Physica D 439 (2022) 133360

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

General breather and rogue wave solutions to the complex short pulse equation

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ARTICLE INFO

Article history:
Received 30 December 2021
Received in revised form 27 April 2022
Accepted 16 May 2022
Available online 2 June 2022
Communicated by Jingsong He

Keywords:
Defocusing complex short pulse equation
Breather
Rogue wave
KP–Toda hierarchy

ABSTRACT

In the present paper, we attempt to construct both the breather and rogue wave solutions to the focusing complex short pulse (fCSP) equation via the KP–Toda reduction method. Following a procedure of reducing the bilinear equations satisfied by tau functions of Kadomtsev–Petviashvili (KP)–Toda hierarchy to the ones of the fCSP equation with nonzero boundary condition, we first deduce the general breather solution of the fCSP equation starting from a specially arranged tau-function of the KP–Toda hierarchy, then we construct and prove the Nth order rogue wave solutions of the fCSP equation and express them in two different but equivalent forms of determinants. The dynamical behaviors of both the breather and rogue wave solutions are illustrated and analyzed.

1. Introduction

Rogue waves, which are initially used for the description of the spontaneous and monstrous ocean surface waves have recently attracted considerable attention on both experimentally and theoretically [1]. Rogue waves have been observed in a variety of physical contexts including optical systems [2–4], Bose–Einstein condensates [5,6], superfluids [7], plasma [8,9], capillary waves [10]. Compared with the stable solitons, rogue waves are the localized structures with the instability and unpredictability [11,12]. A typical model for characterizing the rogue wave is the celebrating nonlinear Schrödinger (NLS) equation. The fundamental rogue wave of the NLS equation is described by Peregrine soliton [13], which is the first-order rogue wave expressed by a rational form with quadratic functions. This rational solution has localized behavior in both space and time, and its maximum amplitude attains three times the constant background. The Peregrine soliton is the limiting case of a breather solution when the period approaches infinity. Since the higher-order rogue waves were discovered by Akhmediev et al. [14], many papers have been devoted to the study of higher-order rogue waves via different methods [15–39]. The higher-order rogue waves were also excited experimentally in a water wave tank [40,41]. In fact, higher-order rogue waves can be treated as the nonlinear superposition of fundamental rogue wave and they are usually expressed in terms of complicated higher-order rational polynomials. These higher-order waves were also localized in both coordinates and could exhibit higher peak amplitudes or multiple intensity peaks.

Recently, a complex short pulse (CSP) equation [42,43]

$$q_{xt} + q + \frac{1}{2}\sigma\left(|q|^2 q_x\right)_x = 0\,, \tag{1}$$

was proposed by one of the authors as an improvement for the (real-valued) short pulse (SP) equation proposed by Schäfer and Wayne [44] to describe the propagation of ultra-short optical pulses in nonlinear media. Here $q = q(x,t)$ is a complex-valued function. In contrast with the short pulse equation, the complex short pulse equation has both the focusing case ($\sigma = 1$) and the defocusing case ($\sigma = -1$) which admits the bright and dark soliton solutions, respectively. The CSP equation can be viewed as an analogue of the NLS equation in the ultra-short regime when the width of optical pulse is of the order 10^{-15} s. As the width of optical pulse is in the order of femtosecond (10^{-15} s), the width of spectrum of this ultra-short pulse is approximately of the order $10^{15}s^{-1}$, the monochromatic assumption to derive the NLS equation is invalid anymore.

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https://doi.org/10.1016/j.physd.2022.133360
0167-2789/

Mathematically, the CSP equation can also be viewed as a reduction of a two-component short pulse equation proposed by Dimakis and Müller-Hoissen [45] and Matsuno [46] independently. It is shown that the CSP Eq. (1) is integrable in the sense that it admits Lax pair, bilinear form and bright-soliton solution form [26,42,47]. It has been shown that the fCSP equation also admits double pole, breather and rogue wave solutions by Darboux transformation method [26,48]. The gauge transformation of the focusing cSPE equation was recently studied in [49]. By formulating the Riemann–Hilbert problem, the inverse scattering transform for the focusing cSPE equation was investigated in [50] and the long-time asymptotic behavior was analyzed in [51]. The multi-dark soliton solution to the defocusing CSP equation was constructed by the generalized Darboux transformation method [43] and the KP hierarchy reduction method [52].

Since the seminal work by Ohta and Yang [22], the KP reduction method has become one of the most effective methods in constructing rogue wave, especially higher order rogue wave solutions. It has been used to find rogue wave solutions to the Ablowitz–Ladik (AL) equation [31], Davey–Stewartson I and II equation [53,54], the Yajima–Oikawa equation [28,55], the derivative long-wave–short-wave interaction model of Newell type [56], the derivative Schrödinger (NLS) equation [57], the three-wave equation [58], Boussinesq equation [59] and the coupled NLS equation (Manakov system) [60].

In the present paper, we are concerned with the general breather and rogue wave solutions of the focusing complex short pulse (fCSP) equation ($\sigma = 1$). Based on the previous work for the dark soliton of the defocusing CSP equation, we are able to construct the general breather and rogue wave solutions to the fCSP equation, which are the main results of this paper. The remainder of this paper is organized as follows. In Section 2, we firstly bilinearize the fCSP equation under nonzero boundary condition and show how a set of bilinear equations of the KP–Toda hierarchy can be reduced to the bilinear equations of the fCSP equation. In Sections 3 and 4, starting from specially arranged tau-functions of exponential type and rational type, we show step by step that how the general breather and rogue wave solutions of the fCSP equation can be reduced. Section 5 is devoted to concluding remarks.

2. Reduction of the KP–Toda hierarchy to the CSP equation

2.1. Bilinearization of the CSP equation

The bilinearization of the fCSP equation is established by the following proposition.

Proposition 1. *By means of the dependent variable transformation*

$$q = \frac{\beta}{2}\frac{g}{f}e^{\mathrm{i}(y+\gamma s/2)}\,, \tag{2}$$

and the hodograph (reciprocal) transformation

$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2(\log f)_s\,, \quad t = -s\,, \tag{3}$$

the CSP equation (1) with $\sigma = -1$ is bilinearized into

$$(D_yD_s + \mathrm{i}D_s + \frac{\gamma}{2}\mathrm{i}D_y)g\cdot f = 0\,, \tag{4}$$

$$\left(D_s^2 + \frac{\beta^2}{8}\right)f\cdot f = \frac{\beta^2}{8}gg^*\,, \tag{5}$$

where D is the Hirota D-operator defined by [61]

$$D_s^nD_y^mf\cdot g = \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'}\right)^n\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m f(y,s)g(y',s')|_{y=y',s=s'}\,.$$

Proof. From the hodograph (reciprocal) transformation and bilinear equations, we could have

$$\frac{\partial x}{\partial y} = -\frac{\gamma}{2} - 2(\log f)_{ys}$$

and

$$\frac{\partial x}{\partial s} = -\frac{\beta^2}{8} - 2(\log f)_{ss} = -\frac{\beta^2}{8}\frac{|g|^2}{f^2} = -\frac{1}{2}|q|^2,$$

by using the bilinear equation (5). This implies

$$\partial_y = \rho\partial_x,\quad \partial_s = -\partial_t - \frac{1}{2}|q|^2\partial_x$$

by define $\frac{\partial x}{\partial y} = \rho$.

Dividing both sides of bilinear equation (4) by f^2, one arrives at

$$\left(\frac{g}{f}\right)_{ys} + 2(\log f)_{ys}\frac{g}{f} + \mathrm{i}\left(\frac{g}{f}\right)_s + \mathrm{i}\frac{\gamma}{2}\left(\frac{g}{f}\right)_y = 0\,. \tag{6}$$

or

$$\left(\frac{g}{f}\right)_{ys} + \mathrm{i}\left(\frac{g}{f}\right)_s + \mathrm{i}\frac{\gamma}{2}\left(\frac{g}{f}\right)_y - \frac{\gamma}{2}\frac{g}{f} = \left(-\frac{\gamma}{2} - 2(\log f)_{ys}\right)\frac{g}{f}\,. \tag{7}$$

Above equation can be written as

$$\rho^{-1} q_{ys} = q\,, \tag{8}$$

or can be converted into

$$\partial_x \left(-\partial_t - \frac{1}{2}|q|^2 \partial_x \right) q = q\,, \tag{9}$$

which is nothing but the focusing CSP equation. □

2.2. The bilinear equations of the KP–Toda hierarchy related to the CSP equation

In this subsection, we first present and prove a set of bilinear equations satisfied by the τ functions of the KP–Toda hierarchy [62] by the following lemma.

Lemma 1. *Let τ_{nkl} be a $N \times N$ determinant defined by*

$$\tau_{nkl} = \det_{1\le i,j\le N}(m_{ij}^{nkl}) = \left| \int \varphi_i^{nkl} \psi_j^{nkl} dx_1 \right|_{1\le i,j\le N}, \tag{10}$$

where φ_i^{nkl} and ψ_j^{nkl} are functions of the variables x_1, x_{-1} t_a, t_b and satisfy the following differential and difference relations:

$$\partial_{x_1}\varphi_i^{nkl} = \varphi_i^{n+1,kl}, \quad \partial_{x_{-1}}\varphi_i^{nkl} = \varphi_i^{n-1,kl}, \tag{11}$$

$$\partial_{t_a}\varphi_i^{nkl} = \varphi_i^{n,k-1,l}, \quad \partial_{t_b}\varphi_i^{nkl} = \varphi_i^{n,k,l-1}, \tag{12}$$

$$\varphi_i^{n,k+1,l} = \varphi_i^{n+1,k} - a\varphi_i^{nkl} \quad \varphi_i^{n,k,l+1} = \varphi_i^{n+1,k} - b\varphi_i^{nkl} \tag{13}$$

$$\partial_{x_1}\psi_j^{nkl} = -\psi_j^{n-1,kl}, \quad \partial_{x_{-1}}\psi_j^{nkl} = -\psi_j^{n+1,kl}, \tag{14}$$

$$\partial_{t_a}\psi_j^{nkl} = -\psi_j^{n,k+1,l}, \quad \partial_{t_b}\psi_j^{nkl} = -\psi_j^{n,k,l+1}, \tag{15}$$

$$\psi_j^{n,k-1,l} = \psi_j^{n-1,k} + a\psi_j^{nkl}, \quad \psi_j^{n,k,l-1} = \psi_j^{n-1,k} + b\psi_j^{nkl}\,. \tag{16}$$

Then τ_{nkl} satisfies the following bilinear equations:

$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right)\tau_{nkl}\cdot\tau_{nkl} = -\tau_{n+1,kl}\tau_{n-1,kl}\,, \tag{17}$$

$$(aD_{t_a} - 1)\tau_{n+1,kl}\cdot\tau_{nkl} = -\tau_{n+1,k-1,l}\tau_{n,k+1,l}\,, \tag{18}$$

$$\left(D_{x_1}(aD_{t_a} - 1) - 2a\right)\tau_{n+1,kl}\cdot\tau_{nkl} = (D_{x_1} - 2a)\tau_{n+1,k-1,l}\cdot\tau_{n,k+1,l}\,, \tag{19}$$

$$\left(D_{x_1}(bD_{t_b} - 1) - 2b\right)\tau_{n+1,kl}\cdot\tau_{nkl} = (D_{x_1} - 2b)\tau_{n+1,k,l-1}\cdot\tau_{nk,l+1}\,. \tag{20}$$

The proof is given in Appendix A by referring to the Grammian technique [61,63].

2.3. Reduction to the fCSP equation

In what follows, we briefly show the procedure of reducing the bilinear equations of extended KP hierarchy (17)–(20) to the bilinear equations (4)–(5). To be specific, if τ_{nkl} satisfies the reduction conditions

$$(\partial_{x_1} + \partial_{x_{-1}})\tau_{nkl} = C_1\tau_{nkl}\,, \tag{21}$$

$$(a^2\partial_{t_a} - \partial_{t_b})\tau_{nkl} = C_2\tau_{nkl}\,, \tag{22}$$

$$\tau_{n-1,k+1,l+1} = \tau_{nkl}\,, \tag{23}$$

then the bilinear equation (17) is reduced to

$$\left(\frac{1}{2}D_{x_1}^2 + 1\right)\tau_{nkl}\cdot\tau_{nkl} = \tau_{n+1,kl}\tau_{n-1,kl}. \tag{24}$$

Moreover, let $b = 1/a$, referring to the bilinear equation (20) and the reduction conditions (22)–(23), we have

$$\left(D_{x_1}(aD_{t_a} - 1) - \frac{2}{a}\right)\tau_{n+1,kl}\cdot\tau_{nkl} = \left(D_{x_1} - \frac{2}{a}\right)\tau_{n,k+1,l}\cdot\tau_{n+1,k-1,l},$$

thus using (18) and (19) we get

$$\left(D_{x_1}(aD_{t_a} - 1) - a - \frac{1}{a}\right)\tau_{n+1,kl}\cdot\tau_{nkl} = \left(-a - \frac{1}{a}\right)\tau_{n+1,k-1,l}\tau_{n,k+1,l}$$

$$= \left(a + \frac{1}{a}\right)(aD_{t_a} - 1)\tau_{n+1,kl}\cdot\tau_{nkl}\,, \tag{25}$$

i.e.,

$$(D_{x_1}(aD_{t_a}-1)-(a^2+1)D_{t_a})\tau_{n+1,kl}\cdot\tau_{nkl}=0\,. \tag{26}$$

Next, if we require

$$\tau^*_{n00}=\tau_{-n,00}.$$

and define

$$f=\tau_{000},\quad g=\tau_{100},$$

we arrive at

$$\left(D_{x_1}D_{t_a}-\frac{1}{a}D_{x_1}-\left(a+\frac{1}{a}\right)D_{t_a}\right)g\cdot f=0\,, \tag{27}$$

$$\left(\frac{1}{2}D^2_{x_1}+1\right)f\cdot f=gg^*\,. \tag{28}$$

Finally, by setting $a=\mathrm{i}\alpha$, $t_a=\alpha y$, $x_1=\beta s/4$ and $\beta(\alpha^2-1)=-2\gamma\alpha$, the above bilinear equations coincide with the bilinear equations (4)–(5).

3. Breather solution to the CSP equation

3.1. Derivation of the general breather solution

To derive the breather solution to the fCSP equation, we give the following two lemmas, whose proofs are given in Appendix B and C, respectively.

Lemma 2. *The generalized tau function τ_{nkl} for the KP–Toda hierarchy defined by*

$$\tau_{nkl}=\left|m^{nkl}_{ij}\right|_{1\le i,j\le N} \tag{29}$$

where

$$m^{nkl}_{ij}=\sum_{m=1}^{2}\sum_{r=1}^{2}\frac{a_{im}b_{jr}}{p_{im}+q_{jr}}\left(-\frac{p_{im}}{q_{jr}}\right)^n\left(-\frac{p_{im}-a}{q_{jr}+a}\right)^k\left(-\frac{p_{im}-b}{q_{jr}+b}\right)^l e^{\xi_{im}+\bar{\xi}_{jr}}, \tag{30}$$

with

$$\xi_{im}=\frac{1}{p_{im}}x_{-1}+p_{im}x_1+\frac{1}{p_{im}-a}t_a+\frac{1}{p_{im}-b}t_b+\xi_{im,0},$$
$$\bar{\xi}_{jr}=\frac{1}{q_{jr}}x_{-1}+q_{jr}x_1+\frac{1}{q_{jr}+a}t_a+\frac{1}{q_{jr}+b}t_b+\bar{\xi}_{jr,0}$$

satisfies the bilinear equations (17)–(20).

Lemma 3. *By imposing the conditions*

$$p_{i1}p_{i2}=1,\quad q_{i1}q_{i2}=1,\quad ab=1\,, \tag{31}$$

then all three reduction relations (21)–(23) are satisfied.

The complex conjugate condition can be achieved by taking $q_{i1}=p^*_{i1}$, $q_{i2}=p^*_{i2}$, $b^*_{im}=a_{im}(m=1,2)$ and $a=\mathrm{i}\alpha$ purely imaginary. In summary, we have the following theorem for the general-breather solution to the fCSP equation.

Theorem 1. *The multi-breather solution of the fCSP equation is given by a parametric form*

$$q=\frac{\beta}{2}\frac{g}{f}e^{\mathrm{i}(y+\gamma s/2)}\,,$$
$$x=-\frac{\gamma}{2}y-\frac{\beta^2}{8}s-2(\log f)_s\,,\quad t=-s\,.$$

where $g=\tau_1$, $f=\tau_0$, $\gamma=-\frac{\beta}{2}(\alpha-\alpha^{-1})$ and τ_n is defined as

$$\tau_n=\left|\sum_{m,r=1}^{2}\frac{a_{im}a^*_{jr}}{p_{im}+p^*_{jr}}\left(-\frac{p_{im}}{p^*_{jr}}\right)^n e^{\xi_{im}+\xi^*_{jr}}\right|_{N\times N}. \tag{32}$$

Here $*$ denotes complex conjugation, p_{i1}, p_{i2} are complex wave numbers satisfying the constraint $p_{i1}p_{i2}=1$, $\xi_{im}=\frac{1}{4}p_{im}\beta s+\frac{\alpha y}{p_{im}-\mathrm{i}\alpha}+\xi_{im,0}$, $i=1,\ldots,N$.

3.2. One- and two-breather solutions

Since $p_{11}p_{12}=1$, let us take $p_{11}=A_1e^{\mathrm{i}\delta_1}$, then $p_{12}=A_1^{-1}e^{-\mathrm{i}\delta_1}$, $q_{11}=A_1e^{-\mathrm{i}\delta_1}$, then $q_{12}=A_1^{-1}e^{\mathrm{i}\delta_1}$. The tau function for one-breather solution can be expressed by

$$
\begin{aligned}
f =& \frac{2A_1(\sinh(L_1)+\cosh(L_1))\cos(L_2-\delta_1)}{A_1^2+1}+\frac{\sec(\delta_1)(\sinh(K_1)+\cosh(K_1))}{2A_1}\\
&+\frac{1}{2}A_1\sec(\delta_1)(\sinh(K_2)+\cosh(K_2)),\\
g =& -\frac{1}{2\left(A_1^3+A_1\right)}\left(\left(A_1^2+1\right)\cos(2\delta_1)\sec(\delta_1)\cosh(K_1)+2\left(A_1^4+1\right)\cosh(L_1)\cos(L_2-\delta_1)+\right.\\
&\cos(\delta_1)\left(\left(A_1^2+1\right)\cos(2\delta_1)\sec^2(\delta_1)\sinh(K_1)+A_1^4\tan^2(\delta_1)(-\sinh(K_2))\right.\\
&-A_1^2\tan^2(\delta_1)\sinh(K_2)+\left(A_1^2+1\right)A_1^2\cos(2\delta_1)\sec^2(\delta_1)\cosh(K_2)\\
&+A_1^4\sinh(K_2)+A_1^2\sinh(K_2)+2A_1^4\tan(\delta_1)\sinh(L_1)\sin(L_2)\\
&\left.\left.+2A_1^4\sinh(L_1)\cos(L_2)+2\tan(\delta_1)\sinh(L_1)\sin(L_2)+2\sinh(L_1)\cos(L_2)\right)\right)\\
&+\frac{\mathbf{i}}{A_1}\left(A_1^2\sin(\delta_1)\sinh(K_2)+A_1^2\sin(\delta_1)\cosh(K_2)-A_1^2\sinh(L_1)\sin(L_2-\delta_1)\right.\\
&-\left(A_1^2-1\right)\cosh(L_1)\sin(L_2-\delta_1)-\sin(\delta_1)\sinh(K_1)\\
&\left.-\sin(\delta_1)\cosh(K_1)+\sinh(L_1)\sin(L_2-\delta_1)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A =& \cos(\delta_1),\quad B=\sin(\delta_1),\\
K_1 =& AA_1\left(\frac{2\alpha y}{A^2A_1^2+(\alpha-A_1B)^2}+\frac{\beta s}{2}\right)+2\mathrm{Re}(\xi_{10}),\\
K_2 =& \frac{1}{2A_1\left(A^2+(\alpha A_1+B)^2\right)}\left(4A^2A_1\mathrm{Re}(\xi_{10})+4A_1\mathrm{Re}(\xi_{10})(\alpha A_1+B)^2\right.\\
&\left.+A^3\beta s+A\left(\alpha A_1^2(\alpha\beta s+4y)+2\alpha A_1\beta Bs+\beta B^2 s\right)\right),\\
L_1 =& A\left(\alpha A_1 y\left(\frac{1}{A^2A_1^2+(\alpha-A_1B)^2}+\frac{1}{A^2+(\alpha A_1+B)^2}\right)+\frac{A_1\beta s}{4}+\frac{\beta s}{4A_1}\right)+2\mathrm{Re}(\xi_{10}),\\
L_2 =& \frac{1}{4}\left(A_1B\left(-\frac{4\alpha y}{A^2A_1^2+(\alpha-A_1B)^2}-\frac{4\alpha y}{A^2+(\alpha A_1+B)^2}+\beta s\right)\right.\\
&\left.+\frac{4\alpha^2 y}{A^2A_1^2+(\alpha-A_1B)^2}-\frac{4\alpha^2A_1^2 y}{A^2+(\alpha A_1+B)^2}+\frac{\beta Bs}{A_1}\right).
\end{aligned}
$$

Several examples are shown in Fig. 1: (a)–(b) is a typical breather solution, (c)–(d) is the Akhmediev breather where p_{11} and p_{12} are real and (e)–(f) is the Kuznetsov–Ma soliton.

The tau functions to two-breather solution is of the form

$$
\tau_n=\left|\sum_{m,r=1}^{2}\frac{1}{p_{im}+p_{jr}^*}\left(-\frac{p_{im}}{p_{jr}^*}\right)^n e^{\xi_{im}+\bar{\xi}_{jr}}\right|_{1\le i,j\le 2} \tag{33}
$$

with

$$
\begin{aligned}
\xi_{im} =& \frac{1}{4}p_{im}\beta s+\frac{\alpha y}{p_{im}-\mathrm{i}\alpha}+\xi_{im,0},\\
\bar{\xi}_{jr} =& \frac{1}{4}p_{jr}^*\beta s+\frac{\alpha y}{p_{jr}^*+ai\alpha}+\xi_{jr,0}^*
\end{aligned}
$$

with parameters: $p_{11}=A_1e^{\mathrm{i}\delta_1}$, then $p_{12}=A_1^{-1}e^{-\mathrm{i}\delta_1}$, $p_{21}=A_2e^{\mathrm{i}\delta_2}$, then $p_{22}=A_2^{-1}e^{-\mathrm{i}\delta_2}$.

A typical example of two breather solution is shown in Fig. 2 with $\alpha=1$, $\beta=2$, $\delta_1=\pi/3$, $\delta_2=0$ $A_1=1$, $A_2=2.0$, in which a regular breather interacts with a Akhmediev breather.

4. General rogue wave solution to the CSP equation

To derive general rogue wave solution, let us introduce

$$
m^{(nkl)}=\frac{1}{p+q}\left(-\frac{p}{q}\right)^n\left(-\frac{p-a}{q+a}\right)^k\left(-\frac{p-b}{q+b}\right)^l e^{\xi+\eta}, \tag{34}
$$

$$
\varphi^{(nkl)}=p^n(p-a)^k(p-b)^l e^{\xi}, \tag{35}
$$

$$
\psi^{(nkl)}=(-q)^{-n}[-(q+a)]^{-k}[-(q+b)]^{-l}e^{\eta}, \tag{36}
$$

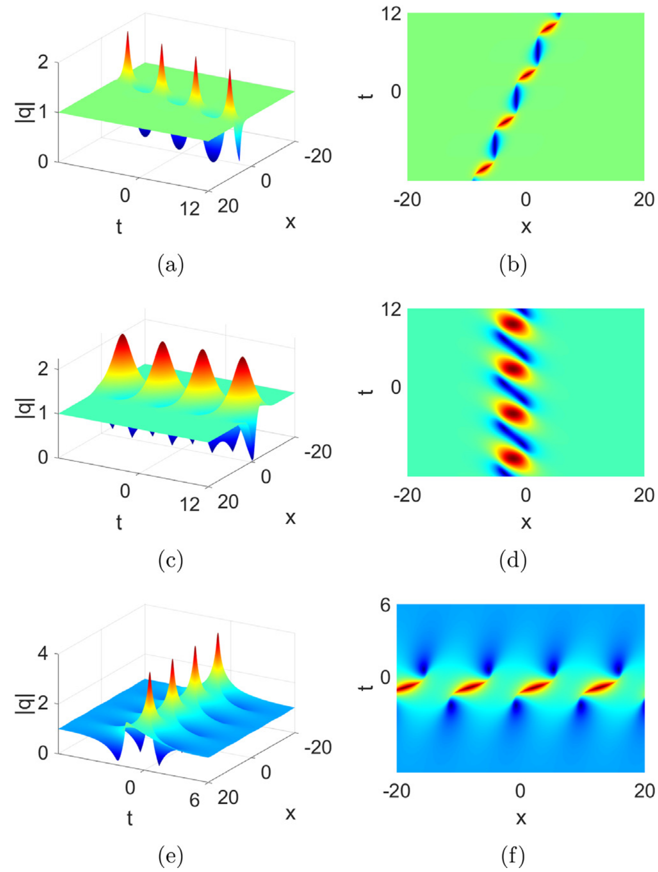

Fig. 1. One-breather solutions to the fCSP equation (a)–(b): $\alpha = 1, \beta = 2, \delta_1 = \pi/3, A_1 = 1$; (c)–(d): Kuznetsov–Ma soliton with $\alpha = -0.35, \beta = 2, \delta_1 = 1.0472, A_1 = 2$; (e)–(f): Akhmediev breather with $\alpha = 1, \beta = 2, \delta_1 = 0, A_1 = 2$.

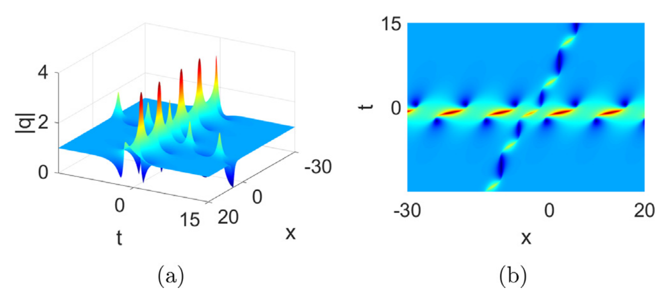

Fig. 2. Interaction between Akhmediev breather in Fig. 1(e) and the breather in Fig. 1(a). (a) profile; (b) contour plot.

with

$$\xi = \frac{1}{p}x_{-1} + px_1 + \frac{1}{p-a}t_a + \frac{1}{p-b}t_b + \xi_0, \tag{37}$$

$$\eta = \frac{1}{q}x_{-1} + qx_1 + \frac{1}{q+a}t_a + \frac{1}{q+b}t_b + \eta_0, \tag{38}$$

where $p, q, \xi_0, \eta_0, a, b$ are complex constants. It is easy to find that these functions satisfy differential and difference rules (11)–(16) without indices i and j.

Then, we define the elements

$$m_{ij}^{(nkl)} = \mathcal{A}_i\mathcal{B}_j m^{(nkl)}, \quad \varphi_i^{(nkl)} = \mathcal{A}_i\varphi^{(nkl)}, \quad \psi_j^{(nkl)} = \mathcal{B}_j\psi^{(nkl)}, \tag{39}$$

where $\mathcal{A}_i$ and $\mathcal{B}_j$ are differential operators with respect to p and q respectively as

$$\mathcal{A}_i = \frac{1}{i!}\left(p\partial_p\right)^i, \quad \mathcal{B}_j = \frac{1}{j!}\left(q\partial_q\right)^j, \tag{40}$$

Since operators $\mathcal{A}_i$ and $\mathcal{B}_j$ commute with operators ∂_{x_1}, $\partial_{x_{-1}}$, ∂_{t_a} and ∂_{t_b}, these functions $m_{ij}^{(nkl)}$, $\varphi_i^{(nkl)}$ and $\psi_j^{(nkl)}$ still obey the differential and difference rules (11)–(16). From Lemma 1, it is known that for an arbitrary sequence of indices $(i_1, i_2, \ldots, i_N; j_1, j_2, \ldots, j_N)$, the determinant

$$\tau_{nkl} = \det_{1\le\nu,\mu\le N}\left(m^{(nkl)}_{i_\nu,j_\mu}\right), \tag{41}$$

satisfies the bilinear equations (17)–(20).

4.1. Dimensional reduction

To realize the dimension reduction, we introduce the following linear differential operators

$$D_{x_1,x_{-1}} = \partial_{x_1} + \partial_{x_{-1}}, \quad D_{t_a,t_b} = a^2\partial_{t_a} - \partial_{t_b}\,. \tag{42}$$

It is straightforward to see that

$$D_{x_1,x_{-1}} m^{(nkl)}_{ij} = \mathcal{A}_i\mathcal{B}_j\left(p + \frac{1}{p} + q + \frac{1}{q}\right) m^{(nkl)}, \tag{43}$$

$$D_{t_a,t_b} m^{(nkl)}_{ij} = \mathcal{A}_i\mathcal{B}_j\left(\frac{a^2}{p-a} - \frac{1}{p-b} + \frac{a^2}{q+a} - \frac{1}{q+b}\right) m^{(nkl)}, \tag{44}$$

and

$$(p\partial_p)^i\left(p+\frac{1}{p}\right) = p + (-1)^i\frac{1}{p}, \tag{45}$$

$$(q\partial_q)^j\left(q+\frac{1}{q}\right) = q + (-1)^j\frac{1}{q}\,. \tag{46}$$

Using the Leibnitz rule, one has

$$D_{x_1,x_{-1}} m^{(nkl)}_{ij} = \sum_{\mu=0}^{i}\frac{1}{\mu!}\left[(p\partial_p)^\mu\left(p+\frac{1}{p}\right)\right] m^{(nkl)}_{i-\mu,j} + \sum_{\nu=0}^{j}\frac{1}{\nu!}\left[(q\partial_q)^\nu\left(q+\frac{1}{q}\right)\right] m^{(nkl)}_{i,j-\nu},$$

which leads to

$$D_{x_1,x_{-1}} m^{(nkl)}_{ij}\Big|_{p=1,q=1} = 2\sum_{\substack{\mu=0,\\ \mu:even}}^{i}\frac{1}{\mu!}\, m^{(nkl)}_{i-\mu,j}\Big|_{p=1,q=1} + 2\sum_{\substack{\nu=0,\\ \nu:even}}^{j}\frac{1}{\nu!}\, m^{(nkl)}_{i,j-\nu}\Big|_{p=1,q=1} \tag{47}$$

by taking $p = 1, q = 1$. Next, we restrict the general determinant (41) to

$$\tilde{\tau}_{nkl} = \det_{1\le i,j\le N}\left(m^{(n,k,l)}_{2i-1,2j-1}\Big|\right)_{p=q=1,b=\frac{1}{a}}\,. \tag{48}$$

By using the relations (45) and (46) as in Ref. [22], we obtain

$$D_{x_1,x_{-1}}\tilde{\tau}_{nkl} = 4N\tilde{\tau}_{nkl}\,. \tag{49}$$

We proceed to prove the second dimension reduction condition (22). For the sake of convenience, we define

$$\mathcal{P}(p) = p + \frac{1}{p}, \quad \mathcal{Q}(q) = q + \frac{1}{q}, \tag{50}$$

$$G_1(p; a, b) = \frac{a^2}{p-a} - \frac{1}{p-b}\,, \quad G_2(q; a, b) = \frac{a^2}{q+a} - \frac{1}{q+b}\,. \tag{51}$$

Notice that $G_1(p; a, b), G_2(q; a, b)$ can be expressed as

$$G_1(p; a, b) = \frac{2a^2}{\mathcal{P} - 2a \pm \sqrt{\mathcal{P}^2-4}} - \frac{2}{\mathcal{P} - 2b \pm \sqrt{\mathcal{P}^2-4}} \equiv F_1(\mathcal{P}),$$

$$G_2(q; a, b) = \frac{2a^2}{\mathcal{Q} + 2a \pm \sqrt{\mathcal{Q}^2-4}} - \frac{2}{\mathcal{Q} + 2b \pm \sqrt{\mathcal{Q}^2-4}} \equiv F_2(\mathcal{Q}),$$

then by using the Faà di Bruno formula we obtain

$$\left(p\partial_p\right)^l F_1(\mathcal{P}) = \sum_{m_1+2m_2+\cdots+lm_l=l}\frac{\frac{d^{\hat{m}}F_1(\mathcal{P})}{d\mathcal{P}^{\hat{m}}}\prod_{j=1}^{l}\left[(p\partial_p)^j\mathcal{P}\right]^{m_j}}{(l!)^{-1}\prod_i^l m_i!(i!)^{m_i}} \tag{52}$$

where $\hat{m} = \sum_{i=1}^{l} m_i$. Moreover, under the conditions $p = q = 1$, one has

$$\left(p\partial_p\right)^l F_1[\mathcal{P}(p)]\Big|_{\substack{p=1\\ b=\frac{1}{a}}} = 0, \quad (l \text{ is odd}), \tag{53}$$

$$\left(p\partial_p\right)^l F_1[\mathcal{P}(p)]\Big|_{\substack{p=1\\ b=\frac{1}{a}}} = \mathcal{C}_{1,l}\left(F_1[\mathcal{P}(p=1)]\right), \quad (l \text{ is even}). \tag{54}$$

By applying the conditions (45) and (46). Similarly, one has

$$\left(q\partial_q\right)^l F_2[\mathcal{Q}(q)]\Big|_{\substack{q=1\\ b=\frac{1}{a}}} = 0, \quad (l \text{ is odd}), \tag{55}$$

$$\left(q\partial_q\right)^l F_2[\mathcal{Q}(q)]\Big|_{\substack{q=1\\ b=\frac{1}{a}}} = \mathcal{C}_{2,l}\left(F_2[\mathcal{Q}(q=1)]\right), \quad (l \text{ is even}). \tag{56}$$

By referring above formulas (53)–(56), it follows that

$$D_{t_a,t_b} m_{ij}^{(nkl)}\Big|_{\substack{p=q=1\\ b=\frac{1}{a}}} = \left(\sum_{\substack{\mu=0,\\ \mu:even}}^{i} \frac{\mathcal{C}_{1,\mu}[G_1(p)]}{\mu!} m_{i-\mu,j}^{(nkl)} + \sum_{\substack{\nu=0,\\ \nu:even}}^{j} \frac{\mathcal{C}_{2,\nu}[G_2(q)]}{\nu!} m_{i,j-\nu}^{(nkl)}\right)_{\substack{p=q=1\\ b=\frac{1}{a}}}. \tag{57}$$

Since $\mathcal{C}_{1,0}[G_1(p=1)] = G_1(p=1)$ and $\mathcal{C}_{2,0}[G_2(q=1)] = G_2(q=1)$, we obtain

$$D_{t_a,t_b}\tilde{\tau}_{nkl}\Big|_{\substack{p=q=1\\ b=\frac{1}{a}}} = \frac{4Na^2}{1-a^2}\tilde{\tau}_{nkl} \tag{58}$$

by using the contiguity relation (57).

4.2. Index reduction

We need to prove the index reduction

$$\tilde{\tau}_{n-1,k+1,l+1} = K^{2N}\tilde{\tau}_{nkl}, \quad K = \frac{1-a}{1+a}. \tag{59}$$

To this end, we define

$$H(p) = \frac{(p-a)(p-b)}{p}, \quad \tilde{H}(q) = \frac{-q}{(q+a)(q+b)}, \tag{60}$$

it then follows

$$\begin{aligned} m_{i,j}^{(n-1,k+1,l+1)} &= A_iB_j\left(\frac{-q}{p}\right)\left(\frac{p-a}{q+a}\right)\left(\frac{p-b}{q+b}\right)m^{(nkl)} \\ &= A_iB_jH(p)\tilde{H}(q)m^{(nkl)} \\ &= \sum_{r=0}^{i}\sum_{s=0}^{j}\frac{1}{r!}\frac{1}{s!}H_r(p)\tilde{H}_s(q)m_{i-r,j-s}^{(nkl)} \end{aligned} \tag{61}$$

where functions $H_r(p)$ and $\tilde{H}_s(q)$ are defined as

$$H_r(p) = (p\partial_p)^r H(p), \quad \tilde{H}_s(q) = (q\partial_q)^s\tilde{H}(q). \tag{62}$$

Introducing two generators

$$\mathcal{L}_1 = \sum_{r=0}^{\infty}\frac{\zeta^r}{r!}(p\partial_p)^r, \quad \mathcal{L}_2 = \sum_{s=0}^{\infty}\frac{\lambda^s}{s!}(q\partial_q)^s, \tag{63}$$

since

$$\mathcal{L}_1 H(p) = H(e^{\zeta}p) = e^{\zeta}p - (a+b) + \frac{ab}{p}e^{-\zeta},$$

$$\mathcal{L}_2\tilde{H}(q) = \tilde{H}(e^{\lambda}q) = \frac{-1}{e^{\lambda}q + (a+b) + \frac{ab}{q}e^{-\lambda}},$$

we can show that $\mathcal{L}_1 H(p)$ and $\mathcal{L}_2\tilde{H}(q)$ are even functions of ζ and λ under the condition $p=q=1, b=1/a$, respectively. Thus $H_{2r-1}(p) = \tilde{H}_{2s-1}(q) = 0$ for all $r \geq 1$. Utilizing these results, we have the relation

$$m_{i,j}^{(n-1,k+1,l+1)}|_{p=q=1,b=1/a} = \sum_{r=0,\, r:even}^{i}\ \sum_{s=0,\, s:even}^{j}\frac{1}{r!}\frac{1}{s!}H_r(p)\tilde{H}_s(q)m_{i-r,j-s}^{(nkl)}|_{p=q=1,b=1/a}.$$

which leads to

$$\left(\tilde{m}_{2i-1,2j-1}^{(n-1,k+1,l+1)}|_{p=q=1,b=1/a}\right)_{1\leq i,j\leq N} = L\left(\tilde{m}_{2i-1,2j-1}^{(nkl)}|_{p=q=1,b=1/a}\right)U, \tag{64}$$

where L is a certain lower triangular matrix with $H_0(p) = H(p)|_{p=q=1,b=1/a}$ on the diagonal and U is a certain upper triangular matrix with $\tilde{H}_0(q) = \tilde{H}(q)|_{p=q=1,b=1/a}$ on the diagonal. Taking determinants to above equation, one obtains

$$\tilde{\tau}_{n-1,k+1,l+1} = [H_0(p)\tilde{H}_0(q)]^N\tilde{\tau}_{nkl} = K^{2N}\tilde{\tau}_{nkl}. \tag{65}$$

4.3. Rogue wave solutions to the dCSP equation

Based on the previous results, the general rogue wave solution to the fCSP equation is given by the following theorem

Theorem 2. *The general rogue wave solution of the fCSP equation is given by* $g = \tilde{\tau}_1, f = \tilde{\tau}_0$ *where* $\tilde{\tau}_n$ *is defined as*

$$\tilde{\tau}_n = \det_{1\le i,j\le N}\left(\tilde{m}^{(n)}_{2i-1,2j-1}\right)_{p=q=1,b=1/a}, \tag{66}$$

where the element of the determinant is defined as

$$\tilde{m}^{(n)}_{ij} = A_i B_j m^{(n)}, \tag{67}$$

with A_i *and* B_j *are defined in* (40) *and*

$$m^{(n)} = \frac{1}{p+q}\left(-\frac{p}{q}\right)^n e^{\frac{\beta(p+q)}{4}s+\left(\frac{1}{p-\mathrm{i}\alpha}+\frac{1}{q+\mathrm{i}\alpha}\right)\alpha y+\xi_0+\xi_0^*}. \tag{68}$$

Therefore, the rogue wave solution is of the parametric form

$$q = \frac{\beta}{2}\frac{g}{f}e^{\mathrm{i}(y+\gamma s/2)},$$

$$x = -\frac{\gamma}{2}y - \frac{\beta^2}{8}s - 2(\log f)_s, \quad t = -s.$$

where $\gamma = -\frac{\beta}{2}(\alpha - \alpha^{-1})$.

The rogue waves of the fCSP equation can also be expressed in terms of Schur polynomials by the following theorem. The elementary Schur polynomials $S_j(\boldsymbol{x})$ are defined via the generating function

$$\sum_{j=0}^{\infty} S_j(\boldsymbol{x})\lambda^j = \exp\left(\sum_{j=1}^{\infty} x_j\lambda^j\right),$$

or more explicitly,

$$S_0(\boldsymbol{x}) = 1, \quad S_1(\boldsymbol{x}) = x_1, \quad S_2(\boldsymbol{x}) = \frac{1}{2}x_1^2 + x_2,$$

and

$$S_j(\boldsymbol{x}) = \sum_{l_1+2l_2+\cdots+ml_m=j}\left(\prod_{j=1}^{m}\frac{x_j^{l_j}}{l_j!}\right),$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$.

Theorem 3. *The tau functions for the general rogue waves of the fCSP equation can also be expressed by* $g = \tilde{\tau}_1, f = \tilde{\tau}_0$ *where*

$$\tilde{\tau}_n = \det_{1\le i,j\le N}\left(m^{(n)}_{2i-1,2j-1}\right) \tag{69}$$

with

$$m^{(n)}_{2i-1,2j-1} = \sum_{\nu=0}^{\min(2i-1,2j-1)}\frac{1}{4^\nu}S_{2i-1-\nu}(\boldsymbol{x}^+(n)+\nu\boldsymbol{\zeta})S_{2j-1-\nu}(\boldsymbol{x}^-(n)+\nu\boldsymbol{\zeta}), \tag{70}$$

where vectors $\boldsymbol{x}^{\pm}(n) = \left(x_1^{\pm}(n), x_2^{\pm}(n), \ldots\right)$ *are defined by*

$$x_1^{\pm}(n) = \frac{\beta}{4}s - \frac{\alpha}{(1\mp \mathrm{i}\alpha)^2}y \pm n, \quad x_{2r}^{\pm} = 0, \tag{71}$$

$$x_{2r+1}^{+} = \alpha_{2r+1}s + \beta_{2r+1}y + a_{2r+1}, \tag{72}$$

$$x_{2r+1}^{-} = \alpha_{2r+1}s + \bar{\beta}_{2r+1}y + \bar{a}_{2r+1}, \tag{73}$$

with α_r, β_r $(r \ge 1)$ *and* $\boldsymbol{\zeta} = (0, \zeta_2, 0, \zeta_4, \ldots)$ *are coefficients from the following expansions*

$$\frac{1}{4}\beta(e^\lambda - 1) = \sum_{r=1}^{\infty}\alpha_r\lambda^r, \quad \frac{\alpha}{e^\lambda - \mathrm{i}\alpha} - \frac{\alpha}{1-\mathrm{i}\alpha} = \sum_{r=1}^{\infty}\beta_r\lambda^r,$$

$$\ln\left(\frac{2}{\lambda}\frac{e^\lambda - 1}{e^\lambda + 1}\right) = \sum_{r=1}^{\infty}\zeta_r\lambda^r,$$

and a_{2r+1} $(r = 1, 2, \ldots)$ *are arbitrary complex parameters.*

The proof can either be given from the expression in Theorem 1 or a delicate limiting process from the general breather solution. In Appendix D, we provide a proof from the breather solution.

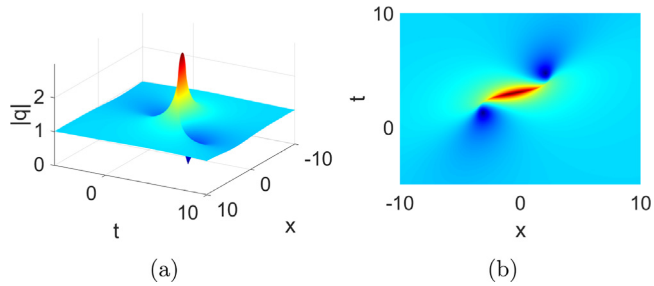

Fig. 3. (a) First-order rouge wave solution with parameter values ($\alpha = 1, \beta = 2$); (b) is the corresponding contour plot.

4.4. Dynamics of rogue waves

The fundamental rogue wave solution can be expressed by

$$q = \frac{\beta}{2}\frac{g}{f}e^{\mathbf{i}(y+\gamma s/2)} = -\frac{\beta}{2}\left(1 + \frac{G_1}{F_1}\right)e^{\mathbf{i}(y+\gamma s/2)} . \tag{74}$$

where

$$G_1 = -16\alpha^4 - 32\alpha^2 - 16 + \mathrm{i}64\alpha^2 y,$$
$$F_1 = \left(\alpha^2+1\right)^2\left(\beta^2 s^2 - 4\beta s + 8\right) + 8\left(\alpha^2 - 1\right)\alpha y(\beta s - 2) + 16\alpha^2 y^2 .$$

where $\gamma = -\frac{\beta}{2}(\alpha - \alpha^{-1})$.

One typical example is illustrated in Fig. 3.

The second order rogue wave solution can be expressed as

$$q = \frac{\beta}{2}\frac{g}{f}e^{\mathbf{i}(y+\gamma s/2)} = \frac{\beta}{2}e^{\mathbf{i}(y+\gamma s/2)}\left(1 + 12\frac{G_2}{F_2}\right) . \tag{75}$$

Here we omit the complicated expressions for G_2 and F_2. Instead,two examples of second-order wave solution are shown in Fig. 4(a)–(d). In Fig. 4(a)–(b), we have single peak and the maximum value of $|q|$ is 5 times the one of plane wave background. In (c)–(d) we observe a triangle pattern. Three examples of second-order wave solution are shown in Fig. 5(a)–(f). When the third order rogue wave forms one peak, the maximum amplitude is 7 times the background wave (see (a)–(b)). Depending on the choices of parameters, the pattern can exhibit either triangle type for large $a_3^{(0)}$ (see (c)–(d)) or pentagon type for large $a_5^{(0)}$ (see (e)–(f)).

5. Concluding remarks

The CSP equation is an analogue of the NLS equation in ultra-short pulse regime and is linked to the complex sine–Gordon equation through a hodograph transformation. In the present work, we constructed general breather and rogue wave solutions to the focusing CSP equation via the KP reduction method. The expressions for the first- and second-rogue wave solutions are given and illustrated in detail. The maximum value of the 1st- and 2nd-order rogue waves are 3 and 5 times of the background, respectively. Different types of the third order rogue waves are plotted, which follows the universal pattern of rogue waves of soliton equations revealed by Bo Yang and Jianke Yang [60] even there is a hodograph transformation involved. In other words, it exhibits triangle, pentagon and other geometric patterns based on the roots of the Yablonskii–Vorob'ev polynomial.

In addition to the CSP equation, we have also proposed semi-discrete CSP (sd-CSP) equation [64,65] and coupled complex short pulse (CCSP) equation [42], which have attracted attention recently. Various soliton solutions including the rogue wave solution to the sd-CSP equation have been studied by Darboux transformation method in [66]. Soliton solutions and inverse scattering transform for the CCSP equation were investigated in [67–69]. It is interesting to investigate the general rogue wave solutions in the sd-CSP and CCSP equations which are analogues of AL equation and Manakov system via the KP reduction method.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Bao-Feng Feng reports financial support was provided by National Science Foundation. Bao-Feng Feng reports a relationship with National Science Foundation that includes: funding grants. Bao-Feng Feng reports a relationship with US Department of the Air Force that includes: funding grants. Ruyun Ma reports a relationship with National Natural Science Foundation of China that includes: funding grants. Yujuan Zhang reports a relationship with National Natural Science Foundation of China that includes: funding grants.

Acknowledgments

We thank Mr. Changyan Shi for helping us with the graphs in this paper. BF's work is partially supported by National Science Foundation (NSF), United States of America under Grant No. DMS-1715991 and U.S. Department of Defense (DoD), Air Force for Scientific Research (AFOSR) under Grant No. W911NF2010276. RM's work is supported by National Natural Science Foundation of China (NSFC) under Grant No. 12061064. YZ's work is supported by NSFC, China under the Grant No. 61807025; Natural Science Foundation of Shannxi, China under Grant No. 2018JQ1065 and the Fundamental Research Funds for the Central Universities, China.

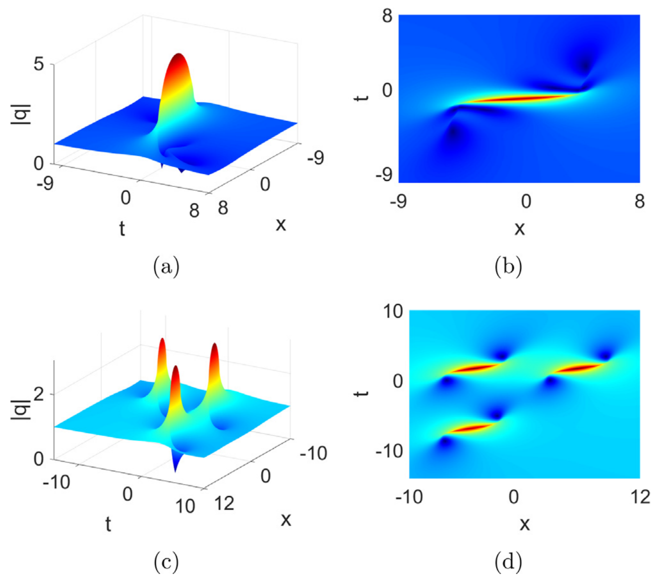

Fig. 4. Second-order rouge wave solutions with parameter values $\alpha = 1, \beta = 2$ and (a) $a_3 = -0.5$, (c) $a_3 = 50$, (b), (d) are the corresponding density plots of (a), (c) respectively.

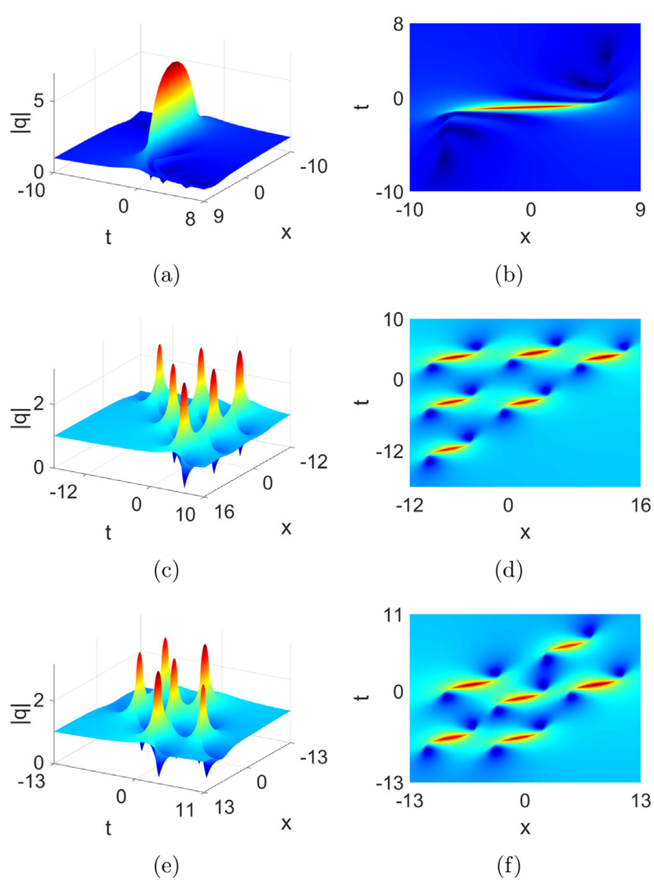

Fig. 5. Third-order rouge wave solutions with parameter values $\alpha = 1, \beta = 2$ and (a) $a_3 = -10.0,\ a_5 = -0.5$; (c) $a_3 = 1000,\ a_5 = 0.0$; (e) $a_3 = 0.0,\ a_5 = 2000$, (b), (d), (f) are the corresponding contour plots of (a), (c) and (e) respectively.

Appendix A

Proof of Lemma 1:

Proof. It is easily shown that m_{ij}^{nkl}, φ_i^{nkl}, ψ_j^{nkl} satisfy

$$\partial_{x_1} m_{ij}^{nkl} = \varphi_i^{nkl}\psi_j^{nkl}, \quad \partial_{x_{-1}} m_{ij}^{nkl} = -\varphi_i^{n-1,kl}\psi_j^{n+1,kl},$$

$$\partial_{t_a} m_{ij}^{nkl} = -\varphi_i^{n,k-1,l}\psi_j^{n,k+1,l},$$

$$m_{ij}^{n+1,kl} = m_{ij}^{nkl} + \varphi_i^{nkl}\psi_j^{n+1,kl}, \quad m_{ij}^{n,k+1,l} = m_{ij}^{nkl} + \varphi_i^{nkl}\psi_j^{n,k+1,l}.$$

Therefore the following differential and difference formulae hold for τ_{nkl},

$$\partial_{x_1}\tau_{nkl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{nkl} \\ -\psi_j^{nkl} & 0 \end{vmatrix}, \quad \partial_{x_{-1}}\tau_{nkl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n-1,kl} \\ \psi_j^{n+1,kl} & 0 \end{vmatrix},$$

$$a\partial_{t_a}\tau_{nkl} = \begin{vmatrix} m_{ij}^{nkl} & a\varphi_i^{n,k-1,l} \\ \psi_j^{n,k+1,l} & 0 \end{vmatrix}, \quad \tau_{n+1,kl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{nkl} \\ -\psi_j^{n+1,kl} & 1 \end{vmatrix},$$

$$\tau_{n-1,kl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n-1,kl} \\ \psi_j^{nkl} & 1 \end{vmatrix}, \quad \tau_{n,k+1,l} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{nkl} \\ -\psi_j^{n,k+1,l} & 1 \end{vmatrix},$$

$$\tau_{n+1,k-1,l} = \begin{vmatrix} m_{ij}^{nkl} & a\varphi_i^{n,k-1,l} \\ \psi_j^{n+1,kl} & 1 \end{vmatrix}, \quad \partial_{x_1}\tau_{n+1,kl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n+1,kl} \\ -\psi_j^{n+1,kl} & 0 \end{vmatrix},$$

$$(\partial_{x_1}+a)\tau_{n,k+1,l} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n+1,kl} \\ -\psi_j^{n,k+1,l} & a \end{vmatrix},$$

$$(\partial_{x_1}\partial_{x_{-1}}-1)\tau_{nkl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n-1,kl} & \varphi_i^{nkl} \\ \psi_j^{n+1,kl} & 0 & -1 \\ -\psi_j^{nkl} & -1 & 0 \end{vmatrix}, \tag{76}$$

$$(a\partial_{t_a}-1)\tau_{n+1,kl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{nkl} & a\varphi_i^{n,k-1,l} \\ -\psi_j^{n+1,kl} & 1 & -1 \\ \psi_j^{n,k+1,l} & -1 & 0 \end{vmatrix}, \tag{77}$$

$$(\partial_{x_1}(a\partial_{t_a}-1)-a)\tau_{n+1,kl} = \begin{vmatrix} m_{ij}^{nkl} & \varphi_i^{n+1,kl} & a\varphi_i^{n,k-1,l} \\ -\psi_j^{n+1,kl} & 0 & -1 \\ \psi_j^{n,k+1,l} & -a & 0 \end{vmatrix}. \tag{78}$$

Applying the Jacobi identity of determinants to the bordered determinants (76)–(78), the three bilinear equations (17)–(19) are satisfied. The bilinear equation (20) can be proved exactly in the same way as Eq. (19). □

Appendix B

The proof of Lemma 2

Proof. By defining

$$\varphi_i^{(n)}(k,l) = \sum_{m=1}^{2} a_{im} p_{im}^n (p_{im}-a)^k (p_{im}-b)^l e^{\xi_{im}},$$

$$\psi_j^{(n)}(k,l) = \sum_{r=1}^{2} b_{jr}(-q_{jr})^{-n}(-(q_{jr}+a))^{-k}(-(q_{jr}+b))^{-l} e^{\bar{\xi}_{jr}},$$

then it is easy to verify that $\left|\int \varphi_i^{(n)}(k,l)\psi_j^{(n)}(k,l)dx_1\right|_{1\le i,j\le N}$ is nothing but τ_{nkl} defined above. Moreover, $\varphi_i^{(n)}(k,l)$ and $\psi_j^{(n)}(k,l)$ satisfy the linear dispersion relations (11)–(16). Therefore, it follows that τ_{nkl} satisfies the bilinear equations (17)–(20). □

Appendix C

The proof of Lemma 3

Proof. Notice that τ_{nkl} can be alternatively expressed as

$$\tau_{nkl}=C\prod_{i=1}^{N}e^{\xi_{i2}+\bar{\xi}_{i2}}\left|m'_{ij}(nkl)\right|$$

where

$$C=\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{a_{i2}b_{j2}}{p_{i2}+q_{j2}}\left(-\frac{p_{i2}}{q_{j2}}\right)^{n}\left(-\frac{p_{i2}-a}{q_{j2}+a}\right)^{k}\left(-\frac{p_{i2}-b}{q_{j2}+b}\right)^{l},$$

and

$$\begin{aligned}m'_{ij}(nkl)=&\ 1+\frac{a_{i1}b_{j1}}{a_{i2}b_{j2}}\frac{p_{i2}+q_{j2}}{p_{i1}+q_{j1}}f^{nkl}(p_{i1},p_{i2})g^{nkl}(q_{j1},q_{j2})e^{\xi_{i1}-\xi_{i2}+\bar{\xi}_{j1}-\bar{\xi}_{j2}}\\&+\frac{b_{j1}}{b_{j2}}\frac{p_{i2}+q_{j2}}{p_{i2}+q_{j1}}g^{nkl}(q_{j1},q_{j2})e^{\bar{\xi}_{j1}-\bar{\xi}_{j2}}+\frac{a_{i1}}{a_{i2}}\frac{p_{i2}+q_{j2}}{p_{i1}+q_{j2}}f^{nkl}(p_{i1},p_{i2})e^{\xi_{i1}-\xi_{i2}},\end{aligned}$$

with

$$f^{nkl}(p_{i1},p_{i2})=\left(\frac{p_{i1}}{p_{i2}}\right)^{n}\left(\frac{p_{i1}-a}{p_{i2}-a}\right)^{k}\left(\frac{p_{i1}-b}{p_{i2}-b}\right)^{l},$$

$$g^{nkl}(q_{j1},q_{j2})=\left(\frac{q_{j1}}{q_{j2}}\right)^{-n}\left(\frac{q_{j1}+a}{q_{j2}+a}\right)^{-k}\left(\frac{q_{j1}+b}{p_{j2}+b}\right)^{-l},$$

$$\begin{aligned}\xi_{i1}-\xi_{i2}&=(p_{i1}-p_{i2})(x_1-\frac{x_{-1}}{p_{i1}p_{i2}})+\left(\frac{1}{p_{i1}-a}-\frac{1}{p_{i2}-a}\right)t_a+\left(\frac{1}{p_{i1}-b}-\frac{1}{p_{i2}-b}\right)t_b,\\ \bar{\xi}_{j1}-\bar{\xi}_{j2}&=(q_{j1}-q_{j2})(x_1-\frac{x_{-1}}{q_{j1}q_{j2}})+\left(\frac{1}{q_{j1}+a}-\frac{1}{q_{j2}+a}\right)t_a+\left(\frac{1}{q_{j1}+b}-\frac{1}{q_{j2}+b}\right)t_b.\end{aligned}$$

So if we impose the reduction condition $p_{i2}=\frac{1}{p_{i1}}$ and $q_{i2}=\frac{1}{q_{i1}}$ then

$$\left(\frac{1}{p_{i1}}-\frac{1}{p_{i2}}\right)=-(p_{i1}-p_{i2}),\quad \left(\frac{1}{q_{i1}}-\frac{1}{q_{i2}}\right)=-(q_{i1}-q_{i2}). \tag{79}$$

Obviously

$$(\partial_{x_1}+\partial_{x_{-1}})m'_{ij}(nkl)=0 \tag{80}$$

which implies

$$(\partial_{x_1}+\partial_{x_{-1}})\tau_{nkl}=C_1\tau_{nkl}\,, \tag{81}$$

Moreover, the conditions $p_{i2}=\frac{1}{p_{i1}}$, $q_{i2}=\frac{1}{q_{i1}}$, $b=\frac{1}{a}$ give

$$a^2\left(\frac{1}{p_{i1}-a}-\frac{1}{p_{i2}-a}\right)=\left(\frac{1}{p_{i1}-b}-\frac{1}{p_{i2}-b}\right), \tag{82}$$

and

$$a^2\left(\frac{1}{q_{i1}+a}-\frac{1}{q_{i2}+a}\right)=\left(\frac{1}{q_{i1}+b}-\frac{1}{q_{i2}+b}\right), \tag{83}$$

thus

$$(a^2\partial_{t_a}-\partial_{t_b})m'_{ij}(nkl)=0 \tag{84}$$

which implies

$$(a^2\partial_{t_a}-\partial_{t_b})\tau_{nkl}=C_2\tau_{nkl}\,, \tag{85}$$

In the last, the conditions $p_{i2}=\frac{1}{p_{i1}}$, $q_{i2}=\frac{1}{q_{i1}}$, $b=\frac{1}{a}$ imply

$$\left(\frac{p_{i2}}{p_{i1}}\right)\left(\frac{p_{i1}-a}{p_{i2}-a}\right)\left(\frac{p_{i1}-b}{p_{i2}-b}\right)=1,$$
$$\left(\frac{q_{i2}}{q_{i1}}\right)\left(\frac{q_{i1}+a}{q_{i2}+a}\right)\left(\frac{q_{i1}+b}{p_{i2}+b}\right)=1$$

which leads

$$f^{nkl}=f^{n-1,k+1,l+1},\quad g^{nkl}=g^{n-1,k+1,l+1}, \tag{86}$$

Thus, the reduction relation (23)$\tau_{n-1,k+1,l+1}=\tau_{n,k,l}$ also holds. □

Appendix D

Proof of Theorem 3:

Proof. The breather solution to the fCSP equation can be written into a slight different form

$$\tau_n = \left|m_{ij}^n\right|_{1\le i,j\le N} \tag{87}$$

where

$$m_{ij}^n = \sum_{m=1}^{2}\sum_{r=1}^{2} \frac{a_{im}b_{jr}(p_{im}+1)(q_{jr}+1)}{p_{im}+q_{jr}} \left(-\frac{p_{im}}{q_{jr}}\right)^n e^{\xi_{im}+\bar{\xi}_{jr}}, \tag{88}$$

under the conditions $p_{i1}p_{i2}=1$, $q_{im}=p_{im}^*$ $(m=1,2)$ with

$$\xi_{im} = \frac{1}{4}p_{im}\beta s + \frac{\alpha y}{p_{im}-\mathrm{i}\alpha}, \quad i=1,\dots,N.$$

The rogue wave solutions can be obtained by taking limit of $p_{11}, p_{12} \to 1$, $p_{21}, p_{22} \to 1$ etc. The detailed procedure is as follows. Let

$$\ln p_{i1} = \lambda_i, \quad \ln p_{i2} = -\lambda_i\,, \tag{89}$$

since

$$\begin{aligned}
\frac{(p_{i1}+1)(q_{j1}+1)}{2(p_{i1}+q_{j1})} &= \frac{1}{1-\frac{(p_{i1}-1)(q_{j1}-1)}{(p_{i1}+1)(q_{j1}+1)}} \\
&= \sum_{\nu=0}^{\infty}\left(\frac{(p_{i1}-1)(q_{j1}-1)}{(p_{i1}+1)(q_{j1}+1)}\right)^{\nu} \\
&= \sum_{\nu=0}^{\infty}\left(\frac{\lambda_i\lambda_j^*}{4}\right)^{\nu}\left(\frac{4}{\lambda_i\lambda_j^*}\frac{(p_{i1}-1)(q_{j1}-1)}{(p_{i1}+1)(q_{j1}+1)}\right)^{\nu},
\end{aligned}$$

then, by multiplying

$$\frac{1}{2}(-1)^n e^{-(\xi+\bar{\xi})}, \quad \xi = \frac{1}{4}\beta s + + \frac{\alpha y}{1-\mathrm{i}\alpha}, \quad \bar{\xi} = \frac{1}{4}\beta s + \frac{\alpha y}{1+\mathrm{i}\alpha},$$

which will not affect the solution, we can expand the first term in m_{ij}^n by

$$\begin{aligned}
&\frac{(p_{i1}+1)(q_{j1}+1)}{2(p_{i1}+q_{j1})}\left(\frac{p_{i1}}{q_{j1}}\right)^n e^{\xi_{i1}-\xi+\bar{\xi}_{j1}-\bar{\xi}} \\
&= \sum_{k,l=0}^{\infty}\left(\sum_{\nu=0}^{min(k,l)}\left(\frac{1}{4}\right)^{\nu} S_{k-\nu}(\boldsymbol{x}^+(n)+\nu\boldsymbol{\zeta})S_{l-\nu}(\boldsymbol{x}^-(n)+\nu\boldsymbol{\zeta})\right)\lambda_i^k(\lambda_j^*)^l\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{(p_{i2}+1)(q_{j1}+1)}{2(p_{i2}+q_{j1})}\left(\frac{p_{i2}}{q_{j1}}\right)^n e^{\xi_{i2}-\xi+\bar{\xi}_{j1}-\bar{\xi}} \\
&= \sum_{k,l=0}^{\infty}\left(\sum_{\nu=0}^{min(k,l)}\left(\frac{1}{4}\right)^{\nu} S_{k-\nu}(\boldsymbol{x}^+(n)+\nu\boldsymbol{\zeta})S_{l-\nu}(\boldsymbol{x}^-(n)+\nu\boldsymbol{\zeta})\right)(-\lambda_i)^k(\lambda_j^*)^l
\end{aligned}$$

$$\begin{aligned}
&\frac{(p_{i1}+1)(q_{j2}+1)}{2(p_{i1}+q_{j2})}\left(\frac{p_{i1}}{q_{j2}}\right)^n e^{\xi_{i1}-\xi+\bar{\xi}_{j2}-\bar{\xi}} \\
&= \sum_{k,l=0}^{\infty}\left(\sum_{\nu=0}^{min(k,l)}\left(\frac{1}{4}\right)^{\nu} S_{k-\nu}(\boldsymbol{x}^+(n)+\nu\boldsymbol{\zeta})S_{l-\nu}(\boldsymbol{x}^-(n)+\nu\boldsymbol{\zeta})\right)\lambda_i^k(-\lambda_j^*)^l
\end{aligned}$$

$$\begin{aligned}
&\frac{(p_{i2}+1)(q_{j2}+1)}{2(p_{i2}+q_{j2})}\left(\frac{p_{i2}}{q_{j2}}\right)^n e^{\xi_{i2}-\xi+\bar{\xi}_{j2}-\bar{\xi}} \\
&= \sum_{k,l=0}^{\infty}\left(\sum_{\nu=0}^{min(k,l)}\left(\frac{1}{4}\right)^{\nu} S_{k-\nu}(\boldsymbol{x}^+(n)+\nu\boldsymbol{\zeta})S_{l-\nu}(\boldsymbol{x}^-(n)+\nu\boldsymbol{\zeta})\right)(-\lambda_i)^k(-\lambda_j^*)^l\,.
\end{aligned}$$

Adding all four terms up, we have

$$\begin{pmatrix}\lambda_i d_i & \lambda_i s_i & \lambda_i^3 d_i & \lambda_i^3 s_i & \cdots\end{pmatrix}\begin{pmatrix}S_0(\boldsymbol{x}_0^+) & & & & \mathbf{0}\\ S_1(\boldsymbol{x}_0^+) & S_0(\boldsymbol{x}_1^+) & & & \\ S_2(\boldsymbol{x}_0^+) & S_1(\boldsymbol{x}_1^+) & S_0(\boldsymbol{x}_2^+) & & \\ S_3(\boldsymbol{x}_0^+) & S_2(\boldsymbol{x}_1^+) & S_1(\boldsymbol{x}_2^+) & S_0(\boldsymbol{x}_3^+) & \\ \vdots & \vdots & \vdots & & \ddots\end{pmatrix}$$

$$\times\begin{pmatrix}1 & & & & \mathbf{0}\\ & 1/4 & & & \\ & & (1/4)^2 & & \\ & & & (1/4)^3 & \\ \mathbf{0} & & & & \ddots\end{pmatrix}$$

$$\times\begin{pmatrix}S_0(\boldsymbol{x}_0^-) & S_1(\boldsymbol{x}_0^-) & S_2(\boldsymbol{x}_0^-) & S_3(\boldsymbol{x}_0^-) & \cdots\\ & S_0(\boldsymbol{x}_1^-) & S_1(\boldsymbol{x}_1^-) & S_2(\boldsymbol{x}_1^-) & \cdots\\ & & S_0(\boldsymbol{x}_2^-) & S_1(\boldsymbol{x}_2^-) & \cdots\\ & & & S_0(\boldsymbol{x}_3^-) & \\ \mathbf{0} & & & & \ddots\end{pmatrix}\begin{pmatrix}\lambda_j^* d_j^*\\ \lambda_j^{*2} s_j^*\\ \lambda_j^{*3} d_j^*\\ \lambda_j^{*3} s_j^*\\ \vdots\end{pmatrix}.$$

Here $a_{i1} - a_{i2} = \lambda_i d_i$ and $a_{i1} + a_{i2} = s_i$ and we use the abbreviations $S_j(\boldsymbol{x}_\nu^+)= S_j(\boldsymbol{x}^+(n) + \nu\boldsymbol{\zeta})$, $S_j(\boldsymbol{x}_\nu^-)= S_j(\boldsymbol{x}^-(n) + \nu\boldsymbol{\zeta})$. Therefore τ_n can be expressed by

$$\left|\begin{pmatrix}\lambda_1 d_1 & \lambda_1 s_1 & \lambda_1^3 d_1 & \lambda_1^3 s_1 & \cdots\\ \lambda_2 d_2 & \lambda_2 s_2 & \lambda_2^3 d_2 & \lambda_2^3 s_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \\ \lambda_N d_N & \lambda_N s_N & \lambda_N^3 d_N & \lambda_N^3 s_N & \cdots\end{pmatrix}\begin{pmatrix}S_0(\boldsymbol{x}_0^+) & & & & \mathbf{0}\\ S_1(\boldsymbol{x}_0^+) & S_0(\boldsymbol{x}_1^+) & & & \\ S_2(\boldsymbol{x}_0^+) & S_1(\boldsymbol{x}_1^+) & S_0(\boldsymbol{x}_2^+) & & \\ S_3(\boldsymbol{x}_0^+) & S_2(\boldsymbol{x}_1^+) & S_1(\boldsymbol{x}_2^+) & S_0(\boldsymbol{x}_3^+) & \\ \vdots & \vdots & \vdots & & \ddots\end{pmatrix}\right.$$

$$\times\begin{pmatrix}1 & & & & \mathbf{0}\\ & 1/4 & & & \\ & & (1/4)^2 & & \\ & & & (1/4)^3 & \\ \mathbf{0} & & & & \ddots\end{pmatrix}\begin{pmatrix}S_0(\boldsymbol{x}_0^-) & S_1(\boldsymbol{x}_0^-) & S_2(\boldsymbol{x}_0^-) & S_3(\boldsymbol{x}_0^-) & \cdots\\ & S_0(\boldsymbol{x}_1^-) & S_1(\boldsymbol{x}_1^-) & S_2(\boldsymbol{x}_1^-) & \cdots\\ & & S_0(\boldsymbol{x}_2^-) & S_1(\boldsymbol{x}_2^-) & \cdots\\ & & & S_0(\boldsymbol{x}_3^-) & \\ \mathbf{0} & & & & \ddots\end{pmatrix}$$

$$\left.\times\begin{pmatrix}\lambda_1^* d_1^* & \lambda_2^* d_2^* & \cdots & \lambda_N^* d_N^*\\ \lambda_1^* s_1^* & \lambda_2^* s_2^* & \cdots & \lambda_N^* s_N^*\\ \lambda_1^{*3} d_1^* & \lambda_2^{*3} d_2^* & \cdots & \lambda_N^{*3} d_N^*\\ \lambda_1^{*3} s_1^* & \lambda_2^{*3} s_2^* & \cdots & \lambda_N^{*3} s_N^*\\ \vdots & \vdots & & \vdots\end{pmatrix}\right|.$$

Now let us take

$$a_{i1} = \frac{1}{2}\Big(1 + \sum_{\nu=1}^{i} p_i^{2\nu-1}\Big), \quad a_{i2} = \frac{1}{2}\Big(1 - \sum_{\nu=1}^{i} p_i^{2\nu-1}\Big).$$

Then we have $d_i = \sum_{\nu=1}^{i} p_i^{2\nu-2}$ and $s_i = 1$. Therefore, the above τ_n is $O(\lambda_1\lambda_1^*\lambda_2\lambda_2^*\cdots\lambda_N\lambda_N^*)$ as $\lambda_i \to 0$ for $1 \le i \le N$. In order to take the lowest order in λ_1, we consider the limit, $\tilde{\tau}(n) := \lim_{\lambda_1\to 0}\tau(n)/(\lambda_1\lambda_1^*)$. In this limit, the leading order becomes $O((\lambda_2\lambda_2^*\cdots\lambda_N\lambda_N^*)^3)$, thus for picking up the lowest order in λ_2, we take the limit, $\lim_{\lambda_2\to 0}\tilde{\tau}(n)/(\lambda_2\lambda_2^*)^3$. So the leading order becomes $O((\lambda_3\lambda_3^*\cdots\lambda_N\lambda_N^*)^5)$. Repeating this procedure, finally we obtain the $\tilde{\tau}_n$ function of rogue wave solution from that of the breather solution. □

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