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SPACE OF RICCI FLOWS (II)—PART B: WEAK COMPACTNESS OF THE FLOWS

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Abstract

Based on the compactness of the moduli of non-collapsed Calabi–Yau spaces with mild singularities, we set up a structure theory for polarized Kähler Ricci flows with proper geometric bounds. Our theory is a generalization of the structure theory of non-collapsed Kähler Einstein manifolds. As applications, we show the convergence of the Kähler Ricci flow in an appropriate topology and prove the partial-C^0-conjecture.

Contents

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1. Introduction

This paper is the continuation of the study in ([**27**]) and ([**30**]). In [**27**], we developed a weak compactness theory for non-collapsed Ricci flows with bounded scalar curvature and bounded half-dimensional curvature integral. This weak compactness theory is applied in [**30**] to show the convergence of the Kähler Ricci flow in complex dimension 2 and its geometric consequences. However, the assumption of half dimensional curvature integral is restrictive. It is not available for high dimensional anti-canonical Kähler Ricci flow, i.e., Kähler Ricci flow on a Fano manifold (M, J), in the class $2\pi c_1(M, J)$. In this paper, by taking advantage of the extra structures from Kähler geometry, we drop this curvature integral condition.

The present paper is inspired by two different sources. One source is the structure theory of Kähler Einstein manifolds which was developed over last 20 years by many people, notably, Anderson, Cheeger, Colding, Tian and more recently, Naber, Donaldson and Sun. The recent progress of the structure theory of Kähler Einstein manifolds supplies many additional tools for our approach. The other source is the seminal work of Perelman on the Ricci flow (c.f. [**49**], [**55**]). Actually, it was pointed out by Perelman already that his idea in [**49**] can be applied to study Kähler Ricci flow. He wrote that

> "*present work has also some applications to the Hamilton–Tian conjecture concerning Kähler–Ricci flow on Kähler manifold with positive first Chern class: these will be discussed in a separate paper*".

We cannot help to wonder how far he will push the subject of Ricci flow if he continued to publicize his works on arXiv. Although "*this separate paper*" never appears, his fundamental estimates of Kähler Ricci flow on Fano manifolds is the base of our present research. Besides Perelman's estimates, we also note that the following technical results in the Ricci flow are important to the formation of this paper over a long period of time: the Sobolev constant estimate by Q.S. Zhang ([**76**]) and R. Ye ([**75**]), and the volume ratio upper bound estimate by Q.S. Zhang ([**78**]) and Chen–Wang ([**31**]). Some other important estimates can be found in the summary of [**26**].

Our key observation is that there is a "canonical neighborhood" theorem for anti-canonical Kähler Ricci flows. The idea of "canonical neighborhood" originates from Theorem 12.1 of Perelman's paper [**49**]. For every 3-dimensional Ricci flow, Perelman showed that the spacetime neighborhood of a high curvature point can be approximated by a κ-solution, which is a model Ricci flow solution. To be precise, a κ-solution is a 3-dimensional, κ-noncollapsed, ancient Ricci flow solution with bounded, nonnegative curvature operator. By definition, it is not clear at all that the moduli of κ-solutions has compactness under (pointed-) smooth topology (modulo diffeomorphisms). Perelman genuinely proved the compactness by delicate use of Hamilton–Ivey estimate and the geometry of nonnegatively curved 3-manifolds. In light of the compactness of the moduli of κ-solutions, by a maximum principle type argument, Perelman developed the "canonical neighborhood" theorem, which is of essential importance to his celebrated solution of the Poincaré conjecture (c.f. [**41**], [**45**], [**5**]).

The idea of "canonical neighborhood" is universal and can be applied in many different geometric settings. In particular, there is a "canonical neighborhood" theorem for the anti-canonical Kähler Ricci flows, where estimates of many quantities, including scalar curvature, Ricci potential and Sobolev constant, are available. Clearly, a "canonical neighborhood" should be a neighborhood in space-time, behaving like a model space-time, which is more or less the blowup limit of the given flow. Therefore, it is natural to expect that the model space-time is the scalar flat Ricci flow solutions, which must be Ricci flat, due to the equation $\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2$, satisfied by the scalar curvature R. For this reason, the model space and model space-time can be identified, since the evolution on time direction is trivial. It is also natural to expect that the model space has a Kähler structure. In other words, the model space should be Kähler Ricci flat space, or Calabi–Yau space. Now the first essential difficulty appears. A good model space should have a compact moduli. For example, in the case of 3-dimensional Ricci flow, the moduli space of κ-solutions, which are the model space-times, has compactness in the smooth topology. However, the moduli space of all the non-collapsed smooth Calabi–Yau space-times is clearly not compact under the smooth topology. A blowdown sequence of Eguchi–Hanson metrics is an easy example. For the sake of compactness, we need to replace the smooth topology by a weaker topology, the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology. At the same time, we also need to enlarge the class of model spaces from complete Calabi–Yau manifolds to the Calabi–Yau spaces with mild singularities (c.f. Definition 2.1), which we denote by $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. Similar to the compactness theorem of Perelman's κ-solutions, we have the compactness of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.

Theorem 1.1 (**Compactness of model moduli**, Chen–Wang [**29**])**.** *$\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ is compact under the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology. In other words, for each sequence of $(X_i,x_i,g_i) \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, by taking subsequence if necessary, we have*

$$(X_i, x_i, g_i) \xrightarrow{\hat{C}^{\infty}} (\bar{X}, \bar{x}, \bar{g}), \tag{1.1}$$

for some $(\bar{X},\bar{x},\bar{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.

Note that the convergence topology in (1.1) was stated as "pointed-Cheeger–Gromov" topology previously in literature, for example, in Chen–Wang [**27**]. We now use extra term $\hat{C}^{\infty}$ to indicate it deals with singularities. Let us say a few more words for its precise meaning. In fact, (1.1) first means that (X_i, x_i, d_i) converges to a pointed-length-space $(\bar{X}, \bar{x}, \bar{d})$, where d_i is the distance structure induced by g_i. The second meaning of (1.1) is that $\bar{X}$ has a regular-singular decomposition $\bar{X} = \mathcal{R}(\bar{X}) \cup \mathcal{S}(\bar{X})$, where the regular part $\mathcal{R}(\bar{X})$ is a smooth manifold equipped with a smooth metric $\bar{g}$, the singular part $\mathcal{S}(\bar{X})$ is a measure ($2n$-dimensional Hausdorff measure) zero set. Locally around each regular point, the metric structure determined by $\bar{g}$ is identical to $\bar{d}$. The regular part $\mathcal{R}(\bar{X})$ has an exhaustion $\cup_{j=1}^{\infty} K_j$ by compact sets K_j. For each compact set $K = K_j$ for some j, one can find diffeomorphisms $\varphi_{K,i}$ from K to $\varphi_{K,i}(K)$, a subset of $\mathcal{R}(X_i)$ such that

$$
\begin{aligned}
&d_i(\varphi_{K,i}(y), x_i) \to \bar{d}(y, \bar{x}), \quad \forall\, y \in K; \\
&\varphi_{K,i}^{*}(g_i) \xrightarrow{C^{\infty}} \bar{g}, \quad \text{on } K.
\end{aligned}
$$

Although in general the global distance structure induced by $\bar{g}$ may not be the same as $\bar{d}$, this difference does not happen whenever the limit space $\bar{X} \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ since $\mathcal{R}(\bar{X})$ is weakly geodesic convex. Clearly, ∞ can be replaced by general positive k and the convergence in the pointed-$\hat{C}^k$-Cheeger–Gromov topology can be defined similarly. So we use $\xrightarrow{\hat{C}^k}$ to denote the (pointed)-$\hat{C}^k$-Cheeger–Gromov topology, i.e., the convergence is in the (pointed)-Gromov–Hausdorff topology, and can be improved to be in C^k-topology (modulo diffeomorphisms) away from singularities. For simplicity of notation, we use $\xrightarrow{P.G.H.}$ to denote the convergence in pointed-Gromov–Hausdorff topology, use $\xrightarrow{G.H.}$ to denote the convergence in Gromov–Hausdorff topology.

The strategy to prove the compactness of $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ follows the same route of the weak compactness theory of Kähler Einstein manifolds, developed by Cheeger, Gromoll, Anderson, Colding, Tian, Naber, etc. However, the analysis foundation on the singular spaces need to be carefully checked, which is discussed in a separate paper [**29**]. Theorem 1.1 is motivated by section 11 of Perelman's seminal paper [**49**], where Perelman proved the compactness of moduli space of κ-solutions and showed that κ-solutions have many properties which are not obvious from definition.

By trivial extension, each $X \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ can be understood as a space-time $X \times (-\infty, \infty)$ satisfying Ricci flow equation. Intuitively, the rescaled space-time structure in a given anti-canonical Kähler Ricci flow should behave similar to that of $X \times (-\infty, \infty)$ for some $X \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$, when the rescaling factor is large enough. In order to make sense that two pointed-space-times are close to each other, we need the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology for space-times, a slight generalization of the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology for metric spaces. When restricted on each time slice, this topology is the same as the usual pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology. Between every two different time slices, there is a natural homeomorphism map connecting them. Therefore, the above intuition can be realized if we can show a blowup sequence of Ricci flow space-times from a given Kähler Ricci flow converges to a limit space-time $X \times (-\infty, \infty)$, in the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology for space-times. However, it is not easy to obtain the homeomorphism maps between different time slices in the limit. Although it is quite obvious to guess that the homeomorphism maps among different time slices are the limit of identity maps, there exists serious technical difficulty to show the existence and regularity of the limit maps. The difficulty boils down to a fundamental improvement of Perelman's pseudolocality theorem (Theorem 10.1 of [49]). Recall that Perelman's pseudolocality theorem says that Ricci flow cannot "quickly" turn an almost Euclidean region into a very curved one. It is a short-time, one-sided estimate in nature. We need to improve it to a long-time, two-sided estimate. Not surprisingly, the rigidity of Kähler geometry plays an essential role for such an improvement. The two-sided, long-time pseudolocality is an estimate in the time direction. Modulo this time direction estimate and the weak compactness in the space direction, we can take limit for a sequence of Ricci flows blown up from a given flow. Then the canonical neighborhood theorem can be set up if we can show that the limit space-time locates in $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, following the same route as that in the proof of Theorem 12.1 of [49].

From the above discussion, it is clear that the strategy to prove the canonical neighborhood theorem is simple. However, the technical difficulty hidden behind this simple strategy is not that simple. We observe that the anti-canonical Kähler Ricci flow has many additional structures, all of them should be used to carry out the proof of the canonical neighborhood theorem. In particular, over every anti-canonical Kähler Ricci flow, there is a natural anti-canonical polarization, which should play an important role, as done in [30]. Although it can be studied in a more general setting, in this paper, however, we shall focus on the flow with pluri-anti-canonical polarizations. We call $\mathcal{LM} = \{(M^n, g(t), J, L, h(t)), t \in (-T, T) \subset \mathbb{R}\}$ a polarized Kähler Ricci flow if

- $\mathcal{M} = \{(M^n, g(t), J), t \in (-T, T)\}$ is a Kähler Ricci flow solution.
- $L = K_M^{-\nu}$ is a pluri-anti-canonical line bundle over M, $h(t)$ is a family of smooth metrics on L whose curvature is $\omega(t)$, the metric form compatible with $g(t)$ and the complex structure J.

Clearly, the first Chern class of L is $[\omega(t)]$, which does not depend on time. So a polarized Kähler Ricci flow stays in a fixed integer Kähler class. The evolution equation of $g(t)$ can be written as

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + \lambda g_{i\bar{j}}, \tag{1.2}$$

where $\lambda = \frac{c_1(M)}{c_1(L)}$. Since the flow stays in the fixed class, we can let $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$. Then $\dot{\varphi}$ is the Ricci potential, i.e.,

$$\sqrt{-1}\partial\bar{\partial}\dot{\varphi} = -Ric + \lambda g. \tag{1.3}$$

Note the choice of φ is unique up to adding a constant. So we can always modify the choice of φ such that $\sup_M \dot{\varphi} = 0$. For simplicity, we denote $\mathscr{K}(n, A)$ as the collection of all the polarized Kähler Ricci flows $\mathcal{LM}$ satisfying the following estimate

$$\begin{cases} T \geq 2, \\ C_S(M) + \frac{1}{\mathrm{Vol}(M)} + |\dot{\varphi}|_{C^1(M)} + |R - n\lambda|_{C^0(M)} \leq A, \end{cases} \tag{1.4}$$

for every time $t \in (-T, T)$. Here C_S means the Sobolev constant, A is a uniform constant. In this paper, we study the structure of polarized Kähler Ricci flows locating in the space $\mathscr{K}(n, A)$. The motivation behind (1.4) arises from the fundamental estimate of diameter, scalar curvature, C^1-norm of Ricci potential, and Sobolev constant along the anti-canonical Kähler Ricci flows (c.f. [**55**], [**76**], [**75**]). Every polarized Kähler Ricci flow solution in $\mathscr{K}(n, A)$ has at least three structures: the metric space structure, the flow structure, the line bundle structure. Same structures can be discussed on the model space-time in $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$. All the structures of a flow in $\mathscr{K}(n, A)$ can be modeled by the corresponding structures in $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$, which is the same meaning as the "canonical neighborhood theorem". We shall compare these structures term by term. Note that κ is the uniform non-collapsing constant determined by the Sobolev constant bound in (1.4). The choice of κ follows from the notation of the famous no-local-collapsing theorem of Perelman [**49**]. More details can be found in Remark 3.32.

Under the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology at time 0, let us compare the metric structure of a flow in $\mathscr{K}(n, A)$ with a Calabi–Yau conifold in $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$. We shall show that $\mathscr{K}(n, A)$ and $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ behaves almost the same in this perspective. Intuitively, one can think that the weak compactness theory of Ricci-flat manifolds and Einstein manifolds are almost the same.

Theorem 1.2 (Metric space estimates). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$. By taking subsequence if necessary, we have*

$$(M_i, x_i, g_i(0)) \xrightarrow{\hat{C}^{\infty}} (\bar{M}, \bar{x}, \bar{g}). \tag{1.5}$$

The limit space $\bar{M}$ has a classical regular-singular decomposition $\mathcal{R} \cup \mathcal{S}$ with the following properties.

- *$(\mathcal{R}, \bar{g})$ is a smooth, open Riemannian manifold. Moreover, $\mathcal{R}$ admits a limit Kähler structure $\bar{J}$ such that $\left(\mathcal{R}, \bar{g}, \bar{J}\right)$ is an open Kähler manifold.*
- *$\mathcal{S}$ is a closed set and $\dim_{\mathcal{M}} \mathcal{S} \leq 2n-4$, where $\dim_{\mathcal{M}}$ means Minkowski dimension (c.f. Definition 2.2 of [**29**]).*
- *Every tangent space of $\bar{M}$ is an irreducible metric cone.*
- *Let v be the volume density, i.e.,*

$$\mathrm{v}(y) = \limsup_{r \to 0} \omega_{2n}^{-1} r^{-2n} |B(y,r)|, \tag{1.6}$$

  *for every point $y \in \bar{M}$. Then a point is regular if and only if $\mathrm{v}(y) = 1$, a point is singular if and only if $\mathrm{v}(y) \leq 1 - 2\delta_0$, where δ_0 is a dimensional constant determined by Anderson's gap theorem (c.f. Lemma 3.1 of [**1**]).*

By definition, we call a point being regular if it has a neighborhood with smooth manifold structure and call a point being singular if it is not regular (c.f. Proposition 4.2 and Remark 4.3). It is important to note the difference between $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ and $\mathscr{K}(n,A)$. We use $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ to denote the space of possible bubbles, or blowup limits. Therefore, every metric space in it is a non-compact one. However, each time slice of flows in $\mathscr{K}(n,A)$ is a compact manifold. The limit space $\bar{M}$ of Theorem 1.2 maybe compact and does not belong to $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. Note also that the weak convexity of $\mathcal{R}(\bar{M})$ is not known without further conditions.

In the study of the line bundle structure of $\mathscr{K}(n,A)$, the Bergman function plays an important role. Actually, for every positive integer k large enough such that L^k is globally generated, we define the Bergman function $\mathbf{b}^{(k)}$ as follows

$$\mathbf{b}^{(k)}(x,t) = \log \sum_{i=0}^{N_k} \left\| S_i^{(k)} \right\|_{h(t)}^2 (x,t), \tag{1.7}$$

where $N_k = \dim_{\mathbb{C}} H^0(M, L^k) - 1$, $\left\{ S_i^{(k)} \right\}_{i=0}^{N_k}$ are orthonormal basis of $H^0(M, L^k)$ under the natural metrics $\omega(t)$ and $h(t)$. Theorem 1.2 means that the metric structure of the center time slice of a Kähler Ricci flow in $\mathscr{K}(n,A)$ can be modeled by non-collapsed Calabi–Yau manifolds with mild singularities. In particular, each tangent space of a point in the limit space is a metric cone. The trivial line bundle structure on metric cone then implies an estimate of line bundle structure of the original manifold, due to delicate use of Hömander's $\bar{\partial}$-estimate, as done by Donaldson and Sun (c.f. [**37**]).

Theorem 1.3 (**Line bundle estimates**). *Suppose $\mathcal{LM} \in \mathscr{K}(n,A)$, then*

$$\inf_{x \in M} \mathbf{b}^{(k_0)}(x,0) \geq -c_0,$$

for some positive number $c_0 = c_0(n,A)$, and positive integer $k_0 = k_0(n,A)$.

In other words, Theorem 1.3 states that there is a uniform partial-C^0-estimate at time $t=0$. This estimate then implies variety structure of limit space, as discussed in [**66**] and [**37**]. Theorem 1.3 can be understood that the line bundle structure of $\mathscr{K}(n,A)$ is modeled after that of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.

Theorem 1.2 and Theorem 1.3 deal only with one time slice. In order to make sense of limit Kähler Ricci flow, we have to compare the limit spaces of different time slices. For example, we choose $x_i \in M_i$, then we have

$$(M_i, x_i, g_i(0)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}), \quad (M_i, x_i, g_i(-1)) \xrightarrow{\hat{C}^\infty} (\bar{M}', \bar{x}', \bar{g}').$$

How are $\bar{M}$ and $\bar{M}'$ related? If $\bar{x}$ is a regular point of $\bar{M}$, can we say $\bar{x}'$ is a regular point of $\bar{M}'$? Note that Perelman's pseudolocality theorem cannot answer this question, due to its short-time, one-sided property. In order to relate different time slices, we need to improve Perelman's pseudolocality theorem to the following long-time, two-sided estimate, which is the technical core of the current paper.

Theorem 1.4 (**Time direction estimates**). *Let $\mathcal{LM} \in \mathscr{K}(n,A)$. Suppose $x_0 \in M$, $\Omega = B_{g(0)}(x_0, r)$, $\Omega' = B_{g(0)}(x_0, \frac{r}{2})$ for some $r \in (0,1)$. At time $t=0$, suppose the isoperimetric constant estimate*

$$\mathbf{I}(\Omega) \geq (1-\delta_0)\mathbf{I}(\mathbb{C}^n)$$

holds for $\delta_0 = \delta_0(n)$, the same constant in Theorem 1.2. Then we have

$$|\nabla^k Rm|(x,t) \leq C_k, \quad \forall\, k \in \mathbb{Z}^{\geq 0}, \quad x \in \Omega', \quad t \in [-1,1],$$

where C_k is a constant depending on n, A, r and k.

Theorem 1.4 holds trivially on each space in $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, when regarded as a static Ricci flow solution. Therefore, it can be understood as the time direction structure, or the flow structure of $\mathcal{LM} \in \mathscr{K}(n,A)$ is similar to that of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. Theorem 1.4 removes the major stumbling block for defining a limit Kähler Ricci flow, since it guarantees that the regular-singular decomposition of the limit space is independent of time. Therefore, there is a natural induced Kähler Ricci flow structure on the regular part of the limit space. We denote its completion by a limit Kähler Ricci flow solution, in a weak sense. Clearly, the limit Kähler Ricci flow naturally inherits a limit line bundle structure, or a limit polarization, on the regular part. Moreover, the limit underlying space does have a variety structure due to Theorem 1.3. With these structures in hand, we are ready to discuss the convergence theorem of polarized Kähler Ricci flows, which is the main structure theorem of this paper (c.f. section 4.5 for meaning of the notations).

Theorem 1.5 (**Weak compactness of polarized flows**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$ satisfying $\mathrm{diam}_{g_i(0)}(M_i) < C$ uniformly or $\sup_{\mathcal{M}_i} |R| \to 0$. By passing to subsequence if necessary, we have*

$$(\mathcal{LM}_i, x_i) \xrightarrow{\hat{C}^\infty} \left(\overline{\mathcal{LM}}, \bar{x}\right), \tag{1.8}$$

where $\overline{\mathcal{LM}}$ is a polarized Kähler Ricci flow solution on an analytic normal variety $\bar{M}$, whose singular set $\mathcal{S}$ has Minkowski codimension at least 4, with respect to each $\bar{g}(t)$. Moreover, if $\bar{M}$ is compact, then it is a projective normal variety with at most log-terminal singularities.

Let us explain a few words of the meaning of (1.8). The limit flow $\overline{\mathcal{LM}}$ exists on $\bar{M} \times (-\bar{T}, \bar{T})$, whose regular part supports a line bundle $(\bar{L}, \bar{h})$ such that the curvature form of $\bar{h}(t)$ is $\bar{\omega}(t)$. Built on the convergence of the time slices at $t = 0$ (c.f. (1.5)), for each compact set $K \subset \mathcal{R}(\bar{M})$, we can find diffeomorphisms $\varphi_{K,i}$ such that

$$\varphi_{K,i}^*(g_i(0)) \xrightarrow{C^\infty} \bar{g}(0).$$

Via the same diffeomorphisms, (1.8) means that we further have

$$\begin{aligned}
&\varphi_{K,i}^*(J_i) \xrightarrow{C^\infty} \bar{J};\\
&\varphi_{K,i}^*(g_i(t)) \xrightarrow{C^\infty} \bar{g}(t), \quad \forall\, t \in (-\bar{T}, \bar{T});\\
&\left(\varphi_{K,i}^*(L_i)\big|_K, h_i(t)\right) \xrightarrow{C^\infty} \left(\bar{L}\big|_K, \bar{h}(t)\right), \quad \forall\, t \in (-\bar{T}, \bar{T}).
\end{aligned}$$

For more details, see the discussion from (4.37) to (4.42).

As it is developed for, our structure theory has applications in the study of anti-canonical Kähler Ricci flows. Due to the fundamental estimate of Perelman and the monotonicity of his μ-functional along each anti-canonical Kähler Ricci flow, we can apply Theorem 1.5 directly and obtain the following theorem.

Theorem 1.6 (**Weak compactness of flows**). *Suppose that the spacetime $\{(M^n, g(t)), 0 \le t < \infty\}$ is an anti-canonical Kähler Ricci flow solution on a Fano manifold (M, J). For every $s > 1$, define*

$$\begin{aligned}
&g_s(t) \triangleq g(t+s),\\
&\mathcal{M}_s \triangleq \{(M^n, g_s(t)), -s \le t \le s\}.
\end{aligned}$$

Then for every sequence $s_i \to \infty$, by taking subsequence if necessary, we have

$$(\mathcal{M}_{s_i}, g_{s_i}) \xrightarrow{\hat{C}^\infty} \left(\bar{\mathcal{M}}, \bar{g}\right), \tag{1.9}$$

where the limit space-time $\bar{\mathcal{M}}$ is a Kähler Ricci soliton flow solution on a Q-Fano normal variety $(\bar{M}, \bar{J})$. Moreover, with respect to each $\bar{g}(t)$, there is a uniform C independent of time such that the r-neighborhood of the singular set $\mathcal{S}$ has measure not greater than Cr^4.

Theorem 1.6 confirms a long-standing conjecture (sometimes called Hamilton–Tian conjecture) concerning the convergence of the Kähler Ricci flow. This conjecture dates back to Hamilton and it was refined by Tian (c.f. Conjecture 9.1. of [**65**] for the precise statement). In fact, Theorem 1.6 provides more information than that was conjectured since it deals with the convergence of the "space-times", rather than time slices of the flow. Historically, the two dimensional case was confirmed by the authors in [**27**]. We note that in a recent paper [**70**], another approach to attack this conjecture in complex dimension 3, based on L^4-bound of Ricci curvature, was presented by Z.L. Zhang and G. Tian. Their work in turn depends on the comparison geometry with integral Ricci bounded, developed by G.F. Wei and P. Petersen ([**50**]). For other important progress in Kähler Ricci flow, we refer interested readers to the following papers (far away from being complete): [**54**], [**76**], [**75**], [**69**], [**77**], [**57**], [**69**], [**56**], [**51**], [**71**], [**60**], as well as references listed therein.

Theorem 1.6 can be used to study the relationship between the existence of Kähler Einstein metrics and the K-stability of the underlying manifolds. By the work of Chen, Donaldson and Sun (c.f. [**19**], [**20**], [**21**] and [**22**]), a long standing stability conjecture, going back to Yau (c.f. Problem 65 of [**74**]) and critically contributed by Tian (c.f [**65**]) and Donaldson (c.f. [**36**]), was confirmed. Theorem 1.6 can be applied to provide an alternative proof of the original solution of the stability conjecture by Chen–Donaldson–Sun. Moreover, the convergence limit in Theorem 1.6 is unique, i.e., does not depend on the choice of $s_i \to \infty$. Such results are proved in a subsequent work Chen–Sun–Wang [**24**].

Theorem 1.7 (**Limit uniqueness and stability**, Theorem 1.2 of Chen–Sun–Wang [**24**])**.** *Suppose $\{(M^n, g(t)), 0 \le t < \infty\}$ is an anti-canonical Kähler Ricci flow solution on a Fano manifold (M, J). There is a unique Gromov–Hausdorff limit $\bar{M}$ of $(M, g(t))$, as a $\mathbb{Q}$-Fano variety endowed with a weak Kähler–Ricci soliton metric. Moreover, if M is K-stable, then $\bar{M}$ is isomorphic to M endowed with a smooth Kähler–Einstein metric. In particular, M admits a Kähler–Einstein metric if it is K-stable.*

As corollaries of Theorem 1.6, we can affirmatively answer some problems raised in [**30**].

Corollary 1.8. *Every anti-canonical Kähler Ricci flow is tamed, i.e., partial-C^0-estimate holds along the flow.*

Corollary 1.9. *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is an anti-canonical Kähler Ricci flow on a Fano manifold M. Then the flow converges to a Kähler Einstein metric if one of the following conditions hold for every large positive integer ν.*

- $\alpha_{\nu,1} > \frac{n}{n+1}$.
- $\alpha_{\nu,2} > \frac{n}{n+1}$ *and* $\alpha_{\nu,1} > \frac{1}{2-\frac{n-1}{(n+1)\alpha_{\nu,2}}}$.

Corollary 1.9 gives rise to a method for searching Fano Kähler Einstein metrics in high dimension, which generalize the 2-dimensional case due to Tian (c.f. [**62**]). The quantities $\alpha_{\nu,k}$ are some algebro-geometric invariant. The interested readers are referred to [**62**] for the precise definition.

Our structure theory can be applied to study a family of Kähler Ricci flows with some uniform initial conditions. In this perspective, we have the following theorem.

Theorem 1.10 (**Partial-C^0-conjecture**). *For every positive constants R_0, V_0, there exists a positive integer k_0 and a positive constant c_0 with the following properties.*

Suppose (M, ω, J) is a Kähler manifold satisfying Ric $\geq R_0$ and $\mathrm{Vol}(M) \geq V_0$, $[\omega] = 2\pi c_1(M, J)$. *Then we have*

$$\inf_{x \in M} \mathbf{b}^{(k_0)}(x) > -c_0.$$

Theorem 1.10 confirms the partial-C^0-conjecture of Tian (c.f. [**64**], [**66**]). The low dimension case ($n \leq 3$) was proved by Jiang ([**40**]), depending on the partial-C^0-estimate along the flow, developed by Chen and Wang ([**27**], [**30**]) in complex dimension 2 and Tian–Zhang ([**70**]) in complex dimension 3. In fact, a more general version of Theorem 1.10 is proved (c.f. Theorem 5.12). As a corollary of Theorem 1.10, we have

Corollary 1.11. *(c.f. [**61**]) The partial-C^0-estimate holds along the classical continuity path.*

Following Corollary 1.8, we obtain the following result, which was originally proved by G. Székelyhidi (c.f. [**61**]) along the classical continuity path.

Corollary 1.12. *Suppose (M, J) is a Fano manifold with Aut(M, J) discrete. If it is stable in the sense of S. Paul (c.f. [**46**]), then it admits a Kähler Einstein metric.*

Because of the solution of stability conjecture by Chen–Donaldson–Sun, we now know a Fano manifold is K-stable if and only if it admits Kähler Einstein metrics. Using the theorem of Székelyhidi (c.f. [**61**]), one can obtain the equivalence of the K-stability and Paul's stability, whenever the underlying manifold has discrete automorphism group. In light of Theorem 1.7 and Corollary 1.12, we obtain a Ricci flow proof of this equivalence.

Let us quickly go over the relationships among the theorems. Theorem 1.1 is the structure theorem of the model space $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. Theorem 1.2, Theorem 1.3 and Theorem 1.4 combined together give the canonical neighborhood structure of the polarized Kähler Ricci flow in $\mathscr{K}(n,A)$, in a strong sense. The main structure theorem in this paper is Theorem 1.5, the weak compactness theorem of polarized Kähler Ricci flows. It is clear that Theorem 1.6 and Theorem 1.10 are direct applications of Theorem 1.5. The proof of Theorem 1.5 is based on the combination of Theorem 1.2, Theorem 1.3 and Theorem 1.4. These three theorems deal with different structures of $\mathscr{K}(n,A)$, including the Ricci flow structure, metric space structure, line bundle structure and variety structure. The importance of these structures decreases in order, for the purpose of developing compactness. However, all these structures are intertwined together. Paradoxically, the proof of the compactness of these structures does not follow the same order, due to the lack of precise estimate of Bergman functions. Instead of proving them in order, we define a concept called "polarized canonical radius", which guarantees the convergence of all these structures under this radius. The only thing we need to do then is to show that this radius cannot be too small. Otherwise, we can apply a maximum principle argument to obtain a contradiction, which essentially arise from the monotonicity of Perelman's reduced volume and localized W-functional.

This paper is organized as follows. In section 2, we quote some results from [**29**] of the model space $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ and the canonical radius with respect to this model space. In section 3, we first set up a forward, long-time pseudolocality theorem based on the existence of partial-C^0-estimate. Motivated by this pseudolocality theorem, we then refine the "canonical radius" to "polarized canonical radius" and discuss the convergence of flow structure and line bundle structure under the assumption that polarized canonical radius is uniformly bounded from below. Finally, at the end of section 3, we use a maximum principle argument to show that there is an a priori bound of the polarized canonical radius. In section 4, we prove Theorem 1.2–1.5, together with some other more detailed properties of the space $\mathscr{K}(n,A)$. At last, in section 5, we develop the structure theory of the anti-canonical Kähler Ricci flows. Applying the structure theory, we prove Theorem 1.6 and Theorem 1.10.

List of notations

- avr: asymptotic volume ratio. Defined in (2.2).
- $\mathbf{b}$: Bergman function of $L = K_M^{-\nu}$. Defined in (3.1).
- $\mathbf{b}^{(k)}$: Bergman function of L^k. Defined in (1.7).
- $\mathbf{cr}$: canonical radius. Defined in Definition 2.9.

- **cvr**: canonical volume radius. Defined in Proposition 2.10.
- $\dim_{\mathcal{H}}$: Hausdorff dimension. First appears in Proposition 3.19.
- $\dim_{\mathcal{M}}$: Minkowski dimension. First appears in Theorem 1.2.
- $\mathcal{D}_r$: points on M where the volume radius is strictly less than r. Defined in (2.12).
- **E**: upper bound of the density estimate of model space. First appears in Theorem 2.7.
- $\mathcal{F}_r$: points on M where the volume radius is at least r. Defined in (2.12).
- $\mathcal{K}(n, A)$: Kähler Ricci flow satisfying (1.4). Defined in Definition 3.14.
- $\mathcal{K}(n, A; r)$: Kähler Ricci flow satisfying (1.4) and $\mathbf{pcr} \geq r$ in the central period. Defined in Definition 3.14.
- $\mathscr{K}\mathscr{S}(n)$: the class of all the complete n-dimensional Calabi–Yau manifolds. Stated before Proposition 2.2.
- $\widetilde{\mathscr{K}\mathscr{S}}(n)$:the class of all n-dimensional Calabi–Yau manifolds with mild singularities. Defined in Definition 2.1.
- $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$: the model space, which is the class of all n-dimensional Calabi–Yau manifolds with mild singularities and at least κ asymptotic volume ratios. Defined in Definition 2.1.
- l: Perelman's reduced distance. First appears in (2.5).
- L: The line bundle polarizing M. First appears in the paragraph before (1.2).
- $\mathcal{L}$: Lagrangian of space-time curves. First appears in (2.4).
- $\mathcal{L}\mathcal{M}$: Polarized Kähler Ricci flow solution. First appears in the paragraph before (1.2).
- $\mathcal{M}$: Kähler Ricci flow solution. First appears in the paragraph before (1.2).
- $\mathcal{M}^t$: the time t slice of the Kähler Ricci flow $\mathcal{M}$. First appears in Proposition 3.12.
- **pcr**: polarized canonical radius. Defined in Definition 3.10.
- p_0: a constant very close to 2. It is chosen as $p_0 = 2 - \frac{1}{1000n}$ in Theorem 2.7.
- $\mathcal{R}$: regular part of the limit space. Defined in (2.23).
- $\mathcal{S}$: singular part of the limit space. Defined in (2.24).
- v: volume density. Defined in (2.1).
- $\mathcal{V}$: Perelman's reduced volume. First appears at (2.7).
- **vr**: volume radius. Defined in Proposition 2.2.
- $\mathcal{Z}_r$: r-neighborhood of the points where $|Rm| > r^{-2}$. Defined in the paragraph before Proposition 4.30.
- $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$: bold symbol Greek letters mean space-time curves. First appears in the discussion before (2.4).
- α, β, γ: Greek letters mean space curves. First appears in the discussion before (2.4).

- $\square$: Heat operator $\partial_t - \Delta$. First appears in Lemma 3.2.
- $\square^*$: Conjugate heat operator $-\partial_t - \Delta + (R - n\lambda)$. First appears in Lemma 3.1.
- $\xrightarrow{\hat{C}^\infty}$: Convergence in smooth pointed-$\hat{C}^\infty$-Cheeger–Gromov topology. First appears in the discussion after Theorem 1.1. It is generalized in the discussion after Theorem 1.5. Similar notation $\xrightarrow{\hat{C}^k}$ is also defined there.
- $\xrightarrow{P.G.H.}$: Convergence in pointed Gromov–Hausdorff topology. First appears in the discussion after Theorem 1.1. Similar notation $\xrightarrow{G.H.}$ is also defined there.

2. Preliminary results

In this section, we collect important results from [**29**].

Definition 2.1. Let $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ be the collection of length spaces (X,g) with the following properties.

1) X has a disjoint regular-singular decomposition $X = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is the regular part, $\mathcal{S}$ is the singular part. A point is called regular if it has a neighborhood which is isometric to a totally geodesic convex domain of some smooth Riemannian manifold. A point is called singular if it is not regular.
2) The regular part $\mathcal{R}$ is a nonempty, open Ricci-flat manifold of real dimension $m = 2n$. Moreover, there exists a complex structure J on $\mathcal{R}$ such that $(\mathcal{R}, g, J)$ is a Kähler manifold.
3) $\mathcal{R}$ is weakly convex, i.e., for every point $x \in \mathcal{R}$, there exists a measure ($2n$-dimensional Hausdorff measure) zero set $\mathcal{C}_x \supset \mathcal{S}$ such that every point in $X \backslash \mathcal{C}_x$ can be connected to x by a unique shortest geodesic in $\mathcal{R}$. For convenience, we call $\mathcal{C}_x$ as the cut locus of x.
4) $\dim_{\mathcal{M}} \mathcal{S} < 2n - 3$, where $\mathcal{M}$ means Minkowski dimension.
5) Let v be the volume density function, i.e.,

$$\mathrm{v}(x) \triangleq \lim_{r \to 0} \frac{|B(x,r)|}{\omega_{2n} r^{2n}}, \tag{2.1}$$

   for every $x \in X$. Then $\mathrm{v} \equiv 1$ on $\mathcal{R}$ and $\mathrm{v} \leq 1 - 2\delta_0$ on $\mathcal{S}$. In other words, the function v is a criterion function for singularity. Here $\delta_0 = \delta_0(n)$ is the Anderson constant.
6) The asymptotic volume ratio $\mathrm{avr}(X) \geq \kappa$. In other words, we have

$$\mathrm{avr}(X) \triangleq \lim_{r \to \infty} \frac{|B(x,r)|}{\omega_{2n} r^{2n}} \geq \kappa, \tag{2.2}$$

   for every $x \in X$.

Let $\widetilde{\mathscr{K}\mathscr{S}}(n)$ be the collection of metric spaces (X,g) with all the above properties except the last one. Since Euclidean space is a special element, we define

$$\widetilde{\mathscr{K}\mathscr{S}}^*(n) \triangleq \widetilde{\mathscr{K}\mathscr{S}}(n)\backslash\{(\mathbb{C}^n, g_{\mathbb{E}})\},$$
$$\widetilde{\mathscr{K}\mathscr{S}}^*(n,\kappa) \triangleq \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)\backslash\{(\mathbb{C}^n, g_{\mathbb{E}})\}.$$

Note that $\mathscr{K}\mathscr{S}(n)$ is the class of all the complete n-dimensional Calabi–Yau (Kahler–Ricci-flat) manifolds, where the classical Cheeger–Colding theory works well. The space $\widetilde{\mathscr{K}\mathscr{S}}(n)$ is an extension of $\mathscr{K}\mathscr{S}(n)$ by including Calabi–Yau manifolds with mild singularities. In [**29**], we develop the structure theory of $\widetilde{\mathscr{K}\mathscr{S}}(n)$. In the study of $\widetilde{\mathscr{K}\mathscr{S}}(n)$, the volume ratio plays a key role. In particular, we have the following properties.

Proposition 2.2 (**Euclidean space by vr**)**.** *Suppose $X \in \widetilde{\mathscr{K}\mathscr{S}}(n)$ and $\mathbf{vr}(x_0) = \infty$ for some $x_0 \in X$, then X is isometric to the Euclidean space $\mathbb{C}^n$. Here $\mathbf{vr}(x_0)$ is the supreme of all radii r such that $\omega_{2n}^{-1} r^{-2n}|B(x_0,r)| \geq 1-\delta_0$.*

Proposition 2.3 (**Rigidity of volume ratio**)**.** *Let $X \in \widetilde{\mathscr{K}\mathscr{S}}(n)$. If for two concentric geodesic balls $B(x_0,r_1) \subset B(x_0,r_2)$ centered at a regular point x_0, we have*

$$\omega_{2n}^{-1} r_1^{-2n}|B(x_0,r_1)| = \omega_{2n}^{-1} r_2^{-2n}|B(x_0,r_2)|, \tag{2.3}$$

then the ball $B(x_0,r_2)$ is isometric to a geodesic ball of radius r_2 in $\mathbb{C}^n$. Furthermore, if $X \in \mathscr{K}\mathscr{S}(n)$, then we can further conclude that X is Euclidean.

A main result of [**29**] is the following compactness theorem.

Theorem 2.4 (**Compactness**, c.f. Theorem 1.1 of [**29**])**.** *$\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ is compact under the pointed-$\hat{C}^\infty$-Cheeger–Gromov topology.*

Moreover, the combination of the 6 defining properties of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ is sufficient to improve the regularity of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.

Theorem 2.5 (**Space regularity improvement**, c.f. Theorem 1.1 of [**29**])**.** *Suppose $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, then $\mathcal{R}$ is strongly convex, and $\dim_{\mathcal{M}} \mathcal{S} \leq 2n-4$. Suppose $x_0 \in \mathcal{S}$, Y is a tangent space of X at x_0. Then Y is a metric cone in $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ with the splitting*

$$Y = \mathbb{C}^{n-k} \times C(Z),$$

for some $k \geq 2$, where $C(Z)$ is a metric cone without lines.

Every space $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ can be regarded as a trivial Ricci flow solution. Therefore, Perelman's celebrated work [**49**] can find its role in the study of X. Let us briefly recall some fundamental functionals defined for the Ricci flow by Perelman.

Suppose $\{(X^m, g(t)), -T \leq t \leq 0\}$ is a Ricci flow solution on a smooth complete Riemannian manifold X of real dimension m. Suppose $x, y \in X$. Suppose $\boldsymbol{\gamma}$ is a space-time curve parameterized by $\tau = -t$ such that

$$\boldsymbol{\gamma}(0) = (x, 0), \quad \boldsymbol{\gamma}(\bar{\tau}) = (y, -\bar{\tau}).$$

Let γ be the space-projection curve of $\boldsymbol{\gamma}$. In other words, we have

$$\boldsymbol{\gamma}(\tau) = (\gamma(\tau), -\tau).$$

By the way, for the simplicity of notations, we always use bold symbol of a Greek character to denote a space-time curve. The corresponding space projection will be denoted by the normal Greek character. Following Perelman, the Lagrangian of the space-time curve $\boldsymbol{\gamma}$ is defined as

$$\mathcal{L}(\boldsymbol{\gamma}) = \int_0^{\bar{\tau}} \sqrt{\tau} \left(R + |\dot{\gamma}|^2\right)_{g(-\tau)} d\tau. \tag{2.4}$$

Among all such $\boldsymbol{\gamma}$'s that connected $(x, 0)$, $(y, -\bar{\tau})$ and parameterized by τ, there is at least one smooth curve $\boldsymbol{\alpha}$ which minimizes the Lagrangian. This curve is called a shortest reduced geodesic. The reduced distance between $(x, 0)$ and $(y, -\bar{\tau})$ is defined as

$$l((x,0),(y,-\bar{\tau})) = \frac{\mathcal{L}(\boldsymbol{\alpha})}{2\sqrt{\bar{\tau}}}. \tag{2.5}$$

Let $V = \dot{\alpha}$. Then V satisfies the equation

$$\nabla_V V + \frac{V}{2\tau} + 2Ric(V, \cdot) + \frac{\nabla R}{2} = 0, \tag{2.6}$$

which is called the reduced geodesic equation. It is easy to check that $\dot{\alpha} = V = \nabla l$. The reduced volume is defined as

$$\mathcal{V}((x,0),\bar{\tau}) = \int_X (4\pi\bar{\tau})^{-\frac{m}{2}} e^{-l} dv. \tag{2.7}$$

It is proved by Perelman that $(4\pi\tau)^{-\frac{m}{2}} e^{-l} dv$, the reduced volume element, is monotonically non-increasing along each reduced geodesic emanating from $(x, 0)$.

Suppose the Ricci flow solution mentioned above is static, i.e., $Ric \equiv 0$. Then it is easy to check that

$$\begin{cases} \mathcal{L}(\boldsymbol{\alpha}) = \frac{d^2(x,y)}{2\sqrt{\bar{\tau}}}, \\ l((x,0),(y,-\bar{\tau})) = \frac{d^2(x,y)}{4\bar{\tau}}, \\ \nabla_V V + \frac{V}{2\tau} = 0, \\ |\dot{\alpha}|^2 = |V|^2 = |\nabla l|^2 = \tau l, \\ \mathcal{V}((x,0),\bar{\tau}) = \int_X (4\pi\bar{\tau})^{-\frac{m}{2}} e^{-\frac{d^2}{4\bar{\tau}}} dv. \end{cases} \tag{2.8}$$

Now we assume $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. By a trivial extension in an extra time direction, we obtain a static, eternal singular Kähler Ricci flow solution. Since distance structure is already known, we can define reduced distance, reduced volume, etc, following equation (2.8). Clearly, this definition coincides with the original one when X is smooth. The following theorem is important to bridge the Cheeger–Colding's structure theory to the Ricci flow theory.

Theorem 2.6 (**Volume ratio and reduced volume**). *Suppose $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, $x \in X$. Let $X \times (-\infty, 0]$ have the obvious static space-time structure. Then we have*

$$\operatorname{avr}(X) = \lim_{\tau \to \infty} \mathcal{V}((x,0),\tau). \tag{2.9}$$

$$\mathrm{v}(x) = \lim_{\tau \to 0} \mathcal{V}((x,0),\tau). \tag{2.10}$$

The compactness of the moduli $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ also implies the following uniform estimates.

Theorem 2.7 (**A priori estimates in model spaces**, c.f. Theorem 1.5 of [**29**]). *Suppose $(X, x_0, g) \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, r is a positive number. Then the following estimates hold.*

1) *Strong volume ratio estimate: $\kappa \le \omega_{2n}^{-1} r^{-2n}|B(x_0,r)| \le 1$.*
2) *Strong regularity estimate: $r^{2+k}|\nabla^k Rm| \le c_a^{-2}$ in $B(x_0, c_a r)$ for every $0 \le k \le 5$ whenever $\mathbf{vr}(x_0) \ge r$.*
3) *Strong density estimate: $r^{2p_0-2n}\displaystyle\int_{B(x_0,r)} \mathbf{vr}(y)^{-2p_0} dy \le \mathbf{E}$.*
4) *Strong connectivity estimate: Every two points $y_1, y_2 \subset B(x_0,r) \cap \mathcal{F}_{\frac{1}{100}c_b r}(X)$ can be connected by a shortest geodesic γ such that $\gamma \subset \mathcal{F}_{\epsilon_b r}(X)$.*

Here, the constants c_a, c_b, ϵ_b all depend on κ and n, the constant p_0 depends only on n and it is very close to 2, say $p_0 = 2 - \frac{1}{1000n}$, the constant $\mathbf{E}$ depends on κ, n and p_0.

Note that p_0 can be chosen as arbitrarily close to 2. We set $p_0 = 2 - \frac{1}{1000n}$ here just for simplicity of notations. The set $\mathcal{F}_r(X)$ is the collection of points y satisfying $\mathbf{vr}(y) \ge r$. See (2.12) for precise definitions and keep in mind that $\mathbf{cr}(X) = \infty$. Among all the estimates listed in Theorem 2.7, we emphasize the following one, originally due to Cheeger–Naber [**14**] for smooth non-collapsing Einstein case.

Proposition 2.8 (**Density estimate of regular points**). *For every $0 < p < 2$, there is a constant $E = E(n,\kappa,p)$ with the following properties.*

Suppose $(X,x,g) \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, r is a positive number. Then we have

$$r^{2p-2n} \int_{B(x,r)} \mathbf{vr}(y)^{-2p} dy \leq E(n,\kappa,p). \tag{2.11}$$

Note that the estimates in Theorem 2.7 hold for each positive r. For a general smooth manifold M, we design a scale to describe how similar a manifold is to the model space.

Definition 2.9. We say that the canonical radius (with respect to model space $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$) of a point $x_0 \in M$ is not less than r_0 if for every $r < r_0$, we have the following properties.

1) Volume ratio estimate: $\kappa \leq \omega_{2n}^{-1} r^{-2n} |B(x_0,r)| \leq \kappa^{-1}$.
2) Regularity estimate: $r^{2+k}|\nabla^k Rm| \leq 4c_a^{-2}$ in the ball $B(x_0, \frac{1}{2}c_a r)$ for every $0 \leq k \leq 5$ whenever $\omega_{2n}^{-1} r^{-2n}|B(x_0,r)| \geq 1-\delta_0$.
3) Density estimate: $r^{2p_0-2n} \displaystyle\int_{B(x_0,r)} \mathbf{vr}^{(r)}(y)^{-2p_0} dy \leq 2\mathbf{E}$, where $p_0 = 2 - \frac{1}{1000n}$.
4) Connectivity estimate: $B(x_0,r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ is $\frac{1}{2}\epsilon_b r$-regularly connected on the scale r. Namely, every two points in $B(x_0,r) \cap \mathcal{F}^{(r)}_{\frac{1}{50}c_b r}(M)$ can be connected by a shortest geodesic $\gamma \subset \mathcal{F}^{(r)}_{\frac{1}{2}\epsilon_b r}(M)$.

Then we define canonical radius of x_0 to be the supreme of all the r_0 with the properties mentioned above. We denote the canonical radius by $\mathbf{cr}(x_0)$. For subset $\Omega \subset M$, we define the canonical radius of Ω as the infimum of all $\mathbf{cr}(x)$ where $x \in \Omega$. We denote this canonical radius by $\mathbf{cr}(\Omega)$.

According to Definition 2.9, the canonical radius of a Kähler–Ricci-flat smooth manifold with asymptotic volume ratio at least κ is ∞. More generally, we can also define the canonical radius on spaces with mild singularities. Then it is not hard to see that $\mathbf{cr}(x) = \infty$ for each $x \in X$ whenever $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.

For each $r \in (0, \mathbf{cr}(M))$, one can decompose the manifold M into r-regular and r-singular part as follows:

$$\mathcal{F}_r \triangleq \mathcal{F}_r^{(\mathbf{cr}(M))}, \quad \mathcal{D}_r \triangleq \{\mathcal{F}_r\}^c, \tag{2.12}$$

where we recall that the set $\mathcal{F}_r^{(\mathbf{cr}(M))}$ is

$$\left\{ x \in M \,\middle|\, \text{there exists } \rho \in (r, \mathbf{cr}(M)),\ \omega_{2n}^{-1}\rho^{-2n}|B(x,\rho)| \geq 1-\delta_0 \right\}.$$

Based on such decomposition, a rough weak compactness can be easily established whenever we have global uniform lower bound $\mathbf{cr} \geq 1$. Let us first list some properties needed to setup the weak compactness. Note that from Proposition 2.10 to Proposition 2.12, we have the common condition that $\mathbf{cr}(M) \geq 1$.

Proposition 2.10. *There is a constant $K = K(n,\kappa)$ with the following properties.*

For each $x \in M$, let $\mathbf{cvr}(x)$ be the supreme of all radius $\rho \in (0, \mathbf{cr}(x))$ such that

$$\omega_{2n}^{-1}\rho^{-2n}|B(x,\rho)| \geq 1-\delta_0.$$

If $r = \mathbf{cvr}(x) < \frac{1}{K}$, then for every $y \in B(x, K^{-1}r)$, we have

$$K^{-1}r \leq \mathbf{cvr}(y) \leq Kr, \tag{2.13}$$

$$\omega_{2n}^{-1}\rho^{-2n}|B(y,\rho)| \geq 1-\frac{1}{100}\delta_0, \quad \forall\, \rho \in (0, K^{-1}r), \tag{2.14}$$

$$|Rm|(y) \leq K^2 r^{-2}, \tag{2.15}$$

$$inj(y) \geq K^{-1}r. \tag{2.16}$$

Proposition 2.11. *For every $r \leq 1$, two points $x, y \in \mathcal{F}_r$ can be connected by a curve $\gamma \subset \mathcal{F}_{\frac{1}{2}\epsilon_b r}$ with length $|\gamma| < 3d(x,y)$.*

Proposition 2.12. *For every $0 < r \leq \rho_0 \leq 1$, $x_0 \in M$, we have*

$$|B(x_0,\rho_0) \cap \mathcal{D}_r| < 4\mathbf{E}\rho_0^{2n-2p_0} r^{2p_0}, \tag{2.17}$$

$$|B(x_0,\rho_0) \cap \mathcal{F}_r| > \left(\kappa\omega_{2n} - 4\mathbf{E}r^{2p_0}\rho_0^{-2p_0}\right)\rho_0^{2n}. \tag{2.18}$$

In particular, there exists at least one point $z \in B(x_0,\rho_0)$ such that

$$\mathbf{cvr}(z) > c_b\rho_0, \tag{2.19}$$

where $c_b = \left(\frac{\kappa\omega_{2n}}{4\mathbf{E}}\right)^{\frac{1}{2p_0}}$.

Then we can easily obtain a rough weak compactness theory. Suppose (M_i, g_i, J_i) is a sequence of Kähler manifolds satisfying $\mathbf{cr}(M_i) \geq r_0$. Let d_i be the length structure induced by g_i. It follows from ball-packing argument that

$$(M_i, x_i, d_i) \xrightarrow{P.G.H.} (\bar{M}, \bar{x}, \bar{d}), \tag{2.20}$$

for some length space $(\bar{M}, \bar{d})$, by taking subsequence if necessary. For each $r < r_0$, define

$$\mathcal{R}_r \triangleq \left\{ \bar{y} \in \bar{M} \,\middle|\, \begin{array}{l} \text{There exists } y_i \in M_i \text{ such that} \\ y_i \to \bar{y} \text{ and } \liminf_{i\to\infty} \mathbf{cvr}(y_i) \geq r \end{array} \right\}, \tag{2.21}$$

$$\mathcal{S}_r \triangleq (\mathcal{R}_r)^c, \tag{2.22}$$

$$\mathcal{R} \triangleq \bigcup_{0<r\leq r_0} \mathcal{R}_r, \tag{2.23}$$

$$\mathcal{S} \triangleq \bigcap_{0<r\leq r_0} \mathcal{S}_r. \tag{2.24}$$

Theorem 2.13 (**Rough weak compactness**). *Suppose* $\mathbf{cr}(M_i) \geq r_0 > 0$ *uniformly and (2.20) holds. Then we have the regular-singular decomposition* $\bar{M} = \mathcal{R} \cup \mathcal{S}$ *with the following properties.*

- *The regular part* $\mathcal{R}$ *is an open, path connected* C^4*-Riemannian manifold. Furthermore, for every two points* $x, y \in \mathcal{R}$*, there exists a curve* γ *connecting* x, y *satisfying*

$$\gamma \subset \mathcal{R}, \quad |\gamma| \leq 3d(x,y). \tag{2.25}$$

- *The singular part* $\mathcal{S}$ *satisfies the Minkowski dimension estimate*

$$\dim_{\mathcal{M}} \mathcal{S} \leq 2n - 2p_0. \tag{2.26}$$

Proposition 2.14 (**Volume convergence**). *The volume (Hausdorff measure of dimension 2n) is continuous under the convergence (2.20), i.e., for every fixed* $\rho_0 > 0$*, we have*

$$|B(\bar{x}, \rho_0)| = \lim_{i \to \infty} |B(x_i, \rho_0)|.$$

Note that the convergence regularity was improved on $\mathcal{R}(\bar{M})$. Therefore, by abuse of notation, we now improve the convergence (2.20) as

$$(M_i, x_i, g_i) \xrightarrow{\hat{C}^4} (\bar{M}, \bar{x}, \bar{g}), \tag{2.27}$$

which means (2.20) together with the extra information that the convergence on $\mathcal{R}(\bar{M})$ happens in C^4-topology modulo diffeomorphisms. It is important to note that **the length structure of $\bar{d}$ is not necessarily equivalent to the length structure induced by $\bar{g}$**. Instead, only a rough equivalence (2.25) is known. However, they will be equivalent when we know that $\mathcal{R}$ is weakly geodesic convex. To show the weak convexity of $\mathcal{R}$, one needs other conditions for (M_i, g_i) such like the Ricci flow condition. The furthermore improvement of this type will be discussed in the next section.

3. Polarized canonical radius(pcr)

In this section, we shall improve the regularity of the limit pace $\bar{M}$ in Theorem 2.13, under the help of Kähler geometry and the Ricci flow. The Ricci flow has reduced volume and local W-functional monotonicity, discovered by Perelman. These monotonicities will be used to show that each tangent space is a metric cone, and the regular part $\mathcal{R}$ is weakly convex, under some natural geometric conditions. However, the weak-compactness we developed in last section only deals with the metric structure. On $\bar{M}$, we cannot see a Ricci flow structure. In order to make use of the intrinsic monotonicity of the Ricci flow, we need a weak compactness of Ricci flows, not just the weak compactness of time slices. However, along the Ricci flow, the metric at different time slices cannot be compared obviously if no estimate of Ricci curvature is known. This is one of the fundamental difficulty to develop the weak compactness theory of the Ricci flows. We overcome this difficulty by taking advantage of the rigidity of Kähler geometry.

3.1. A rough long-time pseudolocality theorem. Suppose $\mathcal{LM} = \{(M^n, g(t), J, L, h(t)), t \in I \subset \mathbb{R}\}$ is a polarized Kähler Ricci flow. Let $\mathbf{b}$ be the Bergman function with respect to $\omega(t)$ and $h(t)$, i.e.,

$$\mathbf{b}(x,t) = \log \sum_{k=0}^{N} \|S_k\|^2_{h(t)}, \tag{3.1}$$

where $N = \dim(H^0(L)) - 1$, $\{S_k\}_{k=0}^N$ are orthonormal holomorphic sections of L in the sense that

$$\int_M \langle S_k, S_l \rangle \omega(t)^n = \delta_{kl}.$$

By pulling back the Fubini–Study metric through the natural holomorphic embedding, we have

$$\tilde{\omega} = \iota^*(\omega_{FS}) = \omega + \sqrt{-1}\partial\bar{\partial}\mathbf{b}.$$

Let $\omega_0 = \omega(0), \mathbf{b}_0 = \mathbf{b}(0)$. Then

$$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi, \quad \tilde{\omega}(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}(\varphi + \mathbf{b}).$$

Clearly, $\varphi(0) = 0$.

In this section, we focus on polarized Kähler Ricci flow $\mathcal{LM}$ satisfying the following estimate

$$\|\dot{\varphi}\|_{C^0(M)} + \|\mathbf{b}\|_{C^0(M)} + \|R\|_{C^0(M)} + |\lambda| + C_S(M) \le B, \tag{3.2}$$

for every time $t \in I$. Let $\mathbf{b}^{(k)}$ be the Bergman function of the line bundle L^k with the naturally induced metric. Then a standard argument implies that

$$\|\dot{\varphi}\|_{C^0(M)} + \left\|\mathbf{b}^{(k)}\right\|_{C^0(M)} + \|R\|_{C^0(M)} + |\lambda| + C_S(M) \le B^{(k)}, \tag{3.3}$$

for a constant $B^{(k)}$ depending on B and k. Define

$$\begin{aligned}
\tilde{\omega}^{(k)} &\triangleq \frac{1}{k}\left(\iota^{(k)}\right)^*(\omega_{FS}),\\
F^{(k)} &\triangleq \Lambda_{\omega_t}\tilde{\omega}_0^{(k)} = n - \Delta\left(\varphi - \mathbf{b}_0^{(k)}\right).
\end{aligned}$$

In this section, the existence time of the polarized Kähler Ricci flow is always infinity, i.e., $I = [0, \infty)$.

Lemma 3.1 (**Integral bound of trace**). *Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2). Suppose u is a positive, backward heat equation solution, i.e.,*

$$\Box^* u = (-\partial_t - \Delta + R - n\lambda)u = 0,$$

and $\int_M udv \equiv 1$. *Then for every* $t_0 > 0$, *we have*

$$\int_0^{t_0} \int_M F^{(k)} udvdt \leq \left(n + 2B^{(k)}\right) t_0 + 2B^{(k)}. \tag{3.4}$$

Proof. For simplicity of notation, we only give a proof of the case $k = 1$ and denote $F = F^{(1)}$. Note that $\mathbf{b} = \mathbf{b}^{(1)}, B = B^{(1)}$. The proof of general k follows verbatim.

Direct calculation shows that

$$\begin{aligned}
\int_0^{t_0} \int_M F udv &= nt_0 - \int_0^{t_0} \int_M \{\Delta(\varphi - \mathbf{b}_0)\} udv \\
&= nt_0 - \int_0^{t_0} \int_M (\varphi - \mathbf{b}_0)(\Delta u) dv \\
&= nt_0 + \int_0^{t_0} \int_M (\varphi - \mathbf{b}_0) \left(\dot{u} - Ru + \lambda nu\right) dv \\
&= nt_0 + \int_0^{t_0} \left[\frac{d}{dt} \left(\int_M (\varphi - \mathbf{b}_0) udv \right) - \int_M \dot{\varphi} udv \right] dt \\
&= nt_0 + \int_M (\varphi - \mathbf{b}_0) udv \Big|_0^{t_0} - \int_0^{t_0} \int_M \dot{\varphi} udvdt.
\end{aligned}$$

Note $|\varphi| \leq Bt_0$ at time t_0, then (3.4) follows from the above inequality and (3.3). q.e.d.

We shall proceed to improve the integral estimate (3.4) of $F^{(k)}$ to point-wise estimate, under local geometry bounds. Before we go into details, let us first fix some notations. Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow solution satisfying (3.2), $x_0 \in M$. In this subsection, we shall always assume

$$\left\{ \begin{array}{l} \Omega \triangleq B_{g(0)}(x_0, r_0), \\ \Omega' \triangleq B_{g(0)}(x_0, (1-\delta) r_0), \\ \Omega'' \triangleq B_{g(0)}(x_0, (1-2\delta) r_0). \end{array} \right. \tag{3.5}$$

Then we define

$$w_0 \triangleq \phi \left(\frac{2(d - 1 + 2\delta)}{\delta} \right), \tag{3.6}$$

where $d = d_{g(0)}(x_0, \cdot)$, ϕ is a cutoff function, which equals one on $(-\infty, 1]$, decreases to 0 on $(1, 2)$. Moreover, $(\phi')^2 \leq 10\phi$. Note that such ϕ exists by considering the behavior of $e^{-\frac{1}{s}}$ around $s = 0$. Clearly, w_0 satisfies

$$\left\{ \begin{array}{lll} |\nabla w_0|^2 \leq \frac{40}{\delta^2} w_0, & & \\ w_0 \equiv 1, & \text{on} & \Omega'', \\ w_0 \equiv 0, & \text{on} & (\Omega')^c. \end{array} \right. \tag{3.7}$$

Lemma 3.2 (Pointwise bound of trace). *Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$, Ω' is defined by (3.5). Suppose $\frac{1}{2}\omega_0 \leq \tilde{\omega}_0^{(k)} \leq 2\omega_0$ on Ω'. Let w be a solution of heat equation $\square w = \left(\frac{\partial}{\partial t} - \Delta\right) w = 0$, initiating from a cutoff function w_0 satisfying (3.7). Then for every $t_0 > 0$ and $y_0 \in M$, we have*

$$F^{(k)}(y_0, t_0) w(y_0, t_0) \leq C, \tag{3.8}$$

where $C = C(B, k, \delta, t_0)$.

Proof. For simplicity of notation, we only give a proof for the case $k = 1$ and denote $F = F^{(1)}$, $B = B^{(1)}$, $H = \frac{40}{\delta^2}$. The proof of general k follows verbatim.

Note that $0 \leq w_0 \leq 1$, since w is the heat solution, it follows from maximum principle that $0 \leq w \leq 1$. On the other hand, according to the choice of w_0, we have $|\nabla w|^2 - Hw \leq 0$ at the initial time. Direct calculation implies that

$$\square \left\{ e^{\lambda t} \left(|\nabla w|^2 - Hw \right) \right\} = -e^{\lambda t} \left\{ |\nabla \nabla w|^2 + |\nabla \bar{\nabla} w|^2 + Hw \right\} \leq 0.$$

Therefore, $|\nabla w|^2 - Hw \leq 0$ is preserved along the flow by maximum principle. In other words, we always have

$$w|\nabla \log w|^2 \leq H, \quad 0 \leq w \leq 1,$$

on the space-time $M \times [0, \infty)$. In light of parabolic Schwarz lemma (c.f. [**58**] and references therein), we obtain

$$\square \log F \leq BF - \lambda.$$

Note that

$$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi = \tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}(\varphi - \mathbf{b}_0) = \tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi},$$

where we denote $\varphi - \mathbf{b}_0$ by $\tilde{\varphi}$ for simplicity of notation. It is obvious that $\dot{\tilde{\varphi}} = \dot{\varphi}$. Direct calculation shows that

$$\begin{aligned}
&\square\tilde{\varphi} = \dot{\varphi} - \Delta\tilde{\varphi} = F - n + \dot{\varphi},\\
&\square(\log F - B\tilde{\varphi}) \leq B(n - \dot{\varphi}) - \lambda \leq B(n + \|\dot{\varphi}\|_{C^0(\mathcal{M})}) + |\lambda| \leq C.
\end{aligned}$$

Let u be the solution of $\square^* u = 0$, starting from a δ-function from (y_0, t_0). Then we calculate

$$\begin{aligned}
&\frac{d}{dt}\int_M Fe^{-B\tilde{\varphi}} wudv\\
&= \int_M \square(Fe^{-B\tilde{\varphi}} w)udv - \int_M Fe^{-B\tilde{\varphi}} w\square^* udv\\
&= \int_M \square(Fe^{-B\tilde{\varphi}} w)udv\\
&= \int_M Fe^{-B\tilde{\varphi}} w \left\{ \square \log(Fe^{-B\tilde{\varphi}} w) - |\nabla \log(Fe^{-B\tilde{\varphi}} w)|^2 \right\} udv
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_M Fe^{-B\tilde{\varphi}} w \left\{\Box \log(Fe^{-B\tilde{\varphi}} w)\right\} udv \\
&= \int_M Fe^{-B\tilde{\varphi}} w \left\{\Box \log(Fe^{-B\tilde{\varphi}}) + \Box \log w\right\} udv \\
&= \int_M Fe^{-B\tilde{\varphi}} w \left\{\Box \log(Fe^{-B\tilde{\varphi}}) + |\nabla \log w|^2\right\} udv \\
&\leq C \int_M Fe^{-B\tilde{\varphi}} wudv + H \int_M Fe^{-B\tilde{\varphi}} udv.
\end{aligned}
$$

It follows that

$$
\frac{d}{dt}\left\{e^{-Ct}\int_M Fe^{-B\tilde{\varphi}} wudv\right\} \leq He^{-Ct}\int_M Fudv.
$$

Integrating the above inequality and applying Lemma 3.1, we have

$$
\begin{aligned}
& e^{-Ct_0} F(y_0, t_0) w(y_0, t_0) e^{-B\tilde{\varphi}(y_0, t_0)} \\
\leq & \int_M Fe^{-B\tilde{\varphi}} wudv\bigg|_{t=0} + H\int_0^{t_0}\int_M Fudvdt \\
\leq & C\int_{\Omega'} Fudv\bigg|_{t=0} + C \\
\leq & C\int_{\Omega'} udv\bigg|_{t=0} + C \leq C.
\end{aligned}
$$

Therefore, (3.8) follows directly from the above inequality. q.e.d.

Lemma 3.3 (**Lower bound of heat solution**). *Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$, notations fixed by (3.5) and (3.6).*

Suppose $\Omega'' \subset B_{g(t)}(x_0, r)$ for some $t > 0$ and $r > 0$. Then in the geodesic ball $B_{g(t)}(x_0, r)$, we have

$$
w(y, t) > c,
$$

for some constant $c = c(n, B, k, \delta, r_0, r, t)$.

Proof. By the construction of w_0 and maximum principle, it is clear that $0 < w \leq 1$ when $t > 0$. Let P be the heat kernel function, then we can write

(3.9)

$$
w(x_0, t) = \int_M P(x_0, t; y, 0) w_0(y) dv_y \geq \int_{\Omega''} P(x_0, t; y, 0) w_0(y) dv_y.
$$

In light of the Sobolev constant bound and scalar curvature bound, one has the on-diagonal bound

$$
\frac{1}{C} t^{-n} \leq P(x, t; x, 0) \leq Ct^{-n},
$$

which combined with the gradient estimate of heat equation (c.f. Theorem 3.3 of [**77**]) implies that

$$P(x,t;y,0) \geq \frac{1}{C} t^{-n},$$

where $C = C(B, d_{g(t)}(x,y))$. Plugging this estimate into (3.9) implies that

$$w(x_0,t) \geq \frac{|\Omega''|}{Ct^n}.$$

Note that C_S bound forces $|\Omega''|$ is bounded from below. Since $0 < w \leq 1$, then (3.9) follows from the above inequality and the gradient estimate of heat equation. q.e.d.

The following two lemmas show that Kähler geometry is much more rigid than Riemannian geometry.

Lemma 3.4 (**Fubini–Study approximation**). *Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$, notations fixed by (3.5) and (3.6).*

Suppose $|Rm| \leq r_0^{-2}$ in Ω at time $t = 0$. Then there exists an integer $k = k(B, r_0, \delta)$ such that

$$\frac{1}{2}\omega_0 \leq \tilde{\omega}_0^{(k)} \leq 2\omega_0 \tag{3.10}$$

on Ω'.

Proof. This follows essentially from the peak section method of Tian. We give a proof here for the convenience of the readers. Further details can be found in [**63**] and [**43**].

Fix arbitrary $x \in \Omega'$, $V \in T_x^{(1,0)}M$ with unit norm. In order to prove (3.10), it suffices to show that

$$\frac{1}{2} \leq \tilde{\omega}_0^{(k)}(V, JV) \leq 2. \tag{3.11}$$

Around x, we can always choose a normal coordinate (K-coordinate, c.f. [**43**]) chart around x such that

$$V = \frac{\partial}{\partial z_1}, \quad g_{i\bar{j}}(x) = \delta_{i\bar{j}}, \quad \frac{\partial^{p_1+p_2+\cdots+p_n}}{\partial z_1^{p_1}\partial z_2^{p_2}\cdots\partial z_n^{p_n}} g_{i\bar{j}}(x) = 0,$$

for any nonnegative integers $p_1, p_2, \cdots, p_n$ with $p = p_1+p_2+p_3+\cdots p_n > 0$. Moreover, there exists a local holomorphic frame e_L of L around x such that the local representation a of the Hermitian metric h has the properties

$$a(x) = 1, \quad \frac{\partial^{p_1+p_2+\cdots+p_n}}{\partial z_1^{p_1}\partial z_2^{p_2}\cdots\partial z_n^{p_n}} a(x) = 0,$$

for any nonnegative integers $p_1, p_2, \cdots, p_n$ with $p = p_1+p_2+p_3+\cdots p_n > 0$.

Suppose $\{S_0^k, \cdots S_{N_k}^k\}$ is an orthonormal basis of $H^0(M, L^k)$, where $N_k = \dim_{\mathbb{C}} H^0(M, L^k) - 1$. Around x, we can write

$$S_0^k = f_0^k e_L, \cdots, S_{N_k}^k = f_{N_k}^k e_L.$$

Rotating basis if necessary (c.f. [63]), we can assume

$$\begin{aligned} &f_i^k(x) = 0, \quad \forall\, i \geq 1, \\ &\frac{\partial f_i^k}{\partial z_j}(x) = 0, \quad \forall\, i \geq j+1. \end{aligned}$$

Recall that

$$\tilde{\omega}^{(k)} = \omega_0 + \frac{1}{k}\sqrt{-1}\partial\bar{\partial}\log\sum_{j=0}^{N_k}\left\|S_j^k\right\|^2 = \frac{1}{k}\sqrt{-1}\partial\bar{\partial}\log\sum_{j=0}^{N_k}|f_j^k|^2.$$

So we have

$$\tilde{\omega}^{(k)}(V, JV) = \frac{1}{k}\frac{\partial^2\log\sum_{j=0}^{N_k}|f_j^k|^2}{\partial z_1\bar{\partial} z_1} = \frac{|\frac{\partial f_1^k}{\partial z_1}|^2}{k|f_0^k|^2}. \tag{3.12}$$

Because of (3.11) and (3.12), the problem boils down to a precise estimate of $\frac{\partial f_1^k}{\partial z_1}$ and f_0^k.

As pointed out by Tian in [63], the peak section method is local in nature. The global information of the underlying manifold is only used in the step of Hörmander's estimate. However, in our case, we have

$$Ric(h) = g, \quad \sqrt{-1}\partial\bar{\partial}\dot{\varphi} + Ric = \lambda g, \quad \lambda > 0.$$

It then follows that

$$Ric(h) + Ric(g) + \sqrt{-1}\partial\bar{\partial}\dot{\varphi} = (1+\lambda)g \geq g.$$

Therefore, Hörmander's L^2-estimate (c.f. Proposition 5.1. of [62] or Proposition 3.1. of [31]) applies and we have

$$\int_M |u|^2 e^{-\dot{\varphi}}\omega^n \leq \int_M |\bar{\partial} u|^2 e^{-\dot{\varphi}}\omega^n.$$

By the uniform bound of $\dot{\varphi}$, we can replace the $e^{-\dot{\varphi}}\omega^n$ by ω^n in the above inequality, up to adjusting a multiplicative constant. Due to the uniformly bounded geometry (up to C^2-norm of g) inside Ω' and the uniform bound of $\sqrt{-1}\partial\bar{\partial}\dot{\varphi} + Ric$ on the whole manifold M, Lemma 1.2 of [63] follows directly and can be written as follows.

For an n-tuple of integers $(p_1, p_2, \cdots, p_n) \in \mathbb{Z}_+^n$ and an integer $p' > p = p_1 + p_2 + \cdots + p_n$, there exists a $k_0 = k_0(n, B, r_0, \delta)$ such that for $k > k_0$, there is a unit norm holomorphic section $S \in H^0(M, L^k)$ satisfying

$$\int_{M\setminus\{|z|^2 \leq \frac{(\log k)^2}{k}\}} \|S\|^2 dv \leq \frac{1}{k^{2p'}}.$$

Then the same argument as in [63] implies that (c.f. Lemma 3.2 of [63])

$$\left|f_0^k(x) - \sqrt{\frac{(n+k)!}{k!}}\left\{1+\frac{1}{2(k+n+1)!}(R(x)-n^2-n)\right\}\right| < \frac{C}{k^2}, \tag{3.13}$$

$$\left|\frac{\partial f_1^k}{\partial z_1}(x) - \sqrt{\frac{(n+k+1)!}{k!}}\left\{1+\frac{1}{2(k+n+1)}(R(x)-n^2-3n-2)\right\}\right| < \frac{C}{k^2}, \tag{3.14}$$

for some $C = C(n, B, r_0, \delta)$. Here R is the complex scalar curvature. Plugging the above estimate into (3.12), we obtain (3.11), whenever k is larger than a big constant, which depends only on n, B, r_0, δ. q.e.d.

Lemma 3.5 (**Liouville type theorem**). *Every complete Kähler Ricci flat metric $\tilde{g}$ on $\mathbb{C}^n$ must be an Euclidean metric if there is a constant C such that*

$$\frac{1}{C}\delta_{i\bar{j}}(z) \le \tilde{g}_{i\bar{j}}(z) \le C\delta_{i\bar{j}}(z), \quad \forall\, z \in \mathbb{C}^n. \tag{3.15}$$

Proof. The original proof of this lemma goes back to the famous paper of E. Calabi [3] and Pogorelov [52] on real Monge Ampère equation. For complex Monge Ampère equation, this is initially due to Riebesehl–Schulz [53] where higher derivatives are used heavily. We say a few words here for the convenience of the readers, using the Schauder estimate of Evans–Krylov.

Actually, it is not difficult to see that the problem boils down to the study of a global pluri-subharmonic function u in $\mathbb{C}^n$ such that

$$\begin{cases} \det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = 1, \\ C^{-1}(\delta_{i\bar{j}}) < \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) < C(\delta_{i\bar{j}}). \end{cases} \tag{3.16}$$

In order to show the metric $\tilde{g}$ is Euclidean, it suffices to show that u is a global quadratic polynomial. Without loss of generality, we may assume that $u(0) = Du(0) = 0$. For every positive integer k, we can define a function $u^{(k)}$ in the unit ball by

$$u^{(k)}(z) = \frac{u(kz)}{k^2}.$$

Clearly, $u^{(k)}$ satisfies (3.16). Note that $\left\|u^{(k)}\right\|_{C^2}$ is uniformly bounded, in the unit ball $B(0,1)$. By standard Evans–Krylov theorem, there exists a uniform constant C such that

$$[D^2 u^{(k)}]_{C^\alpha(B(0,\frac{1}{2}))} \leq C,$$

for every k. Putting back the scaling factor, the above inequality is equivalent to

$$[D^2 u]_{C^\alpha(B(0,\frac{k}{2}))} \leq C k^{-\alpha}, \qquad \forall \quad k = 1, 2, \cdots .$$

Let $k \to \infty$, we have $[D^2 u]_{C^\alpha(\mathbb{C}^n)} = 0$. Therefore, $D^2 u$ is a constant matrix, u is a quadratic polynomial. So we finish the proof. q.e.d.

Proposition 3.6 (**Ball containing relationship implies regularity improvement**). *Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$, notations fixed by (3.5) and (3.6). Suppose $|Rm| \leq r_0^{-2}$ in Ω at time $t = 0$. Moreover, we assume*

$$\Omega'' \subset B_{g(t)}(x_0, r) \subset \Omega', \tag{3.17}$$

for every $0 \leq t \leq t_0$. Then the following estimates hold.

- *In the geodesic ball $B_{g(t)}(x_0, r)$, we have*

$$\frac{1}{C}\omega_0 \leq \omega_t \leq C\omega_0, \tag{3.18}$$

  for some constant $C = C(n, B, k, \delta, r_0, r, t_0)$.
- *In the geodesic ball $B_{g(t)}(x_0, r - \xi)$, we have*

$$|Rm|(x,t)\xi^2 \leq C, \tag{3.19}$$

  for each small ξ and some constant $C = C(n, B, k, \delta, r_0, r, t_0)$.

Proof. Note that Perelman's strong version of pseudolocality theorem, i.e., Theorem 10.3 of [**49**], can be modified and applied here. In fact, the almost Euclidean volume ratio condition in that theorem can be replaced by κ-noncollapsing condition. Since one has injectivity radius estimate when curvature and volume ratio bounds are available, thanks to the work of Cheeger, Gromov and Taylor, in [**13**]. By shrinking the ball to some fixed smaller size, one can get back the condition of almost Euclidean volume ratio. Up to a covering argument, we can apply this strong version pseudolocality theorem to show that $|Rm|$ is uniformly bounded on $\Omega' \times [0, \eta]$ for some positive $\eta = \eta(n, \kappa, \delta)$. Then (3.18) and (3.19) follows trivially. For this reason, we can assume $t_0 > \eta$.

We first prove estimate (3.18). Due to Fubini–Study metrics' approximation, Lemma 3.4, it is clear that one can regard ω_0 and $\tilde{\omega}_0^{(k)}$ as the same metric on Ω'. Therefore, it follows from the combination of Lemma 3.2 and Lemma 3.3 that $F^{(k)}$ is bounded from above, which implies $\Lambda_{\omega_t}\omega_0 \leq C$. Recall that the volume element ω_0^n and ω_t^n are uniformly equivalent, due to the uniform bound of $|R|+|\lambda|$ and the evolution equation

$$\frac{\partial}{\partial t}\log \omega_t^n = n\lambda - R.$$

Consequently, (3.18) follows. We remind the readers that condition (3.17) is used in the above discussion.

We proceed to prove inequality (3.19). Fix L very large. If (3.19) does not hold uniformly, then we can find some space-time point (y_0, s_0) such that $y_0 \in B_{g(s_0)}(x_0, r-\xi)$ and $Q_0 \triangleq |Rm|(y_0, s_0) > 100L^2\xi^{-2}$ is very large. Set $\rho_0 \triangleq d_{g(s_0)}(y_0, x_0)$. On one hand, $\rho_0 < r - \xi$ by the choice of (y_0, s_0). On the other hand, $s_0 > \eta$ for some uniform η due to the application of Perelman's pseudo-locality, as discussed above. Search whether there is a point (x,t) satisfying

$$|Rm|(x,t) > 4Q_0, \quad x \in B_{g(t)}\left(x_0, \rho_0 + LQ_0^{-\frac{1}{2}}\right), \; t \in \left[t_0 - Q_0^{-1}, t_0\right].$$

If there exists such a point, we denote it by (y_1, s_1) and continue the above searching. Inductively, we can find (y_k, s_k). In fact, if (y_{k-1}, s_{k-1}) is defined, then we shall denote $|Rm|(y_{k-1}, s_{k-1})$ by Q_{k-1}, and denote $d_{g(s_{k-1})}(x_0, y_{k-1})$ by ρ_{k-1} and search point (x,t) satisfying

$$\begin{aligned} |Rm|(x,t) > 4Q_{k-1}, \quad & x \in B_{g(t)}\left(x_0, \rho_{k-1} + LQ_{k-1}^{-\frac{1}{2}}\right), \\ & t \in \left[t_{k-1} - Q_{k-1}^{-1}, t_{k-1}\right]. \end{aligned}$$

If there is no such point, we stop the process. Otherwise, we denote such a point by (y_k, s_k) and continue the process. Clearly, we have

$$\begin{aligned} & Q_k = 4^k Q_0 > 100L^2\xi^{-2}, \\ & \rho_k \leq \rho_0 + L\left(Q_0^{-\frac{1}{2}} + \cdots Q_{k-1}^{-\frac{1}{2}}\right) < \rho_0 + 4LQ_0^{-\frac{1}{2}} < r - 0.5\xi, \\ & |s_0 - s_k| = s_0 - s_k \leq Q_0^{-1} + Q_1^{-1} + \cdots Q_{k-1}^{-1} < 2Q_0^{-1} < \frac{\xi^2}{50L^2} << \eta. \end{aligned}$$

Since the process happens in a compact space-time domain with bounded geometry, it must stop after finite steps. Let k be the last (y_k, s_k). We denote it by (y,s) and set $Q = |Rm|(y,s)$ and $\rho = d_{g(s)}(y, x_0)$. Then we have

$$\left\{ \begin{aligned} & Q > 100L^2\xi^{-2}, \\ & \rho < r - 0.5\xi, \\ & s > 0.5\eta, \\ & |Rm|(x,t) < 4Q, \; \forall\, x \in B_{g(t)}(x_0, \rho + LQ^{-\frac{1}{2}}), \; t \in \left[s - Q^{-1}, s\right]. \end{aligned} \right. \tag{3.20}$$

By its choice, we have $d_{g(s)}(x_0, y) = \rho$. We observe that y will stay in $B_{g(t)}(x_0, \rho + 2Q^{-\frac{1}{2}})$ whenever $t \in [s - \frac{1}{5nQ}, s]$. This is an application of section 17 of Hamilton [38], or Lemma 8.3 of Perelman [49]. Actually, let θ_0 be the largest positive number such that y fails to locate in $B_{g(t-\theta_0 Q^{-1})}(x_0, \rho + 2Q^{-\frac{1}{2}})$. Then for each $t \in [s - \theta_0 Q^{-1}, s]$, triangle inequality implies that $B_{g(t)}(y, Q^{-\frac{1}{2}}) \subset B_{g(t)}(x_0, \rho + 3Q^{-\frac{1}{2}})$. Consequently, we have

$$
\begin{aligned}
|Rm|(x,t) &\leq 4Q, \quad \forall\, x \in B_{g(t)}(y, Q^{-\frac{1}{2}}),\\
|Rm|(x,t) &\leq 4Q, \quad \forall\, x \in B_{g(t)}(x_0, Q^{-\frac{1}{2}}).
\end{aligned}
$$

It follows from Lemma 8.3 (b) of Perelman [49] that

$$
\begin{aligned}
&\frac{d}{dt} d(x_0, y) \geq -10nQ^{\frac{1}{2}},\\
\Rightarrow\ & d_{g(s)}(x_0, y) - d_{g(s-\theta_0 Q^{-1})}(x_0, y) \geq -10nQ^{\frac{1}{2}} \cdot \theta_0 Q^{-1}.
\end{aligned}
$$

According to the choice of θ_0, the left hand side of the second inequality is $-2Q^{-\frac{1}{2}}$. It follows that $\theta_0 \geq \frac{1}{5n}$. Now we know that y stays in $B_{g(t)}(x_0, \rho + 2Q^{-\frac{1}{2}})$ for each $t \in [s - \frac{1}{5nQ}, s]$. In view of (3.20) and the fact $L >> 1$, the triangle inequality implies that

$$
|Rm|(x,t) < 4Q, \quad \forall x \in B_{g(t)}(y, 0.5LQ^{-\frac{1}{2}}), \quad t \in \left[s - \frac{1}{5nQ}, s\right].
$$

Let $\tilde{g}(t) = Qg(Q^{-1}t + s)$. We have

$$
\begin{cases}
|\widetilde{Rm}|(y, 0) = 1,\\
|\widetilde{Rm}|(x,t) < 4, \quad \forall x \in B_{\tilde{g}(t)}(y, 0.5L), \quad t \in \left[-\frac{1}{5n}, 0\right].
\end{cases}
$$

Note that $\left[-\frac{1}{5n}, 0\right]$ is a fixed time period. The application of Perelman's pseudo-locality guarantees the existence of such a time period (c.f. (3.20)). Now let $L \to \infty$, we can use the compactness theorem of Hamilton [39] to obtain a limit Ricci flow solution, which is non-flat, Kähler Ricci-flat and non-collapsed on all scales. We remark that the discussion above is nothing but repeating the argument of Claim 1 and Claim 2 in the proof of Perelman's pseudo-locality theorem, i.e, Theorem 10.1 in [49]. Similar argument was also used in the distance estimate of the work of Tian and the second named author [67].

Notice that

$$
\begin{aligned}
B_{g(s)}(y, 0.5LQ^{-\frac{1}{2}}) &\subset B_{g(s)}(x_0, r - 0.5\xi)\\
&\subset B_{g(s)}(x_0, r) \subset \Omega' = B_{g(0)}(x_0, 1-\delta).
\end{aligned}
$$

Therefore, by the same scale blowup at $(y, 0)$, we obtain nothing but $\mathbb{C}^n$. Recall we have (3.18), so we obtain a nontrivial Kähler Ricci flat metric $\tilde{g}_{i\bar{j}}$ on $\mathbb{C}^n$ such that (3.15) holds for some C. This contradicts Lemma 3.5. q.e.d.

The rough estimate (3.18) and (3.19) can be improved when $|R|+|\lambda|$ is very small. When curvature tensor is bounded in the space-time, one can estimate the Ricci curvature in terms of scalar curvature. Let $|R|+|\lambda|$ tend to zero, we see that the Ricci curvature tends to zero at the space-time where $|Rm|$ is bounded. By adjusting ξ if necessary, we obtain that in the limit, $B_{g(t)}(x_0,(1-\xi)r)$ is isometric to $B_{g(0)}(x_0,(1-\xi)r)$ for every $0<t<t_0$. By adjusting ξ and applying Perelman's pseudolocality theorem, we see the convergence at time $t=t_0$ is also smooth since curvature derivatives are all bounded in the ball $B_{g(t_0)}(x_0,(1-\xi)r)$ at time t_0. Further details will appear in Proposition 3.7 and Theorem 3.8.

Proposition 3.7 (**Volume element derivative small implies ball containing relationship**). *For every r_0, T and small ξ, there exists an ϵ with the following property.*

Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$, notations fixed by (3.5) and (3.6). Suppose $|Rm| \leq r_0^{-2}$ in Ω at time $t=0$. If $\sup_{\mathcal{M}}(|R|+|\lambda|)<\epsilon$, then for every $t\in[0,T]$ we have

$$\Omega'' \subset B_{g(t)}\left(x_0, \left(1-\frac{3}{2}\delta\right) r_0\right) \subset \Omega',$$
$$(1-\xi)\,\omega(0) \leq \omega(t) \leq (1+\xi)\,\omega(0), \quad \textit{in } \Omega''.$$

Proof. If the statement was wrong, we can find a tuple (n,B,δ,r_0,T) and $\epsilon_i \to 0$ such that the property does not hold for every $\epsilon_i \to 0$. Without loss of generality, we can assume $r_0=1$.

For each ϵ_i, let $t_i\in[0,T]$ be the critical time of a flow $g_i(t)$ such that the properties hold on $[0,t_i]$. In other words, for every $t\in[0,t_i]$, we have

$$\Omega_i'' \subset B_{g_i(t)}\left(x_i, 1-\frac{3}{2}\delta\right) \subset \Omega_i', \tag{3.21}$$

$$(1-\xi)\,\omega_i(0) \leq \omega_i(t) \leq (1+\xi)\,\omega_i(0), \quad \text{in } \Omega_i''. \tag{3.22}$$

However, for each time $t>t_i$, at least one of the above relations fails to hold. Related to (3.5), here we set

$$\Omega_i \triangleq B_{g_i(0)}(x_i,1), \quad \Omega_i' \triangleq B_{g_i(0)}(x_i,1-\delta), \quad \Omega_i'' \triangleq B_{g_i(0)}(x_i,1-2\delta).$$

We shall show that t_i cannot locate in $[0,T]$ for large i and, therefore, obtain a contradiction.

Note that $|Rm|_{g_i(0)} \leq 1$ at time $t=0$ in the ball $B_{g_i(0)}(x_i,1)$. By the strong version of Perelman's pseudolocality theorem, i.e., Theorem 10.3

of [**49**], one can find a uniform small constant η such that

$$|Rm|_{g_i}(x,t) \leq \frac{\xi}{100n^2}\eta^{-2}, \quad \forall\, x \in B_{g_i(0)}(x_i, 1-\eta),\ t \in [0,\eta^2]. \tag{3.23}$$

The existence of η can be obtained by a contradiction blowup argument. Since metrics evolve by $-Ric + \lambda g$, it follows from (3.23) and the choice of t_i that $\eta^2 \leq t_i \leq T$. Recall that we have the relationship (3.21) by the choice of t_i. Therefore, Proposition 3.6 can be applied to obtain a uniform C, independent of i, such that

$$\frac{1}{C} g_i(0) \leq g_i(t_i) \leq C g_i(0) \tag{3.24}$$

in the ball $B_{g_i(t_i)}(x_i, 1-\frac{3\delta}{2})$. Furthermore, the inequality (3.19) in Proposition 3.6 yields that

$$|Rm|_{g_i}(x,t) \leq \frac{C}{\psi^2}, \quad x \in B_{g(t)}\left(x_i, 1-\frac{3}{2}\delta - \psi\right),\ 0 \leq t \leq t_i,$$

where ψ is a small constant $\psi << \delta$, to be determined. Note that we are in a setting where each geodesic ball's volume ratio is bounded from below, due to the bounds in (1.4). Consequently, injectivity radius has a lower bound (c.f. [**13**]), by shrinking the ball if necessary. Therefore, we can apply Theorem 3.2 of [**73**] to obtain

$$\sup_{\eta^2 \leq t \leq t_i, d_{g_i(t)}(x,x_i) \leq 1-\frac{3}{2}\delta - 2\psi} |Ric|_{g_i}(x,t) \to 0, \quad \text{as } i \to \infty, \tag{3.25}$$

where η is the constant in (3.23). Alternatively, one can apply Lemma 2.1 of [**31**] to obtain the above estimate, with the fact that geodesic balls at different times can be compared due to the Riemannian curvature bound and the evolution equation of the Ricci flow: the metrics evolve by $-Ric + \lambda g$. Since t_i is uniformly bounded by T, the above equation implies (up to a maximum principle type argument of the first violating time if necessary) that

$$\begin{aligned} B_{g_i(\eta^2)}\left(x_i, 1-\frac{3}{2}\delta - 5\psi\right) &\subset B_{g_i(t)}\left(x_i, 1-\frac{3}{2}\delta - 4\psi\right) \\ &\subset B_{g_i(\eta^2)}\left(x_i, 1-\frac{3}{2}\delta - 3\psi\right), \quad \forall\, t \in [\eta^2, t_i). \end{aligned} \tag{3.26}$$

Combining the above relationship with (3.25), we obtain that

$$\sup_{\eta^2 \leq t \leq t_i, d_{g_i(\eta^2)}(x,x_i) \leq 1-\frac{3}{2}\delta - 3\psi} |Ric|_{g_i}(x,t) \to 0, \quad \text{as } i \to \infty. \tag{3.27}$$

By (3.23) and $|R| + |\lambda| \to 0$, we see the metric at $g_i(0)$ and $g_i(\eta^2)$ are almost isometric to each other on the ball $B_{g_i(0)}(x_i, 1-\frac{3}{2}\delta)$. Consequently, we have

$$\begin{aligned}\Omega_i'' = B_{g_i(0)}(x_i, 1-2\delta) &\subset B_{g_i(\eta^2)}\left(x_i, 1-\frac{7}{4}\delta\right)\\ &\subset B_{g_i(t_i)}\left(x_i, 1-\frac{7}{4}\delta+\psi\right) \Subset B_{g_i(t_i)}\left(x_i, 1-\frac{3}{2}\delta\right),\end{aligned}$$

where $\Subset$ means "compactly contained". We claim that we also have

$$B_{g_i(t_i)}\left(x_i, 1-\frac{3}{2}\delta\right) \Subset \Omega_i'.$$

For otherwise, by the choice of t_i, the boundary of $B_{g_i(t_i)}\left(x_i, 1-\frac{3}{2}\delta\right)$ touches the boundary of Ω_i' at time t_i. Therefore, we can find a point y_i satisfying

$$d_{g_i(t_i)}(x_i, y_i) = 1-\frac{3}{2}\delta, \quad d_{g_i(0)}(x_i, y_i) = 1-\delta.$$

Let γ_i be a shortest unit-speed geodesic connecting x_i and y_i, with respect to the metric $g_i(t_i)$. Let $\gamma_i(0) = x_i$ and $\gamma_i(1-\frac{3}{2}\delta) = y_i$. By previous estimates, we see that

$$\gamma_i\left(1-\frac{3}{2}\delta-100\psi\right) \subset B_{g_i(0)}\left(x_i, 1-\frac{3}{2}\delta-50\psi\right).$$

Let α_i be the part of γ_i, connecting $x_i = \gamma_i(0)$ and $\gamma_i(1-\frac{3}{2}\delta-100\psi)$. Let β_i be the remainded part of γ_i, i.e., the part connecting $\gamma_i(1-\frac{3}{2}\delta-100\psi)$ and $y_i = \gamma_i(1-\frac{3}{2}\delta)$. Using $|\cdot|$ to denote the length of curves. It is clear that $|\beta_i|_{g_i(t_i)} = 100\psi$. Note that α_i locates in $B_{g_i(t_i)}(x_i, 1-\frac{3}{2}\delta-100\psi)$. It follows from (3.26), (3.27) and (3.23) that

$$\sup_{\alpha_i\times[\eta^2,t_i]} |Ric|(x,t) \to 0, \quad \text{as } i\to\infty; \qquad \sup_{\alpha_i\times[0,\eta^2]} |Rm|(x,t) \le \frac{\xi}{100n^2}\eta^{-2}.$$

Together with $|R|+|\lambda| \to 0$ as $i \to \infty$, we can compare the length of α_i at time $t = t_i$ and $t = 0$.

$$|\alpha_i|_{g_i(t_i)} = 1-\frac{3}{2}\delta-100\psi, \quad |\alpha_i|_{g_i(0)} \le 1-\frac{3}{2}\delta.$$

However, since $d_{g_i(0)}(x_i, y_i) = 1-\delta$, we have

$$1-\delta \le |\gamma_i|_{g_i(0)} = |\alpha_i|_{g_i(0)} + |\beta_i|_{g_i(0)} \le |\beta_i|_{g_i(0)} + 1 - \frac{3}{2}\delta.$$

It follows that $|\beta_i|_{g_i(0)} \ge \frac{1}{2}\delta$. Recall that $|\beta_i|_{g_i(t_i)} = 100\psi$. Therefore, by mean value theorem, we must have

$$\frac{\sqrt{\langle V, V\rangle_{g_i(0)}}}{\sqrt{\langle V, V\rangle_{g_i(t_i)}}} \ge \frac{\frac{1}{2}\delta}{100\psi} = \frac{\delta}{200\psi},$$

at some point $z_i \in \beta_i$, where V is the unit tangent vector (with respect to $g_i(t_i)$) of β_i at z_i. Since $z_i \in B_{g_i(t_i)}(x_i, 1-\frac{3}{2}\delta)$, one can apply (3.24) to bound the left hand side of the above inequality by $\sqrt{C}$, where C is the constant in (3.24). It follows that $C \geq \frac{\delta^2}{40000\psi^2}$, which is impossible if we choose ψ small enough. Therefore, for i large, we must have

$$\Omega_i'' \Subset B_{g_i(t_i)}\left(x_i, 1-\frac{3}{2}\delta\right) \Subset \Omega_i'.$$

Then we can apply (3.23), (3.25) and the fact that $|R|+|\lambda| \to 0$ to obtain that

$$\left(1-\frac{\xi}{100}\right)\omega_i(0) \leq \omega_i(t) \leq \left(1+\frac{\xi}{100}\right)\omega_i(0), \quad \text{in } \Omega_i'',$$

whenever i large enough. Here $\Omega_i'' = B_{g_i(0)}(x_i, 1-2\delta)$. This means that for large i, we have both (3.21) and (3.22) hold for a short while beyond the time t_i. This contradicts to the choice of time t_i. q.e.d.

By further applying the argument in Proposition 3.6, the following theorem follows directly from the combination of Proposition 3.6 and Proposition 3.7.

Theorem 3.8 (**Rough long-time pseudolocality theorem for polarized Kähler Ricci flow**). *For every group of numbers δ, ξ, r_0, T, there exists an $\epsilon = \epsilon(n, B, \delta, \xi, r_0, T)$ with the following properties.*

Suppose $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (3.2), $x_0 \in M$. Suppose $|Rm| \leq r_0^{-2}$ in Ω at time $t=0$, where $\Omega = B_{g(0)}(x_0, r_0)$. If $\sup_{\mathcal{M}}(|R|+|\lambda|) < \epsilon$, then for every $t \in [0,T]$ we have

$$B_{g(t)}(x_0, (1-2\delta)r_0) \subset \Omega, \tag{3.28}$$

$$|Rm|(\cdot,t) \leq 2r_0^{-2}, \quad \text{in } B_{g(t)}(x_0, (1-2\delta)r_0), \tag{3.29}$$

$$(1-\xi)\,g(0) \leq g(t) \leq (1+\xi)\,g(0), \quad \text{in } B_{g(t)}(x_0, (1-2\delta)r_0). \tag{3.30}$$

3.2. Motivation and definition of pcr. In previous subsection, we see that the assumption (3.2) helps a lot to relate different time slices of the Kähler Ricci flow solution. However, why is this assumption reasonable? This question will be answered in this subsection.

Proposition 3.9 (**Weak continuity of Bergman function**). *There is a big integer constant $k_0 = k_0(n, A)$ and small constant $\epsilon = \epsilon(n, A)$ with the following property.*

Suppose (M, g, J, L, h) is a polarized Kähler manifold, taken out from a polarized Kähler Ricci flow in $\mathscr{K}(n, A)$ as a central time slice. In particular, we have

$$Osc_M \dot{\varphi} + C_S(M) + |\lambda| \leq B, \tag{3.31}$$

where $B = B(n, A)$. *If* $\mathbf{cr}(M) \geq 1$, *then*

$$\sup_{1 \leq k \leq k_0} \mathbf{b}^{(k)}(x) > -k_0, \tag{3.32}$$

whenever $d_{PGH}((M, x, g), (\tilde{M}, \tilde{x}, \tilde{g})) < \epsilon$ *for some space* $(\tilde{M}, \tilde{x}, \tilde{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$.

Proof. Note that the model moduli space $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ has compactness under the pointed Gromov–Hausdorff topology. Actually, this follows from the proof of Theorem 1.1, where the topology can be further improved to the pointed-$\hat{C}^{\infty}$-Cheeger–Gromov topology. This compactness property will be used in the following argument.

Suppose the statement was wrong, then there is a sequence of polarized Kähler manifolds (M_i, x_i, g_i) satisfying (3.31) and spaces $(\tilde{M}_i, \tilde{x}_i, \tilde{g}_i)$ in $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ with the following properties:

$$d_{PGH}((M_i, x_i, g_i), (\tilde{M}_i, \tilde{x}_i, \tilde{g}_i)) < \epsilon_i \to 0,$$
$$\sup_{1 \leq j \leq k_i} \mathbf{b}^{(j)}(x_i) \to -\infty, \quad k_i \to \infty.$$

In light of the compactness of the moduli $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$, i.e., Theorem 1.1, by taking subsequence if necessary, we can find a space $(\bar{M}, \bar{x}, \bar{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ such that

$$\begin{aligned} &\lim_{i \to \infty} d_{PGH}((M_i, x_i, g_i), (\bar{M}, \bar{x}, \bar{g})) \\ = &\lim_{i \to \infty} d_{PGH}((\tilde{M}_i, \tilde{x}_i, \tilde{g}_i), (\bar{M}, \bar{x}, \bar{g})) = 0. \end{aligned}$$

Consequently, we have

$$(M_i, x_i, g_i) \xrightarrow{P.G.H.} (\bar{M}, \bar{x}, \bar{g}); \qquad \sup_{1 \leq j \leq k_i} \mathbf{b}^{(j)}(x_i) \to -\infty, \quad k_i \to \infty. \tag{3.33}$$

Then we shall use the argument of the proof of Theorem 3.2 of [**37**] by Donaldson–Sun to find positive integer $q = q(\bar{x})$, and real numbers $r = r(\bar{x})$, $C = C(\bar{x})$ such that

$$\inf_{y \in B(x_i, r)} \mathbf{b}^{(q)}(y) \geq -C. \tag{3.34}$$

Note that the proof of Theorem 3.2 [**37**] is based on a blowup argument. The essential ingredients there are the convergence theory, the Hömander's estimate, and the fact that each tangent space in the limit space is a good metric cone. By "good" we mean the singular set of the metric cone has Hausdorff codimension strictly greater than 2. It is important to observe that whether the limit space $\bar{M}$ is compact or not does not affect the argument. Basically, this is because of the local property of the Hömander's estimate. Actually, no matter whether

$\bar{M}$ is compact or not, every tangent space of a point on $\bar{M}$ must be non-compact. The contradiction is obtained from the convergence to the good tangent metric cone. With the argument of Theorem 3.2 of [**37**] in mind, we now check the conditions available to us in the current case. Firstly, the canonical radius assumption makes sure that the topology of the convergence can be improved to the pointed-$\hat{C}^4$-Cheeger–Gromov topology. Secondly, by the uniform bound of Sobolev constant and $\|\dot{\varphi}\|_{C^0}$, the general Hörmander's estimate (c.f. section 3 of [**30**] and section 5 of [**72**] for this particular case) can be applied. Thirdly, we know each tangent space at $\bar{x}$ is a good metric cone, by Theorem 2.5 since $(\bar{M}, \bar{x}, \bar{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. Therefore, we can use a contradiction blowup argument, like that in Theorem 3.2 of [**37**], to obtain (3.34). Consequently, we have

$$\mathbf{b}^{(q)}(x_i) \geq -C, \quad \Rightarrow \quad \sup_{j \leq k_i} \mathbf{b}^{(j)}(x_i) \geq -C,$$

which contradicts (3.33), the assumption. q.e.d.

Proposition 3.9 means that the Bergman function has a weak continuity under the pointed-$\hat{C}^4$-Cheeger–Gromov convergence if the limit space is the model space. Inspired by this property, we can define the polarized canonical radius as follows.

Definition 3.10. Suppose (M, g, J, L, h) is a polarized Kähler manifold satisfying (3.31), $x \in M$. We say the polarized canonical radius of x is not less than 1 if

- $\mathbf{cr}(x) \geq 1$.
- $\sup_{1 \leq j \leq 2k_0} \mathbf{b}^{(j)}(x) \geq -2k_0$.

For every $r = \frac{1}{j}, j \in \mathbb{Z}^+$, we say the polarized canonical radius of x is not less than r if the rescaled polarized manifold $\left(M, j^2 g, J, L^j, h^j\right)$ has polarized canonical radius at least 1 at the point x. Fix x, let $\mathbf{pcr}(x)$ be the supreme of all the r with the above property and call it as the polarized canonical radius of x.

We can define the polarized canonical radius of a manifold as the infimum of the polarized canonical radii of all points in that manifold. Similarly, we can define the polarized canonical radius of time slices of a flow. Note that from the above definition, $\mathbf{pcr}$ is always the reciprocal of a positive integer. It could not be zero because of (3.13) in the proof of Lemma 3.4 and the fact that every compact smooth manifold has bounded geometry and positive $\mathbf{cr}$. One can also repeat the argument in the proof of Proposition 3.9 to obtain that the $\mathbf{pcr}$ is always positive. Note that a blowup sequence of (M, g, J, L, h) at a given point $x \in M$ always converges in smooth topology to the standard Euclidean space $(\mathbb{C}^n, g_E, J_E, L_E, h_E)$ with L_E being the trivial line bundle and $h_E = e^{-|z|^2}$. For each sequence of positive integers $k \to \infty$ and the sequence $\left(M, k^2 g, J, L^k, h^k\right)$, we have $\mathbf{b}^{(k)}(x) \to \mathbf{b}^{(E)}(x) = C_n > -2k_0$. Therefore, for large k, we must have $\mathbf{b}^{(k)}(x) \geq -2k_0$. Consequently, the $\mathbf{pcr}(x)$ of $\left(M, k^2 g, J, L^k, h^k\right)$ is at least 1. This means that the $\mathbf{pcr}(x)$ of (M, g, J, L, h) is at least $\frac{1}{k} > 0$.

Under the terminology in Definition 3.10, the continuity of Bergman function implies the following corollary.

Corollary 3.11 (**Weak equivalence of cr and pcr**). *There is a small constant $\epsilon = \epsilon(n, B, \kappa)$ with the following property.*

Suppose (M, g, J, L, h) is a polarized Kähler manifold satisfying (3.31) and $\mathbf{cr}(M) \geq 1$. Then

$$\mathbf{pcr}(x) \geq 1, \tag{3.35}$$

whenever $d_{PGH}((M, x, g), (\tilde{M}, \tilde{x}, \tilde{g})) < \epsilon$ for some space $(\tilde{M}, \tilde{x}, \tilde{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$.

3.3. Kähler Ricci flow with lower bound of pcr. Suppose the polarized canonical radius is uniformly bounded from below, then the convergence theory is much better than that in section 3. This is basically because of the rough long-time pseudolocality theorem, Theorem 3.8.

Proposition 3.12 (**Improving regularity in forward time direction**). *For every $r_0 > 0$, $r \in (0, r_0)$ and $T_0 > 0$, there is an $\epsilon = \epsilon(n, A, r_0, r, T_0)$ with the following properties.*

If $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (1.4) and

$$\mathbf{pcr}(\mathcal{M}^t) \geq r_0, \quad \forall\, t \in [0, T_0], \tag{3.36}$$

then

$$\mathcal{F}_r(M, 0) \subset \bigcap_{0 \leq t \leq T_0} \mathcal{F}_{\frac{r}{K}}(M, t), \tag{3.37}$$

whenever $\sup_{\mathcal{M}}(|R| + |\lambda|) < \epsilon$. Here K is the constant in Proposition 2.10, $\mathcal{M}^t$ is $(M, g(t))$, the time t slice of the flow $\mathcal{M}$.

Proof. It follows directly from Theorem 3.8, the long time pseudolocality theorem for polarized Kähler Ricci flow with partial-C^0-estimate. q.e.d.

Proposition 3.13 (**Improving regularity in backward time direction**). *For every $r_0 > 0$, $r \in (0, r_0)$ and $T_0 > 0$, there is an $\epsilon = \epsilon(n, A, r_0, r, T_0)$ with the following properties.*

If $\mathcal{LM}$ is a polarized Kähler Ricci flow satisfying (1.4) and (3.36), then

$$\bigcup_{0 \le t \le T_0} \mathcal{F}_r(M,t) \subset \mathcal{F}_{\frac{r}{K}}(M,0), \tag{3.38}$$

whenever $\sup_{\mathcal{M}}(|R|+|\lambda|) < \epsilon$.

Proof. At time 0, $\mathcal{F}_r(M,0) \subset \mathcal{F}_{\frac{r}{K}}(M,0)$ trivially. Suppose $t_0 > 0$ is the first time such that (3.38) start to fail. It suffices to show that $t_0 > T_0$ whenever ϵ is small enough. Otherwise, at time $t_0 \in (0,T_0]$, we can find a point $x_0 \in (\partial \mathcal{F}_{\frac{r}{K}}(M,0)) \cap (\partial \mathcal{F}_r(M,t_0))$. In other words, we have

$$\mathbf{cvr}(x_0,0) = \frac{r}{K}, \quad \mathbf{cvr}(x_0,t_0) = r.$$

In particular, we have

$$\left|B_{g(0)}\left(x_0, \frac{r}{K}\right)\right|_0 = (1-\delta_0)\,\omega_{2n}\left(\frac{r}{K}\right)^{2n}. \tag{3.39}$$

Let ξ be a small number which will be fixed later. Let $\Omega_\xi(x_0,t_0)$ be the subset of unit sphere of tangent space of $T_{x_0}(M,g(t_0))$ such that every geodesic (under metric $g(t_0)$) emanating from x_0 along the direction in $\Omega_\xi(x_0,t_0)$ does not hit points in $\mathcal{D}_\xi(M,0)$ before distance $\frac{r}{K}$. By canonical radius assumption, $|Rm|_{g(t_0)}$ is uniformly bounded in $B_{g(t_0)}(x_0,\frac{r}{K})$. By long-time pseudolocality theorem (c.f. Proposition 3.13), $B_{g(t_0)}(x_0,\frac{r}{K^3})$ has empty intersection with $\mathcal{D}_\xi(M,0)$ when $\xi << \frac{r}{K^3}$. Note that every geodesic (emanating from x_0) entering $\mathcal{D}_\xi(M,0)$ must hit $\partial\mathcal{D}_\xi(M,0)$ first, where $\mathbf{cvr}(\cdot,0)=\xi$. So every point in $\partial\mathcal{D}_\xi(M,0)$ will be uniformly regular at time t_0, in light of the long-time pseudolocality. At time t_0, observing from x_0, the set which stays behind $\partial\mathcal{D}_\xi(M,0)$ must have small measure. Since $B_{g(t_0)}(x_0,\frac{r}{K})$ has uniformly bounded curvature, it is clear that $\Omega_\xi(x_0,t_0)$ is an almost full measure subset of S^{2n-1}. Actually, we have

$$|\Omega_\xi(x_0,t_0)| \ge 2n\omega_{2n}\cdot\left(1 - C\xi^{2p_0}\right),$$

whenever ϵ is sufficiently small. On the other hand, we see that every geodesic (under metric $g(t_0)$) emanating from $\Omega_\xi(x_0,t_0)$ is almost geodesic at time $t=0$ (under metric $g(0)$), when ϵ small enough. Therefore, $|B_{g(0)}(x_0,\frac{r}{K})|_0$ is almost not less than $|B_{g(t_0)}(x_0,\frac{r}{K})|_{t_0}$. Note that the volume ratio of $B_{g(t_0)}(x_0,\frac{r}{K})$ is at least $(1-\frac{\delta_0}{100})\omega_{2n}$. Suppose we choose ξ small (according to δ_0) and ϵ very small (based on ξ,δ_0,A,T_0), we obtain

$$\left|B_{g(0)}\left(x_0,\frac{r}{K}\right)\right|_0 \ge \left(1-\frac{\delta_0}{2}\right)\omega_{2n}\left(\frac{r}{K}\right)^{2n},$$

which contradicts (3.39). q.e.d.

Definition 3.14. Let $\mathscr{K}(n,A)$ be the collection of polarized Kähler Ricci flows satisfying (1.4). For every $r \in (0,1]$, define

$$\mathscr{K}(n,A;r) \triangleq \{\mathcal{LM} \,|\mathcal{LM} \in \mathscr{K}(n,A), \mathbf{pcr}(M \times [-1,1]) \geq r\}.$$

Clearly, we have

$$\mathscr{K}(n,A) \supset \mathscr{K}(n,A;r_1) \supset \mathscr{K}(n,A;r_2),$$

whenever $1 \geq r_2 > r_1 > 0$. Since every polarized Kähler Ricci flow $\mathcal{LM} \in \mathscr{K}(n,A)$ has a smooth compact underlying manifold, we see $\mathcal{LM} \in \mathscr{K}(n,A;r)$ for some very small r, which depends on $\mathcal{LM}$. Therefore, it is clear that

$$\bigcup_{0<r<1} \mathscr{K}(n,A;r) = \mathscr{K}(n,A).$$

Fix $r > 0$, we shall first make clear the structure of $\mathscr{K}(n,A;r)$ under the help of polarized canonical radius. Then we show that the canonical radius can actually been bounded a priori. In other words, there exists a uniform small constant $\hbar$ (Planck scale) such that

$$\mathscr{K}(n,A) = \mathscr{K}(n,A;\hbar),$$

which will be proved in Theorem 3.44.

Proposition 3.15 (**Limit space-time with static regular part**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n,A)$ satisfies the following properties.*

- $\mathbf{pcr}(M_i \times [-T_i, T_i]) \geq r_0$ *for each* i.
- $\lim_{i\to\infty} \sup_{\mathcal{M}_i}(|R| + |\lambda|) = 0$.

Suppose $x_i \in M_i$ and $\lim_{i\to\infty} \mathbf{cvr}(x_i, 0) > 0$, then

$$(M_i, x_i, g_i(0)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}). \tag{3.40}$$

Moreover, we have

$$(M_i, x_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}), \tag{3.41}$$

for every $t \in (-\bar{T}, \bar{T})$, where $\bar{T} = \lim_{i\to\infty} T_i > 0$. In particular, the limit space does not depend on time.

Proof. It follows from the combination of Proposition 3.12 and Proposition 3.13 that the limit space does not depend on time. From the definition of canonical radius, the convergence locate in $\hat{C}^4$-topology for each time. However, this can be improved to $\hat{C}^\infty$-topology. Actually, if $\bar{y}$ is a regular point of $\bar{M}$ (c.f. the definition in Theorem 2.13), then we can find $y_i \in M_i$ such that $y_i \to \bar{y}$ and $\mathbf{cvr}(y_i, 0) \geq \eta$ uniformly for some $\eta > 0$, in light of the definition of regular points in Theorem 2.13 and its proof. It follows from Proposition 3.13 that

$$\inf_{t\in[-1,0]} \mathbf{cvr}(y_i,t) \geq K^{-1}\eta,$$

for all large i. By second property, or regularity estimate of canonical radius (c.f. Definition 2.9), we know

$$|Rm|(z,t) \leq CK^{-4}\eta^{-2}, \quad \forall\, z \in B_{g_i(t)}(y_i, K^{-2}\eta),\ t \in [-1,0].$$

Note that $R \to 0$, which implies $|Ric| \to 0$ when we have $|Rm|$-bound in a bigger ball (c.f. the $|Ric| \leq \sqrt{|Rm||R|}$-type estimate in [**73**]). In particular, we have

$$B_{g_i(0)}(y_i, 0.1K^{-2}\eta) \subset B_{g_i(t)}(y_i, K^{-2}\eta),$$

for all $t \in [-0.5, 0]$. Hence, we obtain

$$|Rm|(z,t) \leq CK^{-4}\eta^{-2}, \tag{3.42}$$

for every $z \in B_{g_i(0)}(y_i, 0.1K^{-2}\eta)$, $t \in [-0.5,0]$. Then we can apply Shi's estimate to obtain that $|\nabla^k Rm| \leq C_k$ on $B_{g_i(0)}(y_i, 0.01K^{-2}\eta)$ for each positive integer k. This is enough to set up a uniform sized harmonic coordinate chart around y_i (with respect to metric $g_i(0)$) and all the metric tensor and its derivatives are uniformly bounded (c.f. Hamilton [**39**]). Clearly, the convergence around $\bar{y}$ happens in the pointed-C^∞-topology. Since $\bar{y}$ is an arbitrary regular point, we see that the convergence to $\bar{M}$ is in pointed-$\hat{C}^\infty$-Cheeger–Gromov topology.

Note that we currently do not know whether $\bar{M}$ locates in the model space $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. However, we do know that $\bar{M} = \mathcal{R}(\bar{M}) \cup \mathcal{S}(\bar{M})$. The regular part is a smooth Ricci-flat manifold, due to the smooth convergence and $|Ric| \to 0$ on regular part. The singular part satisfies the Minkowski dimension bound (c.f. (2.26) in Theorem 2.13):

$$\dim_{\mathcal{M}} \mathcal{S} \leq 2n - 2p_0 < 2n - 4 + \frac{2}{1000n} < 2n - 4 + \frac{2}{2n-1}, \tag{3.43}$$

where we used the fact that p_0 is very close to 2.

Claim 3.16 (**Good version of Lipschitz function**). *Every bounded function $f \in N_0^{1,2}(\bar{M})$ with finite $\|\nabla f\|_{L^\infty(\bar{M})}$ and Lipschitz on $\mathcal{R}(\bar{M})$ has a good version $\tilde{f}$ such that*

$$f(x) = \tilde{f}(x), \quad \forall\, x \in \mathcal{R}(\bar{M}), \tag{3.44}$$

$$\sup_{\bar{M}} |\nabla \tilde{f}| \leq \|\nabla f\|_{L^\infty(\bar{M})}, \tag{3.45}$$

where the inequality (3.45) can be understood as

$$|\tilde{f}(x) - \tilde{f}(y)| \leq \|\nabla f\|_{L^\infty(\bar{M})} \cdot d(x,y), \quad \forall\, x, y \in \bar{M}.$$

This is a flow property, so we assume $\lambda = 0$ without loss of generality. For simplicity of notation, we also assume that $\|\nabla f\|_{L^\infty(\bar{M})} = 1$ and the support of f is contained in $B(\bar{x}, 1)$. Note that these assumptions can always be achieved up to rescaling argument. Let $\chi_\epsilon = \phi(\frac{d(x,\mathcal{S})}{\epsilon})$ be the cutoff function where ϕ is a smooth cutoff function such that $\phi \equiv 1$ on $(2,\infty)$ and $\phi \equiv 0$ on $(-\infty,1)$ and $|\nabla\phi| \leq 2$, $\phi' \leq 0$. Then $\chi_\epsilon f$ is a Lipschitz function with compact support. By the smooth convergence away from singularity (c.f. Proposition 3.13 and the discussion around inequality (3.42)), we can regard $\chi_\epsilon f$ as a Lipschitz function on $(M_i, g_i(-\delta))$, denoted by $f_{\epsilon,i}$, where $\delta = \epsilon^{\frac{1}{n}}$. Starting from $f_{\epsilon,i}$, we solve the heat equation until time $t=0$ and obtain a function $h_{\epsilon,i} = f_{\epsilon,i}(0)$, together with the metric evolving by the Ricci flow. Then we have

$$h_{\epsilon,i}(x) = \int_{M_i} w(x,y,-\delta) f_{\epsilon,i}(y) dv_y, \quad \forall\, x \in M_i,$$

where w is the fundamental solution of $\square^* w = (\partial_\tau - \Delta + R)w = 0$. Recall that $\int_{M_i} w dv \equiv 1$ and $|f_{\epsilon,i}| \leq C$ uniformly, we have

$$\begin{aligned} |h_{\epsilon,i}|(x) &= \left|\int_{M_i} w(x,y,-\delta) f_{\epsilon,i}(y) dv_y\right| \\ &\leq \sup_{M_i} |f_{\epsilon,i}| \int_{M_i} w(x,y,-\delta) dv_y = \sup_{M_i} |f_{\epsilon,i}| \leq C. \end{aligned} \tag{3.46}$$

Direct calculation shows that

$$\square |\nabla f_{\epsilon,i}|^2 = (\partial_t - \Delta)\, |\nabla f_{\epsilon,i}|^2 = -2|\nabla\nabla f_{\epsilon,i}|^2 \leq 0.$$

It follows that

$$\begin{aligned} &|\nabla h_{\epsilon,i}|^2(x) - \int_{M_i} w(x,y,-\delta) |\nabla f_{\epsilon,i}|^2(y) dv_y \\ &= -2\int_{-\delta}^{0} \int_{M_i} w(x,y,t) |\nabla\nabla f_{\epsilon,i}|^2 dv_y dt \leq 0. \end{aligned}$$

Consequently, we have

$$\begin{aligned} |\nabla h_{\epsilon,i}|^2(x) &\leq \int_{M_i} w(x,y,-\delta) |\nabla f_{\epsilon,i}|^2(y) dv_y \\ &= \int_{\Omega_i \backslash A_i} w(x,y,-\delta) |\nabla f_{\epsilon,i}|^2(y) dv_y \\ &\quad + \int_{A_i} w(x,y,-\delta) |\nabla f_{\epsilon,i}|^2(y) dv_y, \end{aligned} \tag{3.47}$$

where A_i is the set where the pull back of χ_ϵ achieves values in $(0,1)$, Ω_i is the support of the pull back function f_i. Note that $|\nabla f_{\epsilon,i}|(x) \leq 1+\xi$ for arbitrary small, but fixed ξ, whenever i is large enough and $x \in \Omega_i \backslash A_i$. On A_i, we have $|\nabla f_{\epsilon,i}| \leq C\epsilon^{-1}$ for some universal constant C. Note that $\Omega_i \subset B_{g_i(0)}(x_i, 1)$, the canonical assumption then implies the density estimate $|A_i| \leq C\epsilon^{2p_0}$. Recall that we have the heat kernel (hence, conjugate heat kernel estimate) estimate $w(x,y,-\delta) \leq C\delta^{-n}$

for some universal constant C. Plugging these inequalities into (3.47), we obtain

$$\begin{aligned}
&|\nabla h_{\epsilon,i}|^2(x)\\
&\leq \int_{\Omega_i\backslash A_i} w(x,y,-\delta)|\nabla f_{\epsilon,i}|^2(y)dv_y + \int_{A_i} w(x,y,-\delta)|\nabla f_{\epsilon,i}|^2(y)dv_y\\
&\leq (1+\xi)\int_{\Omega_i\backslash A_i} w(x,y,-\delta)dv_y + C\epsilon^{-2}\cdot C\delta^{-n}\cdot |A_i|\\
&\leq (1+\xi)\int_{M_i} w(x,y,-\delta)dv_y + C\epsilon^{2p_0-2}\delta^{-n}\\
&\leq (1+\xi) + C\epsilon^{2p_0-2}\delta^{-n} \leq 1+\xi+C\epsilon^{2-\frac{1}{500n}}\delta^{-n},
\end{aligned}$$

where we used the fact $\epsilon < 1$ and inequality (3.43) in the last step. Recall that $\delta = \epsilon^{\frac{1}{n}}$ and let $\xi = \epsilon^{1-\frac{1}{500n}} << \sqrt{\epsilon}$, we then have

$$|\nabla h_{\epsilon,i}|^2(x) \leq 1+\sqrt{\epsilon}. \tag{3.48}$$

Moreover, if $\bar{z}$ is a regular point of $\bar{M}$, i.e., $\bar{z}\in\mathcal{R}(\bar{M})$. Let $z_i \in M_i$ and $z_i \to \bar{z}$. Then we have

$$\begin{aligned}
&|h_{\epsilon,i}(z_i) - f_{\epsilon,i}(z_i)|\\
&= \left|\int_{M_i} w(z_i,y,-\delta)\left\{f_{\epsilon,i}(y) - f_{\epsilon,i}(z_i)\right\}dv_y\right|\\
&\leq \int_{B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})} w(z_i,y,-\delta)\left|f_{\epsilon,i}(y) - f_{\epsilon,i}(z_i)\right|dv_y\\
&\quad + \int_{M_i\backslash B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})} w(z_i,y,-\delta)\left|f_{\epsilon,i}(y) - f_{\epsilon,i}(z_i)\right|dv_y.
\end{aligned}$$

Note that $\bar{z}$ is regular, we can assume that the regularity scale (for example, **cvr**) of each z_i is much larger than $\delta = \epsilon^{\frac{1}{n}}$, if we choose ϵ small enough. Clearly, $B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})\cap A_i = \emptyset$, which implies the Lipschitz constant of $f_{\epsilon,i}$ on $B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})$ is uniformly bounded by C. Recall that $\int_{M_i} wdv \equiv 1$. Therefore, we have

$$|h_{\epsilon,i}(z_i) - f_{\epsilon,i}(z_i)| \leq C\delta^{\frac{1}{4}} + C\int_{M_i\backslash B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})} w(z_i,y,-\delta)dv_y. \tag{3.49}$$

The last term of the above inequality is a small term which can be absorbed in $C\delta^{\frac{1}{4}}$. Actually, let ψ be a cutoff function such that $\psi\equiv 0$ on $B_{g_i(0)}(z_i,0.5\delta^{\frac{1}{4}})$, $\psi\equiv 1$ on $M_i\backslash B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})$. Moreover, we have

$$|\nabla\psi|^2 + |\Delta\psi| \leq C\delta^{-\frac{1}{2}}.$$

This can be done since $\delta^{\frac{1}{4}}$ is much less than the regularity scale of z_i. Now we extend ψ to be a function on space-time by letting $\psi(x,t) =$

$\psi(x)$. Due to Proposition 3.13 and the discussion before (c.f. inequality (3.42)), we obtain $|\nabla\psi|^2+|\Delta\psi|\leq C\delta^{-\frac{1}{2}}$ on $M_i\times[-\delta,0]$. Consequently, we obtain

$$\frac{d}{dt}\int_{M_i}\psi(y,t)w(z_i,y,t)dv_y=\int_{M_i}(w\Box\psi-\psi\Box^*w)dv_y=-\int_{M_i}w\Delta\psi dv_y.$$

As w converges to the δ-function at z_i as t approaches 0, $\psi(z_i,0)=0$, we have

$$\begin{aligned}&0-\int_{M_i}\psi(y,-\delta)w(z_i,y,-\delta)dv_y\\&=-\int_{-\delta}^0\int_{M_i}w\Delta\psi dv_ydt\geq -C\delta^{-\frac{1}{2}}\int_{-\delta}^0\int_{M_i}wdv_ydt=-C\delta^{\frac{1}{2}},\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{M_i\backslash B_{g_i(0)}(z_i,\delta^{\frac{1}{4}})}\psi(y,-\delta)w(z_i,y,-\delta)dv_y\\&\leq\int_{M_i}\psi(y,-\delta)w(z_i,y,-\delta)dv_y\leq C\delta^{\frac{1}{2}}.\end{aligned}$$

Plugging the above inequality into (3.49), and noticing that $\delta=\epsilon^{\frac{1}{n}}$, we obtain

$$|h_{\epsilon,i}(z_i)-f_{\epsilon,i}(z_i)|\leq C\delta^{\frac{1}{4}}\leq C\epsilon^{\frac{1}{4n}}. \tag{3.50}$$

It follows from the combination of (3.46), (3.48) and (3.50) that there is a limit function h_ϵ on $\bar{M}$. Let $\epsilon=2^{-i}\to 0$, up to a diagonal sequence argument, we can assume that $h_{2^{-i},i}$ converges to a limit function h, which satisfies

$$\begin{aligned}&\sup_{\bar{M}}|h|\leq C,\quad \sup_{\bar{M}}|\nabla h|\leq 1=\|\nabla f\|_{L^\infty(\bar{M})},\\&h(x)=f(x),\quad \forall\, x\in\mathcal{R}(\bar{M}).\end{aligned}$$

In particular, h is a good version of f. We finish the proof of Claim 3.16.

Based on Claim 3.16, the proof of (3.41) follows from the standard technique used in the proof of the Cheeger–Gromoll splitting lemma in [**29**]. Actually, for each $t\neq 0$, we already know that $(M_i,x_i,g_i(t))$ converges in the pointed Gromov–Hausdorff topology to some $(\bar{M}',\bar{x}',\bar{g}')$. We only need to show that $\bar{M}'$ is isometric to $\bar{M}$. By Proposition 3.13 and the fact $|R|+|\lambda|\to 0$, we know that there is a natural identification map between $\mathcal{R}(\bar{M})$ and $\mathcal{R}(\bar{M}')$, which contain a common point $\bar{x}$. In the following discussion, we shall show that this identification map can be extended to an isometry between $\bar{M}$ and $\bar{M}'$.

Let $\bar{y},\bar{z}$ be two regular points of $\bar{M}$. Clearly, $\bar{y}$ and $\bar{z}$ can also be regarded as regular points on $\bar{M}'$. We omit the identification map for the simplicity of notations. Suppose $d_{\bar{g}}(\bar{y},\bar{z})=D>0$. We can construct a function χ on $\bar{M}'$ as follows

$$\chi(x)=\begin{cases}\max\{D-d_{\bar{g}}(x,\bar{y}),0\}, & \text{if} \quad x\in\mathcal{R}(\bar{M}'),\\ 0, & \text{if} \quad x\in\mathcal{S}(\bar{M}').\end{cases} \tag{3.51}$$

Fix point $x\in\mathcal{R}(\bar{M}')\backslash B_{\bar{g}'}(\bar{y},3D)$, every smooth curve connecting x and $\bar{y}$ has length as least $3D$. In light of inequality (2.25) in Theorem 2.13 (applying to both $\bar{g}$ and $\bar{g}'$), we know that

$$\min\{d_{\bar{g}}(x,\bar{y}),d_{\bar{g}'}(x,\bar{y})\}\geq D,$$

which implies that $\chi(x)=0$ by definition equation (3.51). We remark that inequality (2.25) together with the high codimension of $\mathcal{S}$ implies that $\chi\in N_0^{1,2}(\bar{M}')$ and we have

$$\|\nabla\chi\|_{L^\infty(\bar{M}')}=\|\nabla\chi\|_{L^\infty(\mathcal{R}(\bar{M}'))}=\|\nabla\chi\|_{L^\infty(\mathcal{R}(\bar{M}))}=1.$$

Then we can apply Claim 3.16 to obtain a good version $\tilde{\chi}$ of χ. In particular, we have

$$\begin{aligned} d_{\bar{g}}(\bar{y},\bar{z})=D=|\chi(\bar{y})-\chi(\bar{z})|&=|\tilde{\chi}(\bar{y})-\tilde{\chi}(\bar{z})|\\ &\leq d_{\bar{g}'}(\bar{y},\bar{z})\|\nabla\chi\|_{L^\infty(\bar{M}')}\leq d_{\bar{g}'}(\bar{y},\bar{z}).\end{aligned} \tag{3.52}$$

Similarly, by reversing the role of $\bar{g}'$ and $\bar{g}$ when we choose the test function, we obtain that

$$d_{\bar{g}'}(\bar{y},\bar{z})\leq d_{\bar{g}}(\bar{y},\bar{z}). \tag{3.53}$$

By the arbitrary choice of $\bar{y},\bar{z}$, we know the identity map between $\mathcal{R}(\bar{M})$ and $\mathcal{R}(\bar{M}')$ is an isometry map by (3.52) and (3.53). Since $\mathcal{R}(\bar{M})$ is dense in $\bar{M}$, $\mathcal{R}(\bar{M}')$ is dense in $\bar{M}'$, we obtain $\bar{M}$ and $\bar{M}'$ are isometric to each other by taking metric completion. Consequently, (3.41) follows from (3.40). q.e.d.

In Proposition 3.15, we show that the limit flow exists and is static in the regular part, whenever we have $|R|+|\lambda|\to 0$. It is possible that the limit points in the singular part $\mathcal{S}$ are moving as time evolves. However, this possibility will be ruled out finally (c.f. Proposition 4.23).

3.3.1. Tangent structure of the limit space. In this subsection, we shall show that the tangent space of each point in the limit space has a metric cone structure, provided polarized canonical radius is uniformly bounded below. Basically, the cone structure is induced from the local-ized W-functional's monotonicity. Up to a parabolic rescaling, we can assume $\lambda=0$ without loss of generality.

Proposition 3.17 (**Local W-functional**). *Let $\mathcal{LM}_i\in\mathscr{K}(n,A;r_0)$ and $\sup_{\mathcal{M}_i}(|R|+|\lambda|)\to 0$.*

Let u_i be the fundamental solution of the backward heat equation $\left[-\frac{\partial}{\partial t} - \triangle + R\right] u_i = 0$ based at the space-time point $(x_i, 0)$. Then u_i converges to a limit positive solution $\bar{u}$ on $\mathcal{R} \times (-1, 0]$, i.e.,

$$\left[-\frac{\partial}{\partial t} - \Delta + R\right] \bar{u} = 0.$$

Moreover, we have

$$\iint_{\mathcal{R}\times(-1,0]} 2|t| \left| Ric + \nabla\nabla \bar{f} + \frac{\bar{g}}{2t}\right|^2 \bar{u} dv_{\bar{g}} dt \leq C, \tag{3.54}$$

where $C = C(n, A)$, $\bar{u} = (4\pi|t|)^{-n} e^{-\bar{f}}$.

Proof. This is a flow property and has nothing to do with polarization. So we can assume $\lambda = 0$ for simplicity.

Fix $r > 0$. Choose a point $\bar{y} \in \mathcal{R}_r$ and a time $\bar{t} < 0$. Without loss of generality, we assume that there is a sequence of points (y_i, t_i) converging to $(\bar{y}, \bar{t})$. Note that $d_{g_i(0)}(y_i, x_i)$ is uniformly bounded. It is not hard to see that u_i is uniformly bounded around (y_i, t_i). Actually, let w_i be the heat equation $\square w_i = (\partial_t - \Delta) w_i = 0$, starting from a δ-function at (y_i, t_i). Then by the heat kernel estimate of Cao–Zhang (c.f. [7]), we obtain the on-diagonal bound

$$\frac{1}{C}|t_i|^{-n} < w_i(y_i, 0) < C|t_i|^{-n},$$

for some uniform constant C. Then the gradient estimate of Cao–Hamilton–Zhang (c.f. [77], [6]) and the fact $d_{g_i(0)}(y_i, x_i) < C$ implies that $|\log w_i(x_i, 0)|$ is uniformly bounded. Note that $w_i(x_i, 0) = u_i(y_i, t_i)$ since the integral $\int_M u_i v_i d\mu$ does not depend on time. Therefore, we have

$$\frac{1}{C} \leq u_i(y_i, t_i) = w_i(x_i, 0) \leq C, \tag{3.55}$$

where C depends on $|t_i|$ and $d_{g_i(0)}(y_i, x_i)$. It clearly works uniformly for a fixed-sized space-time neighborhood of (y_i, t_i), where curvatures are uniformly bounded. Then standard regularity argument from heat equation shows that all derivatives of u_i are uniformly bounded around (y_i, t_i). Therefore, there is a limit positive solution $\bar{u}$ around $(\bar{y}, \bar{t})$. By the arbitrary choice of $r, \bar{y}, \bar{t}$. It is clear that there is a smooth heat solution $\bar{u}$ defined on $\mathcal{R} \times (-1, 0)$.

By Perelman's calculation, for each flow g_i, we have

$$\begin{aligned} &\int_{-1}^{0} \int_{M_i} 2|t| \left| Ric_{g_i} + \nabla\nabla f_i + \frac{g_i}{2t}\right|^2 u_i dv_{g_i} dt \\ &= -\mu(M_i, g_i(t_i), 1) \leq C, \end{aligned} \tag{3.56}$$

since Sobolev constant is uniformly bounded. By passing to limit, (3.54) follows. q.e.d.

Theorem 3.18 (**Tangent cone structure**). *Suppose $\mathcal{LM}_i$ is a sequence of polarized Kähler Ricci flow solutions in $\mathscr{K}(n, A; r_0)$, $x_i \in M_i$. Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$, $\bar{y}$ be an arbitrary point of $\bar{M}$. Then every tangent space of $\bar{M}$ at $\bar{y}$ is an irreducible metric cone.*

Proof. Suppose $\hat{Y}$ is a tangent space of $\bar{M}$ at the point $\bar{y}$, i.e., there are scales $r_k \to 0$ such that

$$(\hat{Y}, \hat{y}, \hat{g}) = \lim_{k\to\infty} (\bar{M}, \bar{y}, \bar{g}_k), \tag{3.57}$$

where $\bar{g}_k = r_k^{-2}\bar{g}$. By taking subsequence if necessary, we can assume $(\hat{Y}, \hat{y}, \hat{g})$ as the limit space of $(M_{i_k}, y_{i_k}, \tilde{g}_{i_k})$ where $\tilde{g}_{i_k} = r_i^{-2} g_{i_k}(0)$. Denote the regular part of $\hat{Y}$ by $\mathcal{R}(\hat{Y})$. Then on the space-time $\mathcal{R}(\hat{Y}) \times (-\infty, 0]$, there is a smooth limit backward heat solution $\hat{u}$. Recall that $\hat{u}$ is positive by Proposition 3.17. For every compact subset $K \subset \mathcal{R}(\hat{Y})$ and positive number H, it follows from Cheeger–Gromov convergence and the estimate (3.56) that

$$\iint_{K\times[-H,0]} 2|t| \left|Ric + \nabla\nabla\hat{f} + \frac{\hat{g}}{2t}\right|^2 \hat{u}\,dvdt = 0.$$

Note the scaling invariance of $\hat{u}dv$ and $|t|\left|Ric + \nabla\nabla\hat{f} + \frac{\hat{g}}{2t}\right|^2 dt$. Actually, if the above equality fails for some K and H, then by definition of tangent space and the integral accumulation, we shall obtain the left hand side of (3.56) is infinity and obtain a contradiction. Then by the arbitrary choice of K and H, we arrive

$$\iint_{\mathcal{R}(\hat{Y})\times(-\infty,0]} 2|t| \left|Ric + \nabla\nabla\hat{f} + \frac{\hat{g}}{2t}\right|^2 \hat{u}\,dvdt = 0.$$

Note that $\mathcal{R}(\hat{Y})$ is Ricci flat. So there is a smooth function $\hat{f}$ defined on $\mathcal{R}(\hat{Y}) \times (-\infty, 0]$ such that

$$\nabla\nabla\hat{f} + \frac{\hat{g}}{2t} \equiv 0. \tag{3.58}$$

The above equation means that $\nabla\hat{f}$ is a conformal Killing vector field, when restricted on each time slice $t < 0$. It follows from the work of Cheeger–Colding (c.f. [**10**]) that there is a local cone structure around each regular point. We shall show that a global cone structure can be obtained due to the high co-dimension of the singular set $\mathcal{S}$ and the Killing property arises from (3.58). The basic techniques we shall use in our proof is very similar to that in the proof of Lemma 2.31, Lemma 2.34 of [**29**] and Proposition 3.15.

Let's first list the excellent properties of $\hat{f}$. Recall that $\hat{f}$ satisfies the following differential equation on $\mathcal{R} \times (-\infty, 0)$ from the limit process.

$$\hat{f}_t = -\Delta\hat{f} + |\nabla\hat{f}|^2 - R - \frac{n}{t} = |\nabla\hat{f}|^2. \tag{3.59}$$

On the other hand, it follows from (3.58) that

$$\nabla\left(t|\nabla \hat{f}|^2+\hat{f}\right)=2tHess_{\hat{f}}(\nabla \hat{f},\cdot)+\nabla \hat{f}=-\nabla \hat{f}+\nabla \hat{f}\equiv 0.$$

So we have $t|\nabla \hat{f}|^2+\hat{f}=C(t)$, whose time derivatives calculation yields that

$$\begin{aligned} C'(t)&=|\nabla \hat{f}|^2+2t\left\langle \nabla \hat{f},\nabla \hat{f}_t\right\rangle+\hat{f}_t=2|\nabla \hat{f}|^2+2t\left\langle \nabla \hat{f},\nabla|\nabla \hat{f}|^2\right\rangle\\ &=2|\nabla \hat{f}|^2+4tHess_{\hat{f}}\left(\nabla \hat{f},\nabla \hat{f}\right)=2|\nabla \hat{f}|^2-2|\nabla \hat{f}|^2=0, \end{aligned}$$

where we repeatedly used (3.59) and (3.58). Therefore, $t|\nabla \hat{f}|^2+\hat{f}\equiv C$ on $\mathcal{R}\times(-\infty,0)$. Replacing $\hat{f}$ by $\hat{f}+C$ if necessary, we can assume that $t|\nabla \hat{f}|^2+\hat{f}\equiv 0$, which implies that

$$\left(t\hat{f}\right)_t=\hat{f}+t\hat{f}_t=\hat{f}+t|\nabla \hat{f}|^2\equiv 0. \tag{3.60}$$

Consequently, we have

$$\hat{f}(x,t)=\frac{-1}{t}\hat{f}(x,-1), \quad \forall\, x\in\mathcal{R}(\hat{Y}), \tag{3.61}$$

$$\left|\nabla\sqrt{\hat{f}(x,t)}\right|=\frac{1}{2|t|}. \tag{3.62}$$

We remark that the above discussion is nothing but the application of general property of gradient shrinking solitons (c.f. Chapter 4 of Chow–Lu–Ni [**32**]), in the special case that $Ric\equiv 0$.

Intuitively, a space which is both Ricci-flat and is a gradient shrinking soliton must be a metric cone. This can be easily proved if the underlying space is smooth. In our current situation, due to high codimension of $\mathcal{S}$, the cone structure can be established using the technique developed in section 2. Suppose $\hat{Y}$ is a metric cone based at $\hat{y}$, then we should have

$$\hat{f}=\frac{d^2}{4|t|}, \tag{3.63}$$

where d is the distance to the origin. This will be confirmed in the following discussion. The cone structure of $\hat{Y}$ will be established together with equality (3.63). The basic idea to prove (3.63) is to compare the level sets of $\hat{f}$ with geodesic balls, with more and more preciseness. Note that similar ideas to estimate distance will be essentially used in section 5.3 (c.f. Lemma 4.20). We remark that our proof could be much simpler if we use Lemma 3.23, which is independent (c.f. Remark 3.24). For example, the application of Lemma 3.23 directly implies that $\hat{f}$ must achieve minimum only at base point $\hat{y}$ (see step 3 below), since $\hat{f}$ is a strictly convex function in regular part $\mathcal{R}$ and can be regarded as a continuous function on $\hat{Y}$ (c.f. step 1 below). Here we want to give a self-contained proof, using only the good property of $\hat{f}$ to improve the regularity of $\hat{Y}$.

We divide the proof of (3.63) into four steps.

Step 1. $\hat{f}$ is a nonnegative, continuous, proper function which achieves minimum value 0 *at $\hat{y}$.*

Let us focus our attention on time slice $t=-1$ for a while. Denote $\hat{f}(x,-1)$ by $\hat{f}(x)$ for simplicity of notation. It is not hard to observe that $\hat{f}(x)$ is weakly proper. In other words, we have

$$\lim_{\mathcal{R}(\hat{Y})\ni x\to\infty} \hat{f}(x)=\infty. \tag{3.64}$$

For otherwise, we can find a sequence of points $z_i \in \mathcal{R}(\hat{Y})$ such that $d(z_i,\hat{y})\to\infty$ and $\hat{f}(z_i)\leq D$ for some positive number D. Note that $\hat{f}$ is uniformly bounded from below in the ball $B(z_i,1)$. Actually, for every smooth point $x\in B(z_i,1)$, we can find a smooth curve γ connecting x to z_i such that $|\gamma|\leq 3d(x,z_i)$. This is an application of inequality (2.25) in Theorem 2.13. Note that the canonical radius is very large in the current situation. Parametrize γ by arc length and let $\gamma(0)=z_i$ and $\gamma(L)=x$. Then $|\gamma|=L\leq 3$. Along the curve γ, by (3.62), we have

$$\frac{d}{ds}\sqrt{\hat{f}}(\gamma(s))=\left\langle \nabla\sqrt{\hat{f}},\dot{\gamma}(s)\right\rangle \leq \left|\nabla\sqrt{\hat{f}}\right|=\frac{1}{2}.$$

Integration of the above inequality implies that

$$\begin{aligned}\hat{f}(x)=\hat{f}(\gamma(L))&\leq\left(\frac{1}{2}L+\hat{f}(\gamma(0))\right)^2=\left(\frac{1}{2}L+\hat{f}(z_i)\right)^2\\ &\leq (1.5+D)^2\leq 2(1+D)^2.\end{aligned}\tag{3.65}$$

The above inequality holds for every regular point $x\in B(z_i,1)$. In particular, we know $\int_{B(z_i,1)} e^{-\hat{f}}dv$ is uniformly bounded from below by some C^{-1}. Consequently, we have

$$\int_{B(z_i,1)}\hat{u}dv=(4\pi)^{-n}\int_{B(z_i,1)}e^{-\hat{f}}dv\geq\frac{1}{C},$$

for some uniform constant C depending on κ and D. Up to reselecting a subsequence if necessary, we can assume that all $B(z_i,1)$ are disjoint to each other. Then we have

$$C\geq\sum_{i=1}^{\infty}\int_{B(z_i,1)}\hat{u}dv\geq\infty,$$

which is impossible. This contradiction establishes the proof of (3.64). Note that in the above discussion, we already know that the function $\hat{f}$ is bounded on $B\cap\mathcal{R}(\hat{Y})$ for each fixed geodesic ball B, by the application of the proof of (3.65). Consequently, we have uniform gradient estimate of $\hat{f}$ in $B\cap\mathcal{R}(\hat{Y})$ by (3.60), since $t=-1$. The locally Lipschitz condition guarantees that $\hat{f}$ can be extended as a continuous function on whole $\hat{Y}$. Actually, let $\bar{z}$ be a singular point on $\hat{Y}$. Suppose a_k and b_k are two sequences of regular points in $\mathcal{R}(\hat{Y})$ converging to $\bar{z}$. Clearly, $d(a_k, b_k) \to 0$. By inequality (2.25) in Theorem 2.13, we can find a smooth curve $\gamma_k \subset \mathcal{R}(\hat{Y})$ connecting a_k, b_k such that $|\gamma_k| < 3d(a_k, b_k) \to 0$. The bound of $|\nabla \hat{f}|$ then implies that $|\hat{f}(a_k) - \hat{f}(b_k)| \to 0$. So we can define $\hat{f}(\bar{z}) \triangleq \lim_{y \to \bar{z}, y \in \mathcal{R}(\hat{Y})} \hat{f}(y)$ without ambiguity (c.f. Proposition 2.29 of [**29**] for similar discussion). Therefore, from now on we can regard $\hat{f}$ as a continuous function on $\hat{Y}$, rather than only on $\mathcal{R}(\hat{Y})$. Clearly, the previous discussion implies that $\hat{f}$ is proper. Namely, we have

$$\lim_{\hat{Y} \ni x \to \infty} \hat{f}(x) = \infty. \tag{3.66}$$

Consequently, the minimum value of $\hat{f}$ can be achieved at some point $\hat{z}$. The above discussion can be trivially extended for the function $\hat{f}(\cdot, t)$ for each $t \in (-\infty, 0)$. So we know $\hat{f}(\cdot, t)$ is a continuous proper function, which achieves minimum value at $\hat{z}$ also, by (3.61). Furthermore, it is also clear that (3.61) and the first part of (3.60) can be extended to hold on whole $\hat{Y} \times (-\infty, 0)$. Then we observe that

$$\hat{f}(\hat{y}, t) = \min_{x \in \hat{Y}} \hat{f}(x, t) = 0, \quad \forall\, t \in (-\infty, 0). \tag{3.67}$$

Actually, following the discussion around inequality (3.55), we can use the on-diagonal estimate of Cao–Zhang and the gradient estimate of Cao–Hamilton–Zhang to obtain that

$$(4\pi|t|)^{-n} e^{-\hat{f}(x,t) - C} = \hat{u}(x, t) \geq \frac{1}{C}|t|^{-n}, \quad \forall\, x \in B\left(\hat{y}, \sqrt{|t|}\right) \cap \mathcal{R}(\hat{Y}),$$

where we used the fact that we adjusted $\hat{f}$ globally by adding a constant to obtain (3.60). By the continuity of $\hat{f}$, the above inequality implies that

$$\frac{\hat{f}(\hat{y}, -1)}{|t|} = \hat{f}(\hat{y}, t) \leq C, \quad \Rightarrow \quad \hat{f}(\hat{y}, -1) \leq C|t|, \quad \forall\, t \in (-\infty, 0).$$

This forces that $\hat{f}(\hat{y}, -1) = 0$. Recall that $\hat{f}$ is a nonnegative function by (3.60), so we obtain $\min_{x \in \hat{Y}} \hat{f}(x, -1) = 0$. Then (3.67) follows from the extended version of (3.61). So we finish Step 1.

Step 2. Unit level set of $\hat{f}$ is comparable with unit geodesic ball centered at $\hat{y}$.

For each nonnegative number a, we define

$$\Omega_a \triangleq \{x \in \hat{Y} | \hat{f}(x, -1) \leq a^2\}.$$

According to this definition, we immediately know that $\hat{y} \in \Omega_0$. Furthermore, by (3.61), it is clear that

$$\Omega_a = \{x \in \hat{Y} | \hat{f}(x,t) \le |t|^{-1}a^2\}, \quad \forall\, t \in (-\infty, 0).$$

Note that Ω_1 is bounded by the properness of $\hat{f}(\cdot) = \hat{f}(\cdot, -1)$. For simplicity, we assume that $\Omega_1 \subset B(\hat{y}, 0.5H)$ for some $H > 0$. On the other hand, applying the gradient estimate of $\sqrt{\hat{f}}$, i.e., (3.62), and the smooth curve length estimate (2.25), we have

$$\sqrt{\hat{f}(x)} \le \sqrt{\hat{f}(\hat{y})} + \frac{1}{2} \cdot 4 \cdot H \le 2H, \quad \forall\, x \in B(\hat{y}, H),$$

which means that $B(\hat{y}, H) \subset \Omega_{2H}$. Let $D = 2H$, we have the following relationships in short:

$$\Omega_1 \subset B(\hat{y}, 0.25D) \subset B(\hat{y}, 0.5D) \subset \Omega_D. \tag{3.68}$$

Equation (3.68) can be regarded as the first step to improve (3.66) and (3.67). In order to obtain the estimates of general level sets of $\hat{f}$, we need to use the conformal Killing equation (3.58). We observe that the space-time vector field $(-\nabla\hat{f}, \frac{\partial}{\partial t}) = (-\frac{\hat{r}}{2}\frac{\partial}{\partial \hat{r}}, \frac{\partial}{\partial t}) = (-0.5\hat{r}\partial_{\hat{r}}, \partial_t)$, as the "lift" of the conformal Killing vector field $-\nabla\hat{f}$ (c.f. (3.58)), has many excellent properties. First, direct calculation (c.f. (3.59)) shows that

$$\frac{d}{dt}\hat{f} = \hat{f}_t - |\nabla\hat{f}|^2 \equiv 0, \tag{3.69}$$

along the integral curve of this space-time vector field. Second, it follows from (3.58) that

$$L_{(-\nabla\hat{f}, \frac{\partial}{\partial t})}\{|t|\hat{g}\} = 0, \tag{3.70}$$

where L means Lie derivative. Now we can regard $\mathcal{R}(\hat{Y}) \times (-\infty, 0)$ as a Riemannian manifold, equipped with metric $|t|\hat{g}(t) + dt^2$ (c.f. section 6 of Perelman [**49**]). Then $(-\nabla\hat{f}, \frac{\partial}{\partial t})$ is really a Killing vector field.

Step 3. $\hat{f}$ and $d(\hat{y}, \cdot)$ have the same unique minimum value point $\hat{y}$.

In other words, the infimum of $\hat{f}$ must be 0 and it is only achieved at base point $\hat{y}$. We shall use Killing vector field to generate quasi-isometric diffeomorphisms. Then an application of the technique, i.e., bounding distance by choosing good Lipschitz functions, used in the proof Lemma 2.31, Lemma 2.34 of [**29**] and Proposition 3.15 will imply the diameter bound for general level sets Ω_a. For small a, we shall show that $\operatorname{diam}\Omega_a$ is also small. Then Ω_0 has diameter 0 and consists of only one point $\hat{y}$, which is of course the unique minimum point of $d(\hat{y}, \cdot)$. Actually, if one only want to show $\Omega_0 = \{\hat{y}\}$, then there is a shortcut by using the uniform convexity of $\hat{f}$ on $\mathcal{R}(\hat{Y})$ (i.e., equation (3.58)), the homogeneity of $\hat{f}$ in time direction (i.e., equation (3.61)), the fact $\int_{\hat{Y}} e^{-\hat{f}} dv < C$ and the application of Lemma 3.23. We leave the details to interested readers. In the following paragraph, we shall show $\Omega_0 = \{\hat{y}\}$ together with the construction of the cone structure.

Killing vector property together with high codimension of $\mathcal{S}$ implies metric product rigidity, as we have done in Lemma 2.31, Lemma 2.34 of [**29**]. We repeat the discussion here again for the convenience of the readers. Fix each positive integer k. We claim that there is a bounded closed set $E_k \subset \hat{Y}$ satisfying $\dim_{\mathcal{M}}(E_k) < 2n-2$. Furthermore, for each $t \in [-2^{-k}, -2^{-k}]$, we have a family of smooth diffeomorphism $\varphi_{k,t}$ from $\Omega_D \backslash E_k$ to $\Omega_{\sqrt{2^k|t|}D} \backslash E_k$ with

$$\varphi_k^*(\hat{g})(z) = 2^k|t|\hat{g}(z), \quad \forall\, t \in [-2^{-k}, -2^{-k}],\ z \in \Omega_{\sqrt{2^k|t|}D} \backslash E_k. \tag{3.71}$$

The set E_k can be constructed similarly as the set E_k in the proof of Claim 2.32 of [**29**]. Now the Killing vector field ∇b^+ is replaced by the space-time "Killing" (c.f. (3.70)) vector field $(-\nabla \hat{f}, \partial_t)$. Let's describe more details about the construction of E_k. Actually, fixing a small positive number ξ, we define the set $E^-_{k,\xi}$ to be

$$\begin{aligned} \{x \in \Omega_D | &\text{flow line of } (-\nabla \hat{f}, \partial_t) \text{ passing through } (x, -2^{-k}) \\ &\text{hits } \mathcal{D}_\xi \text{ at some } t \in (-2^k, -2^{-k})\}. \end{aligned} \tag{3.72}$$

The minus sign in $E^-_{k,\xi}$ indicates that we are flowing backward along the space-time integral curve of $(-\nabla \hat{f}, \partial_t)$, since $-2^{-k} > t$ for each $t \in (-2^k, -2^{-k})$. Note that the intersection point to $\mathcal{D}_\xi$ locates in a uniformly bounded set. This can be simply proved as follows. Let $(y, -\tau)$ be the first point on $\mathcal{D}_\xi$. By (3.69) and (3.61), we have

$$\begin{aligned} &\frac{\hat{f}(y,-1)}{\tau} = \hat{f}(y,-\tau) = \hat{f}(x,-2^{-k}) = 2^k \hat{f}(x,-1) \leq 2^k \cdot D^2, \\ &\Rightarrow \hat{f}(y,-1) \leq 2^k \tau D^2 \leq 4^k D^2, \end{aligned}$$

which means that $y \in \Omega_{2^k D}$, a uniformly bounded set by the properness of $\hat{f}$. By high Minkowski codimension of $\mathcal{S}$ and the application of the Killing condition (3.70), similar argument for equation (2.54) in Claim 2.32 of [**29**] implies that

$$|E^-_{k,\xi}| \leq C\xi^{2p_0 - 1 - \epsilon},$$

where p_0 is the constant appeared in (3.43), i.e., $\dim_{\mathcal{M}} \mathcal{S} < 2n - 2p_0$, C may depends on ϵ also. Let $\xi_i \to 0$ and define $E^-_k = \cap_{i=1}^{\infty} E^-_{k,\xi_i}$. We obtain a measure-zero closed set E^-_k. Moreover, same as (2.55) in the proof of Claim 2.32 of [**29**], the ξ-neighborhood of E^-_k is contained in $E^-_{k,C\xi}$ for some uniform constant C. Then the above volume estimate implies that $\dim_{\mathcal{M}} E^-_k \leq 2n - 2p_0 + 1 < 2n - 2$. Now we reverse the direction. Similar to the definition of $E^-_{k,\xi}$ in (3.72), we can define $E^+_{k,\xi}$

as follows:

$$\{x \in \Omega_{2^k D} | \text{flow line of } (-\nabla \hat{f}, \partial_t) \text{ passing through } (x, -2^k)$$
$$\text{hits } \mathcal{D}_\xi \text{ at some } t \in (-2^k, -2^{-k})\}.$$

Clearly, the plus sign in $E^+_{k,\xi}$ indicates that we are flowing forward along the space-time integral curve of $(-\nabla \hat{f}, \partial_t)$, since $t > -2^k$ for each $t \in (-2^k, -2^{-k})$. Suppose we start from $(x, -2^k)$ outside $\mathcal{D}_\xi$ and the flow line of $(-\nabla \hat{f}, \partial_t)$ enters $\mathcal{D}_\xi$ at some $(y, -\tau)$. We know $\hat{f}(y, -\tau) = \hat{f}(x, -2^k)$ since the flow preserves $\hat{f}$-value. Then we have

$$\hat{f}(y, -1) = \tau \hat{f}(y, -\tau) = \tau \hat{f}(x, -2^k) = 2^{-k} \tau \hat{f}(x, -1) \leq \hat{f}(x, -1) \leq 4^k D^2.$$

Consequently, $y \in \Omega_{2^k D}$. Therefore, the forward flow is also restricted in a bounded domain when we start from a point $(x, -2^k)$ satisfying $x \in \Omega_{2^k D}$. Applying high codimension of $\mathcal{S}$ and Killing condition again, we know $E^+_{k,\xi}$ has volume bounded by $C\xi^{2p_0 - 1 - \epsilon}$. Let $\xi_i \to 0$ and set E^+_k to be $\cap_{i=1}^\infty E^+_{k,\xi}$. We know E^+_k is a bounded closed set satisfying $\dim_{\mathcal{M}} E^+_k < 2n - 2$. Now we define

$$E_k \triangleq E^+_k \cup E^-_k. \tag{3.73}$$

Then each E_k is a closed bounded set satisfying $\dim_{\mathcal{M}} E_k < 2n - 2$. According to their definitions and the above discussion, we know that there is a family of diffeomorphism $\varphi_{k,t}$, parametrized by $t \in [-2^k, -2^{-k}]$, from $\Omega_D \backslash E_k$ to $\Omega_{\sqrt{2^k |t|} D} \backslash E_k$, generated by the integral curve of $(-\nabla \hat{f}, \partial_t)$. It is clear that (3.71) follows from the integration of (3.70). The above argument is almost the same as that in the proof of Claim 2.32 in Lemma 2.31 of [**29**]. In particular, the argument for the proof of equation (2.53) of [**29**] is more or less repeated here. We remind the readers that weak convexity of $\mathcal{R}$ is not used in the proof of equation (2.53) of [**29**]. Only the high codimension of $\mathcal{S}$ and the Killing vector properties are used.

Now we are ready to use the existence of the diffeomorphism (c.f. discussion around (3.71)) $\varphi_{k,-2^k} : \Omega_D \backslash E_k \to \Omega_{2^k D} \backslash E_k$ to relate the estimate of general Ω_a to (3.68). We are particularly interested in the sets Ω_a for small a's. Without loss of generality, let $a = 2^{-k}$. Fix some points $x, y \in \Omega_{2^{-k}} \backslash E_k$. Denote $\rho = d(x, y)$. Similar to (3.51) in the proof of Proposition 3.15, we choose a function

$$\tilde{\chi} \triangleq \max\{\rho - d(\cdot, x), 0\}. \tag{3.74}$$

Note that $x, y \in \Omega_{2^{-k}} \subset \Omega_1 \subset B(\hat{y}, 0.25D)$ by (3.68), which forces that $\rho = d(x, y) < 0.25D$. Also by (3.68), we know that $\tilde{\chi}$ is supported in $B(x, \rho) \subset B(x, 0.25D) \subset B(\hat{y}, 0.5D) \subset \Omega_D$. Let φ be the diffeomorphism generated by integrating $(-\nabla \hat{f}, \partial_t)$ from $t = -2^{-k}$ to $t = -2^k$.

In other words, $\varphi = \varphi_{k,-2^{-k}}$. Using φ, we can push forward the function $\tilde{\chi}$ to obtain

$$\varphi_*(\tilde{\chi})(z) \triangleq \tilde{\chi}(\varphi^{-1}(z)), \quad \forall\, z \in \Omega_{2^k D} \backslash E_k.$$

Clearly, $\varphi_*(\tilde{\chi})$ is supported on $\Omega_{2^k D} \backslash E_k$ with

$$\|\nabla \varphi_*(\tilde{\chi})\|_{L^\infty(\hat{Y})} \leq 2^{-k} \|\nabla \tilde{\chi}\|_{L^\infty(\hat{Y})} = 2^{-k},$$

in light of (3.71) and $t = -2^k$. By the high codimension of E_k, we know that $\varphi_*(\tilde{\chi})$ is an $N_0^{1,2}$-function, which has a good version such that $\sup_{\hat{Y}} |\nabla \varphi_*(\tilde{\chi})| \leq \|\nabla \varphi_*(\tilde{\chi})\|_{L^\infty(\hat{Y})}$, due to the high codimension of $\mathcal{S}$ (c.f. Claim 3.16). For simplicity of notation, we still denote the new version of $\tilde{\chi}$ by $\tilde{\chi}$. Note that the values of $\tilde{\chi}(x)$ and $\tilde{\chi}(y)$ are independent of the different versions, since x, y are away from E_k. Recall that $x, y \in \Omega_{2^{-k}}$. Integration of (3.69) implies that

$$\hat{f}(x, -2^k) = 2^{-k} \hat{f}(x, -1) = 4^{-k} \hat{f}(x, 2^{-k}) \leq 4^k \cdot 4^{-k} = 1.$$

Therefore, $\varphi(x) \in \Omega_1$. Similarly, we also know $\varphi(y) \in \Omega_1$. Combining the previous inequalities and use (3.68) again, we obtain that

$$\begin{aligned} 0.5D \geq d(\varphi(x), \varphi(y)) &\geq \frac{|\varphi_*(\tilde{\chi})(\varphi(x)) - \varphi_*(\tilde{\chi})(\varphi(y))|}{\sup_{\hat{Y}} |\nabla \varphi_*(\tilde{\chi})|} \\ &\geq \frac{|\varphi_*(\tilde{\chi})(\varphi(x)) - \varphi_*(\tilde{\chi})(\varphi(y))|}{\|\nabla \varphi_*(\tilde{\chi})\|_{L^\infty(\hat{Y})}} \geq \frac{|\tilde{\chi}(x) - \tilde{\chi}(y)|}{2^{-k}}. \end{aligned}$$

Recall that $\tilde{\chi}(x) = \rho$ and $\tilde{\chi}(y) = 0$ by (3.74). It follows from the above inequality that

$$\rho = d(x, y) \leq 0.5D \cdot 2^{-k} = 2^{-1-k} D, \tag{3.75}$$

which is independent of the choice of $x, y \in \Omega_{2^{-k}} \backslash E_k$. Recall that $\Omega_{2^{-k}} \backslash E_k$ is dense in $\Omega_{2^{-k}}$. So we have

$$\operatorname{diam} \Omega_{2^{-k}} = \operatorname{diam}\{\Omega_{2^{-k}} \backslash E_k\} \leq 2^{-1-k} D.$$

Consequently, $\lim\limits_{k \to \infty} \operatorname{diam}(\Omega_{2^{-k}}) = 0$. Since $\Omega_0 = \bigcap_{1 \leq k < \infty} \overline{\Omega_{2^{-k}}}$, we know that Ω_0 consists of only one point $\{\hat{y}\}$.

Step 4. The level sets of $\hat{f}$ coincide the geodesic balls centered at $\hat{y}$. Define

$$\hat{r}(x) \triangleq \sqrt{4 \hat{f}(x, -1)} = \sqrt{4 \hat{f}(x)}, \quad d(x) \triangleq d(x, \hat{y}). \tag{3.76}$$

Recall that in the standard Euclidean case, $\hat{f} = \frac{d^2}{4}$ and $\hat{r} = d$. Our destination (3.63) is equivalent to the equation $\hat{r} - d \equiv 0$. Clearly, we have $|\nabla \hat{r}| = \frac{|\nabla \hat{f}|}{\sqrt{\hat{f}}} = 1$. Recall that (c.f. (3.73)) each E_k is a bounded closed set with $\dim_{\mathcal{M}} E_k < 2n - 2$. Let $E = \cup_{k=1}^\infty E_k$. Then it is clear that E is measure-zero and $\hat{Y} \backslash E$ is dense in $\hat{Y}$. Note that $\hat{Y} \backslash E$ has a cone structure, as every point $x \in \hat{Y} \backslash E$ can be flowed to $\hat{y}$ along the integral curve of $\nabla \hat{f} = \frac{1}{2}\hat{r}\partial_{\hat{r}}$ without hitting singularities (c.f. Section 1 of [**10**]). Let $x \in \mathcal{R}(\hat{Y})$ and $a = \hat{r}(x) > 0$, we can find $x_k \in \mathcal{R}(\hat{Y})\backslash E$ approaching y. Every point x_k can be flowed to a point nearby $\hat{y}$. So we obtain

$$d(x) = d(x, \hat{y}) \leq \lim_{k\to\infty} d(x_k, \hat{y}) \leq \lim_{k\to\infty} \hat{r}(x_k) \leq \hat{r}(x). \tag{3.77}$$

On the other hand, we can construct a function χ as

$$\chi(x) \triangleq \max\{a - \hat{r}(x), 0\},$$

which is supported on a bounded set $\Omega_{0.5a}$. Clearly, χ is Lipschitz. By the high codimension of $\mathcal{S}$, by replacing χ with a new version if necessary, we can assume $\sup_{\hat{Y}} |\nabla\chi| \leq \|\nabla\chi\|_{L^\infty(\hat{Y})} \leq 1$. Note the values at x and y does not depend on the choice of versions since they are regular points. Therefore, we have

$$\begin{aligned} d(x, y) \geq \frac{|\chi(x) - \chi(y)|}{\sup_{\hat{Y}} |\nabla\chi|} &\geq= \frac{|\chi(x) - \chi(y)|}{\|\nabla\chi\|_{L^\infty(\hat{Y})}} \\ &= |\chi(x) - \chi(y)| = |\chi(y)| \geq a - |\hat{r}(y)|, \end{aligned}$$

for every $y \in \mathcal{R}(\hat{Y})$. Let y approach $\hat{y}$ in $\mathcal{R}(\hat{Y})\backslash E$, we obtain

$$d(x) \geq a = \hat{r}(x),$$

which together with (3.77) yields that

$$d(x) = \hat{r}(x), \tag{3.78}$$

for arbitrary $x \in \mathcal{R}(\hat{Y})\backslash\{\hat{y}\}$. Since both d and $\hat{r}$ are uniformly Lipschitz, equation (3.78) holds for every $y \in \hat{Y}$ by continuity and density reason. In particular, the relationship (3.68) can be improved to the following one:

$$\Omega_a = B(\hat{y}, 2a), \quad \forall\, a \geq 0.$$

This confirms our expectation. Clearly, (3.63) follows from the combination of (3.76) and the extended version of (3.78). The proof of (3.63) is complete.

From the discussion in Step 4 of the proof of (3.63), we already know that $\hat{Y}\backslash\{E \cup \{\hat{y}\}\}$ has a local cone structure, which induces the global cone structure of $\hat{Y}$ by taking completion. In view of (2.25) in Proposition 2.13, we know $\mathcal{R}(\hat{Y})$ is path connected. Therefore, the cone $\hat{Y}$ is irreducible, i.e., $\hat{Y}\backslash\{\hat{y}\}$ is path connected. Therefore, we obtain the global cone structure from the local cone structure, due to the high codimension of the singular set $\mathcal{S}$ and the Killing property arisen from (3.58), as we claimed. q.e.d.

3.3.2. Improved estimates in $\mathscr{K}(n,A;r_0)$. In this subsection, we shall improve the limit space structure by the fact that every tangent space is a metric cone. For simplicity, we assume $r_0=1$ if we do not mention otherwise.

Proposition 3.19 (**Improvement of codimension estimate of $\mathcal{S}$**). *Suppose $\mathcal{LM}_i$ is a sequence of polarized Kähler Ricci flow solutions in $\mathscr{K}(n,A;1)$, $x_i \in M_i$. Let $(\bar{M},\bar{x},\bar{g})$ be the limit space of $(M_i,x_i,g_i(0))$. Let $\mathcal{S}$ be the singular part of $\bar{M}$. Then*

$$\dim_{\mathcal{M}} \mathcal{S} \leq 2n-2p_0, \quad \dim_{\mathcal{H}} \mathcal{S} \leq 2n-4, \tag{3.79}$$

where $\dim_{\mathcal{M}}$ *is the Minkowski dimension,* $\dim_{\mathcal{H}}$ *is the Hausdorff dimension.*

Proof. The Minkowski dimension estimate follows from Theorem 2.13. Recall that we are in a situation where canonical radius is uniformly bounded from below. Therefore, there is a gap between local behavior of singular point and regular point. In particular, if one tangent space is Euclidean space, then the base point has a neighborhood with smooth manifold structure. This follows from the volume convergence (c.f. Proposition 2.14) and the *regularity estimate* in the definition of canonical radius (c.f. Definition 2.9). One can find the detailed argument in the proof of Proposition 4.2, where only polarized canonical radius lower bound is used. Note that each iterated tangent space (away from vertex) is also a tangent space, and, hence, force a tangent cone with more splitting directions. Consequently, we can use induction to show that every tangent cone's singularity has an integer Hausdorff dimension (c.f. [**11**]). However, the Minkowski dimension of singularity is at most $2n-2p_0$. This forces that every tangent cone's singularity has Hausdorff dimension $2n-4$ at most, which in turn implies $\dim_{\mathcal{H}} \mathcal{S} \leq 2n-4$. q.e.d.

After we set up the tangent cone structure, we can improve Proposition 2.11.

Proposition 3.20 (**Improvement of regular curve estimate**). *Same conditions as in Proposition 3.19.*

For every two points $x,y \in \mathcal{R}$ and every small positive number $\epsilon>0$, there exists a rectifiable curve connecting x,y such that

- *γ locates in $\mathcal{R}$.*
- *$|\gamma| \leq (1+\epsilon)d(x,y)$.*

Proof. The proof is very similar to the proof of Proposition 2.11. The basic idea is to use the tangent cone structure, i.e., Theorem 3.18 to improve Proposition 2.11.

First, every point in $\bar{M}$ has a cone-like neighborhood.

To be more precise, fix $\epsilon > 0$, for every point $z \in \bar{M}$, there is a radius r_z, depending on z and ϵ, with the following property:

For every point $v \in B(z, r_z)$, one can find a curve α such that

- Initial point of α locates in $B(z, \epsilon d(v,z))$, end point of α locates in $B(v, \epsilon d(v,z))$.
- $\alpha \subset \mathcal{R}, \quad |\alpha| < (1+\epsilon)d(v,z)$.

The existence of r_z can be obtained by application of Theorem 3.18 and a contradiction blowup argument. Actually, if for some z such r_z does not exist, we can find $v_i \to z$ such that corresponding α_i does not exist. Blowup by $d^{-2}(v_i, z)$, we obtain a tangent cone M_∞ with vertex z_∞ and a point v_∞ on the unit sphere of the cone. By the density of regular part in the tangent cone, we have a regular point $\tilde{v}_\infty \in B(v_\infty, 0.5\epsilon)$. The cone structure guarantees that the shortest geodesic connecting $\tilde{v}_\infty$ to z_∞, which we denote by $\overline{z_\infty \tilde{v}_\infty}$, has regular interior (c.f. (3.78)). Denote the intersection of $\overline{z_\infty \tilde{v}_\infty}$ and $M_\infty \backslash B(z_\infty, 0.5\epsilon)$ by α_∞. Then α_∞ is a compact curve and locates in the regular part of M_∞. By the uniform convergence around α_∞, we obtain a curve α_i with the desired property before we arrive limit. Contradiction.

Second, we can find a good covering of each shortest geodesic by cone-like neighborhoods.

Fix any two points $x, y \in \mathcal{R}$. Let β be a shortest geodesic connecting x, y. Since $\bigcup_{z \in \beta} B(z, \frac{1}{4} r_z)$ is a cover of a compact curve β, we can find a finite covering. Starting from this finite covering, by deleting redundant extra balls from x to y (e.g., using the "greedy algorithm"), we obtain a covering $\cup_{i=1}^N B(z_i, \frac{1}{4} r_{z_i})$ with the following properties.

- z_i's are ordered by their distance to x.
- Each point on β locates in at most two balls. If a point on β is contained in two balls, then these two balls must be "adjacent". In other words, if $z \in \beta \cap B(z_k, \frac{1}{4} r_k) \cap B(z_l, \frac{1}{4} r_l)$, then $|k - l| = 1$.
- Every pair of "adjacent" balls have nonempty intersection, i.e., if $|k-l| = 1$, then $B(z_k, \frac{1}{4} r_k) \cap B(z_l, \frac{1}{4} r_l) \neq \emptyset$.

Third, based on the good covering, one can construct approximation curve.

Now we have a covering of β by $\cup_{k=0}^N B(z_k, \frac{1}{4} r_k)$ with the property mentioned in the second step. Without loss of generality, we further assume $z_0 = x, z_N = y$. For each $0 \leq k \leq N-1$, let β_k be the part of β connecting z_k and z_{k+1}, let d_k be the length of β_k. Then we have

$$d_k = d(z_k, z_{k+1}) < \frac{1}{4} r_k + \frac{1}{4} r_{k+1} \leq \frac{1}{2} \max\{r_k, r_{k+1}\}.$$

Hence, either z_{k+1} locates in the cone-like neighborhood of z_k, or z_k locates in the cone-like neighborhood of z_{k+1}. No matter what case happens, we can find an approximation curve $\alpha_k \subset \mathcal{R}$, whose two ends locate in the ϵd_k neighborhood of z_k and z_{k+1}, satisfying $|\alpha_k| < (1+\epsilon)d_k$. According to this choice, the end point of α_{k-1} and the initial point of α_k have distance bounded by $\epsilon(d_k + d_{k-1})$, whenever $1 \leq k \leq N-1$. So they can be connected by a curve $\gamma_k \subset \mathcal{R}$ with $|\gamma_k| \leq 3\epsilon(d_k + d_{k-1})$, due to Proposition 2.11. For the boundary case, it is not hard to see that $z_0 = x$ can be connected to the initial point of α_0 by $\gamma_0 \subset \mathcal{R}$ and $|\gamma_0| < 3\epsilon d_0$. Similarly, $z_N = y$ can be connected to the end point of α_{N-1} by $\gamma_N \subset \mathcal{R}$ and $|\gamma_N| < 3\epsilon d_{N-1}$. Concatenating all the curves α_k and γ_k, we obtain a curve $\gamma \subset \mathcal{R}$ connecting x, y and satisfying

$$\begin{aligned}
|\gamma| &= \sum_{k=0}^{N-1} |\alpha_k| + \sum_{k=0}^{N} |\gamma_k| \\
&\leq \left(\sum_{k=0}^{N-1} (1+\epsilon) d_k \right) + \left(3\epsilon d_0 + \sum_{k=0}^{N-2} 3\epsilon (d_k + d_{k+1}) + 3\epsilon d_{N-1} \right) \\
&= (1+\epsilon) \sum_{k=0}^{N-1} d_k + 6\epsilon \sum_{k=0}^{N-1} d_k = (1+7\epsilon)|\beta| = (1+7\epsilon) d(x,y).
\end{aligned}$$

Replacing ϵ by 0.1ϵ at the beginning, we then find a curve γ satisfying the requirement. q.e.d.

Lemma 3.21 (**Rough estimate of reduced distance**). *There is an $\epsilon = \epsilon(n, A)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A; 1)$, $x, y \in M$ and $r = d_0(x, y) < 1$. Suppose $y \in \mathcal{F}_{\frac{\epsilon_b}{2} r}(M, 0)$. Then we have

$$l((x,0),(y,-r^2)) < 100, \tag{3.80}$$

whenever $\sup_{\mathcal{M}}(|R| + |\lambda|) < \epsilon$.

Proof. Let $y_0 = y$. According to the construction in Proposition 3.14 of [**29**], there exists a point $y_1 \in \partial B_{g(0)}(x, \frac{r}{2}) \cap \mathcal{F}_{\frac{\epsilon_b r}{4}}(M, 0)$ and a curve $\gamma_1 \subset \mathcal{F}_{\frac{\epsilon_b^2}{8} r}(M, 0)$ connecting y_0, y_1, with length less than $\frac{9}{2} r$. Suppose $|R| + |\lambda|$ is small enough, then $\gamma_1 \subset \bigcap_{-r^2 \leq t \leq 0} \mathcal{F}_{\frac{\epsilon_b^2 r}{16}}(M, t)$. So γ_1 can be lifted as a space-time curve connecting $(y_1, -\frac{r^2}{4})$ and $(y_0, -r^2)$. Reparameterizing γ_1 by τ, after a proper adjustment, we have

$$\int_{\frac{r^2}{4}}^{r^2} \sqrt{\tau} |\dot{\gamma}_1|^2_{g(-\tau)} d\tau < 100 r.$$

Following the same procedure, we can find γ_2 connecting y_1 to $y_2 \in \partial B_{g(0)}(x, \frac{r}{4}) \cap \mathcal{F}_{\frac{\epsilon_b r}{8}}(M, 0)$ with $\gamma_2 \subset \bigcap_{-\frac{r^2}{4} \leq t \leq 0} \mathcal{F}_{\frac{\epsilon_b^2 r}{32}}(M, t)$. By a proper reparameterization of τ, we can regard γ_2 as a space-time curve connecting $(y_1, -\frac{r^2}{4})$ and $(y_2, -\frac{r^2}{16})$, and it satisfies the estimate

$$\int_{\frac{r^2}{16}}^{\frac{r^2}{4}} \sqrt{\tau}|\dot{\gamma}_2|^2_{g(-\tau)} d\tau < 100 \cdot \frac{r}{2}.$$

Note that there is no need to choose a new ϵ because of the rescaling property of $|R| + |\lambda|$. Repeating this process, we can find curve γ_k connecting $(y_k, -\frac{r^2}{4^k})$ and $(y_{k+1}, -\frac{r^2}{4^{k+1}})$. Concatenating all γ_k's together, we obtain a space-time curve γ connecting $(x, 0)$ and $(y, -r^2)$ such that

$$\int_0^{r^2} \sqrt{\tau}|\dot{\gamma}|^2_{g(-\tau)} d\tau < 100 \sum_{k=0}^{\infty} \frac{r}{2^k} = 200r.$$

It follows that

$$l((x,0),(y,-r^2)) < \frac{200r}{2\sqrt{r^2}} = 100. \qquad \text{q.e.d.}$$

Lemma 3.22 (**Most shortest reduced geodesics avoid high curvature part**). *For every group of numbers $0 < \xi < \eta < 1 < H$, there is a big constant $C = C(n, A, \eta, H)$ and a small constant $\epsilon = \epsilon(n, A, H, \eta, \xi)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A; 1)$, $x \in \mathcal{F}_\eta(M, 0)$. Let Ω_ξ be the collection of points $z \in M$ such that there exists a shortest reduced geodesic $\boldsymbol{\beta}$ connecting $(x, 0)$ and $(z, -1)$ satisfying

$$\beta \cap \mathcal{D}_\xi(M, 0) \neq \emptyset. \tag{3.81}$$

Then

$$|B_{g(0)}(x, H) \cap \mathcal{F}_\eta(M, 0) \cap \Omega_\xi| < C\xi^{2p_0 - 1}, \tag{3.82}$$

whenever $\sup_{\mathcal{M}}(|R| + |\lambda|) < \epsilon$.

Proof. This is a flow property, we assume $\lambda = 0$ without loss of generality.

From the argument in Lemma 3.21, it is not hard to obtain the following bound

$$l((x,0),(z,-1)) < C, \quad \forall\, z \in B_{g(0)}(x, H) \cap \mathcal{F}_\eta(M, 0), \tag{3.83}$$

where $C = C(\eta, H)$. Suppose $z \in B_{g(0)}(x, H) \cap \mathcal{F}_\eta(M, 0)$, $\boldsymbol{\beta}$ is a shortest reduced geodesic connecting $(x, 0)$ and $(z, -1)$. Let β be the corresponding space curve. Note that x and z, the two end points of β, locate outside of $\mathcal{D}_\xi(M, 0)$. Therefore, if $z \in \Omega_\xi$, then (3.81) is satisfied. In other words, the shortest reduced geodesic connecting $(x, 0)$ and $(z, -1)$ cannot avoid the "high curvature" part $\mathcal{D}_\xi(M, 0)$. By continuity, we have

$$\beta \cap \partial\mathcal{F}_\xi(M, 0) = \beta \cap \partial\mathcal{D}_\xi(M, 0) \neq \emptyset.$$

Let τ_a be the first time $\boldsymbol{\beta}$ escape from $\mathcal{F}_{K^{-1}\eta}$, τ_b be the last time such that $\boldsymbol{\beta}(\tau)$ re-enter $\mathcal{F}_{K^{-1}\eta}$, when we move along backward time direction. Here K is the constant defined in Proposition 2.10. To be more precise, we define

$$\tau_a \triangleq \sup\left\{\tau \,\middle|\, \beta(s) \in \mathcal{F}_{K^{-1}\eta}, \quad \forall\, s \in (0,\tau)\right\},$$
$$\tau_b \triangleq \inf\left\{\tau \,\middle|\, \beta(s) \in \mathcal{F}_{K^{-1}\eta}, \quad \forall\, s \in (\tau,1)\right\}.$$

By the choice of x and z, it is clear that $0 < \tau_a < \tau_b < 1$. We can further estimate τ_a and τ_b uniformly. Actually, since l is achieved by $\boldsymbol{\beta}$ and is bounded by C, it follows from the definition of l (c.f. equation (2.4) and (2.5)) that

$$\int_0^1 \sqrt{\tau}\left(R + |\dot{\beta}|^2\right)_{g(-\tau)} d\tau < C. \tag{3.84}$$

Note that $\beta(\tau) \in \mathcal{F}_{\eta}(M,0)$ whenever $\tau \in (0,\tau_a) \cup (\tau_b, 1)$. In view of Proposition 3.15, we have the metric equivalence

$$0.5g(x,0) < g(x,-\tau) < 2g(x,0),$$

for all $x \in \mathcal{F}_{K^{-1}\eta}(M,0)$ and $\tau \in (0,1)$. Recalling that $|R|$ is uniformly small. Then (3.84) implies that

$$\int_0^{\tau_a} \sqrt{\tau}|\dot{\beta}|^2_{g(0)} d\tau < C, \quad \int_{\tau_b}^1 \sqrt{\tau}|\dot{\beta}|^2_{g(0)} d\tau < C.$$

It follows from Proposition 2.10 and the above inequality that

$$\begin{aligned}
\frac{\eta}{C} &< d_{g(0)}(x, \beta(\tau_a)) \leq \int_0^{\tau_a} |\dot{\beta}|_{g(0)} d\tau \\
&< \left(\int_0^{\tau_a} \sqrt{\tau}|\dot{\beta}|^2_{g(0)} d\tau\right)^{\frac{1}{2}} \left(\int_0^{\tau_a} \frac{1}{\sqrt{\tau}} d\tau\right)^{\frac{1}{2}} < C\tau_a^{\frac{1}{4}},
\end{aligned} \tag{3.85}$$

$$\begin{aligned}
\frac{\eta}{C} &< d_{g(0)}(\beta(\tau_b), z) \leq \int_{\tau_b}^1 |\dot{\beta}|_{g(0)} d\tau \\
&< \left(\int_{\tau_b}^1 \sqrt{\tau}|\dot{\beta}|^2_{g(0)} d\tau\right)^{\frac{1}{2}} \left(\int_{\tau_b}^1 \frac{1}{\sqrt{\tau}} d\tau\right)^{\frac{1}{2}} < C\sqrt{1-\sqrt{\tau_b}},
\end{aligned} \tag{3.86}$$

where $C = C(\eta, H, K)$. Consequently, we have

$$\tau_a > \frac{\eta^4}{C}, \quad 1 - \tau_b = (1-\sqrt{\tau_b})(1+\sqrt{\tau_b}) \geq 1 - \sqrt{\tau_b} \geq \frac{\eta^2}{C}.$$

This means that $[\tau_a, \tau_b] \subset \left[\frac{\eta^4}{C}, 1 - \frac{\eta^2}{C}\right]$. Define $\bar{\tau}$ as

$$\bar{\tau} \triangleq \max\{\tau | \beta(\tau) \in \mathcal{D}_{\xi}(M,0)\}. \tag{3.87}$$

Clearly, we have $\beta(\bar{\tau}) \in \partial\mathcal{D}_{\xi}(M,0) = \partial\mathcal{F}_{\xi}(M,0)$. Since $\xi < K^{-1}\eta$, we have $\bar{\tau} \in [\tau_a, \tau_b]$ for continuity reason. Consequently, we know

$$\frac{\eta^4}{C} < \bar{\tau} < 1 - \frac{\eta^2}{C}, \tag{3.88}$$

for some $C = C(\eta, H, K)$, whenever $\xi < K^{-1}\eta$ and $|R|$ very small.

Beyond the estimate of $\bar{\tau}$, there are more estimates around $\boldsymbol{\beta}(\bar{\tau})$. In light of the choice of $\bar{\tau}$, we have $\beta(\tau) \in \mathcal{F}_{\xi}(M,0)$ for each $\tau \in [\bar{\tau}, 1]$. By (3.83), we have uniform rough bound of the reduced distance from $(x,0)$ to $(z,-1)$. Noting that R may be negative and $|R|$ is very small, we have

$$\int_{\bar{\tau}}^{1} \sqrt{\tau}\left(R + |\dot{\beta}|^2\right)_{g(-\tau)} d\tau < 1 + \int_{0}^{1} \sqrt{\tau}\left(R + |\dot{\beta}|^2\right)_{g(-\tau)} d\tau < C(\eta, H).$$

Following the route of (3.85), noting that metrics $g(0)$, $g(-\tau)$ and $g(-1)$ are all uniformly equivalent on $\beta(\tau)$ whenever $\tau \in [\bar{\tau}, 1]$, we have

$$\begin{aligned} d_{g(0)}(z, \beta(\bar{\tau})) &\leq \int_{\bar{\tau}}^{1} |\dot{\beta}|_{g(0)} d\tau \leq 2\int_{\bar{\tau}}^{1} |\dot{\beta}|_{g(-1)} d\tau \\ &< 2\left(\int_{\bar{\tau}}^{1} \sqrt{\tau}|\dot{\beta}|^2_{g(-1)} d\tau\right)^{\frac{1}{2}} \left(\int_{\bar{\tau}}^{1} \frac{1}{\sqrt{\tau}} d\tau\right)^{\frac{1}{2}} < C. \end{aligned}$$

Note that $d_{g(0)}(z,x) < H$. Triangle inequality then implies that

$$d_{g(0)}(\beta(\bar{\tau}), x) < F, \tag{3.89}$$

for some F independent of ξ when $|R| + |\lambda|$ small enough.

The purpose of this paragraph is to estimate $\dot{\beta}(\bar{\tau})$. Recalling the reduced geodesic equation (2.6):

$$\nabla_V V + \frac{V}{2\tau} + 2Ric(V, \cdot) + \frac{\nabla R}{2} = 0,$$

where $V = \dot{\beta}$. It follows that along the reduced geodesic $\boldsymbol{\beta}$, we have

$$\begin{aligned} &\frac{d}{d\tau}|\dot{\beta}|^2 = 2\langle \nabla_{\dot{\beta}}\dot{\beta}, \dot{\beta}\rangle + 2Ric(\dot{\beta}, \dot{\beta}) = -\frac{|\dot{\beta}|^2}{\tau} - 2Ric(\dot{\beta}, \dot{\beta}) - \langle \nabla R, \dot{\beta}\rangle, \\ &\frac{d}{d\tau}\left\{\tau|\dot{\beta}|^2\right\} = -\tau\left\{2Ric(\dot{\beta}, \dot{\beta}) + \langle \nabla R, \dot{\beta}\rangle\right\}. \end{aligned}$$

Note that $\beta(\tau) \in \mathcal{F}_{\xi}(M,0)$ for each $\tau \in [\bar{\tau}, 1]$. By Proposition 3.12, we can assume $\beta(\tau) \in \mathcal{F}_{K^{-1}\xi}(M, -\tau)$. It follows that $|Ric|$ and $|\nabla R|$ are uniformly small whenever $|R|$ globally very small. Therefore, the above equation implies

$$\left|\frac{d}{d\tau}\left(\tau|\dot{\beta}|^2 + 1\right)\right| < \theta\left(\tau|\dot{\beta}|^2 + 1\right), \quad \forall\, \tau \in (\bar{\tau}, 1), \tag{3.90}$$

for some small constant θ depending on ξ and $\sup_{\mathcal{M}} |R|$. Moreover, $\theta \to 0$ if $\sup_{\mathcal{M}} |R| \to 0$ and ξ is fixed. Integrating (3.90) and using (3.88), we obtain

$$\tau|\dot{\beta}|^2 + 1 > e^{-\theta} \left(\bar{\tau}|\dot{\beta}|^2_{g(-\bar{\tau})} + 1 \right).$$

It follows that

$$\begin{aligned}
&\int_{\bar{\tau}}^{1} \sqrt{\tau}|\dot{\beta}|^2_{g(-\tau)} d\tau \\
&= \int_{\bar{\tau}}^{1} \frac{1}{\sqrt{\tau}} \tau|\dot{\beta}|^2_{g(-\tau)} d\tau \geq \int_{\bar{\tau}}^{1} \left\{ e^{-\theta} \left(\bar{\tau}|\dot{\beta}|^2_{g(-\bar{\tau})} + 1 \right) - 1 \right\} d\tau \\
&= e^{-\theta}|\dot{\beta}|^2_{g(-\bar{\tau})} \bar{\tau}(1-\bar{\tau}) + (e^{-\theta} - 1)(1-\bar{\tau}).
\end{aligned}$$

In view of (3.84) and the fact that $|R|$ is very small, we know the left hand side of the above inequality is bounded above by $C = C(\eta, H)$. Since θ is very small, $\bar{\tau} \in \left[\frac{\eta^4}{C}, 1 - \frac{\eta^2}{C}\right]$ by (3.88), the above inequality yields that

$$\left| \dot{\beta}(\bar{\tau}) \right|_{g(-\bar{\tau})} < C, \tag{3.91}$$

where $C = C(\eta, H, K)$ is independent of ξ. Note that $\boldsymbol{\beta}(\tau) = (\beta(\tau), -\tau)$, the space-time tangent vector of $\boldsymbol{\beta}$ is $(\dot{\beta}, -1)$. Intuitively, (3.91) can be understood that the "angle" between the space-time tangent and the space tangent form a positive "angle" which is uniformly bounded below.

Note that the reduced volume element $(4\pi\tau)^{-n}e^{-l}dv$ is decreasing along $\boldsymbol{\beta}$. Up to a perturbation, $\partial\mathcal{F}_\xi(M, 0)$ can be regarded as a smooth hypersurface in M satisfying

$$\left| \partial\mathcal{F}_\xi(M, 0) \cap B_{g(0)}(x, F) \right|_{\mathcal{H}^{2n-1}} \leq C\xi^{2p_0 - 1}, \tag{3.92}$$

for some $C = C(n, A, F)$, $F = F(\eta, H)$ is the constant in (3.89). Consequently, $\partial\mathcal{F}_\xi(M, 0) \times [-1, 0]$ can be regarded as a hypersurface in the space-time. Recall that Ω_ξ is the collection of points $z \in M$ such that there exists a shortest reduced geodesic $\boldsymbol{\beta}$ connecting $(x, 0)$ and $(z, -1)$ satisfying (3.81). By reduced geodesic theory (c.f. Section 7 of [**49**] and the corresponding sections in [**41**] for more details), the following results are known.

(a). For every $z \in M$, $(z, -1)$ can be connected to $(x, 0)$ by a shortest reduced geodesic.

(b). For every $z \in M \backslash E$, $(z, -1)$ can be connected to $(x, 0)$ by a unique shortest reduced geodesic, where E is a measure-zero set and is called the $\mathcal{L}$-cut-locus.

Therefore, we can define a projection map φ as follows.

$$\begin{aligned} \varphi : B_{g(0)}(x,H) \cap \mathcal{F}_\eta(M,0) \cap \{\Omega_\xi \backslash E\} &\mapsto \partial\mathcal{F}_\xi(M,0) \times [-1,0], \\ z &\mapsto \boldsymbol{\beta}(\bar{\tau}). \end{aligned} \tag{3.93}$$

For simplicity, let $\Omega = B_{g(0)}(x,H) \cap \mathcal{F}_\eta(M,0) \cap \{\Omega_\xi \backslash E\}$. The reduced distance bound (3.83) and the entering-time bound (3.88) implies that the reduced volume element $(4\pi\tau)^{-n}e^{-l}dv$ along $\boldsymbol{\beta}$ is uniformly equivalent to dv, whenever $\tau \in [\bar{\tau}, 1]$. Since $(4\pi\tau)^{-n}e^{-l}dv$ is monotone along $\boldsymbol{\beta}$, we can regard dv as almost monotone, up to multiplying a uniform constant C. Therefore, we have

$$|\Omega|_{\mathcal{H}^{2n}} = \int_\Omega 1 dv \le C\int_\Omega e^{-l(z)}dv_z \le \int_{\varphi(\Omega)} \bar{\tau}^{-n}e^{-l(y)}dv_y \le C\int_{\varphi(\Omega)} dv_y,$$

where $y = \varphi(x)$. Note that inequality (3.91) can be regarded as an "angle" bound, since $\dot{\boldsymbol{\beta}} = (\dot{\beta}, -1)$. The uniform bound of $|\dot{\beta}|$ guarantees that $dv_y \le C|d\sigma_y \wedge dt|$ where $d\sigma_y$ is the "area" element of $\partial\mathcal{F}_\xi$. Then we have

$$\begin{aligned} |\Omega|_{\mathcal{H}^{2n}} &\le C\int_{\varphi(\Omega)} |d\sigma_y \wedge dt| \le C\int_{\left\{\partial\mathcal{F}_\xi(M,0)\cap B_{g(0)}(x,F)\right\}\times[-1,0]} |d\sigma_y \wedge dt| \\ &= C\left|\left\{\partial\mathcal{F}_\xi(M,0) \cap B_{g(0)}(x,F)\right\} \times [-1,0]\right|_{\mathcal{H}^{2n}} \\ &\le C\left|\partial\mathcal{F}_\xi(M,0) \cap B_{g(0)}(x,F)\right|_{\mathcal{H}^{2n-1}}, \end{aligned}$$

where we used the almost product structure of $\partial\mathcal{F}_\xi \times [-1,0]$ in the last step. Note that $C = C(\eta, H, K) = C(n, A, \eta, H)$ since K is determined by n, A (c.f. Proposition 2.10). Recall that

$$\Omega = B_{g(0)}(x,H) \cap \mathcal{F}_\eta(M,0) \cap \{\Omega_\xi \backslash E\}.$$

Plugging (3.92) into the above inequality, we obtain (3.82). q.e.d.

Note that in Lemma 3.22, for every point

$$z \in \left\{B_{g(0)}(x,H) \cap \mathcal{F}_\eta(M,0)\right\} \backslash \{\Omega_\xi \cup E\},$$

there is a unique shortest reduced geodesic connecting $(z,-1)$ to $(x,0)$ and avoiding $\mathcal{D}_\xi(M,0)$. If $z \in \left\{B_{g(0)}(x,H) \cap \mathcal{F}_\eta(M,0) \cap E\right\} \backslash \Omega_\xi$, then every shortest reduced geodesic connecting $(z,-1)$ to $(x,0)$ must avoid $\mathcal{D}_\xi(M,0)$. However, we may not have uniqueness.

Now we pass Lemma 3.22 to limit and have the following property.

Lemma 3.23 (Rough weak convexity by reduced geodesics). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A; 1)$ satisfies*

$$\lim_{i\to\infty}\left(\frac{1}{T_i} + \frac{1}{\mathrm{Vol}(M_i)} + \sup_{\mathcal{M}_i}(|R| + |\lambda|)\right) = 0. \tag{3.94}$$

Suppose $x_i \in M_i$. Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$, $\mathcal{R}$ be the regular part of $\bar{M}$ and $\bar{x} \in \mathcal{R}$. Suppose $\bar{t} < 0$ is a fixed number.

Then every $(\bar{z},\bar{t})$ can be connected to $(\bar{x},0)$ by a smooth reduced geodesic, whenever $\bar{z}$ is away from a closed measure-zero set.

Proof. Without loss of generality, let $\bar{t}=-1$ and $\lambda=0$. We use $\bar{g}(0)$ as the default metric on the limit space.

Since $\bar{x}\in\mathcal{R}$, it locates in $\mathcal{R}_{\eta_0}$ for some $\eta_0\in(0,1)$, where we used the notation defined in equation (2.21). Fix $\eta\in(0,\eta_0)$. Let $E_{\eta,\xi}$ be the closure of the limit set of $B_{g_i(0)}\left(x_i,\eta^{-1}\right)\cap\mathcal{F}_\eta(M_i,0)\cap\Omega_\xi(M_i)$, which we denote by $E'_{\eta,\xi}(M_i)$ for simplicity. Suppose $\bar{z}\in E_{\eta,\xi}$ is the limit of some sequence $z_i\in E'_{\eta,\xi}(M_i)$. Then it is easy to see that

$$\bar{z}\in\overline{B(\bar{x},\eta^{-1})}\cap\mathcal{R}_\eta(\bar{M}).$$

For each i, there is a shortest reduced geodesic $\boldsymbol{\beta}_i$ connecting $(x_i,0)$ to $(z_i,-1)$ and passing through $\mathcal{D}_\xi(M_i,0)$. Let $\boldsymbol{\beta}$ be the limit of $\boldsymbol{\beta}_i$. Note that $\boldsymbol{\beta}$ may pass through singularity. The largest τ such that $\boldsymbol{\beta}(\tau)$ comes out of $\mathcal{S}_\xi$ (c.f. equation (2.22) for notations) is denoted by $\bar{\tau}$ (c.f. equation (3.87)). By (3.88), i.e., $\frac{\eta^4}{C}<\bar{\tau}<1-\frac{\eta^2}{C}$, we know $\bar{\tau}$ is uniformly bounded away from 0 and 1. Moreover, $d(\bar{x},\beta(\bar{\tau}))$ is uniformly bounded by some constant F (c.f. inequality (3.89)), the value $|\dot{\beta}(\bar{\tau})|$ is uniformly bounded by inequality (3.91). By taking limit on $\bar{M}$, we see that for every point $\bar{z}$ (no matter whether it is a limit of points in $E'_{\eta,\xi}(M_i)$), we can find a shortest reduced geodesic $\boldsymbol{\beta}$ connecting $(\bar{x},0)$ and $(\bar{z},-1)$, with $\bar{\tau}$ satisfying (3.88) and $\beta(\bar{\tau})$ locating in $B(\bar{x},F)$ for some uniform constant F, and $|\dot{\beta}(\bar{\tau})|$ uniformly bounded by C. Note that both C and F are independent of ξ.

As a closure, $E_{\eta,\xi}$ is clearly a closed set. Note that

$$E_{\eta,\xi}\subset\overline{B(\bar{x},\eta^{-1})\cap\mathcal{R}_\eta}\subset B(\bar{x},2\eta^{-1})\cap\overset{\circ}{\mathcal{R}}_{0.5\eta},$$

which is an open smooth manifold. Therefore, $E_{\eta,\xi}$ is measurable.

Suppose $\bar{z}_a,\bar{z}_b$ are two points in $\bar{E}_{\eta,\xi}$. Tracing their origin and use the shortest property, it is clear that $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}_b$ have no intersection except $(\bar{x},0)$, where $\boldsymbol{\beta}_a$ is a shortest reduced geodesic connecting $(\bar{x},0)$ to $(\bar{z}_a,-1)$, $\boldsymbol{\beta}_b$ is a shortest reduced geodesic connecting $(\bar{x},0)$ to $(\bar{z}_b,-1)$. Similar to (3.93) in the proof of Lemma 3.22, we now define a multi-valued projection map $\tilde{\varphi}$ from $E_{\eta,\xi}$ to $\partial\mathcal{R}_\xi$ as follows:

$$\begin{aligned}\tilde{\varphi}: E_{\eta,\xi}&\mapsto\partial\mathcal{R}_\xi\times[-1,0],\\ z&\mapsto\{\boldsymbol{\beta}(z),\ \boldsymbol{\beta}\text{ is a shortest reduced geodesic connecting}\\ &\qquad\qquad(z,-1)\text{ to }(\bar{x},0)\text{ with }\beta\cap\mathcal{S}_\xi\neq\emptyset\}.\end{aligned}$$

Following the argument at the end of the proof of Lemma 3.22, we have

$$|E_{\eta,\xi}|_{\mathcal{H}^{2n}}=\int_{E_{\eta,\xi}}1dv\le C\int_{E_{\eta,\xi}}e^{-l(z)}dv_z\le\int_{\tilde{\varphi}(E_{\eta,\xi})}\bar{\tau}^{-n}e^{-l(y)}dv_y,$$

where $(y,-\bar{\tau})=\boldsymbol{\beta}(\bar{\tau})$ for some $\boldsymbol{\beta}$ connecting $(\bar{x},0)$ to $(z,-1)$ satisfying $\beta\cap\mathcal{S}_{\xi}\neq\emptyset$. Note that the last inequality holds even if $\tilde{\varphi}$ is multi-valued. Starting from the above step, the remainder argument exactly follows from the proof of (3.82). Consequently, we have

$$|E_{\eta,\xi}|\leq C\xi^{2p_0-1}, \tag{3.95}$$

for some C independent of ξ. Note that $E_{\eta_1,\xi_1}\subset E_{\eta_2,\xi_2}$ whenever

$$0<\xi_1<\xi_2, \quad 0<\eta_2\leq\eta_1.$$

Then we define

$$E_{\eta}\triangleq\bigcap_{\xi\in(0,\eta)}E_{\eta,\xi}. \tag{3.96}$$

In light of (3.95), we see that E_{η} is a closed subset of $\overline{B(\bar{x},\eta^{-1})\cap\mathcal{R}_{\eta}(\bar{M})}$ with measure zero. Suppose

$$\bar{z}\in\left\{B(\bar{x},\eta^{-1})\cap\mathcal{R}_{\eta}(\bar{M})\right\}\backslash E_{\eta}=\bigcup_{\xi\in(0,\eta)}\left\{\left\{B(\bar{x},\eta^{-1})\cap\mathcal{R}_{\eta}(\bar{M})\right\}\backslash E_{\eta,\xi}\right\},$$

then $\bar{z}\in B(\bar{x},\eta^{-1})\cap\mathcal{R}_{\eta}(\bar{M})\backslash E_{\eta,\xi}$ for some $\xi\in(0,\eta)$. By the smooth flow convergence on $\mathcal{F}_{\xi}(M_i,0)\times[-1,0]$ (c.f. Proposition 3.15) and the definition of $E_{\eta,\xi}$, we obtain that $(\bar{z},-1)$ can be connected to $(\bar{x},0)$ by some shortest smooth reduced geodesic contained in $\mathcal{R}_{\xi}(\bar{M})\times[-1,0]$. Moreover, every smooth shortest reduced geodesic connecting $(\bar{x},0)$ and $(\bar{z},-1)$ are uniformly ξ-regular. To be more precise, every point $\bar{z}\in\left\{B(\bar{x},\eta^{-1})\cap\mathcal{R}_{\eta}(\bar{M})\right\}\backslash E_{\eta}$ satisfies the following property:

$(\bar{z},-1)$ *can be connected to* $(\bar{x},0)$ *by a shortest smooth reduced geodesic* $\boldsymbol{\beta}$. *In other words, for every other smooth reduced geodesic* $\boldsymbol{\gamma}$ *with the same ends, we have* $\mathcal{L}(\boldsymbol{\gamma})\geq\mathcal{L}(\boldsymbol{\beta})$.

Now we define

$$E\triangleq\bigcup_{k\in\{1,2,\cdots\}}E_{2^{-k}\eta_0}\backslash\left\{B(\bar{x},2^{k-2}\eta_0^{-1})\cap\mathring{\mathcal{R}}_{2^{-k+2}\eta_0}\right\}. \tag{3.97}$$

The η_0 above is some fixed positive number. According to this definition, every regular point locates in finitely many closed sets

$$E_{2^{-k}\eta_0}\backslash\left\{B(\bar{x},2^{k-2}\eta_0^{-1})\cap\mathring{\mathcal{R}}_{2^{k-2}\eta_0}\right\}.$$

The reason we choose to define E in the way of (3.97) is to obtain the closedness of $E\cup\mathcal{S}$. Note that if we simply define E to be the union of all $E_{2^{-k}\eta_0}$, then $E\cup\mathcal{S}$ may not be closed set. It is possible to obtain points in $E_{2^{-k}\eta_0}$ converging to a regular point. However, from the discussion in the above paragraph, it is clear that for every regular point, one can find a small closed ball regular neighborhood $\bar{B}$ where every point (with time $t=-1$) can be connected to $(\bar{x},0)$ away from a closed set $E_{\bar{B}}=E_{\eta}\cap\bar{B}$, where η depends on $\bar{B}$. Taking a countable, locally finite cover of $\mathcal{R}$ by such $\bar{B}$'s and let E' be the union of such $E_{\bar{B}}$. Then E' is measure zero and relatively closed in $\mathcal{R}$. The choice of E in (3.97) follows the same idea, with the covering of $\mathcal{R}$ being written down explicitly.

It follows from (3.97) that E is the union of countably many measure-zero sets. Consequently, E is measure-zero. Fix arbitrary $\bar{z} \in \mathcal{R} \backslash E$. Because $\bar{z} \in \mathcal{R}$, we see that $\bar{z} \in B(\bar{x}, \eta^{-1}) \cap \mathcal{R}_{\eta}(\bar{M})$ for some $\eta > 0$. Accordingly, we can find k_0 very large such that

$$\bar{z} \in B(\bar{x}, 2^{k_0}\eta_0^{-1}) \cap \mathcal{R}_{2^{-k_0}\eta_0}(\bar{M}).$$

Now using $\bar{z} \notin E$ and the decomposition of E in (3.96), we have

$$\begin{aligned} & \bar{z} \notin E_{2^{-k_0}\eta_0} \backslash \left\{ B(\bar{x}, 2^{k_0-2}\eta_0^{-1}) \cap \mathring{\mathcal{R}}_{2^{-k_0+2}\eta_0} \right\} \\ \Leftrightarrow \quad & \bar{z} \in \left\{ B(\bar{x}, 2^{k_0-2}\eta_0^{-1}) \cap \mathring{\mathcal{R}}_{2^{-k_0+2}\eta_0} \right\} \backslash E_{2^{-k_0}\eta_0}, \\ \Rightarrow \quad & \bar{z} \in \left\{ B(\bar{x}, 2^{k_0}\eta_0^{-1}) \cap \mathcal{R}_{2^{-k_0}\eta_0} \right\} \backslash E_{2^{-k_0}\eta_0}. \end{aligned}$$

Then it follows from our discussion in the previous paragraph that $(\bar{z}, -1)$ can be connected to $(\bar{x}, 0)$ by a shortest smooth reduced geodesic in $\mathcal{R}(\bar{M}) \times [-1, 0]$.

It is not hard to see that $E \cup \mathcal{S}$ is a closed set, which will be proved in this paragraph. Suppose z_i is a sequence of points in E. Without loss of generality, we can assume

$$z_i \in E_{2^{-k}\eta_0} \backslash \left\{ B(\bar{x}, 2^{k-2}\eta_0^{-1}) \cap \mathcal{R}_{2^{k-2}\eta_0} \right\},$$

where $k = k(i)$. Let z be a limit point of z_i. There are two possibilities (by taking subsequence if necessary):

- $z \in \mathcal{S}$.
- $z \in \mathcal{R}$. Then $z \in \mathcal{R}_{2\eta} \cap B(\bar{x}, 0.5\eta^{-1})$ for some $\eta > 0$. Therefore, we can assume $z_i \in \mathcal{R}_{\eta} \cap B(\bar{x}, \eta^{-1})$ for large i. This forces that $k(i)$ is uniformly bounded. By taking subsequence if necessary, we can assume that $z_i \in E_{2^{-k}\eta_0} \backslash \left\{ B(\bar{x}, 2^{k-2}\eta_0^{-1}) \cap \mathring{\mathcal{R}}_{2^{k-2}\eta_0} \right\}$ for a fixed k. By closedness of each E_{η}, we see that
$$z \in E_{2^{-k}\eta_0} \backslash \left\{ B(\bar{x}, 2^{k-2}\eta_0^{-1}) \cap \mathring{\mathcal{R}}_{2^{k-2}\eta_0} \right\} \subset E.$$

Therefore, we conclude that $z \in E \cup \mathcal{S}$. Note that $\mathcal{S}$ is a closed set and has measure ($2n$-Hausdorff measure) zero. Then we obtain $E \cup \mathcal{S}$ is a closed measure-zero set.

Clearly, away from the closed measure-zero set $E \cup \mathcal{S}$, every point $\bar{z} \in \bar{M}$ satisfies the following property: $(\bar{z}, -1)$ *can be connected to* $(\bar{x}, 0)$ *by a shortest smooth reduced geodesic.* q.e.d.

Remark 3.24. The development from Lemma 3.21 to Lemma 3.23 is parallel, or independent to the development from Proposition 3.17 to Proposition 3.20. Our key observation is that the limit space has weakly convex regular part, which essentially arises from the weak convexity of $\mathcal{R} \times [-1, 0]$ in terms of reduced geodesics. Actually, there exists an independent proof of Proposition 3.20 in Appendix C of [**28**].

By natural projection to the time slice $t = 0$, we obtain the following property.

Proposition 3.25 (**Weak convexity by Riemannian geodesics**). *Same conditions as in Lemma 3.23. Then away from a measure-zero set, every point in $\mathcal{R}$ can be connected to $\bar{x}$ with a unique smooth shortest geodesic. Consequently, $\mathcal{R}$ is weakly convex.*

Proof. Fix $\bar{x} \in \mathcal{R}$ and let E be the measure-zero set constructed in the proof of Lemma 3.23. Therefore, $(\bar{y}, -1)$ can be connected to $(\bar{x}, 0)$ by a smooth shortest reduced geodesic $\boldsymbol{\beta}$, with space projection curve β, whenever $\bar{y} \in \mathcal{R} \backslash E$. For our purpose of weak convexity, it suffices to show that each β is a smooth shortest geodesic connecting $\bar{x}$ and $\bar{y}$. Actually, it follows from reduced geodesic equation on Ricci-flat manifold (c.f. equations (2.8)) that $\mathcal{L}(\boldsymbol{\beta}) = \dfrac{1}{2}|\beta|^2$, where $|\beta|$ is the length of β. Since both $\bar{x}$ and $\bar{y}$ are regular, for each small $\epsilon > 0$, we can find a smooth geodesic γ such that $|\gamma| < d_0(\bar{x}, \bar{y}) + \epsilon$, by Proposition 3.20. Because the limit space-time is static, we can lift γ to be a space-time curve $\boldsymbol{\gamma}$ such that $\mathcal{L}(\boldsymbol{\gamma}) = \frac{|\gamma|^2}{2}$. Using the shortest property of $\boldsymbol{\beta}$ and the construction of γ, we have

$$\frac{|\beta|^2}{2} = \mathcal{L}(\boldsymbol{\beta}) \le \mathcal{L}(\boldsymbol{\gamma}) = \frac{|\gamma|^2}{2} < \frac{(d_0(\bar{x}, \bar{y}) + \epsilon)^2}{2}, \quad \Rightarrow \quad |\beta| < d_0(\bar{x}, \bar{y}) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, we have $|\beta| \le d_0(\bar{x}, \bar{y})$, which means $|\beta| = d_0(\bar{x}, \bar{y})$ and β is a shortest Riemannian geodesic.

By adjusting E to a bigger measure zero set E' if necessary, we obtain the uniqueness of geodesics from $\bar{y}$ to $\bar{x}$ for each $\bar{y} \in \mathcal{R} \backslash E'$. This follows from standard Riemannian geometry argument since $E' \backslash E \subset \mathcal{R}$. q.e.d.

By the correspondence between smooth Riemannian geodesic and smooth reduced geodesic (c.f. the discussion in Section 2.7 of [**29**]), it is clear (from the proof of Proposition 3.25) now that most smooth reduced geodesics obtained in Lemma 3.23 are shortest among all smooth reduced geodesics. Furthermore, the rough estimate in Lemma 3.21 can be improved as the following proposition.

Proposition 3.26 (**Continuity of reduced distance**). *Same conditions as in Lemma 3.23. Suppose $(y_i, t_i) \in \mathcal{M}_i$ converges to $(\bar{y}, \bar{t})$,* *which is regular and $\bar{t} < 0$. Then we have*

$$\lim_{i\to\infty} l((x_i,0),(y_i,t_i)) = \frac{d_0^2(\bar{x},\bar{y})}{4|\bar{t}|} = l((\bar{x},0),(\bar{y},\bar{t})), \tag{3.98}$$

where l is Perelman's reduced distance. Therefore, reduced distance is continuous function under Cheeger–Gromov topology whenever $\bar{y}$ is regular.

Proof. Without loss of generality, we assume $t_i \equiv -1$, $d_0(x_i,y_i) \equiv 1$. We first show

$$\lim_{i\to\infty} l((x_i,0),(y_i,t_i)) \le \frac{1}{4}. \tag{3.99}$$

If x_i are uniformly regular, then there is a limit smooth geodesic connecting $\bar{x}$ and $\bar{y}$, which can be lifted to a smooth reduced geodesic connecting $(\bar{x},0)$ and $(\bar{y},-1)$ with reduced length $\frac{1}{4}$. Then (3.99) follows trivially. So we focus on the case when $\bar{x}$ is a singular point. Choose a smooth point $\bar{z}$ very close to $\bar{x}$, say δ-away from $\bar{x}$ under metric $\bar{g}(0)$. From Lemma 3.21, the reduced length from $(x_i,0)$ to $(z_i,-\delta^2)$ is uniformly less than 100. So we have space-time curves $\boldsymbol{\alpha}_i$ connecting these two points such that

$$\int_0^{\delta^2} \sqrt{\tau}|\dot{\alpha}_i|^2 d\tau < 200\delta.$$

Note that $(\bar{z},-\delta^2)$ and $(\bar{y},-1)$ can be connected by a space-time curve $\boldsymbol{\beta}$ such that

$$\int_{\delta^2}^1 \sqrt{\tau}|\dot{\beta}|^2 d\tau < \frac{1}{2} + 100\delta,$$

if δ is small enough. So for large i, we have space-time curve $\boldsymbol{\beta}_i$ connecting $(z_i,-\delta^2)$ and $(y_i,-1)$ such that

$$\int_{\delta^2}^1 \sqrt{\tau}|\dot{\beta}_i|^2 d\tau < \frac{1}{2} + 200\delta.$$

Concatenating $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ to obtain $\boldsymbol{\gamma}_i$ such that

$$\int_{\delta^2}^1 \sqrt{\tau}|\dot{\gamma}_i|^2 d\tau < \frac{1}{2} + 400\delta,$$

which implies $l((x_i,0),(y_i,-1)) < \frac{1}{4} + 200\delta$ for large i. Thus, (3.99) follows by letting $i \to \infty$ and $\delta \to 0$.

Then we show the equality holds. Otherwise, there exists a small ϵ such that

$$\lim_{i\to\infty} l((x_i,0),(y_i,-1)) < \frac{1}{4} - \epsilon.$$

Note that $(y_i,-1)$ is uniformly regular. So we can find small δ such that

$$l((x_i,0),(z,-1-\delta^2))<\frac{1}{4}-\frac{1}{2}\epsilon, \quad \forall\, z\in B_{g(-1-\delta^2)}(y_i,\epsilon\delta).$$

By Lemma 3.23, we obtain a point $(\bar{z},-1-\delta^2)$, which can be connected to $(\bar{x},0)$ by a smooth reduced geodesic, with reduced length smaller than $\frac{1}{4}-\frac{1}{2}\epsilon$. Projecting this reduced geodesic to time zero slice, we obtain a curve connecting $\bar{x}$ and $\bar{z}$ with

$$d_0^2(\bar{x},\bar{y})<4(1+\delta^2)\cdot\left(\frac{1}{4}-\frac{1}{2}\epsilon\right)=(1+\delta^2)(1-2\epsilon)<1-\epsilon,$$

if we choose δ sufficiently small. This is impossible since $d_0(\bar{x},\bar{y})=1$. Therefore, we have

$$\lim_{i\to\infty} l((x_i,0),(y_i,-1))=\frac{1}{4}. \qquad \text{q.e.d.}$$

Since singular set has measure zero, it is clear that

$$\mathcal{V}((\bar{x},0),|\bar{t}|)\leq\lim_{i\to\infty}\mathcal{V}((x_i,0),|\bar{t}|), \tag{3.100}$$

where the "lim" of the right hand side of the above inequality should be understood as "lim sup". We shall improve the above inequality as equality.

Lemma 3.27 (**Major part of reduced volume**). *For every positive η and H, there exists an $\epsilon=\epsilon(n,A,\eta,H)$ with the following properties.*

Suppose $\mathcal{LM}\in\mathscr{K}(n,A)$, $x\in\mathcal{F}_\eta(M,0)$. Then we have

$$\left|\mathcal{V}((x,0),1)-(4\pi)^{-n}\int_{B_{g(0)}(x,H)}e^{-l}dv\right|\leq 2a(H), \tag{3.101}$$

whenever $\sup_{\mathcal{M}}(|R|+|\lambda|)<\epsilon$. *Here a is a positive function defined as*

$$a(H)\triangleq(4\pi)^{-n}\int_{\{|\vec{w}|>\frac{H}{100}\}\subset\mathbb{R}^{2n}}e^{-\frac{|\vec{w}|^2}{4}}dw. \tag{3.102}$$

Proof. The line bundle structure is not used in the following proof. So up to a parabolic rescaling if necessary, we can assume $\lambda=0$.

For every $y\in M$, there is at least one shortest reduced geodesic $\boldsymbol{\gamma}$ connecting $(x,0)$ and $(y,-1)$. By standard ODE theory, the limit $\lim\limits_{\tau\to 0}\sqrt{\tau}\gamma'(\tau)$ is unique as a vector in T_xM, which is called the reduced tangent vector of $\boldsymbol{\gamma}$. Away from a measure-zero set, every $(y,-1)$ can be connected to $(x,0)$ by a unique shortest reduced geodesic. For simplicity for our argument, we may assume this measure-zero set is empty, since measure-zero set does not affect integral at all. So there is a natural injective map from M to T_xM, by mapping y to the corresponding reduced tangent vector $\vec{w}$. We define

$$\Omega(H) \triangleq \{y \in M||\vec{w}| > H\}.$$

It follows from the monotonicity of reduced element along reduced geodesic that

$$\int_{\Omega(H)} (4\pi)^{-n} e^{-l} dv \leq \int_{\{|\vec{w}|>H\}\subset \mathbb{R}^{2n}} (4\pi)^{-n} e^{-\frac{|\vec{w}|^2}{4}} dw.$$

Choose $\xi < \eta$, with size to be determined. Suppose $\boldsymbol{\gamma}$ is a reduced geodesic connecting $(x,0)$ to $(y,-1)$ for some $y \in M$. It is clear that $\gamma(0)$ is in the interior part of $\mathcal{F}_\xi(M,0)$. Let τ to be the first time such that $\gamma(\tau)$ touches the boundary of $\mathcal{F}_\xi(M,0)$. Then we see that $\boldsymbol{\gamma}([0,\tau])$ locates in a space-time domain with uniformly bounded geometry, Ricci curvature very small. In particular, the reduced distance between $(x,0)$ and $\boldsymbol{\gamma}(\tau)$ is comparable to the length of $\vec{w}$, which is the reduced tangent vector of $\boldsymbol{\gamma}$ at $(x,0)$. If $|\vec{w}| < H$, then we see that

$$\frac{H^2}{4} > \frac{|\vec{w}|^2}{4} \sim \frac{d^2_{g(0)}(x,\gamma(\tau))}{4\tau} > \frac{c_a^2\eta^2}{100\tau}, \quad \Rightarrow \quad \tau > \frac{c_a^2\eta^2}{25H^2}.$$

Note that $\boldsymbol{\gamma}([0,\tau])$ is in a space-time region where Ricci curvature is almost flat, geometry is uniformly bounded. So the lower bound of τ and the upper bounded of $|\vec{w}|$ imply an upper bound of $d_{g(0)}(x,\gamma(\tau))$. Say $d_{g(0)}(x,\gamma(\tau)) < H'$.

Around $\boldsymbol{\gamma}$, there is a natural projection (induced by reduced geodesic) from the space-time hypersurface $\partial\mathcal{F}_\xi(M,0) \times [-1, -\frac{c_a^2\eta^2}{25H^2}]$, to the time slice $M \times \{-1\}$. At point $\boldsymbol{\gamma}(\tau)$, $\boldsymbol{\gamma}$ has space-time tangent vector $(\gamma', -1)$, with $\tau|\gamma'(\tau)|^2$ is almost less than $\frac{H^2}{4}$. Together with the lower bound of τ, we obtain an upper bound of $|\gamma'(\tau)|$. Up to a constant depending on H, η, the volume element of $\partial\mathcal{F}_\xi(M,0) \times [-1, -\frac{c_a^2\eta^2}{25H^2}]$ is comparable to the reduced volume element $(4\pi\tau)^{-n}e^{-l}$ of M, around the point $\boldsymbol{\gamma}(\tau)$. Note that the reduced volume element is monotone along each reduced geodesic. This implies that the projection map mentioned above "almost" decreases weighted hypersurface volume element, if we equip $\{B(x,H') \cap \partial\mathcal{F}_\xi(M,0)\} \times [-1, -\frac{\eta^2}{4H^2}]$ with the natural weighted volume element $e^{-l}|d\sigma \wedge dt|$. Let Ω'_ξ be the collection of all y's such that $(y,-1)$ cannot be connected to $(x,0)$ by a shortest reduced geodesic $\boldsymbol{\gamma}$ which locates completely in $\mathcal{F}_\xi(M,0) \times [-1,0]$. Then we have

$$\begin{aligned}\int_{\Omega'_\xi} e^{-l}(4\pi\tau)^{-n} dv &\leq C \int_{\frac{c_a^2\eta^2}{25H^2}}^1 \int_{B(x,H')\cap\partial\mathcal{F}_\xi(M,0)} e^{-l} d\sigma d\tau \\ &\leq C \int_{B(x,H')\cap\partial\mathcal{F}_\xi(M,0)} d\sigma \leq C\xi^{2p_0-1},\end{aligned}$$

where $C = C(n, H, H', \eta) = C(n, H, \eta)$. By choosing ξ small enough, we have

$$\int_{\Omega'_\xi} e^{-l}(4\pi\tau)^{-n} dv \le (4\pi)^n a(H). \tag{3.103}$$

Note that

$$\Omega_{100H} \cap B_{g(0)}(x, H) \subset \Omega'_\xi, \qquad M \backslash (\Omega'_\xi \cup B_{g(0)}(x, H)) \subset \Omega_{\frac{H}{100}}.$$

Therefore, recalling the definition of reduced volume (2.7), we have

$$\begin{aligned}
&(4\pi)^n \mathcal{V}((x, 0), 1) = \int_M e^{-l} dv \\
&= \int_{M \backslash (\Omega'_\xi \cup B_{g(0)}(x,H))} e^{-l} dv + \int_{\Omega'_\xi} e^{-l} dv + \int_{B_{g(0)}(x,H) \backslash \Omega'_\xi} e^{-l} dv \\
&\le \int_{|\vec{w}| > \frac{H}{100}} e^{-\frac{|\vec{w}|^2}{4}} dw + \int_{\Omega'_\xi} e^{-l} dv + \int_{B_{g(0)}(x,H)} e^{-l} dv \\
&\le \int_{|\vec{w}| > \frac{H}{100}} e^{-\frac{|\vec{w}|^2}{4}} dw + C\xi^{2p_0 - 1} + \int_{B_{g(0)}(x,H)} e^{-l} dv \\
&\le 2(4\pi)^n a(H) + \int_{B_{g(0)}(x,H)} e^{-l} dv.
\end{aligned}$$

Then (3.101) follows from the above inequality directly. q.e.d.

Lemma 3.27 is related to Corollary 6.82 of [**45**].

Lemma 3.28 (**Uniform continuity of reduced volume**). *Suppose $\mathcal{M} = \{(M, g(t)), -\tau \le t \le 0\}$ is an unnormalized Kähler Ricci flow solution. Suppose x, y are two points in M, $d = d_{g(0)}(x, y)$. Then we have*

$$|\mathcal{V}((x, 0), \tau) - \mathcal{V}((y, 0), \tau)| < (4n + 1)(e^{\frac{d}{2}} - 1). \tag{3.104}$$

In particular, the reduced volume changes uniformly continuously with respect to the base point.

Proof. Recall the definition of reduced volume (2.7):

$$\mathcal{V}((x, 0), \tau) = (4\pi\tau)^{-n} \int_M e^{-l} dv.$$

Let x move along a unit speed Riemannian geodesic α, with respect to the metric $g(0)$. Let $x = \alpha(0)$, s be parameter of α, $\vec{u} = \alpha'$. For simplicity of notation, we denote $\mathcal{V}((\alpha(s), 0), \tau)$ by $\mathcal{V}_s$. It can be calculated directly the first variation of l is $\langle \vec{u}, \vec{w} \rangle$ where $\vec{w}$ is the tangent vector of the reduced geodesic at time $t = 0$. Therefore, we have

$$\begin{aligned}
&\left|\frac{d}{ds}\mathcal{V}((\alpha(s),0),\tau)\right| \\
&= \left|(4\pi\tau)^{-n}\int_M \langle \vec{u},\vec{w}\rangle e^{-l}dv\right| \leq (4\pi\tau)^{-n}\int_M \frac{1+|\vec{w}|^2}{2}e^{-l}dv \\
&= \frac{1}{2}\mathcal{V} + \frac{1}{2}\int_{\mathbb{R}^{2n}} |\vec{w}|^2 e^{-\frac{|\vec{w}|^2}{4}} J dw \leq \frac{1}{2}\mathcal{V} + \frac{(4\pi)^{-n}}{2}\int_{\mathbb{R}^{2n}} |\vec{w}|^2 e^{-\frac{|\vec{w}|^2}{4}} dw,
\end{aligned}$$

where J is the Jacobian determinant of the reduced exponential map, which is always not greater than 1, due to Perelman's argument in Section 7 of [**49**]. Plugging the identity

$$(4\pi)^{-n}\int_{\mathbb{R}^{2n}} |\vec{w}|^2 e^{-\frac{|\vec{w}|^2}{4}} dw = 4n,$$

into the above inequality implies $\left|\frac{d}{ds}\mathcal{V}\right| \leq \frac{1}{2}\mathcal{V} + 2n$, which can be integrated as

$$(-\mathcal{V}_0 + 4n)(1 - e^{-\frac{s}{2}}) \leq \mathcal{V}_s - \mathcal{V}_0 \leq (\mathcal{V}_0 + 4n)(e^{\frac{s}{2}} - 1).$$

Note that $0 < \mathcal{V}_0 \leq 1, s > 0$. So we obtain

$$|\mathcal{V}_s - \mathcal{V}_0| \leq (4n+1)(e^{\frac{s}{2}} - 1),$$

which yields (3.104) by letting $s = d$. q.e.d.

The above argument clearly works for every Riemannian Ricci flow.

Note that the reduced volume is continuous for geodesic balls of each fixed scale under the Cheeger–Gromov convergence. Combining this continuity together with the estimate in Lemma 3.27 and Lemma 3.28, we can improve (3.100) as an equality.

Proposition 3.29 (**Continuity of reduced volume**). *Same conditions as in Lemma 3.23, $\bar{t} < 0$ is a finite number. Then we have*

$$\mathcal{V}((\bar{x},0),|\bar{t}|) = \lim_{i\to\infty} \mathcal{V}((x_i,0),|\bar{t}|). \tag{3.105}$$

Then we can study the gap property of the singularities.

Proposition 3.30 (**Gap of local volume density**). *Same conditions as in Theorem 3.18.*

Suppose $\bar{y} \in \mathcal{S}(\bar{M})$, then we have

$$\mathrm{v}(\bar{y}) = \lim_{r\to 0} \omega_{2n}^{-1} r^{-2n} |B(\bar{y},r)| \leq 1 - 2\delta_0. \tag{3.106}$$

Proof. Due to the tangent cone structure (c.f. Theorem 2.6), we have

$$\mathrm{v}(\bar{y}) = \lim_{r\to 0} \omega_{2n}^{-1} r^{-2n} |B(\bar{y},r)| = \lim_{r\to 0} \mathcal{V}((\bar{y},0),r^2). \tag{3.107}$$

Let $y_i \to \bar{y}$ under the metric $g_i(0)$. By rearranging points and taking subsequences if necessary, we can assume y_i has the "local minimum" canonical volume radius ρ_i.

The rearrangement is a standard point-picking technique. In fact, since $\bar{y}$ is a singular point, it is clear that $r_i = \mathbf{cvr}(y_i, 0) \to 0$. Since everything is done at time slice $t = 0$, we shall drop the time in the following argument. Fix $L \geq 1$ and i, we search if y_i is the point such that

$$\mathbf{cvr}(y) < 0.5\mathbf{cvr}(y_i), \quad \forall\, y \in B(y_i, Lr_i).$$

If so, we stop. Otherwise, we can find a point $z \in B(y_i, Lr_i)$ such that $\mathbf{cvr}(z) < 0.5\mathbf{cvr}(y_i)$. Denote such z by $y_i^{(1)}$ and set $r_i^{(1)} = \mathbf{cvr}\left(y_i^{(1)}\right)$. We then repeat the previous process for $y_i^{(1)}$ and $r_i^{(1)}$. To search points in the ball $B\left(y_i^{(1)}, Lr_i^{(1)}\right)$ with $\mathbf{cvr} < 0.5r_i^{(1)}$. If no such points exist, we stop. Otherwise, we find such a point and denote it by $y_i^{(2)}$ and set $r_i^{(2)} = \mathbf{cvr}\left(y_i^{(2)}\right)$. Note this process happens in a compact set since

$$d\left(y_i^{(k)}, y_i\right) < L\left(r_i + r_i^{(1)} + \cdots + r_i^{(k)}\right) < 2Lr_i.$$

Each $\mathcal{LM}_i$ is smooth. Therefore, the process above must stop at some finite step k. Denote $z_i = y_i^{(k)}$ and $\rho_i = \mathbf{cvr}(z_i)$. Then we have

$$\mathbf{cvr}(y) > 0.5\rho_i, \quad \forall\, y \in B(z_i, L\rho_i).$$

Note that $L\rho_i \to 0$ as $i \to \infty$. Therefore, the limit of z_i and the limit of y_i are the same point $\bar{y}$. Then we let $L = L_k \to \infty$ and take diagonal sequence if necessary, we can guarantee that $L_i\rho_i \to 0$ and $\rho_i \to 0$ simultaneously. Thus, we obtain z_i such that

$$\mathbf{cvr}(y) > 0.5\rho_i, \quad \forall\, y \in B(z_i, L_i\rho_i); \qquad \lim_{i\to\infty} z_i = \bar{y}.$$

Therefore, we can regard z_i as the rearrangement of y_i, with the property that each z_i achieve the "local minimum" of $\mathbf{cvr}$.

By rescaling ρ_i to 1, we obtain new Ricci flows $\tilde{g}_i$. Taking limit of $(M_i, y_i, \tilde{g}_i(0))$, we have a complete, Ricci flat eternal Ricci flow solution. It is not hard to see the limit space is not Euclidean. For otherwise, each geodesic ball's volume ratio, under metric $\tilde{g}_\infty(0)$, is exactly the Euclidean volume ratio ω_{2n}. Following from the volume convergence and the definition of the canonical volume radius, it is clear that the canonical volume radius of the rescaled flow is strictly greater than 1 which contradicts to our assumption. So it has normalized asymptotic volume ratio less than $1 - 2\delta_0$, according to Anderson's gap theorem. Then the infinity tangent cone structure implies the asymptotic reduced volume is the same as the asymptotic reduced volume ratio. So it is at most $1 - 2\delta_0$. Therefore, there exists a big constant H such that

$$\mathcal{V}_{\tilde{g}_i}((y_i, 0), H) < 1 - 2\delta_0.$$

Note that $H\rho_i^2 < r$ for each fixed r and the corresponding large i. Recall the scaling invariant property of reduced volume, we can apply the reduced volume monotonicity to obtain

$$\mathcal{V}_{g_i}((y_i,0),r^2) \leq \mathcal{V}_{\tilde{g}_i}((y_i,0),\rho_i^{-2}r^2) \leq \mathcal{V}_{\tilde{g}_i}((y_i,0),H) < 1-2\delta_0.$$

The continuity of reduced volume (Proposition 3.29) then implies that

$$\mathcal{V}((\bar{y},0),r^2) \leq 1-2\delta_0,$$

for each $r>0$, which in turn yields

$$\lim_{r\to 0} \mathcal{V}((\bar{y},0),r^2) \leq 1-2\delta_0. \tag{3.108}$$

Then (3.106) follows from the combination of (3.107) and (3.108). q.e.d.

Theorem 3.31 (**Metric structure of a blowup limit**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n,A;1)$ satisfies (3.94), $x_i \in M_i$. Let $(\bar{M},\bar{x},\bar{g})$ be the limit space of $(M_i,x_i,g_i(0))$. Then $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$.*

Proof. We only need to check $\bar{M}$ satisfies all the 6 properties required in the definition of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, i.e., Definition 2.1. In fact, the 1st property follows directly from definition. The 2nd property follows from the fact that $\mathcal{R}$ is scalar flat and satisfies Kähler Ricci flow equation, by Proposition 3.15. The 3rd property, weak convexity of $\mathcal{R}$ is shown in Proposition 3.25. The 4th property, codimension estimate of singularity follows from Proposition 3.19. The 5th property, gap estimate, follows from Proposition 3.30. The 6th property, asymptotic volume ratio estimate can be obtained by the condition $Vol(M_i) \to \infty$, Sobolev constant uniformly bounded, and the volume convergence, Proposition 2.14. Note that $\kappa = \kappa(n,C_S)$ is in general much smaller than ω_{2n}. More details can be found in Remark 3.32. So we have checked all the properties needed to define $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ are satisfied by $\bar{M}$. In other words, $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$. q.e.d.

Remark 3.32. It is known in the literature of the Ricci flow that a noncollapsing constant κ can be determined by dimension and the L^2-Sobolev constant C_S of a closed manifold (M^{2n},g), whenever scalar curvature is uniformly bounded. Actually, it follows from the observation of Klaus Ecker (c.f. Lemma 8 of Cao–Sesum [**4**]) that $\mu(g,\tau)$ is uniformly bounded from below for each $\tau \in (0, \frac{1}{\sup_M |R|})$, where $\mu(g,\tau)$ is the functional of Perelman. Then the κ-noncollapsing of each geodesic ball $B(y,r)$ follows from the argument of Perelman (c.f. Remark 13.13 of Kleiner–Lott [**41**]), whenever $|R|r^2 \leq 1$. Note that $|R| \to 0$ by (3.94), we obtain uniform κ-noncollapsing for each fixed r on $(M_i,g_i(0))$, whenever i large enough. In other words, for each $r>0$, with respect to the metric $g_i(0)$, we have

$$\inf_{y \in M_i} \frac{|B(y,r)|}{\omega_{2n} r^{2n}} \geq \kappa,$$

for large i.

Since $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, it is clear that $\mathbf{cr}(\bar{x}) = \infty$. Therefore, we have $\mathbf{vr}(\bar{x}) = \mathbf{cvr}(\bar{x})$ by definition.

Proposition 3.33. *Same conditions as in Theorem 3.31. Let* $\bar{r} = \lim_{i \to \infty} \mathbf{cr}(x_i)$*. Then we have*

$$\min\{\bar{r}, \mathbf{vr}(\bar{x})\} = \lim_{i \to \infty} \mathbf{cvr}(x_i). \tag{3.109}$$

Proof. We divide the proof in three cases according to the value of $\min\{\bar{r}, \mathbf{vr}(\bar{x})\}$.

Case 1. $\min\{\bar{r}, \mathbf{vr}(\bar{x})\} = 0$.

Otherwise, there exists a positive number ρ_0 such that

$$\lim_{i \to \infty} \mathbf{cvr}(x_i) \geq \rho_0.$$

Therefore, $\bar{x} \in \mathcal{R}_{\rho_0} \subset \mathcal{R}$, which in turn implies that $\mathbf{vr}(\bar{x}) > 0$. Consequently, we have $\min\{\bar{r}, \mathbf{vr}(\bar{x})\} > 0$. Contradiction.

Case 2. $\min\{\bar{r}, \mathbf{vr}(\bar{x})\} = \infty$.

In this case, $\mathbf{vr}(\bar{x}) = \infty$. By the gap theorem in the space $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, we see that $\bar{M}$ is the Euclidean space $\mathbb{C}^n$. Therefore, for each $H > 0$, we have $\omega_{2n}^{-1} H^{-2n} |B(x_i, H)|$ converges to 1, the normalized volume ratio of $\mathbb{C}^n$. Since $\bar{r} = \lim_{i \to \infty} \mathbf{cr}(x_i) = \infty$, this means that $\mathbf{cvr}(x_i) \geq H$ for large i by the volume convergence. Since H is chosen arbitrarily, we obtain $\lim_{i \to \infty} \mathbf{cvr}(x_i) = \infty$.

So the remainder case is that $\min\{\bar{r}, \mathbf{vr}(\bar{x})\}$ is a finite positive number. Two more subcases can be divided.

Case 3(a). $\min\{\bar{r}, \mathbf{vr}(\bar{x})\} < \bar{r}$.

Let $H = \mathbf{vr}(\bar{x})$, a finite number in this case. Clearly, $\bar{x}$ is a regular point and the normalized volume ratio of the ball $B(\bar{x}, H)$ is $1 - \delta_0$. Clearly, $B(\bar{x}, H)$ cannot be a isometric to an Euclidean ball. Therefore, by the rigidity of $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ (c.f. Proposition 2.3), we see that

$$\begin{aligned}
\omega_{2n}^{-1} r^{-2n} |B(\bar{x}, r)| &> 1 - \delta_0, \quad \forall\, r \in (0, H), \\
\omega_{2n}^{-1} r^{-2n} |B(\bar{x}, r)| &< 1 - \delta_0, \quad \forall\, r \in (H, \bar{r}).
\end{aligned}$$

Then the volume convergence implies that $\lim_{i \to \infty} \mathbf{cvr}(x_i) = H$.

Case 3(b). $\min\{\bar{r}, \mathbf{vr}(\bar{x})\} = \bar{r}$.

In this case, we see that the normalized volume ratio of $B(\bar{x}, \bar{r})$ is at least $1 - \delta_0$. Also, we see that $\bar{x}$ is a regular point. Same argument as in the previous case, we see that

$$\omega_{2n}^{-1} r^{-2n} |B(\bar{x}, r)| > 1 - \delta_0, \quad \forall\, r \in (0, \bar{r}).$$

Therefore, for every fixed $r \in (0, \bar{r})$, the volume convergence implies that $\lim_{i \to \infty} \mathbf{cvr}(x_i) \geq r$. Consequently, we have $\lim_{i \to \infty} \mathbf{cvr}(x_i) \geq \bar{r}$ by the arbitrariness of r. On the other hand, the definition of $\mathbf{cvr}(x_i)$ implies that

$$\lim_{i \to \infty} \mathbf{cvr}(x_i) \leq \lim_{i \to \infty} \mathbf{cr}(x_i) = \bar{r}.$$

Therefore, we obtain $\lim_{i \to \infty} \mathbf{cvr}(x_i) = \bar{r}$. q.e.d.

Corollary 3.34. *Same conditions as in Theorem 3.31. Then for each $r \in (0, 1)$, we have*

$$\mathcal{F}_r(\bar{M}) = \mathcal{R}_r(\bar{M}). \tag{3.110}$$

In particular, for each $0 < r < 1 < H < \infty$, we have

$$B(x_i, H) \cap \mathcal{F}_r(M_i) \xrightarrow{G.H.} B(\bar{x}, H) \cap \mathcal{F}_r(\bar{M}).$$

Moreover, this convergence can be improved to take place in C^∞-topology, i.e.,

$$B(x_i, H) \cap \mathcal{F}_r(M_i) \xrightarrow{C^\infty} B(\bar{x}, H) \cap \mathcal{F}_r(\bar{M}). \tag{3.111}$$

Corollary 3.35. *Same conditions as in Theorem 3.31, $0 < H \leq 3$. Then we have*

$$\lim_{i \to \infty} \int_{B(x_i, H)} \mathbf{vr}^{(1)}(y)^{-2p_0} dy \leq H^{2n - 2p_0} \mathbf{E}. \tag{3.112}$$

Proof. Fix two positive scales r_1, r_2 such that $0 < r_2 < r_1 < 1$.

$$\begin{aligned} &\int_{B(x_i, H) \cap \mathcal{F}_{r_1}} \mathbf{vr}^{(1)}(y)^{-2p_0} dy \\ &\leq r_1^{-2p_0} |B(x_i, H) \cap \mathcal{F}_{r_1}| \leq r_1^{-2p_0} |B(x_i, H)|. \end{aligned} \tag{3.113}$$

Fix arbitrary $r \in (0, 1)$, then we have

$$\begin{aligned} &\lim_{i \to \infty} \int_{B(x_i, H) \cap (\mathcal{F}_{r_2} \backslash \mathcal{F}_{r_1})} \mathbf{vr}^{(1)}(y)^{-2p_0} dy \\ &= \int_{B(\bar{x}, H) \cap (\mathcal{F}_{r_2} \cap \mathcal{F}_{r_1})} \mathbf{vr}^{(1)}(y)^{-2p_0} dy. \end{aligned}$$

Note that

$$\begin{aligned}
&\int_{B(\bar{x},H)\cap(\mathcal{F}_{r_2}\cap\mathcal{F}_{r_1})} \mathbf{vr}^{(1)}(y)^{-2p_0}dy\\
\leq &\int_{B(\bar{x},H)\cap(\mathcal{F}_{r_2}\cap\mathcal{F}_{r_1})} \min\{\mathbf{vr},1\}^{-2p_0}dy\\
< &\int_{B(\bar{x},H)\cap(\mathcal{F}_{r_2}\cap\mathcal{F}_{r_1})} \left\{1+\mathbf{vr}(y)^{-2p_0}\right\}dy\\
< &\int_{B(\bar{x},H)} \left\{1+\mathbf{vr}(y)^{-2p_0}\right\}dy\\
< &|B(\bar{x},H)| + H^{2n-2p_0}E(n,\kappa,p_0).
\end{aligned}$$

It follows that

$$\begin{aligned}
(3.114)\qquad &\lim_{i\to\infty}\int_{B(x_i,H)\cap(\mathcal{F}_{r_2}\backslash\mathcal{F}_{r_1})} \mathbf{vr}^{(1)}(y)^{-2p_0}dy\\
\leq &|B(\bar{x},H)| + H^{2n-2p_0}E(n,\kappa,p_0).
\end{aligned}$$

Note that $\mathcal{S}\cap\overline{B(\bar{x},H)}$ is a compact set with Hausdorff dimension at most $2n-4$, which is strictly less than $2n-2p_0$. By the definition of Hausdorff dimension, for every small number ξ, we can find finite cover $\cup_{j=1}^{N_\xi}B(\bar{y}_j,\rho_j)$ of $\mathcal{S}\cap\overline{B(\bar{x},H)}$, such that $\displaystyle\sum_{j=1}^{N_\xi}|\rho_j|^{2n-2p_0}<\xi$. By the finiteness of this cover, we can choose an r_2 very small such that $\cup_{j=1}^{N_\xi}B(\bar{y}_j,\rho_j)$ is a cover of $\mathcal{D}_{r_2}\cap\overline{B(\bar{x},H)}$. Therefore, for large i, we have a finite cover $\cup_{j=1}^{N_\xi}B(y_{i,j},\rho_j)$ of the set $\mathcal{D}_{r_2}(M_i)\cap\overline{B(x_i,H)}$ such that $\displaystyle\sum_{j=1}^{N_i}|\rho_{i,j}|^{2n-2p_0}<\xi$. Combining this with the canonical radius density estimate, we have

$$\begin{aligned}
(3.115)\quad \int_{B(x_i,H)\cap\mathcal{D}_{r_2}} \mathbf{vr}^{(1)}(y)^{-2p_0}dy &\leq \sum_{j=1}^{N_i}\int_{B(y_{i,j},\rho_{i,j})} \mathbf{vr}^{(\rho_{i,j})}(y)^{-2p_0}dy\\
&\leq 2\mathbf{E}\sum_{j=1}^{N_i}|\rho_{i,j}|^{2n-2p_0} < 2\mathbf{E}\xi.
\end{aligned}$$

Putting (3.113), (3.114) and (3.115) together, we have

$$\begin{aligned}
&\int_{B(x_i,H)} \mathbf{vr}^{(1)}(y)^{-2p_0}dy\\
\leq &\int_{B(x_i,H)\cap\mathcal{F}_{r_1}} \mathbf{vr}^{(1)}(y)^{-2p_0}dy \quad + \int_{B(x_i,H)\cap(\mathcal{F}_{r_2}\backslash\mathcal{F}_{r_1})} \mathbf{vr}^{(1)}(y)^{-2p_0}dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B(x_i,H)\cap\mathcal{D}_{r_2}} \mathbf{vr}^{(1)}(y)^{-2p_0}dy\\
&\le r_1^{-2p_0}|B(x_i,H)|+|B(\bar{x},H)|+H^{2n-2p_0}E(n,\kappa,p_0)+2\mathbf{E}\xi.
\end{aligned}
$$

Taking limit on both sides and then letting $\xi \to 0, r_1 \to 1$, we have

$$
\begin{aligned}
\lim_{i\to\infty}\int_{B(x_i,H)} \mathbf{vr}^{(1)}(y)^{-2p_0} &\le 2|B(\bar{x},H)|+H^{2n-2p_0}E(n,\kappa,p_0)\\
&\le \left(2\omega_{2n}H^{2p_0}+E(n,\kappa,p_0)\right)H^{2n-2p_0}\\
&\le \left(2\cdot 9^{p_0}\omega_{2n}+E(n,\kappa,p_0)\right)H^{2n-2p_0},
\end{aligned}
$$

where we used the fact that $H \le 3$ in the last step. Then (3.112) follows from the definition of $\mathbf{E}$. q.e.d.

Proposition 3.36. *Same conditions as in Theorem 3.31. Suppose $1 \le H < \infty$. Then we have*

$$
\lim_{i\to\infty}\sup_{1\le\rho\le H}\omega_{2n}^{-1}\rho^{-2n}|B(x_i,\rho)| < \kappa^{-1}, \tag{3.116}
$$

where $g_i(0)$ is the default metric. In other words, for every large i, the volume ratio estimate holds on $(M_i, x_i, g_i(0))$ for every scale $\rho \in (0, H]$.

Proof. We argue by contradiction. If (3.116) were false, by taking subsequence if necessary, one can assume that there exists $\rho_i \in [1, H]$ such that $\omega_{2n}^{-1}\rho_i^{-2n}|B(x_i,\rho_i)| > \kappa^{-1}$. Recall that we are in the situation that $\mathbf{cr}$ is bounded from below by 1. Let $\bar{\rho}$ be the limit of ρ_i, then by the volume continuity in the pointed-$\hat{C}^4$-Cheeger–Gromov convergence, we see that

$$
\omega_{2n}^{-1}\bar{\rho}^{-2n}|B(\bar{x},\bar{\rho})| \ge \kappa^{-1}. \tag{3.117}
$$

However, since $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, we know $\omega_{2n}^{-1}\bar{\rho}^{-2n}|B(\bar{x},\bar{\rho})| \le 1$, which contradicts (3.117). q.e.d.

Proposition 3.37. *Same conditions as in Theorem 3.31. Suppose $1 \le H < \infty$. For every large i, the regularity estimate holds on $(M_i, x_i, g_i(0))$ for every scale $\rho \in (0, H]$.*

Proof. If the statement were false, then by taking subsequence if necessary, we can assume there exists $\rho_i \in (0, H]$ such that the regularity estimates fail on the scale ρ_i, i.e., the following two inequalities hold simultaneously.

$$
\omega_{2n}^{-1}\rho_i^{-2n}|B(x_i,\rho_i)| > 1-\delta_0, \tag{3.118}
$$

$$
\max_{0\le k\le 5}\left\{\rho_i^{2+k}\sup_{B(x_i,\frac{1}{2}c_a\rho_i)}|\nabla^k Rm|\right\} > 4c_a^{-2}. \tag{3.119}
$$

Clearly, $\rho_i \in [1, H]$ by the fact $\mathbf{cr}(x_i, 0) \geq 1$. Let $\bar{\rho}$ be the limit of ρ_i. Then we have

$$\omega_{2n}^{-1}\bar{\rho}^{-2n}|B(\bar{x}, \bar{\rho})| \geq 1 - \delta_0. \tag{3.120}$$

Since $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$, (3.120) implies

$$\max_{0 \leq k \leq 5} \left\{ \bar{\rho}^{2+k} \sup_{B(\bar{x}, c_a \bar{\rho})} |\nabla^k Rm| \right\} < c_a^{-2},$$

which contradicts (3.119) in light of the smooth convergence (c.f. Proposition 3.15). q.e.d.

Proposition 3.38. *Same conditions as in Theorem 3.31,* $1 \leq H \leq 2$. *Then we have*

$$\lim_{i \to \infty} \sup_{1 \leq \rho \leq H} \rho^{2p_0 - 2n} \int_{B(x_i, \rho)} \mathbf{vr}^{(\rho)}(y)^{-2p_0} dy \leq \frac{3}{2}\mathbf{E}. \tag{3.121}$$

In particular, the density estimate holds on $(M_i, x_i, g_i(0))$ *for every scale* $\rho \in (0, H]$ *and each large* i.

Proof. Since $\mathbf{vr}^{(\rho)} \geq \mathbf{vr}^{(1)}$ whenever $\rho \geq 1$, in order to show (3.121), it suffices to show

$$\lim_{i \to \infty} \sup_{1 \leq \rho \leq H} \rho^{2p_0 - 2n} \int_{B(x_i, \rho)} \mathbf{vr}^{(1)}(y)^{-2p_0} dy \leq \frac{3}{2}\mathbf{E}. \tag{3.122}$$

We argue by contradiction. If (3.122) were false, by taking subsequence if necessary, one can assume that there exists $\rho_i \in [1, H]$ such that

$$\rho_i^{2p_0 - 2n} \int_{B(x_i, \rho_i)} \mathbf{vr}^{(1)}(y)^{-2p_0} dy > \frac{3}{2}\mathbf{E},$$
$$\Rightarrow \int_{B(x_i, \rho_i)} \mathbf{vr}^{(1)}(y)^{-2p_0} dy \geq \frac{3}{2}\mathbf{E}\rho_i^{2n - 2p_0}.$$

Let $\bar{\rho}$ be the limit of ρ_i. Fix ϵ arbitrary small positive number, then we have

$$\int_{B(x_i, \bar{\rho} + \epsilon)} \mathbf{vr}^{(1)}(y)^{-2p_0} dy > \frac{3}{2}\mathbf{E}\rho_i^{2n - 2p_0} > \frac{5}{4}\mathbf{E}(\bar{\rho} + \epsilon)^{2n - 2p_0}, \tag{3.123}$$

for large i. Note that $\bar{\rho} + \epsilon < 3$, so (3.123) contradicts (3.112). q.e.d.

Proposition 3.39. *Same conditions as in Theorem 3.31. Suppose* $1 \leq H < \infty$. *Then for every large* i, *the connectivity estimate holds on* $(M_i, x_i, g_i(0))$ *for every scale* $\rho \in (0, H]$.

Proof. By the canonical radius assumption, we know the connectivity estimate holds for every scale $\rho \in (0, 1]$.

If the statement were false, then by taking subsequence if necessary, we can assume that for each i, there is a scale $\rho_i \in [1, H]$ such that the connectivity estimate fails on the scale ρ_i. In other words, $\mathcal{F}_{\frac{1}{50}c_b \rho_i} \cap$

$B(x_i,\rho_i)$ is not $\frac{1}{2}\epsilon_b\rho_i$-regularly connected. So there exist points $y_i, z_i \in \mathcal{F}_{\frac{1}{50}c_b\rho_i} \cap B(x_i,\rho_i)$ which cannot be connected by a curve $\gamma \subset \mathcal{F}_{\frac{1}{2}\epsilon_b\rho_i}$ satisfying $|\gamma| \leq 2d(y_i,z_i)$. By the canonical radius assumption, it is clear that $\rho_i \in [1,H]$, $d(y_i,z_i) \in [1,2H]$. Let $\bar{\rho}$ be the limit of ρ_i, $\bar{y}$ and $\bar{z}$ be the limit of y_i and z_i, respectively. Clearly, we have $\bar{y}, \bar{z} \in \mathcal{R}_{\frac{1}{50}c_b\bar{\rho}} \subset \mathcal{F}_{\frac{1}{100}c_b\bar{\rho}}(\bar{M})$. Since $\bar{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, we can find a shortest geodesic $\bar{\gamma}$ connecting $\bar{y}$ and $\bar{z}$ such that $\bar{\gamma} \subset \mathcal{F}_{\epsilon_b\bar{\rho}}$. Note that the limit set of $\mathcal{F}_{\frac{1}{50}c_b\rho_i} \cap B(x_i,\rho_i)$ falls into $\mathcal{F}_{\frac{1}{100}c_b\bar{\rho}}$. Moreover, this convergence takes place in the smooth topology (c.f. Corollary 3.34). So by deforming $\bar{\gamma}$ if necessary, we can construct a curve γ_i which locates in $\mathcal{F}_{\frac{1}{2}\epsilon_b\rho_i}$ and $|\gamma_i| < \frac{3}{2}d(\bar{y},\bar{z}) < 3d(y_i,z_i)$. The existence of such a curve contradicts the choice of the points y_i and z_i. q.e.d.

Combining Proposition 3.36 to 3.39, we obtain a weak-semi-continuity of canonical radius.

Theorem 3.40 (**Weak continuity of canonical radius**). *Same conditions as in Theorem 3.31. Then we have* $\lim_{i\to\infty} \mathbf{cr}(\mathcal{M}_i^0) = \infty$.

Proof. If the statement were wrong, then we can find a sequence of polarized Kähler Ricci flow solutions $\mathcal{LM}_i \in \mathscr{K}(n,A;1)$ satisfying (3.94) and

$$\lim_{i\to\infty} \mathbf{cr}(\mathcal{M}_i^0) = H < \infty. \tag{3.124}$$

Here $\mathbf{cr}(\mathcal{M}_i^0) = \mathbf{cr}(M, g_i(0))$. We remind the readers that $\mathbf{cr}$ is defined in Definition 2.9, $\mathscr{K}(n,A;1)$ is defined in Definition 3.14.

For each $\mathcal{M}_i$, we can find a point x_i such that

$$\mathbf{cr}(x_i,0) = \mathbf{cr}(x_i, g_i(0)) \leq \frac{3}{2}\mathbf{cr}(\mathcal{M}_i^0)$$

by definition. So we have

$$\lim_{i\to\infty} \mathbf{cr}(x_i,0) \leq \frac{3}{2}H < \infty. \tag{3.125}$$

In light of Proposition 3.36, Proposition 3.37, Proposition 3.38 and Proposition 3.39, we see that there exists an $N = N(H)$ such that for every $i > N$, we have volume ratio estimate, regularity estimate, density estimate and connectivity estimate hold on each scale $\rho \in (0, 2H]$. Therefore, by definition, we obtain that $\lim_{i\to\infty} \mathbf{cr}(x_i,0) \geq 2H$, which contradicts (3.125). q.e.d.

Corollary 3.41 (**Weak continuity of canonical volume radius**). *Same conditions as in Theorem 3.31. Then we have*

$$\mathbf{vr}(\bar{x}) = \lim_{i\to\infty} \mathbf{cvr}(x_i).$$

Proof. It follows from the combination of Proposition 3.33 and Theorem 3.40. q.e.d.

Theorem 3.42 (**Weak continuity of polarized canonical radius**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A; 0.5)$ satisfies (3.94). Then*

$$\mathbf{pcr}(\mathcal{M}_i^0) \geq 1,$$

for i large enough.

Proof. Note that in Theorem 3.31 and Theorem 3.40, the condition $\mathcal{LM}_i \in \mathscr{K}(n, A; 1)$ can be replaced by $\mathcal{LM}_i \in \mathscr{K}(n, A; r_0)$ for arbitrary $r_0 \in (0, 1)$. The existence of a fixed r_0 allows us to use the weak compactness theorem and then the proof follows verbatim. The exact value of r_0 is not important in the argument.

Now the theorem follows from the combination of Theorem 3.31, Theorem 3.40 (for the case $r_0 = 0.5$) and Corollary 3.11. q.e.d.

3.4. A priori lower bound of pcr. We shall use a maximum principle type argument to show that the polarized canonical radius cannot be too small. The technique used in the following proof is inspired by the proof of Theorem 12.1 of [**49**].

Proposition 3.43 (**A priori lower bound of pcr**). *There is a uniform integer constant $j_0 = j_0(n, A)$ with the following property.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, then

$$\mathbf{pcr}(\mathcal{M}^t) \geq \frac{1}{j_0}, \tag{3.126}$$

for every $t \in [-1, 1]$.

Proof. Suppose for some positive integer j_0, (3.126) fails at time $t_0 \in [-1, 1]$. Then we check whether

$$\mathbf{pcr}(\mathcal{M}^t) \geq \frac{1}{2j_0}$$

on the interval $[t_0 - \frac{1}{2j_0}, t_0 + \frac{1}{2j_0}]$. If so, stop. Otherwise, choose t_1 to be such a time and continue to check if $\mathbf{pcr}(\mathcal{M}^t) \geq \frac{1}{4j_0}$ on the interval $[t_1 - \frac{1}{4j_0}, t_1 + \frac{1}{4j_0}]$. In each step, we shrink the scale to one half of the scale in the previous step. Note this process will never escape the time interval $[-2, 2]$ since

$$|t_k - t_0| < \frac{1}{j_0}\left(\frac{1}{2} + \frac{1}{4} + \cdot + \frac{1}{2^k}\right) < \frac{1}{j_0} < 1, \quad |t_k| < |t_0| + 1 \leq 2.$$

By compactness of the underlying manifold, it is clear that the process stops after finite steps. So we can find t_k such that

$$\frac{1}{2^{k+1}j_0} \leq \mathbf{pcr}(\mathcal{M}^{t_k}) < \frac{1}{2^k j_0},$$

and $\mathbf{pcr}(\mathcal{M}^t) \geq \frac{1}{2^{k+1}j_0}$ for every $t \in [t_k - \frac{1}{2^{k+1}j_0}, t_k + \frac{1}{2^{k+1}j_0}]$. Translating the flow and rescaling it by constant $4^k j_0^2$, we obtain a new polarized Kähler Ricci flow $\widetilde{\mathcal{LM}} \in \mathscr{K}(n, A)$ such that

$$\begin{cases} \mathbf{pcr}(\widetilde{\mathcal{M}}^0) < 1, \\ \mathbf{pcr}(\widetilde{\mathcal{M}}^t) \geq \frac{1}{2}, \quad \forall\, t \in [-2^{k-1}j_0, 2^{k-1}j_0], \\ |R| + |\lambda| < \frac{A}{4^k j_0^2} < \frac{A}{j_0^2}, \quad \text{on } \widetilde{\mathcal{M}}, \\ \frac{1}{T} + \frac{1}{\mathrm{Vol}(M)} < \frac{1}{2^{k-1}j_0} + \frac{A}{j_0^2}, \quad \text{on } \widetilde{\mathcal{M}}. \end{cases} \tag{3.127}$$

In other words, $\widetilde{\mathcal{LM}} \in \mathscr{K}(n, A; 0.5)$ and $|R| + |\lambda| + \frac{1}{T} + \frac{1}{\mathrm{Vol}(M)}$ very small.

Now we return to the main proof. If the statement fails, after adjusting, translating and rescaling, we can find a sequence of polarized Kähler Ricci flow $\widetilde{\mathcal{LM}_i} \in \mathscr{K}(n, A; 0.5)$ satisfying

$$\begin{cases} \mathbf{pcr}\left(\mathcal{M}_i^0\right) < 1, \\ \frac{1}{T_i} + \frac{1}{\mathrm{Vol}(M_i)} + \sup_{\widetilde{\mathcal{M}}_i}(|R| + |\lambda|) \to 0, \end{cases}$$

which contradicts Theorem 3.42. q.e.d.

Let $\hbar = \frac{1}{j_0}$. Then we have the following fact.

Theorem 3.44 (**Homogeneity on small scales**). *For some small positive number $\hbar = \hbar(n, A)$, we have*

$$\mathscr{K}(n, A) = \mathscr{K}\left(n, A; \hbar\right). \tag{3.128}$$

4. Structure of polarized Kähler Ricci flows

In this section, we shall study the structure of polarized Kähler Ricci flows belong to $\mathscr{K}(n, A)$. In light of Theorem 3.44, it is known that $\mathscr{K}(n, A) = \mathscr{K}\left(n, A; \hbar\right)$. Therefore, we do have a uniform lower bound of polarized canonical radius for every flow in $\mathscr{K}(n, A)$.

4.1. Local metric, flow, and line bundle structure. The purpose of this subsection is to set up estimates related to the local metric structure, flow structure and line bundle structure of every flow in $\mathscr{K}(n, A)$. In particular, we shall prove Theorem 1.2 and Theorem 1.3.

Proposition 4.1 (**Kähler tangent cone**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$ is a sequence of polarized Kähler Ricci flows. Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$. Then for each $\bar{y} \in \bar{M}$, every tangent space of $\bar{M}$ at $\bar{y}$ is an irreducible metric cone. Moreover, this metric cone can be extended as an eternal, possibly singular Ricci flow solution.*

Proof. It follows from Theorem 3.44 and Theorem 3.18 that every tangent space is an irreducible metric cone. From the proof of Theorem 3.18, it is clear that the tangent cone can be extended as an eternal, static Ricci flow solution. q.e.d.

Proposition 4.2 (**Regularity equivalence**). *Same conditions as in Proposition 4.1, $\bar{y} \in \bar{M}$. Then the following statements are equivalent.*

1) *One tangent space of $\bar{y}$ is $\mathbb{C}^n$.*
2) *Every tangent space of $\bar{y}$ is $\mathbb{C}^n$.*
3) *$\bar{y}$ has a neighborhood with C^4-manifold structure.*
4) *$\bar{y}$ has a neighborhood with C^∞-manifold structure.*
5) *$\bar{y}$ has a neighborhood with C^ω-manifold (real analytic manifold) structure.*

Proof. It is obvious that $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. So it suffices to show $1 \Rightarrow 5$ to close the circle. Suppose $\bar{y}$ has a tangent space which is isometric to $\mathbb{C}^n$. So we can find a sequence $r_k \to 0$ such that

$$(\bar{M}, \bar{y}, r_k^{-2}\bar{g}) \overset{P.G.H.}{\longrightarrow} (\mathbb{C}^n, 0, g_{Euc}).$$

So for large k, the unit ball $B_{r_k^{-2}\bar{g}}(\bar{y}, 1)$ has volume ratio almost the Euclidean one. Fix such a large k, we see that $B_{\bar{g}}(\bar{y}, r_k)$ has almost Euclidean volume ratio. It follows from volume convergence that

$$\mathbf{cvr}(y_i, 0) \geq r_k,$$

for large i, where $y_i \in M_i$ and $y_i \to \bar{y}$ as $(M_i, x_i, g_i(0))$ converges to $(\bar{M}, \bar{x}, \bar{g})$. By the regularity improving property of canonical volume radius, there is a uniform small constant c such that $B(y_i, cr_k)$ is diffeomorphic to the same radius Euclidean ball in $\mathbb{C}^n$ and the metrics on $B(y_i, cr_k)$ is C^2-close to the Euclidean metric. Then one can apply the backward pseudolocality (c.f. Theorem 4.7) to obtain higher order derivative estimate for the metrics. Therefore, $B(y_i, \frac{1}{2}cr_k)$ will converge in smooth topology to a limit smooth geodesic ball $B(\bar{y}, \frac{1}{2}cr_k)$. Moreover, it is clear that geometry is uniformly bounded in a space-time neighborhood containing $B(y_i, \frac{1}{2}cr_k) \times [-c^2 r_k^2, 0]$, by shrinking c if necessary. So we obtain a limit Kähler Ricci flow solution on $B(\bar{y}, \frac{1}{4}cr_k) \times [-\frac{1}{4}c^2 r_k^2, 0]$. It follows from the result of Kotschwar (c.f. [**42**]), that $B(\bar{y}, \frac{1}{4}cr_k)$ is actually an analytic manifold, which is the desired neighborhood of $\bar{y}$. So we finish the proof of $1 \Rightarrow 5$ and close the circle. q.e.d.

Remark 4.3. By Proposition 4.2, our initial non-classical definition of regularity is proved to be the same as the classical one.

Proposition 4.4 (**Volume density gap**). *Same conditions as in Proposition 4.1, $\bar{y} \in \bar{M}$. Then $\bar{y}$ is singular if and only if*

$$\limsup_{r \to 0} \frac{|B(\bar{y}, r)|}{\omega_{2n} r^{2n}} \leq 1 - 2\delta_0. \tag{4.1}$$

Proof. If (4.1) holds, then every tangent cone of $\bar{y}$ cannot be $\mathbb{C}^n$, so $\bar{y}$ is singular. If $\bar{y}$ is singular, then every tangent space of $\bar{y}$ is an irreducible metric cone in the model space $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ with vertex a singular point, it follows from the gap property of $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ that asymptotic volume ratio of such a metric cone must be at most $1-2\delta_0$. Then (4.1) follows from the volume convergence and a scaling argument. q.e.d.

Proposition 4.5 (**Regular-Singular decomposition**). *Same conditions as in Proposition 4.1, $\bar{M}$ has the regular-singular decomposition $\bar{M}=\mathcal{R}\cup\mathcal{S}$. Then the regular part $\mathcal{R}$ admits a natural Kähler structure $\bar{J}$. The singular part $\mathcal{S}$ satisfies the estimate* $\dim_{\mathcal{H}}\mathcal{S}\leq 2n-4$.

Proof. The existence of $\bar{J}$ on $\mathcal{R}$ follows from smooth convergence, due to the backward pseudolocality (c.f. Theorem 4.7) and Shi's estimate. The Hausdorff dimension estimate of $\mathcal{S}$ follows from the combination of Proposition 3.19 and Theorem 3.44. q.e.d.

Therefore, Theorem 1.2 follows from the combinations from Proposition 4.1 to Proposition 4.5. Now we are going to discuss more delicate properties of the moduli space $\widetilde{\mathscr{K}}(n,A)$.

Proposition 4.6 (**Improved regularity in two time directions**). *There is a small positive constant $c=c(n,A)$ with the following properties.*

Suppose $\mathcal{LM}\in\mathscr{K}(n,A)$, $x_0\in M$. Let $r_0=\min\{\mathbf{cvr}(x_0,0),1\}$. Then we have

$$r^{2+k}|\nabla^k Rm|(x,t)\leq\frac{C_k}{c^{2+k}},$$

for every $k\in\mathbb{Z}^+$, $x\in B_{g(0)}(x_0,cr_0)$, $t\in[-c^2r^2,c^2r^2]$. Here C_k is a constant depending on n,A and k.

Proof. Otherwise, there exists a fixed positive integer k_0 and a sequence of $c_i\to 0$ such that

$$(c_ir_i)^{2+k_0}|\nabla^{k_0}Rm|(y_i,t_i)\to\infty, \tag{4.2}$$

for some $y_i\in B_{g_i(0)}(x_i,r_i)$, $t_i\in[-c_ir_i^2,c_ir_i^2]$, where

$$r_i=\min\{\mathbf{cvr}(x_i,0),1\}.$$

Let $\tilde{g}_i(t)=(c_ir_i)^{-2}g_i((c_ir_i)^2t+t_i)$. Then we have

$$\mathbf{cvr}_{\tilde{g}_i}(y_i,0)=(c_ir_i)^{-1}\to\infty.$$

Note that $\mathbf{pcr}_{\tilde{g}_i}(y_i,0)\geq\min\{\hbar(c_ir_i)^{-1},1\}\geq 1$. It is also clear that for the flows $\tilde{g}_i$, $|R|+|\lambda|\to 0$. Therefore, Proposition 3.15 can be applied to obtain

$$(M_i,y_i,\tilde{g}_i(0))\xrightarrow{\hat{C}^\infty}(\hat{M},\hat{y},\hat{g}). \tag{4.3}$$

However, it follows from Theorem 3.31 and Corollary 3.41 that

$$(\hat{M},\hat{y},\hat{g})\in\widetilde{\mathscr{KS}}(n,\kappa),\quad \mathbf{cvr}(\hat{y})=\infty.$$

In light of the gap property, Proposition 2.2, we know that $\hat{M}$ is isometric to $\mathbb{C}^n$. So the convergence (4.3) can be rewritten as

$$(M_i, y_i, \tilde{g}_i(0)) \xrightarrow{C^\infty} (\mathbb{C}^n, 0, g_{Euc}).$$

In particular, $|\nabla^{k_0} Rm|_{\tilde{g}_i}(y_i, 0) \to 0$, which is the same as

$$(c_i r_i)^{2+k_0} |\nabla^{k_0} Rm|(y_i, t_i) \to 0.$$

This contradicts the assumption (4.2). q.e.d.

Perelman's pseudolocality theorem says that an almost Euclidean domain cannot become very singular in a short time. His almost Euclidean condition is explained as isoperimetric constant close to that of the Euclidean one. In our special setting, we can reverse this theorem, i.e., an almost Euclidean domain cannot become very singular in the reverse time direction for a short time period.

Theorem 4.7 (**Two-sided pseudolocality**). *There is a small positive constant $\xi = \xi(n, A)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x_0 \in M$. Let

$$\Omega = B_{g(0)}(x_0, r),\ \Omega' = B_{g(0)}(x_0, \frac{r}{2}),$$

for some $0 < r \leq 1$. Suppose $\mathbf{I}(\Omega) \geq (1-\delta_0)\mathbf{I}(\mathbb{C}^n)$ at time $t = 0$, then

$$(\xi r)^{2+k} |\nabla^k Rm|(x, t) \leq C_k,$$

for every $k \in \mathbb{Z}^{\geq 0}$, $x \in \Omega'$, $t \in [-\xi^2 r^2, \xi^2 r^2]$. Here C_k is a constant depending on n, A and k.

Proof. Note that each geodesic ball contained in Ω has volume ratio at least $(1-\delta_0)\omega_{2n}$. Then the theorem follows directly from Proposition 4.6. q.e.d.

After we obtain the bound of geometry, we can go further to study the evolution of potential functions.

Theorem 4.8 (**Two-sided pseudolocality of the potential**). *Same conditions as in Theorem 4.7. Let ω_B be a smooth metric form in $2\pi c_1(M, J)$ and denote ω_t by $\omega_B + \sqrt{-1}\partial\bar{\partial}\varphi(\cdot, t)$. Suppose $\varphi(x_0, 0) = 0$ and $Osc_\Omega \varphi(\cdot, 0) \leq H$. Let $\Omega'' = B_{g(0)}(x_0, \frac{r}{4})$. Then we have*

$$(\xi r)^{-2+k} \|\varphi(\cdot, t)\|_{C^k(\Omega'', \omega_t)} \leq C_k, \tag{4.4}$$

for every $k \in \mathbb{Z}^{\geq 0}$, $t \in \left[-\frac{\xi^2}{2} r^2, \frac{\xi^2}{2} r^2\right]$. Here C_k depends on k, n, A, ξ and $\frac{H}{r^2}$.

Proof. Up to rescaling, we may assume $\xi r = 1$.

Note that φ and $\dot{\varphi}$ satisfy the equations

$$\begin{cases} \dot{\varphi} = \log \frac{\omega_t^n}{\omega_B^n} + \varphi + \dot{\varphi}(\cdot, 0), \\ -\sqrt{-1}\partial\bar{\partial}\dot{\varphi} = Ric - \lambda g. \end{cases}$$

It follows from Theorem 4.7 that geometry is uniformly bounded in $\Omega' \times [-\xi r^2, \xi r^2]$. The trace form of the second equation in the above list is $-\Delta\dot{\varphi} = R - n\lambda$. Therefore, the regularity theory of Laplacian operator applies and we have uniform bound of $\|\dot{\varphi}\|_{C^k}$ in a neighborhood of $\Omega'' \times [-\frac{\xi}{2}r^2, \frac{\xi}{2}r^2]$. Up to a normalization, we can rewrite the first equation as

$$\log \frac{(\omega_t - \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} = \varphi - \dot{\varphi} + \dot{\varphi}(\cdot, 0).$$

On Ω', the metric $g(0)$ and $g(t)$ are uniformly equivalent in each C^k-topology. So it is clear that $\|\dot{\varphi} - \dot{\varphi}(\cdot, 0)\|_{C^k(\Omega')}$ are uniformly bounded, for each k, with respect to metric $g(t)$. Since all higher derivatives of curvature are uniformly bounded on Ω', (4.4) follows from standard Monge–Ampère equation theory and bootstrapping argument. q.e.d.

Theorem 4.9 (**Improved regularity of potentials**). *Suppose that $\mathcal{LM} \in \mathscr{K}(n, A)$, $\mathbf{cvr}(M, 0) = r_0$. Let ω_B be a smooth metric in $[\omega_0]$ such that*

$$\frac{1}{2}\omega_B \leq \omega_0 \leq 2\omega_B. \tag{4.5}$$

Let $\omega_0 = \omega_B + \sqrt{-1}\partial\bar{\partial}\varphi$. Suppose $\int_M \varphi\omega_0^n = 0$ and $Osc_M\varphi \leq H$. Then we have

$$\|\varphi\|_{C^k(M,\omega_B)} \leq C_k, \quad \forall\, k \in \mathbb{Z}^{\geq 0}, \tag{4.6}$$

where C_k depends on k, ω_B, n, A, r_0 and H.

Proof. Since $\mathbf{cvr}(M, 0) = r_0 > 0$, we see that all the possible ω_0's form a compact set under the smooth topology. In other words, ω_0 has uniformly bounded geometry in each regularity level. Fix a positive integer $k_0 \geq 4$. Therefore, around each point $x \in M$, one can find a coordinate chart Ω, with uniform size, such that

$$\omega_0 = \omega_{Euc} + \sqrt{-1}\partial\bar{\partial}f, \quad \|f\|_{C^{k_0}(\Omega,\omega_{Euc})} \leq 0.01.$$

Note that in Ω, the connection terms of the metric ω_0 are pure derivatives $f_{i\bar{j}l}$, which are uniformly bounded. Similarly, all derivatives of connection terms can be expressed as high order pure derivatives of f. Therefore, up to order $k_0 - 3$, the derivatives of connections are uniformly bounded. It is clear that the metric ω_0 and ω_{Euc} are uniformly equivalent. By the covariant derivatives' bounds $\|\varphi\|_{C^k(M,\omega_0)} \leq C_k$, the bounds of connection derivatives yield that

$$\|\varphi\|_{C^k(\Omega,\omega_{Euc})} \leq C_k, \quad \forall\, 0 \leq k \leq k_0 - 1. \tag{4.7}$$

In other words, we have uniform bound for every order pure derivatives of φ, up to order k_0-1. Together with the choice assumption of Ω, we have

$$\|f-\varphi\|_{C^k(\Omega,\omega_{Euc})} \leq C_k, \quad \forall\, 0 \leq k \leq k_0-1.$$

Therefore, the connection derivatives of metric ω_B in Ω are uniformly bounded, up to order k_0-4. Consequently, the pure derivative bound (4.7) implies

$$\|\varphi\|_{C^k(\Omega,\omega_B)} \leq C_k, \quad \forall\, 0 \leq k \leq k_0-1,$$

since ω_B is a fixed smooth, compact metric with every level of regularity. Clearly, the above constant C_k depends on k, n, A, r_0, ω_B and H. Recall that the size of Ω is uniformly bounded from below, (M,ω_B) is a compact manifold. Consequently, a standard covering argument implies (4.6) for each $k \leq k_0-1$. In the end, we free k_0 and finish the proof. q.e.d.

In Ricci-flat theory, a version of Anderson's gap theorem says that regularity can be improved in the center of a ball if the volume ratio of the unit ball is very close to the Euclidean one. In our special setting, this gap theorem has a reduced volume version.

Theorem 4.10 (Gap of reduced volume). *There is a constant $\delta_0' \in (0,\delta_0]$ and a small constant η with the following property.*
Suppose $\mathcal{LM} \in \mathscr{K}(n,A)$, $x_0 \in M$, $0 < r \leq 1$. If

$$\mathcal{V}((x_0,0),r^2) \geq 1-\delta_0',$$

then we have

$$\mathbf{cvr}(x_0,0) \geq \eta r. \tag{4.8}$$

Proof. If $\lambda=0$, reduced volume is monotone. If λ is bounded, then reduced volume is almost monotone. A simple calculation shows that $\mathcal{V}((x_0,0),\rho^2) \geq 1-\delta_0$ for all $0<\rho \leq r^2$ whenever $\mathcal{V}((x_0,0),r^2) \geq 1-\delta_0'$ for some $0<r\leq 1$. Therefore, without loss of generality, we may assume $\lambda=0$ and $\delta_0'=\delta_0$ in the proof.

If the statement was wrong, there exists a sequence of

$$\eta_i \to 0,\ 0<r_i \leq 1,\ x_i \in M_i,$$

and corresponding Kähler Ricci flows satisfying

$$\begin{cases} \mathcal{V}((x_i,0),r_i^2) \geq 1-\delta_0, \\ \mathbf{cvr}(x_i,0) < \eta_i r_i. \end{cases}$$

By the monotonicity of reduced volume, we have

$$\begin{cases} \mathcal{V}((x_i,0),H\eta_i^2 r_i^2) \geq 1-\delta_0, \\ \mathbf{cvr}(x_i,0) < \eta_i r_i, \end{cases}$$

for each fixed H and large i. Let $\tilde{g}_i(t) = (\eta_i r_i)^{-2} g((\eta_i r_i)^2 t)$. It is clear that

$$\mathbf{cvr}_{\tilde{g}_i}(x_i, 0) = 1. \tag{4.9}$$

The canonical radius of $\tilde{g}_i$ tends to infinity, $|R| + |\lambda| \to 0$. Similar to the proof of Proposition 4.6, we have the convergence:

$$(M_i, x_i, \tilde{g}_i(0)) \xrightarrow{\hat{C}^\infty} (\hat{M}, \hat{x}, \hat{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa).$$

The limit space $\hat{M}$ can be extended to a static eternal Kähler Ricci flow solution. Moreover, Proposition 3.29 can be applied here and guarantees the reduced volume convergence.

$$\mathcal{V}((\hat{x}, 0), H) = \lim_{i \to \infty} \mathcal{V}_{\tilde{g}_i}((x_i, 0), H) = \lim_{i \to \infty} \mathcal{V}((x_i, 0), H(\eta_i r_i)^2) \geq 1 - \delta_0.$$

Note that H is arbitrary. By the homogeneity of reduced volume at infinity, Theorem 2.6, we see that

$$\mathrm{avr}(\hat{M}) = \lim_{H \to \infty} \mathcal{V}((\bar{x}, 0), H) \geq 1 - \delta_0.$$

So Proposition 2.2 applies to force $\hat{M}$ to be isometric to be $\mathbb{C}^n$. In particular, $\mathbf{vr}(\hat{x}) = \infty$. It follows from Corollary 3.41 that

$$\lim_{i \to \infty} \mathbf{cvr}_{\tilde{g}_i}(x_i, 0) = \infty,$$

which contradicts (4.9). q.e.d.

According to Theorem 4.10, one can define a concept of reduced volume radius for the purpose of improving regularity. Clearly, other regularity radius can also be defined. However, it seems all of them are equivalent. For simplicity, we shall not compare all of them, but only prove an example case: the equivalence of harmonic radius and canonical volume radius. The proof of other cases are verbatim. Following [**1**], for each $x_0 \in (M^m, g)$, we define harmonic radius of x_0 to be the largest r such that the ball $B(x_0, r)$ has a harmonic coordinate $\{x^i\}_{i=1}^m$ satisfying

$$\frac{1}{2}\delta_{ij} \leq g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \leq 2\delta_{ij},$$
$$r^{\frac{3}{2}} \|g_{ij}\|_{C^{1,\frac{1}{2}}} \leq 2.$$

We denote the harmonic radius of x_0 by $\mathbf{hr}(x_0, g)$. If g is the time slice $g(t)$ in a flow, we shall denote $\mathbf{hr}(x_0, g(t))$ by $\mathbf{hr}(x_0, t)$.

Proposition 4.11 (**Equivalence of regularity radii**). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x \in M$. Suppose* $\max\{\mathbf{hr}(x, 0), \mathbf{cvr}(x, 0)\} \leq 1$*, then we have*

$$\frac{1}{C}\mathbf{hr}(x, 0) \leq \mathbf{cvr}(x, 0) \leq C\mathbf{hr}(x, 0),$$

for some uniform constant $C = C(n, A)$.

Proof. Clearly, $\mathbf{cvr}(x,0) \leq C\mathbf{hr}(x,0)$ follows from the C^5-regularity property of canonical volume radius. It suffices to show

$$\frac{1}{C}\mathbf{hr}(x,0) \leq \mathbf{cvr}(x,0).$$

However, since $\mathbf{cr}(x,0) \geq \hbar$, it is clear from definition that

$$\mathbf{cvr}(x,0) \geq \frac{1}{C}\min\{\mathbf{hr}(x,0),\hbar\}.$$

If $\mathbf{hr}(x,0) \leq \hbar$, then we are done. Otherwise, we have $\hbar < \mathbf{hr}(x,0) \leq 1$. It follows that

$$\mathbf{cvr}(x,0) \geq \frac{1}{C}\hbar \geq \frac{\hbar}{C}\mathbf{hr}(x,0) \geq \frac{1}{C'}\mathbf{hr}(x,0).$$

So we finish the proof. q.e.d.

Theorem 4.12 (**Improved density estimate**). *For arbitrary small ϵ, arbitrary $0 \leq p < 2$, there is a constant $\delta = \delta(n, A, p)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x \in M$. Then under the metric $g(0)$, we have

$$\log \frac{\int_{B(x,r)} \mathbf{cvr}^{-2p} dv}{E(n,\kappa,p) r^{2n-2p}} < \epsilon, \tag{4.10}$$

whenever $r < \delta$. Here the number $E(n,\kappa,p)$ is defined in Proposition 2.8.

Proof. We argue by contradiction. Note that every blowup limit is in $\widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$ (c.f. Theorem 3.31). Then a contradiction can be obtained by the weak continuity of **cvr** (c.f. Corollary 3.41) if the statement of this theorem does not hold. q.e.d.

Note that $E(n,\kappa,0) = \omega_{2n}$. So we are led to the volume ratio estimate immediately.

Corollary 4.13 (**Volume-ratio estimate**). *For arbitrary small ϵ, there is a constant $\delta = \delta(n, A)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x \in M$. Then under the metric $g(0)$, we have

$$\log \frac{|B(x,r)|}{\omega_{2n} r^{2n}} < \epsilon, \tag{4.11}$$

whenever $r < \delta$.

In the Kähler Ricci flow setting, Corollary 4.13 improves the volume ratio estimates in [**78**] and [**31**] (c.f. Remark 1.1 of [**31**]). Note that the integral (4.10) can be used to show that for every $p \in (0, 2)$, there is a $C = C(n, A, p)$ such that the volume of the r-neighborhood of $\mathcal{S}$ in a unit ball is bounded by Cr^{2p} (c.f. Theorem 2.13), where $\mathcal{S}$ is the singular part of a limit space. By the definition of Minkowski dimension (c.f. Definition 2.2 of [**29**]), we can improve Proposition 4.5 as follows.

Corollary 4.14 (**Minkowski dimension of singular set**). *Same conditions as in Proposition 4.1, $\bar{M}$ has the regular-singular decomposition $\bar{M} = \mathcal{R} \cup \mathcal{S}$. Then* $\dim_{\mathcal{M}} \mathcal{S} \leq 2n - 4$.

In [**73**], the second author developed an estimate of the type

$$|Ric| \leq \sqrt{|Rm||R|},$$

where $\sqrt{|Rm|}$ should be understood as the reciprocal of a regular scale. Due to the improving regularity property of canonical volume radius, it induces the estimate $|Ric| \leq \frac{\sqrt{|R|}}{\mathbf{cvr}}$ pointwisely. By the uniform bound of scalar curvature and Theorem 4.12, the following estimate is clear now.

Corollary 4.15 (**Ricci curvature estimate**). *There is a constant $C = C(n, A, r_0, p)$ with the following property.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x_0 \in M$, $0 < r \leq r_0$, $0 < p < 2$. Then under the metric $g(0)$, we have

$$r^{2p-2n} \int_{B(x_0,r)} |Ric|^{2p} dv < C. \tag{4.12}$$

Corollary 4.15 localizes the L^{2p}-curvature estimate of [**70**] in a weak sense, since (4.12) only holds for $p < 2$. If $n = 2$, (4.12) also holds for $p = 2$, since the finiteness of singularity guarantees that one can choose good cutoff functions. We believe that the same localization result hold for $p = 2$ even if $n > 2$.

We return to the canonical neighborhood theorems in the introduction, Theorem 1.2, Theorem 1.3 and Theorem 1.4. However, Theorem 1.4 is not completely local. Actually, Theorem 4.7 is enough to show the local flow structure of $\mathscr{K}(n, A)$ can be approximated by $\mathscr{K}\mathscr{S}(n, \kappa)$. In light of its global properties, the proof of Theorem 1.4 is harder and is postponed to section 5.5. On the other hand, Theorem 1.2 and Theorem 1.3 are local. We now close this subsection by proving Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. It follows directly from the combination of Proposition 4.1, Proposition 4.2, Proposition 4.4, and Proposition 4.5. q.e.d.

Proof of Theorem 1.3. It follows from Theorem 3.44, Definition 3.10 and a scaling argument. q.e.d.

4.2. Local variety structure. We focus on the variety structure of the limit space in this subsection. We essentially follow the argument in [**37**], with slight modification.

Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$. Let $(\bar{M}, \bar{x}, \bar{g})$ be a pointed-Gromov–Hausdorff limit of $(M_i, x_i, g_i(0))$. Since $\bar{M}$ may be non-compact, the limit line bundle $\bar{L}$ may have infinitely many orthogonal holomorphic sections. Therefore, in general, we cannot expect to embed $\bar{M}$ into a projective space of finite dimension by the complete linear system of $\bar{L}$. However, when we focus our attention to the unit geodesic ball $B(\bar{x}, 1)$, we can choose some holomorphic sections of $\bar{L}$, peaked around $\bar{x}$, to embed $B(\bar{x}, 1)$ into $\mathbb{CP}^N$ for a finite N.

Actually, for every $\epsilon > 0$, we can find an ϵ-net of $B(\bar{x}, 2)$ such that every point in this net has canonical volume radius at least $c_0\epsilon$. For each point y in this ϵ-net, we have a peak section s_y, which is a holomorphic section such that $\|s(y)\|$ achieves the maximum among all unit L^2-norm holomorphic sections $s \in H^0(\bar{M}, \bar{L})$. By the partial-$C^0$-estimate argument (c.f. [**30**] for the flow case with weak convergence), we can assume that $\|s_y\|^2$ is uniformly bounded below in $B(y, 2\epsilon)$.

On the other hand, by the choice of y, $B(y, \eta\epsilon)$ has a smooth manifold structure for some $\eta = \eta(n)$. Therefore, we can choose n holomorphic sections of $\bar{L}^k$ such that these sections are the local deformation of $z_1, z_2, \cdots, z_n$. Here k is a positive integer proportional to ϵ^{-2}. Put these holomorphic sections together with s_y^k, we obtain $(n+1)$-holomorphic sections of $\bar{L}^k$ based at the point y. Let y run through all points in the ϵ-net and collect all the holomorphic sections based at y, we obtain a set of holomorphic sections $\{s_i\}_{i=0}^N$ of $\bar{L}^k$. Let $\{\tilde{s}_i\}_{i=0}^N$ be the orthonormal basis of $\mathit{span}\{s_0, s_1, \cdots, s_N\}$. We define the Kodaira map ι as follows.

$$\begin{aligned} \iota : B(0,2) &\mapsto \mathbb{CP}^N, \\ x &\mapsto [\tilde{s}_0(x) : \tilde{s}_1(x) : \cdots : \tilde{s}_N(x)]. \end{aligned}$$

This map is well defined. In fact, for every $z \in B(\bar{x}, 1)$, we can find a point y in the ϵ-net and $z \in B(y, 2\epsilon)$, then $\|s_y\|^2(z) > 0$ by the partial-C^0-estimate. It forces that $\tilde{s}_j(z) \neq 0$ for some j. Since k is proportional to ϵ^{-2}, we can just let $\epsilon = \frac{1}{\sqrt{k}}$ without loss of generality. In the following argument, by saying "raise the power of line bundle" from k_1 to k_2, we simultaneously means the underlying ϵ-net is strengthened from a $\frac{1}{\sqrt{k_1}}$-net to a $\frac{1}{\sqrt{k_2}}$-net.

Lemma 4.16. *Suppose $w \in \iota(B(\bar{x}, 1))$, then $\iota^{-1}(w) \cap \overline{B(\bar{x}, 1)}$ is a finite set.*

Proof. Let $y \in \iota^{-1}(w) \cap \overline{B(\bar{x}, 1)}$. It is clear that $\iota^{-1}(w)$ is contained in a ball centered at y with fixed radius, say 10ϵ. Therefore, $\iota^{-1}(w)$ is a bounded, closed set and, therefore, compact. Let F be a connected component of $\iota^{-1}(w)$. Then $\iota(F)$ is a connected, compact subvariety of $\mathbb{C}^N$, and, consequently, is a point. Note that $\iota(F)$ is always a connected set no matter how do we raise the power of ι. On the other hand, $\iota(F)$

will contain more than one point if F is not a single point, after we raise power high enough. These force that F can only be a point. Since $\iota^{-1}(w) \cap \overline{B(\bar{x}, 1)}$ is compact, it must be union of finite points. q.e.d.

Denote $\iota(\overline{B(\bar{x}, 1)})$ by W. Then W is a compact set and locally can be extended as an analytic variety. By dividing W into different components, one can apply induction argument as that in [**37**]. Following verbatim the argument of Proposition 4.10, Lemma 4.11 of [**37**], one can show that ι is an injective, non-degenerate embedding map on $B(\bar{x}, 1)$, by raising power of $\bar{L}$ if necessary. Furthermore, since being normal is a local property, one can improve Lemma 4.12 of [**37**] as follows.

Lemma 4.17. *By raising power if necessary, W is normal at the point $\iota(y)$ for every $y \in B(\bar{x}, \frac{1}{2})$.*

Under the help of parabolic Schwarz lemma and heat flow localization technique (c.f. Section 4.1 and Proposition 4.37), we can parallelly generalize Proposition 4.14 of [**37**] as follows.

Lemma 4.18. *Suppose $y \in B(\bar{x}, \frac{1}{2}) \cap \mathcal{S}$, then $\iota(y)$ is a singular point of W.*

It follows from the proof of Proposition 4.15 of [**37**] that there always exist a holomorphic form Θ on $\mathcal{R} \cap B(\bar{x}, 1)$ such that

$$\int_{\mathcal{R} \cap B(\bar{x}, 1)} \Theta \wedge \bar{\Theta} < \infty.$$

This means that every singular point $y \in \iota(B(\bar{x}, \frac{1}{2})) \cap W$ is log-terminal.

Combining all the previous lemmas, we have the following structure theorem.

Theorem 4.19 (**Analytic variety structure**)**.** *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$, $(\bar{M}, \bar{x}, \bar{g})$ is a pointed Gromov–Hausdorff limit of $(M_i, x_i, g_i(0))$. Then $\bar{M}$ is an analytic space with normal, log terminal singularities.*

4.3. Distance estimates. In this subsection, we shall develop the distance estimate along polarized Kähler Ricci flow in terms of the estimates from line bundle.

Lemma 4.20. *Suppose (M, L) is a polarized Kähler manifold satisfying the following conditions*

- $|B(x, r)| \geq \kappa \omega_{2n} r^{2n}, \quad \forall x \in M, 0 < r < 1.$
- $|\mathbf{b}| \leq 2c_0$ *where* $\mathbf{b}$ *is the Bergman function.*
- $\|\nabla S\| \leq C_1$ *for every L^2-unit section $S \in H^0(M, L)$.*

For every positive number a, define $\Omega(x,a)$ as the path-connected component containing x of the set

$$\left\{ z \,\middle|\, \|S\|^2(z) \geq e^{-2a-2c_0}, \|S\|^2(x) = e^{\mathbf{b}(x)}, \int_M \|S\|^2 dv = 1 \right\}. \tag{4.13}$$

Then we have

$$B\left(x,r\right) \subset \Omega(x,a) \subset B(x,\rho), \tag{4.14}$$

for some $r = r(n,\kappa,c_0,C_1,a)$ and $\rho = \rho(n,\kappa,c_0,C_1,a)$.

Proof. Define $r \triangleq \frac{1-e^{-a}}{C_1 e^{a+c_0}}$. Recall that $\|S\|(x) \geq e^{-c_0}$. By the gradient bound of S, it is clear that every point in $B(x,r)$ satisfies $\|S\| \geq e^{-a-c_0}$. In other words, we have

$$B(x,r) \subset \Omega(x,a).$$

On the other hand, we can cover $\Omega(x,a)$ by finite balls $B(x_i,2r)$ such that each $x_i \in \Omega(x,a)$ and different $B(x_i,r)$'s are disjoint to each other. Again, the gradient bound of S implies that $\|S\| \geq e^{-2a-c_0}$ in each $B(x_i,r)$. Then we have

$$N\kappa\omega_{2n}r^{2n} \leq \sum_{i=1}^{N} |B(x_i,r)| \leq |\Omega(x,2a)| \leq e^{4a+2c_0}.$$

For every $z \in \Omega(x,a)$, we have

$$d(x,z) \leq 4Nr \leq \frac{4e^{4a+2c_0}}{\kappa\omega_{2n}r^{2n-1}} = \frac{4e^{4a+2c_0}}{\kappa\omega_{2n}} \cdot \frac{C_1^{2n-1}e^{(2n-1)(a+c_0)}}{(1-e^{-a})^{2n-1}}.$$

Let ρ be the number on the right hand side of the above inequality. Then it is clear that

$$\Omega(x,a) \subset B(x,\rho).$$

So we finish the proof. q.e.d.

Lemma 4.20 implies that the level sets of peak holomorphic sections are comparable to geodesic balls. However, the norm of peak holomorphic section has stability under the Kähler Ricci flows in $\mathscr{K}(n,A)$. Therefore, one can compare distances at different time slices in terms the values of norms of a same holomorphic sections.

Lemma 4.21. *There exists a small constant $\epsilon_0 = \epsilon_0(n,A)$ such that the following properties are satisfied.*

Suppose $\mathcal{LM} \in \mathscr{K}(n,A)$, then we have

$$B_{g(t_1)}(x,\epsilon_0) \subset B_{g(t_2)}\left(x,\epsilon_0^{-1}\right), \tag{4.15}$$

whenever $t_1,t_2 \in [-1,1]$.

Proof. Without loss of generality, we only need to show (4.15) for time $t_1 = 0, t_2 = 1$. Because of Theorem 1.3 and Moser iteration, we can assume $|\mathbf{b}| \leq 2c_0$ for some $c_0 = c_0(n, A)$. By Moser iteration technique, we can also assume $\|\nabla S\| \leq C_1$ for every unit L^2-norm holomorphic section of L (c.f. Lemma 5.1 of [**72**] and Lemma 3.2 of [**30**]). Note that $e^{\mathbf{b}(x)}$ is the maximum value of $\|S\|^2$ among all unit L^2-norm holomorphic sections of L. So we can choose ϵ small enough such that

$$\|S\|(z) \geq \frac{1}{2}e^{-c_0}, \quad \forall\, z \in B(x, 2\epsilon),$$

for some unit holomorphic section S. Note ϵ can be chosen uniformly, say $\epsilon = \frac{e^{-c_0}}{4C_1}$.

Fix S and define

$$\Omega \triangleq \left\{ z \middle| \|S\|_0(z) \geq \frac{1}{2}e^{-c_0} \right\}, \qquad \tilde{\Omega} \triangleq \left\{ z \middle| \|S\|_1(z) \geq \frac{1}{4}e^{-c_0 - A} \right\}.$$

Without loss of generality, we can assume both Ω and $\tilde{\Omega}$ are path-connected. Otherwise, just replace them by the corresponding path-connected part containing z. It follows from definition that $B(x, \epsilon) \subset \Omega$. In view of the volume element evolution equation, it is also clear that $\Omega \subset \tilde{\Omega}$.

Note that S is a unit section at time $t = 0$. At time $t = 1$, its L^2-norm locates in $[e^{-2A}, e^{2A}]$. So we have

$$e^{2A} \geq \int_M \|S\|_1^2 dv > |\tilde{\Omega}|_1 \frac{1}{16} e^{-2c_0 - 2A}, \quad \Rightarrow \quad |\tilde{\Omega}|_1 < 16 e^{2c_0 + 4A}.$$

Now we can follow the covering argument in the previous lemma to show a diameter bound of $\tilde{\Omega}$ under the metric $g(1)$. In fact, we can cover $\tilde{\Omega}$ by finite geodesic balls $B(x_i, 2\epsilon)$ such that $x_i \in \tilde{\Omega}$ and all different $B(x_i, \epsilon)$'s are disjoint to each other. Clearly, each geodesic ball $B(x_i, \epsilon)$ has volume at least $\kappa \omega_{2n} \epsilon^{2n}$, where $\kappa = \kappa(n, A)$. Let N be the number of balls, then

$$N \kappa \omega_{2n} \epsilon^{2n} \leq \sum_{i=1}^{N} |B(x_i, 1)| \leq |\tilde{\Omega}| \leq 16 e^{2c_0 + 4A}.$$

Therefore, under metric $g(1)$, we obtain

$$\operatorname{diam} \Omega \leq \operatorname{diam} \tilde{\Omega} \leq 4N\epsilon \leq \frac{64 e^{2c_0 + 4A}}{\kappa \omega_{2n} \epsilon^{2n-1}}.$$

Recall that $B_{g(0)}(x, \epsilon) \subset \Omega \subset \tilde{\Omega}$, $\epsilon = \frac{e^{-c_0}}{4C_1}$. So under metric $g(1)$, we have

$$\operatorname{diam} B_{g(0)}(x, \epsilon) \leq \operatorname{diam} \tilde{\Omega} \leq \frac{4^{2n+2} e^{(2n+3)c_0 + 4A} C_1^{2n-1}}{\kappa \omega_{2n}}.$$

Define

$$\epsilon_0 \triangleq \min\left\{\frac{e^{-c_0}}{4C_1}, \frac{\kappa\omega_{2n}}{4^{2n+2}e^{(2n+3)c_0+4A}C_1^{2n-1}}\right\}. \tag{4.16}$$

Note that ϵ_0 depends only on n, A. Then we have

$$\operatorname{diam} B_{g(0)}(x, \epsilon_0) \leq \epsilon_0^{-1},$$

which implies

$$B_{g(0)}(x, \epsilon_0) \subset B_{g(1)}(x, \epsilon_0^{-1}).$$

So we finish the proof. q.e.d.

Lemma 4.22. *For every r small, there is a δ with the following property.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$. Suppose $|R| + |\lambda| < \delta$ on $M \times [-1, 1]$, then we have

$$B_{g(t_1)}(x, \epsilon_0 r) \subset B_{g(t_2)}(x, \epsilon_0^{-1} r), \tag{4.17}$$

for every $t_1, t_2 \in [-1, 1]$. Here ϵ_0 is the constant in Lemma 4.21.

Proof. We proceed by a contradiction argument.

Again, it suffices to show (4.17) for $t_1 = 0$ and $t_2 = 1$. By adjusting r if necessary, we can also make a rescaling by integer factor. Up to rescaling, (4.17) is the same as

$$B_{g(0)}(x, \epsilon_0) \subset B_{g(r^{-2})}(x, \epsilon_0^{-1}). \tag{4.18}$$

Suppose the statement of this lemma was wrong. Then there is an $r_0 > 0$ and a sequence of points $x_i \in M_i$ such that

$$B_{g_i(0)}(x_i, \epsilon_0) \not\subset B_{g_i(r_0^{-2})}(x_i, \epsilon_0^{-1}).$$

However, $|R| + |\lambda| \to 0$ in C^0-norm as $i \to \infty$. So we can take a limit

$$(M_i, x_i, g_i(0)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}).$$

As usual, we can find a regular point $\bar{z} \in \bar{M}$ near $\bar{x}$. Let $z_i \in M_i$ and $z_i \to \bar{z}$ as the above convergence happens. Then we can extend the above convergence to each time slice.

$$(M_i, z_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{z}, \bar{g}), \quad \forall\, t \in [0, r_0^{-2}].$$

Note that $\bar{x}$ may be a singular point of $\bar{M}$. So in the above convergence, we only have x_i converges to $\bar{x}(t)$, which may depends on time t. Lemma 4.21 guarantees that $\bar{x}(t)$ is not at infinity.

Note that $Osc_M \dot{\varphi}$ is scaling invariant and, consequently, uniformly bounded by condition inequality (1.4). From the polarized Kähler Ricci flow solution condition (1.3), we have

$$\Delta\dot{\varphi} = -R + n\lambda,$$

whose right hand side is tending to zero. Therefore, $\dot{\varphi}$ converges to a limit bounded function which is harmonic function on $\mathcal{R}(\bar{M})$, the regular part of $\bar{M}$. Such a function must be a constant by Liouville-type theorem (c.f. Corollary 2.25 of [**29**]). Actually, a bounded harmonic function on $\mathcal{R}(\bar{M})$ will automatically be a bounded Lipschitz function on $\bar{M}$, by Proposition 2.29 of [**29**]. Applying normalization condition, the limit function must be zero on $\bar{M} \times [0, r_0^{-2}]$. Therefore, the limit line bundle $\bar{L}$ admits a limit metric which does not evolve along time. Therefore, for a fixed holomorphic section $\bar{S}$ and a fixed level value, the level sets of $\left\|\bar{S}\right\|^2$ does not depend on time.

Choose S_i be the peak section of L_i at x_i, with respect to the metrics at time $t = 0$. By the choice of ϵ_0, it is clear that $\|S_i\|_t \geq \frac{1}{2}e^{-c_0}$ on the ball $B(x_i, \epsilon_0)$. In other words, we have

$$B(x_i, \epsilon_0) \subset \Omega_{i,t} \triangleq \left\{ z \,\middle|\, \|S_i\|_t(z) \geq \frac{1}{2}e^{-c_0} \right\}.$$

Without loss of generality, we can assume $\Omega_{i,t}$ is path connected. Clearly, each $\Omega_{i,t}$ has uniformly bounded diameter, due to Lemma 4.20. Let $\bar{\Omega}$ be the limit set of $\Omega_{i,0}$. Clearly, $\bar{z} \in \bar{\Omega}$. Then the above discussion implies that $\bar{\Omega}$ is actually the limit set of each $\Omega_{i,t}$.

Let $\bar{y}$ be the limit point of y_i, which is a point in $B_{g_i(0)}(x_i, \epsilon_0)$ and start to escape $B(x_i, \epsilon_0^{-1})$ at time t_i, which converges to $\bar{t}$. So we obtain

$$\bar{y} \in B_{\bar{g}(0)}(\bar{x}, \epsilon_0), \quad d(\bar{y}, \bar{x}(\bar{t})) = \epsilon_0^{-1}. \tag{4.19}$$

Recall that

$$\frac{d}{dt}dv = (n\lambda - R)dv, \quad \frac{d}{dt}h(t) = \frac{d}{dt}\{e^{-\varphi}h(0)\} = -\dot{\varphi}h(t).$$

Since $|R| + |\lambda| \to 0$ and $|\dot{\varphi}| \to 0$, the volume element of the underlying manifold and the line bundle metric are all almost static when time evolves. Then it is easy to see that y_i can never escape $\Omega_{i,t}$. So $\bar{y} \in \bar{\Omega}$. Similarly, we know $\bar{x}(\bar{t}) \in \bar{\Omega}$. Therefore, at time $\bar{t}$, we have

$$d(\bar{y}, \bar{x}(\bar{t})) \leq \operatorname{diam} \bar{\Omega}.$$

Note that the argument in the proof of Lemma 4.20 holds for the polarized singular manifold $(\bar{M}, \bar{L})$, due to the high codimension of $\mathcal{S}(M)$ and the gradient bound of each S_i. Since $\int_{\bar{M}} \left\|\bar{S}\right\|^2 dv$ is uniformly bounded from above by 1, we can follow the proof of Lemma 4.20 to show that

$$\operatorname{diam}(\bar{\Omega}) \leq \rho(n, \kappa, c_0, C_1, \log 2) < \epsilon_0^{-1}$$

by the choice of ϵ_0 in (4.16). Consequently, we have $d(\bar{y}, \bar{x}(\bar{t})) < \epsilon_0^{-1}$, which contradicts (4.19). q.e.d.

Based on Lemma 4.22, we can improve Proposition 3.15. Namely, under the condition $|R| + |\lambda| \to 0$, the limit flow is static, even on the singular part. Clearly, due to Theorem 3.44, we do not need the assumption of lower bound of polarized canonical radius anymore.

Proposition 4.23 (**Static limit space-time**). *Suppose* $\mathcal{LM}_i \in \mathscr{K}(n, A)$ *satisfies*

$$\lim_{i\to\infty} \sup_{\mathcal{M}_i}(|R| + |\lambda|) = 0.$$

Suppose $x_i \in M_i$. *Then*

$$(M_i, x_i, g_i(0)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}).$$

Moreover, we have

$$(M_i, x_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}, \bar{g}),$$

for every $t \in (-\bar{T}, \bar{T})$, *where* $\bar{T} = \lim_{i\to\infty} T_i > 0$. *In other words, the identity maps between different time slices converge to the limit identity map.*

As a direct application, we obtain the bubble structure of a given family of polarized Kähler Ricci flows.

Theorem 4.24 (**Space-time structure of a bubble**). *Suppose* $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$, $t_i \in (-T_i, T_i)$, *and* $r_i \to 0$. *Suppose* $\widetilde{\mathcal{M}_i}$ *is the adjusting of* $\mathcal{M}_i$ *by shifting time* t_i *to* 0 *and then rescaling the space-time by the factors* r_i^{-2}, *i.e.,* $\tilde{g}_i(t) = r_i^{-2} g(r_i^2 t + t_i)$. *Suppose* $r_i^{-2} \max\{|t_i - T_i|, |t_i + T_i|\} = \infty$. *Then we have*

$$(M_i, x_i, \tilde{g}_i(t)) \xrightarrow{\hat{C}^\infty} (\hat{M}, \hat{x}, \hat{g}),$$

for each time $t \in (-\infty, \infty)$ *with* $\hat{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$.

Theorem 4.24 means that the space-time structure of $\hat{M} \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ is the model for the space-time structures around (x_i, t_i), up to proper rescaling. Therefore, Theorem 4.24 is an improvement of Theorem 3.31, where we only concern the metric structure.

In view of Proposition 4.23, it is not hard to see that distance is a uniform continuous function of time in $\mathscr{K}(n, A)$.

Theorem 4.25 (**Uniform continuity of distance function**). *Suppose* $\mathcal{LM} \in \mathscr{K}(n, A)$, $x, y \in M$. *Suppose* $d_{g(0)}(x, y) < 1$. *Then for every small* ϵ, *there is a* $\delta = \delta(n, A, \epsilon)$ *such that*

$$|d_{g(t)}(x, y) - d_{g(0)}(x, y)| < \epsilon,$$

whenever $|t| < \delta$.

Proof. We argue by contradiction. Suppose the statement was wrong, we can find an $\bar{\epsilon} > 0$ and a sequence of flows violating the statement for time $|t_i| \to 0$. Around x_i, in the ball $B_{g_i(0)}\left(x_i, \frac{\epsilon_0^2\bar{\epsilon}}{10}\right)$, we can find x_i' which are uniform regular at time $t = 0$, where ϵ_0 is the same constant in Lemma 4.22 and Lemma 4.21. Namely, x_∞', the limit point of x_i' is a regular point in the limit space. By two-sided pseudolocality, Theorem 4.7, it is clear that x_i' is also uniform regular at time $t = t_i$. Similarly, we can choose y_i'. By virtue of triangle inequality and Lemma 4.22, we obtain

$$d_{g_i(0)}(x_i', y_i') - \frac{\epsilon_0^2\bar{\epsilon}}{5} \le d_{g_i(0)}(x_i, y_i) \le d_{g_i(0)}(x_i', y_i') + \frac{\epsilon_0^2\bar{\epsilon}}{5},$$
$$d_{g_i(t_i)}(x_i', y_i') - \frac{\bar{\epsilon}}{5} \le d_{g_i(t_i)}(x_i, y_i) \le d_{g_i(t_i)}(x_i', y_i') + \frac{\bar{\epsilon}}{5}.$$

By argument similar to that in Proposition 3.20, it is clear that

$$\lim_{i\to\infty} d_{g_i(t_i)}(x_i', y_i') = \lim_{i\to\infty} d_{g_i(0)}(x_i', y_i').$$

Then it follows that

$$\begin{aligned}&\lim_{i\to\infty} d_{g_i(0)}(x_i, y_i) - \frac{(1+\epsilon_0^2)}{5}\bar{\epsilon}\\ &\le \lim_{i\to\infty} d_{g_i(t_i)}(x_i, y_i) \le \lim_{i\to\infty} d_{g_i(0)}(x_i, y_i) + \frac{(1+\epsilon_0^2)}{5}\bar{\epsilon}.\end{aligned}$$

In particular, for large i, we have

$$\left|d_{g_i(0)}(x_i, y_i) - d_{g_i(t_i)}(x_i, y_i)\right| < \frac{(1+\epsilon_0^2)}{5}\bar{\epsilon} < \bar{\epsilon},$$

which contradicts our assumption. q.e.d.

4.4. Volume of high curvature neighborhood. In this subsection, we shall develop the flow version of the volume estimate of Donaldson and the first author (c.f. [**17**], [**18**], see also [**14**]).

Proposition 4.26 (Kähler cone complex splitting). *Same conditions as in Proposition 4.1, $\bar{y} \in \bar{M}$. Suppose $\hat{Y}$ is a tangent cone of $\bar{M}$ at $\bar{y}$, then there is a fixed nonnegative integer k such that*

$$\hat{Y} = C(Z) \times \mathbb{C}^{n-k}, \tag{4.20}$$

where $C(Z)$ is a metric cone without straight line. A point in $\bar{M}$ is regular if and only if one of the tangent cone is $\mathbb{C}^n$.

Proof. By definition of tangent cone, one can find a sequence of numbers $r_i \to 0$. Taking subsequence if necessary, let $\tilde{g}_i(t) = r_i^{-2} g_i(r_i^2 t)$, then we have

$$(M_i, y_i, \tilde{g}_i(0)) \xrightarrow{\hat{C}^\infty} (\hat{Y}, \hat{y}, \hat{g}).$$

By compactness, we see that $(\hat{Y}, \hat{y}, \hat{g}) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$. On the other hand, it is a metric cone, which is the tangent space of itself at the origin. So $\hat{Y}$ has the decomposition (4.20), by Theorem 2.5. q.e.d.

Proposition 4.27 (**Kähler tangent cone rigidity**, c.f. Theorem 2 of [**18**]). *Suppose that $\mathcal{LM}_i \in \mathscr{K}(n, A)$. Suppose $x_i \in M_i$ and $(\bar{M}, \bar{x}, \bar{g})$ is a limit space of $(M_i, x_i, g_i(0))$. Let $\hat{Y}$ be a tangent space of $\bar{M}$ satisfying*

$$\hat{Y} = \left(\mathbb{C}^k/\Gamma\right) \times \mathbb{C}^{n-k}, \quad \Gamma \subset U(k).$$

Then $\hat{Y}$ satisfies the splitting (4.20) for $k = 2$ or $k = 0$.

Proof. Clearly, $k = 0$ if and only if the base point is regular. So it suffices to show that for every singular tangent space we have $k = 2$. By Proposition 4.26, we only need to rule out the case $k \geq 3$. However, this follows from the rigidity of complex structure on the smooth annulus in $\mathbb{C}^k/\Gamma$, where Γ is a finite group of holomorphic isometry of $\mathbb{C}^k$, when $k \geq 3$. Note that $[\omega_i] = c_1(L_i)$, which is an integer class. Therefore, the proof follows verbatim as that in [**18**]. Note that Ricci curvature uniformly bounded condition in [**18**] is basically used to guarantee the pointed-$\hat{C}^4$-Cheeger–Gromov convergence. In our case, the convergence can be obtained from Theorem 3.44. q.e.d.

Proposition 4.28 (**Existence of holomorphic slicing**). *Suppose $\hat{Y} \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ is a metric cone satisfying the splitting (4.20). Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x \in M$. If $(M, x, g(0))$ is very close to $(\hat{Y}, \hat{y}, \hat{g})$, i.e., the pointed-Gromov–Hausdorff distance*

$$d_{PGH}((M, x, g(0)), (\hat{Y}, \hat{y}, \hat{g})) < \epsilon,$$

for sufficiently small ϵ, which depends on $n, A, \hat{Y}$, then there exists a holomorphic map

$$\Psi = (u_{k+1}, u_{k+2}, \cdots, u_n) : B(x, 10) \mapsto \mathbb{C}^{n-k}$$

satisfying

$$|\nabla \Psi| \leq C(n, A), \tag{4.21}$$

$$\sum_{k+1 \leq i,j \leq n} \int_{B(x,10)} |\delta_{ij} - \langle \nabla u_i, \nabla u_j \rangle| \, dv \leq \eta(n, A, \epsilon), \tag{4.22}$$

where η is a small number such that $\lim_{\epsilon \to 0} \eta = 0$.

Proof. It follows from the argument in [**37**] that the constant section 1 of the trivial bundle over $\hat{Y}$ can be "pulled back" as a non-vanishing holomorphic section of L over $B(x, 10)$, up to a finite lifting of power of L. Therefore, we can regard L as a trivial bundle over $B(x, 10)$ without loss of generality. Let S_0 be the pull-back of the constant 1 section. In particular, S_0 is a non-vanishing holomorphic section on $B(x, 10)$. On

$B(x,10)$, every holomorphic section S of L can be written as $S = uS_0$ for a holomorphic function u and $\|S\|_h^2 = |u|^2\|S_0\|_h^2$.

From the splitting (4.20), there exist natural coordinate holomorphic functions $\{z_j\}_{j=k+1}^n$ on $\hat{Y}$. Same as [**37**], one can apply Hörmander's estimate to construct $\{S_j\}_{j=k+1}^n$, which are holomorphic sections of L. Each S_j can be regarded as an "approximation" of z_j, although they have different base spaces. Let $u_j = \frac{S_j}{S_0}$ for each $j \in \{k+1, \cdots, n\}$. Then we can define a holomorphic map Ψ from $B(x,10)$ to $\mathbb{C}^{n-k}$ as follows

$$\Psi(y) \triangleq (u_{k+1}, u_{k+2}, \cdots, u_n).$$

Note that each S_i is a holomorphic section of L with L^2-norm bounded from two sides, according to its construction. Using metrics induced by h and condition (1.3), direct calculation shows that

$$\begin{aligned}\Delta\|\nabla S\|^2 &= \|\nabla\nabla S\|^2 - (n+2)\|\nabla S\|^2 + n\|S\|^2 + R_{i\bar{j}}\bar{S}_{,\bar{i}}S_{,j}\\ &= \|\nabla\nabla S\|^2 + \{\lambda - (n+2)\}\|\nabla S\|^2 + n\|S\|^2 - \dot{\varphi}_{i\bar{j}}\bar{S}_{,\bar{i}}S_{,j}.\end{aligned}$$

In light of (1.4), $\dot{\varphi}$ is bounded and there exists a uniform Sobolev constant. Then Moser iteration (c.f. Lemma 3.2 of [**30**] and Lemma 5.1 of [**72**]) implies that there exists a uniform bound $\|\nabla S_i\|_h^2 \leq C(n,A)$, which implies (4.21) when restricted on $B(x,10)$. Moreover, on $B(x,10)$, by smooth convergence, it is not hard to see that $\langle \nabla u_i, \nabla u_j\rangle$ can pointwisely approximate δ_{ij} away from singularities of $\hat{Y}$, in any accuracy level when $\epsilon \to 0$. This approximation together with (4.21) yields (4.22). q.e.d.

Theorem 4.29 (**Weak monotonicity of curvature integral**). *There exists a small constant $\epsilon = \epsilon(n,A)$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n,A)$. Suppose $x \in M$, $0 < r \leq 1$. Then under the metric $g(0)$, we have

$$\sup_{B(x,\frac{1}{2}r)} |Rm| \leq r^{-2}, \tag{4.23}$$

whenever $r^{4-2n}\int_{B(x,r)} |Rm|^2 dv \leq \epsilon$.

Proof. Up to rescaling, we can assume $r = 1$ without loss of generality. If the statement was wrong, we can find a sequence of points $x_i \in M_i$ such that

$$\int_{B(x_i,1)} |Rm|^2 dv \to 0, \qquad \sup_{B(x_i,\frac{1}{2})} |Rm| \geq 1,$$

where the default metric is $g_i(0)$, the time zero metric of a flow g_i, in the moduli space $\mathscr{K}(n,A)$. By the smooth convergence at places when curvature uniformly bounded, it is clear that the above conditions imply that

$$\int_{B(x_i,1)} |Rm|^2 dv \to 0, \qquad \sup_{B(x_i,\frac{3}{4})} |Rm| \to \infty.$$

Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$. Then $B(\bar{x}, \frac{3}{4})$ contains at least one singularity $\bar{y}$. Without loss of generality, we can assume $\bar{x}$ is a singular point. Note that $B(\bar{x}, \frac{1}{4})$ is a flat manifold away from singularities. So every tangent space of $\bar{M}$ at $\bar{x}$ is a flat metric cone. Let $\hat{Y}$ be one of such a flat metric cone. By taking subsequence if necessary, we can assume

$$(M_i, x_i, \tilde{g}_i(0)) \xrightarrow{\hat{C}^\infty} (\hat{Y}, \hat{x}, \hat{g}),$$

for some flow metrics $\tilde{g}_i$ satisfying $\tilde{g}_i(t) = r_i^{-2} g_i(r_i^2 t)$, $r_i \to 0$. Since $\hat{Y}$ is a flat metric cone, in light of Proposition 4.27, we have the splitting

$$\hat{Y} = (\mathbb{C}^2/\Gamma) \times \mathbb{C}^{n-2}.$$

Let $(M, x, \tilde{g})$ be one of $(M_i, x_i, \tilde{g}_i(0))$ for some large i. Because of Proposition 4.28, we can construct a holomorphic map $\Psi : B(x, 10) \to \mathbb{C}^{n-2}$ satisfying (4.21) and (4.22). Then we can follow the slice argument as in [**12**] and [**9**]. Our argument will be simpler since our slice functions are holomorphic rather than harmonic.

Actually, for generic $\vec{z} = (z_3, z_4, \cdots, z_n)$ satisfying $|\vec{z}| < 0.1$, we know $\Psi^{-1}(\vec{z}) \cap B(x, 5)$ is a complex surface with boundary. Clearly, $\Psi^{-1}((S^3/\Gamma) \times \{\vec{z}\})$ is close to $(S^3/\Gamma) \times \vec{z}$, if we regard S^3/Γ as the unit sphere in $\mathbb{C}^2/\Gamma$. Deform the preimage a little bit if necessary, we can obtain a $\partial\Omega$ which bounds a complex surface Ω. By coarea formula and the bound of $|\nabla\Psi|$, it is clear that for generic Ω obtained in this way, we have

$$\int_{\Omega} |Rm|^2 d\sigma \to 0.$$

Consider the restriction of TM on Ω. Let c_2 be a form representing the second Chern class of the tangent bundle TM, obtained from the Kähler metric $\tilde{g}(0)$ from the classical way. Let $\hat{c}_2$ be the corresponding differential character with value in $\mathbb{R}/\mathbb{Z}$. Since the point-wise norm of c_2 is bounded by $|Rm|^2$, it is clear that

$$\hat{c}_2(\partial\Omega) = \int_{\Omega} c_2 \quad (mod\ \mathbb{Z}) \to 0. \tag{4.24}$$

On the other hand, since $\partial\Omega$ converges to S^3/Γ, we have

$$\hat{c}_2(\Omega) \to \frac{1}{|\Gamma|}. \tag{4.25}$$

Therefore, the combination of (4.24) and (4.25) forces that $|\Gamma| = 1$. This is impossible since $|\Gamma| \geq 2$ by our assumption that $\hat{Y}$ is a singular metric cone. q.e.d.

From now on to the end of this subsection, we use $g(0)$ as the default metric. Similar to the definition in [**17**], for any small r, let $\mathcal{Z}_r$ be the r-neighborhood of the points where $|Rm| > r^{-2}$. Recall the definition equation (2.12), we denote $\mathcal{F}_r$ as the collection of points whose canonical volume radii are greater than r, $\mathcal{D}_r$ as the complement of $\mathcal{F}_r$. Under these notations, we have the following property.

Proposition 4.30 (**Equivalence of singular neighborhoods**). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $0 < r < \hbar$. Then at time zero, we have*

$$\mathcal{D}_{cr} \subset \mathcal{Z}_r \subset \mathcal{D}_{\frac{1}{c}r}, \tag{4.26}$$

for some small constant $c = c(n, A)$.

Proof. Let us first prove $\mathcal{D}_{cr} \subset \mathcal{Z}_r$. Suppose the statement was wrong, we can find a sequence $c_i \to 0$ and flows in $\mathscr{K}(n, A)$ such that $\mathcal{D}_{c_i r_i} \not\subset \mathcal{Z}_{r_i}$ for some $r_i < \hbar$. Choose $x_i \in \mathcal{D}_{c_i r_i} \cap \mathcal{Z}_{r_i}^c$. Let ρ_i be the canonical volume radius of x_i. Rescale the flow such that the canonical volume radius at x_i becomes 1. Taking limit, we will obtain a smooth flat space in $\widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$, which is nothing but $\mathbb{C}^n$. Therefore, the canonical volume radii of the base points x_i should tend to infinity, which is a contradiction.

Then we prove $\mathcal{Z}_r \subset \mathcal{D}_{\frac{1}{c}r}$. Suppose $x \in \mathcal{Z}_r$, then $|Rm|(y) \geq r^{-2}$ for some $y \in B(x, r)$. By the regularity improving property of canonical volume radius, it is clear that $\mathbf{cvr}(x) \leq \frac{2}{c_a} r$. In other words, $x \in \mathcal{D}_{\frac{2}{c_a} r}$. q.e.d.

Theorem 4.31 (**Volume estimates of high curvature neighborhood**). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$. Under the metric $g(0)$, we have*

$$|\mathcal{Z}_r| \leq Cr^4,$$

where C depends on n, A and the upper bound of $\int_M |Rm|^2 dv$.

Proof. Because of Proposition 4.30, it suffices to show $|\mathcal{D}_{cr}| \leq Cr^4$. In light of Theorem 4.29, if

$$r^{4-2n} \int_{B(x,r)} |Rm|^2 dv < \epsilon,$$

for some $r < \hbar$, then $x \in \mathcal{F}_{cr}$. In other words, if $x \in \mathcal{D}_{cr}(M, 0)$, then it is forced that

$$r^{4-2n} \int_{B(x,r)} |Rm|^2 dv \geq \epsilon.$$

Let $\bigcup_{i=1}^N B(x_i, 2r)$ be a finite cover of $\mathcal{D}_{cr}$ such that

- $x_i \in \mathcal{D}_{cr}$.
- $B(x_i, r)$ are disjoint to each other.

Then we can bound N as follows.

$$N\epsilon r^{2n-4} \leq \sum_{i=1}^{N} \int_{B(x_i,r)} |Rm|^2 dv \leq \int_M |Rm|^2 dv \leq H.$$

Consequently, we have

$$|\mathcal{D}_{cr}(M)| \leq \sum_{i=1}^{N} |B(x_i, 2r)| \leq \frac{H}{\epsilon} r^{4-2n} \kappa^{-1} \omega_{2n} (2r)^{2n} \leq Cr^4.$$

Since both κ and ϵ depends only on n and A. It is clear that $C = C(n, A, H)$ where H is the upper bound of $\int_M |Rm|^2 dv$. q.e.d.

Corollary 4.32 (Volume estimates of singular neighborhood). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$. Suppose $\int_{M_i} |Rm|^2 dv \leq H$ uniformly under the metric $g_i(0)$. Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$. Let $\mathcal{S}_r$ be the set defined in (2.22), then we have*

$$|\mathcal{S}_r| \leq Cr^4,$$

for each small r and some constant $C = C(n, A, H)$. In particular, we have the estimate of Minkowski dimension of the singularity

$$\dim_{\mathcal{M}} \mathcal{S} \leq 2n - 4.$$

Following [**30**], the space $\bar{M} = \mathcal{R} \cup \mathcal{S}$ is called a metric-normal Q-Fano variety if there exists a homeomorphic map $\varphi : \bar{M} \to Z$ for some Q-Fano normal variety Z such that $\varphi|_{\mathcal{R}}$ is a biholomorphic map. Moreover, $\dim_{\mathcal{M}} \mathcal{S} \leq 2n - 4$.

Theorem 4.33 (Limit structure). *Suppose that $\mathcal{LM}_i \in \mathscr{K}(n, A)$. Under the metric $g_i(0)$, suppose*

$$\mathrm{Vol}(M_i) + \int_{M_i} |Rm|^2 dv \leq H, \tag{4.27}$$

for some uniform H. Let $(\bar{M}, \bar{x}, \bar{g})$ be the limit space of $(M_i, x_i, g_i(0))$. Then $\bar{M}$ is a compact metric-normal Q-Fano variety.

Proof. It follows from (4.27) and the non-collapsing that $diam(M_i)$ is uniformly bounded. So the limit space $\bar{M}$ is compact. Due to Theorem 1.3, the partial C^0-estimate, one can follow the argument in [**37**] to show that $\bar{M}$ is a Q-Fano, normal variety. The metric-normal property follows from Corollary 4.32. q.e.d.

Based on the estimates developed in this subsection, we can easily prove Corollary 1.8 and Corollary 1.9 in the introduction.

Proof of Corollary 1.8 and Corollary 1.9. It follows from the combination of Theorem 4.33, Corollary 4.32 of this paper and main results in [**30**]. Note that the line bundle metric choice in this paper is equivalent to that in [**30**], due to the bound of $\dot{\varphi}$. q.e.d.

4.5. Singular Kähler Ricci flows. In this subsection, we shall relate the different limit time slices, without the assumption of $|R| + |\lambda| \to 0$. We shall further improve regularity, by estimates essentially arising from complex analysis of holomorphic sections.

We want to compare ω_t, the Kähler Ricci flow metrics, and $\tilde{\omega}_t$, the evolving Bergman metrics. We first show that $\tilde{\omega}_t$ is very stable when t evolves.

Lemma 4.34. *Suppose $G(t)$ is a family of $(N+1)\times(N+1)$ matrices parameterized by $t \in [-1,1]$. Suppose $G(0) = Id$, $\dot{G}(0) = B$. Let $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_N$ be the real eigenvalues of the Hermitian matrix $B + \bar{B}^\tau$. If we regard G as a holomorphic map from $\mathbb{CP}^N$ to $\mathbb{CP}^N$, then we have*

$$(\lambda_0 - \lambda_N)\omega_{FS} \le \left.\frac{d}{dt} G(t)^*(\omega_{FS})\right|_{t=0} \le (\lambda_N - \lambda_0)\omega_{FS}. \tag{4.28}$$

Proof. Let $\{z_i\}_{i=0}^N$ be the homogeneous coordinate of $\mathbb{CP}^N$. Let $G = G(t)$. Then we have

$$\begin{aligned}
\omega_{FS} &= \sqrt{-1}\partial\bar{\partial}\log(|z_0|^2 + |z_1|^2 + \cdots |z_N|^2)\\
&= \sqrt{-1}\left\{\frac{\partial z_i \wedge \bar{\partial}\bar{z}_i}{|z|^2} + \frac{(z_i\bar{\partial}\bar{z}_i)\wedge(\bar{z}_j\partial z_j)}{|z|^4}\right\},\\
G^*(\omega_{FS}) &= \sqrt{-1}\partial\bar{\partial}\log(|\tilde{z}_0|^2 + |\tilde{z}_1|^2 + \cdots |\tilde{z}_N|^2)\\
&= \sqrt{-1}\left\{\frac{\partial \tilde{z}_i \wedge \bar{\partial}\bar{\tilde{z}}_i}{|\tilde{z}|^2} + \frac{(\tilde{z}_i\bar{\partial}\bar{\tilde{z}}_i)\wedge(\bar{\tilde{z}}_j\partial \tilde{z}_j)}{|\tilde{z}|^4}\right\},
\end{aligned}$$

where $\tilde{z}_i = G_{ij}z_j$. Let $\{w_1, \cdots, w_N\}$ be local coordinate. At point z, the matrix of ω_{FS} is

$$E_0 = J\left(\frac{Id}{|z|^2} - \frac{\bar{z}^\tau z}{|z|^4}\right)\bar{J}^\tau = JF_0\bar{J}^\tau,$$

where J is an $N\times(N+1)$ matrix which is the Jacobi matrix $\left(\frac{\partial z_j}{\partial w_\alpha}\right)$. The matrix of $\omega_{G^*\omega_{FS}}$ is

$$E_t = JG\left(\frac{Id}{|\tilde{z}|^2} - \frac{\bar{\tilde{z}}^\tau \tilde{z}}{|\tilde{z}|^4}\right)\bar{G}^\tau\bar{J}^\tau = JF_t\bar{J}^\tau.$$

Clearly, we have

$$\begin{aligned}
&\left.\frac{d}{dt}F_t\right|_{t=0}\\
&= B\left(\frac{Id}{|z|^2} - \frac{\bar{z}^\tau z}{|z|^4}\right) + \left(\frac{Id}{|z|^2} - \frac{\bar{z}^\tau z}{|z|^4}\right)\bar{B}^\tau - \frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^4}Id
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^6}\bar{z}^\tau z-\frac{\bar{B}^\tau\bar{z}^\tau z+\bar{z}^\tau zB}{|z|^4}\\
=&\left\{\frac{B+\bar{B}^\tau}{|z|^2}-\frac{(B+\bar{B}^\tau)\bar{z}^\tau z+\bar{z}^\tau z(B+\bar{B}^\tau)}{|z|^4}+\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^6}\bar{z}^\tau z\right\}\\
&-\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^4}F_0\\
\triangleq\ & M-\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^4}F_0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{d}{dt}E_t\Big|_{t=0}&=\frac{d}{dt}\left\{JG\left(\frac{Id}{|\tilde{z}|^2}-\frac{\bar{\tilde{z}}^\tau\tilde{z}}{|\tilde{z}|^4}\right)\bar{G}^\tau\bar{J}^\tau\right\}\Big|_{t=0}\\
&=J\left(M-\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^2}F_0\right)\bar{J}^\tau.
\end{aligned}
$$

It is easy to check that

$$
zM\bar{z}^\tau=0,\quad zF_0\bar{z}^\tau=0,\quad z\left(M-\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^4}F_0\right)\bar{z}^\tau=0.
$$

Without loss of generality, we can assume $B+\bar{B}^\tau$ is a diagonal matrix $\mathrm{diag}(\lambda_0,\lambda_1,\cdots,\lambda_N)$. Let $v=(v_0,v_1,\cdots,v_N)$ be a vector in $\mathbb{C}^{N+1}$ satisfying

$$
z\bar{v}^\tau=\bar{v}_0z_0+\bar{v}_1z_1+\cdots+\bar{v}_Nz_N=0.
$$

Then it is clear that

$$
vM\bar{v}^\tau=\frac{v(B+\bar{B}^\tau)\bar{v}^\tau}{|z|^2},\quad vF_0\bar{v}^\tau=\frac{|v|^2}{|z|^2}.
$$

Therefore, we have

$$
\begin{aligned}
&v\left(M-\frac{z(B+\bar{B}^\tau)\bar{z}^\tau}{|z|^2}F_0\right)\bar{v}^\tau\\
=&\frac{1}{|z|^4}\big\{(\lambda_0|v_0|^2+\cdots+\lambda_N|v_N|^2)(|z_0|^2+\cdots+|z_N|^2)\\
&-(\lambda_0|z_0|^2+\cdots\lambda_N|z_N|^2)(|v_0|^2+\cdots+|v_N|^2)\big\}\\
=&\frac{1}{|z|^4}\big\{\left[(\lambda_0-\lambda_0)|z_0|^2+(\lambda_0-\lambda_1)|z_1|^2+\cdots+(\lambda_0-\lambda_N)|z_N|^2\right]|v_0|^2\\
&+\left[(\lambda_1-\lambda_0)|z_0|^2+(\lambda_1-\lambda_1)|z_1|^2+\cdots+(\lambda_1-\lambda_N)|z_N|^2\right]|v_1|^2\\
&+\cdots\\
&+\left[(\lambda_N-\lambda_0)|z_0|^2+(\lambda_N-\lambda_1)|z_1|^2+\cdots+(\lambda_N-\lambda_N)|z_N|^2\right]|v_N|^2\big\}\\
\le&(\lambda_N-\lambda_0)\frac{|v|^2}{|z|^2}.
\end{aligned}
$$

Similarly, we have

$$v\left(M-\frac{z(B+\bar{B}^{\tau})\bar{z}^{\tau}}{|z|^2}F_0\right)\bar{v}^{\tau}\geq(\lambda_0-\lambda_N)\frac{|v|^2}{|z|^2}.$$

Note that $zF_0\bar{v}^{\tau}=0$. Therefore, we can apply the orthogonal decomposition with respect to F_0 to obtain that for every vector $f=(f_0,f_1,\cdots,f_N)\in\mathbb{C}^{N+1}$, we have

$$\left|f\left(M-\frac{z(B+\bar{B}^{\tau})\bar{z}^{\tau}}{|z|^2}F_0\right)\bar{f}^{\tau}\right|\leq(\lambda_N-\lambda_0)\frac{|f|^2}{|z|^2}=(\lambda_N-\lambda_0)fF_0\bar{f}^{\tau}.$$

Let $u\in T_z^{(1,0)}\mathbb{CP}^N$. Then we have

$$\begin{aligned}\langle u,u\rangle_{\omega_{FS}}&=(uJ)F_0\overline{(uJ)}^{\tau},\\ \left|\langle u,u\rangle_{\frac{d}{dt}G^*(\omega_{FS})\big|_{t=0}}\right|&=\left|(uJ)\left(M-\frac{z(B+\bar{B}^{\tau})\bar{z}^{\tau}}{|z|^2}F_0\right)\overline{uJ}^{\tau}\right|\\ &\leq(\lambda_N-\lambda_0)(uJ)F_0\overline{(uJ)}^{\tau}\\ &\leq(\lambda_N-\lambda_0)\langle u,u\rangle_{\omega_{FS}}.\end{aligned}$$

By the arbitrary choice of u, then (4.28) follows directly from the above inequality. q.e.d.

Lemma 4.35. *Suppose $\mathcal{LM}\in\mathscr{K}(n,A)$. Let $\tilde{\omega}_t$ be the pull back of the Fubini–Study metric by orthonormal basis of L with respect to ω_t and h_t. Then we have the evolution inequality of $\tilde{\omega}_t$:*

$$-2A\tilde{\omega}_t\leq\frac{d}{dt}\tilde{\omega}_t\leq 2A\tilde{\omega}_t. \tag{4.29}$$

Proof. Without loss of generality, it suffices to show (4.29) at time $t=0$.

Suppose $\{s_i\}_{i=0}^N$ is an orthonormal basis at time 0, $\{\tilde{s}_i\}_{i=0}^N$ is an orthonormal basis at time t. They are related by $\tilde{s}_i=s_jG_{ji}$. Fix e_L a local representation of the line bundle L around a point x so that locally we have $s_j=z_je_L$ and $\tilde{s}_j=\tilde{z}_je_L=z_jG_{ji}e_L$. Then we have

$$\begin{aligned}\tilde{\omega}_0&=\sqrt{-1}\partial\bar{\partial}\log\left(|z_0|^2+|z_1|^2+\cdots+|z_N|^2\right),\\ \tilde{\omega}_t&=\sqrt{-1}\partial\bar{\partial}\log\left(|\tilde{z}_0^2|+|\tilde{z}_1|^2+\cdots+|\tilde{z}_N|^2\right).\end{aligned}$$

Let ι be the Kodaira embedding map induced by $\{s_i\}_{i=0}^N$ at time 0. Then it is clear that

$$\tilde{\omega}_0=\iota^*\omega_{FS},\quad \tilde{\omega}_t=\iota^*(G^*\omega_{FS}).$$

Therefore, we have

$$\frac{d}{dt}\tilde{\omega}_t\bigg|_{t=0}=\iota^*\left(\frac{d}{dt}G^*(\omega_{FS})\bigg|_{t=0}\right).$$

So (4.29) is reduced to the estimate

$$-2A\omega_{FS} \leq \frac{d}{dt}G(t)^*(\omega_{FS})\bigg|_{t=0} \leq 2A\omega_{FS}. \tag{4.30}$$

However, note that

$$\delta_{ik} = G_{ij}\bar{G}_{kl}\int_M \langle s_j, s_l\rangle_{h_t}\frac{\omega_t^n}{n!}.$$

Taking derivative on both sides at time 0 and denote $\dot{G}$ by B, we obtain

$$0 = B_{ik} + \bar{B}_{ki} + \int_M (-\dot{\varphi} + n\lambda - R)\langle s_i, s_k\rangle_{h_0}\frac{\omega_0^n}{n!}.$$

Therefore, for every $v \in \mathbb{C}^{N+1}$, the following inequality holds.

$$\begin{aligned} &\left|v_i(B_{ij} + \bar{B}_{ji})\bar{v}_j\right| \\ &= \left|-v_i\bar{v}_j\int_M (-\dot{\varphi} + n\lambda - R)\langle s_i, s_j\rangle_{h_0}\frac{\omega_0^n}{n!}\right| \leq A|v|^2. \end{aligned} \tag{4.31}$$

In particular, each eigenvalue of the Hermitian matrix $B + \bar{B}^\tau$ has absolute value bounded by A. Then (4.30) follows from Lemma 4.34. q.e.d.

In view of Lemma 4.35, the following property is obvious now.

Proposition 4.36 (Bergman metric equivalence along time). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$. Then we have*

$$e^{-2A|t|}\tilde{\omega}_0 \leq \tilde{\omega}_t \leq e^{2A|t|}\tilde{\omega}_0. \tag{4.32}$$

In general, we cannot hope a powerful estimate like (4.32) holds for metrics ω_t, since such an estimate will imply the Ricci curvature is uniformly bounded by A. However, if we only focus on points regular enough, then we do have a similar weaker estimate.

Proposition 4.37 (Flow metric equivalence along time). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $x \in \mathcal{F}_r(M, 0)$. Then we have*

$$\frac{1}{C}\omega_0(x) \leq \omega_t(x) \leq C\omega_0(x), \tag{4.33}$$

for every $t \in [-1, 1]$. Here C is a constant depending only on n, A and r.

Proof. Recall that Theorem 1.3 is already proved at the end of Section 4.1. In light of Theorem 1.3, up to raising the power of line bundle if necessary, we may assume **b** is uniformly bounded from below. On the other hand, for each holomorphic section $S \in H^0(M, L)$ satisfying the normalization condition $\int_M \|S\|^2 dv \triangleq \int_M \|S\|_h^2 dv = 1$, it follows from direct calculation that

$$\Delta \|S\|^2 = \|\nabla S\|^2 - n\|S\|^2 \geq -n\|S\|^2.$$

Moser iteration then implies that $\|S\|^2 \leq C$ point-wisely. Using the expression (1.7), we then know $\mathbf{b}$ is uniformly bounded from above also. Therefore, with out loss of generality, we can assume $\mathbf{b}$ is uniformly bounded.

By short time two-sided pseudolocality, Theorem 4.7 and rescaling, it suffices to show (4.33) for $t=-1$ and $t=1$. At time 0, it is clear that $\omega_0(x)$ and $\tilde{\omega}_0(x)$ are uniformly equivalent. The volume form ω_0^n is uniformly equivalent to ω_t^n. By the stability of $\tilde{\omega}$, inequality (4.32), it suffices to prove the following two inequalities hold at point x.

$$\Lambda_{\omega_1} \tilde{\omega}_0 \leq C, \tag{4.34}$$

$$\Lambda_{\omega_0} \tilde{\omega}_{-1} \leq C. \tag{4.35}$$

We shall prove the above two inequalities separately.

Let w_0 be defined as that before Lemma 3.3. Let w be the solution of $\square w = 0$, initiating from w_0. By the heat kernel estimate and the uniform upper bound of diameter of $B_{g(0)}(x, r)$ under metric $g(t)$ (c.f. Lemma 4.21), we see that $w(x, 1)$ is uniformly bounded away from 0. Then Lemma 3.2 applies and we obtain that

$$\Lambda_{\omega_1(x)} \tilde{\omega}_0(x) = F(x, 1) \leq \frac{C}{w(x, 1)} < C.$$

So we finish the proof of (4.34). The proof of (4.35) is similar. Modulo time shifting, the only difference is that we do not know whether x is very regular at time $t = -1$, so the construction of initial value of a heat equation may be a problem. However, due to Proposition 2.12, we can always find a point $y_0 \in \mathcal{F}_{c_b \hbar}(M, -1) \cap B_{g(-1)}(x, \hbar)$. Consider the heat equation w', starting from a cutoff function supported around y_0 at time $t = -1$. In light of uniform diameter bound of $B_{g(-1)}(y_0, \hbar)$ under the metric $g(0)$, $w'(x, 0)$ is uniformly bounded away from 0. So we can follow the proof of Lemma 3.2 to obtain that

$$\Lambda_{\omega_0(x)} \tilde{\omega}_{-1}(x) < \frac{C}{w'(x, 0)} < C.$$

Therefore, (4.35) is proved. q.e.d.

Note that due to the two-sided pseudolocality, Theorem 4.7, we now can use blowup argument, taking for granted that every convergence in regular part takes place in smooth topology. Therefore, we can use the blowup argument in the proof of Proposition 3.6, based on the Liouville type theorem, Lemma 3.5. Then the following corollary follows directly from Proposition 4.37.

Corollary 4.38 (Long-time regularity improvement in two time directions). *Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$, $r > 0$, then*

$$\mathcal{F}_r(M, 0) \subset \bigcap_{-1 \le t \le 1} \mathcal{F}_\delta(M, t),$$

for some $\delta = \delta(n, A, r)$.

Now we are ready to prove Theorem 1.4, the long-time, two-sided pseudolocality theorem.

Proof of Theorem 1.4. It follows from the combination of Corollary 4.38 and Proposition 4.6. q.e.d.

Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$. Then for each time $t \in [-1, 1]$ we have

$$(M_i, x_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}(t), \bar{x}(t), \bar{g}(t)). \tag{4.36}$$

Let us see how are the two time slice limits $\bar{M}(0)$ and $\bar{M}(1)$ related. Clearly, by Theorem 1.4, the regular parts of $\bar{M}(0)$ and $\bar{M}(1)$ can be identified. The relations among the singular parts at different time slices are more delicate. For simplicity, we denote $(\bar{M}(0), \bar{x}(0), \bar{g}(0))$ by $(\bar{M}, \bar{x}, \bar{g})$, denote $(\bar{M}(1), \bar{x}(1), \bar{g}(1))$ by $(\bar{M}', \bar{x}', \bar{g}')$. Let us also assume $\mathrm{Vol}(M_i)$ is uniformly bounded. Then it is clear that both $\bar{M}$ and $\bar{M}'$ are compact by the uniform non-collapsing caused by Sobolev constant bound. In light of the uniform partial-C^0-estimate along the flow, without loss of generality, we can assume that the Bergman function $\mathbf{b}$ is uniformly bounded below. By the fundamental estimates in [**37**], we obtain that the map

$$Id_0 : \quad (\bar{M}, \bar{x}, \bar{g}) \to (\bar{M}, \bar{x}, \tilde{\bar{g}})$$

is a homeomorphism. Recall that $(\bar{M}, \bar{x}, \tilde{\bar{g}})$ is the limit of $(M_i, x_i, \tilde{g}_i(0))$, where $\tilde{g}_i$ is the pullback of Fubini–Study metric. Similarly, we have another homeomorphism map at time $t = 1$.

$$Id_1 : \quad (\bar{M}', \bar{x}', \bar{g}') \to (\bar{M}', \bar{x}', \tilde{\bar{g}}').$$

By Proposition 4.36, the pullback Fubini–Study metrics $\tilde{g}_i(t)$ are uniformly equivalent for $t \in [-1, 1]$. It follows that there is a Lipschitz map Id_{01} between two time slices, for the pullback Fubini–Study metrics:

$$Id_{01} : \quad (\bar{M}, \bar{x}, \tilde{\bar{g}}) \to (\bar{M}', \bar{x}', \tilde{\bar{g}}').$$

Combining the previous steps and letting $\Psi = Id_1^{-1} \circ Id_{01} \circ Id_0$, we obtain that the map

$$\Psi : (\bar{M}, \bar{x}, \bar{g}) \to (\bar{M}', \bar{x}', \bar{g}')$$

is a homeomorphism. By analyzing each component identity map, it is clear that $\Psi|_{\mathcal{R}(\bar{M})}$, where $\mathcal{R}(\bar{M})$ is the regular part of $\bar{M}$, maps $\mathcal{R}(\bar{M})$

to $\mathcal{R}(\bar{M}')$, as a biholomorphic map. Similarly, $\Psi|_{\mathcal{S}(\bar{M})}$ is a homeomorphism to $\mathcal{S}(\bar{M}')$. Therefore, the variety structure of the $\bar{M}(t)$ does not depend on time. We remark that the compactness of $\bar{M}$ is not essentially used here. If $\bar{M}$ is noncompact, the above argument go through formally if we replace the target embedding space $\mathbb{CP}^N$ by $\mathbb{CP}^\infty$. This formal argument can be made rigorous by applying delicate localization technique. However, in our applications, $\bar{M}$ is always compact except it is a bubble, i.e., a blowup limit. In this situation, we have the extra condition $|R| + |\lambda| \to 0$, then Ψ can be easily chosen as identity map, due to Proposition 4.23.

From the above discussion, it is clear that the topology structure and variety structure of $\bar{M}(t)$ does not depend on time. So we just denote $\bar{M}(t)$ by $\bar{M}$. Then we can denote the convergence (4.36) by

$$(M_i, x_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}, \bar{x}(t), \bar{g}(t)), \tag{4.37}$$

for each t. Hence, the limit family of metric spaces can be regarded as a family of evolving metrics on the limit variety. Therefore, the above convergence at each time t can be glued together to obtain a global convergence

$$\{(M_i, x_i, g_i(t)), -T_i < t < T_i\} \xrightarrow{\hat{C}^\infty} \left\{(\bar{M}, \bar{x}, \bar{g}(t)), -\bar{T} < t < \bar{T}\right\}, \tag{4.38}$$

where $\bar{T} = \lim\limits_{i\to\infty} t_i$. Clearly, $\bar{g}(t)$ satisfies the Kähler Ricci flow equation on the regular part of $\bar{M}$. Recall that we typically denote the Kähler Ricci flow $\{(M_i, x_i, g_i(t)), -T_i < t < T_i\}$ by $\mathcal{M}_i$. Then we obtain the convergence of Kähler Ricci flows (with base points):

$$(\mathcal{M}_i, x_i) \xrightarrow{\hat{C}^\infty} (\bar{\mathcal{M}}, \bar{x}). \tag{4.39}$$

If we further know the underlying space $\bar{M}$ is compact, then the notation can be even simplified as

$$\mathcal{M}_i \xrightarrow{\hat{C}^\infty} \bar{\mathcal{M}}. \tag{4.40}$$

Remark 4.39. The limit flow $\bar{\mathcal{M}}$ can be regarded as an intrinsic Kähler Ricci flow on the normal variety $\bar{M}$. Actually, it is already clear that $\bar{\mathcal{M}}$ is at least a weak super solution of Ricci flow, in the sense of R.J. McCann and P.M. Topping ([**44**]). From the point of view of Kähler geometry, when restricted to the potential level, the flow $\bar{\mathcal{M}}$ coincides with the weak Kähler Ricci flow solution defined by Song and Tian ([**56**]), if $\bar{M}$ is compact.

If we also consider the convergence of the line bundle structure, we can obviously generalize the convergence in (4.39) as

$$(\mathcal{LM}_i, x_i) \xrightarrow{\hat{C}^\infty} \left(\overline{\mathcal{LM}}, \bar{x}\right), \qquad \text{if } \bar{M} \text{ is non-compact,} \tag{4.41}$$

$$\mathcal{LM}_i \xrightarrow{\hat{C}^\infty} \overline{\mathcal{LM}}, \qquad \text{if } \bar{M} \text{ is compact.} \tag{4.42}$$

With these notations, we can formulate our compactness theorem as follows.

Theorem 4.40 (**Polarized flow limit**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$, $x_i \in M_i$. Then we have*

$$(\mathcal{LM}_i, x_i) \xrightarrow{\hat{C}^\infty} \left(\overline{\mathcal{LM}}, \bar{x}\right),$$

where $\overline{\mathcal{LM}}$ is a polarized Kähler Ricci flow solution on an analytic normal variety $\bar{M}$. Moreover, if $\bar{M}$ is compact, then it is a projective normal variety.

Notice that we have already proved Theorem 1.5 now.

Proof of Theorem 1.5. The limit polarized flow on variety follows from the combination of Theorem 4.40 and Theorem 4.19. The Minkowski dimension estimate of the singular set follows from Corollary 4.14. q.e.d.

The properties of the limit spaces can be improved if extra conditions are available.

Proposition 4.41 (**KE limit**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$ satisfies*

$$\int_{-T_i}^{T_i} \int_{M_i} |R - n\lambda| dv dt \to 0. \tag{4.43}$$

Then $\overline{\mathcal{LM}}$ is a static, polarized Kähler Ricci flow solution. In other words, $\bar{g}(t) \equiv \bar{g}(0)$ and, consequently, are Kähler Einstein metric.

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$ and $\lambda > 0$. Then it is clear that $c_1(M) > 0$, or M is Fano. Note that for every Fano manifold, we have a uniform bound $c_1^n(M) \leq C(n)$ (c.f. [**34**]). This implies that

$$\frac{1}{A} \leq \mathrm{Vol}(M) = c_1^n(L) = \lambda^{-n} c_1^n(M) \leq C\lambda^{-n}.$$

So λ is bounded away from above. If we assume λ is bounded away from zero, then $\mathrm{Vol}(M) = c_1^n(L)$ is uniformly bounded. Consequently, $\mathrm{diam}(M)$ is uniformly bounded by non-collapsing, due to the Sobolev constant bound. Therefore, if we have a sequence of $\mathcal{LM}_i \in \mathscr{K}(n, A)$ with $\lambda_i > \lambda_0 > 0$, we can always assume

$$\lambda_i \to \bar{\lambda} > 0, \quad \mathcal{LM} \xrightarrow{\hat{C}^\infty} \overline{\mathcal{LM}},$$

without considering the base points.

Proposition 4.42 (**KRS limit**). *Suppose $\mathcal{LM}_i \in \mathscr{K}(n, A)$ satisfies*

$$\lambda_i > \lambda_0 > 0, \quad \mu\left(M_i, g(T_i), \frac{\lambda_i}{2}\right) - \mu\left(M_i, g(-T_i), \frac{\lambda_i}{2}\right) \to 0, \tag{4.44}$$

where μ is Perelman's W-functional. Suppose $\overline{\mathcal{LM}}$ is the limit of $\mathcal{LM}$. Then $\overline{\mathcal{M}}$ is a gradient shrinking Kähler Ricci soliton. In other words, there is a smooth real valued function $\hat{f}$ defined on $\mathcal{R}(\bar{M}) \times (-\bar{T}, \bar{T})$ such that

$$\hat{f}_{jk} = \hat{f}_{\bar{j}\bar{k}} = 0, \quad R_{j\bar{k}} + \hat{f}_{j\bar{k}} - \hat{g}_{j\bar{k}} = 0. \tag{4.45}$$

Proof. Without loss of generality, we may assume $\lambda_i = 1$. Let $\mathcal{LM} \in \mathscr{K}(n, A)$. At time $t = 1$, let u be the minimizer of Perelman's μ-functional. Then solve the backward heat equation

$$\square^* u = (-\partial_t - \Delta + R - n\lambda)u = 0.$$

Let f be the function such that $(2\pi)^{-n}e^{-f} = u$. Then we have

$$\begin{aligned}
&\int_{-1}^{1}\int_M (2\pi)^{-n}\left\{\left|R_{j\bar{k}} + f_{j\bar{k}} - g_{j\bar{k}}\right|^2 + |f_{jk}|^2 + |f_{\bar{j}\bar{k}}|^2\right\} e^{-f} dv \\
&\le \mu\left(M, g(1), \frac{1}{2}\right) - \mu\left(M, g(-1), \frac{1}{2}\right) \\
&\le \mu\left(M, g(T), \frac{1}{2}\right) - \mu\left(M, g(-T), \frac{1}{2}\right) \to 0.
\end{aligned}$$

At time $t = 1$, f has good regularity estimate for it is a solution of an elliptic equation. For $t \in (-1, 1)$, we have estimate of f from heat kernel estimate. It is not hard to see that, on the space-time domain $\mathcal{R} \times (-1, 1)$, f converges to a limit function $\hat{f}$ satisfying (4.45). Clearly, the time interval of $(-1, 1)$ can be replaced by $(-a, a)$ for every $a \in (1, \bar{T})$. For each a, we have a limit function $\hat{f}^{(a)}$, which satisfies equation (4.45) and, therefore, has enough a priori estimates. Then let $a \to \bar{T}$ and take diagonal sequence limit, we obtain a limit function $\hat{f}^{(\bar{T})}$ which satisfies (4.45) on $\mathcal{R} \times (-\bar{T}, \bar{T})$. Without loss of generality, we still denote $\hat{f}^{(\bar{T})}$ by $\hat{f}$. Then $\hat{f}$ satisfies (4.45) on $\mathcal{R} \times (-\bar{T}, \bar{T})$. q.e.d.

Remark 4.43. It is an interesting problem to see whether $(\bar{M}, \bar{g}(0))$ is a conifold in Theorem 4.40. This question has affirmative answer when we know $(\bar{M}, \bar{g}(0))$ has Einstein regular part, following the proof of Theorem 2.5 and Proposition 3.25. In particular, the limit spaces in Proposition 4.41 and Proposition 4.23 are Kähler Einstein conifolds, in the sense of Chen–Wang (c.f. Definition 1.2 of [**29**]).

5. Applications

In this section, we will focus on the applications of our structure theory to the study of anti-canonical Kähler Ricci flows.

5.1. Convergence of Kähler Ricci flows. Based on the structure theory, Theorem 1.6 can be easily proved.

Proof of Theorem 1.6. In view of the fundamental estimate of Perelman (c.f. [**55**]), in order (1.4) to hold, we only need a Sobolev constant bound, which was proved by Q. Zhang (c.f. [**76**]) and R. Ye (c.f. [**75**]). Therefore, the truncated flow sequences locate in $\mathscr{K}(n, A)$ for a uniform A. It follows from Theorem 1.5 that the limit Kähler Ricci flow exists on a compact projective normal variety. The limit normal variety is Q-Fano since it has a limit anti-canonical polarization. According to Proposition 4.42, the boundedness and monotonicity of Perelman's μ-functional force the limit flow to be a Kähler Ricci soliton. The volume estimate of r-neighborhood of $\mathcal{S}$ follows from Corollary 4.32 and estimate (3.34) of [**29**]. q.e.d.

We continue to discuss applications beyond Theorem 1.6. The following property is well known to experts, we write it down here for the convenience of the readers.

Proposition 5.1 (**Connectivity of limit moduli**). *Suppose $\mathcal{M} = \{(M^n, g(t)), 0 \le t < \infty\}$ is an anti-canonical Kähler Ricci flows on Fano manifold (M, J). Let $\mathscr{M}$ be the collection of all the possible limit space along this flow. Then $\mathscr{M}$ is connected.*

Proof. If the statement was wrong, we have two limit spaces $\bar{M}_a$ and $\bar{M}_b$, locating in different connected components of $\mathscr{M}$. Let $\mathscr{M}_a$ be the connected component containing $\bar{M}_a$. Since $\mathscr{M}_a$ is a connected component, it is open and closed. So its closure $\overline{\mathscr{M}_a}$ is the same as $\mathscr{M}_a$. Clearly, $\mathscr{M}_a$ is compact under the Gromov–Hausdorff topology. Define

$$d(X, \mathscr{M}_a) \triangleq \inf_{Y \in \mathscr{M}_a} d_{GH}(X, Y), \tag{5.1}$$

$$\eta_a \triangleq \inf_{X \in \mathscr{M} \backslash \mathscr{M}_a} d(X, \mathscr{M}_a). \tag{5.2}$$

Clearly, $\eta_a > 0$ by the compactness of $\mathscr{M}_a$ and the fact that $\mathscr{M}_a$ is a connected component.

Without loss of generality, we can assume $(M, g(t_i))$ converges to $\bar{M}_a$, $(M, g(s_i))$ converges to $\bar{M}_b$, for $t_i \to \infty$ and $s_i > t_i$. For simplicity of notation, we denote $(M, g(t_i))$ by M_{t_i}, $(M, g(s_i))$ by M_{s_i}. For large i, we have

$$d_{GH}(M_{t_i}, \bar{M}_a) < \frac{\eta_a}{100}, \quad d_{GH}(M_{s_i}, \bar{M}_b) < \frac{\eta_a}{100}. \tag{5.3}$$

In particular, the above inequalities imply that

$$d(M_{t_i}, \mathscr{M}_a) < \frac{\eta_a}{100}, \quad d(M_{s_i}, \mathscr{M}_a) > \frac{99}{100}\eta_a.$$

By continuity of the flow, we can find $\theta_i \in (t_i, s_i)$ such that

$$d(M_{\theta_i}, \mathscr{M}_a) = \frac{1}{2}\eta_a,$$

whose limit form is

$$d(\bar{M}_c, \mathscr{M}_a) = \frac{1}{2}\eta_a, \tag{5.4}$$

where $\bar{M}_c$ is the limit of M_{θ_i}. However, (5.4) contradicts with (5.2) and the fact $\eta_a > 0$. q.e.d.

Proposition 5.1 can be generalized as follows.

Proposition 5.2 (**KRS limit moduli**). *Suppose that*

$$\mathcal{M}_s = \{(M_s^n, g_s(t)), 0 \le t < \infty, s \in X\}$$

is a smooth family of anti-canonical Kähler Ricci flows on Fano manifolds (M_s, J_s), where X is a connected parameter space. We call $(\bar{M}, \bar{g})$ as a limit space if $(\bar{M}, \bar{g})$ is the Gromov–Hausdorff limit of $(M, g_{s_i}(t_i))$ for some $t_i \to \infty$ and $s_i \to \bar{s} \in X$.

Suppose $f(s) = \lim_{t\to\infty} \mu\left(g_s(t), \frac{1}{2}\right)$ is an upper semi-continuous function on X. Then we have the following properties.

- *Every limit space is a Kähler Ricci soliton.*
- *Let $\widetilde{\mathscr{M}}$ be the collection of all the limit spaces. Then $\widetilde{\mathscr{M}}$ is connected under the Gromov–Hausdorff topology.*

Proof. We shall only show that every limit space is a Kähler Ricci soliton. The connectedness of $\widetilde{\mathscr{M}}$ can be proved almost the same as Proposition 5.1. So we leave the details to the readers.

Suppose $s_i \to \bar{s}$. Fix ϵ, we can choose T_ϵ such that

$$\mu\left(g_{\bar{s}}(T_\epsilon), \frac{1}{2}\right) > f_{\bar{s}} - \epsilon.$$

By the smooth convergence of $g_{s_i}(T_\epsilon)$ and the upper semi-continuity of f, we have

$$\mu\left(g_{s_i}(T_\epsilon), \frac{1}{2}\right) > f_{s_i} - \epsilon,$$

for large i. Recall that $t_i \to \infty$. Therefore, it follows from the monotonicity of Perelman's functional that

$$\mu\left(g_{s_i}(T_\epsilon), \frac{1}{2}\right) < \mu\left(g_{s_i}(t_i - 1), \frac{1}{2}\right) < \lim_{t\to\infty} \mu\left(g_{s_i}(t), \frac{1}{2}\right) = f_{s_i}.$$

Hence, we have

$$0 \le \mu\left(g_{s_i}(t_i + 1), \frac{1}{2}\right) - \mu\left(g_{s_i}(t_i - 1), \frac{1}{2}\right) < \epsilon,$$

for large i. By the arbitrary choice of ϵ, we obtain

$$\lim_{i\to\infty}\left\{\mu\left(g_{s_i}(t_i+1),\frac{1}{2}\right)-\mu\left(g_{s_i}(t_i-1),\frac{1}{2}\right)\right\}=0.$$

Therefore, $(M, g_{s_i}(t_i))$ converges to a Kähler Ricci soliton, in light of Proposition 4.42. q.e.d.

The gap between singularity and regularity in Theorem 1.2 has a global version as follows.

Proposition 5.3 (**Gap around smooth KE**). *Suppose $(\tilde{M}, \tilde{g}, \tilde{J})$ is a compact, smooth Kähler Einstein manifold which belongs to $\mathscr{K}(n, A)$ when regarded as a trivial polarized Kähler Ricci flow solution. Then there exists an $\epsilon = \epsilon(n, A, \tilde{g})$ with the following properties.*

Suppose $\mathcal{LM} \in \mathscr{K}(n, A)$ and $d_{GH}((\tilde{M}, \tilde{g}), (M, g(0))) < \epsilon$, then we have

$$\mathbf{cvr}(M, g(0)) > \frac{1}{2}\mathbf{cvr}(\tilde{M}, \tilde{g}).$$

Proof. It follows from the continuity of canonical volume radius under the $\hat{C}^{\infty}$-Cheeger–Gromov convergence (c.f. Proposition 3.33 and Corollary 3.41). q.e.d.

Proposition 5.3 means that there is no singular limit space around any given smooth Kähler Einstein manifold. Clearly, the single smooth Kähler Einstein manifold in this Proposition can be replaced by a family of smooth Kähler Einstein manifolds with bounded geometry. The gap between smooth and singular Kähler Einstein metrics can be conveniently used to carry out topology argument.

Theorem 5.4 (**Convergence of KRF family**). *Suppose that*

$$\mathcal{M}_s = \{(M_s^n, g_s(t), J_s), 0 \le t < \infty, s \in X\}$$

is a smooth family of anti-canonical Kähler Ricci flows on Fano manifolds (M_s, J_s), where X is a connected parameter space. Moreover, we assume that

- *The Mabuchi's K-energy is bounded from below along each flow.*
- *Smooth Kähler Einstein metrics in all adjacent complex structures (c.f. Definition 1.4 of [**23**]) have uniformly bounded Riemannian curvature.*

Let Ω be the collection of s such that the flow g_s has bounded Riemannian curvature. Then $\Omega = \emptyset$ or $\Omega = X$.

Proof. It suffices to show that Ω is both open and closed in X.

The openness follows from the stability of Kähler Ricci flow around a given smooth Kähler Einstein metric, due to Sun and Wang (c.f. [**59**]). Suppose $s \in \Omega$, then the flow g_s converges to some Kähler Einstein manifold (M', g', J'), which is the unique Kähler Einstein metric in its small smooth neighborhood. By continuous dependence of flow on the initial data, and the stability of Kähler Ricci flow in a very small neighborhood of (M', g', J'), it is clear that s has a neighborhood consisting of points in Ω. Therefore, Ω is an open subset of X.

The closedness follows from Proposition 5.2. Suppose $s_i \in \Omega$ and $s_i \to \bar{s} \in X$. Due to the fact that the Mabuchi's K-energy is bounded from below along each Kähler Ricci flow we are concerning now, the limit Perelman functional is always the same (c.f. [**23**]). Therefore, we can apply Proposition 5.2 to show that every limit space is a possibly singular Kähler Einstein. However, along every g_{s_i}, we obtain a smooth limit Kähler Einstein manifold (M', g', J'), which has uniformly bounded curvature, as a Kähler Einstein manifold in an adjacent complex structure. Note that the diameter of M' is uniformly bounded by Myers theorem. The volume of M' is a topological constant. Therefore, the geometry of (M', g') are uniformly bounded. By a generalized version of Proposition 5.3, (M', g', J') is uniformly bounded away from singular Kähler Einstein metrics. Due to Proposition 5.2, the connectedness of $\mathscr{M}$ forces that the flow $g_{\bar{s}}$ must converge to a smooth (M', g', J'). In particular, $g_{\bar{s}}$ has bounded curvature. Therefore, $\bar{s} \in \Omega$ and Ω is closed. q.e.d.

The two assumptions in Theorem 5.4 seem to be artificial. However, if J_s is a trivial family or a test configuration family, by the unique degeneration theorem of Chen–Sun (c.f. [**23**]), all the smooth Kähler Einstein metrics form an isolated family, then the second condition is satisfied automatically. On the other hand, by the existence of Kähler Einstein metrics in the weak sense, one can also obtain the lower bound of Mabuchi's K-energy (c.f. [**2**], [**35**], [**16**]). Consequently, Theorem 5.4 can be applied to these special cases and obtain the following corollaries.

Corollary 5.5 (**Convergence to given KE**, c.f. Tian–Zhu [**68**], Collins–Székelyhidi [**33**])**.** *Suppose (M, J) is a Fano manifold with a Kähler Einstein metric g_{KE}. Then every anti-canonical Kähler Ricci flow on (M, J) converges to (M, g_{KE}, J).*

Proof. Let ω_{KE} be the Kähler Einstein metric form. Then every metric form ω can be written as $\omega_{KE} + \sqrt{-1}\partial\bar{\partial}\varphi$ for some smooth function φ. Define

$$\omega_s = \omega_{KE} + s\sqrt{-1}\partial\bar{\partial}\varphi, \quad s \in [0, 1].$$

It follows from Theorem 5.4 that the Kähler Ricci flow from every ω_s has bounded curvature, and, consequently, converges to ω_{KE}, by the uniqueness theorem of Chen–Sun (c.f. [**23**]). In particular, the flow start from ω converges to ω_{KE}. q.e.d.

Corollary 5.6 (**Convergence of a test configuration**)**.** *Suppose* **M** *is a smooth test configuration, i.e., a family of Fano manifolds*

(M_s, J_s) parameterized by s in unit disk $D \subset \mathbb{C}^1$ with a natural $\mathbb{C}^*$-action. Suppose each fiber is smooth and the central fiber (M_0, g_0, J_0) admits Kähler Einstein metric (M_0, g_{KE}, J_0). Then each Kähler Ricci flow starting from (M_s, g_s, J_s) for arbitrary $s \in D$ converges to (M_0, g_{KE}, J_0).

Proof. Theorem 5.4 can be applied for $X = D$. The central Kähler Ricci flow converges by Corollary 5.5. Therefore, the Kähler Ricci flow on each fiber has bounded curvature and converge to some smooth Kähler Einstein metric, which can only be (M_0, g_{KE}, J_0), due to the uniqueness theorem of Chen–Sun again. q.e.d.

Remark 5.7. Corollary 5.5 was announced by G. Perelman. The first written proof was given by Tian–Zhu in [**68**] whenever there is no non-trivial holomorphic vector field. The general case was proved by Collins–Székelyhidi in [**33**]. The strategy of Corollary 5.5 was inspired by that in [**69**]. Corollary 5.5-Corollary 5.6 have the corresponding Kähler Ricci soliton versions. These generalizations will be discussed in a separate paper.

5.2. Degeneration of Kähler Ricci flows. In this subsection, we shall prove Theorem 1.10 and related corollaries.

The following Theorem is due to Jiang (c.f. [**40**]). It is a generalization of the estimate of Perelman (c.f. [**55**]).

Theorem 5.8 (**Generalization of Perelman's estimate**). *Suppose that*

$$\mathcal{M} = \{(M^n, g(t), J), 0 \le t < \infty\}$$

is an anti-canonical Kähler Ricci flow solution satisfying

$$\left\|Ric^-\right\|_{C^0(M)} + |\log \mathrm{Vol}(M)| + C_S(M, g(0)) \le F \tag{5.5}$$

at time $t = 0$. Then we have

$$|R| + |\nabla\dot{\varphi}|^2 \le \frac{C}{t^{n+1}}, \tag{5.6}$$

for some constant $C = C(n, F)$.

Note that (5.6) implies a uniform bound of diameter at each time $t > 0$, by the uniform bound of Perelman's functional. Then one can easily deduce a uniform bound (depending on t) of $\|\dot{\varphi}\|_{C^1(M)}$. Combing this with the Sobolev constant estimate along the flow (c.f. [**76**], [**75**]), we see that

$$\|R\|_{C^0(M)} + \|\dot{\varphi}\|_{C^1(M)} + C_S(M, g(t)) \le C(n, F, t), \tag{5.7}$$

for each $t > 0$. Therefore, away from the initial time, we can always apply our structure theory.

Theorem 5.9 (**Weak convergence with initial time**). *Suppose $\mathcal{M}_i = \{(M_i^n, g_i(t), J_i), 0 \leq t < \infty\}$ is a sequence of anti-canonical Kähler Ricci flow solutions, whose initial time slices satisfy estimate (5.5) uniformly. Then we have*

$$(\mathcal{M}_i, g_i) \xrightarrow{G.H.} (\bar{\mathcal{M}}, \bar{g}), \tag{5.8}$$

where the limit is a weak Kähler Ricci flow solution on a Q-Fano normal variety $\bar{M}$, for time $t > 0$. Moreover, the convergence can be improved to be in the $\hat{C}^\infty$-Cheeger–Gromov topology for each $t > 0$, i.e.,

$$(M_i, g_i(t)) \xrightarrow{\hat{C}^\infty} (\bar{M}(t), \bar{g}(t)), \tag{5.9}$$

for each $t > 0$.

Clearly, if (M_i, g_i) is a sequence of almost Kähler Einstein metrics in the anti-canonical classes (c.f. [**67**]), then $(\bar{M}(0), \bar{g}(0))$ and $(\bar{M}(1), \bar{g}(1))$ are isometric to each other, due to Proposition 4.41 and the estimate in [**67**]. In this particular case, it is easy to see that partial-C^0-estimate holds uniformly at time $t = 0$ for each i, at least intuitively. Actually, by the work Jiang [**40**], it is now clear that partial-C^0-estimate at time $t = 0$ only requires a uniform Ricci lower bound.

Note that the evolution equation of the anti-canonical Kähler Ricci flow is

$$\dot{\varphi} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - u_\omega, \tag{5.10}$$

where u_ω is the Ricci potential satisfying the normalization condition $\int_M e^{-u_\omega} \frac{\omega^n}{n!} = (2\pi)^n$. By maximum principle and Green function argument, we have the following property (c.f. [**40**]).

Proposition 5.10 (**Potential equivalence**). *Suppose that*

$$\mathcal{M} = \{(M^n, g(t), J), 0 \leq t < \infty\}$$

is an anti-canonical Kähler Ricci flow solution satisfying (5.5). At time $t = 0$, let $\varphi = 0$ and u_ω satisfy the normalization condition. Then we have

$$C(1 - e^t) \leq \varphi \leq Ce^t, \tag{5.11}$$

for a constant $C = C(n, F)$.

Let $\mathbf{b}(\cdot, t)$ be the Bergman function at time t. By definition, at point $x \in M$ and time $t = 0$, we can find a holomorphic section $S \in H^0(M, K_M^{-1})$ such that

$$\int_M \|S\|_{h(0)}^2 \frac{\omega^n}{n!} = 1, \quad \mathbf{b}(x, 0) = \log \|S\|_{h(0)}^2(x).$$

Note that $\|S\|_{h(1)}^2 = \|S\|_{h(0)}^2 e^{-\varphi(1)}$. By (5.11), it is clear that $\|S\|_{h(1)}^2$ and $\|S\|_{h(0)}^2$ are uniformly equivalent. On the other hand, $\Delta \|S\|^2 \geq -n\|S\|^2$.

At time $t = 0$, applying Moser iteration implies that $\|S\|^2_{h(0)} \leq C$. Hence, we obtain $\|S\|^2_{h(1)} \leq C$. At time $t = 1$, let $\tilde{S}$ be the normalization of S, i.e., $\tilde{S} = \lambda S$ such that $\int_M \left\|\tilde{S}\right\|^2_{h(1)} \frac{\omega_1^n}{n!} = 1$. Then we have

$$\lambda^{-2} = \int_M \|S\|^2_{h(1)} \frac{\omega_1^n}{n!} \leq C.$$

It follows that

$$\begin{aligned}
\mathbf{b}(x,1) &\geq \log \left\|\tilde{S}\right\|^2_{h(1)}(x) = \log \|S\|^2_{h(1)}(x) + \log \lambda^2 \\
&= \log \|S\|^2_{h(0)}(x) - \varphi(1) + \log \lambda^2 \\
&= \mathbf{b}(x,0) - \varphi(1) + \log \lambda^2 \\
&\geq \mathbf{b}(x,0) - C.
\end{aligned}$$

By reversing time, we can obtain a similar inequality with reverse direction. Same analysis applies to $\mathbf{b}^{(k)}$ for each positive integer k. So we have the following property.

Proposition 5.11 (**Bergman function equivalence**). *Suppose that*

$$\mathcal{M} = \{(M^n, g(t), J), 0 \leq t < \infty\}$$

is an anti-canonical Kähler Ricci flow solution satisfying (5.5). For each positive integer k, there exists $C = C(n, F, k)$ such that

$$\mathbf{b}^{(k)}(x,0) - C \leq \mathbf{b}^{(k)}(x,1) \leq \mathbf{b}^{(k)}(x,0) + C, \tag{5.12}$$

for all $x \in M$.

In view of Theorem 5.9, partial-C^0-estimate holds at time $t = 1$, which induces the partial-C^0-estimate at time $t = 0$, by Proposition 5.11. Therefore, the following theorem is clear now.

Theorem 5.12 (**Partial-C^0-estimate at initial time**). *Suppose $\mathcal{M} = \{(M^n, g(t), J), 0 \leq t < \infty\}$ is an anti-canonical Kähler Ricci flow solution satisfying (5.5). Then*

$$\inf_{x \in M} \mathbf{b}^{(k_0)}(x,0) \geq -c_0,$$

for some positive integer $k_0 = k_0(n, F)$ and positive number $c_0 = c_0(n, F)$.

By the Sobolev constant estimates for manifolds with uniform positive Ricci curvature, it is clear that Theorem 1.10 follows from Theorem 5.12 directly. It is also clear that Corollary 1.11 follows from Theorem 5.12.

The proof of Corollary 1.12 is known in literature (c.f. [**61**]), provided the partial-C^0-estimate along the Kähler Ricci flow. We shall be sketchy here. In fact, due to the work of S. Paul ([**47**], [**48**]) and the argument in section 6 of Tian and Zhang ([**70**]), one obtains that the I-functional is bounded along the flow. Then the Kähler Ricci flow converges to a Kähler–Einstein metric, on the same Fano manifold.

It is an interesting problem to study the K-stability through the Kähler Ricci flow. Based on Theorem 1.6, the weak compactness of polarized Kähler Ricci flow, we are able to give an alternative Kähler Ricci flow proof of the stability theorem (Yau's conjecture) of Chen–Donaldson–Sun. Interested readers are referred to [**24**] for the details.

References

[1] M. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., 102(2):429–445, 1990. MR1074481, Zbl 0711.53038.

[2] S. Bando, T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, In: Oda, T. (ed.) Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure. Math., vol. 10, Amsterdam: North-Holland and Tokyo: Kinokuniya 1987. MR0946233, Zbl 0641.53065.

[3] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J., 5:105–126, 1958. MR0106487, Zbl 0113.30104.

[4] H. Cao, N. Sesum, *A compactness result for Kähler Ricci solitons*, Adv. Math., 211(2):794–818, 2007. MR2323545, Zbl 1127.53055.

[5] H. Cao, X.P. Zhu, *A complete proof of the Poincarè and geometrization conjectures—application of the Hamilton–Perelman theory of the Ricci flow*, Asian J. Math., 10(2):165–492, 2006. MR2233789, Zbl 1200.53057.
Erratum to "A Complete Proof of the Poincaré and Geometrization Conjectures – Application of the Hamilton–Perelman theory of the Ricci Flow", Asian Journal of Math., 10:663–664, 2006. MR2282358, Zbl 1200.53058.

[6] X. Cao, R. Hamilton, *Differential Harnack estimates for time-dependent heat equations with potentials*, Geom. Funct. Anal., 19(4):989–1000, 2009. MR2570311, Zbl 1183.53059.

[7] X. Cao, Q.S. Zhang, *The conjugate heat equation and ancient solutions of the Ricci flow*, Adv. Math., 228(5):2891–2919, 2011. MR2838064, Zbl 1238.53038.

[8] J. Cheeger, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Publications of the Scuola Normale Superiore, Edizioni della Normale, October 1, 2001. MR2006642, Zbl 1055.53024.

[9] J. Cheeger, *Integral Bounds on curvature, elliptic estimates and rectifiability of singular sets*, GAFA, 13:20–72, 2003. MR1978491, Zbl 1086.53051.

[10] J. Cheeger, T.H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Annals. Math., 144(1):189–237, 1996. MR1405949, Zbl 0865.53037.

[11] J. Cheeger, T.H. Colding, *On the structure of spaces with Ricci curvature bounded below.I*, J. Differential Geometry, 45:406–480, 1997. MR1484888, Zbl 0902.53034.

[12] J. Cheeger, T.H. Colding, G. Tian, *On the Singularities of Spaces with Bounded Ricci Curvature*, GAFA, Geom. Funct. Anal., 12:873–914, 2002. MR1937830, Zbl 1030.53046.

[13] J. Cheeger, M. Gromov, M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Diff. Geom., 17:15–53, 1982. MR0658471, Zbl 0493.53035.

[14] J. Cheeger, A. Naber, *Lower bounds on Ricci curvature and quantitative behavior of singular sets*, Invent. Math., 191(2):321–339, 2013. MR3010378, Zbl 1268.53053.

[15] T.H. Colding, *Shape of manifolds with positive Ricci curvature*, Invent. Math., 124:175–191, 1996. MR1369414, Zbl 9871.53027.

[16] X.X. Chen, *Space of Kähler metrics(IV)—On the lower bound of the K-energy*, arXiv:0809.4081.

[17] X.X. Chen, S. Donaldson, *Volume estimates for Kähler–Einstein metrics: the three dimensional case*, J. Differential Geom., 93(2):175–189, 2013. MR3024304, Zbl 1279.32020.

[18] X.X. Chen, S. Donaldson, *Volume estimates for Kähler–Einstein metrics and rigidity of complex structures*, J. Differential Geom., 93(2):191–201, 2013. MR3024305, Zbl 1281.32019.

[19] X.X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics and Stability*, IMRN, 2014(8):2119–2125. MR3194014, Zbl 1331.32011.

[20] X.X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, I: Approximation of metrics with cone singularities*, JAMS, 28(1):183–197. MR3264766, Zbl 1312.53096.

[21] X.X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, II: limits with cone angle less than* 2π, JAMS, 28(1):199–234. MR3264767, Zbl 1312.53097.

[22] X.X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, III: limits as cone angle approaches* 2π *and completion of the main proof*, JAMS, 28(1):235–278. MR3264768, Zbl 1311.53059.

[23] X.X. Chen, S. Sun, *Calabi flow, Geodesic rays, and uniqueness of constant scalar curvature Kähler metrics*, Ann. of Math. (2), 180(2):407–454, 2014. MR3224716, Zbl 1307.53058.

[24] X.X. Chen, S. Sun, B. Wang, *Kähler Ricci flow, Kähler Einstein metric, and* K*-stability*, Geom. Topol., 22(6):3145–3173, 2018. MR3858762, Zbl 06945124.

[25] X.X. Chen, B. Wang, *Kähler Ricci flow on Fano Surfaces(I)*, Math. Z., 270(1–2):577–587, 2012. MR2875850, Zbl 1237.53066.

[26] X.X. Chen, B. Wang, *Remarks on Kähler Ricci flow*, J. Geom. Anal., 20(2):335–353, 2010. MR2579513, Zbl 1185.53075.

[27] X.X. Chen, B. Wang, *Space of Ricci flows (I)*, Comm. Pure Appl. Math., 65(10):1399–1457, 2012. MR2957704, Zbl 1252.53076.

[28] X.X. Chen, B. Wang, *Space of Ricci flows (II)*, arXiv:1405.6797.

[29] X.X. Chen, B. Wang, *Space of Ricci flows (II)—Part A: compactness of the moduli of model spaces*, Forum of Mathematics, Sigma(2017), vol. 5. MR3739253, Zbl 06824442.

[30] X.X. Chen, B. Wang, *Kähler Ricci flow on Fano manifolds(I)*, J. Eur. Math. Soc., 14(6):2001–2038, 2012. MR2984594, Zbl 1257.53094.

[31] X.X. Chen, B. Wang, *On the conditions to extend Ricci flow (III)*, IMRN, 2013(10):2349–2367. MR3061942, Zbl 1317.53082.

[32] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci Flow*, Graduate Studies in Mathematics, 77, American Mathematical Society, Providence, RI; Science Press, New York, 2006. MR2274812, Zbl 1118.53001.

[33] T.C. Collins, G. Székelyhidi, *The twisted Kähler Ricci flow*, J. Reine Angew. Math., 716:179–205, 2016. MR3518375, Zbl 1357.53076.

[34] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR1841091, Zbl 0978.14001.

[35] W.Y. Ding, G. Tian, *Kähler–Einstein metrics and the generalized Futaki invariants*, Invent. Math., 110(2):315–335, 1992. MR1185586, Zbl 0779.53044.

[36] S.K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Diff. Geom., 62(2):289–349, 2002. MR1988506, Zbl 1074.53059.

[37] S.K. Donaldson, S. Sun, *Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math., 213(1):63–106, 2014. MR3261011, Zbl 1318.53037.

[38] R.S. Hamilton, *Formation of singularities in the Ricci flow*, Surveys in Diff. Geom., 2:7–136, 1995. MR1375255, Zbl 0867.53030.

[39] R.S. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. J. Math., 117(3):545–572, 1995. MR1333936, Zbl 0840.53029.

[40] W.S. Jiang, *Bergman Kernel along the Kähler Ricci flow and Tian's conjecture*, J. Reine Angew. Math., 717:195–226, 2016. MR3530538, Zbl 1345.53069.

[41] B. Kleiner, J. Lott, *Notes on Perelman's papers*, Geometry and Topology, 12(5):2587–2855, 2008. MR2460872, Zbl 1204.53033.

[42] B. Kotschwar, *A local version of Bando's theorem on the real-analyticity of solutions to the Ricci flow*, Bull. Lond. Math. Soc., 45(1):153–158, 2013. MR3033963, Zbl 1259.53065.

[43] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch*, Amer. J. Math., 122(2):235–273, 2000. MR1749048, Zbl 0972.53042.

[44] R.J. McCann, P.M. Topping, *Ricci flow, entropy and optimal transportation*, Amer. J. Math., 132(3):711–730, 2010. MR2666905, Zbl 1203.53065.

[45] J. Morgan, G. Tian, *Ricci flow and the Poincaré conjecture*, Clay mathematics monographs, 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR2334563, Zbl 1179.57045.

[46] S. Paul, *Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics*, Ann. of Math. (2), 175(1):255–296, 2012. MR2874643, Zbl 1243.14038.

[47] S. Paul, *A Numerical Criterion for K-Energy maps of Algebraic Manifolds*, arXiv:1210.0924.

[48] S. Paul, *Stable Pairs and Coercive Estimates for The Mabuchi Functional*, arXiv:1308.4377.

[49] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[50] V.P. Petersen, G.F. Wei, *Relative volume comparison with integral curvature bounds*, GAFA, 7(6):1031–1045, 1997. MR1487753, Zbl 0910.53029.

[51] D.H. Phong, J. Sturm, *On stability and the convergence of the Kähler–Ricci flow*, J. Differential Geom., 72(1):149–168, 2006. MR2215459, Zbl 1125.53048.

[52] A.V. Pogorelov, *The Minkowski multidimensional problem. Translated from the Russian by Vladimir Oliker and Introduction by Louis Nirenberg*, Scripta Series in Mathematics. Washington, D.C., Winston, 1978. MR0478079, Zbl 0387.53023.

[53] D. Riebesehl, F. Schulz, *A priori estimates and a Liouville theorem for complex Monge–Ampère equations*, Math. Z., 186(1):57–66, 1984. MR0735051, Zbl 0566.32013.

[54] N. Sesum, *Convergence of a Kähler Ricci flow*, Math. Res. Lett., 12(5–6):623–632, 2005. MR2189226, Zbl 1087.53063.

[55] N. Sesum, G. Tian, *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman) and some applications*, J. Inst. Math. Jussieu, 7(3):575–587, 2008. MR2427424, Zbl 1147.53056.

[56] J. Song, G. Tian, *The Kähler–Ricci flow through singularities*, Invent. Math., 207(2):519–595, 2017. MR3595934, Zbl 06685346.

[57] J. Song, B. Weinkove, *Contracting exceptional divisors by the Kähler–Ricci flow*, Duke Math. J., 162(2):367–415, 2013. MR3018957, Zbl 1266.53063.

[58] J. Song, B. Weinkove, *Lecture notes on the Kähler Ricci flow*, arXiv:1212.3653.

[59] S. Sun, Y.Q. Wang, *On the Kähler Ricci flow near a Kähler Einstein metric*, J. Reine Angew. Math., 699:143–158, 2015. MR3305923, Zbl 1314.53123.

[60] G. Székelyhidi, *The Kähler–Ricci flow and K-polystability*, Amer. J. Math., 132(4):1077–1090, 2010. MR2663648, Zbl 1206.53075.

[61] G. Székelyhidi, *The partial C^0-estimate along the continuity method*, JAMS., 29(2):537–560, 2016. MR3454382, Zbl 1335.53098.

[62] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., 101(1):101–172. MR1055713, Zbl 0716.32019.

[63] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom., 32(1):99–130, 1990. MR1064867, Zbl 0706.53036.

[64] G. Tian, *Kähler–Einstein metrics on algebraic manifolds*, Proc. of Int. Congress of Math., Kyoto, 1990, Vol. I, 587–598(1991). MR1159246, Zbl 0747.53038.

[65] G. Tian, *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math., 130(1):1–37, 1997. MR1471884, Zbl 0892.53027.

[66] G. Tian, *Existence of Einstein metrics on Fano manifolds*, Metric and differential geometry: The Jeff Cheeger Anniversary volume, X. Dai and X. Rong, edt., Prog. Math., 297:119–159, 2012. MR3220441, Zbl 1250.53044.

[67] G. Tian, B. Wang, *On the structure of almost Einstein manifolds*, JAMS, 28(4):1169–1209, 2015. MR3369910, Zbl 1320.53052.

[68] G. Tian, X.H. Zhu, *Convergence of Kähler Ricci flow*, JAMS., 20(3):675–699, 2007. MR2291916, Zbl 1185.53078.

[69] G. Tian, X.H. Zhu, *Convergence of the Kähler Ricci flow on Fano manifolds*, J. Reine Angew. Math., 678:223–245, 2013. MR3056108, Zbl1276.14061.

[70] G. Tian, Z.L. Zhang, *Regularity of Kähler Ricci flows on Fano manifolds*, Acta Math., 216(1):127–176, 2016. MR3508220, Zbl 1356.53067.

[71] V. Tosatti, *Kähler Ricci flow on stable Fano manifolds*, J. Reine Angew. Math., 640:67–84, 2010. MR2629688, Zbl 1189.53069.

[72] B. Wang, *Ricci flow on orbifold*, arXiv:1003.0151.

[73] B. Wang, *On the Conditions to Extend Ricci Flow(II)*, IMRN, 2012(14):3192–3223. MR2946223, Zbl 1251.53040.

[74] S.T. Yau, *Open problems in geometry*, Differential geometry: partial differential equations on manifolds, Los Angeles, CA, 1990, 1–28, Proc. Symp. Pure

Math., vol.54, Part 1, Amer. Math. Soc., Providence, RI, 1993. MR1216573, Zbl 0801.53001.

[75] R.G. Ye, *The logarithmic Sobolev inequality along the Ricci flow: The case $\lambda_0(g_0) = 0$*, Commun. Math. Stat., 2(3–4):363–368, 2014. MR3326237, Zbl 1316.53080.

[76] Q.S. Zhang, *A uniform Sobolev inequality under Ricci flow*, IMRN, 2007(17), article ID rnm056. MR2354801, Zbl 1141.53064.

[77] Q.S. Zhang, *Some gradient estimates for the heat equation on domains and for an equation by Perelman*, IMRN, 2006(15), article ID 92314. MR2250008, Zbl 1123.35006.

[78] Q.S. Zhang, *Bounds on volume growth of geodesic balls under Ricci flow*, Math. Res. Lett., 19(1):245–253, 2012. MR2923189, Zbl 1272.53056.

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