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Equivariant chromatic localizations and commutativity

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Abstract
In this paper, we study the extent to which Bousfield and finite localizations relative to a thick subcategory of equivariant finite spectra preserve various kinds of highly structured multiplications. Along the way, we describe some basic, useful results for analyzing categories of acyclics in equivariant spectra, and we show that Bousfield localization with respect to an ordinary spectrum (viewed as an equivariant spectrum with trivial action) always preserves equivariant commutative ring spectra.

Keywords Bousfield localization · Equivariant homotopy · Chromatic homotopy

1 Introduction

Bousfield localization is a fundamental tool in modern algebraic topology. The ability to focus on pieces of the stable homotopy category allows in many cases for more conceptual or algebraic descriptions and computations. Just as in ordinary algebra, classical Bousfield localization is always a lax monoidal functor, preserving commutative ring objects and allowing one to talk about localizations in categories of modules or algebras.

Bousfield localization plays an equally important role in equivariant homotopy, but here, the functors need not preserve commutative ring spectra. This was originally shown by McClure for the Greenlees–May Tate spectrum [17], and in work with Hopkins, we showed sufficient conditions for when a general Bousfield localization preserves equivariant commutative ring spectra [12]. Moreover, we showed that Bousfield localization always preserves algebras over a trivial E_∞ operad (so an E_∞ operad viewed as a G-equivariant operad by endowing it with a trivial action), and in work

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with Blumberg, we verified that this is sufficient to have a good symmetric monoidal catgegory of modules [6]. Thus essentially all of the desired classical properties hold.

For equivariant homotopy, however, we can ask for more. If R is a genuine equivariant commutative ring spectrum, then the category of R-modules has a natural G-symmetric monoidal enhancement. More generally, if R is an algebra over a linear isometries operad, then the category of R-modules inherits those norms which the linear isometries operad parameterizes. It is therefore a natural question to see if a particular Bousfield localization preserves these richer structures.

In this paper, we study Bousfield and finite localizations for equivariant chromatic localizations. Using Balmer's notion of the spectrum of a tensor triangulated category, Balmer and Sanders determined the prime spectrum of the category of G-spectra for finite groups G and describe the topology up to a small indeterminacy [3]. Barthel, Hausmann, Naumann, Nikolaus, Noel, and Stapleton resolve this ambiguity for finite abelian groups [4]. Coupled with the natural Balmer–Zariski topology on this spectrum, this provides a complete classification of the thick subcategories of finite G-spectra. Associated to any such thick subcategory are corresponding Bousfield and finite localizations, and these are our primary focus. In particular, we prove in Theorem 5.2 below sufficient conditions for these localizations to preserve various operadic algebras.

Along the way, we provide several tools which are helpful in analyzing equivariant Bousfield localizations. The role of geometric fixed points here cannot be overstated, as it provides elegant (and surprisingly checkable) reformulations of what it means for a G-spectrum Z to be acyclic. This has several amusing consequences for the kinds of spectra which arose in the solution to the Kervaire invariant one problem which we could not resist including.

Notation and conventions

In all that follows, G will denote a fixed finite group. In general, the letters H, J and K will be reserved for subgroups of G. Capital letters close to X in the alphabet will denote G-spaces, while capital letters close to T will denote G-sets. Spectra will be often denoted with letters like E (or Z when the role as an acyclic is being stressed).

By "genuine equivariant equivalences", we will always mean equivariant maps which induce a weak equivalence on all fixed points. In particular, "equivariantly contractible" spectra are those for which all fixed points are [weakly] contractible.

Category names and assumptions

We work in the category of genuine G-spectra, and all of our statements are implicitly homotopical. For concreteness, the reader is invited to use orthogonal G-spectra, where all the needed homotopical properties were checked in [11, Appendix B]. The category of genuine G-spectra will be denoted $\mathcal{S}p^G$ and the category of spectra will be denoted $\mathcal{S}p$. For either of these, the full subcategory of compact objects will be indicated by a subscript "c".

The category of genuine equivariant commutative ring spectra (the commutative monoids in one of the good symmetric monoidal model categories of G-spectra) will be denoted $\mathcal{C}omm^G$.

The category of finite G-sets and G-equivariant maps will be denoted $\mathcal{S}et^G$.

Familiar functors

The geometric fixed points functor will be denoted Φ^G.

If $H \subset G$, then i_H^* will denote the restriction functor from G-spectra to H-spectra. The functor Φ^H will also be used to denote the composite functor $\Phi^H \circ i_H^*$ on genuine G-spectra.

2 Equivariant commutativity

2.1 The norm and geometric fixed points

One of the most important tools developed in the solution of the Kervaire invariant one problem was a homotopically meaningful norm functor

$$N_H^G \colon \mathcal{S}p^H \to \mathcal{S}p^G.$$

This is a strong symmetric monoidal left Quillen functor, and on equivariant commutative ring spectra, it participates in a Quillen adjunction as the left adjoint to the forgetful functor:

$$N_H^G \colon \mathcal{C}omm^H \rightleftarrows \mathcal{C}omm^G \colon i_H^*.$$

In particular, for any G-equivariant commutative ring spectrum R, there is a canonical map of G-equivariant commutative ring spectra

$$N_H^G i_H^* R \to R.$$

It is the requirement that a localization play nicely with these maps that confounds equivariant Bousfield localization. From a homotopical point of view, these are extra structure which we must control the behavior of. In our analysis of various trivial and chromatic localizations, we will also need to understand the geometric fixed points of the norm functor. Luckily, this is easily determined by a kind of generalized diagonal.

Lemma 2.1 ([11, Proposition B.209], [1, Proposition 2.19]) *For any K, $H \subset G$ and for any H-spectrum E, the diagonal gives an equivalence of spectra*

$$\bigwedge_{g \in K\backslash G/H} \Phi^{K^g \cap H} E \xrightarrow{\simeq} \Phi^K N_H^G E.$$

It is conceptually convenient to also include notation for several other endo-functors of the category of G-spectra.

Definition 2.2 For any $H \subset G$ and for any G-spectrum E, let

$$N^{G/H}(E) := N_H^G i_H^* E.$$

If

$$T = G/H_1 \amalg \cdots \amalg G/H_n$$

is a finite G-set, then let

$$N^T E := \bigwedge_{i=1}^{n} N^{G/H_i} E.$$

Remark 2.3 Although the definition as given involves choices of orbit decompositions, one can make this coordinate free by defining the T-norm as the symmetric monoidal pushforward of the constant $B_G T$-shaped diagram with value E, where $B_G T$ is the translation category of T, mirroring the original discussion in [11, Appendix A.3].

2.2 N_∞ operads and algebras

The failure of equivariant Bousfield localization to preserve commutative ring spectra should be viewed as a peculiarity of the monoidal model structure: in genuine G-spectra, the commutative monoids have not only a homotopy coherent commutative multiplication (an ordinary E_∞ structure) but also coherent norm maps relating the value of the ring at various subgroups. Classically, this is packaged via a G-E_∞ operad. In work with Blumberg, we generalized the notion of a G-E_∞ operad to cover all kinds of coherently commutative multiplications with some norms on genuine G-spectra. We briefly review the relevant details now.

Definition 2.4 [7, Defintion 3.7] An N_∞ operad is an operad $\mathcal{O}$ in G-spaces such that

(1) The space $\mathcal{O}_0$ is G-contractible,
(2) the action of Σ_n on $\mathcal{O}_n$ is free, and
(3) the space $\mathcal{O}_n$ is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{e\}$.

There is a purely combinatorial way to package the collection of subgroups which show up in the families for an N_∞ operad, and this is closely connected to the structure of algebras over the operad. For this, recall that a symmetric monoidal coefficient system is a contravariant functor from the orbit category of G to the category of symmetric monoidal categories and strong monoidal functors. The prototype of such a symmetric monoidal category is $\underline{\mathcal{S}et}$, for which

$$\underline{\mathcal{S}et}(G/H) := \mathcal{S}et^H.$$

Definition 2.5 An indexing system $\underline{\mathcal{O}}$ is a full, symmetric monoidal sub-coefficient system of $\underline{\mathcal{S}et}$ such that

(1) For all orbits G/H, $\underline{\mathcal{O}}(G/H)$ is closed under finite limits, and
(2) if $H/K \in \underline{\mathcal{O}}(G/H)$ and $T \in \underline{\mathcal{O}}(G/K)$, then $H \times_K T \in \underline{\mathcal{O}}(G/H)$.

In particular, any indexing system contains all trivial sets and is closed under passage to subobjects.

Associated to any N_∞ operad $\mathcal{O}$ is an indexing system $\underline{\mathcal{O}}$.

Definition 2.6 [7, Definition 4.3] Let $\mathcal{O}$ be an N_∞ operad. Let T be a finite H-set of cardinality n, classified by a map $H \to \Sigma_n$, and let Γ_T be the graph of this homomorphism. Then T is in $\underline{\mathcal{O}}(G/H)$ if and only if $\mathcal{O}_n^{\Gamma_T} \simeq *$. If $T \in \underline{\mathcal{O}}(G/H)$, then we say that T **is admissible**.

This construction gives an equivalence of categories, so we will henceforth ignore the distinction between an N_∞ operad and an indexing system.

Theorem 2.7 *The assignment $\mathcal{O} \mapsto \underline{\mathcal{O}}$ gives a fully-faithful embedding of the homotopy category of N_∞ operads into the poset of indexing systems [7, Theorem 3.24] which is essentially surjective [18], [9, Corollary IV], [10, Section 4].*

Since the poset of indexing systems has an initial object $\underline{\mathcal{O}}^{tr}$ consisting of the indexing system of sets with a trivial action, there is a homotopy initial N_∞ operad. This is just an ordinary, non-equivariant E_∞ operad viewed as a G-operad by endowing it with a trivial G-action. Thus any $\mathcal{O}$-algebra has a canonical coherently commutative multiplication, since it is an algebra over a trivial E_∞ operad. The role of the indexing system here is to parameterize the additional norms present in an $\mathcal{O}$-algebra.

Theorem 2.8 [7, Lemma 6.6] *If H/K is an admissible H-set for $\mathcal{O}$, and R is an $\mathcal{O}$-algebra in spectra, then we have a contractible space of maps*

$$N_K^H i_K^* R \to i_H^* R.$$

3 Equivariant Bousfield classes

3.1 Equivariant localizing subcategories

Definition 3.1 If E is a G-spectrum, let $\mathcal{Z}_E^G$ denote the category of E-acyclics: the full subcategory of $\mathcal{S}p^G$ consisting of all Z such that $E \wedge Z$ is equivariantly contractible.

If we are working non-equivariantly, then the acyclics will be denoted simply $\mathcal{Z}_E$ with no superscript.

Since geometric fixed points detect weak equivalences and are strong symmetric monoidal, this gives another, conceptually simpler way to understand membership in $\mathcal{Z}_E$.

Proposition 3.2 *A G-spectrum Z is a G-equivariant E-acyclic spectrum if and only if for all $H \subset G$, $\Phi^H(Z)$ is a non-equivariant $\Phi^H(E)$-acyclic spectrum:*

$$\mathcal{Z}_E^G = \bigcap_{H \subset G} \left(\Phi^H\right)^{-1} (\mathcal{Z}_{\Phi^H E}).$$

Proof A genuine G-spectrum E' is equivariantly contractible if and only if for all $H \subset G$, $\Phi^H(E')$ is contractible. Thus if E is a fixed G-spectrum and Z is any other G-spectrum, then $E \wedge Z$ is contractible if and only if for all $H \subset G$,

$$\Phi^H(E \wedge Z) \simeq \Phi^H(E) \wedge \Phi^H(Z) \simeq *.$$

□

The category of E-acyclics is also an equivariant subcategory in that it is closed under restriction and induction.

Proposition 3.3 *For all $H \subset G$, we have natural inclusions*

$$i_H^* \mathcal{Z}_E^G \subset \mathcal{Z}_{i_H^* E}^H \text{ and } G_+ \wedge_H \mathcal{Z}_{i_H^* E}^H \subset \mathcal{Z}_E^G.$$

Proof The first inclusion is obvious, since i_H^* is a strong symmetric monoidal functor. For the second, let Z be in $\mathcal{Z}_{i_H^* E}$. The Frobenius relation

$$\left(G_+ \wedge_H Z\right) \wedge E \simeq G_+ \wedge_H (Z \wedge i_H^* E)$$

then shows that $G_+ \wedge_H Z$ is E-acyclic. □

Corollary 3.4 *For all $H \subset G$ and $Z \in \mathcal{Z}_E^G$, we have*

$$G/H_+ \wedge Z \in \mathcal{Z}_E^G.$$

3.2 Application: acyclics for Kervaire spectra

Proposition 3.2 gives a way to readily determine the acyclics for the kinds of chromatic spectra which arose in the proof of the Kervaire invariant one problem. In particular, we can determine the acyclics for any of the spectra which arise as particular localizations of the norms of the Landweber–Araki Real bordism spectrum $MU_{\mathbb{R}}$.

Recall from [11, Section 5.4.2] that if $G = C_{2^n}$, then there are classes

$$\bar{r}_i^G \in \pi_{(2^i-1)\rho_2} N_{C_2}^G MU_{\mathbb{R}}$$

such that

$$\Phi^{C_{2^n}}\left(N_{C_2}^G MU_{\mathbb{R}}\left[(N_{C_2}^G \bar{r}_i^G)^{-1}\right]\right)$$

is contractible [11, Proposition 5.50]. More generally, if $C_2 \subset H \subset G$, then the unit of the norm-forget adjunction on commutative ring spectra produces a canonical map

$$N_{C_2}^H MU_{\mathbb{R}} \to i_H^* N_{C_2}^G MU_{\mathbb{R}}.$$

Via the unit map, we can then have elements

$$\bar{r}_i^H \in \pi_{(2^i-1)\rho_2} N_{C_2}^G MU_{\mathbb{R}}$$

such that

$$\Phi^H\left(N_{C_2}^G MU_{\mathbb{R}}\left[(N_{C_2}^G \bar{r}_i^H)^{-1}\right]\right)$$

is contractible.

Proposition 3.5 *Let $G = C_{2^n}$. Let $\bar{D}$ be any class in $\pi_{m\rho_{2^n}} N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ such that for all $C_2 \subset H \subset G$, there is a j_H such that $N_{C_2}^{C_{2^n}} \bar{r}_{j_H}^H$ divides $\bar{D}$. Finally, let M be any module over the commutative ring spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}[\bar{D}^{-1}]$. Then*

$$\mathcal{Z}_M^G = \left(i_{\{e\}}^*\right)^{-1} \mathcal{Z}_{i_{\{e\}}^* M}.$$

Proof The conditions ensure that all non-trivial geometric fixed points of M are contractible. The result then follows from Proposition 3.2. □

Corollary 3.6 *If M is a wedge of spectra M_i, each of which is a module over $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}[\bar{D}_i^{-1}]$ for some $\bar{D}_i$ as in Proposition 3.5, then*

$$\mathcal{Z}_M^G = \left(i_{\{e\}}^*\right)^{-1} \mathcal{Z}_{i_{\{e\}}^* M}.$$

We deduce an immediate application to the Real Morava K-theories and Johnson–Wilson theories introduced by Hu–Kriz and studied extensively by Kitchloo–Wilson [14,16].

Corollary 3.7 *The equivariant Bousfield classes of $E_{\mathbb{R}}(n)$, of $MU_{\mathbb{R}}[\bar{v}_n^{-1}]$, and of $K_{\mathbb{R}}(0) \vee \cdots \vee K_{\mathbb{R}}(n)$ agree.*

Proof By Corollary 3.6, the Bousfield class of $K_{\mathbb{R}}(0) \vee \cdots \vee K_{\mathbb{R}}(n)$ is determined by the underlying spectrum:

$$i_e^*\left(K_{\mathbb{R}}(0) \vee \cdots \vee K_{\mathbb{R}}(n)\right) \simeq K(0) \vee \cdots \vee K(n).$$

Similarly, a direct application of Proposition 3.5 shows that the Bousfield classes of $E_{\mathbb{R}}(n)$ and $MU_{\mathbb{R}}[\bar{v}_n^{-1}]$ are also determined by the underlying spectra, which are $E(n)$ and $MU_{\mathbb{R}}[\bar{v}_n^{-1}]$, respectively. The result is now classical. □

3.3 Localizations of $\mathcal{O}$-algebras

Since the smash product is associative, we know that $\mathcal{Z}_E^G$ is always a tensor ideal of $\mathcal{S}p^G$. In particular, it is a non-unital symmetric monoidal subcategory of $\mathcal{S}p^G$. This gives another way to interpret Proposition 3.3.

Proposition 3.8 *For any G-spectrum E, the assignment*

$$G/H \mapsto \underline{\mathcal{Z}}_E(G/H) := \mathcal{Z}^H_{i_H^* E} \subset \mathcal{S}p^H$$

defines a non-unital symmetric monoidal sub-coefficient system of $\underline{\mathcal{S}p}$.

This reformulation allows us to most easily state the sufficient conditions for a localization to preserve $\mathcal{O}$-algebra structures for an N_∞ operad $\mathcal{O}$. The sufficiency of the following theorem was proved in [12]; the thesis of White built upon this in a more general context and also showed necessity as well [19,20].

Theorem 3.9 [12, Theorem 7.3], [19, Section 5] *Let $\mathcal{O}$ be an N_∞ operad, and let L be a Bousfield localization on G-spectra. If for every subgroup $H \subset G$ and for every admissible $T \in \underline{\mathcal{O}}(G/H)$ the category of acyclics is closed under N^T then L preserves $\mathcal{O}$-algebras.*

In this paper, we are also concerned with finite localizations (which are always known to be smashing). Here, the same result holds; the proof is identical.

Theorem 3.10 *Let $\mathcal{O}$ be an N_∞ operad, and let $\mathcal{V}$ be a thick subcategory of $\mathcal{S}p^G$. If for every $H \subset G$ and for every admissible $T \in \underline{\mathcal{O}}(G/H)$, N^T restricts to an endofunctor of $\mathcal{V}$, then the finite localization $L^f_{\mathcal{V}}$ preserves $\mathcal{O}$-algebras.*

Proposition 3.11 *If $\mathcal{V}$ is the thick subcategory generated by an object E, then the conditions of Theorem 3.10 are met provided $N^T(E) \in \mathcal{V}$ for all admissible T.*

Proof This is essentially [11, Proposition B.170]. In short, the norms commute with sifted colimits, and there is a formula for describing the norm of a cofiber in terms of the norms of the pieces. This reduces checking for a general object in the thick subcategory to checking for the generator. □

Since categories of acyclics are always non-unital symmetric monoidal subcategories, and since the equivariant thick subcategories are tensor ideals, we conclude that these localizations always preserve at least the trivial E_∞-structure.

Corollary 3.12 *If L is any Bousfield or finite localization on G-spectra, then L preserves trivial E_∞ algebras.*

Although this is less structured than we might like, it is enough structure to guarantee a good, symmetric monoidal category of modules.

Corollary 3.13 [6, Theorem 1.1] *Let $\mathcal{O}$ be an N_∞ operad. If R is an $\mathcal{O}$-algebra in G-spectra, and if L is any Bousfield or finite localization, then there is a symmetric monoidal category of $L(R)$-modules.*

The richer structure in a general $\mathcal{O}$-algebra translates to a richer structure on the category of modules for an $\mathcal{O}$-algebra R.

Corollary 3.14 [6, Section 5.2] *Let $\mathcal{O}$ be an N_∞ operad of the homotopy type of the linear isometries operad for a G-universe U. If R is an $\mathcal{O}$-algebra in G-spectra, and if L is any Bousfield or finite localization which preserves $\mathcal{O}$-algebras, then there is an $\underline{\mathcal{O}}$-symmetric monoidal category of $L(R)$-modules.*

Remark 3.15 There is also a very exciting ∞-categorical approach to the norm functors and various kinds of G-symmetric monoidal enhancements which arise in equivariant homotopy theory due to ongoing work of Barwick, Dotto, Glasman, Nardin, and Shah [5]. This will elegantly remove the "linear isometries" hypothesis, giving $\underline{\mathcal{O}}$-symmetric monoidal categories of modules over any $\mathcal{O}$-algebra.

3.4 Pushfowards and localization

There is an interesting family of localizations which always preserves all of the desired multiplicative structure: localizations with respect to an ordinary ring spectrum viewed as a G-spectrum with a trivial action. We begin with a classical observation.

Proposition 3.16 *If $R \to S$ is a map of ring spectra, then*

$$\mathcal{Z}_R \subset \mathcal{Z}_S \subset \mathcal{S}p.$$

Proposition 3.17 *If E is a ring spectrum, then for any sub-conjugate $K \subset H$, we have*

$$\mathcal{Z}_{E^H} \subset \mathcal{Z}_{E^K} \subset \mathcal{S}p.$$

Proof For any subgroup H, there is a map of equivariant ring spectra

$$E \to \mathrm{Map}_H(G_+, i_H^* E).$$

Applying fixed points gives a map of ring spectra $E^G \to E^H$, giving the result. □

Corollary 3.18 *The assignment of G/H to the Bousfield class of E^H is a contravariant functor from the orbit category to the Bousfield lattice of spectra.*

The fixed points functor is a categorical right adjoint, with left adjoint the "push forward".

Definition 3.19 Let i_* denote the push-forward functor

$$i_* \colon \mathcal{S}p \to \mathcal{S}p^G,$$

which is the left-adjoint to the G-fixed points functor.

While in general it is very difficult to determine the fixed points of a smash product, when one of the factors is in the image of the pushforward, we can readily do so. In particular, we can simply move the fixed points past the smash product in this case.

Proposition 3.20 [15] *If E' is a spectrum and E is a G-spectrum, then we have a natural equivalence of spectra*

$$(E' \wedge E)^G \simeq E' \wedge E^G.$$

In particular,

$$(i_*E')^G \simeq E' \wedge (S^0)^G.$$

Corollary 3.21 *For any subgroup H of G, for any ordinary spectrum E', and for any G-spectrum E,*

$$\Phi^H(i_*E' \wedge E) \simeq E' \wedge \Phi^H(E).$$

Combined with Proposition 3.2, this gives another way to understand the acyclics for i_*E.

Proposition 3.22 *A G-spectrum Z is i_*E-acyclic if and only if for all $H \subset G$, $\Phi^H(Z)$ is E-acyclic:*

$$\mathcal{Z}^G_{i_*E} = \bigcap_{H \subset G} \left(\Phi^H\right)^{-1}(\mathcal{Z}_E).$$

Proposition 3.22 gives a readily checkable collection of criteria for acyclicity. In particular, since the geometric fixed points of the norm is well-understood, this quickly gives the following.

Theorem 3.23 *If E is any non-equivariant spectrum, then L_{i_*E} and the associated $L^f_{i_*E}$ preserve G-equivariant commutative rings.*

Proof Theorems 3.9 and 3.10 show that a sufficient condition for L_{i_*E} or $L^f_{i_*E}$ to preserve commutative rings is for the category of acyclics to be closed under all norms. In other words, we need to show that if $Z \in \underline{\mathcal{Z}}_{i_*E}(G/H)$, and $H \subset K \subset G$, then

$$N_H^K Z \in \underline{\mathcal{Z}}_{i_*E}(G/K).$$

G plays no role in this, since $i_K^*(i_*E) = i_*E$, so it suffices to check this for $K = G$. Proposition 3.22 then shows that it suffices to show that if Z is an H-acyclic, then for all subgroups K of G, we have

$$\Phi^K\left(N_H^G Z\right) \in \mathcal{Z}_E.$$

Lemma 2.1 shows that we have an equivalence

$$\Phi^K\left(N_H^G Z\right) \simeq \bigwedge_{g \in K\backslash G/H} \Phi^{K^g \cap H} Z.$$

Proposition 3.22 then tells us again that since Z is an H-acyclic for i_*E, we know that for all subgroups J of H that $\Phi^J(Z)$ is E-acyclic. In particular, for any K and any double coset, we know $\Phi^{K^g \cap H} Z$ is E-acyclic, giving the result. □

This gives us a nice selection of equivariant chromatic types that preserve commutative ring spectra.

Corollary 3.24 *If V is any type n-spectrum that is not type $(n+1)$ (at some prime p if $n > 0$), then the finite chromatic localization $L^f_{i_*V}$ preserves commutative ring spectra.*

These equivariant chromatic types are the first one considers, as they are lifted directly from the unstable information. Work of Balmer and Sanders describes all of the thick subcategories equivariantly, and we turn now to understanding which of their localizations preserve $\mathcal{O}$-algebras.

4 Review of equivariant thick subcategories

The equivariant thick subcategories of $\mathcal{S}p_c^G$ have been classified by Balmer–Sanders using Balmer's notion of the spectrum of a tensor triangulated category [3], where again $\mathcal{S}p_c^G$ is the full subcategory of compact objects in G-spectra. The Balmer spectrum of a tensor triangulated category should be thought of as an extension of the classical Zariski spectrum to a context which formally looks like the derived category of modules over a ring [2]. Balmer describes a notion of a "prime" tensor triangulated ideal, and these form the points in his Zariski spectrum. Out of this space, one can recover the thick subcategories of the (essentially small) tensor triangulated category.

The heart of the Balmer–Sanders result is that the geometric fixed points functors, being a tensor triangulated functor, induces maps

$$\mathrm{Spec}(\Phi^H)\colon\ \mathrm{Spec}(\mathcal{S}p_c) \to \mathrm{Spec}(\mathcal{S}p_c^G)$$

for all subgroups $H \subset G$. The "Thick subcategory theorem" of Hopkins–Smith determines all of the prime ideals in $\mathcal{S}p_c$.

Definition 4.1 Let p be a prime. For each $0 \leq m < \infty$, let $K(m, p)$ denote a Morava K-theory of height m at the prime p. Finally, for $m \geq 1$, define a full subcategory of finite spectra by

$$C_{m,p} := \big\{X \mid K(m-1, p)_*(X) = 0\big\}.$$

Theorem 4.2 [13] *The prime ideals in $\mathcal{S}p$ are given by $C_{m,p}$ for all primes p and all natural numbers $m \geq 1$. Their inclusions and intersections are as follows:*

(1) For all primes p and q, $C_{1,p} = C_{1,q}$, which is the category of torsion finite spectra.
(2) If $m < m'$, then $C_{m',p} \subset C_{m,p}$.
(3) If p and q are distinct primes and m and n are greater than one, then $C_{m,p}$ is not contained in $C_{n,q}$.

Definition 4.3 [3, Definition 4.1] For each subgroup H of G, each prime p, and each natural number m, let

$$\mathcal{P}(H, m, p) := \big(\Phi^H\big)^{-1} C_{m,p}.$$

Theorem 4.4 [3, Theorem 4.9] *For any finite group G, the spectrum of $\mathcal{S}p_c^G$ is*

$$\mathrm{Spec}(\mathcal{S}p_c^G) = \{\mathcal{P}(H, m, p) \mid H \subset G, m \in \mathbb{N}, p \text{ prime}\}.$$

For a fixed subgroup H and prime p, we have the usual chromatic inclusions: if $n \leq m$, then

$$\mathcal{P}(H, m, p) \subset \mathcal{P}(H, n, p).$$

Balmer–Sanders determine almost all of the other inclusions, and Barthel–Hausmann–Naumann–Nikolaus–Noel–Stapleton determined all remaining ones in the case of abelian groups. This gives a complete description of the topology.

Theorem 4.5 [4, Corollary 1.3] *When G is an abelian group, $K \subset H \subset G$, p is a prime, and $1 \leq n < \infty$, then the minimal i such that*

$$\mathcal{P}(K, n, p) \subset \mathcal{P}(H, n - i, p)$$

is the p-rank of H/K, the dimension of the largest elementary abelian p-group in H/K.

The topology on the Balmer–Zariski spectrum gives a complete classification of the thick subcategories. These are determined by the notion of "support" for an element.

Definition 4.6 If $X \in \mathcal{S}p^G$, then the **support** of X is the set of prime ideals not containing X:

$$\mathrm{Supp}(X) = \{\mathfrak{p} \mid X \notin \mathfrak{p}\}.$$

The **vanishing locus** of X is the complement of the support:

$$\mathrm{V}(X) = \mathrm{Spec}(\mathcal{S}p^G) - \mathrm{Supp}(X) = \{\mathfrak{p} \mid X \in \mathfrak{p}\}.$$

Thick subcategories are determined by the support. Thick subcategories are equivalent to the condition that their support be a Thomason closed subset.

Proposition 4.7 *Let X and Y be finite G-spectra. Then the thick subcategory generated by X contains Y if and only if*

$$\mathrm{Supp}(Y) \subset \mathrm{Supp}(X),$$

or equivalently

$$\mathrm{V}(X) \subset \mathrm{V}(Y).$$

5 Chromatic localizations and structured multiplications

5.1 General results for arbitrary *G*

We now restrict attention to determining conditions which guarantee that the Bousfield and finite localizations with respect to an equivariant thick subcategory preserve $\mathcal{O}$-algebras.

Theorem 5.1 *Let $\mathcal{X}$ be a thick subcategory of $\mathcal{S}p_c^G$. Then the norm N_K^H preserves $\mathcal{X}$ if for all $\mathcal{P}(J, m, p) \in \mathrm{V}(\mathcal{X})$ with $J \subset H$, there exists an $h \in H$ such that*

$$\mathcal{P}(K^h \cap J, m, p) \in \mathrm{V}(\mathcal{X}).$$

Proof By Proposition 4.7, this is equivalent to

$$\mathrm{V}(i_H^* X) \subset \mathrm{V}(N_K^H i_H^* X).$$

Thus a sufficient condition is that if $K(m, p)_* \Phi^J X = 0$, then

$$K(m, p)_* \Phi^J N_K^H i_K^* X = 0.$$

By Lemma 2.1, we have an isomorphism

$$K(m, p)_* \left(\Phi^J N_K^H i_K^* X\right) \cong$$
$$K(m, p)_* \left(\bigwedge_{KhJ \in K\backslash H/J} \Phi^{J^h \cap K} X \right) \cong \bigotimes_{KhJ \in K\backslash H/J} K(m, p)_* (\Phi^{J^h \cap K} X)$$

This vanishes if and only if there is an $h \in H$ such that $\mathcal{P}(J^h \cap H, m, p)$ is in the vanishing locus of X, as desired. □

As an immediate corollary, we deduce sufficient conditions for chromatic Bousfield and finite localizations to preserve $\mathcal{O}$-algebras, by Theorems 3.9 and 3.10.

Theorem 5.2 *The Bousfield and finite localizations with respect to a thick subcategory $\mathcal{X}$ preserve $\mathcal{O}$ algebras if for all $H/K \in \mathcal{O}(G/H)$ and for all $\mathcal{P}(J, m, p) \in \mathrm{V}(\mathcal{X})$ with $J \subset H$, there exists an $h \in H$ such that*

$$\mathcal{P}(K^h \cap J, m, p) \in \mathrm{V}(\mathcal{X}).$$

Remark 5.3 In some sense, Theorems 4.5 and 5.1 describe closure conditions in opposite directions. Theorem 4.5 describes how a prime being in a thick subcategory guarantees the inclusions of certain primes for larger groups. Theorem 5.1 shows how a prime being in a thick subcategory closed under norms guarantees the inclusion of certain primes for smaller groups.

5.2 Application: the p-local cases for $G = C_{p^n}$

When G is a cyclic group of order a power of a fixed prime p, then we can reduce the conditions from Theorem 5.2 to a collection of inequalities. These, together with the inclusions of Theorem 4.5, greatly cuts down the number of possible localizations which preserve commutative rings.

For simplicity, we restrict attention to the p-local subcategory. Localized at primes $q \neq p$, the category of C_{p^n} spectra splits, and the problem essentially becomes algebra. This splitting is in general not a splitting of G-symmetric monoidal categories, however, so it is not immediately clear how to couple this with the localization results above. Work of Böhme exactly addresses this point [8].

For $G = C_{p^n}$, a p-local thick subcategory is completely determined by a finite collection of extended natural numbers (so a natural number or infinity).

Definition 5.4 Let $\mathcal{X}$ be a p-local thick subcategory of $\mathcal{S}p_c^{C_{p^n}}$. For each $0 \leq k \leq n$, let

$$\ell_k = \max\left\{\ell \mid \mathcal{P}(C_{p^k}, \ell, p) \in \mathrm{V}(\mathcal{X})\right\}$$

provided this set is non-empty, and if for no ℓ is $\mathcal{P}(C_{p^k}, \ell, p)$ in the vanishing locus of $\mathcal{X}$, then let $\ell_k = 0$. Denote this sequence of extended integers by $\vec{\ell}(\mathcal{X})$.

Conversely, given a sequence $\vec{\ell} = (\ell_0, \ldots, \ell_n)$, let $\mathcal{X}_{\vec{\ell}}$ be defined by

$$\mathrm{V}(\mathcal{X}_{\vec{\ell}}) = \bigcup_{i=0}^{n} \bigcup_{j=1}^{\ell_i} \{\mathcal{P}(C_{p^i}, j, p)\},$$

where if $\ell_i = 0$, then that union is empty.

By definition, if $\mathcal{X}$ is a p-local thick subcategory of $\mathcal{S}p_c^{C_{p^n}}$, then

$$\mathcal{X} = \mathcal{X}_{\vec{\ell}(\mathcal{X})}.$$

However, not every sequence of integers works to give thick subcategories (as there is an implicit closure condition here). The inclusions of Barthel–Hausmann–Naumann–Nikolaus–Noel–Stapleton (described above as Theorem 4.5) provide constraints connecting the ℓ_i for various i.

Theorem 5.5 *For all $0 \leq i < j \leq n$, if $\mathcal{X}_{\vec{\ell}}$ is a p-local thick subcategory, then we have inequalities*

$$\ell_i \leq \ell_j + 1.$$

Proof The p-rank of any cyclic group is 1, so for any $i < j$ and for any m, Theorem 4.5 shows that we have inclusions

$$\mathcal{P}(C_{p^i}, m, p) \subset \mathcal{P}(C_{p^j}, m-1, p).$$

□

Applying the analysis for Theorem 5.2, we deduce the following.

Theorem 5.6 *Let $\mathcal{X}$ be a p-local, thick subcategory of $\mathcal{S}p_c^{C_{p^n}}$. The norm $N_{C_{p^k}}^{C_{p^j}}$ preserves $\mathcal{X}$ if*

$$\ell_k \geq \ell_{k+1}, \ldots, \ell_j.$$

Proof Since every subgroup is normal and the subgroups are nested, this is an immediate application of Theorem 5.1. □

Putting this all together gives a condition for chromatic localizations to preserve commutative ring spectra.

Corollary 5.7 *Let $\mathcal{X}$ be a p-local, thick subcategory of $\mathcal{S}p_c^{C_{p^n}}$. Then the finite and Bousfield localizations nullifying $\mathcal{X}$ preserve commutative ring spectra if for all $0 \leq i < j \leq n$,*

$$\ell_j \leq \ell_i \leq \ell_j + 1.$$

Corollary 5.8 *Let*

$$\vec{\ell}_k = (m, \ldots, m, m-1, \ldots, m-1),$$

where we have k copies of m. Then the Bousfield and finite localizations nullifying $\mathcal{X}_{\vec{\ell}_k}$ preserve commutative ring spectra.

The case that the sequence is constant (i.e. $k = n+1$) is one we have already studied: these are the thick subcategories generated by the pushforward for a type $(n+1)$-complex. The other cases are new.

As a final corollary, the finite localizations by construction are smashing. Thus all of these results can be restated in terms of certain structured multiplications on the localized sphere spectrum.

Corollary 5.9 *For $G = C_{p^n}$, let $\vec{\ell}$ denote a sequence of extended integers such that for all $0 \leq i < j \leq n$, we have*

$$\ell_j \leq \ell_i \leq \ell_j + 1,$$

Then the chromatically localized spheres

$$L^f_{\mathcal{X}_{\vec{\ell}}} S^0$$

are C_{p^n}-equivariant commutative ring spectra.

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