



Integrable semi-discretization of a modified short wave equation

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ABSTRACT

In this paper, we are concerned with one of the integrable equations proposed by Hone et al. (2018). We firstly show that it can be derived from the Tzitzeica equation through hodograph or reciprocal transformations and it is linked to the short wave limits of the DP and Novikov equations, respectively, through Miura transformations. Based on the semi-discrete analogue of CKP-Toda hierarchy, we construct the integrable semi-discretization of what we call the modified short wave equation and provide its multi-soliton solution.

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1. Introduction

Recently, Hone et al. proposed two new integrable equations [1]

$$u_{tx} = u - \frac{1}{2}uu_x - 2uu_x^2 - u^2u_{xx}, \quad (1)$$

$$u_{tx} = 3u - uu_{xx} - \frac{1}{4}u_x^2. \quad (2)$$

The first equation can be rewritten as

$$m_t + u^2m_x + 2uu_xm = 0, \quad m = (1 - 2u_x)^{1/3}(1 + u_x)^{2/3}. \quad (3)$$

while the second one can be recast into

$$m_t + um_x + \frac{3}{2}u_xm = 0, \quad m = 2 - u_{xx}. \quad (4)$$

by differentiating with respect to x on both sides. Eq. (4) can be viewed as a short wave limit of the Novikov equation [2,3], while another equation among the list in [1]

$$m_t + um_x + 3u_xm = 0, \quad m = 1 - u_{xx}, \quad (5)$$

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can be viewed as the short wave limit of the Depasperi–Procesi equation [4]. As shown in subsequent section, there is a Miura transformation between the solution of Eq. (3) and the ones of Eqs. (5) and (4). Therefore, we name Eq. (3) as modified short wave (mSW) equation similar to the relation between the KdV equation and modified KdV equation.

Discrete integrable systems are more fundamental and universal than continuous integrable systems, recently, much attention has been paid to the study of discrete integrable systems [5]. The main purpose of the present paper is to construct integrable semi-discretization of the mSW equation (3). In Section 2, we show that the mSW equation can be derived from the Tzitzeica equation through hodograph or reciprocal transformations and it is linked to the short wave limits of the DP and Novikov equations through Miura transformations. In Section 3, based on a semi-discrete analogue of a set of bilinear equations originated from the period 3 reduction of the C -type two-dimensional Toda lattice, we derive an integrable semi-discrete analogue of the mSW equation (4) and provide its N -soliton solution in the form of determinant. We conclude the paper in Section 4 by some remarks and comments.

2. Derivation of the modified short wave solution

The bilinear equations of the two-dimensional Toda lattice (2DTL) [6,7] may be written as

$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right)\tau_n \cdot \tau_n = -\tau_{n-1}\tau_{n+1}. \quad (6)$$

where D is the Hirota D -operator defined by [8]

$$D_s^n D_y^m f \cdot g = \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m f(y, s)g(y', s')|_{y=y', s=s'}.$$

As shown in [9], through a two-step reductions: C -type and period 3 reductions, the tau-function sequence becomes a cycle of 3 with the pattern $\{\tau_0, \tau_1, \tau_1\}$ such that we end up with only a pair of bilinear equations

$$\left(\frac{1}{2}D_s D_y - 1\right)G \cdot G = -F^2, \quad \left(\frac{1}{2}D_s D_y - 1\right)F \cdot F = -FG. \quad (7)$$

by defining $\tau_0 = G$, $\tau_1 = F$ and $x_1 = y$, $x_{-1} = s$. In addition, by defining $r = G/F$, above two bilinear equations are equivalent to the following Tzitzeica equation [10]

$$(\ln r)_{ys} = r - r^{-2}, \quad (8)$$

which was originally derived in the context of geometry. About seventy years later, it was rediscovered in the context of soliton theory [11].

The solution of the Tzitzeica equation, or the tau-function of period 3 reduction of CKP-Toda hierarchy is given by the following determinant with even order

$$\tau_n = \left| C_{ij} + \frac{1}{p_i + p_j} \left(-\frac{p_i}{p_j}\right)^n e^{\xi_i + \xi_j} \right|_{1 \leq i, j \leq 2N}, \quad (9)$$

where

$$C_{ij} = \delta_{j, 2N+1-i} c_i, \quad c_{2N+1-i} = c_i, \quad \xi_i = p_i y + \frac{1}{p_i} s + \xi_0, \quad p_{2N+1-i}^3 + p_i^3 = 0, \quad (1 \leq i \leq N). \quad (10)$$

It is noted that $C_{ij} = C_{ji}$. Furthermore, it is shown in [9,12] that the tau function $F = \tau_1$ in 2DTL can be expressed as a square of a pfaffian, i.e.,

$$F = cf^2, \quad (11)$$

where

$$c = \left(\prod_{i=1}^{2N} 2p_i \right)^{-1}, \quad f = \text{Pf}(1, 2, \dots, 2N)$$

with

$$(i, j) = \delta_{j, 2N+1-i} 2c_i \frac{p_{2N+1-i}^2}{p_i} + \frac{p_i - p_j}{p_i + p_j} e^{\xi_i + \xi_j}.$$

It is also shown in [12] that G and f satisfy the following bilinear equation

$$D_{1,2,s}^2 G \cdot f = 0 \quad (12)$$

where $D_{m,n,s}$ is the generalized Hirota D -operator defined by

$$D_{m,n,s}^j f(s) \cdot g(s) = \frac{\partial^j}{\partial x^j} f(s + mx) g(s - nx) \Big|_{x=0}.$$

Based on the results listed above, we could derive the SW equation (3), which is given by the following theorem.

Theorem 2.1. *The mSW equation (3) is derived from a set of bilinear equations (7) and (12) through the dependent variable transformation*

$$u = \left(\ln \frac{G}{F} \right)_s, \quad (13)$$

and a hodograph (reciprocal) transformation

$$x = 3y - (\ln GF^2)_s, \quad t = s. \quad (14)$$

Furthermore, the multi-soliton solution is given by the tau-function $G = \tau_0$ and $F = \tau_1$ where τ_n is given in (9) with constraints given by (10)

Proof. The pair of bilinear equations (7) can be rewritten as

$$(\ln F)_{sy} = 1 - \frac{G}{F}, \quad (\ln G)_{sy} = 1 - \frac{F^2}{G^2}. \quad (15)$$

Subtracting above two equations, one obtains exactly the Tzitzeica equation (8). Notice that Eq. (12) is equivalent to

$$4f_{ss}G - 4f_s g_s + f G_{ss} = 0, \quad (16)$$

which can be rewritten as

$$4 \frac{f_{ss}}{f} - 4 \frac{f_s}{f} \frac{G_s}{G} + \frac{G_{ss}}{G} = 0, \quad (17)$$

by dividing Gf on both sides. From the hodograph (reciprocal) transformation (14), we have

$$\frac{\partial x}{\partial y} = 3 - (\ln GF^2)_{sy} = 2r + r^{-2} = \rho^{-1}, \quad (18)$$

such that $\partial x = \rho \partial y$. From the transformation (13), the Tzitzeica equation actually implies

$$u_y = r - r^{-2}. \quad (19)$$

Eliminating r from Eqs. (18) and (19) by a little bit tricky, one obtains

$$\rho^3 = \frac{1}{27} (1 - 2\rho u_y) (1 + \rho u_y)^2 = \frac{1}{27} (1 - 2u_x) (1 + u_x)^2. \quad (20)$$

Next let us calculate $\frac{\partial x}{\partial s}$ by inserting Eq. (17)

$$\frac{\partial x}{\partial s} = -(\ln GF^2)_{ss} = -\left(\frac{G_{ss}}{G} - \frac{G_s^2}{G^2}\right) - 4\left(\frac{f_{ss}}{f} - \frac{f_s^2}{f^2}\right) = \frac{G_s^2}{G^2} - 4\frac{f_s}{f}\frac{G_s}{G} + 4\frac{f_s^2}{f^2} = ((\ln G)_s - (\ln F)_s)^2 = u^2. \quad (21)$$

It follows

$$\partial_s = \partial_t + u^2 \partial_x. \quad (22)$$

The compatibility condition $\frac{\partial}{\partial s} \frac{\partial x}{\partial y} = \frac{\partial}{\partial y} \frac{\partial x}{\partial s}$ gives

$$(\rho^{-1})_s = (u^2)_y, \quad (23)$$

by referring to (18) and (21). Eq. (23) can be written

$$(\ln \rho)_s = -2\rho u u_y = -2u u_x. \quad (24)$$

In the last, by defining $m = (1 - 2u_x)^{1/3}(1 + u_x)^{2/3}$ and using (20) and (22), Eq. (24) is converted into

$$(\partial_t + u^2 \partial_x) \ln m = -2u u_x, \quad (25)$$

which is nothing but the mSW equation (3). $\square$

Remark 2.2. We should point out here that the bilinear equation (12) implies a Miura transformation which links the solution of the mSW equation (3) to the solutions for the short wave limits of the DP equation (5) and the Novikov equation (4), respectively. In addition, the multi-soliton solution given here agrees with the one given by Matsuno [13].

Lemma 2.3. The generalized bilinear equation (12) implies the Miura transformation

$$u_{SWDP} = \frac{1}{3}(u^2 + u_s), \quad u_{SWNK} = \frac{1}{3}(u^2 - 2u_s), \quad (26)$$

where u_{SWNK} and u_{SWDP} represent the solutions to Eqs. (4) and (5) respectively, which are given by

$$u_{SWDP} = -(\ln F)_{ss}, \quad u_{SWNK} = -(\ln G)_{ss}. \quad (27)$$

Proof.

$$u = \left(\ln \frac{G}{F}\right)_s = (\ln G - 2 \ln f)_s = \frac{G_s}{G} - 2\frac{f_s}{f}. \quad (28)$$

By using the relation (17), it is easy to show that

$$-u^2 - u_s = 6(\ln f)_{ss} = 3(\ln F)_{ss}, \quad (29)$$

$$-u^2 + 2u_s = 3(\ln G)_{ss}, \quad (30)$$

which gives the Miura transformation (27). $\square$

3. Integrable semi-discretization of the mSW equation

In [14], we have constructed a semi-discrete analogue of bilinear equations (7) and (11) for the period 3 reduction of the CKP-Toda (BKP-Toda) hierarchy, which is of the form

$$(D_s - 2b)G_{l+1} \cdot G_l = -2bF_l^2, \quad (31)$$

$$(D_s - b)f_{l+1} \cdot f_l = -bc'G_{l+1}, \quad (32)$$

$$f_l^2 = c'F_l, \quad (33)$$

under the constraint

$$p_i^3(1 - b^2 p_{2N+1-i}^2) + p_{2N+1-i}^3(1 - b^2 p_i^2) = 0. \quad (34)$$

Here b is a real constant and c' is a real constant. G_l and F_l are two $2N \times 2N$ Gram-type determinants defined by

$$G_l = \left| m_{ij}(l) \right|_{1 \leq i, j \leq 2N}, \quad F_l = \left| m'_{ij}(l) \right|_{1 \leq i, j \leq 2N}, \quad (35)$$

where

$$m_{ij}(l) = C_{ij} + \frac{1}{p_i + p_j} \varphi_i^{(0)}(l) \varphi_j^{(0)}(l), \quad m'_{ij}(l) = C_{ij} - \frac{p_j}{p_i} \frac{1}{p_i + p_j} \frac{1 + bp_i}{1 - bp_j} \varphi_i^{(0)}(l) \varphi_j^{(0)}(l),$$

with

$$C_{ij} = \delta_{j, 2N+1-i} c_i, \quad c_{2N+1-i} = c_i, \quad \varphi_i^{(n)}(l) = p_i^n \left(\frac{1 + bp_i}{1 - bp_i} \right)^l e^{\xi_i}, \quad \xi_i = p_i^{-1} s + \xi_{i0}.$$

f_l is a pfaffian defined by $f_l = \text{Pf}(1, 2, \dots, 2N)_l$ whose elements are

$$\text{Pf}(i, j)_l = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \varphi_i^{(0)}(l) \varphi_j^{(0)}(l).$$

It is shown in [14] that $C_{ij} = C_{ji}$ and $c_{ij} = -c_{ji}$. It should be pointed out that, by fixing l , the following bilinear equation

$$D_{1,2,s}^2 G_l \cdot f_l = 0 \quad (36)$$

holds same as Eq. (12). Based on above results, we can construct integrable semi-discretization for the mSW equation, which is given by the following theorem.

Theorem 3.1. *The semi-discrete analogue of the modified short wave equation (3)*

$$\begin{cases} \frac{dm_l}{ds} = \frac{m_l}{3} \left(u_{l+1} - u_l - 2 \frac{u_{l+1}^2 - u_l^2}{\delta_l} - \frac{u_l^2 - u_{l-1}^2}{\delta_{l-1}} \right), \\ \frac{d\delta_l}{ds} = u_{l+1}^2 - u_l^2, \end{cases} \quad (37)$$

is determined from the bilinear equations (31)–(33) through discrete hodograph transformation

$$x_l = 6lb - (\ln G_l F_l^2)_s, \quad t = s \quad (38)$$

and a dependent variable transformation

$$u_l = \left(\ln \frac{G_l}{F_l} \right)_s. \quad (39)$$

Here

$$\delta_l = x_{l+1} - x_l, \quad \Delta u_l = \frac{u_{l+1} - u_l}{2b}, \quad (40)$$

$$m_l = \left(\frac{G_{l+1}}{G_l} \frac{f_l}{f_{l+1}} (1 + \rho_l \Delta u_l) (1 + \rho_{l-1} \Delta u_{l-1}) (1 - 2\rho_l \Delta u_l) \right)^{\frac{1}{3}}. \quad (41)$$

Moreover, the multi-soliton solution is given by the determinants G_l and F_l given in (35).

Proof. First, we rewrite Eqs. (31) and (32) into

$$\left(\ln \frac{G_{l+1}}{G_l} \right)_s - 2b = -2b \frac{F_l^2}{G_{l+1} G_l}, \quad (42)$$

and

$$\left(\ln \frac{f_{l+1}}{f_l} \right)_s - b = -bc' \frac{G_{l+1}}{f_{l+1}f_l}, \quad (43)$$

respectively. Eq. (43) can also be rewritten as

$$\left(\ln \frac{F_{l+1}}{F_l} \right)_s - 2b = -2bc' \frac{G_{l+1}}{f_{l+1}f_l}. \quad (44)$$

It follows from (38) and (40) that the nonuniform mesh δ_l is

$$\delta_l = 6b - \left(\ln \frac{G_{l+1}F_{l+1}^2}{G_lF_l^2} \right)_s = 2b \frac{F_l^2}{G_{l+1}G_l} + 4bc' \frac{G_{l+1}}{f_{l+1}f_l} \quad (45)$$

with the use of (42) and (44).

Introducing an auxiliary variable $r_l = G_l/F_l$, Eq. (45) becomes

$$\frac{\delta_l}{2b} = 2 \frac{f_{l+1}}{f_l} r_{l+1} + \frac{f_l^2}{f_{l+1}^2} \frac{1}{r_l r_{l+1}}. \quad (46)$$

Differentiating Eq. (45) with respect to s , one obtains

$$\frac{d\delta_l}{ds} = - \left(\ln \frac{G_{l+1}F_{l+1}^2}{G_lF_l^2} \right)_{ss} = u_{l+1}^2 - u_l^2 \quad (47)$$

by referring to (36) similar to the continuous case.

Subtracting (44) from (42), we get

$$\left(\ln \frac{G_{l+1}F_l}{G_lF_{l+1}} \right)_s = 2bc' \frac{G_{l+1}}{f_{l+1}f_l} - 2b \frac{F_l^2}{G_{l+1}G_l}, \quad (48)$$

which is

$$\left(\ln \frac{r_{l+1}}{r_l} \right)_s = 2b \left(\frac{f_{l+1}}{f_l} r_{l+1} - \frac{f_l^2}{f_{l+1}^2} \frac{1}{r_l r_{l+1}} \right). \quad (49)$$

By using the variable transformation (39), the above equation is actually

$$\Delta u_l = \frac{u_{l+1} - u_l}{2b} = \frac{f_{l+1}}{f_l} r_{l+1} - \frac{f_l^2}{f_{l+1}^2} \frac{1}{r_l r_{l+1}}. \quad (50)$$

Combining (46) with (50), we obtain

$$r_{l+1} = \frac{1}{3} \frac{f_l}{f_{l+1}} \left(\frac{\delta_l}{2b} + \Delta u_l \right). \quad (51)$$

Defining $\rho_l^{-1} = \frac{\delta_l}{2b}$ and substituting Eq. (51) and its alternative form by shifting from $l+1$ to l into Eq. (46), one obtains

$$\rho_l^2 \rho_{l-1} = \frac{1}{27} \frac{f_{l+1}f_{l-1}}{f_l^2} (1 + \rho_l \Delta u_l)(1 + \rho_{l-1} \Delta u_{l-1})(1 - 2\rho_l \Delta u_l), \quad (52)$$

or

$$m_l^3 = \frac{216b^3}{\delta_l^2 \delta_{l-1}} \frac{r_{l+1}}{r_l}. \quad (53)$$

Taking the logarithmic derivative on both sides of (53), we have

$$\frac{dm_l}{ds} = \frac{m_l}{3} ((\ln r_{l+1})_s - (\ln r_l)_s - 2(\ln \delta_l)_s - (\ln \delta_{l-1})_s) = \frac{m_l}{3} \left(u_{l+1} - u_l - 2 \frac{u_{l+1}^2 - u_l^2}{\delta_l} - \frac{u_l^2 - u_{l-1}^2}{\delta_{l-1}} \right). \quad (54)$$

Eqs. (47) and (54) constitute the semi-discrete analogue of the mSW equation. $\square$

Finally, let us consider the continuous limit as $b \rightarrow 0$. The dependent variable u is a function of l and s . Meanwhile, we regard it as a function of x and t , where x is the space coordinate at l -th lattice point and t is the time, defined by $x = x_0 + \sum_{j=0}^{l-1} \delta_j$, $t = s$. Then in the continuous limit, $b \rightarrow 0$ ($\delta_l \rightarrow 0$), we have

$$u_{l+1} - u_l \rightarrow 0, \quad \frac{u_{l+1}^2 - u_l^2}{\delta_l} \rightarrow (u^2)_x, \quad \frac{u_l^2 - u_{l-1}^2}{\delta_{l-1}} \rightarrow (u^2)_x,$$

$$m_l^3 = \frac{G_{l+1}}{G_l} \frac{f_l}{f_{l+1}} (1 + \rho_l \Delta u_l)(1 + \rho_{l-1} \Delta u_{l-1})(1 - 2\rho_l \Delta u_l) \rightarrow (1 + u_x)^2(1 - 2u_x).$$

On the other hand, since

$$\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} \frac{\partial \delta_j}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} (u_{j+1}^2 - u_j^2) \rightarrow u^2,$$

it then follows

$$\partial_s = \partial_t + \frac{\partial x}{\partial s} \partial_x \rightarrow \partial_t + u^2 \partial_x.$$

Consequently, Eq. (54) converges to

$$(\partial_t + u^2 \partial_x)m = \frac{m}{3}(-3(u^2)_x) = -2uu_x m \quad (55)$$

which is exactly the mSW equation (3).

4. Concluding remarks

In this paper, we have shown the reciprocal link between a modified short wave equation and the Tzitzeica equation. We also show that the multi-soliton solutions of the short wave limits of the DP and Novikov equations are connected to the solution of the mSW equation by Miura transformations. Based on the semi-discrete analogue of period 3 reduction of C-type KP-Toda lattice, we have constructed an integrable semi-discretization of the mSW equation. Meanwhile, its N -soliton solution in terms of determinant is also provided. We should point out that, most recently, an integrable semi-discrete analogue of the short wave limit of the Novikov equation has been constructed by the same authors [15]. Similar to our previous results [16–18], the semi-discrete analogue of the mSW equation can be served as an integrable numerical scheme, the so-called self-adaptive moving mesh method, for numerical simulations.

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