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MONOTONE SOLUTIONS OF THE MASTER EQUATION FOR MEAN FIELD GAMES WITH IDIOSYNCRATIC NOISE*

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Abstract. We introduce a notion of a weak solution of the master equation without idiosyncratic noise in mean field game theory and establish its existence, uniqueness up to a constant, and consistency with classical solutions when it is smooth. We work in a monotone setting and rely on Lions' Hilbert space approach. For the first-order master equation without idiosyncratic noise, we also give an equivalent definition in the space of measures and establish the well-posedness.

Key words. mean field games, master equation, weak solutions

MSC codes. 35D40, 35R15

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Introduction. We introduce a notion of a weak solution (called a "monotone solution") of the master equation (ME) in the mean field games (MFG) theory for first- and second-order models in a monotone setting and without idiosyncratic noise. Using Lions' Hilbert space approach, we show that the solution exists, is unique up to additive constants, and, when it is smooth, classical. For the first-order ME without idiosyncratic noise, we also give an equivalent definition in the space of measures and establish its well-posedness. The arguments do not use any regularity on the solutions which are known only in the presence of idiosyncratic noise.

The ME in the MFG theory was introduced by Lions in his courses at Collège de France [34]. Lions also introduced in [34] the Hilbertian approach and proved the existence of a classical solution under suitable structure conditions on the coupling function (monotonicity) and Hamiltonian (convexity in the gradient variable).

Defining a notion of well-posed weak solutions for the ME in MFG is one of the important problems in the theory.

A step in this direction is a recent paper of Bertucci [9] on finite state models which introduced the notion of monotone solutions for MFG with finite state space and studied its well-posedness. The work of [9], which is based on a uniqueness technique developed by Lions in [34], brought to bear techniques from the theory of viscosity solutions although the actual notion of solution is not related to them. The very recent work [10] by Bertucci extends [9] to the continuous state space and for several noise structures, and relies on a regularity assumption on the solution which is known only for problems with idiosyncratic noise. Here we borrow the terminology of "monotone solution" from [9, 10].