v_k(\mathbf{a}_k(v^*)) \geq k - 1$$

for all k . This implies there exists $\varepsilon > 0$ so that $v_k(\mathbf{m}_\xi) \geq \varepsilon$ for all $k > 1$. Since

$$c := \text{lct}(X, \Delta; \mathbf{m}_\xi) = \inf_{w \in \text{Val}_X^*} \frac{A_{X, \Delta}(w)}{w(\mathbf{m}_\xi)},$$

we also have $v_k(\mathbf{m}_\xi) \leq A_{X, \Delta}(v_k)/c = 1/c$ for all k .

Observe that

$$V := \{v \in \text{Val}_X, | v(\mathbf{m}_\xi) \in [\varepsilon, 1/c] \text{ and } A_{X, \Delta}(v) \leq 1\} \subseteq \text{Val}_X$$

is a compact subset of the valuation space by [BdFFU15, Thm 3.1]. Since any compact subset of the valuation space is also sequentially compact [Poi13] (see also the proof of [LX20, Prop 3.9]), there exists a subsequence $(v_{k_j})_j$ so that the limit $w^* = \lim_{j \rightarrow \infty} v_{k_j}$ exists in Val_X . We will proceed to show $w^* = v^*$.

By the lower semicontinuity of the log discrepancy function, $A_{X,\Delta}(w^*) \leq 1$. For any positive integers k , there is an inclusion $\mathfrak{a}_m(v^*)^{[k/m]} \subseteq \mathfrak{a}_k(v^*)$. This implies $[k/m]v_k(\mathfrak{a}_m(v^*)) \geq v_k(\mathfrak{a}_k(v^*)) \geq k - 1$. Therefore,

$$w^*(\mathfrak{a}_m(v^*)) = \lim_{j \rightarrow \infty} v_{k_j}(\mathfrak{a}_m(v^*)) \geq \lim_{j \rightarrow \infty} \frac{k_j - 1}{[k_j/m]} = m.$$

From the latter inequality, we see $w^*(\mathfrak{a}_\bullet(v^*)) \geq 1$. Hence, $w^* \geq v^*$ holds by [JM12 Lem. 2.4]. Therefore, $S(w^*) \geq S(v^*)$ and equality holds if and only if $w^* = v^*$ by [BJ20 Prop. 3.15].

Since v^* computes the stability threshold, $\frac{A_{X,\Delta}(v^*)}{S(v^*)} \leq \frac{A_{X,\Delta}(w^*)}{S(w^*)}$. Using that $A_{X,\Delta}(w^*) \leq 1 = A_{X,\Delta}(v^*)$, we conclude $S(w^*) = S(v^*)$. Hence, $w^* = v^*$ and $v^* = \lim_{j \rightarrow \infty} v_{k_j}$.

Since S is lower semicontinuous on the valuation space $\liminf_{j \rightarrow \infty} S(v_{k_j}) \geq S(v^*)$. Hence, $\limsup_{j \rightarrow \infty} \frac{A_{X,\Delta}(v_{k_j})}{S(v_{k_j})} \leq \frac{A_{X,\Delta}(v^*)}{S(v^*)} = \delta(X, \Delta)$. By (2.2), the limit exists and equals $\delta(X, \Delta)$. \square

4. CONSTRUCTIBILITY

4.1. Invariance of volumes. To prove Theorem 1.1, we will need a constructibility result for the functions S and T when the valuation varies in a family.

Consider the following setup: Let $(X, \Delta) \rightarrow B$ be a \mathbb{Q} -Gorenstein family of log Fano pairs with B smooth. Let D be an effective \mathbb{Q} -divisor such that $D \sim_{B, \mathbb{Q}} -K_{X/B} - \Delta$, $\text{Supp}(D)$ does not contain a fiber, and $(X_b, \Delta_b + D_b)$ is lc for all $b \in B$.

Proposition 4.1. *If $(X, \Delta + D) \rightarrow B$ admits a fiberwise log resolution $g : Y \rightarrow X$ and F is a toroidal divisor with respect to $\text{Exc}(g) + \text{Supp}(g_*^{-1}\Delta)$ satisfying $A_{X, \Delta + D}(F) < 1$, then*

$$S_{X_b, \Delta_b}(F_b) \quad \text{and} \quad T_{X_b, \Delta_b}(F_b)$$

are independent of $b \in B$.

The *toroidal* condition in the above theorem means F is an exceptional divisor of a sequence of toroidal blowups of Y with respect to $\text{Exc}(g) + \text{Supp}(g_*^{-1}\Delta)$. This is equivalent to the condition that $\text{ord}_F \in \text{QM}(Y, \text{Exc}(g) + \text{Supp}(g_*^{-1}\Delta))$.

The result is a consequence of the deformation invariance of log plurigenera in smooth families [HMX13 Thm. 1.8], whose proof is based on comparing the relative MMP over B and the MMP for individual fibers.

Proof. By shrinking B , we may assume B is affine. By repeatedly blowing up the center of F on Y , we may assume F is a prime divisor on Y . We fix $t \in \mathbb{Q}_{>0}$ and aim to show

$$\text{vol}(-g_b^*(K_{X_b} + \Delta_b) - tF_b) \tag{4.1}$$

is independent of $b \in B$.

Let Γ_1 and Γ_2 be the effective \mathbb{Q} -divisors without common components in their support such that

$$K_Y + \Gamma_1 = g^*(K_X + \Delta + D) + \Gamma_2$$

and $g_*\Gamma_1 = \Delta + D$. Note that $\text{Supp}(\Gamma_1 + \Gamma_2)$ is relative snc over B and $d := \text{coeff}_F(\Gamma_1) = 1 - A_{X, \Delta+D}(F) > 0$. By inversion of adjunction we know that $(X, \Delta + D)$ is log canonical. In particular, Γ_1 has coefficients in $[0, 1]$.

Since $-K_{X/B} - \Delta$ is f -ample, we may use Bertini's Theorem to find an effective \mathbb{Q} -divisor $H \sim_{B, \mathbb{Q}} -(d/t)(K_{X/B} + \Delta)$ such that $\Gamma_1 + g^*H - dF$ has coefficients in the interval $[0, 1]$ and $\text{Supp}(\Gamma_1 + g^*H - dF)$ is relative snc over B after possibly shrinking B . Applying [HMX13, Thm. 1.8 (3)] gives that

$$\text{vol}(K_{Y_b} + (\Gamma_1)_b + g_b^*H_b - dF_b) \quad (4.2)$$

is independent of $b \in B$. Observe that

$$\begin{aligned} K_Y + \Gamma_1 + g^*H - dF &\sim_{B, \mathbb{Q}} g^*(K_{X/B} + \Delta + D + H) - dF + \Gamma_2 \\ &\sim_{B, \mathbb{Q}} -(d/t)g^*(K_{X/B} + \Delta) - dF + \Gamma_2 \end{aligned}$$

and, hence,

$$K_{Y_b} + (\Gamma_1)_b + g_b^*H_b - dF_b \sim_{\mathbb{Q}} (d/t)(-g_b^*(K_{X_b} + \Delta_b) - tF_b + (t/d)(\Gamma_2)_b).$$

For any sufficiently divisible $m \in \mathbb{Z}_{>0}$, every effective Cartier divisor $G \in |m(-g_b^*(K_{X_b} + \Delta_b) - tF_b + (t/d)(\Gamma_2)_b)|$ satisfies that $G + mtF_b \in |m(-g_b^*(K_{X_b} + \Delta_b) + (t/d)(\Gamma_2)_b)|$. Since $(\Gamma_2)_b$ is g_b -exceptional, we know that

$$|m(-g_b^*(K_{X_b} + \Delta_b) + (t/d)(\Gamma_2)_b)| = g_b^*|m(-K_{X_b} - \Delta_b)| + (mt/d)(\Gamma_2)_b.$$

Hence $G + mtF_b \geq (mt/d)(\Gamma_2)_b$. Since $F_b \not\subset \text{Supp}(\Gamma_2)$, we know that $G - (mt/d)(\Gamma_2)_b$ is an effective Cartier divisor in $|m(-g_b^*(K_{X_b} + \Delta_b) - tF_b)|$. Thus we have

$$\begin{aligned} \text{vol}(-g_b^*(K_{X_b} + \Delta_b) - tF_b) &= \text{vol}(-g_b^*(K_{X_b} + \Delta_b) - tF_b + (t/d)(\Gamma_2)_b) \\ &= (t/d)^n \text{vol}(K_{Y_b} + \Gamma_b + g_b^*H_b - dF_b). \end{aligned}$$

Hence, (4.1) is independent of $b \in B$. Since this holds for each $t \in \mathbb{Q}_{>0}$, $S_{X_b, \Delta_b}(F_b)$ and $T_{X_b, \Delta_b}(F_b)$ are also independent of $b \in B$. \square

For each closed point $b \in B$, we consider the infimum of $\frac{A}{S}$ and $\frac{A}{T}$ over the lc centers of $(X_b, \Delta_b + D_b)$. Specifically, we set

$$a_b := \inf_v \frac{A_{X_b, \Delta_b}(v)}{T_{X_b, \Delta_b}(v)} \quad \text{and} \quad d_b := \inf_v \frac{A_{X_b, \Delta_b}(v)}{S_{X_b, \Delta_b}(v)}, \quad (4.3)$$

where the infima run through $v \in \text{Val}_{X_b}^*$ with $A_{X_b, \Delta_b + D_b}(v) = 0$. Using Proposition 4.1, we can understand how these values vary in families.

Proposition 4.2. *If $(X, \Delta + D) \rightarrow B$ admits a fiberwise log resolution, then a_b and d_b are independent of the closed point $b \in B$. Furthermore, each infimum is a minimum and achieved by a quasi-monomial valuation.*

Proof. Let $g : Y \rightarrow X$ be a fiberwise log resolution of $(X, \Delta + D)$ and define Γ by the formula

$$K_Y + \Gamma = g^*(K_X + \Delta + D).$$

Since g is a fiberwise log resolution, $\text{Supp}(\Gamma)$ is snc and each stratum of $\text{Supp}(\Gamma)$ is smooth with irreducible fibers over B . Write $\Gamma^{=1}$ for the sum of the prime divisors with coefficient 1 in Γ .

For $b \in B$, $K_{Y_b} + \Gamma_b = g^*(K_{X_b} + \Delta_b + D_b)$ and the valuations on X_b that are lc places of $(X_b, \Delta_b + D_b)$ are precisely the valuations in $\text{QM}(Y_b, \Gamma_b^{=1})$ by Lemma 2.3. Note that A , S , and T are continuous on $\text{QM}(Y_b, \Gamma_b^{=1})$ (see Proposition 2.4 for S and T) and homogeneous of degree 1. Therefore, $\frac{A}{T}$ and $\frac{A}{S}$ are continuous and homogeneous of degree zero on $\text{QM}(Y_b, \Gamma_b^{=1})^*$. This implies that the functions achieve minima on the latter set.

Next, observe that there is a natural isomorphism of simplicial cone complexes

$$\text{QM}(Y, \Gamma^{=1}) \simeq \text{QM}(Y_b, \Gamma_b^{=1}),$$

since the stratum of $\text{Supp}(\Gamma^{=1})$ are smooth over B and have irreducible fibers. To finish the proof, we will show that the map

$$\text{QM}(Y, \Gamma^{=1})^* \xrightarrow{\sim} \text{QM}(Y_b, \Gamma_b^{=1})^* \xrightarrow{\frac{A}{S}} \mathbb{R}_{>0}$$

is independent of $b \in B$ and the same holds for $\frac{A}{T}$.

It is clear that $\text{QM}(Y, \Gamma^{=1})^* \xrightarrow{\sim} \text{QM}(Y_b, \Gamma_b^{=1})^* \xrightarrow{\frac{A}{S}} \mathbb{R}_{>0}$ is independent of $b \in B$, since

$$(K_Y - g^*(K_X + D))|_{X_b} = K_{Y_b} - g_b^*(K_{X_b} + D_b).$$

Additionally, Proposition 4.1 implies $\text{QM}(Y, \Gamma^{=1})^* \xrightarrow{\sim} \text{QM}(Y_b, \Gamma_b^{=1})^* \xrightarrow{\frac{S}{T}} \mathbb{R}_{>0}$ is independent of $b \in B$ along the rational points of $\text{QM}(Y, \Gamma^{=1})^*$ and the same holds for T . Since S and T are continuous on cones, the statement holds for all points in $\text{QM}(\Gamma^{=1})$ and the proof is complete. \square

4.2. Constructibility of thresholds. We are now ready to prove that the stability threshold and global log canonical threshold are constructible in families.

Proposition 4.3. *If $(X, \Delta) \rightarrow B$ is a \mathbb{Q} -Gorenstein family of log Fano pairs over a normal variety B , then the functions*

$$B \ni b \mapsto \min\{\alpha(X_b, \Delta_b), 1\} \quad \text{and} \quad B \ni b \mapsto \min\{\delta(X_b, \Delta_b), 1\}$$

are constructible.

Proof. We only prove the statement for the stability threshold, since the statement for the global log canonical threshold follows from the same argument, but with Corollary 3.6 replaced by Corollary 3.7.

Fix a positive integer r so that $r(K_X + \Delta)$ is a Cartier divisor. Next, apply Corollary 3.6 to find a positive integer N so that the following holds: if $b \in B$ is a closed point and $\delta(X_b, \Delta_b) \leq 1$, then

$$\delta(X_b, \Delta_b) = \inf_v \frac{A_{X_b, \Delta_b}(v)}{S_{X_b, \Delta_b}(v)}$$

where the infimum runs through all divisorial valuations $v \in \text{Val}_{X_b}^*$ that are lc places of an N -complement. Notice that for such a valuation v , there exists $D_b \in \frac{1}{rN} | -rN(K_{X_b} + \Delta_b) |$ such that $(X_b, \Delta_b + D_b)$ is lc and $A_{X_b, \Delta_b + D_b}(v) = 0$.

To parameterize such boundaries, observe that $f_* \mathcal{O}_X(-rN(K_{X/B} + \Delta))$ commutes with base change, since Kawamata-Viehweg vanishing implies $H^1(X_b, \mathcal{O}_{X_b}(-rN(K_{X_b} + \Delta_b))) = 0$ for all $b \in B$. Set

$$W := \mathbb{P}(f_* \mathcal{O}_X(-rN(K_{X/B} + \Delta))^*) \rightarrow B$$

and note that, for $b \in B$, the $k(b)$ -points of W_b are in bijection with divisors in $| -rN(K_{X_b} + \Delta_b) |$. Let H be the universal divisor on $X \times_B W$ under this correspondence and set $D := \frac{1}{rN} H$. By the lower semicontinuity of the log canonical threshold, the locus

$$Z := \{w \in W \mid \text{lct}(X_w, \Delta_w; D_w) = 1\}$$

is locally closed in W . The scheme Z together with the \mathbb{Q} -divisor D_Z on $X_Z := X \times_B Z$ parameterizes boundaries of the desired form.

For a closed point $z \in Z$, set

$$d_z := \inf \left\{ \frac{A_{X_z, \Delta_z}(v)}{S_{X_z, \Delta_z}(v)} \mid v \in \text{Val}_{X_z}^* \text{ and } A_{X_z, \Delta_z + D_z}(v) = 0 \right\},$$

By the above discussion, if $b \in B$ is closed, then $\min\{1, \delta(X_b, \Delta_b)\}$ equals the infimum of $\{1\} \cup \{d_z \mid z \in Z_b\}$.

Now, choose a locally closed decomposition $Z = \cup_{i=1}^r Z_i$ so that each Z_i is smooth and there is an étale map $Z'_i \rightarrow Z_i$ such that $(X_{Z'_i}, \Delta_{Z'_i} + D_{Z'_i})$ admits a fiberwise log resolution. For a closed point $z \in Z_i$, d_z is independent of $z \in Z_i$ (by Proposition 4.2) and we denote the value by $d^{(i)}$. Hence, for a closed point $b \in B$, $\min\{1, \delta(X_b, \Delta_b)\}$ is the minimum of $\{1\} \cup \{d^{(i)} \mid b \in \pi(Z_i)\}$. Therefore, we may write $B = \cup_j B_j$ as a finite union of constructible subsets such that $B_j \ni b \mapsto \min\{1, \delta(X_{\bar{b}}, \Delta_{\bar{b}})\}$ is constant on closed points. Since the latter function is lower semicontinuous [BL18], it must be constant on all scheme theoretic points and the proof is complete. \square

Proof of Theorem 1.1. Proposition 4.3 implies that the functions are constructible. The main result of [BL18] implies they are lower semicontinuous. \square

Remark 4.4. To deduce Theorem 1.1, we do not need the full strength of [BL18]. Indeed, we only need that the functions $\min\{1, \alpha\}$ and $\min\{1, \delta\}$ are weakly lower semicontinuous, which means that they do not increase under specialization.

Proof of Corollary 1.2. Since a log Fano pair is K-semistable if and only if $\delta \geq 1$, $B^\circ = \{b \in B \mid \delta(X_{\bar{b}}, \Delta_{\bar{b}}) \geq 1\}$. Theorem 1.1 implies that the latter set is a Zariski open subset of B . \square

Proof of Theorem 1.3. The proof of [BX19, Cor. 1.4], but with the openness of uniform K-stability replaced by the openness of K-semistability (Corollary 1.2), implies $\mathfrak{X}_{V,n}^{\text{Kss}}$ is an Artin stack of finite type over k . With that step complete, we may apply

[ABHLX20, Cor. 1.2] to see $\mathfrak{X}_{V,n}^{\text{Kss}}$ admits a good moduli space. By [ABHLX20, Rem. 2.2], k -points of the good moduli space are in bijection with closed k -points of $\mathfrak{X}_{V,n}^{\text{Kss}}$.

It remains to show a k -point $[X] \in \mathfrak{X}_{V,n}^{\text{Kss}}$ is closed if and only if X is K-polystable. By [LWX19] any K-semistable \mathbb{Q} -Fano variety degenerates to a uniquely determined K-polystable \mathbb{Q} -Fano variety via a special test configuration. Hence, if $[X]$ is closed, X must be K-polystable. Next, assume X is K-polystable. Choose a closed k -point $[X_0]$ in the closure of $[X]$. Since $[X_0]$ is closed, it is K-polystable. Therefore, [BX19] implies $X \simeq X_0$ and, hence, $[X]$ is closed. This completes the proof. \square

As a consequence of Corollary [1.2] and Theorem [1.3], we deduce the openness of K-stability and constructibility of K-polystability.

Theorem 4.5. *Let $(X, \Delta) \rightarrow B$ be a \mathbb{Q} -Gorenstein family of log Fano pairs over a normal base B . Then the set*

$$B^{\text{Ks}} := \{b \in B \mid (X_{\bar{b}}, \Delta_{\bar{b}}) \text{ is K-stable}\}$$

is a Zariski open subset of B . Moreover, the set

$$B^{\text{Kps}} := \{b \in B \mid (X_{\bar{b}}, \Delta_{\bar{b}}) \text{ is K-polystable}\}$$

is a constructible subset of B .

Proof. For simplicity, we assume $\Delta = 0$ as the proof for the general case follows similarly by replacing the K-moduli space of \mathbb{Q} -Fano varieties by log Fano pairs (see e.g. [XZ20, Theorem 2.21]). We may also assume that B is irreducible. Let $n := \dim X_b$ and $V := (-K_{X_b})^n$ for a closed point $b \in B$. By Corollary [1.2] there is an open subset B° of B parametrizing K-semistable fibers. Denote by $X^\circ := X \times_B B^\circ$. We take the Artin stack $\mathfrak{X}_{n,V}^{\text{Kss}}$ and its separated good moduli space $X_{n,V}^{\text{Kps}}$ from Theorem [1.3]. Since the Artin stack $\mathfrak{X}_{n,V}^{\text{Kss}}$ represents the moduli functor of K-semistable \mathbb{Q} -Fano varieties, there is a morphism $\phi : B^\circ \rightarrow \mathfrak{X}_{n,V}^{\text{Kss}}$ whose pull-back of the universal family over $\mathfrak{X}_{n,V}^{\text{Kss}}$ gives $X^\circ \rightarrow B^\circ$.

As shown in the proof of Theorem [1.3], K-polystable \mathbb{Q} -Fano varieties of dimension n and volume V corresponds to closed points of the stack $\mathfrak{X}_{n,V}^{\text{Kss}}$. Let x be such a closed point with stabilizer group G_x which is reductive by [ABHLX20]. Since $\mathfrak{X}_{n,V}^{\text{Kss}}$ is a global quotient stack, by the Luna étale slice theorem [ABHLX20, Remark 2.11], there exist an affine scheme $\text{Spec}(A)$ with an action of G_x , a G_x -fixed closed point $w \in \text{Spec}(A)$, and a Cartesian diagram

$$\begin{array}{ccc} [\text{Spec}(A)/G_x] & \xrightarrow{f} & \mathfrak{X}_{n,V}^{\text{Kss}} \\ \downarrow & & \downarrow \pi \\ \text{Spec}(A) // G_x & \longrightarrow & X_{n,V}^{\text{Kps}} \end{array}$$

such that $f([w]) = [x]$, $\text{Spec}(A) // G_x$ is an étale neighborhood of $\pi(x)$, f sends closed points to closed points, and f induces an isomorphism of stabilizer groups at

closed points. In particular, we know that a geometric point $y \in \operatorname{Spec}(A)$ is GIT stable (resp. GIT polystable) if and only if $f([y])$ represents a K-stable (resp. K-polystable) \mathbb{Q} -Fano variety. Since GIT stable locus is open (see [MFK94, Chapter 1.4] or [Muk03, Proposition 5.15]), and openness is an étale-local property, we know that B^{Ks} is an open subset of B° . Similarly, it is a well-known fact that the GIT polystable locus is constructible. Here we give a brief proof of this fact. Denote by $Z := \operatorname{Spec}(A)$ and $\Phi : Z \rightarrow Z // G_x$ the good quotient morphism. Let $Z_r := \{z \in Z \mid \dim \operatorname{Stab}(z) \geq r\}$. Then it is clear that Z_r is a closed subset of Z . Let Z^{ps} be the GIT polystable locus of Z . Then by similar argument to [Muk03, Proposition 5.13], we know that $Z^{\text{ps}} = \bigcup_{r=0}^{\dim G_x} (Z_r \setminus \Phi^{-1}(\Phi(Z_{r+1})))$. Hence Z^{ps} is constructible in Z . Since constructibility is an étale-local property, we have that B^{Kps} is a constructible subset of B° . The proof is finished. \square

4.3. Existence of valuations computing the stability threshold. A consequence of the proof of Theorem 1.1 is the following result. In Theorem A.7 we will obtain properties of valuations computing $\delta \leq 1$.

Theorem 4.6. *If (X, Δ) is a log Fano pair with $\delta(X, \Delta) \leq 1$, then there exists a quasi-monomial valuation $v \in \operatorname{Val}_X^*$ computing $\delta(X, \Delta)$.*

The existence of valuations computing the stability threshold was previously proven in [BJ20] using the generic limit construction. The proof below is entirely different.

Proof. Fix a positive integer r so that $r(K_X + \Delta)$ is a Cartier divisor and apply Corollary 3.6 to find a positive integer N so that $\delta(X, \Delta) = \inf_v \frac{A_{X, \Delta}(v)}{S(v)}$, where the infimum runs through all valuations $v \in \operatorname{DivVal}_X$ that are lc places of an N -complement. For such a valuation v , there exists $D \in \frac{1}{rN}|-rN(K_X + \Delta)|$ such that $(X, \Delta + D)$ is lc and $A_{X, \Delta + D}(v) = 0$. We proceed to parameterize such \mathbb{Q} -divisors.

Set $W := \mathbb{P}(H^0(X, \mathcal{O}_X(-rN(K_X + \Delta)))^*)$. Write H for the universal divisor on $X \times W$ parameterizing divisors in $|-rN(K_X + \Delta)|$ and set $D := \frac{1}{rN}H$. By the lower semicontinuity of the log canonical threshold, the locus

$$Z := \{w \in W \mid \operatorname{lct}(X_w, \Delta_w; D_w) = 1\}$$

is locally closed in W . Hence, the \mathbb{Q} -divisor D_Z on $X \times Z$ parameterizes boundaries of the desired form.

For each closed point $z \in Z$, set $d_z := \inf_v \frac{A_{X, \Delta}(v)}{S(v)}$, where the infimum runs through all $v \in \operatorname{Val}_{X_z}$ such that $A_{X, \Delta + D_z}(v) = 0$. This infimum is a minimum by Proposition 4.2 and is achieved by a quasi-monomial valuation v_z^* . By the above discussion, $\delta(X, \Delta)$ equals $\inf\{d_z \mid z \in Z\}$.

Now, choose a locally closed decomposition $Z = \bigcup_{i=1}^r Z_i$ so that each Z_i is smooth and there is an étale map $Z'_i \rightarrow Z_i$ such that $(X_{Z'_i}, \Delta_{Z'_i} + D_{Z'_i})$ admits a fiberwise log resolution. For closed point $z \in Z_i$, d_z is independent of $z \in Z_i$ by Proposition 4.2. Therefore, d_z takes finitely many values and we can find $z_0 \in Z$ such that $\delta(X, \Delta) = d_{z_0}$ and is computed by $v_{z_0}^*$. \square

Remark 4.7. The proof of the existence of valuations computing the stability threshold in [BJ20] requires that the base field be uncountable. The assumption is not needed in the above proof.

APPENDIX A. K-STABILITY AND COMPLEMENTS

In the appendix, we will combine the cone construction and results from [Xu20] to further use complements to study the K-stability of a log Fano pair (X, Δ) . During our investigation, we will relate degenerations of the log Fano pair (X, Δ) to valuations centered on the vertex of the cone. This idea was first introduced in [Li17] and then extended in [LX20, LWX21] in the study of relations between K-stability and normalized volume. It is a powerful technique and works particularly well for studying the valuations v which computes $\delta(X, \Delta) = 1$ (see e.g. [BX19]). Following the spirit of [Li17, LX20, LWX21], we obtain results on the log Fano pair (X, Δ) by applying the local results from [Xu20] to cone singularities. We note that [Xu20] and the current paper have a similar strategy via the local-to-global principle. Both papers use the existence of bounded complements proved in [Bir19].

A.1. Test configuration and lc places. In [Tia97, Don02], the K-(semi,poly)stability of a log Fano pair is defined by looking at the sign of the *generalized Futaki invariant* of every *test configuration*. For definitions and more background, see e.g. [LX14, BHJ17].

Definition A.1. Let (X, Δ) be a log Fano pair. A test configuration (resp. semiample test configuration) $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$ of (X, Δ) is said to be *weakly special* if $(\mathcal{X}, \Delta_{\text{tc}} + \mathcal{X}_0)$ is log canonical and $\mathcal{L} \sim_{\mathbb{Q}} -K_{\mathcal{X}} - \Delta_{\text{tc}}$ is ample (resp. semiample) over \mathbb{A}^1 . A finite set of (possibly trivial) \mathbb{Z} -valued divisorial valuations $\{v_1, \dots, v_d\} \subset \text{Val}_X$ is called a *weakly special collection* if there exists a weakly special semiample test configuration $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$ of (X, Δ) such that $v_i = v_{\mathcal{X}_0^{(i)}}$ (see [BHJ17, Def. 4.4]) where $\{\mathcal{X}_0^{(i)}\}_{i=1}^d$ are all irreducible components of \mathcal{X}_0 . A *prime divisor* E over (X, Δ) is said to be *weakly special* if there exists a weakly special test configuration $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$ of (X, Δ) with \mathcal{X}_0 irreducible such that $v_{\mathcal{X}_0} = c \cdot \text{ord}_E$ for some $c \in \mathbb{Z}_{>0}$.

Theorem A.2. Let n be a positive integer and $I \subset \mathbb{Q} \cap [0, 1]$ a finite set. Then there exists a positive integer $N = N(n, I)$ satisfying the following:

If (X, Δ) is an n -dimensional log Fano pair such that coefficients of Δ belong to I , then a finite set of \mathbb{Z} -valued divisorial valuations $\{v_1, \dots, v_d\} \subset \text{Val}_X$ is a weakly special collection if and only if there exists an N -complement Δ^+ of (X, Δ) such that each v_i is an lc place of (X, Δ^+) .

Note that a special case of Theorem [A.2] on weakly special divisors with small β -invariants is proved in [ZZ19, Theorem 3.10] independently.

Our argument is a refinement of [LWX21, Proof of Lemma 3.4]. In particular, we track the integral structure of test configurations and complements.

We first show that any weakly special collection of \mathbb{Z} -valued divisorial valuations $\{v_1, \dots, v_d\}$ are lc places of a common \mathbb{Q} -complement. We will use the following cone construction: Fix a positive integer m such that $L := -m(K_X + \Delta)$ is Cartier. Let $Z := C(X, L)$ be the affine cone over X with polarization L . Denote by $o \in Z$ the cone vertex. Let Γ be the Zariski closure of the pull-back of Δ under the projection $Z \setminus \{o\} \rightarrow X$. Denote by $w_0 := \text{ord}_{X_0}$ the canonical valuation in Val_Z where X_0 is the exceptional divisor of blowing up the cone vertex o in Z . Assume $v_i = c_i \cdot \text{ord}_{E_i}$ with $c_i \in \mathbb{Z}_{\geq 0}$. Denote by $k_0 := m \cdot \max_{1 \leq i \leq d} \{A_{X, \Delta}(v_i)\}$. For each divisor E_i over X and each integer $k > k_0$, we consider the divisorial valuation $w_{i,k}$ on Z as a quasi-monomial combination of ord_{X_0} and $\text{ord}_{E_{i,\infty}}$ with weights $(1 - \frac{mA_{X, \Delta}(v_i)}{k}, \frac{c_i}{k})$ where $E_{i,\infty}$ is the pull-back of E_i under the projection $Z \setminus \{o\} \rightarrow X$. Since $mA_{X, \Delta}(E_i)$ is a positive integer, we know that the value group of $w_{i,k}$ is generated by 1 and $\frac{c_i}{k}$. Thus for any $k > k_0$ there is a prime divisor $E_{i,k}$ over Z centered at o such that $\frac{k}{\gcd(k, c_i)} \cdot w_{i,k} = \text{ord}_{E_{i,k}}$.

Proposition A.3. *Suppose $\{v_1, \dots, v_d\}$ is a weakly special collection of \mathbb{Z} -valued divisorial valuations over (X, Δ) . Then for any k sufficiently large, there exists a proper birational morphism $\mu_k : W_k \rightarrow Z$ from a normal variety W_k such that*

- (1) μ_k is an isomorphism over $Z \setminus \{o\}$ and $\mu_k^{-1}(o) = \cup_{i=1}^d E_{i,k}$;
- (2) $(W_k, (\mu_k)_*^{-1}\Gamma + \sum_{i=1}^d E_{i,k})$ is log canonical;
- (3) $-(K_{W_k} + (\mu_k)_*^{-1}\Gamma + \sum_{i=1}^d E_{i,k})$ is semiample over Z .

Proof. Let $(\mathcal{X}, \Delta_{\text{tc}})$ be the weakly special semiample test configuration corresponding to $\{v_i\}_{i=1}^d$. For $L = -m(K_X + \Delta)$, we denote by

$$R = \oplus_{j=0}^{\infty} R_j := \oplus_{j=0}^{\infty} H^0(X, jL).$$

It is clear that $Z = \text{Spec}(R)$. Let us take an ample model $\rho : \mathcal{X} \rightarrow \mathcal{X}'$ of $-K_{\mathcal{X}} - \Delta_{\text{tc}}$ over \mathbb{A}^1 . Denote by $\Delta'_{\text{tc}} := \rho_* \Delta_{\text{tc}}$, then it is clear that $(\mathcal{X}', \Delta'_{\text{tc}})$ is a weakly special test configuration of (X, Δ) . After reindexing we can assume that ρ precisely contracts $\mathcal{X}_0^{(i)}$ for $d' < i \leq d$ where d' is the number of irreducible components of \mathcal{X}'_0 .

Before constructing $\mu_k : W_k \rightarrow Z$, we first construct $\mu'_k : W'_k \rightarrow Z$ such that (1), (2), and (3) hold as well after replacing (μ_k, W_k, d) and “semiample” by (μ'_k, W'_k, d') and “ample”, respectively. We denote these new statements by (1'), (2'), and (3').

For part (1'), consider the following \mathbb{Z} -filtration of R

$$\mathcal{F}^p R_j := \{s \in R_j \mid v_i(s) \geq p + mA_{X, \Delta}(v_i)j \text{ for any } 1 \leq i \leq d'\}. \quad (\text{A.1})$$

By [BHJ17] Propositions 2.15, 4.11, and Lemma 5.17], we know that the filtration $\mathcal{F}^\bullet R$ is finitely generated,

$$\mathcal{X}' \cong \text{Proj} \bigoplus_{j=0}^{\infty} \bigoplus_{p=-\infty}^{\infty} t^{-p} \mathcal{F}^p R_j, \quad (\text{A.2})$$

and $-m(K_{\mathcal{X}'/\mathbb{A}^1} + \Delta'_{\text{tc}})$ corresponds to $\mathcal{O}(1)$ under the grading of j . Consider the ideal sequence $I_p := \oplus_{j=0}^{\infty} \mathcal{F}^{p-jk} R_j \subset R$ for $p \in \mathbb{Z}$. By (A.1) we see that $\mathcal{F}^{p-jk} R_j = R_j$,

whenever $j \geq p/(k - mA_{X,\Delta}(v_i))$ for all $1 \leq i \leq d'$. So I_p is cosupported at o if $p > 0$, and $I_p = R$ if $p \leq 0$. Since $\mathcal{F}^\bullet R$ is finitely generated and multiplicative, we see that $\bigoplus_{p=0}^\infty I_p$ is also a finitely generated R -algebra. Let $W'_k := \text{Proj}_R \bigoplus_{p=0}^\infty I_p$ with $\mu'_k : W'_k \rightarrow Z$ the projection morphism.

Next we show that $I_p = \bigcap_{i=1}^{d'} \mathfrak{a}_{p/k}(w_{i,k})$. Since $w_{i,k}$ is \mathbb{G}_m -invariant, its valuation ideals are graded. Hence it suffices to verify the above equality for all homogeneous elements. Let $s \in R_j$ be a homogeneous element. From the definition of $w_{i,k}$ we know

$$w_{i,k}(s) = \left(1 - \frac{mA_{X,\Delta}(v_i)}{k}\right)j + \frac{c_i}{k} \text{ord}_{E_i}(s) = \frac{1}{k}(jk - mA_{X,\Delta}(v_i)j + v_i(s)).$$

Thus $w_{i,k}(s) \geq p/k$ if and only if $v_i(s) \geq p - jk + mA_{X,\Delta}(v_i)j$, which implies $I_p = \bigcap_{i=1}^{d'} \mathfrak{a}_{p/k}(w_{i,k})$. Since for each $p \geq 0$ the ideal I_p is integrally closed as a finite intersection of valuation ideals, we know that W'_k is normal. Besides, by [BHJ17 Lemma 5.17] if any i is dropped from the intersection on the right-hand side of (A.1) we would not get $\mathcal{F}^p R_j$. Hence [BHJ17 Theorem 1.10] implies that for p sufficiently divisible, the set of Rees valuations of I_p is given by $\{\frac{k}{p}w_{i,k}\}_{i=1}^{d'}$. Thus $\mu'_k : W'_k \rightarrow Z$ precisely extracts $\bigcup_{i=1}^{d'} E'_{i,k}$ where $E'_{i,k}$ is the birational transform of $E_{i,k}$ and we confirm part (1').

For part (3'), we know that

$$\begin{aligned} A_{Z,\Gamma}(w_{i,k}) &= \left(1 - \frac{mA_{X,\Delta}(v_i)}{k}\right) A_{Z,\Gamma}(X_0) + \frac{c_i}{k} A_{Z,\Gamma}(E_{i,\infty}) \\ &= \left(1 - \frac{mA_{X,\Delta}(v_i)}{k}\right) \frac{1}{m} + \frac{1}{k} A_{X,\Delta}(v_i) = \frac{1}{m}. \end{aligned}$$

Hence $A_{Z,\Gamma}(E_{i,k}) = \frac{k}{m \gcd(k, c_i)}$. Straight computation shows

$$K_{W'_k} + (\mu'_k)^{-1} \Gamma + \sum_{i=1}^{d'} E'_{i,k} = \mu'^*_k(K_Z + \Gamma) + \sum_{i=1}^{d'} A_{Z,\Gamma}(E_{i,k}) E'_{i,k} \sim_{\mu'_k, \mathbb{Q}} \sum_{i=1}^{d'} \frac{k E'_{i,k}}{m \gcd(k, c_i)}. \quad (\text{A.3})$$

From the above discussion on Rees valuations, we know that $\mathcal{O}_{W'_k}(-1) = \sum_{i=1}^{d'} \frac{E'_{i,k}}{\gcd(k, c_i)}$ is anti-ample over Z . Thus $-(K_{W'_k} + (\mu'_k)^{-1} \Gamma + \sum_{i=1}^{d'} E'_{i,k}) \sim_{\mu'_k, \mathbb{Q}} \frac{k}{m} \mathcal{O}_{W'_k}(1)$ is ample over Z which confirms part (3').

For part (2'), notice that the ideal sequence I_\bullet induces a \mathbb{G}_m -equivariant degeneration $\mathcal{Z} \rightarrow \mathbb{A}^1$ of $o \in Z$, where $\mathcal{Z} := \text{Spec } \bigoplus_{p \in \mathbb{Z}} t^{-p} I_p$. Then we have the central fiber $Z_0 = \text{Spec}(\text{gr}_{I_\bullet} R)$ where

$$\text{gr}_{I_\bullet} R := \bigoplus_{p=0}^\infty I_p / I_{p+1} = \bigoplus_{p=-\infty}^\infty \bigoplus_{j=0}^\infty \mathcal{F}^{p-jk} R_j / \mathcal{F}^{p-jk+1} R_j. \quad (\text{A.4})$$

Here we are using the fact that $\mathcal{F}^{p-jk}R_j = R_j$ whenever $p \leq 0$. From (A.2) we know that

$$\mathcal{X}'_0 \cong \text{Proj} \bigoplus_{j=0}^{\infty} \bigoplus_{p=-\infty}^{\infty} \mathcal{F}^p R_j / \mathcal{F}^{p+1} R_j \quad (\text{A.5})$$

and $-m(K_{\mathcal{X}'_0} + \Delta'_{\text{tc},0})$ corresponds to $\mathcal{O}(1)$ under the grading of j . It is clear that $\text{gr}_{I_\bullet} R$ is isomorphic to $\bigoplus_{j=0}^{\infty} \bigoplus_{p=-\infty}^{\infty} \mathcal{F}^p R_j / \mathcal{F}^{p+1} R_j$ up to a grading shift. Let Γ_Z be the effective \mathbb{Q} -divisor on Z as the Zariski closure of $\Gamma \times (\mathbb{A}^1 \setminus \{0\})$. Denote by $\Gamma_0 := \Gamma_Z|_{Z_0}$ the degeneration of Γ to Z_0 . Then (Z_0, Γ_0) is semi-log canonical (slc) since it is isomorphic to the affine cone over the slc pair $(\mathcal{X}'_0, \Delta'_{\text{tc},0})$ with the polarization $-m(K_{\mathcal{X}'_0} + \Delta'_{\text{tc},0})$. Thus we know that $(Z, \Gamma_Z, \xi; \eta)$ is a weakly special test configuration of (Z, Γ, ξ) in the sense of [LWX21, Definition 2.14] where ξ (resp. η) is the vector field on Z (resp. Z) induced by the grading of j (resp. of p). We will follow the idea of [LWX21, Proof of Lemma 2.21(2)] to show log canonicity of $(W'_k, (\mu'_k)^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$.

Denote by $E' := \sum_{i=1}^{d'} \frac{E'_{i,k}}{\gcd(k, c_i)}$ and $E'_{\text{red}} := \sum_{i=1}^{d'} E'_{i,k}$. From the proof of part (3') we know that $E' = \mathcal{O}_{W'_k}(-1)$ is anti-ample over Z . Let l be a sufficiently divisible positive integer such that lE' is Cartier on W'_k . The test configuration $(Z, \Gamma_Z, \xi; \eta)$ has the natural \mathbb{G}_m -action generated by η . Consider the μ_l -action on (Z, Γ_Z) where $\mu_l < \mathbb{G}_m$ is the multiplicative group of l -th roots of unity. Let $(Z', \Gamma_{Z'}) := (Z, \Gamma_Z)/\mu_l$. By construction, we have that $Z' := \text{Spec} \bigoplus_{p \in \mathbb{Z}} t^{-p} I_{lp} \rightarrow \mathbb{A}^1_t$, such that the quotient map $\sigma : Z \rightarrow Z'$ is a lifting of the map $\mathbb{A}^1_t \rightarrow \mathbb{A}^1_t, t \mapsto t^l$. Clearly σ is étale away from the central fibers. Since $Z_0 = \text{Spec} \bigoplus_{p \in \mathbb{Z}_{\geq 0}} I_p / I_{p+1}$, we know that $Z_0/\mu_l = \text{Spec} \bigoplus_{p \in \mathbb{Z}_{\geq 0}} I_{lp} / I_{lp+1}$, and $\text{Supp}(Z_0/\mu_l) = \text{Supp}(Z'_0)$.

Next, we show that $Z_0/\mu_l \cong C_a(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}}(-lE'|_{E'_{\text{red}}}))$ where $C_a(X, L)$ represents the affine cone over X with polarization L (see [Kol13, Section 3.1]). Indeed, from the equality $I_p = \bigcap_{i=1}^{d'} \mathfrak{a}_{p/k}(w_{i,k})$ we see that $I_p = (\mu'_k)_* \mathcal{O}_{W'_k}(\lfloor -pE' \rfloor)$. Since lE' is Cartier and $\lceil E' \rceil = E'_{\text{red}}$, we know that $\lfloor -(lp+1)E' \rfloor = -lpE' - E'_{\text{red}}$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{O}_{W'_k}(-lpE' - E'_{\text{red}}) \rightarrow \mathcal{O}_{W'_k}(-lpE') \rightarrow \mathcal{O}_{E'_{\text{red}}}(-lpE'|_{E'_{\text{red}}}) \rightarrow 0. \quad (\text{A.6})$$

Since l is sufficiently divisible and $-E'$ is ample over Z , we have $R^1(\mu'_k)_* \mathcal{O}_{W'_k}(-lpE' - E'_{\text{red}}) = 0$ for $p \geq 1$ by Serre vanishing. Thus taking $(\mu'_k)_*$ of (A.6) yields a short exact sequence

$$0 \rightarrow I_{lp+1} \rightarrow I_{lp} \rightarrow H^0(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}}(-lpE'|_{E'_{\text{red}}})) \rightarrow 0,$$

i.e. $I_{lp}/I_{lp+1} \cong H^0(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}}(-lpE'|_{E'_{\text{red}}}))$ when $p \geq 1$. If $p = 0$, then the above arguments give an injection $I_0/I_1 \hookrightarrow H^0(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}})$ which implies that they are isomorphic as $h^0(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}}) = 1$ by reducedness of E'_{red} .

Since $(Z, \Gamma_Z, \xi; \eta)$ is weakly special, we know that $(Z, \Gamma_Z + Z_0)$ is log canonical. In particular, we know that Z_0 is reduced and so is $Z_0/\mu_l = (Z'_0)_{\text{red}}$. Since the quotient map $\sigma : Z \rightarrow Z'$ is étale away from central fibers, we have that $K_Z + \Gamma_Z + Z_0 = \sigma^*(K_{Z'} + \Gamma_{Z'} + (Z'_0)_{\text{red}})$. Therefore, the quotient $(Z', \Gamma_{Z'} + (Z'_0)_{\text{red}})$ is also log canonical

by [KM98, Proposition 5.20]. By adjunction we know that $(Z_0, \Gamma_0)/\mu_l$ is slc. This implies that the base $(E'_{\text{red}}, \Gamma_{E'_{\text{red}}})$ is slc where $K_{E'_{\text{red}}} + \Gamma_{E'_{\text{red}}} = (K_{W'_k} + (\mu'_k)_*^{-1}\Gamma + E'_{\text{red}})|_{E'_{\text{red}}}$. By inversion of adjunction, the pair $(W'_k, (\mu'_k)_*^{-1}\Gamma + E'_{\text{red}})$ is log canonical. This proves part (2').

So far we have proven (1'), (2'), and (3') for $\mu'_k : W'_k \rightarrow Z$. In order to construct $\mu_k : W_k \rightarrow Z$, we will show that $E_{i',k}$ is an lc place of $(W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ for any $d' < i' \leq d$. By [A.3], we know

$$\begin{aligned} A_{W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k}}(w_{i',k}) &= A_{Z, \Gamma}(w_{i',k}) - w_{i',k} \left(\sum_{i=1}^{d'} \frac{k E'_{i,k}}{m \gcd(k, c_i)} \right) \\ &= \frac{1}{m} (1 - k w_{i',k}(\mathcal{O}_{W'_k}(-1))) . \end{aligned} \quad (\text{A.7})$$

Indeed, since $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$ is the pull-back test configuration of $(\mathcal{X}', \Delta'_{\text{tc}}; \mathcal{L}')$, by [BHH17, Lemma 2.13] they define the same filtration, i.e.

$$\mathcal{F}^p R_j = \{s \in R_j \mid v_i(s) \geq p + m A_{X, \Delta}(v_i) j \quad \text{for any } 1 \leq i \leq d\}.$$

Similar to the arguments above, we have $I_p = \cap_{i=1}^d \mathfrak{a}_{p/k}(w_{i,k})$. Hence $w_{i,k}(\mathcal{O}_{W'_k}(-1)) = \frac{1}{k}$ for any $1 \leq i \leq d$. This together with [A.7] implies $w_{i',k}$ is an lc place of $(W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ for any $d' < i' \leq d$. It is clear that all non-klt centers of $(W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ are contained in $\cup_{i=1}^{d'} E'_{i,k}$, thus W'_k is of Fano type over Z . Then [BCHM10] implies that there exists a projective birational morphism $\rho_k : W_k \rightarrow W'_k$ from a normal variety W_k such that $\text{Exc}(\rho_k) = \cup_{d' < i' \leq d} E_{i',k}$. Moreover, we know that $K_{W_k} + (\mu_k)_*^{-1}\Gamma + \sum_{i=1}^d E_{i,k}$ is the log pull-back of $K_{W'_k} + (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k}$ since ρ_k only extracts lc places of the latter. By taking $\mu_k := \mu'_k \circ \rho_k$, it is easy to see that (1), (2), and (3) are all satisfied. Thus the proof is finished. \square

Proposition A.4. *There exists a positive integer $N_1 = N_1(n, I)$ such that the following holds: for any weakly special collection of \mathbb{Z} -valued divisorial valuations $\{v_1, \dots, v_d\}$ over (X, Δ) where $\dim(X) = n$ and coefficients of Δ belongs to I , and any $k \gg 1$, there exists a local N_1 -complement Γ_k^+ of $o \in (Z, \Gamma)$ such that $E_{i,k}$ is an lc place of (Z, Γ_k^+) for any $1 \leq i \leq d$.*

Proof. By applying the boundedness of relative complements [Bir19, Theorem 1.8] to the morphism $\mu_k : W_k \rightarrow Z$ constructed in Proposition [A.3] there exists an N_1 -complement Θ_k of $(W_k, (\mu_k)_*^{-1}\Gamma + \sum_{i=1}^d E_{i,k})$ over $o \in Z$ where N_1 only depends on the dimension n and the coefficient set I . Then $\Gamma_k^+ := (\mu_k)_*(\Theta_k)$ is a local N_1 -complement of $o \in (Z, \Gamma)$ such that $E_{i,k}$ is an lc place for any $1 \leq i \leq d$. \square

Proof of Theorem [A.2]. Let Γ_k^+ be the N_1 -complement as in Proposition [A.4]. Then we know that

$$A_{Z, \Gamma_k^+}(w_{i,k}) = \frac{\gcd(k, c_i)}{k} A_{Z, \Gamma_k^+}(\text{ord}_{E_{i,k}}) = 0.$$

Let r be the Gorenstein index of $o \in (Z, \Gamma)$. Then

$$rN_1(\Gamma_k^+ - \Gamma) \sim rN_1(K_Z + \Gamma_k^+) - rN_1(K_Z + \Gamma) \sim 0.$$

Thus we have $\Gamma_k^+ = \Gamma + \frac{1}{rN_1}\text{div}(f_k)$ where $f_k \in \mathcal{O}_{o,Z}$. It is then clear that

$$A_{Z,\Gamma}(w_{i,k}) = \frac{w_{i,k}(f_k)}{rN_1}.$$

By definition we know that $w_{i,k} \geq (1 - \frac{mA_{X,\Delta}(v_i)}{k})w_0$. On the other hand, for any $f \in R_j$ it is clear that

$$w_{i,k}(f) = \left(1 - \frac{mA_{X,\Delta}(v_i)}{k}\right)j + \frac{c_i}{k}v_i(f) \leq j + \frac{c_i mT(v_i)}{k}j = \left(1 + \frac{c_i mT(v_i)}{k}\right)w_0(f).$$

Hence there exists a sequence of positive numbers $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$ such that $(1 - \epsilon_k)w_0 \leq w_{i,k} \leq (1 + \epsilon_k)w_0$ for any $1 \leq i \leq d$. This implies

$$A_{Z,\Gamma}(w_0) = \lim_{k \rightarrow \infty} A_{Z,\Gamma}(w_{i,k}) \leq \liminf_{k \rightarrow \infty} \frac{(1 + \epsilon_k)w_0(f_k)}{rN_1} = \liminf_{k \rightarrow \infty} \frac{w_0(f_k)}{rN_1}.$$

However, since (Z, Γ_k^+) is lc, we always have $A_{Z,\Gamma}(w_0) \geq \frac{w_0(f_k)}{rN_1}$ for $k \gg 1$. Then $w_0(f_k)$ being an integer implies that

$$A_{Z,\Gamma}(w_0) = \frac{w_0(f_k)}{rN_1} \text{ for } k \gg 1.$$

Therefore, w_0 is also an lc place of (Z, Γ_k^+) for $k \gg 1$. Denote by $\Gamma'_k := \Gamma + \frac{1}{rN_1}\text{div}(\mathbf{in}(f_k))$ where $\mathbf{in}(f_k)$ is the initial degeneration of f_k . Then by [dFEM10, Theorem 3.1] we know that (Z, Γ'_k) is also lc. Furthermore, by lower semicontinuity of the log discrepancy function, we know that both w_0 and $w_{i,k}$ are still lc places of (Z, Γ'_k) for $k \gg 1$. Hence by taking a \mathbb{G}_m -equivariant resolution, we see that

$$A_{Z,\Gamma'_k}(w_{i,k}) = \left(1 - \frac{mA_{X,\Delta}(v_i)}{k}\right)A_{Z,\Gamma'_k}(w_0) + \frac{c_i}{k}A_{Z,\Gamma'_k}(\text{ord}_{E_{i,\infty}})$$

which implies that $E_{i,\infty}$ is an lc place of (Z, Γ'_k) as well. Since Γ'_k is \mathbb{G}_m -invariant, it is the cone of some \mathbb{Q} -divisor Δ'_k on X . Hence, we know that E_i is an lc place of Δ'_k which is a \mathbb{Q} -complement of (X, Δ) . Then by an easy generalization of Theorem 3.5 to the case with multiple divisors over X , we may replace Δ'_k by an N -complement Δ_k^+ whose lc places still contain E_i for any $k \gg 1$ and any $1 \leq i \leq d$. This finishes proving the “only if” part. The “if” part follows from Proposition A.5. \square

Proposition A.5. *Let (X, Δ) be a log Fano pair. Let $\{v_1, \dots, v_d\}$ be a set of \mathbb{Z} -valued divisorial valuations in Val_X . If $\{v_i\}_{i=1}^d$ is contained in the set of lc places of some \mathbb{Q} -complement Δ^+ , then it is a weakly special collection.*

Proof. Let $v_i = c_i \cdot \text{ord}_{E_i}$. Then similar as before, we have the affine cone $o \in (Z, \Gamma)$ over (X, Δ) . For any $k \gg 1$ we have divisorial valuation $w_{i,k}$ and prime divisor $E_{i,k}$ over Z such that $w_{i,k} = \frac{\gcd(k, c_i)}{k} \cdot \text{ord}_{E_{i,k}}$. Denote by Γ^+ the Zariski closure of the

pull-back of Δ^+ under the projection $Z \setminus \{o\} \rightarrow X$. Then it is clear that $w_{i,k}$ is an lc place of (Z, Γ^+) for any $1 \leq i \leq d$ and any $k \gg 1$. Hence by [BCHM10], there exists a \mathbb{G}_m -equivariant projective birational morphism $\tilde{\mu}_k : \tilde{W}_k \rightarrow Z$ from a normal \mathbb{Q} -factorial variety \tilde{W}_k such that the following properties hold.

- The exceptional divisors of $\tilde{\mu}_k$ is $\cup_{i=1}^d \tilde{E}_{i,k}$ where $\tilde{E}_{i,k}$ is the birational transform of $E_{i,k}$;
- \tilde{W}_k is of Fano type over Z ;
- $(\tilde{W}_k, (\tilde{\mu}_k)_*^{-1}\Gamma^+ + \sum_{i=1}^d \tilde{E}_{i,k})$ is a log canonical crepant model of (Z, Γ^+) .

By [BCHM10], we could run the \mathbb{G}_m -equivariant $-(K_{\tilde{W}_k} + (\tilde{\mu}_k)_*^{-1}\Gamma + \sum_{i=1}^d \tilde{E}_{i,k})$ -MMP over Z and this MMP yields a birational contraction $\rho_k : \tilde{W}_k \dashrightarrow W'_k$ where W'_k is the log canonical model. For simplicity let us assume that $\rho_k : \tilde{W}_k \dashrightarrow W'_k$ precisely contracts $\tilde{E}_{i',k}$ for $d' < i' \leq d$. Denote by $E'_{i,k} := (\rho_k)_* \tilde{E}_{i,k}$ for $1 \leq i \leq d'$.

Next we will show that $\mu'_k : W'_k \rightarrow Z$ satisfies (1'), (2'), and (3') in the proof of Proposition A.3. Since $\tilde{\mu}_k$ is isomorphic in codimension 1 over $Z \setminus \{o\}$ as $c_Z(\tilde{E}_{i,k}) = o$, so is μ'_k . Since W'_k is the log canonical model, we have that $-(K_{W'_k} + (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ is ample over Z , which implies that μ'_k is an isomorphism over $Z \setminus \{o\}$ as $-(K_{W'_k} + (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})|_{W'_k \setminus \mu'^{-1}(o)} = \mu'^*_k(-(K_Z + \Gamma)|_{Z \setminus \{o\}})$. And $(W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ is log canonical since there is a \mathbb{Q} -complement. Thus (1'), (2'), and (3') in the proof of Proposition A.3 hold for μ'_k from the above arguments.

Next we construct the weakly special test configuration $(\mathcal{X}', \Delta'_{\text{tc}}; \mathcal{L}')$ by essentially reversing the argument in the proof of Proposition A.3. By the proof of Proposition A.3, we know that

$$-(K_{W'_k} + (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k}) \sim_{\mu'_k, \mathbb{Q}} -\sum_{i=1}^{d'} \frac{k}{m \gcd(k, c_i)} E'_{i,k} \quad (\text{A.8})$$

is ample over Z . Hence by taking valuation ideals of the Rees valuations of μ'_k , we know that $W'_k \cong \text{Proj}_Z \oplus_{p=0}^{\infty} I_p$ where $I_p := \cap_{i=1}^{d'} \mathfrak{a}_{p/k}(w_{i,k})$ is an ideal sequence on Z cosupported at o . Since W'_k is the log canonical model of $-(K_{\tilde{W}_k} + (\tilde{\mu}_k)_*^{-1}\Gamma + \sum_{i=1}^d \tilde{E}_{i,k}) \sim_{\tilde{\mu}_k, \mathbb{Q}} -\sum_{i=1}^d \frac{k}{m \gcd(k, c_i)} \tilde{E}_{i,k}$, we also have that $I_p = \cap_{i=1}^d \mathfrak{a}_{p/k}(w_{i,k})$. Hence the proof of Proposition A.3 implies that $E_{i',k}$ is an lc place of $(W'_k, (\mu'_k)_*^{-1}\Gamma + \sum_{i=1}^{d'} E'_{i,k})$ for any $d' < i' \leq d$. Consider the \mathbb{Z} -filtration $\mathcal{F}^\bullet R$ of R defined as $\mathcal{F}^p R_j := I_{p+jk} \cap R_j$. Then by the proof of Proposition A.3 we have

$$\begin{aligned} \mathcal{F}^p R_j &= \{s \in R_j \mid v_i(s) \geq p + mA_{X,\Delta}(v_i)j \quad \text{for any } 1 \leq i \leq d\} \\ &= \{s \in R_j \mid v_i(s) \geq p + mA_{X,\Delta}(v_i)j \quad \text{for any } 1 \leq i \leq d'\}. \end{aligned} \quad (\text{A.9})$$

Similar to the proof of Proposition A.3, denote by $\mathcal{Z} := \text{Spec } \oplus_{p \in \mathbb{Z}} t^{-p} I_p$ as the \mathbb{G}_m -equivariant degeneration of Z over \mathbb{A}^1 . Let $\Gamma_{\mathcal{Z}}$ be the effective \mathbb{Q} -divisor on \mathcal{Z} as the Zariski closure of $\Gamma \times (\mathbb{A}^1 \setminus \{0\})$. Then \mathcal{Z} is normal by integral closedness of I_p .

Let $E' := \sum_{i=1}^{d'} \frac{E'_{i,k}}{\gcd(k, c_i)}$ and $E'_{\text{red}} := \sum_{i=1}^{d'} E'_{i,k}$. Let l be a sufficiently divisible positive integer such that lE' is Cartier on W'_k . Let $(\mathcal{Z}', \Gamma_{\mathcal{Z}'}):= (\mathcal{Z}, \Gamma_{\mathcal{Z}})/\mu_l$. From the proof of Proposition A.3 we know that Z_0/μ_l is isomorphic to the affine cone $C_a(E'_{\text{red}}, \mathcal{O}_{E'_{\text{red}}}(-lE'|_{E'_{\text{red}}}))$. Since $(W'_k, (\mu'_k)^{-1}\Gamma + E'_{\text{red}})$ is log canonical, by adjunction we know that $(E'_{\text{red}}, \Gamma_{E'_{\text{red}}})$ is slc where $\Gamma_{E'_{\text{red}}}$ is the corresponding different divisor. Moreover, by A.8 we have

$$-(K_{E'_{\text{red}}} + \Gamma_{E'_{\text{red}}}) \sim_{\mathbb{Q}} -(K_{W'_k} + (\mu'_k)^{-1}\Gamma + E'_{\text{red}})|_{E'_{\text{red}}} \sim_{\mathbb{Q}} -\frac{k}{m}E'|_{E'_{\text{red}}}$$

which is ample. Thus $(E'_{\text{red}}, \Gamma_{E'_{\text{red}}})$ is a slc log Fano pair which implies that the affine cone $(Z_0, \Gamma_0)/\mu_l$ is also slc. In particular, we know $Z_0/\mu_l = (Z'_0)_{\text{red}}$ as it is reduced. Since Z_0/μ_l is reduced, we know that Z_0 is generically reduced, which implies that Z_0 is reduced as it is S_1 by [BHJ17 Proposition 2.6(ii)]. Since the quotient map $\sigma : \mathcal{Z} \rightarrow \mathcal{Z}'$ is étale away from central fibers, we have that $K_{\mathcal{Z}} + \Gamma_{\mathcal{Z}} + Z_0 = \sigma^*(K_{\mathcal{Z}'} + \Gamma_{\mathcal{Z}'} + (Z'_0)_{\text{red}})$. Since $((Z'_0)_{\text{red}}, \Gamma_{\mathcal{Z}'|_{(Z'_0)_{\text{red}}}}) \cong (Z_0, \Gamma_0)/\mu_l$ is slc, inversion of adjunction implies that $(\mathcal{Z}', \Gamma_{\mathcal{Z}'} + (Z'_0)_{\text{red}})$ is log canonical, which implies that $(\mathcal{Z}, \Gamma_{\mathcal{Z}} + Z_0)$ is log canonical by [KM98 Proposition 5.20]. Thus $(\mathcal{Z}, \Gamma_{\mathcal{Z}}, \xi; \eta)$ is a weakly special test configuration of (Z, Γ, ξ) where ξ (resp. η) is the vector field on Z (resp. \mathcal{Z}) induced by the grading of j (resp. of p). By adjunction we know that (Z_0, Γ_0) is slc.

Next, we consider the test configuration $(\mathcal{X}', \Delta'_{\text{tc}}; \mathcal{L}')$ of (X, Δ) by setting $\mathcal{X}' := \text{Proj} \bigoplus_{j=0}^{\infty} \bigoplus_{p=-\infty}^{\infty} t^{-p} \mathcal{F}^p R_j$ and $\mathcal{L}' = -m(K_{\mathcal{X}'} + \Delta'_{\text{tc}})$. Then by A.4 and A.5 we know that (Z_0, Γ_0) is isomorphic to the affine cone over $(\mathcal{X}'_0, \Delta'_{\text{tc},0}; \mathcal{L}'_0)$. Since (Z_0, Γ_0) is slc, we know that $(\mathcal{X}'_0, \Delta'_{\text{tc},0}; \mathcal{L}'_0)$ is also slc, and hence $(\mathcal{X}', \Delta'_{\text{tc}}; \mathcal{L}')$ is weakly special by inversion of adjunction. Moreover, $v_i = v_{\mathcal{X}'_0(i)}$ for any $1 \leq i \leq d'$ where $(\mathcal{X}'_0(i))_{1 \leq i \leq d'}$ are all the irreducible components of \mathcal{X}'_0 .

Finally we construct the desired semiample test configuration $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$ by extracting certain divisors over \mathcal{X}' . Let F_i be the prime divisor over $X \times \mathbb{A}^1$ as the quasi-monomial combination of $X \times \{0\}$ and $E_i \times \mathbb{A}^1$ with weights $(1, c_i)$. Then it is clear that $\text{ord}_{F_i}|_{K(X)} = v_i$. We claim that $F_{i'}$ is an lc place of $(\mathcal{X}', \mathcal{X}'_0 + \Delta'_{\text{tc}})$ for any $d' < i' \leq d$. Let \mathcal{Y} be the total space of a test configuration of $(X; L)$ dominating \mathcal{X}' and $X \times \mathbb{A}^1$ such that $F_{i'}$ is a divisor on \mathcal{Y} for any $d' < i' \leq d$. Denote by $\pi_1 : \mathcal{Y} \rightarrow \mathcal{X}'$ and $\pi_2 : \mathcal{Y} \rightarrow X \times \mathbb{A}^1$ the projection morphisms. Set $\mathcal{D} := \pi_1^* \mathcal{L}' - \pi_2^* \mathcal{L}_{\mathbb{A}^1}$ where $\mathcal{L}_{\mathbb{A}^1} := -m(K_{X \times \mathbb{A}^1} + \Delta \times \mathbb{A}^1)$. By [BHJ17 Lemmas 2.13 and 5.17] and A.9, we have that $\text{ord}_{F_{i'}}(\mathcal{D}) = -mA_{X, \Delta}(v_{i'})$. On the other hand, from the definition of \mathcal{D} we see that

$$\text{ord}_{F_{i'}}(\mathcal{D}) = m(A_{\mathcal{X}', \mathcal{X}'_0 + \Delta'_{\text{tc}}}(F_{i'}) - A_{X \times \mathbb{A}^1, X \times \{0\} + \Delta \times \mathbb{A}^1}(F_{i'})).$$

Since $A_{X \times \mathbb{A}^1, X \times \{0\} + \Delta \times \mathbb{A}^1}(F_{i'}) = A_{X, \Delta}(v_{i'})$, we know that $A_{\mathcal{X}', \mathcal{X}'_0 + \Delta'_{\text{tc}}}(F_{i'}) = 0$. Thus the claim is proved. By [BCHM10], we can extract the divisors $\{F_{i'}\}_{d' < i' \leq d}$ over \mathcal{X}' to obtain the desired weakly special semiample test configuration $(\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})$. This finishes the proof. \square

Remark A.6. Applying to the case with a prime divisor, we see that a prime divisor E over (X, Δ) is weakly special if and only if $\{\text{ord}_E\}$ is a weakly special collection, which is the same as E being an lc place of some N -complement.

A.2. Valuations computing the stability threshold. In this section, we show any valuation computing $\delta \leq 1$ is the lc place of a bounded complement. The result may be viewed as a stronger version of Proposition 3.8

Theorem A.7. *Let n be a natural number and $I \subseteq \mathbb{Q}$ a finite set. There exists a positive integer $N := N(n, I)$ depending only on n and I satisfying the following:*

Assume (X, Δ) is an n -dimensional log Fano pair such that coefficients of Δ belong to I and $\delta(X, \Delta) \leq 1$. If $v \in \text{Val}_X^$ computes $\delta(X, \Delta)$, then v is quasi-monomial and is an lc place of an N -complement.*

Proof. Let (X, Δ) be an n -dimensional log Fano pair such that coefficients of Δ belong to I . Assume $\delta(X, \Delta) \leq 1$ and $v \in \text{Val}_X^*$ computes the stability threshold. By [BJ20, Prop. 4.8], v is the unique valuation (up to scaling) computing $\text{lct}(X, \Delta; \mathbf{a}_\bullet(v))$ [BJ20, Prop. 4.8]. Hence, v is quasi-monomial by [Xu20].

To prove the second part of Theorem A.7, we will again use the cone construction. Fix a positive integer r so that $L := -r(K_X + \Delta)$ is a Cartier divisor and set $R := R(X, L)$. Let $Z = \text{Spec}(R)$ denote the cone over X with respect to the polarization L , $o \in Z$ the vertex of the cone, and Γ the \mathbb{Q} -divisor on Z defined by pulling back Δ .

For each $t \in \mathbb{R}_{\geq 0}$, we consider the valuation $v_t \in \text{Val}_Z$ defined by

$$v_t(f) = \min\{tv(f_m) + m \mid f_m \neq 0\},$$

where $f = \sum f_m$ and each $f_m \in R_m$. The valuation v_t is quasi-monomial, since v is quasi-monomial, and satisfies $A_{Z, \Gamma}(v_t) = \frac{1}{r} + tA_{X, \Delta}(v)$ (see the proof of [Li17, Lemma 6.14]).

Lemma A.8. *For any $t \in \mathbb{R}_{> 0}$, $\text{lct}(Z, \Gamma; \mathbf{a}_\bullet(v_t)) = A_{Z, \Gamma}(v_t)$.*

Proof. Since the inequality $\text{lct}(Z, \Gamma; \mathbf{a}_\bullet(v_t)) \leq A_{Z, \Gamma}(v_t)$ always holds, it suffices to show the reverse inequality. Pick any $\varepsilon > 0$. We will proceed to show $\text{lct}(Z, \Gamma; \mathbf{a}_\bullet(v_t)) \geq A_{Z, \Gamma}(v_t) - \varepsilon$.

Claim: For any $\varepsilon' > 0$, there exists a \mathbb{Q} -complement Δ^+ of (X, Δ) such that $A_{X, \Delta^+}(v) < \varepsilon'$.

To prove the claim, for each m divisible by r choose an m -basis type divisor B_m such that $S_m(v) = v(B_m)$. If we set $c_m = \min\{1, \delta_m(X, \Delta)\}$, then $(X, \Delta + c_m B_m)$ is lc by the definition of δ_m and

$$A_{X, \Delta + c_m B_m}(v) = A_{X, \Delta}(v) - c_m v(B_m) = A_{X, \Delta}(v) - c_m S_m(v).$$

Since $S_m(v) \rightarrow S(v)$ and $c_m \rightarrow \delta(X, \Delta)$ as $m \rightarrow \infty$, we see

$$\lim_{m \rightarrow \infty} A_{X, \Delta + c_m B_m}(v) = A_{X, \Delta}(v) - \delta(X, \Delta)S(v),$$

which is zero since v computes $\delta(X, \Delta)$.

Therefore, we may find m so that $A_{X,\Delta+c_mB_m}(v) < \varepsilon'$. Since $-K_X - \Delta$ is ample, we may choose $H \in |-K_X - \Delta|_{\mathbb{Q}}$ so that $(X, \Delta + c_mB_m + (1 - c_m)H)$ remains lc [KM98, Lem. 5.17.2]. Hence, $\Delta^+ := \Delta + c_mB_m + (1 - c_m)H$ is a \mathbb{Q} -complement of (X, Δ) and satisfies $A_{X,\Delta^+}(v) \leq A_{X,\Delta+c_mB_m}(v) < \varepsilon'$.

By the above claim, we may choose a \mathbb{Q} -complement Δ^+ of (X, Δ) such that $A_{X,\Delta^+}(v) < \varepsilon/t$. Since $\Delta^+ - \Delta \sim_{\mathbb{Q}} -K_X - \Delta$, there exists a positive integer m and $f \in H^0(X, \mathcal{O}_X(mL))$ such that $\Delta^+ = \Delta + \frac{1}{mr}\{f = 0\}$.

Since (X, Δ^+) is lc and $K_X + \Delta^+ \sim_{\mathbb{Q}} 0$, the pair $(Z, \Gamma + \frac{1}{mr}\{f = 0\})$ is lc [Kol13, Lem. 3.1]. Using that $A_{Z,\Gamma}(v_t) = \frac{1}{r} + tA_{X,\Delta}(v)$ and $v_t(f) = m + tv(f)$, we see

$$A_{Z,\Gamma+\frac{1}{mr}\{f=0\}}(v_t) = t(A_{X,\Delta}(v) - \frac{1}{mr}v(f)) = tA_{X,\Delta^+}(v) < \varepsilon.$$

Hence, if we set $s := v_t(f)$, then

$$s = mr(A_{Z,\Gamma}(v_t) - A_{Z,\Gamma+\frac{1}{mr}\{f=0\}}(v_t)) \geq mr(A_{Z,\Gamma}(v_t) - \varepsilon). \quad (\text{A.10})$$

To estimate the asymptotic lct, observe that $f^{[p/s]} \in \mathfrak{a}_p(v_t)$ for each positive integer p . Hence,

$$\text{lct}(Z, \Gamma; \mathfrak{a}_p(v_t)) \geq \text{lct}(Z, \Gamma; (f^{[p/s]})) \geq \frac{1}{[p/s]mr}$$

where the last inequality uses that $(Z, \Gamma + \frac{1}{mr}\{f = 0\})$ is lc. Therefore,

$$\text{lct}(Z, \Gamma; \mathfrak{a}_{\bullet}(v_t)) = \lim_{p \rightarrow \infty} p \cdot \text{lct}(Z, \Gamma; \mathfrak{a}_p(v_t)) \geq \lim_{p \rightarrow \infty} \frac{p}{[p/s]mr} = \frac{s}{mr}.$$

After referring back to (A.10), we see $\text{lct}(Z, \Gamma; \mathfrak{a}_{\bullet}(v_t)) \geq A_{Z,\Gamma}(v_t) - \varepsilon$. \square

Lemma A.9. *There exists a positive integer M such that the following holds: for each positive integer k , there exists $f^{(k)} \in R$ such that the pair*

$$(Z, \Gamma + \frac{1}{rM}\{f^{(k)} = 0\})$$

is lc in a neighborhood of $o \in Z$ and $v_{1/k}$ is an lc place of the pair.

Proof. Fix a positive integer k . Choose a log resolution $W \rightarrow Z$ of (Z, Γ) and local coordinates y_1, \dots, y_q at a point $\eta \in W$ such that $v_{1/k}$ may be written as v_{α} for some $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{R}_{\geq 0}^q$ (see (2.1) for the definition). After replacing W with a higher model and choosing new local coordinates, we may assume $\alpha_1, \dots, \alpha_q$ are linearly independent over \mathbb{Q} .

Note that $v_{1/k}$ computes $\text{lct}(Z, \Gamma; \mathfrak{a}_{\bullet}(v_{1/k}))$, since $A_{Z,\Gamma}(v_{1/k}) = \text{lct}(Z, \Gamma; \mathfrak{a}_{\bullet}(v_{1/k}))$ by Lemma A.8 and the equality $v_{1/k}(\mathfrak{a}_{\bullet}(v_{1/k})) = 1$. Hence, [LX18, Rem. 2.52] implies $v_{1/k}$ admits a weak lc model in the sense of [LX18, Def. 2.49]. In particular, the argument in *loc. cit.* implies there exists a proper birational morphism $\rho : W^{\text{wlc}} \rightarrow Z$, prime divisors S_1, \dots, S_q on W^{wlc} with $\text{ord}_{S_j} = v_{\beta^{(j)}}$ for some $\beta^{(j)} \in \mathbb{Z}_{\geq 0}^q$ such that

- (i) $(W^{\text{wlc}}, \rho_*^{-1}\Gamma + \sum S_i)$ is lc,
- (ii) $-K_{W^{\text{wlc}}} - \rho_*^{-1}\Gamma - \sum S_i$ is ρ -nef, and
- (iii) α lies in the convex cone generated by $\beta^{(1)}, \dots, \beta^{(q)}$.

Applying [Bir19, Thm. 1.8], we may find a positive integer M , dependent only on $\dim(Z)$ and the coefficients on Γ , and an effective \mathbb{Q} -divisor $\Gamma_{W^{\text{wlc}}}^+ \geq \rho_*^{-1}(\Gamma) + \sum S_j$ such that, in a neighborhood of $\rho^{-1}(o)$, $(W^{\text{wlc}}, \Gamma_{W^{\text{wlc}}}^+)$ is lc and $M(K_{W^{\text{wlc}}} + \Gamma_{W^{\text{wlc}}}^+) \sim_Z 0$. Set $\Gamma^+ := \rho_*(\Gamma_{W^{\text{wlc}}}^+)$. Observe that

$$\Gamma^+ \geq \Gamma \quad \text{and} \quad K_{W^{\text{wlc}}} + \Gamma_{W^{\text{wlc}}}^+ = \rho^*(K_Z + \Gamma^+).$$

Hence, in a neighborhood of $o \in Z$, (Z, Γ^+) is lc and $M(K_Z + \Gamma^+) \sim 0$. Additionally, each S_j is an lc place of Γ^+ .

To see $v_{1/k}$ is an lc place of (Z, Γ^+) , note that

$$A_{Z, \Gamma^+}(w) = A_{Z, \Gamma}(w) - w(\Gamma^+ - \Gamma) \quad \text{for any } w \in \text{Val}_Z.$$

Since $Y \rightarrow Z$ is a log resolution of (Z, Γ) , $A_{Z, \Gamma}$ is linear on our simplicial cone. Additionally, $w \mapsto -w(\Gamma^+ - \Gamma)$ is convex on the cone by [BFJ08, Lem. 1.10]. Using that α lies in the convex cone generated by $\beta^{(1)}, \dots, \beta^{(q)}$ and $A_{Z, \Gamma^+}(v_{\beta^{(j)}}) = 0$ for each $j = 1, \dots, q$, we see $A_{Z, \Gamma^+}(v_{1/k}) \leq 0$. Since (Z, Γ^+) is lc, we conclude $A_{Z, \Gamma^+}(v_{1/k}) = 0$.

Finally, note that

$$rM(\Gamma^+ - \Gamma) = rM(K_Z + \Gamma^+) - rM(K_Z + \Gamma) \sim 0$$

at $o \in Z$. Hence, we may find $f^{(k)} \in R$ such that $\Gamma^+ - \Gamma$ agrees with $\frac{1}{Mr}\{f^{(k)} = 0\}$ in a neighborhood of $o \in Z$, which completes the proof of the lemma. \square

For each positive integer k , consider the lc pair $(Z, \Gamma + \frac{1}{Mr}\{f^{(k)} = 0\})$ constructed above. Repeating the proof of Theorem A.2, we see that if $k \gg 0$, then $\Delta^+ := \Delta + \frac{1}{Mr}\{\text{in}(f^{(k)}) = 0\}$ is a \mathbb{Q} -complement of (X, Δ) with v an lc place.

To show v is the lc place of an $N := N(n, I)$ complement, let $\pi : Y \rightarrow X$ be a log resolution of (X, Δ^+) and write Δ_Y^+ for the \mathbb{Q} -divisor satisfying

$$K_Y + \Delta_Y^+ = \pi^*(K_X + \Delta^+).$$

By Lemma 2.3, the lc places of (X, Δ^+) coincide with the simplicial cone complex $\text{QM}(Y, (\Delta_Y^+)^{=1})$.

Choose a sequence of divisorial valuations $(v_j)_j$ in $\text{QM}(Y, (\Delta_Y^+)^{=1})$ converging to v . Since each v_j is divisorial and an lc place of (X, Δ^+) , Theorem 3.5 implies there exists a positive integer $N := N(n, I)$, depending only on n and I , such that each v_j is the lc place of an N complement. Hence, for each j , we may choose an N -complement Δ_j^+ with $A_{X, \Delta_j^+}(v_j) = 0$.

Set $D_j := \Delta_j^+ - \Delta$ and write $\varphi_{D_j} : \text{QM}(Y, (\Delta_Y^+)^{=1}) \rightarrow \mathbb{R}$ for the function defined by $v \mapsto v(D_j)$. Since each D_j is an element of $\frac{1}{rN}| - rN(K_X + \Delta)|$, the set of functions $\{\varphi_{D_j} \mid j \in \mathbb{N}\}$ is finite by Lemma 2.5. Therefore, after replacing $(v_j)_j$ with a subsequence, we may find an individual N -complement Δ_*^+ such that $A_{X, \Delta_*^+}(v_j) = 0$ for all j . Using that $v = \lim_j v_j$, we conclude $A_{X, \Delta_*^+}(v) = 0$. \square

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