



PAPER

On mix-norms and the rate of decay of correlations

To cite this article: Bryan W Oakley *et al* 2021 *Nonlinearity* **34** 3762

View the [article online](#) for updates and enhancements.

You may also like

- [On the relation between Lyapunov exponents and exponential decay of correlations](#)
Julia Slipantschuk, Oscar F Bandtlow and Wolfram Just
- [Analysis of decay chains of superheavy nuclei produced in the \$^{249}\text{Bk}+^{48}\text{Ca}\$ and \$^{245}\text{Am}+^{48}\text{Ca}\$ reactions](#)
V B Zlokazov and V K Utyonkov
- [Quantum Anti-Zeno Effect in Nuclear Decay](#)
Ming Ji, , Chang Xu et al.

On mix-norms and the rate of decay of correlations

Bryan W Oakley¹, Jean-Luc Thiffeault^{1,*}  and Charles R Doering²

¹ Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, United States of America

² Center for the Study of Complex Systems, Department of Mathematics and Department of Physics, University of Michigan, Ann Arbor, MI 48109, United States of America

E-mail: boakley@wisc.edu, jeanluc@math.wisc.edu and doering@umich.edu

Received 23 February 2020, revised 30 October 2020

Accepted for publication 14 January 2021

Published 17 June 2021



CrossMark

Abstract

Two quantitative notions of mixing are the decay of correlations and the decay of a mix-norm—a negative Sobolev norm—and the intensity of mixing can be measured by the rates of decay of these quantities. From duality, correlations are uniformly dominated by a mix-norm; but can they decay asymptotically faster than the mix-norm? We answer this question by constructing an observable with correlation that comes arbitrarily close to achieving the decay rate of the mix-norm. Therefore the mix-norm is the sharpest rate of decay of correlations in both the uniform sense and the asymptotic sense. Moreover, there exists an observable with correlation that decays at the same rate as the mix-norm if and only if the rate of decay of the mix-norm is achieved by its projection onto low-frequency Fourier modes. In this case, the function being mixed is called *q-recurrent*; otherwise it is *q-transient*. We use this classification to study several examples and raise questions for future investigations.

Keywords: mix-norms, decay of correlations, mixing, dynamical systems

Mathematics Subject Classification numbers: 37A25 and 76F25.

(Some figures may appear in colour only in the online journal)

*Author to whom any correspondence should be addressed.

Recommended by Dr Mark F Demers.

1. Introduction

Consider a spatially-periodic mean-zero function $f^t(\mathbf{x}) = f(t, \mathbf{x})$ bounded uniformly in $L^2(\mathbb{T}^d)$ for all $t > 0$. For example, $f(t, \mathbf{x})$ might be a solution to the advection-diffusion equation

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = D \Delta f, \quad (1.1)$$

with $f^0 \in L^2(\mathbb{T}^d)$ and smooth divergence-free velocity field $\mathbf{u}(t, \mathbf{x})$. We may also consider $D = 0$ in equation (1.1), in which case it is the transport equation. Another example, in the context of dynamical systems, is when an initial condition $f^0 \in L^2$ is transported by an area-preserving map M via the transfer operator $f^{n+1} = f^n \circ M^{-1}$.

Decay of the correlation function $C_t(g) = |\langle f^t, g \rangle|$ as $t \rightarrow \infty$ for observables g in $L^2(\mathbb{T}^d)$ corresponds to mixing of f^t as $t \rightarrow \infty$ [32]. Mathew *et al* [25] introduced the $H^{-1/2}$ norm as another criterion to quantify mixing, and Lin *et al* [19] extended this to any negative Sobolev (e.g., H^{-q}) norm and showed that correlations decay to zero if and only if any such ‘mix-norm’ decays to zero. That is,

$$\lim_{t \rightarrow \infty} \langle f^t, g \rangle = 0 \quad \forall g \in L^2 \Leftrightarrow \lim_{t \rightarrow \infty} \|f^t\|_{H^{-q}} = 0, \quad \text{for any } q > 0.$$

Mix-norms are well-suited to quantification of mixing efficiencies [9, 18, 23, 31, 33–36], to lower bounds on the rate of mixing in general [15, 21, 22], and to analyzing mixing [24, 26, 27, 37]. Moreover, such negative Sobolev spaces provide a natural setting for a discussion of enhanced dissipation and relaxation [1, 4, 6–8, 12, 16, 17]. Mathew *et al* [25] introduced the mix-norm in the context of spatial averages over strips, and made the connection to weak convergence (see also [39]).

While mix-norms are well-adapted to the PDE context, correlations and weak convergence are more commonly studied in the context of ergodic theory. A central question, then, is the quantitative relationship between decay rate of correlations and decay of mix-norms. This is the central focus of this paper where we will work in a setting where the evolution of a function $f^t(\mathbf{x})$ is given, arising from the continuous-time solution of a PDE or in discrete times from an iterated map.

When studying a collection of functions converging to zero as $t \rightarrow \infty$, such as $|\langle f^t, g \rangle|$ for $g \in X$ where $X \subset L^2$ is some Banach space, there are several reasonable ways to define a rate of decay:

- We can consider a *uniform* upper bound [2, 5, 10, 20, 30].
- We can say that each function is $\mathcal{O}(\varrho)$ ³ where $\varrho(t)$ is some rate function [38]. This lifts the tail of the rate function by multiplying by a constant that depends on $g \in X$.
- We can instead lift the tail of the rate function by translation and say that each function is bounded above by a translation of some rate function [11].

We summarize as follows (for concreteness, fix some $q > 0$ and consider $X = H^q(\mathbb{T}^d)$):

- Correlations decay at the *uniform* rate $r(t)$ for $g \in H^q$ if

$$|\langle f^t, g \rangle| \leq r(t) \|g\|_{H^q} \quad \forall g \in H^q. \quad (1.2)$$

³ We say that $a(t) = \mathcal{O}(b(t))$ as $t \rightarrow \infty$ if there are T, M so that $|a(t)| \leq Mb(t)$ for $t > T$. For $b(t) > 0$, this is equivalent to $\limsup_{t \rightarrow \infty} |a(t)|/b(t) = C \in [0, \infty)$. Moreover, we say $a(t) = o(b(t))$ if $C = 0$.

(b) Correlations decay at the *asymptotic* rate $\varrho(t)$ for $g \in H^q$ if

$$|\langle f^t, g \rangle| = \mathcal{O}(\varrho), \text{ for each } g \in H^q. \quad (1.3)$$

That is,

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\varrho(t)} = C_g \in [0, \infty). \quad (1.4)$$

(c) Correlations decay at the *translational* rate $\lambda(t)$ for $g \in H^q$ if for each $g \in H^q$ there exists $\tau_g \in \mathbb{R}$ such that for all $t > \tau_g$ we have

$$|\langle f^t, g \rangle| \leq \lambda(t - \tau_g) \|g\|_{H^q}. \quad (1.5)$$

From considering duality in section 2, we find that the smallest uniform rate is the mix-norm $\|f^t\|_{H^{-q}}$. Since any uniform rate trivially satisfies the definitions of asymptotic rate and translational rate, the question is whether there is a ϱ (or λ) that decays faster than $\|f^t\|_{H^{-q}}$. We answer this question by showing that we cannot have $\varrho = o(\|f^t\|_{H^{-q}})$. Similarly, given the additional assumption that $\limsup_{t \rightarrow \infty} \lambda(t - \tau)/\lambda(t)$ is finite, we cannot have $\lambda = o(\|f^t\|_{H^{-q}})$. We remark that this growth condition on λ is satisfied by power law and exponential functions⁴.

We prove the above facts by constructing an observable $g \in H^q$ such that $|\langle f^t, g \rangle|$ decays arbitrarily closely to the mix-norm. Namely, for any positive $h(t) = o(\|f^t\|_{H^{-q}})$ there is a $g \in H^q$ such that $|\langle f^t, g \rangle|$ is big-O but not little-O of h . Note that this is not the same as asymptotic equivalence because the correlation may be much smaller than h at certain times.

Let $P_I f^t$ denote the projection of f^t onto the Fourier modes I . We say f^t is *q-recurrent* (otherwise it is *q-transient*) if there is a finite set I where $\|P_I f^t\|_{H^{-q}}$ is big-O but not little-O of $\|f^t\|_{H^{-q}}$. Heuristically, in this case, the decay of the mix-norm is characterized by $P_I f^t$. We prove f^t is *q-recurrent* if and only if there is a $g \in H^q$ such that $|\langle f^t, g \rangle|$ is big-O but not little-O of $\|f^t\|_{H^{-q}}$. Therefore, *q-recurrence* is the criterion for the existence of a correlation that obtains the decay rate of the mix-norm.

In section 2 we introduce the key definitions and main theorems. Section 3 contains examples, and sections 4 and 5 contains the full proofs of the theorems.

2. Overview

Throughout, it will be more convenient to work with the homogeneous Sobolev spaces $\dot{H}^\alpha$ for $\alpha \in \mathbb{R}$. Since the torus $\mathbb{T}^d$ is a compact manifold, Poincaré's inequality applies [14] so that the H^α norm and $\dot{H}^\alpha$ norm are equivalent for mean-zero functions. For $\alpha > 0$, the $\dot{H}^{-\alpha}$ norm is typically defined via the duality equation

$$\|f\|_{\dot{H}^{-\alpha}} = \sup_{g \in \dot{H}^\alpha} \frac{|\langle f, g \rangle|}{\|g\|_{\dot{H}^\alpha}}.$$

However, there is an equivalent definition [13] for all $\alpha \in \mathbb{R}$. Let $f_k = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \mathbf{k}} d\mathbf{x}$ denote the Fourier coefficients of $f(\mathbf{x})$. Then

⁴ The growth condition is not satisfied by functions that decay faster than exponentially, such as $\lambda(t) = e^{-t^2}$. In this case, λ is not asymptotically equivalent to its own translation: for large t , we see $e^{-(t-\tau)^2} = e^{2\tau t - \tau^2} e^{-t^2} \gg e^{-t^2}$. For λ not satisfying the growth condition, a translation of λ is much larger than a constant multiple of λ . If there is a correlation bounded by a translation of λ but not a constant multiple of λ , then $\lambda = o(\varrho)$.

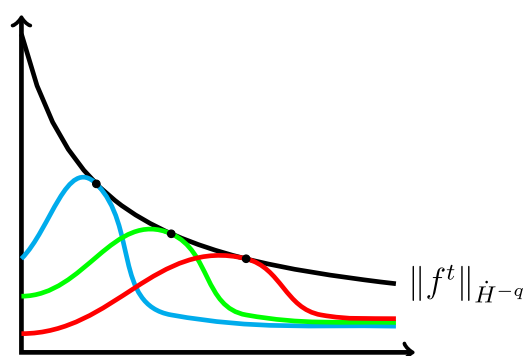


Figure 1. Plotted above are the mix-norm and correlations with different choices of g , demonstrating that the mix-norm is the envelope over $|\langle f^t, g \rangle|$ with $\|g\|_{\dot{H}^q} = 1$.

$$\|f\|_{\dot{H}^\alpha} = \left(\sum_{\mathbf{k} \neq \mathbf{0}} k^{2\alpha} |f_{\mathbf{k}}|^2 \right)^{1/2},$$

where $k^2 = |\mathbf{k}|^2 = |k_1|^2 + \dots + |k_d|^2$. We will typically omit the $\mathbf{k} \neq \mathbf{0}$ under the sum since $f_{\mathbf{0}} = 0$ for mean-zero functions.

Similarly, correlations have a simple expression. Since the trigonometric functions $\{e^{2\pi i \mathbf{x} \cdot \mathbf{k}}\}$ provide an orthonormal basis for $L^2(\mathbb{T}^d)$, the Fourier transform is a unitary map to $\ell^2(\mathbb{Z}^d)$. Therefore the Fourier transform preserves the inner product [13] and we have Plancherel's theorem:

$$\langle f, g \rangle = \sum_{\mathbf{k}} f_{\mathbf{k}} \bar{g}_{\mathbf{k}} \quad \forall f, g \in L^2(\mathbb{T}^d).$$

Now say $q > 0$ and consider $g \in \dot{H}^q$ for the rest of this paper. For time-dependent $f^t(\mathbf{x})$, the duality equation implies $|\langle f^t, g \rangle| \leq \|f^t\|_{\dot{H}^{-q}} \|g\|_{\dot{H}^q}$ for all t . Moreover, fix $t = t_0$ and take g with Fourier coefficients

$$g_{\mathbf{k}} = f_{\mathbf{k}}^{t_0} k^{-2q} \|f^{t_0}\|_{\dot{H}^{-q}}^{-1}. \quad (2.1)$$

Then $\|g\|_{\dot{H}^q} = 1$ and Plancherel's theorem gives $\langle f^{t_0}, g \rangle = \|f^{t_0}\|_{\dot{H}^{-q}}$. The correlation achieves the mix-norm at the time t_0 . Since t_0 is arbitrary, we see that $\|f^t\|_{\dot{H}^{-q}}$ is the envelope of the set of functions $|\langle f^t, g \rangle|$ with $\|g\|_{\dot{H}^q} = 1$, as in figure 1. This shows the point-wise smallest uniform rate of decay of correlations, in the sense of equation (1.2), is the mix-norm $\|f^t\|_{\dot{H}^{-q}}$.

Using only duality, the most that can be said about the relationship between the rate of decay of a correlation and the rate of decay of the mix-norm is that

$$|\langle f^t, g \rangle| = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}}) \quad \text{for each } g \in \dot{H}^q.$$

However, each correlation could decay strictly faster than the mix-norm as illustrated in figure 1. We are then led to ask if such a situation is possible.

When is $|\langle f^t, g \rangle| = o(\|f^t\|_{\dot{H}^{-q}})$ for each $g \in \dot{H}^q$? To answer this question, we must construct functions $g \in \dot{H}^q$ such that the correlations $|\langle f^t, g \rangle|$ decay as slowly as possible. To do this, we first classify f^t as either q -recurrent or q -transient as follows.

For a set $I \subset \mathbb{Z}^d$ let

$$P_I f^t = \sum_{k \in I} f_k^t e^{2\pi i x \cdot k}$$

denote the projection of f^t onto the Fourier modes $k \in I$. Then

$$\|P_I f^t\|_{\dot{H}^{-q}}^2 = \sum_{k \in I} k^{-2q} |f_k^t|^2$$

measures the amount of mix-norm supported on I . We often refer to this as the *Fourier energy* contained in I . This notion of energy is q -dependent, though the q will usually be clear from the context.

Definition 1. We say f^t is q -recurrent if there exists a finite set $I \subset \mathbb{Z}^d$ such that

$$\limsup_{t \rightarrow \infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{\|f^t\|_{\dot{H}^{-q}}} > 0. \quad (2.2)$$

Functions that are not q -recurrent will be called q -transient.

Remark. We emphasize that q -recurrence is a property of f^t which encompasses both the stirring action and the initial condition coupled together. To clarify, in the context of the advection-diffusion equation (1.1), q -recurrence is a property of a particular realization of $\mathbf{u}$ and f^0 taken together—it is *not* just a property of the vector field $\mathbf{u}$. Flows which don't mix, like $\mathbf{u} = \text{const.}$, will trivially induce a q -recurrent trajectory f^t for any initial condition and any q . However, this stability does not hold in general. In example 2 of section 3, we will see that a given f^t may be q -recurrent for some (larger) values of q and q -transient for other (smaller) q . Additionally, we will find that a given flow $\mathbf{u}$ may induce a q -recurrent evolution f^t for some initial conditions and q -transient for others.

From inequality (2.2) and the trivial bound $\|P_I f^t\|_{\dot{H}^{-q}} \leq \|f^t\|_{\dot{H}^{-q}}$ we see that $\|P_I f^t\|_{\dot{H}^{-q}}$ is big-O but not little-O of $\|f^t\|_{\dot{H}^{-q}}$. Unpacking the definition of limit supremum offers another interpretation: there exists $\delta > 0$ and a sequence $t_m \rightarrow \infty$ where

$$\|P_I f^{t_m}\|_{\dot{H}^{-q}} \geq \delta \|f^{t_m}\|_{\dot{H}^{-q}}. \quad (2.3)$$

This means that there is at least a δ fraction of the mix-norm supported on I at arbitrarily large times. As time progresses the Fourier energy could move off of I , but we can always find a future time t_{m+1} where a proportion δ of the mix-norm is again on I . In other words, some Fourier energy always returns to populate the spatial scales in I . In this case, test functions g with coefficients for $k \in I$ similar to that in equation (2.1) will match well with f^t at times t_m (after possibly taking a subsequence) so that in section 4 we can prove the following theorem, a central result of our paper.

Remark. The case $\|f^{t_0}\|_{\dot{H}^{-q}} = 0$ is degenerate in the context of the advection-diffusion equation and dynamical systems. In those settings, if the mix-norm is zero at any finite time it will remain zero for future times. In such a case where f^t is eventually zero, the mix-norm and correlations will trivially decay at the same rate, the zero function. Otherwise, if the mix-norm is nonzero for a sequence $t_m \rightarrow \infty$, we may drop the times where the mix-norm is zero so that $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Without loss of generality, to simplify the presentation of our results, we will take this as an assumption in the theorems below.

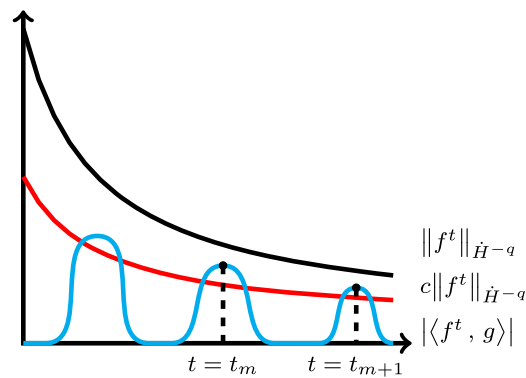


Figure 2. There exists $g \in \dot{H}^q$ where $|\langle f^t, g \rangle|$ does not decay faster than $c\|f^t\|_{\dot{H}^{-q}}$.

Theorem 1. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Then f^t is q -recurrent if and only if there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Equivalently, there is a function $g \in \dot{H}^q$, a constant $c > 0$, and a sequence $t_m \rightarrow \infty$ where

$$|\langle f^{t_m}, g \rangle| \geq c \|f^{t_m}\|_{\dot{H}^{-q}}. \quad (2.4)$$

Remark. As demonstrated in figure 2, it is possible that $|\langle f^t, g \rangle|$ is small at times $t \neq t_m$ and so we do not show asymptotic equivalence. We interpret our result as demonstrating that the correlation does not decay asymptotically faster than the mix-norm in the sense that $|\langle f^t, g \rangle|$ is big-O but not little-O of $\|f^t\|_{\dot{H}^{-q}}$. From this theorem, the answer to our previously posed question ‘when is $|\langle f^t, g \rangle| = o(\|f^t\|_{\dot{H}^{-q}})$ for each $g \in \dot{H}^q$?’ is exactly when f^t is q -transient.

This naturally prompts us to ask if f^t is q -transient and we carefully choose $g \in \dot{H}^q$, how slowly can we make $|\langle f^t, g \rangle|$ decay? The following theorem answers this question.

Theorem 2. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h(t)} > 0.$$

Remark. Theorems 1 and 2 do not require f^t to be bounded uniformly in $L^2(\mathbb{T}^d)$ in time, nor for the mix-norm to decay to zero. Additionally, theorem 2 is valid whether f^t is q -recurrent or q -transient.

The proof of theorem 2 is deferred until section 5, but we present the idea behind the proof now. If f^t is q -recurrent, then the proof is accomplished by a result similar to theorem 1. For q -transient functions, the proof relies on the construction of sets I_m and times t_m satisfying certain properties, the first being that we want the finite disjoint sets $I_m \in \mathbb{Z}^d$ to capture a large amount of the Fourier energy at time t_m . We can do this since q -transience ensures that we can wait for the next time t_{m+1} where a proportion of the Fourier energy moves off of I_m and never comes back. Then by choosing the Fourier coefficients of g on I_m to agree with f^t at time t_m , we can guarantee that $|\langle f^t, g \rangle|$ will be large at time t_m . Hence, the function g in theorem 2 which gives the slowly decaying $|\langle f^t, g \rangle|$ has Fourier coefficients

$$g_k = \begin{cases} f_k^{t_m} k^{-2q} \|f^{t_m}\|_{\dot{H}^{-q}}^{-2} h(t_m), & k \in I_m; \\ 0, & \text{otherwise;} \end{cases} \quad (2.5)$$

where I_m are disjoint and $\|P_{I_m} f^{t_m}\|_{\dot{H}^{-q}}$ captures a nonzero proportion of $h(t_m)$, similar to inequality (2.3). These coefficients are similar to those of equation (2.1) except with an extra factor of $h/\|f\|_{\dot{H}^{-q}}$. This factor is needed so that we can satisfy the second property we require from the sets I_m and times t_m : by taking a subsequence, we can use the fact that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$ to make the g_k decay fast enough as $k \rightarrow \infty$ to have $g \in \dot{H}^q$. Hence, although correlations may not achieve the decay rate of the mix-norm, they may achieve the decay rate of h .

These theorems allow us to show the result outlined in the introduction. The following corollary reveals it is not possible to find a ϱ or λ (under a given growth condition) that is little-O of the mix-norm.

Corollary 1.

(a) For any ϱ satisfying equation (1.3), we have

$$\limsup_{t \rightarrow \infty} \frac{\varrho(t)}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

(b) For λ satisfying equation (1.5) and $\limsup_{t \rightarrow \infty} \lambda(t - \tau)/\lambda(t)$ finite for any $\tau \in \mathbb{R}$, we have

$$\limsup_{t \rightarrow \infty} \frac{\lambda(t)}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Proof of corollary 1. Seeking contradiction we suppose there is a $\varrho(t)$ satisfying equation (1.3) such that $\varrho(t) = o(\|f^t\|_{\dot{H}^{-q}})$. Choosing $h(t) = \sqrt{\varrho(t)\|f^t\|_{\dot{H}^{-q}}}$, the geometric mean of ϱ and the mix-norm, we see

$$\limsup_{t \rightarrow \infty} \frac{h}{\|f^t\|_{\dot{H}^{-q}}} = \limsup_{t \rightarrow \infty} \sqrt{\frac{\varrho}{\|f^t\|_{\dot{H}^{-q}}}} = 0. \quad (2.6)$$

Then theorem 2 assures there is a $g \in \dot{H}^q$ with

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h} > 0.$$

Then we have arrived at a contradiction:

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h} \leq \limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\varrho} \cdot \limsup_{t \rightarrow \infty} \frac{\varrho}{h} = 0$$

since $\limsup_{t \rightarrow \infty} |\langle f^t, g \rangle|/\varrho$ is finite by equation (1.3) and $\limsup_{t \rightarrow \infty} \varrho/h = 0$ as in equation (2.6).

A similar argument gives us the second half of the corollary. In this case choose $h(t) = \sqrt{\lambda(t)\|f^t\|_{\dot{H}^{-q}}}$ and apply theorem 2 to produce a test function g which comes with a τ_g from equation (1.5). Then we have another contradiction:

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h(t)} \leq \limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\lambda(t - \tau_g)} \cdot \limsup_{t \rightarrow \infty} \frac{\lambda(t - \tau_g)}{\lambda(t)} \cdot \limsup_{t \rightarrow \infty} \frac{\lambda(t)}{h(t)} = 0$$

since $\limsup_{t \rightarrow \infty} \lambda(t - \tau_g)/\lambda(t)$ is finite by hypothesis. $\square$

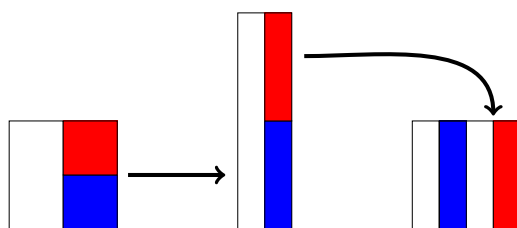


Figure 3. The baker's map.

In section 3 we present an example of a q -transient function and alter it to send energy down the spectrum *less* efficiently, resulting in q -recurrence for a range of q . We then include diffusion at every time step, demonstrating a transition to q -recurrence for all $q > 0$. We include a numerical example and provide intuition about how to recognize when f^t is q -recurrent. Finally, we prove the theorems in generality in sections 4 and 5.

3. Examples

Example 1 (baker's map and q -transience). Let B be the baker's map, the area preserving transformation of the domain $[0, 1]^2$ as pictured in figure 3. For the y -independent initial function $f^0(x, y) = 2 \cos(2\pi x)$, applying the baker's map simply doubles the frequency in the x direction. After n applications of the baker's map we have $f^n = f^0 \circ B^{-n} = 2 \cos(2\pi 2^n x)$. As a result, the Fourier coefficients have the simple expression

$$f_k^n = \begin{cases} 1 & k_1 = 2^n, k_2 = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This is a one dimensional action on Fourier coefficients $f_k^n = f_{k_1, 0}^n$ via an infinite dimensional matrix $A_{k\ell}$ as

$$f_k^{n+1} = \sum_{\ell} A_{k\ell} f_{\ell}^n \quad (3.1)$$

where

$$(A_{k\ell}) = \begin{pmatrix} & 1 & 2 & 3 & 4 & \dots \\ \begin{matrix} 1 \\ \vdots \end{matrix} & \begin{matrix} \text{row 2} \\ \vdots \end{matrix} & \begin{matrix} \text{row 3} \\ \vdots \end{matrix} & \begin{matrix} \text{row 4} \\ \vdots \end{matrix} & \begin{matrix} \text{row 5} \\ \vdots \end{matrix} & \dots \end{pmatrix}$$

$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{matrix}$

is populated by 1's along a subdiagonal of slope -2 and 0's everywhere else.

Table 1. Nonvanishing Fourier coefficients of f^n defined by equation (3.1), for $f^0(x) = 2 \cos(2\pi x)$.

	$k = 1$	2	3	4	5	6	7	8	...
f_k^0	1								
f_k^1	a	b							
f_k^2	a^2	ab		b					
f_k^3	a^3	a^2b		ab				b	

The entire mix-norm is supported on just one Fourier mode and given any finite set $I \in \mathbb{Z}^d$, it is clear that, as n increases, the Fourier energy will move off of I and never return. Therefore f^n is q -transient $\forall q > 0$. $\triangle$

Example 2 (baker-like action and q -recurrence). We now alter the previous example so that the energy of f^n is sent down the spectrum *less* effectively, the result being a q -recurrent function (if q is large enough). This time, consider the action on the Fourier coefficients of $f^n(x)$ as in equation (3.1) via the infinite dimensional matrix

$$(\tilde{A}_{k\ell}) = \begin{pmatrix} & 1 & 2 & 3 & 4 & \dots \\ a & & & & & \\ b & & & & & \\ & & & & & \\ & & 1 & & & \\ & & & & & \\ & & & 1 & & \\ & & & & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{matrix}$$

where $a, b > 0$ are constants such that $a^2 + b^2 = 1$.

Remark. It is not evident that the current example is still a dynamical systems example. That is, we do not know that there is a map $T : [0, 1]^d \rightarrow [0, 1]^d$ so that $f^{n+1} = f^n \circ T$ and $f_k^{n+1} = \sum_{\ell} \tilde{A}_{k\ell} f_{\ell}^n$. Moreover, such a map might not be injective, surjective, or unique. For example, B^{-1} from the previous example is not injective when thought of as a map on $[0, 1]$, but it is when we allow it to move through another dimension ($d = 2$). For the present example, taking $a, b = \sqrt{2}/2$, the initial function $f^0(x) = 2 \cos(2\pi x)$ is transformed to $f^1(x) = \sqrt{2}(a \cos(2\pi x) + b \cos(4\pi x))$ after one time step. Since the range of f^0 and f^1 are not the same set, any such T cannot be surjective. In this example, we see the map is also not measure preserving. Lastly, if f^n is constant, then T can be any map. These caveats notwithstanding, we consider the current example as a study of possible ways for energy to move down the spectrum and proceed with an analysis.

We show the coefficients of f^n for initial function $f^0(x) = 2 \cos(2\pi x)$ in table 1. Heuristically, the energy starts concentrated on the $k = 1$ mode and subsequently splits between modes $k = 1, 2$ so that L^2 norm is preserved. After that the $k = 1$ mode continues to donate a proportion b of its energy to $k = 2$ and the energy on $k = 2$ is transported down the spectrum at the same rate as the baker's map ($k = 2^n$).

Notice that f^n is mean-zero because it is the finite sum of cosine functions with full period. We can compute the L^2 norm and find $\|f^n\|_{L^2} = 1 \ \forall n$. Hence $A_{k\ell}$ is a unitary map on ℓ^2 by the polarization identity. Therefore f^n is bounded uniformly in L^2 and so f^n satisfies the hypothesis of the theorems in section 2.

Consider the contribution to the mix-norm from mode $k = 1$

$$\mathcal{E}_1^n = \|P_{k=1}f^n\|_{H^{-q}}^2 = \sum_{k=1} k^{-2q} |f_k^n|^2 = |f_1^n|^2 = a^{2n},$$

and compare it to the contributions from modes $k > 1$ (a geometric sum):

$$\begin{aligned} \mathcal{E}_{k>1}^n &= \|P_{k>1}f^n\|_{H^{-q}}^2 = \sum_{k>1} k^{-2q} |f_k^n|^2 \\ &= \begin{cases} c_{q,a} (a^{2n} - 2^{-2qn}), & \text{for } a \neq 2^{-q}; \\ b^2 a^{2n}, & \text{for } a = 2^{-q}; \end{cases} \end{aligned}$$

where

$$c_{q,a} = \frac{b^2}{a^{2q} - 1}.$$

For $a > 2^{-q}$, we see that $\mathcal{E}_1^n \sim \mathcal{E}_{k>1}^n$ as $n \rightarrow \infty$ and the mode $k = 1$ captures a non-zero proportion of the mix-norm for arbitrarily large n . Therefore f^n is q -recurrent for $q > \log_2(1/a)$.

For $a \leq 2^{-q}$, $\mathcal{E}_1^n = o(\mathcal{E}_{k>1}^n)$ so the mode $k = 1$ does not capture a non-zero proportion of the mix-norm for arbitrarily large n . This suggests that f^n is q -transient for $q \leq \log_2(1/a)$. To prove q -transience, we need to show the same holds for an arbitrary finite set I . Take $I = [0, 2^R]$ for some $R \in \mathbb{N}$. For $n > R$,

$$\|P_I f^n\|_{H^{-q}}^2 = c_{R,q,a} a^{2n},$$

where

$$c_{R,q,a} = \begin{cases} 1 + c_{q,a} (1 - a^{-2R} 2^{-2qR}), & \text{for } a \neq 2^{-q}; \\ 1 + b^2 R, & \text{for } a = 2^{-q} \end{cases}$$

and we conclude that

$$\lim_{n \rightarrow \infty} \frac{\|P_I f^n\|_{H^{-q}}^2}{\|f^n\|_{H^{-q}}^2} = \begin{cases} c_{R,q,a} (1 + c_{q,a})^{-1}, & \text{for } a > 2^{-q}; \\ 0, & \text{for } a \leq 2^{-q}. \end{cases}$$

Therefore f^n is q -transient for $q \leq \log_2(1/a)$ and q -recurrent for $q > \log_2(1/a)$. $\triangle$

Remark. In the above example, an initial function with $f_1^0 = 0$ has a trajectory f^n that is q -transient. This is because the coefficients do not see the alterations we have made to the baker's map and are sent down the spectrum at the same rate as the baker's map ($k = 2^n$). We emphasize that q -recurrence is a property of a particular realization of the flow and initial condition taken together—a given flow may induce a q -recurrent evolution f^t for some initial conditions and q -transient for others.

Example 3 (baker-like action with diffusion). We use the same matrix $\tilde{A}_{k\ell}$ and initial condition as in example 2 but now we also include diffusion. Without diffusion the conclusion

from the previous section was that f^n is q -recurrent if q was large enough. We now show that with diffusion, f^n is q -recurrent for all q . Along the way, we show the rate of decay of the Sobolev norm $\|f^n\|_{H^\beta}$ is independent of $\beta \in \mathbb{R}$.

Let

$$\gamma_k = \exp(-\kappa(2\pi k)^2) \quad (3.2)$$

and update the Fourier coefficients according to

$$f_k^{n+1} = \sum_{\ell} \gamma_k \tilde{A}_{k\ell} f_{\ell}^n, \quad (3.3)$$

where $\tilde{A}_{k\ell}$ is the matrix defined in example 2. This matrix multiplication will result in *pulsed diffusion* with diffusion constant κ . We display the coefficients of f^n in table 2.

The amount of mix-norm found on mode $k = 1$,

$$\mathcal{E}_1^n = \|P_{k=1} f^n\|_{H^\beta}^2 = \sum_{k=1} k^{2\beta} |f_k^n|^2 = |f_1^n|^2 = (a\gamma_1)^{2n}, \quad (3.4)$$

is asymptotically equivalent to the amount found on modes $k > 1$ because

$$\begin{aligned} \mathcal{E}_{k>1}^n &= \|P_{k>1} f^n\|_{H^\beta}^2 = \sum_{k>1} k^{2\beta} |f_k^n|^2 \\ &= \sum_{\ell=1}^n (2^\ell)^{2\beta} ((a\gamma_1)^{n-\ell} b \prod_{s=1}^{\ell} \gamma_{2^s})^2 \\ &= (a\gamma_1)^{2n} \sum_{\ell=1}^n (2^\ell)^{2\beta} ((a\gamma_1)^{-\ell} b \prod_{s=1}^{\ell} \gamma_{2^s})^2, \end{aligned}$$

where the factor

$$c_{n,\beta,a,\kappa} := \sum_{\ell=1}^n (2^\ell)^{2\beta} ((a\gamma_1)^{-\ell} b \prod_{s=1}^{\ell} \gamma_{2^s})^2 \quad (3.5)$$

limits to a finite constant $c_{\beta,a,\kappa}$ as $n \rightarrow \infty$. We see that $c_{n,\beta,a,\kappa}$ converges since the factor

$$\gamma_{2^\ell} = \exp(-\kappa(2\pi 2^\ell)^2) \quad (3.6)$$

dominates the terms in the sum to render the sum convergent. Lastly, notice that $c_{\beta,a,\kappa} \rightarrow \infty$ as $\beta \rightarrow \infty$ or $a \rightarrow 0$. We conclude that f^n is q -recurrent for all $q \in \mathbb{R}$ and, moreover, that all of the Sobolev norms decay at the same rate. $\triangle$

Example 4 (sine flow). Lastly we consider a computational example, the random sine flow, which is a simple model flow that is empirically quite efficient at mixing [29, 35]. The sine flow is a two-dimensional time-periodic flow with a full period consisting of the shear flow

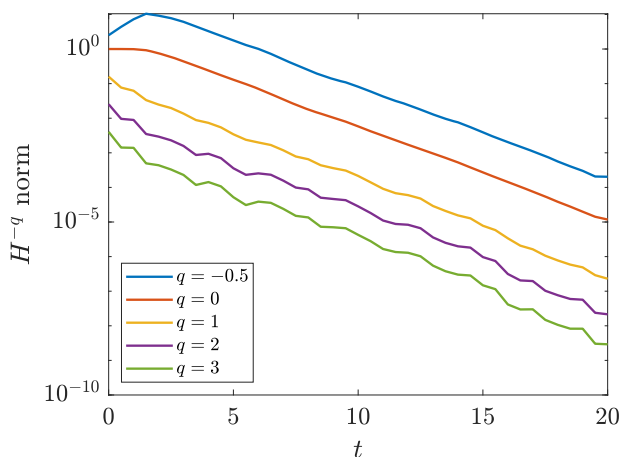
$$\mathbf{u}_1(t, x) = \sqrt{2} (0, \sin(2\pi x + \psi_1)), \quad 0 \leq t < 1/2, \quad (3.7a)$$

followed by

$$\mathbf{u}_2(t, y) = \sqrt{2} (\sin(2\pi y + \psi_2), 0), \quad 1/2 \leq t < 1, \quad (3.7b)$$

Table 2. Nonvanishing Fourier coefficients of f^n defined by equation (3.3), for $f^0(x) = 2 \cos(2\pi x)$.

	$k = 1$	2	3	4	5	6	7	8	...
f_k^0	1								
f_k^1	$a\gamma_1$	$b\gamma_2$							
f_k^2	$a^2\gamma_1^2$	$a\gamma_1 b\gamma_2$		$b\gamma_2\gamma_4$					
f_k^3	$a^3\gamma_1^3$	$a^2\gamma_1^2 b\gamma_2$		$a\gamma_1 b\gamma_2\gamma_4$				$b\gamma_2\gamma_4\gamma_8$	

**Figure 4.** For the advection-diffusion equation (1.1) with u given by the random sine flow (3.7), the rate of decay of the mix-norms is independent of q . The initial condition is $f^0(x) = \sqrt{2} \cos(2\pi x)$, and the diffusivity is $D = 10^{-5}$.

with $(x, y) \in [0, 1]^2$ and periodic spatial boundary conditions. Here ψ_1 and ψ_2 are random phases, uniformly distributed in $[0, 2\pi]$, chosen independently at every period. Unlike the pulsed diffusion in example 3, diffusion acts continuously by solving the advection–diffusion equation (1.1) with diffusivity $D = 10^{-5}$. We display $\|f^t\|_{H^{-q}}$ for various q in figure 4, for initial condition $f^0(x) = \sqrt{2} \cos(2\pi x)$, and observe that the mix-norms all decay at the same rate, at least within numerical fluctuations. $\triangle$

In general, if f^t is q -recurrent then the decay rate of the mix-norm is independent of q in the following sense:

Theorem 3. *If f^t is q -recurrent, then it is also q' -recurrent for any $q' > q$. Moreover, we have*

$$\limsup_{t \rightarrow \infty} \frac{\|f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Then together with the trivial estimate

$$\|f^t\|_{\dot{H}^{-q'}} \leq \|f^t\|_{\dot{H}^{-q}} \quad (3.8)$$

we conclude that $\|f^t\|_{\dot{H}^{-q'}}$ is big- O but not little- O of $\|f^t\|_{\dot{H}^{-q}}$.

Proof. Since f^t is q -recurrent, there is a finite set I such that

$$0 < \limsup_{t \rightarrow \infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{\|f^t\|_{\dot{H}^{-q}}}. \quad (3.9)$$

Say $I \subset [-R, R]$ for some $R \in \mathbb{N}$. Then

$$\|P_I f^t\|_{\dot{H}^{-q}} \leq R^{(q'-q)} \|P_I f^t\|_{\dot{H}^{-q'}}. \quad (3.10)$$

Putting together equations (3.8)–(3.10) we obtain

$$0 < R^{(q'-q)} \limsup \frac{\|P_I f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q'}}}.$$

We conclude f^t is q' -recurrent. Moreover, the trivial estimate $\|P_I f^t\|_{\dot{H}^{-q'}} \leq \|f^t\|_{\dot{H}^{-q'}}$ together with equations (3.9) and (3.10) imply

$$0 < R^{(q'-q)} \limsup \frac{\|f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q}}}. \quad (3.11)$$

□

One question for further investigation is whether a converse to the above theorem exists. That is, can we conclude f^t is q -recurrent for a range of q if the mix-norms decay at the same rate for the those q ? Another question concerns the transition from q -transient to q -recurrent when including pulsed diffusion. Does q -recurrence imply an introduction of the Batchelor scale and anomalous dissipation [3, 27, 28]?

4. Proof of theorem 1

We begin by generalizing the definition of q -recurrent functions to the notion of ‘(q,h)-recurrent’ functions.

Definition 2. For positive functions $h(t)$, we say f^t is (q,h)-recurrent if there exists a finite set $I \subset \mathbb{Z}^d$ such that

$$\limsup_{t \rightarrow \infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{h} > 0. \quad (4.1)$$

Functions that are not (q,h)-recurrent are called (q,h)-transient.

Lemma 1. If f^t is (q,h)-recurrent, then there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h} > 0.$$

Moreover, $g \in \dot{H}^\beta$ for any $\beta \in \mathbb{R}$ with

$$\|g\|_{\dot{H}^\beta}^2 = 2 \sum_{k \in I} k^{2(\beta-q)}. \quad (4.2)$$

Proof. There exists a constant $c > 0$ and a sequence of times $t_m \rightarrow \infty$ where

$$\sum_{k \in I} |f_k^{t_m}|^2 k^{-2q} \geq c^2 h^2(t_m). \quad (4.3)$$

Recall the signum function

$$\operatorname{sgn} x = \begin{cases} 1, & x \geq 0; \\ -1, & x < 0. \end{cases} \quad (4.4)$$

Now notice that for each fixed time t_m , $\{f_k^{t_m}\}_{k \in I}$ is a list of $|I|$ numbers in $\mathbb{C}$. Write $f_k^{t_m} = a_k^{t_m} + ib_k^{t_m}$ where $a_k^{t_m}$ and $b_k^{t_m}$ are real. Then $f_k^{t_m}$ is found in one of the four quadrants of the complex plane, depending on the two possibilities for $\operatorname{sgn} a_k^{t_m}$ and two possibilities for $\operatorname{sgn} b_k^{t_m}$. Thus, $\{f_k^{t_m}\}_{k \in I}$ has $4^{|I|}$ possible states. Since we have an infinite sequence of times t_m , one of these states must occur infinitely many times. By taking a subsequence t_{m_ℓ} , we can ensure $\{f_k^{t_{m_\ell}}\}_{k \in I}$ is the same state for all ℓ . Let $\{(c_k, d_k)\}_{k \in I}$ encode this state, meaning that $c_k = \operatorname{sgn} a_k^{t_{m_\ell}}$ and $d_k = \operatorname{sgn} b_k^{t_{m_\ell}}$ for all ℓ . We see that $a_k^{t_{m_\ell}} c_k = |a_k^{t_{m_\ell}}|$ and $b_k^{t_{m_\ell}} d_k = |b_k^{t_{m_\ell}}|$ for all ℓ . Let

$$g_k = \begin{cases} (c_k + id_k) k^{-q}, & k \in I; \\ 0, & \text{otherwise.} \end{cases} \quad (4.5)$$

Notice that $g \in \dot{H}^\beta$ because

$$\|g\|_{\dot{H}^\beta}^2 = \sum |g_k|^2 k^{2\beta} = \sum_{k \in I} (|c_k|^2 + |d_k|^2) k^{-2q} k^{2\beta} = 2 \sum_{k \in I} k^{2(\beta-q)} < \infty \quad (4.6)$$

since I is a finite set. We have

$$\begin{aligned} \sum f_k^{t_{m_\ell}} \bar{g}_k &= \sum_{k \in I} (a_k^{t_{m_\ell}} + ib_k^{t_{m_\ell}}) (c_k - id_k) k^{-q} \\ &= \sum_{k \in I} (a_k^{t_{m_\ell}} c_k + b_k^{t_{m_\ell}} d_k + i(b_k^{t_{m_\ell}} c_k - a_k^{t_{m_\ell}} d_k)) k^{-q} \\ &= \sum_{k \in I} (|a_k^{t_{m_\ell}}| + |b_k^{t_{m_\ell}}| + i(b_k^{t_{m_\ell}} c_k - a_k^{t_{m_\ell}} d_k)) k^{-q}. \end{aligned}$$

We conclude that

$$\begin{aligned} |\langle f^{t_{m_\ell}}, g \rangle| &\geq \operatorname{Re} \sum f_k^{t_{m_\ell}} \bar{g}_k \\ &= \sum_{k \in I} (|a_k^{t_{m_\ell}}| + |b_k^{t_{m_\ell}}|) k^{-q} \\ &\geq \sum_{k \in I} \sqrt{|a_k^{t_{m_\ell}}|^2 + |b_k^{t_{m_\ell}}|^2} k^{-q} \\ &= \sum_{k \in I} |f_k^{t_{m_\ell}}| k^{-q}, \end{aligned}$$

as desired. Lastly, we use dominance of ℓ^2 by ℓ^1 together with equation (4.3) to conclude $|\langle f^{t_{m_\ell}}, g \rangle| \geq c h(t_{m_\ell}), \forall m_\ell$. $\square$

Lemma 1 characterizes the behavior of (q,h)-recurrent functions. We now develop the tool we need to further analyze (q,h)-transient functions.

Lemma 2. *Let f^t be (q,h)-transient for some positive $h = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$. For any δ with $0 < \delta < 1$, there exist sets $I_i = \{\mathbf{k} | J_{i-1} < |\mathbf{k}| \leq J_i\}$ with $J_0 = -1$ and a sequence of times $T_i \rightarrow \infty$ satisfying the following.*

(a) *The set I_i captures a significant proportion of the Fourier energy at time T_i , so that*

$$\sum_{\mathbf{k} \in I_i} |f_{\mathbf{k}}^{T_i}|^2 k^{-2q} \geq (1 - \delta) \|f^{T_i}\|_{\dot{H}^{-q}}^2. \quad (4.7)$$

(b) *Enough of the Fourier energy does not return to lower frequency modes, so that*

$$\sum_{|\mathbf{k}| \leq J_{i-1}} |f_{\mathbf{k}}^t|^2 k^{-2q} \leq \delta h^2(t) \quad \text{for all } t \geq T_i. \quad (4.8)$$

Proof. *Base case: $i = 1$. Since (by definition of absolute convergence)*

$$\lim_{J \rightarrow \infty} \sum_{|\mathbf{k}| \leq J} |f_{\mathbf{k}}^0|^2 k^{-2q} = \|f^0\|_{\dot{H}^{-q}}^2 \quad (4.9)$$

there is a J_1 such that

$$\sum_{|\mathbf{k}| \leq J_1} |f_{\mathbf{k}}^0|^2 k^{-2q} \geq (1 - \delta) \|f^0\|_{\dot{H}^{-q}}^2. \quad (4.10)$$

We see $I_1 = \{\mathbf{k} | J_0 < |\mathbf{k}| \leq J_1\}$ where $J_0 = -1$ and $T_1 = 0$ therefore trivially satisfy equations (4.7) and (4.8).

Induction step. Suppose that we are given J_{i-2}, J_{i-1} and T_{i-1} that satisfy equations (4.7) and (4.8). Since $h = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$, there is a $c > 0$ and a time T so that $h(t) \leq c \|f^t\|_{\dot{H}^{-q}} \forall t \geq T$.

Recall that f^t is (q,h)-transient and so, for any finite set I ,

$$\limsup_{t \rightarrow \infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{h} = 0. \quad (4.11)$$

That is, given finite set I , then for any constant ϵ there is T_ϵ so that

$$\sum_{\mathbf{k} \in I} |f_{\mathbf{k}}^t|^2 k^{-2q} \leq \epsilon h^2(t) \quad \text{for all } t \geq T_\epsilon. \quad (4.12)$$

Take $I = \{\mathbf{k} \text{ s.t. } |\mathbf{k}| \leq J_{i-1}\}$ and $\epsilon = \min(\delta, \delta/c^2)$. We conclude that there exists T_i with $T_i \geq T_{i-1} + 1$ and $T_i \geq T$ such that equation (4.8) is satisfied. Moreover, we have

$$\sum_{|\mathbf{k}| \leq J_{i-1}} |f_{\mathbf{k}}^{T_i}|^2 k^{-2q} < (\delta/c^2) h^2(T_i) \leq \delta \|f^{T_i}\|_{\dot{H}^{-q}}^2. \quad (4.13)$$

Hence,

$$\sum_{|\mathbf{k}| > J_{i-1}} |f_{\mathbf{k}}^{T_i}|^2 k^{-2q} > (1 - \delta) \|f^{T_i}\|_{\dot{H}^{-q}}^2. \quad (4.14)$$

That is,

$$\lim_{J \rightarrow \infty} \sum_{J_{i-1} < |k| \leq J} |f_k^{T_i}|^2 k^{-2q} = C \|f^{T_i}\|_{\dot{H}^{-q}}^2, \quad (4.15)$$

where $C > (1 - \delta)$. From the definition of the limit, it follows that there is a J_i large enough so that for $I_i = \{k | J_{i-1} < |k| < J_i\}$ we have

$$\sum_{k \in I_i} |f_k^{T_i}|^2 k^{-2q} \geq (1 - \delta) \|f^{T_i}\|_{\dot{H}^{-q}}^2 \quad (4.16)$$

which is equation (4.7). $\square$

Having developed all of the tools we will need, we now prove theorem 1 and, in the next section, theorem 2.

Theorem 1. *Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Then f^t is q -recurrent if and only if there is a function $g \in \dot{H}^q$ such that*

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Proof. The forward direction is a special case of lemma 1 with $h(t) = \|f^t\|_{\dot{H}^{-q}}$. We assume f^t is q -transient and show $\langle f^t, g \rangle = o(\|f^t\|_{\dot{H}^{-q}})$ for all $g \in \dot{H}^q$. We already know $\langle f^t, g \rangle = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$ for all $g \in \dot{H}^q$, so

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{\dot{H}^{-q}}} < \infty, \quad \forall g \in \dot{H}^q. \quad (4.17)$$

Seeking a contradiction, we suppose there exists a $g \in \dot{H}^q$ such that

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{\dot{H}^{-q}}} = C > 0. \quad (4.18)$$

There is a sequence $t_n \rightarrow \infty$ such that

$$|\langle f^{t_n}, g \rangle| \geq \frac{C}{2} \|f^{t_n}\|_{\dot{H}^{-q}}, \quad \forall n. \quad (4.19)$$

We will show that equation (4.19) implies that the Fourier coefficients of g decay too slowly for g to be in $\dot{H}^q$. Since $g \in \dot{H}^q$, we can choose δ small enough that $\frac{C}{2} - 2\sqrt{\delta}\|g\|_{\dot{H}^q} = C_0 > 0$.

Applying lemma 2 with $h(t) = \|f^t\|_{\dot{H}^{-q}}$, there exist sets $I_i = \{k | J_{i-1} < |k| \leq J_i\}$ and a sequence of times $T_i \rightarrow \infty$ (without loss of generality, say that $\{T_i\}$ is a subsequence of $\{t_n\}$ above) such that we have equations (4.7) and (4.8). Equation (4.7) implies

$$\sum_{|k| > J_i} |f_k^{T_i}|^2 k^{-2q} \leq \delta \|f^{T_i}\|_{\dot{H}^{-q}}^2. \quad (4.20)$$

Note that

$$\begin{aligned} \langle f^{T_i}, g \rangle &= \sum_{j \geq 1} \sum_{k \in I_j} f_k^{T_i} \bar{g}_k \\ &= \sum_{k \in I_i} f_k^{T_i} \bar{g}_k + E \end{aligned} \quad (4.21)$$

and we can bound the error using the Cauchy–Schwarz inequality

$$E = \sum_{k \notin I_i} f_k^{T_i} \bar{g}_k \leq \left(\sum_{k \notin I_i} |f_k^{T_i}|^2 k^{-2q} \right)^{1/2} \left(\sum_{k \notin I_i} |g_k|^2 k^{2q} \right)^{1/2}. \quad (4.22)$$

Applying equation (4.7) of lemma 2, we have

$$|E| \leq \sqrt{\delta} \|f^{T_i}\|_{\dot{H}^{-q}} \|g\|_{\dot{H}^q}. \quad (4.23)$$

and therefore

$$|\langle f^{T_i}, g \rangle| \leq \left| \sum_{k \in I_i} f_k^{T_i} \bar{g}_k \right| + \sqrt{\delta} \|f^{T_i}\|_{\dot{H}^{-q}} \|g\|_{\dot{H}^q}. \quad (4.24)$$

Putting together equations (4.19) and (4.24), we have

$$\left(\frac{C}{2} - \sqrt{\delta} \|g\|_{\dot{H}^q} \right) \|f^{T_i}\|_{\dot{H}^{-q}} \leq \left| \sum_{k \in I_i} f_k^{T_i} \bar{g}_k \right|. \quad (4.25)$$

Since $g \in \dot{H}^q$, we can choose δ small enough that

$$C_0 := \left(\frac{C}{2} - \sqrt{\delta} \|g\|_{\dot{H}^q} \right) > 0. \quad (4.26)$$

Applying Cauchy–Schwarz to the right-hand side of equation (4.25) we have

$$C_0 \|f^{T_i}\|_{\dot{H}^{-q}} \leq \left(\sum_{k \in I_i} |f_k^{T_i}|^2 k^{-2q} \right)^{1/2} \left(\sum_{k \in I_i} |g_k|^2 k^{2q} \right)^{1/2} \quad (4.27)$$

$$\leq \|f^{T_i}\|_{\dot{H}^{-q}} \left(\sum_{k \in I_i} |g_k|^2 k^{2q} \right)^{1/2}. \quad (4.28)$$

Therefore

$$\sum_{k \in I_i} |g_k|^2 k^{2q} \geq C_0^2, \quad \forall i. \quad (4.29)$$

This shows that the coefficients of g are large on sets I_i and we have

$$\|g\|_{\dot{H}^q}^2 = \sum_i \sum_{k \in I_i} |g_k|^2 k^{2q} \geq \sum_i C_0^2 = \infty. \quad (4.30)$$

We conclude that g is not in $\dot{H}^q$ —a contradiction. $\square$

5. Proof of theorem 2

Theorem 2. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \rightarrow \infty} \frac{|\langle f^t, g \rangle|}{h(t)} > 0.$$

Proof of theorem 2. If f^t is (q,h)-recurrent, then we are done by lemma 1, so say f^t is (q,h)-transient. Take some $\delta < 1/3$ and apply lemma 2 to construct sets $\{I_i\}_{i=1}^\infty$ and a sequence $\{T_i\}_{i=1}^\infty$. Let $\{T_{i_\ell}\}_{\ell=1}^\infty$ be a subsequence of $\{T_i\}_{i=1}^\infty$ satisfying

$$\sum_{\ell > L} \left(\frac{h(T_{i_\ell})}{\|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}} \right)^2 \leq \delta^2 \left(\frac{h(T_{i_L})}{\|f^{T_{i_L}}\|_{\dot{H}^{-q}}} \right)^2. \quad (5.1)$$

and

$$\sum \left(\frac{h(T_{i_\ell})}{\|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}} \right)^2 \leq \delta. \quad (5.2)$$

This can be done since $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$. Let g be the function with Fourier coefficients given by

$$g_k = \begin{cases} f_k^{T_{i_\ell}} k^{-2q} \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{i_\ell}) & k \in I_{i_\ell}; \\ 0 & \text{otherwise.} \end{cases} \quad (5.3)$$

Equation (5.2) allows us to conclude that $g \in \dot{H}^q$:

$$\begin{aligned} \|g\|_{\dot{H}^q}^2 &= \sum |g_k|^2 k^{2q} \\ &= \sum_{i_\ell} \sum_{k \in I_{i_\ell}} \left| f_k^{T_{i_\ell}} \right|^2 k^{-4q} \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-4} h^2(T_{i_\ell}) k^{2q} \\ &= \sum_{i_\ell} \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-4} h^2(T_{i_\ell}) \left(\sum_{k \in I_{i_\ell}} \left| f_k^{T_{i_\ell}} \right|^2 k^{-2q} \right) \\ &\leq \sum_{i_\ell} \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-2} h^2(T_{i_\ell}). \end{aligned}$$

We will now finish the proof by showing $|\langle f^{T_{i_\ell}}, g \rangle| \geq (1 - 3\delta) h(T_{i_\ell})$. We begin with some notation. Split the following sum into two parts:

$$\begin{aligned} \langle f^{T_{i_\ell}}, g \rangle &= \sum_{j_\ell} \sum_{k \in I_{j_\ell}} f_k^{T_{i_\ell}} \overline{f_k^{T_{j_\ell}}} k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}) \\ &= S^{T_{i_\ell}} + E^{T_{i_\ell}}, \end{aligned}$$

where $S^{T_{i_\ell}}$ is the sum when $j_\ell = i_\ell$:

$$\begin{aligned} S^{T_{i_\ell}} &= \sum_{j_\ell = i_\ell} \sum_{\mathbf{k} \in I_{i_\ell}} f_{\mathbf{k}}^{T_{i_\ell}} \overline{f_{\mathbf{k}}^{T_{i_\ell}}} k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}) \\ &= \sum_{\mathbf{k} \in I_{i_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right|^2 k^{-2q} \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{i_\ell}), \end{aligned}$$

and $E^{T_{i_\ell}}$ is the sum over $j_\ell \neq i_\ell$:

$$E^{T_{i_\ell}} = \sum_{j_\ell \neq i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} f_{\mathbf{k}}^{T_{i_\ell}} \overline{f_{\mathbf{k}}^{T_{j_\ell}}} k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}). \quad (5.4)$$

The idea is that $g_{\mathbf{k}}$ is constructed to agree well with $f_{\mathbf{k}}^{T_{i_\ell}}$ when $\mathbf{k} \in I_{i_\ell}$. We will show that $S^{T_{i_\ell}}$ dominates the error $E^{T_{i_\ell}}$. Consider $j_\ell < i_\ell$ and $j_\ell > i_\ell$ separately in equation (5.4); taking absolute value, we have

$$|E^{T_{i_\ell}}| \leq \sum_{j_\ell \neq i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right| \left| f_{\mathbf{k}}^{T_{j_\ell}} \right| k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}) \quad (5.5)$$

and let $E_1^{T_{i_\ell}}$ be the sum over $j_\ell < i_\ell$:

$$E_1^{T_{i_\ell}} := \sum_{j_\ell < i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right| \left| f_{\mathbf{k}}^{T_{j_\ell}} \right| k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}). \quad (5.6)$$

Similarly define $E_2^{T_{i_\ell}}$ to be the sum over $j_\ell > i_\ell$:

$$E_2^{T_{i_\ell}} := \sum_{j_\ell > i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right| \left| f_{\mathbf{k}}^{T_{j_\ell}} \right| k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}). \quad (5.7)$$

We now bound $E_1^{T_{i_\ell}}$ using the Cauchy–Schwarz inequality:

$$\begin{aligned} E_1^{T_{i_\ell}} &\leq \left(\sum_{j_\ell < i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right|^2 k^{-2q} \right)^{1/2} \left(\sum_{j_\ell < i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{j_\ell}} \right|^2 k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-4} h^2(T_{j_\ell}) \right)^{1/2} \\ &\leq \left(\sum_{j_\ell < i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} \left| f_{\mathbf{k}}^{T_{i_\ell}} \right|^2 k^{-2q} \right)^{1/2} \left(\sum_{j_\ell < i_\ell} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h^2(T_{j_\ell}) \right)^{1/2}. \end{aligned}$$

We use equation (4.8) from lemma 2 to bound the first factor and equation (5.2) to bound the second factor:

$$E_1^{T_{i_\ell}} \leq (\delta h^2(T_{i_\ell}))^{1/2} (\delta)^{1/2} = \delta h(T_{i_\ell}). \quad (5.8)$$

We similarly bound $E_2^{T_{i_\ell}}$:

$$\begin{aligned}
E_2^{T_{i_\ell}} &\leq \left(\sum_{j_\ell > i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} |f_{\mathbf{k}}^{T_{i_\ell}}|^2 k^{-2q} \right)^{1/2} \left(\sum_{j_\ell > i_\ell} \sum_{\mathbf{k} \in I_{j_\ell}} |f_{\mathbf{k}}^{T_{j_\ell}}|^2 k^{-2q} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-4} h^2(T_{j_\ell}) \right)^{1/2} \\
&\leq \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}} \left(\sum_{j_\ell > i_\ell} \|f^{T_{j_\ell}}\|_{\dot{H}^{-q}}^{-2} h^2(T_{j_\ell}) \right)^{1/2}.
\end{aligned}$$

Using equation (5.1), we find

$$E_2^{T_{i_\ell}} \leq \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}} \left(\delta^2 \left(\frac{h(T_{i_\ell})}{\|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}} \right)^2 \right)^{1/2} = \delta h(T_{i_\ell}) \quad (5.9)$$

and therefore

$$\langle f^{T_{i_\ell}}, g \rangle = S^{T_{i_\ell}} + E^{T_{i_\ell}} \geq S^{T_{i_\ell}} - E_1^{T_{i_\ell}} - E_2^{T_{i_\ell}} \geq S^{T_{i_\ell}} - \delta h(T_{i_\ell}) - \delta h(T_{i_\ell}).$$

Again using equation (4.7) from lemma 2 that the set I_{i_ℓ} captures a large proportion of the Sobolev norm, we conclude

$$\begin{aligned}
\langle f^{T_{i_\ell}}, g \rangle &\geq (1 - \delta) \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^2 \|f^{T_{i_\ell}}\|_{\dot{H}^{-q}}^{-2} h(T_{i_\ell}) - 2\delta h(T_{i_\ell}) \\
&= (1 - 3\delta) h(T_{i_\ell}).
\end{aligned}$$

□

Acknowledgments

The authors thank Gautam Iyer for helpful discussions and Georg Gottwald for asking the question that prompted this research. This work was supported in part by NSF Award DMS-1813003.

ORCID iDs

Jean-Luc Thiffeault  <https://orcid.org/0000-0001-7724-7966>

References

- [1] Bedrossian J and He S 2020 Inviscid damping and enhanced dissipation of the boundary layer for 2D Navier–Stokes linearized around Couette flow in a channel *Commun. Math. Phys.* **379** 177–226
- [2] Bedrossian J, Blumenthal A and Punshon-Smith S 2019 Almost-sure exponential mixing of passive scalars by the stochastic Navier–Stokes equations (arXiv:1905.03869)
- [3] Bedrossian J, Blumenthal A and Punshon-Smith S 2019 The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics (arXiv:1911.11014)
- [4] Bedrossian J, Blumenthal A and Punshon-Smith S 2021 Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection–diffusion by stochastic Navier–Stokes *Probab. Theory and Related Fields* **179** 777–834
- [5] Chernov N I 1998 Markov approximations and decay of correlations for Anosov flows *Ann. Math.* **147** 269–324
- [6] Constantin P, Kiselev A, Ryzhik L and Zlatoš A 2008 Diffusion and mixing in fluid flow *Ann. Math.* **168** 643–74

- [7] Zelati M C 2020 Stable mixing estimates in the infinite Péclet number limit *J. Func. Anal.* **279** 108562
- [8] Zelati M C, Delgadino M G and Elgindi T M 2019 On the relation between enhanced dissipation time-scales and mixing rates *Comm. Pure Appl. Math.* **73** 1205–44
- [9] Doering C R and Thiffeault J-L 2006 Multiscale mixing efficiencies for steady sources *Phys. Rev. E* **74** 025301(R)
- [10] Dolgopyat D 1998 On decay of correlations in Anosov flows *Ann. Math.* **147** 357–90
- [11] Elgindi T M and Zlatoš A 2019 Universal mixers in all dimensions *Adv. Math.* **356** 106807
- [12] Feng Y and Iyer G 2019 Dissipation enhancement by mixing *Nonlinearity* **32** 1810–51
- [13] Real G B F 1999 *Analysis: Modern Techniques and Their Applications* 2nd edn (New York: Wiley)
- [14] Hebey E 2000 *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities* 1st edn (New York: Courant)
- [15] Iyer G, Kiselev A and Xu X 2014 Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows *Nonlinearity* **27** 973–85
- [16] Kiselev A and Xu X 2016 Suppression of chemotactic explosion by mixing *Arch. Rational Mech. Anal.* **222** 1077–112
- [17] Kiselev A, Shterenberg R and Zlatoš A 2008 Relaxation enhancement by time-periodic flows *Indiana Univ. Math. J.* **57** 2137–52
- [18] Lin Z, Božová K and Doering C R 2010 Models & measures of mixing & effective diffusion *Discr. Cont. Dyn. Sys.* **28** 259–74
- [19] Lin Z, Thiffeault J-L and Doering C R 2011 Optimal stirring strategies for passive scalar mixing *J. Fluid Mech.* **675** 465–76
- [20] Liverani C 1995 Decay of correlations *Ann. Math.* **142** 239–301
- [21] Lunasin E, Lin Z, Novikov A, Mazzucato A and Doering C R 2012 Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows *J. Math. Phys.* **53** 115611
- [22] Lunasin E, Lin Z, Novikov A, Mazzucato A and Doering C R 2013 Erratum: ‘optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows’ *J. Math. Phys.* **54** 079903
- [23] Marcotte F and Caulfield C P 2018 Optimal mixing in two-dimensional stratified plane Poiseuille flow at finite Péclet and Richardson numbers *J. Fluid Mech.* **853** 359–85
- [24] Mathew G, Mezić I and Petzold L 2003 A multiscale measure for mixing and its applications *Proc. Conf. on Decision and Control* (Piscataway, NJ: IEEE)
- [25] Mathew G, Mezić I and Petzold L 2005 A multiscale measure for mixing *Phys. D* **211** 23–46
- [26] Mathew G, Mezić I, Grivopoulos S, Vaidya U and Petzold L 2007 Optimal control of mixing in Stokes fluid flows *J. Fluid Mech.* **580** 261–81
- [27] Miles C J and Doering C R 2018 Diffusion-limited mixing by incompressible flows *Nonlinearity* **31** 2346–59
- [28] Miles C J and Doering C R 2018 A shell model for optimal mixing *J. Nonlinear Sci.* **28** 2153–86
- [29] Pierrehumbert R T 1994 Tracer microstructure in the large-eddy dominated regime *Chaos, Solit. Fractals* **4** 1091–110
- [30] Pollicott M 1981 On the rate of mixing of Axiom A flows *Invent. Math.* **81** 413–26
- [31] Shaw T A, Thiffeault J-L and Doering C R 2007 Stirring up trouble: multi-scale mixing measures for steady scalar sources *Phys. D* **231** 143–64
- [32] Sturman R, Ottino J M and Wiggins S 2006 *The Mathematical Foundations of Mixing: The Linked Twist Map as a Paradigm in Applications: Micro to Macro, Fluids to Solids* (Cambridge: Cambridge University Press)
- [33] Thiffeault J-L 2012 Using multiscale norms to quantify mixing and transport *Nonlinearity* **25** R1–R44
- [34] Thiffeault J-L and Doering C R 2011 The mixing efficiency of open flows *Phys. D* **240** 180–6
- [35] Thiffeault J-L, Doering C R and Gibbon J D 2004 A bound on mixing efficiency for the advection-diffusion equation *J. Fluid Mech.* **521** 105–14
- [36] Vermach L and Caulfield C P 2018 Optimal mixing in three-dimensional plane Poiseuille flow at high Péclet number *J. Fluid Mech.* **850** 875–923
- [37] Yao Y and Zlatoš A 2017 Mixing and un-mixing by incompressible flows *J. Eur. Math. Soc.* **19** 1911–48
- [38] Young L-S 1999 Recurrence times and rates of mixing *Isr. J. Math.* **110** 153–88
- [39] Zillinger C 2019 On geometric and analytic mixing scales: comparability and convergence rates for transport problems *Pure Appl. Anal.* **1** 543–70