The Derivative Yajima–Oikawa System: Bright, Dark Soliton and Breather Solutions

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In this paper, we study the derivative Yajima–Oikawa (YO) system which describes the interaction between long and short waves (SWs). It is shown that the derivative YO system is classified into three types which are similar to the ones of the derivative nonlinear Schrödinger equation. The general N-bright and N-dark soliton solutions in terms of Gram determinants are derived by the combination of the Hirota's bilinear method and the Kadomtsev–Petviashvili hierarchy reduction method. Particularly, it is found that for the dark soliton solution of the SW component, the magnitude of soliton can be larger than the nonzero background for some parameters, which is usually called anti-dark soliton. The asymptotic analysis of two-soliton solutions shows that for both kinds of soliton only elastic collision exists and each soliton results in phase shifts in the long and SWs. In addition, we derive two types of breather solutions from the different reduction, which contain the homoclinic orbit and Kuznetsov–Ma breather solutions as special cases. Moreover, we propose a new (2+1)-dimensional derivative Yajima–Oikawa system and present its soliton and breather solutions.

1. Introduction

The long wave (LW)–short wave (SW) resonance interaction describes a resonant interaction process which takes place between a LW and a

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SW when the phase velocity of the former exactly or almost matches the group velocity of the latter. The theoretical investigation of this nonlinear resonance interaction originated from the study of the dynamics of Langmuir and ion acoustic waves in plasma by Zakharov [1]. In the case of LW propagating in only one direction, the Zakharov system reduces to the Yajima–Oikawa (YO) system [2]

$$\mathrm{i}\hat{S}_t + \hat{S}_{xx} + \hat{S}\hat{L} = 0, \tag{1}$$

$$\hat{L}_t = 2\sigma(|\hat{S}|^2)_x, \qquad \sigma^2 = 1, \tag{2}$$

which was shown to be integrable by using inverse scattering transform (IST) and admits multisoliton solutions [2, 3].

Based on the general theory established by Benney for the interaction between the SW and LW [4], Newell presented an exactly solvable model via IST [5, 6]

$$\mathrm{i}S_t + S_{xx} + \mathrm{i}SL_x + SL^2 - 2\sigma S|S|^2 = 0, \tag{3}$$

$$L_t = 2\sigma(|S|^2)_x, \tag{4}$$

where $S = S(x,t)$ represents the envelope of the SW and $L = L(x,t)$ is the amplitude of the LW. This system was found to be related to the YO system (1) and (2) through the appropriate gauge transformation or Muira transformation [7]

$$\hat{L} = \mathrm{i}L_x + L^2 - 2\sigma|S|^2, \qquad \hat{S} = S, \qquad \hat{S}^* = 2\mathrm{i}S_x^* + 2S^*L. \tag{5}$$

Thus, such system can be called the derivative Yajima–Oikawa (DYO) system. The complete Painlevé integrability of the DYO system was checked by the Weiss–Tabor–Carnevale approach in [8]. Ling et al. constructed the Darboux transformation for the DYO system and found a closed multisoliton solution formula [9, 10]. By applying the dressing method, soliton solutions including cusp solution for the DYO system were derived by using the properties of Cauchy matrix [11]. Geng et al. provided the algebro-geometric constructions of quasi-periodic flows of the DYO system and their explicit theta function representations [12].

The derivative nonlinear Schrödinger (DNLS) equations are the well-known integrable models with a variety of physical applications. Among these DNLS equations, there are three representative types, namely, the Kaup–Newell equation (DNLS-I) [13]

$$\mathrm{i}q_t + q_{xx} \pm 4\mathrm{i}(|q|^2 q)_x = 0, \tag{6}$$

the Chen–Lee–Liu equation (DNLS-II) [14]

$$\mathrm{i}q_t + q_{xx} \pm 4\mathrm{i}|q|^2 q_x = 0, \tag{7}$$

and the Gerdjikov–Ivanov equation (DNLS-III) [15]

$$\mathrm{i}q_t + q_{xx} \mp 4\mathrm{i}q^2 q_x^* + 8|q|^4 q = 0. \tag{8}$$

The relations of three types of the DNLS equations (6)–(8) were given through the gauge transformations. Further, the DNLS equations (6)–(8) correspond to special cases of the generalized DNLS equation [16] ($\epsilon = \mp 4$, ∓ 2 and 0, respectively)

$$\mathrm{i}\tilde{q}_t + \tilde{q}_{xx} - 2\mathrm{i}\epsilon|\tilde{q}|^2\tilde{q}_x - 2\mathrm{i}(\epsilon \pm 2)\tilde{q}^2\tilde{q}_x^* + (\epsilon \pm 2)(\epsilon \pm 4)\tilde{q}|\tilde{q}|^4 = 0. \tag{9}$$

Indeed, one can generate (9) from the Gerdjikov–Ivanov equation (8) via the gauge transformation $\tilde{q} = q\mathrm{e}^{\mathrm{i}\epsilon\int^x |q|^2 dx'}$, in which the continuity equation of DNLS equation (8), i.e.,

$$(|q|^2)_t = \left[\mathrm{i}(q^*\partial_x q - q\partial_x q^*) \pm 2|q|^4\right]_x \tag{10}$$

is used.

The DYO system (3) and (4) is similar to the DNLS equation of Gerdjikov–Ivanov type, because the first Eq. (3) has the higher order nonlinearity term SL^2. In the same spirit of the DNLS equtions, we can derive different types of the DYO systems. First, the second Eq. (4) itself is a continuity equation. Thus, one can obtain the generalized DYO system

$$\mathrm{i}\tilde{S}_t + \tilde{S}_{xx} - 2\mathrm{i}c\tilde{S}_x\tilde{L} - \mathrm{i}(c-1)\tilde{S}\tilde{L}_x + 2\sigma(c-1)\tilde{S}|\tilde{S}|^2 - (c^2-1)\tilde{S}\tilde{L}^2 = 0, \tag{11}$$

$$\tilde{L}_t = 2\sigma(|\tilde{S}|^2)_x. \tag{12}$$

by defining new fields through the following gauge transformation

$$\tilde{S} = S\mathrm{e}^{\mathrm{i}c\mathcal{T}}, \qquad \tilde{S}^* = S^*\mathrm{e}^{-\mathrm{i}c\mathcal{T}}, \qquad \tilde{L} = L, \qquad \mathcal{T} = \int^x L dx', \tag{13}$$

where c is a constant.

For the particular case $c = 1$, Eqs. (11) and (12) reduce to the following equation given in [5]

$$\mathrm{i}\tilde{S}_t + \tilde{S}_{xx} - 2\mathrm{i}\tilde{S}_x\tilde{L} = 0, \tag{14}$$

$$\tilde{L}_t = 2\sigma(|\tilde{S}|^2)_x, \tag{15}$$

which is similar to the DNLS equation of Chen–Lee–Liu type. The DYO system (14) and (15) is linked to the YO system (1) and (2) via the transformation:

$$\hat{L} = \mathrm{i}\tilde{L}_x + \tilde{L}^2 - 2\sigma|\tilde{S}|^2, \qquad \hat{S} = \tilde{S}\mathrm{e}^{-\mathrm{i}\int_{-\infty}^x \tilde{L}dx'}, \qquad \hat{S}^* = 2\mathrm{i}\tilde{S}_x^*\mathrm{e}^{\mathrm{i}\int_{-\infty}^x \tilde{L}dx'}. \tag{16}$$

For the particular case $c = -1$, Eqs. (11) and (12) become

$$\mathrm{i}\tilde{S}_t + \tilde{S}_{xx} + 2\mathrm{i}(\tilde{S}\tilde{L})_x - 4\sigma\tilde{S}|\tilde{S}|^2 = 0, \tag{17}$$

$$\tilde{L}_t = 2\sigma(|\tilde{S}|^2)_x, \tag{18}$$

in which the first Eq. (17) contains two types of derivative nonlinear term that is analogue to the DNLS equation of Kau–Newell type. The relation between this DYO system (17) and (18) and the YO system (1) and (2) is given by the transformation:

$$\hat{L} = \mathrm{i}\tilde{L}_x + \tilde{L}^2 - 2\sigma|\tilde{S}|^2, \qquad \hat{S} = \tilde{S}\mathrm{e}^{-\mathrm{i}\int_{-\infty}^{x}\tilde{L}dx'},$$
$$\hat{S}^* = 2\left(\mathrm{i}\tilde{S}^*_x + 2\tilde{S}^*\tilde{L}\right)\mathrm{e}^{\mathrm{i}\int_{-\infty}^{x}\tilde{L}dx'}. \tag{19}$$

It is noted that general N-soliton solutions to three types of DNLS equations can be expressed by same tau functions for bright and dark solitons, respectively, as shown in [17–19]. Here, we also show that above three types of DYO systems share same tau functions for bright and dark soliton solutions in subsequent sections.

On the other hand, from the DYO system (3) and (4), one can derive a second continuity equation of the form

$$\partial_t(\sigma L^2 - 2|S|^2) - 2\partial_x\left[\mathrm{i}\left(SS^*_x - S^*S_x\right) + 2L|S|^2\right] = 0. \tag{20}$$

Similar to the first case, we can define new fields through the transformation as follows:

$$\tilde{S} = S\mathrm{e}^{\mathrm{i}c\mathcal{T}}, \qquad \tilde{S}^* = S^*\mathrm{e}^{-\mathrm{i}c\mathcal{T}}, \qquad \tilde{L} = L, \qquad \mathcal{T} = \int^x \sigma L^2 - 2|S|^2 dx'. \tag{21}$$

This gauge transformation leads to the DYO system with more higher order nonlinearity and derivative nonlinear terms

$$\mathrm{i}\tilde{S}_t + \tilde{S}_{xx} - 2\sigma\tilde{S}|\tilde{S}|^2 + \mathrm{i}\tilde{S}\tilde{L}_x + 4\mathrm{i}c(|\tilde{S}|^2)_x\tilde{S} - 2\mathrm{i}c\sigma\tilde{L}(\tilde{L}\tilde{S})_x$$
$$+ \tilde{S}[(\tilde{L} + 2c|\tilde{S}|^2)^2 - c\tilde{L}^4] = 0, \tag{22}$$

$$\tilde{L}_t = 2\sigma(\tilde{S}\tilde{S}^*)_x. \tag{23}$$

The purpose of this paper is to construct the N-bright and dark soliton (for the SW component) solutions of the DYO system in the framework of the bilinear approach and the Kadomtsev–Petviashvili (KP) hierarchy reduction. Specifically, both N-bright and dark soliton solutions of the DYO system expressed in determinants are derived. Based on the soliton solution, we investigate the properties of one-soliton solutions of bright and dark types. In particular, it is shown that when the SW takes dark soliton soliton, both dark and anti-dark solitons appear on the nonzero background under the different parameters' conditions. We also perform the asymptotic analysis of two-soliton solutions for both cases and discuss the collision dynamics of the SW and LW components. In addition, two kinds of breather solutions are derived by considering the reduction differed from reductions for soliton solutions. Such two kinds of breather solutions are linked by $p_{2k-1} \to -\mathrm{i}p_{2k-1}$, $p_{2k} \to \mathrm{i}p_{2k}$, $\bar{p}_{2k-1} \to -\mathrm{i}\bar{p}_{2k-1}$, and $\bar{p}_{2k} \to \mathrm{i}\bar{p}_{2k}$, in which the homoclinic orbit and Kuznetsov–Ma breather solutions are two special cases, respectively. Moreover, a (2+1)-dimensional DYO system which, to our knowledge, is a new integrable two-dimensional analogue of the (1+1)-dimensional DYO system, is proposed in the process of the derivation of breather solutions.

The rest of the paper is organized as follows. In Sections 2 and 3, the bright and dark soliton solutions in terms of determinants of the DYO system are constructed from the reduction of the KP hierarchy. The soliton properties and collision behaviors are analyzed in detail. Section 4 is devoted to breather solutions of the DYO system, which is the relatively new results in the literature. The paper is summarized in Section 5. Appendices A and B provide the proofs of Lemmas 1 and 2, respectively.

2. The bright N-soliton solution

In this section, we construct the bright soliton solution for the DYO system under the boundary condition $S \to 0, L \to 0$ because $|x| \to \infty$. The DYO system is first transformed to a set of bilinear equations, and then we show such bilinear equations can be obtained from the reduction of the two-component KP hierarchy.

2.1. Bilinearization

By means of the dependent variable transformations

$$S = \frac{g}{f}, \qquad L = \mathrm{i}\frac{\partial}{\partial x} \ln \frac{f^*}{f}, \tag{24}$$

the DYO system (3) and (4) is converted to the following bilinear form

$$(\mathrm{i}D_t + D_x^2)g \cdot f = 0, \tag{25}$$

$$\mathrm{i}D_t f \cdot f^* = D_x^2 f \cdot f^*, \tag{26}$$

$$\mathrm{i}D_t f \cdot f^* = -2\sigma |g|^2, \tag{27}$$

where g and f are complex-valued functions and * denotes the complex conjugation hereafter. The Hirota's bilinear differential operators are defined by

$$D_x^n D_t^m (a \cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')\bigg|_{x=x',t=t'},$$

where n and m are nonnegative integers. Here, substitution of (24) into the gauge transformation (13) yields the bright soliton solutions for the second kind of DYO system (14) and (15) with the form

$$\tilde{S} = \frac{g}{f^*}, \qquad \tilde{L} = \mathrm{i}\frac{\partial}{\partial x} \ln \frac{f^*}{f} \tag{28}$$

and for third kind of DYO system (17) and (18) with the form

$$\tilde{S} = \frac{gf^*}{f^2}, \qquad \tilde{L} = \mathrm{i}\frac{\partial}{\partial x} \ln \frac{f^*}{f}. \tag{29}$$

2.2. The N-bright soliton solution

We first give N-bright soliton to the DYO system by the following theorem.

THEOREM 1. *The tau functions satisfying the bilinear equations* (25)–(27) *are given by the determinants f, g, f^*, and g^* where*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \qquad f^* = \begin{vmatrix} A' & I \\ -I & B \end{vmatrix}, \tag{30}$$

$$g = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{C} & 0 \end{vmatrix}, \qquad g^* = -\begin{vmatrix} A' & I & \mathbf{0}^T \\ -I & B & \mathbf{C}^{*T} \\ -\Phi^* & \mathbf{0} & 0 \end{vmatrix}. \tag{31}$$

Here, I is an $N \times N$ identity matrix, A, A', and B are $N \times N$ matrices whose entries are

$$a_{ij} = \frac{p_j^*}{p_i + p_j^*} \mathrm{e}^{\xi_i + \xi_j^*}, \qquad a'_{ij} = -\frac{p_i}{p_i + p_j^*} \mathrm{e}^{\xi_i + \xi_j^*}, \qquad b_{ij} = -\frac{2\sigma \alpha_i^* \alpha_j}{p_i^{*2} - p_j^2},$$

and $\mathbf{0}$ is a N-component zero-row vector, Φ and $\mathbf{C}$ are N-component row vectors given by

$$\Phi = (\mathrm{e}^{\xi_1}, \ldots, \mathrm{e}^{\xi_N}), \qquad \mathbf{C} = (\alpha_1, \ldots, \alpha_N),$$

with $\xi_i = p_i x + \mathrm{i} p_i^2 t + \xi_{i0}$.

The above bright N-soliton solution is characterized by $3N$ complex parameters p_i, α_i, and $\xi_{i0} (i = 1, \ldots, N)$. The former parameters p_i determine the amplitude and velocity of the solitons, whereas the latter ones α_i and ξ_{i0} determine the polarizations and the envelope phases of the solitons.

2.3. Proof of the bright N-soliton solution

LEMMA 1. *The following bilinear equations in the extended KP hierarchy*

$$\left(D_{x_2} - D_{x_1}^2\right) \tau_{1,-1}^{0,0} \cdot \tau_{0,0}^{0,0} = 0, \tag{32}$$

$$\left(D_{x_1}^2 + D_{x_2}\right)\tau_{0,0}^{0,0} \cdot \tau_{0,0}^{0,-1} = 0, \tag{33}$$

$$D_{y_1}\tau_{0,0}^{0,0} \cdot \tau_{0,0}^{0,-1} = -\tau_{1,-1}^{0,0}\tau_{-1,1}^{0,-1}, \tag{34}$$

have the Gram-type determinant solutions

$$\tau_{0,0}^{0,0} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \qquad \tau_{0,0}^{0,-1} = \begin{vmatrix} A' & I \\ -I & B \end{vmatrix}, \tag{35}$$

$$\tau_{1,-1}^{0,0} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}, \qquad \tau_{-1,1}^{0,-1} = \begin{vmatrix} A' & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}, \tag{36}$$

where I is an $N \times N$ identity matrix, A, A', and B are $N \times N$ matrices whose entries are

$$a_{ij} = \frac{\bar{p}_j}{p_i + \bar{p}_j}\mathrm{e}^{\xi_i+\bar{\xi}_j}, \qquad a'_{ij} = -\frac{p_i}{p_i + \bar{p}_j}\mathrm{e}^{\xi_i+\bar{\xi}_j}, \qquad b_{ij} = \frac{1}{q_i + \bar{q}_j}\mathrm{e}^{\eta_i+\bar{\eta}_j},$$

and $\mathbf{0}$ is a N-component zero-row vector, Φ, Ψ, $\bar{\Phi}$, and $\bar{\Psi}$ are N-component row vectors given by

$$\Phi = (\mathrm{e}^{\xi_1}, \ldots, \mathrm{e}^{\xi_N}), \qquad \bar{\Phi} = (\mathrm{e}^{\bar{\xi}_1}, \ldots, \mathrm{e}^{\bar{\xi}_N}), \qquad \Psi = (\mathrm{e}^{\eta_1}, \ldots, \mathrm{e}^{\eta_N}),$$
$$\bar{\Psi} = (\mathrm{e}^{\bar{\eta}_1}, \ldots, \mathrm{e}^{\bar{\eta}_N}),$$

with

$$\xi_i = p_i x_1 + p_i^2 x_2 + \xi_{i0}, \qquad \bar{\xi}_j = \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_{j0},$$
$$\eta_i = q_i y_1 + \eta_{i0}, \qquad \eta_j = \bar{q}_j y_1 + \bar{\eta}_{j0}.$$

Here, p_i, $\bar{p}_j$, q_i, $\bar{q}_j$, ξ_{i0}, $\bar{\xi}_{j0}$, η_{i0}, and $\bar{\eta}_{j0}$ are complex parameters.

The proof is given in Appendix A.

Now, we consider the reduction of above bilinear equations (32)–(34) in the extended KP hierarchy to the bilinear equations (25)–(27), by which the N-bright soliton solution can be derived. First, we conduct the dimension reduction. To this end, by performing row and column operations, one can rewrite tau functions $\tau_{0,0}^{0,0}$ and $\tau_{0,0}^{0,-1}$ as

$$\tau_{0,0}^{0,0} = \begin{vmatrix} \tilde{A} & I \\ -I & \tilde{B} \end{vmatrix}, \qquad \tau_{0,0}^{0,-1} = \begin{vmatrix} \tilde{A}' & I \\ -I & \tilde{B} \end{vmatrix} \tag{37}$$

where $\tilde{A}$, $\tilde{A}'$, and $\tilde{B}$ are $N \times N$ matrices whose elements given by

$$\tilde{a}_{ij} = \frac{\bar{p}_j}{p_i + \bar{p}_j}, \qquad \tilde{a}'_{ij} = -\frac{p_i}{p_i + \bar{p}_j}, \qquad \tilde{b}_{ij} = \frac{1}{q_i + \bar{q}_j}\mathrm{e}^{\eta_i+\bar{\eta}_j+\bar{\xi}_i+\xi_j},$$

with

$$\eta_i + \bar{\xi}_i = q_i y_1 + \bar{p}_i x_1 - \bar{p}_i^2 x_2 + \eta_{i0} + \bar{\xi}_{i0},$$
$$\bar{\eta}_j + \xi_j = \bar{q}_j y_1 + p_j x_1 + p_j^2 x_2 + \bar{\eta}_{j0} + \xi_{j0}.$$

Imposing the constraints on parameters

$$q_i = -\frac{\bar{p}_i^2}{2\sigma}, \qquad \bar{q}_i = \frac{p_i^2}{2\sigma}, \tag{38}$$

the following relations hold

$$\left(\partial_{y_1} - \frac{1}{2\sigma}\partial_{x_2}\right)\tau_{0,0}^{0,0} = 0, \qquad \left(\partial_{y_1} - \frac{1}{2\sigma}\partial_{x_2}\right)\tau_{0,0}^{0,-1} = 0, \tag{39}$$

by which, the bilinear equation (34) reduces to

$$D_{x_2}\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{0,-1} = -2\sigma\tau_{1,-1}^{0,0}\tau_{-1,1}^{0,-1}. \tag{40}$$

Next, we consider the complex conjugate reduction. By assuming x_1 is real, x_2, y_1 are pure imaginary and letting $p_i^* = \bar{p}_i$, $\xi_{i0}^* = \bar{\xi}_{i0}$, and $\eta_{i0}^* = \bar{\eta}_{i0}$, it can be verified that

$$a_{ij}^* = -a'_{ji}, \qquad b_{ij} = -b_{ji}^*. \tag{41}$$

Therefore, if we can define

$$f = \tau_{0,0}^{0,0}, \qquad f^* = \tau_{0,0}^{0,-1}, \qquad g = \tau_{1,-1}^{0,0}, \qquad g^* = -\tau_{-1,1}^{0,-1}, \tag{42}$$

then the bilinear equations (32), (33), and (40) become

$$\left(D_{x_2} - D_{x_1}^2\right) g\cdot f = 0, \tag{43}$$

$$\left(D_{x_1}^2 + D_{x_2}\right) f\cdot f^* = 0, \tag{44}$$

$$D_{x_2} f\cdot f^* = 2\sigma g g^*. \tag{45}$$

Furthermore, by applying the variable transformations

$$x_1 = x, \qquad x_2 = \mathrm{i}t, \tag{46}$$

i.e.,

$$\partial_{x_1} = \partial_x, \qquad \partial_{x_2} = -\mathrm{i}\partial_t. \tag{47}$$

Equations (43)–(45) are nothing but the bilinear equations of the DYO system (25)–(27). Under the above variable transformations, the variable y_1 becomes a dummy variable, which can basically be treated as a constant. Consequently, we could let $\mathrm{e}^{\eta_i} = \alpha_i^*$, $\mathrm{e}^{\bar{\eta}_i} = \alpha_i$ $(i = 1, \ldots, N)$ and define $\bar{\Psi} = C$ and $\Psi = C^*$, and then we arrive at Theorem 1 which gives N-bright soliton solution of the DYO system.

2.4. *The bright one- and two-soliton solutions*

In this subsection, we investigate the properties of the one- and two-bright soliton solutions. First, we redefine $\xi_i' = \xi_i + \tilde{\xi}_i'$ with $\alpha_i = \exp(\tilde{\xi}_i')$, and further assume the complex parameters p_i and $\tilde{\xi}_i'$ as

$$p_i = a_i + \mathrm{i}b_i, \qquad \xi_{i0}' = \gamma_{i0} + \mathrm{i}\zeta_{i0}, \qquad i = 1, 2, \tag{48}$$

where a_i, b_i, γ_{i0}, and ζ_{i0} are real constants. Then, the variables ξ_i' $(i = 1, 2)$ can be expressed by

$$\xi_i' = \gamma_i + \mathrm{i}\zeta_i, \qquad \gamma_i = a_i(x - 2b_i t) + \gamma_{i0}, \qquad \zeta_i = b_i x + \left(a_i^2 - b_i^2\right) t + \zeta_{i0}, \quad i = 1, 2. \tag{49}$$

2.4.1. The bright one-soliton solution. By taking $N = 1$ in (30) and (31), we get the tau functions for the bright one-soliton solution

$$f_1 = 1 + \frac{2\sigma\alpha_1\alpha_1^* p_1^*}{(p_1 + p_1^*)^2(p_1 - p_1^*)}\mathrm{e}^{\xi_1+\xi_1^*}, \qquad g_1 = \alpha_1 \mathrm{e}^{\xi_1} \tag{50}$$

or

$$f_1 = 1 + \frac{\sigma(a_1 - \mathrm{i}b_1)}{4\mathrm{i}b_1 a_1^2}\mathrm{e}^{2\gamma_1}, \qquad g_1 = \mathrm{e}^{\gamma_1+\mathrm{i}\zeta_1}, \tag{51}$$

in terms of the parameters defined by (48) and (49). These tau functions yield the one-bright soliton solution

$$S = \frac{\mathrm{e}^{\gamma_1+\mathrm{i}\zeta_1}}{1 + \frac{\sigma(a_1-\mathrm{i}b_1)}{4\mathrm{i}b_1a_1^2}\mathrm{e}^{2\gamma_1}}, \qquad L = -\frac{\sigma\mathrm{e}^{2\gamma_1}}{b_1\left|1 + \frac{\sigma(a_1-\mathrm{i}b_1)}{4\mathrm{i}b_1a_1^2}\mathrm{e}^{2\gamma_1}\right|^2}. \tag{52}$$

The square of the modulus of S and L can be written as:

$$|S|^2 = \frac{2|b_1|a_1^2}{\sqrt{a_1^2 + b_1^2}}\frac{1}{\cosh(2\gamma_1 + 2\delta) - \frac{\sigma|b_1|}{\sqrt{a_1^2+b_1^2}}}, \tag{53}$$

$$L = -\frac{2\mathrm{sgn}(b_1)\sigma a_1^2}{\sqrt{a_1^2 + b_1^2}}\frac{1}{\cosh(2\gamma_1 + 2\delta) - \frac{\sigma|b_1|}{\sqrt{a_1^2+b_1^2}}}, \tag{54}$$

with $\mathrm{e}^{4\delta} = \frac{a_1^2+b_1^2}{16b_1^2a_1^4}$. Thus, the SW and LW have the amplitudes A_S and A_L given by

$$A_S = \sqrt{2|b_1|\left(\sqrt{a_1^2 + b_1^2} + \sigma|b_1|\right)} = \sqrt{\frac{|v|}{2}\left(\sqrt{4a_1^2 + v^2} + \sigma|v|\right)}, \tag{55}$$

$$A_L = 2\left(\sqrt{a_1^2 + b_1^2} + \sigma|b_1|\right) = \sqrt{4a_1^2 + v^2} + \sigma|v|, \tag{56}$$

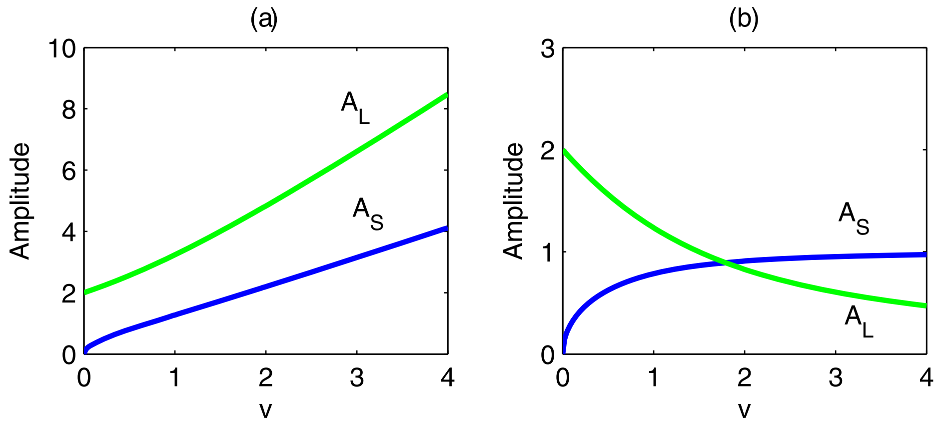

Figure 1. The amplitude–velocity relations for $a_1 = 1$: (a) $\sigma = 1$ and (b) $\sigma = -1$.

which also implies the amplitude–velocity relations with the velocity $v = 2b_1$. Without loss of generality, we discuss the amplitude–velocity relations for the case of the velocity $v > 0$. As a result, one can find that for a fixed value of the real part of p_1 (a_1), both A_S and A_L are increasing functions of v when $\sigma = 1$, whereas A_S is an increasing function of v and A_L is a decreasing function of v when $\sigma = -1$. The amplitude–velocity relations are displayed in Fig. 1.

In the case of $\sigma = 1$, it is necessary to point out that the amplitudes A_S and A_L remain finite in the limit of $a_1 \to 0$. Indeed, when a_1 approaches to zero, the one-bright soliton in (52) becomes an algebraic soliton of the form

$$S = \pm \frac{e^{i[b_1(x-b_1t)+\zeta_{10}]}}{x - 2b_1t + x_0 + \frac{i}{2b_1}}, \qquad L = -\frac{1}{b_1\left|x - 2b_1t + x_0 + \frac{i}{2b_1}\right|^2}, \tag{57}$$

where we have put $e^{2\gamma_{10}} = e^{2a_1x_0 - 2\delta}$. The similar structure of the solution can be found for the bright soliton solution of the DNLS-type equations [13, 20].

2.4.2. The bright two-soliton solution. By taking $N = 2$ in (30) and (31), we get the tau functions for the bright two-soliton solution

$$\begin{aligned} f_2 = 1 &+ c_{11^*}e^{\xi_1+\xi_1^*} + c_{21^*}e^{\xi_2+\xi_1^*} + c_{12^*}e^{\xi_1+\xi_2^*} \\ &+ c_{22^*}e^{\xi_2+\xi_2^*} + c_{121^*2^*}e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \end{aligned} \tag{58}$$

$$g_2 = \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2} + c_{121^*}e^{\xi_1+\xi_2+\xi_1^*} + c_{122^*}e^{\xi_1+\xi_2+\xi_2^*}, \tag{59}$$

where

$$c_{ij^*} = \frac{2\sigma\alpha_i\alpha_j^* p_j^*}{(p_i + p_j^*)^2(p_i - p_j^*)}, \qquad c_{12i^*} = (p_2 - p_1)\left(\frac{\alpha_2 c_{1i^*}}{p_2 + p_i^*} - \frac{\alpha_1 c_{2i^*}}{p_1 + p_i^*}\right),$$

$$c_{121^*2^*} = |p_1 - p_2|^2\left[\frac{c_{11^*}c_{22^*}}{(p_1 + p_2^*)(p_2 + p_1^*)} - \frac{c_{12^*}c_{21^*}}{(p_1 + p_1^*)(p_2 + p_2^*)}\right].$$

With the tau functions provided above, we are able to analyze the asymptotic properties of two-bright soliton solution (58) and (59). To this end, we choose $a_1 > 0$, $a_2 > 0$ for convenience and assume $b_1 > b_2 > 0$, then the two-soliton solution (58) and (59) has the following asymptotic forms.

(a) Before collision ($t \to -\infty$):

(i) soliton 1 ($\gamma_1 \simeq 0$, $\gamma_2 \to -\infty$):

$$S \simeq \frac{e^{\gamma_1 + i\zeta_1}}{1 + \frac{2\sigma p_1^*}{(p_1+p_1^*)^2(p_1-p_1^*)}e^{2\gamma_1}} \equiv S_1(\gamma_1, \zeta_1), \tag{60}$$

$$L \simeq i\frac{d}{dx}\ln\frac{1 + \frac{2\sigma p_1}{(p_1+p_1^*)^2(p_1^*-p_1)}e^{2\gamma_1}}{1 + \frac{2\sigma p_1^*}{(p_1+p_1^*)^2(p_1-p_1^*)}e^{2\gamma_1}} \equiv L_1(\gamma_1); \tag{61}$$

(ii) soliton 2 ($\gamma_2 \simeq 0$, $\gamma_1 \to +\infty$):

$$S \simeq \frac{\frac{(p_1+p_2)(p_1-p_2)^2}{(p_2-p_1^*)(p_2+p_1^*)^2}e^{\gamma_2 + i\zeta_2}}{1 + \frac{2\sigma p_2^*}{(p_2+p_2^*)^2(p_2^*-p_2)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_2}} \equiv S_2(\gamma_2 + \Delta\gamma_{12}, \zeta_2 + \Delta\zeta_2), \tag{62}$$

$$L \simeq i\frac{d}{dx}\ln\frac{1 + \frac{2\sigma p_2}{(p_2+p_2^*)^2(p_2^*-p_2)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_2}}{1 + \frac{2\sigma p_2^*}{(p_2+p_1^*)^2(p_2-p_2^*)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_2}} \equiv L_2(\gamma_2 + \Delta\gamma_2). \tag{63}$$

(b) After collision ($t \to +\infty$):

(i) soliton 1 ($\gamma_1 \simeq 0$, $\gamma_2 \to +\infty$):

$$S \simeq \frac{\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}e^{\gamma_1 + i\zeta_1}}{1 + \frac{2\sigma p_1^*}{(p_1+p_1^*)^2(p_1-p_1^*)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_1}} \equiv S_1(\gamma_1 + \Delta\gamma_{12}, \zeta_1 + \Delta\zeta_1), \tag{64}$$

$$L \simeq i\frac{d}{dx}\ln\frac{1 + \frac{2\sigma p_1}{(p_1+p_1^*)^2(p_1^*-p_1)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_1}}{1 + \frac{2\sigma p_1^*}{(p_1+p_1^*)^2(p_1-p_1^*)}\left|\frac{(p_1+p_2)(p_1-p_2)^2}{(p_1-p_2^*)(p_1+p_2^*)^2}\right|^2 e^{2\gamma_1}} \equiv L_1(\gamma_1 + \Delta\gamma_1); \tag{65}$$

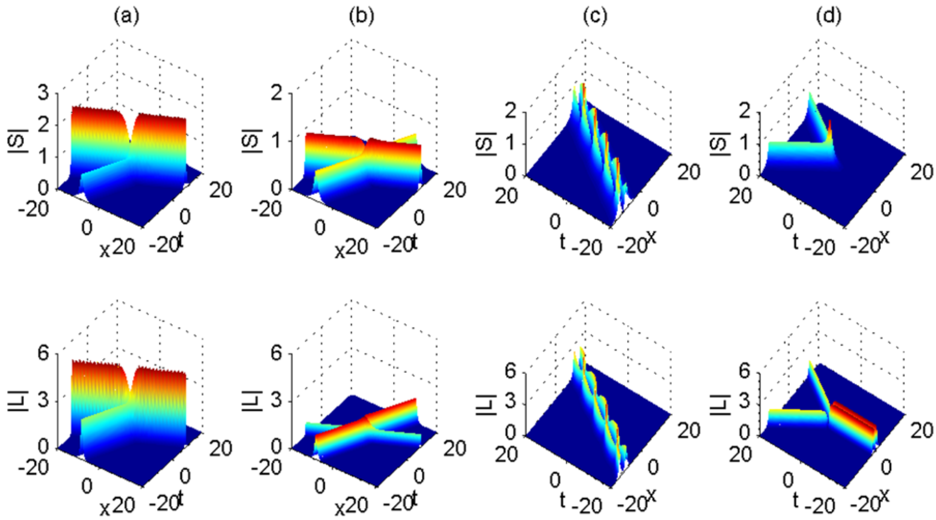

Figure 2. The interaction of two bright soliton with the parameters $\alpha_1 = \alpha_2 = 1$, $p_1 = 1 + \mathrm{i}$, and $p_2 = 1 + \frac{1}{4}\mathrm{i}$: (a) $\sigma = 1$ and (b) $\sigma = -1$; (c) the bound state with $\sigma = 1$, $p_1 = 1 + \frac{1}{3}\mathrm{i}$, and $p_2 = \frac{3}{5} + \frac{1}{3}\mathrm{i}$; (d) the resonant soliton with $\sigma = -1$, $p_1 = -p_2 = 1 + \frac{1}{3}\mathrm{i}$.

(ii) soliton 2 ($\gamma_2 \simeq 0$, $\gamma_1 \to -\infty$):

$$S \simeq \frac{\mathrm{e}^{\gamma_2 + \mathrm{i}\zeta_2}}{1 + \frac{2\sigma p_2^*}{(p_2 + p_2^*)^2 (p_2 - p_2^*)} \mathrm{e}^{2\gamma_2}} \equiv S_2(\gamma_2, \zeta_2), \tag{66}$$

$$L \simeq \mathrm{i}\frac{d}{dx} \ln \frac{1 + \frac{2\sigma p_2}{(p_2 + p_2^*)^2 (p_2^* - p_2)} \mathrm{e}^{2\gamma_2}}{1 + \frac{2\sigma p_2^*}{(p_2 + p_2^*)^2 (p_2 - p_2^*)} \mathrm{e}^{2\gamma_2}} \equiv L_2(\gamma_2). \tag{67}$$

The quantities in the above expressions are defined by

$$\Delta\gamma_{12} = \ln \left| \frac{(p_1 + p_2)(p_1 - p_2)^2}{(p_1 - p_2^*)(p_1 + p_2^*)^2} \right|,$$

$$\Delta\zeta_1 = \arg \left[\frac{(p_1 + p_2)(p_1 - p_2)^2}{(p_1 - p_2^*)(p_1 + p_2^*)^2} \right], \qquad \Delta\zeta_2 = \arg \left[\frac{(p_1 + p_2)(p_1 - p_2)^2}{(p_2 - p_1^*)(p_2 + p_1^*)^2} \right].$$

From above asymptotic results, two solitons remain their shape after the collision; thus, they undertake elastic collision. However, the solitons undergo phase shifts in the interaction process. Specifically, the first soliton has the positive phase shifts $(\Delta\gamma_{12}, \Delta\zeta_1)$ and the second one has the negative phase shifts $(-\Delta\gamma_{12}, -\Delta\zeta_2)$ in the SW, while the phase shifts of two solitons in the LW are given by $\Delta\gamma_{12}$ and $-\Delta\gamma_{12}$.

Figure 2 depicts two bright soliton interaction with different parameters. As shown in Fig. 2(a) and (b), the SW always exhibits the regular collision of solitons, in which the larger soliton propagates faster than the smaller soliton. The solitons in the LW also undergo the same interaction process for $\sigma = 1$. However, in the case of $\sigma = -1$, the LW has the collision in which the smaller soliton moves faster than the larger soliton.

Besides, the bound soliton state belongs to one class of special multi-soliton solutions, in which multiple solitons move with the same velocity. For the bright two-soliton solution of the DYO system, one needs to restrict $b_1 = b_2$ to obtain the bright two-soliton bound state. Such a bound state is illustrated in Fig. 2(c). In addition, by choosing $p_1 = -p_2$, one can find that $c_{121^*} = c_{122^*} = c_{121^*2^*} = 0$ in (58) and (59) and the bright two-soliton features the special localized structure, namely, resonant soliton. Indeed, more abundant resonant soliton solutions were given in [9]. We display this resonant soliton of V-Y type in Fig. 2(d).

3. The *N*-dark soliton solution

In this section, we derive the dark soliton solution for the DYO system. To this end, it is necessary to consider the nonvanishing boundary condition $S \to \rho e^{i[\alpha x - (\alpha^2 + 2\sigma\rho^2)t]}$, $L \to 0$ because $|x| \to \infty$ with ρ and α being real constants. Then, it is shown that the DYO system is transformed to another set of the bilinear equations, which can be reduced from the single-component KP hierarchy but with shifted singular points.

3.1. Bilinearization

To derive the dark soliton solution, we apply the dependent variable transformations

$$S = \rho e^{i[\alpha x - (\alpha^2 + 2\sigma\rho^2)t]} \frac{g}{f}, \qquad L = i \frac{d}{dx} \ln \frac{f^*}{f}, \tag{68}$$

which convert the DYO system (3) and (4) to the following bilinear equations

$$\left(iD_t + 2i\alpha D_x + D_x^2\right) g \cdot f = 0, \tag{69}$$

$$iD_t f \cdot f^* = D_x^2 f \cdot f^*, \tag{70}$$

$$iD_t f \cdot f^* = 2\sigma\rho^2(|f|^2 - |g|^2). \tag{71}$$

Similar to the bright soliton solution, by substituting (24) into the gauge transformation (13), one can give the dark soliton solutions for the second kind of DYO system (14) and (15) in the form

$$\tilde{S} = \rho e^{i[\alpha x - \alpha^2 t]} \frac{g}{f^*}, \qquad \tilde{L} = i \frac{\partial}{\partial x} \ln \frac{f^*}{f},$$

and for the third kind of DYO system (17) and (18) in the form

$$\tilde{S} = \rho \mathrm{e}^{\mathrm{i}[\alpha x - (\alpha^2 + 4\sigma\rho^2)t]} \frac{g f^*}{f^2}, \qquad \tilde{L} = \mathrm{i} \frac{\partial}{\partial x} \ln \frac{f^*}{f}.$$

3.2. The N-dark soliton solution

The N-dark soliton solution to the DYO system is given by the following theorem:

THEOREM 2. *The tau functions satisfying bilinear equations* (69)–(71) *are given by the determinants f and g where*

$$f = \left| \delta_{ij} - \frac{\mathrm{i} p_j^*}{p_i + p_j^*} \mathrm{e}^{\xi_i + \xi_j^*} \right|_{N \times N}, \tag{72}$$

$$g = \left| \delta_{ij} + \frac{\mathrm{i} p_j^*}{p_i + p_j^*} \left(\frac{p_i - \mathrm{i}\alpha}{p_i^* + \mathrm{i}\alpha} \right) \mathrm{e}^{\xi_i + \xi_j^*} \right|_{N \times N}, \tag{73}$$

with $\xi_i = p_i x + \mathrm{i} p_i^2 t + \xi_{i0}$. Here, p_i, ξ_{i0} are complex constants, and these parameters satisfy the constraint condition:

$$\frac{2\mathrm{i}\sigma\alpha\rho^2}{|p_i - \mathrm{i}\alpha|^2} = p_i - p_i^*. \tag{74}$$

Note that if $p_i = p_{i,R} + \mathrm{i} p_{i,I}$, we can solve the real part of p_i:

$$p_{i,R} = \pm \left[\frac{\sigma\alpha\rho^2}{p_{i,I}} - (p_{i,I} - \alpha)^2 \right]^{\frac{1}{2}}. \tag{75}$$

Thus, the dark N-soliton solution involves $N+2$ real parameters $p_{i,I} (i = 1, \ldots, N)$, α, ρ, and N complex parameters $\xi_{i0} (i = 1, \ldots, N)$. The parameters p_i determine the amplitude and the velocity of the solitons, whereas the parameters ξ_{i0} determine the phase of the solitons.

3.3. Proof of the N-dark soliton solution

LEMMA 2. *The following bilinear equations in the extended KP hierarchy*

$$\left(D_{x_2} - 2a D_{x_1} - D_{x_1}^2 \right) \tau_{n,k+1} \cdot \tau_{n,k} = 0, \tag{76}$$

$$\left(D_{x_2} + D_{x_1}^2 \right) \tau_{n,k} \cdot \tau_{n+1,k} = 0, \tag{77}$$

$$(a D_{x_{-1}} + 1) \tau_{n,k} \cdot \tau_{n+1,k} = \tau_{n,k+1} \tau_{n+1,k-1}, \tag{78}$$

where a is a complex constant, n and k are integers, and have the Gram-type determinant solutions

$$\tau_{n,k} = \left| m_{ij}^{n,k} \right|_{1 \le i,j \le N}, \tag{79}$$

where the entries of the determinant are given by

$$m_{ij}^{n,k} = \delta_{ij} + \frac{\mathrm{i} p_i}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n \left(-\frac{p_i - a}{\bar{p}_j + a} \right)^k \mathrm{e}^{\xi_i + \bar{\xi}_j},$$

with

$$\xi_i = \frac{1}{p_i - a} x_{-1} + p_i x_1 + p_i^2 x_2 + \xi'_{i0},$$

$$\bar{\xi}_j = \frac{1}{\bar{p}_j + a} x_{-1} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}'_{j0}.$$

Here, p_i, $\bar{p}_j$, ξ'_{i0}, $\bar{\xi}'_{j0}$, and a are complex parameters.

The proof is given in Appendix B.

Now, we turn to perform the reduction of above bilinear equations (76)–(78) in the extended KP hierarchy to the bilinear equations (69)–(71) and derive the dark soliton solution. For the dimension reduction, it is easy to show that if p_i and $\bar{p}_i$ satisfy the condition

$$\frac{1}{p_i - a} + \frac{1}{\bar{p}_i + a} = \frac{1}{2\sigma a \rho^2} \left(p_i^2 - \bar{p}_i^2 \right), \tag{80}$$

or

$$\frac{2\sigma a \rho^2}{(p_i - a)(\bar{p}_i + a)} = (p_i - \bar{p}_i), \tag{81}$$

the following relation holds

$$\left(a \partial_{x_{-1}} - \frac{1}{2\sigma\rho^2} \partial_{x_2} \right) \tau_{n,k} = 0, \tag{82}$$

by which the bilinear equation (78) reduces to

$$\left(D_{x_2} + 2\sigma\rho^2 \right) \tau_{n,k} \cdot \tau_{n+1,k} = 2\sigma\rho^2 \tau_{n,k+1} \tau_{n+1,k-1}. \tag{83}$$

Next, we perform the complex conjugate reduction. Assuming x_1, x_{-1} to be real, x_2, $a(= \mathrm{i}\alpha)$ pure imaginary, and $\bar{p}_i = p_i^*$, $\bar{\xi}'_{i0} = \xi_{i0}^{'*}$, one can verify

$$\left(m_{ij}^{-n-1,-k} \right)^* = m_{ji}^{n,k}. \tag{84}$$

Therefore, we have

$$\tau_{-1,0} = \tau_{0,0}^*, \qquad \tau_{-1,1} = \tau_{0,-1}^*. \tag{85}$$

For the sake of convenience, we define

$$f = \tau_{-1,0}, \qquad g = \tau_{-1,1}, \qquad f^* = \tau_{0,0}, \qquad g^* = \tau_{0,-1}. \tag{86}$$

Therefore, the bilinear equations (76), (77), and (83) become

$$\left(D_{x_2} - 2aD_{x_1} - D_{x_1}^2\right) g \cdot f = 0, \tag{87}$$

$$\left(D_{x_2} + D_{x_1}^2\right) f \cdot f^* = 0, \tag{88}$$

$$\left(D_{x_2} + 2\sigma\rho^2\right) f \cdot f^* = 2\sigma\rho^2 g g^*. \tag{89}$$

Finally, by using the same variable transformations (46), the above bilinear equations (87)–(89) become ones of the DYO system (69)–(71). Moreover, the variable x_{-1} becomes a dummy variable, and then we could let $\frac{1}{p_i - i\alpha}x_{-1} + \xi'_{i0} = \xi_{i0}$ and $\frac{1}{p_i^* + a}x_{-1} + \xi'^*_{i0} = \xi^*_{i0}$. In summary, we construct the dark soliton solution to the DYO system as stated in Theorem 2.

3.4. *The one- and two-dark soliton solutions*

In this subsection, we give explicit solutions to the one- and two-dark soliton solutions and study their properties. Similarly, we rewrite the complex parameters p_i and ξ'_i as

$$p_i = a_i + \mathrm{i}b_i, \qquad \xi_{i0} = \vartheta_{i0} + \mathrm{i}\chi_{i0}, \qquad i = 1, 2, \tag{90}$$

where a_i, b_i, ϑ_{i0}, and χ_{i0} are real constants, and then the variables ξ_i $(i = 1, 2)$ are taken in the form

$$\xi_i = \vartheta_i + \mathrm{i}\chi_i, \qquad \vartheta_i = a_i(x - 2b_i t) + \vartheta_{i0}, \qquad \chi_i = b_i x + \left(a_i^2 - b_i^2\right) t + \chi_{i0},$$
$$i = 1, 2. \tag{91}$$

3.4.1. The one-dark soliton solution. By taking $N = 1$ in (72) and (73), we obtain the tau functions for the dark one-soliton solution

$$f_1 = 1 - \frac{\mathrm{i}p_1^*}{p_1 + p_1^*}\mathrm{e}^{\xi_1 + \xi_1^*}, \qquad g_1 = 1 + \frac{\mathrm{i}p_1^*}{p_1 + p_1^*}\left(\frac{p_1 - \mathrm{i}\alpha}{p_1^* + \mathrm{i}\alpha}\right)\mathrm{e}^{\xi_1 + \xi_1^*}, \tag{92}$$

or

$$f_1 = 1 - \frac{b_1 + \mathrm{i}a_1}{2a_1}\mathrm{e}^{2\vartheta_1}, \qquad g_1 = 1 + \frac{b_1 + \mathrm{i}a_1}{2a_1}\left[\frac{a_1 + \mathrm{i}(b_1 - \alpha)}{a_1 - \mathrm{i}(b_1 - \alpha)}\right]\mathrm{e}^{2\vartheta_1}, \tag{93}$$

in terms of the parameters defined by (90) and (91). These tau functions lead to the one-dark soliton solution for the SW and LW components

$$S = \rho \mathrm{e}^{\mathrm{i}[\alpha x - (\alpha^2 + 2\sigma\rho^2)t]} \frac{1 + \frac{b_1 + \mathrm{i}a_1}{2a_1}\left[\frac{a_1 + \mathrm{i}(b_1 - \alpha)}{a_1 - \mathrm{i}(b_1 - \alpha)}\right]\mathrm{e}^{2\vartheta_1}}{1 - \frac{b_1 + \mathrm{i}a_1}{2a_1}\mathrm{e}^{2\vartheta_1}}, \tag{94}$$

$$L = \frac{-2a_1 e^{2\vartheta_1}}{\left(1 - \frac{b_1 + \mathrm{i}a_1}{2a_1} e^{2\vartheta_1}\right)\left(1 - \frac{b_1 - \mathrm{i}a_1}{2a_1} e^{2\vartheta_1}\right)}. \tag{95}$$

The square of the modulus of S and L reads

$$|S|^2 = \rho^2 \left[1 + \frac{2\alpha a_1^2 \mathrm{sgn}(a_1)}{\left[a_1^2 + (b_1 - \alpha)^2\right]\sqrt{a_1^2 + b_1^2}} \frac{1}{\cosh(2\vartheta_1 + 2\delta') - \frac{b_1 \mathrm{sgn}(a_1)}{\sqrt{a_1^2 + b_1^2}}} \right], \tag{96}$$

$$L = -\frac{2a_1^2 \mathrm{sgn}(a_1)}{\sqrt{a_1^2 + b_1^2}} \frac{1}{\cosh(2\vartheta_1 + 2\delta') - \frac{b_1 \mathrm{sgn}(a_1)}{\sqrt{a_1^2 + b_1^2}}}, \tag{97}$$

with $e^{4\delta'} = \frac{a_1^2 + b_1^2}{4a_1^2}$. This suggests that if $\alpha a_1 < 0$, then the SW takes the form of a dark soliton, whereas if $\alpha a_1 > 0$, it becomes an anti-dark soliton on a constant background $S = \rho$, which means that the amplitude of the soliton is larger than ρ. Besides, if we define

$$e^{2\vartheta_0} = \frac{\sqrt{a_1^2 + b_1^2}}{-2a_1}, \qquad e^{2\mathrm{i}\phi} = \frac{b_1 + \mathrm{i}a_1}{\sqrt{a_1^2 + b_1^2}}, \qquad e^{2\mathrm{i}\phi^+} = \frac{a_1 + \mathrm{i}(b_1 - \alpha)}{a_1 - \mathrm{i}(b_1 - \alpha)}, \tag{98}$$

the dark one-soliton solution of the SW and LW can be written as:

$$S = \frac{\rho}{2} e^{\mathrm{i}[\alpha x - (\alpha^2 + 2\sigma\rho^2)t]} \left[1 + e^{2\mathrm{i}\phi^+} + (e^{2\mathrm{i}\phi^+} - 1)\tanh(\vartheta_1 + \vartheta_0 + \mathrm{i}\phi) \right], \tag{99}$$

$$L = \frac{2a_1^2}{\sqrt{a_1^2 + b_1^2}} \frac{1}{\cosh(2\vartheta_1 + 2\vartheta_0) + \frac{b_1}{\sqrt{a_1^2 + b_1^2}}}. \tag{100}$$

Thus, the phase of the SW acquires shifts in the amount of $2\phi^+$ but the LW's phase shift is zero because ϑ_1 varies from $-\infty$ to $+\infty$.

Note that the constraint condition (75) satisfies

$$a_1^2 = \frac{\sigma\alpha\rho^2}{b_1} - (b_1 - \alpha)^2 = \frac{2\sigma\alpha\rho^2}{v} - \frac{(v - 2\alpha)^2}{4} > 0. \tag{101}$$

We then find that there are two cases for the real parameter b_1:

(i) for $\sigma = 1$, the real b_1 lies in the interval $b_{1,min} < b_1 < b_{1,max}$ where

$$b_{1,max} = \frac{(\mu_1 + 2\alpha)^2}{6\mu_1}, \qquad \mu_1 = \left[4\alpha\Lambda_1 + 12\sqrt{3\alpha^2\rho^2\Lambda_1}\right]^{1/3}, \qquad \alpha > 0, \tag{102}$$

$$b_{1,min} = \frac{(\nu_1 + 2\alpha)^2}{6\nu_1}, \qquad \nu_1 = \left[4\alpha\Lambda_1 - 12\sqrt{3\alpha^2\rho^2\Lambda_1}\right]^{1/3}, \qquad \alpha < 0, \tag{103}$$

with $\Lambda_1 = 27\rho^2 - 2\alpha^2$ and $-\frac{3}{2}\sqrt{3\rho^2} \le \alpha \le \frac{3}{2}\sqrt{3\rho^2}$.

(ii) for $\sigma = -1$, the real b_1 lies in the interval $b'_{1,min} < b_1 < b'_{1,max}$ where

$$b'_{1,max} = \frac{(\mu'_1 + 2\alpha)^2}{6\mu'_1}, \qquad \mu'_1 = \left[-4\alpha\Lambda_2 + 12\sqrt{3\alpha^2\rho^2\Lambda_2}\right]^{1/3}, \qquad \alpha < 0, \tag{104}$$

$$b'_{1,min} = \frac{(\nu'_1 + 2\alpha)^2}{6\nu'_1}, \qquad \nu'_1 = \left[-4\alpha\Lambda_2 - 12\sqrt{3\alpha^2\rho^2\Lambda_2}\right]^{1/3}, \qquad \alpha > 0, \tag{105}$$

with $\Lambda_2 = 27\rho^2 + 2\alpha^2$.

In what follows, we discuss the amplitude–velocity relations for the dark one-soliton. Without loss of generality, we consider the case of the velocity $v = 2b_1 > 0$.

Case 1: $\alpha > 0$. In this case, Eq. (75) implies $\sigma = 1$ and $0 < v < v_{1,max} = 2b_{1,max}$.

(i) $a_1 > 0$, anti-dark soliton for the SW. The amplitude–velocity relations are given by

$$A_S = \frac{\sqrt{2}}{2}\sqrt{2\rho^2 + 2v\sqrt{\frac{\alpha}{v}(v^2 + 2\rho^2 - \alpha v)} + v^2} - \rho, \tag{106}$$

$$A_L = 2\sqrt{\frac{\alpha}{v}(v^2 + 2\rho^2 - \alpha v)} + v. \tag{107}$$

As can be seen from Fig. 3(a), A_S is an increasing function, whereas A_L is a decreasing function in the interval $0 < v < v_1$ and increasing function in $v_1 < v < v_{1,max}$, where v_1 is a critical velocity satisfied by

$$v_1^5 - 2\alpha v_1^4 + 2\rho^2 v_1^3 + 4\alpha\rho^2 v_1^2 - 4\alpha\rho^2 = 0.$$

Figure 4 shows the profiles of the one-soliton for the SW and LW with $\alpha > 0$ and $\sigma = 1$. The profiles a and b in the SW represent anti-dark soliton, which exhibits the behavior of the bright soliton, but it differs from the usual bright soliton due to the nonzero background.

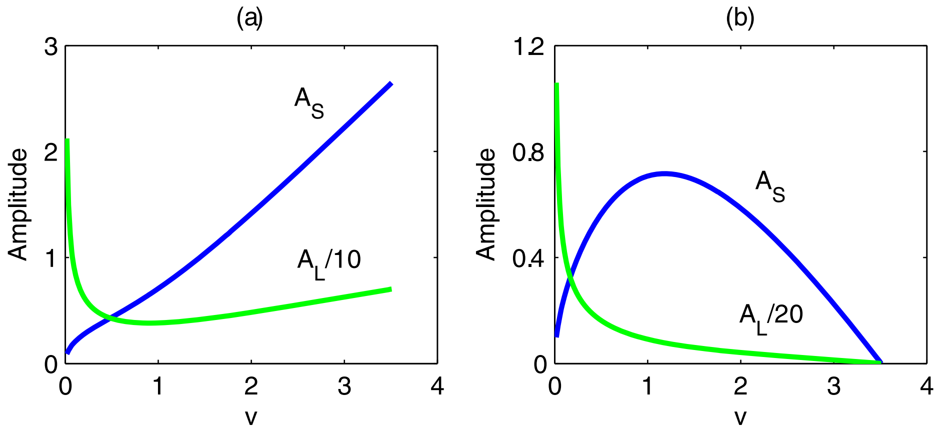

Figure 3. The amplitude–velocity relations for $\sigma = \alpha = \rho = 1$: (a) the anti-dark soliton for the SW and (b) the dark soliton for the SW.

(ii) $a_1 < 0$, dark soliton for the SW. The amplitude–velocity relation has the form

$$A_S = \rho - \frac{\sqrt{2}}{2}\sqrt{2\rho^2 - 2v\sqrt{\frac{\alpha}{v}(v^2 + 2\rho^2 - \alpha v)} + v^2}, \tag{108}$$

$$A_L = 2\sqrt{\frac{\alpha}{v}(v^2 + 2\rho^2 - \alpha v)} - v. \tag{109}$$

As shown in Fig. 3(b), A_L is a decreasing function, whereas A_S is an increasing function in the interval $0 < v < v_2$ and decreasing function in $v_2 < v < v_{1,max}$, where v_2 is a critical velocity satisfied by

$$4v_2^3 - 5\alpha v_2^2 + 2\alpha^2 v_2 - 2\alpha\rho^2 = 0.$$

Case 2: $\alpha < 0$. In this case, Eq.(75) suggests $\sigma = -1$ and $0 < v < v_{2,max} = 2b'_{1,max}$.

(i) $a_1 > 0$, dark soliton for the SW. The amplitude–velocity relations read

$$A_S = \rho - \frac{\sqrt{2}}{2}\sqrt{2\rho^2 - 2v\sqrt{\frac{\alpha}{v}(v^2 - 2\rho^2 - \alpha v)} - v^2}, \tag{110}$$

$$A_L = 2\sqrt{\frac{\alpha}{v}(v^2 - 2\rho^2 - \alpha v)} + v. \tag{111}$$

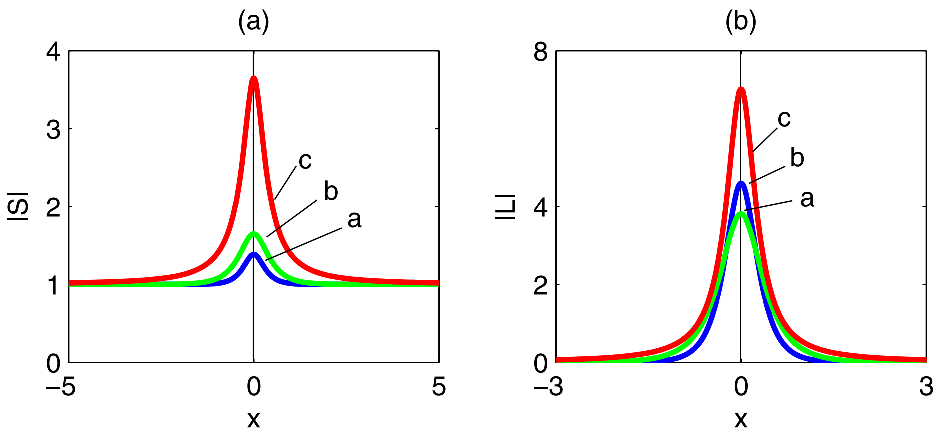

Figure 4. Profiles of the amplitudes of soliton for $\sigma = \alpha = \rho = 1$ at $t = 0$: (a) $b_1 = 0.2$; (b) $b_1 = \frac{v_1}{2} = 0.45$; and (c) $b_1 = b_{1,max} = 1.75$. Profile c is the algebraic soliton.

As depicted in Fig. 5(a), A_L is a decreasing function, whereas A_S is an increasing function in the interval $0 < v < v_3$ and decreasing function in $v_3 < v < v_{2,max}$, where v_3 is a critical point $v_3 = \alpha + \sqrt{\alpha^2 + 2\rho^2}$. It is noted that this point leads to black soliton for the SW.
Figure 6 displays the profiles of the one-soliton for the SW and LW with $\alpha < 0$ and $\sigma = -1$. The profiles (a) and (b) of the SW represent the usual dark soliton that the center intensity is lower than the background. Particularly, the profile (b) shows a black soliton for the SW component.

(ii) $a_1 < 0$, anti-dark soliton for the SW. The amplitude–velocity relations take the form

$$A_S = \frac{\sqrt{2}}{2}\sqrt{2\rho^2 + 2v\sqrt{\frac{\alpha}{v}(v^2 - 2\rho^2 - \alpha v)} - v^2} - \rho, \tag{112}$$

$$A_L = 2\sqrt{\frac{\alpha}{v}(v^2 - 2\rho^2 - \alpha v)} - v. \tag{113}$$

As is illustrated in Fig. 5(b), A_L is a decreasing function, whereas A_S is an increasing function in the interval $0 < v < v_4$ and decreasing function in $v_4 < v < v_{2,max}$, where v_4 is a critical velocity satisfied by

$$4v_4^3 - 5\alpha v_4^2 + 2\alpha^2 v_4 + 2\alpha\rho^2 = 0.$$

Similar to the bright soliton, at the limit value of the wave width, the algebraic soliton can be produced from the soliton of hyperbolic type.

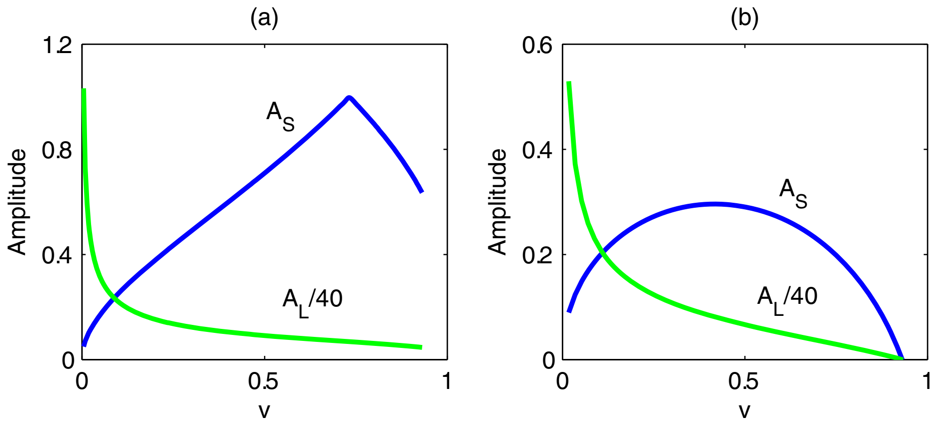

Figure 5. The amplitude–velocity relations for $\sigma = \alpha = -\rho = -1$: (a) the dark soliton for the SW and (b) the anti-dark soliton for the SW.

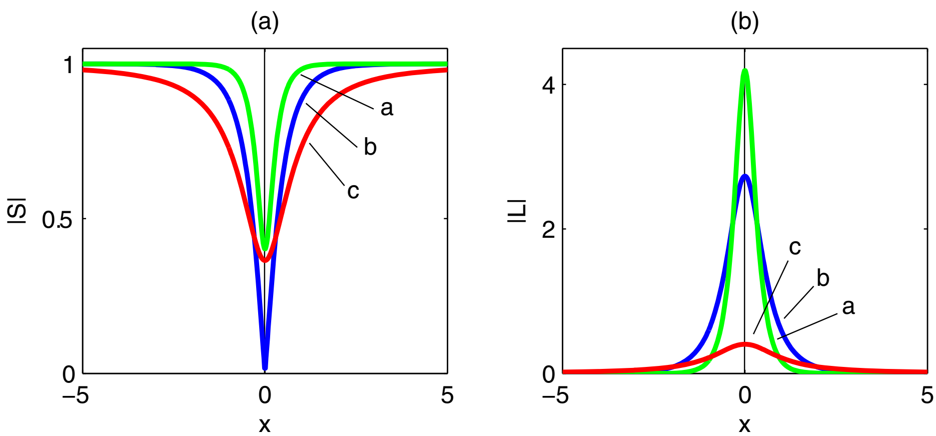

Figure 6. Profiles of the amplitudes of soliton for $\sigma = \alpha = -\rho = -1$ at $t = 0$: (a) $b_1 = 0.2$; (b) $b_1 = \frac{v_3}{2} = 0.37$; and (c) $b_1 = b'_{1,max} = 0.47$. Profile c is the algebraic soliton and for the SW profile b is black soliton.

Indeed, if we put $\vartheta_{10} = x_0 - \delta'$ in (93), then the following expansion formulae hold for small a_1:

$$f_1 \sim 1 - \mathrm{sgn}(a_1)\mathrm{sgn}(b_1)\left[1 + 2a_1\left(x - 2b_1 t + x_0 + \frac{\mathrm{i}}{2b_1}\right)\right] + O\left(a_1^2\right), \quad (114)$$

$$g_1 \sim 1 - \mathrm{sgn}(a_1)\mathrm{sgn}(b_1)\left[1 + 2a_1\left(x - 2b_1 t + x_0 + \frac{\mathrm{i}}{2b_1}\frac{\alpha + b_1}{\alpha - b_1}\right)\right] + O\left(a_1^2\right), \quad (115)$$

Thus, we know that only $\mathrm{sgn}(a_1)\mathrm{sgn}(b_1) = 1$ or $a_1 b_1 > 0$ yields the algebraic soliton:

(i) For $\sigma = 1$, one obtains the anti-dark algebraic soliton for the SW component

$$
\begin{aligned}
S &= \rho e^{\mathrm{i}[\alpha x-(\alpha^2+2\rho^2)t]}\frac{x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}\frac{\alpha+b_1}{\alpha-b_1}}{x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}},\\
L &= -\frac{1}{b_1}\frac{1}{\left|x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}\right|^2},
\end{aligned}
\tag{116}
$$

which can be realized in two cases: (a) $a_1 > 0, b_1 > 0, \alpha > 0$ then $b_1 = b_{1,max}$; (b) $a_1 < 0, b_1 < 0, \alpha < 0$ then $b_1 = b_{1,min}$.

(ii) For $\sigma = -1$: we have the dark algebraic soliton for the SW component

$$
\begin{aligned}
S &= \rho e^{\mathrm{i}[\alpha x-(\alpha^2-2\rho^2)t]}\frac{x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}\frac{\alpha+b_1}{\alpha-b_1}}{x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}},\\
L &= -\frac{1}{b_1}\frac{1}{\left|x-2b_1t+x_0+\frac{\mathrm{i}}{2b_1}\right|^2},
\end{aligned}
\tag{117}
$$

which can be achieved in two cases: (a) $a_1 > 0, b_1 > 0, \alpha < 0$ then $b_1 = b'_{1,max}$; (b) $a_1 < 0, b_1 < 0, \alpha > 0$ then $b_1 = b'_{1,min}$.

The similar structure of the solution can be also found for the dark soliton solution of the DNLS-type equations [19].

3.4.2. The two-dark soliton solution. By taking $N = 2$ in (72) and (73), we have the tau functions for the dark two-soliton solution

$$f_2 = 1 + d_{11^*}e^{\xi_1+\xi_1^*} + d_{22^*}e^{\xi_2+\xi_2^*} + d_{11^*}d_{22^*}\Omega_{12}e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \tag{118}$$

$$g_2 = 1 + d_{11^*}K_1e^{\xi_1+\xi_1^*} + d_{22^*}K_2e^{\xi_2+\xi_2^*} + d_{11^*}d_{22^*}K_1K_2\Omega_{12}e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \tag{119}$$

with

$$d_{ii^*} = -\frac{\mathrm{i}p_i^*}{p_i+p_i^*}, \qquad K_i = -\frac{p_i-\mathrm{i}\alpha}{p_i^*+\mathrm{i}\alpha}, \qquad \Omega_{12} = \frac{|p_1-p_2|^2}{|p_1+p_2^*|^2}.$$

Because of the analysis of one-soliton solution, the SW allows dark and anti-dark soliton solution. Therefore, two-soliton solution of the SW can be classified into three types, i.e., dark-dark solitons, dark-anti-dark solitons, and anti-dark-anti-dark solitons. Here, we mainly investigate the asymptotic behavior of dark-dark solitons in the SW. To do so, we only discuss this interaction process for $\alpha < 0$. Furthermore, we choose $a_1 > 0$, $a_2 > 0$ and assume $b_1 > b_2 > 0$, then the dark two-soliton solution (118) and (119) takes the following asymptotic forms.

(a) Before collision ($t \to -\infty$):
(i) soliton 1 ($\vartheta_1 \simeq 0$, $\vartheta_2 \to -\infty$):

$$S \simeq \rho \mathrm{e}^{\mathrm{i}[\alpha x-(\alpha^2+2\sigma\rho^2)t]} \frac{1+d_{11^*}K_1\mathrm{e}^{\xi_1+\xi_1^*}}{1+d_{11^*}\mathrm{e}^{\xi_1+\xi_1^*}} \equiv S_1(\vartheta_1), \quad (120)$$

$$L \simeq \mathrm{i}\frac{d}{dx}\ln\frac{1-d_{1^*1}\mathrm{e}^{\xi_1+\xi_1^*}}{1+d_{11^*}\mathrm{e}^{\xi_1+\xi_1^*}} \equiv L_1(\vartheta_1). \quad (121)$$

(ii) soliton 2 ($\vartheta_2 \simeq 0$, $\vartheta_1 \to +\infty$):

$$S \simeq \rho \mathrm{e}^{\mathrm{i}[\alpha x-(\alpha^2+2\sigma\rho^2)t+2\phi_1]} \frac{1+d_{22^*}K_2\mathrm{e}^{2\vartheta_2+2\Delta\vartheta_{12}}}{1+d_{22^*}\mathrm{e}^{2\vartheta_2+2\Delta\vartheta_{12}}} \equiv \mathrm{e}^{2\mathrm{i}\phi_1}S_2(\vartheta_2+\Delta\vartheta_{12}), \quad (122)$$

$$L \simeq \mathrm{i}\frac{d}{dx}\ln\frac{1-d_{2^*2}\Omega_{12}\mathrm{e}^{2\vartheta_2}}{1+d_{22^*}\Omega_{12}\mathrm{e}^{2\vartheta_2}} \equiv L_2(\vartheta_2+\Delta\vartheta_{12}). \quad (123)$$

(b) After collision ($t \to +\infty$):
(i) soliton 1 ($\vartheta_1 \simeq 0$, $\vartheta_2 \to +\infty$):

$$S \simeq \rho \mathrm{e}^{\mathrm{i}[\alpha x-(\alpha^2+2\sigma\rho^2)t]} K_2 \frac{1+d_{11^*}K_1\Omega_{12}\mathrm{e}^{\xi_1+\xi_1^*}}{1+d_{11^*}\Omega_{12}\mathrm{e}^{\xi_1+\xi_1^*}} \equiv \mathrm{e}^{2\mathrm{i}\phi_2}S_1(\vartheta_1+\Delta\vartheta_{12}), \quad (124)$$

$$L \simeq \mathrm{i}\frac{d}{dx}\ln -\frac{d_{2^*2}}{d_{22^*}}\frac{1-d_{1^*1}\Omega_{12}\mathrm{e}^{2\vartheta_1}}{1+d_{11^*}\Omega_{12}\mathrm{e}^{2\vartheta_1}} \equiv L_1(\vartheta_1+\Delta\vartheta_{12}), \quad (125)$$

(ii) soliton 2 ($\vartheta_2 \simeq 0$, $\vartheta_1 \to -\infty$):

$$S \simeq \rho \mathrm{e}^{\mathrm{i}[\alpha x-(\alpha^2+2\sigma\rho^2)t]} \frac{1+d_{22^*}K_2\mathrm{e}^{\xi_2+\xi_2^*}}{1+d_{22^*}\mathrm{e}^{\xi_2+\xi_2^*}} \equiv S_2(\vartheta_2), \quad (126)$$

$$L \simeq \mathrm{i}\frac{d}{dx}\ln\frac{1-d_{2^*2}\mathrm{e}^{\xi_2+\xi_2^*}}{1+d_{22^*}\mathrm{e}^{\xi_2+\xi_2^*}} \equiv L_2(\vartheta_2). \quad (127)$$

Here, the quantities in the above expressions are given by $\Omega_{12} = \mathrm{e}^{2\Delta\vartheta_{12}}$, $K_1 = \mathrm{e}^{2\mathrm{i}\phi_1}$, and $K_2 = \mathrm{e}^{2\mathrm{i}\phi_2}$.

Based on the asymptotic expressions, as the analysis of the amplitude of one-soliton, we know that only elastic collision takes place in the SW and LW components. In the interaction process, each soliton in the SW component suffers the phase shifts $(+2\phi_2, +\Delta\vartheta_{12})$ and $(-2\phi_1, -\Delta\vartheta_{12})$ respectively, and both solitons suffer the phase shifts $+\Delta\vartheta_{12}$ and $-\Delta\vartheta_{12}$ in the LW component where

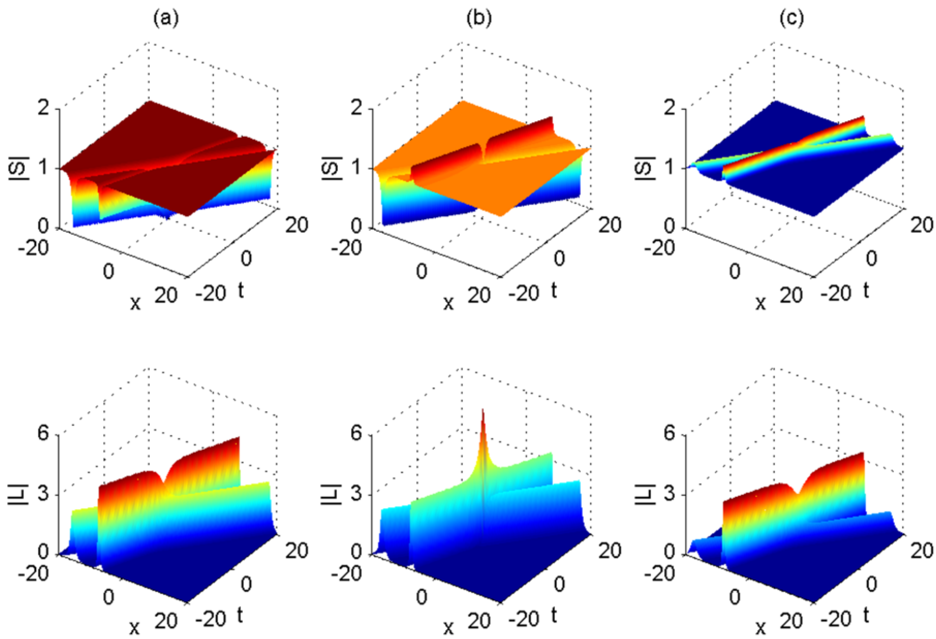

Figure 7. The interaction of two soliton with the parameters $\sigma = \alpha = -\rho = -1$, $b_1 = 0.4$ and $b_2 = 0.2$: (a) dark-dark soliton for the SW $a_1 = 0.73, a_2 = 1.89$; (b) dark-anti-dark soliton for the SW $a_1 = 0.73, a_2 = -1.89$; and (c) anti-dark-anti-dark soliton for the SW $a_1 = -0.73, a_2 = -1.89$.

$$\phi_i = \arg\left[-\frac{p_i - \mathrm{i}\alpha}{p_i^* + \mathrm{i}\alpha}\right], \qquad \Delta\vartheta_{12} = \ln\frac{|p_1 - p_2|}{|p_1 + p_2^*|},$$
$$-\Delta\vartheta_{12} = -\ln\frac{|p_1 - p_2|}{|p_1 + p_2^*|}. \tag{128}$$

In Fig. 7, dark-dark solitons, dark-anti-dark solitons, and anti-dark-anti-dark solitons for the SW are illustrated with the different parameters. In addition, under the constraint conditions (74) or (75), one cannot get different value of a_i for the same value of b_i; thus, there does not exist bound state and resonant soliton for the dark-soliton in the DYO system.

4. The breather solution

4.1. A (2+1)-dimensional derivative YO system

To construct the breather solution of the DYO system, we need to consider the same transformations under nonvanishing boundary condition, same as the dark soliton in previous section, which lead to the same bilinear equations.

Starting from the bilinear equations (76)–(78) in KP hierarchy, we define

$$f = \tau_{-1,0}, \qquad g = \tau_{-1,1}, \qquad \hat{f} = \tau_{0,0}, \qquad \hat{g} = \tau_{0,1}, \tag{129}$$

and introduce independent variable transformations

$$x_1 = x, \qquad x_2 = \mathrm{i}y, \qquad x_{-1} = 2\mathrm{i}a\sigma\rho^2(t-y), \tag{130}$$

i.e.,

$$\partial_{x_1} = \partial_x, \qquad \partial_{x_2} = -\mathrm{i}(\partial_t + \partial_y), \qquad \partial_{x_{-1}} = -\frac{\mathrm{i}}{2a\sigma\rho^2}\partial_t, \tag{131}$$

and then a set of bilinear equations can be obtained:

$$\left(\mathrm{i}D_t + \mathrm{i}D_y + 2aD_x + D_x^2\right) g\cdot f = 0, \tag{132}$$

$$\left(\mathrm{i}D_t + \mathrm{i}D_y - D_x^2\right) f\cdot \hat{f} = 0, \tag{133}$$

$$\left(\mathrm{i}D_t - 2\sigma\rho^2\right) f\cdot \hat{f} = -2\sigma\rho^2 g\hat{g}, \tag{134}$$

which admit the following determinant solutions

$$f = \left|\delta_{ij} - \frac{\mathrm{i}\bar{p}_j}{p_i+\bar{p}_j}\mathrm{e}^{\xi_i+\bar{\xi}_j}\right|_{N\times N}, \; g = \left|\delta_{ij} - \frac{\mathrm{i}\bar{p}_j}{p_i+\bar{p}_j}\left(-\frac{p_i-a}{\bar{p}_j+a}\right)\mathrm{e}^{\xi_i+\bar{\xi}_j}\right|_{N\times N}, \tag{135}$$

$$\hat{f} = \left|\delta_{ij} + \frac{\mathrm{i}p_i}{p_i+\bar{p}_j}\mathrm{e}^{\xi_i+\bar{\xi}_j}\right|_{N\times N}, \; \hat{g} = \left|\delta_{ij} + \frac{\mathrm{i}p_i}{p_i+\bar{p}_j}\left(-\frac{\bar{p}_j+a}{p_i-a}\right)\mathrm{e}^{\xi_i+\bar{\xi}_j}\right|_{N\times N}, \tag{136}$$

with

$$\xi_i = \frac{2\sigma a\rho^2}{p_i-a}(t-y) + p_i x + \mathrm{i}p_i^2 y + \xi'_{i0},$$
$$\bar{\xi}_j = \frac{2\sigma a\rho^2}{\bar{p}_j+a}(t-y) + \bar{p}_j x_1 - \mathrm{i}\bar{p}_j^2 y + \bar{\xi}'_{j0}.$$

If we set the complex conjugate condition

$$\hat{f} = f^*, \qquad \hat{g} = g^*, \tag{137}$$

and $a = \mathrm{i}\alpha$, the above bilinear equations become

$$\left(\mathrm{i}D_t + \mathrm{i}D_y + 2\mathrm{i}\alpha D_x + D_x^2\right) g \cdot f = 0, \tag{138}$$

$$\left(\mathrm{i}D_t + \mathrm{i}D_y - D_x^2\right) f \cdot f^* = 0, \tag{139}$$

$$\left(\mathrm{i}D_t - 2\sigma\rho^2\right) f \cdot f^* = -2\sigma\rho^2 g g^*. \tag{140}$$

By using the following dependent variable transformations

$$S = \rho \mathrm{e}^{\mathrm{i}[\alpha x + \beta y - (\alpha^2 + \beta + 2\sigma\rho^2)t]} \frac{g}{f}, \qquad L = \mathrm{i}\frac{d}{dx} \ln \frac{f^*}{f}, \qquad U = \mathrm{i}\frac{d}{dy} \ln \frac{f^*}{f}, \tag{141}$$

a $(2+1)$-dimensional derivative YO system can be derived from bilinear Eqs. (138)–(140),

$$\mathrm{i}(S_t + S_y) + S_{xx} + \mathrm{i}SL_x + SL^2 - 2\sigma S|S|^2 - US = 0, \tag{142}$$

$$L_t = 2\sigma\left(|S|^2\right)_x, \quad U_x = L_y. \tag{143}$$

If we insert $\bar{p}_i = p_i^*$ for $i = 1, 2, \ldots, N$, it is easy to verify that Eq.(137) is satisfied. Therefore, the $(2+1)$-dimensional derivative YO system (142) and (143) has multisoliton solution (141), and (135) and (136) with $\bar{p}_i = p_i^*$ for $i = 1, 2, \ldots, N$.

To find the breather solution, we assume that the integer N in (135) and (136) is even, i.e., $N = 2M$, and then tau functions f, $\hat{f}$, g, and $\hat{g}$ can be rewritten as:

$$f = \Delta_0\Delta_1 \left| \frac{\delta_{ij}(-1)^i}{\bar{p}_i \mathrm{e}^{\xi_i + \bar{\xi}_i}} - \frac{(-1)^i \mathrm{i}}{p_i + \bar{p}_j} \right|_{N\times N} \equiv \Delta_0\Delta_1 \left|(F_{ij})_{1\le i,j\le N}\right|, \tag{144}$$

$$\hat{f} = \Delta_0\bar{\Delta}_1 \left| \frac{\delta_{ij}(-1)^{i+1}}{p_i \mathrm{e}^{\xi_i + \bar{\xi}_i}} + \frac{(-1)^{i+1} \mathrm{i}}{p_i + \bar{p}_j} \right|_{N\times N} \equiv \Delta_0\bar{\Delta}_1 \left|(\hat{F}_{ij})_{1\le i,j\le N}\right|, \tag{145}$$

$$\begin{aligned} g &= \Delta_0\Delta_1\Delta_2 \left| \frac{\delta_{ij}(-1)^i}{\bar{p}_i \mathrm{e}^{\xi_i + \bar{\xi}_i}} \left(-\frac{\bar{p}_i + a}{p_i - a}\right) - \frac{(-1)^i \mathrm{i}}{p_i + \bar{p}_j} \right|_{N\times N} \\ &\equiv \Delta_0\Delta_1\Delta_2 \left|(G_{ij})_{1\le i,j\le N}\right|, \end{aligned} \tag{146}$$

$$\hat{g} = \frac{\Delta_0\bar{\Delta}_1}{\Delta_2} \left| \frac{\delta_{ij}(-1)^{i+1}}{p_i \mathrm{e}^{\xi_i + \bar{\xi}_i}} \left(-\frac{p_i - a}{\bar{p}_i + a}\right) + \frac{(-1)^{i+1} \mathrm{i}}{p_i + \bar{p}_j} \right|_{N\times N} \equiv \frac{\Delta_0\bar{\Delta}_1}{\Delta_2} \left|(\hat{G}_{ij})_{1\le i,j\le N}\right|, \tag{147}$$

with

$$\Delta_0 = \mathrm{e}^{\sum_{i=1}^{N}\xi_i+\bar{\xi}_i}, \qquad \Delta_1 = \prod_{k=1}^{M}(-\bar{p}_{2k-1}\bar{p}_{2k}),$$

$$\bar{\Delta}_1 = \prod_{k=1}^{M}(-p_{2k-1}p_{2k}), \qquad \Delta_2 = \prod_{i=1}^{N}\left(\frac{p_i-a}{\bar{p}_i+a}\right).$$

Further, we conduct the dimension reduction, and then the bilinear equations (138)–(140) reduce to the bilinear form of the (1+1)-dimensional DYO system (69)–(71). The derivation of the breather solution corresponds to different reduction from same bilinear members in KP hierarchy as the construction of dark solutions. As shown in the subsequent section, there exist two types of breather solution.

4.2. *The breather I solution to (1+1)-dimensional DYO system*

The breather I solution for the (1+1)-dimensional DYO system is given in the following theorem:

THEOREM 3. *The tau functions satisfying bilinear equations* (69)–(71) *are given by the determinants* $f = |F_{k,l}|$ *and* $g = |G_{k,l}|$, *where the matrix elements are defined by*

$$F_{k,k} = \begin{pmatrix} -\frac{1}{(\mathrm{i}\omega_k+\Omega_k)\mathrm{e}^{\zeta_k}} + \frac{1}{2\omega_k} & -\frac{\mathrm{i}}{\Omega_k+\Omega_k^*} \\ -\frac{\mathrm{i}}{\Omega_k+\Omega_k^*} & -\frac{1}{(\mathrm{i}\omega_k+\Omega_k^*)\mathrm{e}^{\zeta_k^*}} + \frac{1}{2\omega_k} \end{pmatrix}, \tag{148}$$

$$G_{k,k} = \begin{pmatrix} \frac{1}{(\mathrm{i}\omega_k+\Omega_k)\mathrm{e}^{\zeta_k}}\left(\frac{\Omega_k+\mathrm{i}\alpha+\mathrm{i}\omega_k}{\Omega_k+\mathrm{i}\alpha-\mathrm{i}\omega_k}\right) + \frac{1}{2\omega_k} & -\frac{\mathrm{i}}{\Omega_k+\Omega_k^*} \\ -\frac{\mathrm{i}}{\Omega_k+\Omega_k^*} & -\frac{1}{(\mathrm{i}\omega_k+\Omega_k^*)\mathrm{e}^{\zeta_k^*}}\left(\frac{\Omega_k^*-\mathrm{i}\alpha+\mathrm{i}\omega_k}{\Omega_k^*-\mathrm{i}\alpha-\mathrm{i}\omega_k}\right) + \frac{1}{2\omega_k} \end{pmatrix}, \tag{149}$$

$$F_{k,l} = G_{k,l} = \begin{pmatrix} \frac{\mathrm{i}}{\mathrm{i}(\omega_k-\omega_l)-(\Omega_k-\Omega_l)} & \frac{\mathrm{i}}{\mathrm{i}(\omega_k-\omega_l)-(\Omega_k+\Omega_l^*)} \\ \frac{\mathrm{i}}{\mathrm{i}(\omega_k-\omega_l)-(\Omega_k^*+\Omega_l)} & \frac{\mathrm{i}}{\mathrm{i}(\omega_k+\omega_l)-(\Omega_k^*-\Omega_l^*)} \end{pmatrix}, \tag{150}$$

with $\zeta_k = 2\mathrm{i}\omega_k x - \frac{4\mathrm{i}\sigma\alpha\rho^2\omega_k}{(\mathrm{i}\Omega_k-\alpha)^2-\omega_k^2}t + \zeta_{k,0}$. *Here,* Ω_k, $\zeta_{k,0}$ *are complex parameters and* ω_k *are real parameters for* $k = 1, 2\ldots, M$, *and these parameters satisfy the constraint condition:*

$$\frac{\mathrm{i}\sigma\alpha\rho^2}{(\mathrm{i}\Omega_k-\alpha)^2-\omega_k^2} + \Omega_k = 0. \tag{151}$$

Proof. By taking

$$p_{2k-1} = \mathrm{i}\omega_k - \Omega_k, \qquad p_{2k} = -\mathrm{i}\omega_k + \Omega_k^*, \qquad \bar{p}_{2k-1} = \mathrm{i}\omega_k + \Omega_k,$$
$$\bar{p}_{2k} = -\mathrm{i}\omega_k - \Omega_k^*, \tag{152}$$

and $a = \mathrm{i}\alpha$, $\xi'_{2k-1,0} = \xi'^{*}_{2k,0} \equiv \xi_{k,0}$, $\bar{\xi}'_{2k-1,0} = \bar{\xi}'^{*}_{2k,0} \equiv \eta_{k,0}$, $\zeta_{k,0} = \xi_{k,0} + \eta_{k,0}$, where $\Omega_k, \xi_{k,0}, \eta_{k,0}, \zeta_{k,0}$ are complex parameters and ω_k are real parameters for $k = 1, 2 \ldots, M$, we have

$$\xi_{2k-1} + \bar{\xi}_{2k-1} = \xi_{2k}^* + \bar{\xi}_{2k}^* = 2\mathrm{i}\omega_k x + 4\omega_k \left[\frac{\mathrm{i}\sigma\alpha\rho^2}{(\mathrm{i}\Omega_k - \alpha)^2 - \omega_k^2} + \Omega_k \right] y$$
$$- \frac{4\mathrm{i}\sigma\alpha\rho^2\omega_k}{(\mathrm{i}\Omega_k - \alpha)^2 - \omega_k^2} t + \zeta_{k,0}, \tag{153}$$

and Δ_0 is real, $\bar{\Delta}_1 = \Delta_1^*$, and $\Delta_2^{-1} = \Delta_2^*$.

Let

$$\tilde{F} = H_1 F^T H_1^T, \qquad \tilde{G} = H_1 G^T H_1^T, \tag{154}$$

where ^{T} denotes the transposition, and H_1 is an antisymmetric $2M \times 2M$ matrix:

$$H_1 = \begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_1 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

one can find

$$\tilde{F}_{ij}^* = \hat{F}_{ij}, \qquad \tilde{G}_{ij}^* = \hat{G}_{ij}, \tag{155}$$

and

$$|\tilde{F}| = \left| H_1 F^T H_1^T \right| = |F|, \qquad |\tilde{G}| = \left| H_1 G^T H_1^T \right| = |G|. \tag{156}$$

Considering the gauge freedom of tau functions and noting that $|\Delta_2| = 1$, we can redefine $f = |F|$, $\hat{f} = |\hat{F}|$, $g = |G|$, and $\hat{g} = |\hat{G}|$, and then the complex conjugate condition (137) is satisfied and the breather solution is obtained for the $(2+1)$-dimensional DYO bilinear equations (138)–(140). Further, inserting the dimensional reduction condition (151), the bilinear form of the $(1+1)$-dimensional DYO system (69)–(71) holds and we get the first kind of breather solution as given in Theorem 3. ■

4.2.1. The breather I solution for $M = 1$ and $M = 2$. By taking $M = 1$, the tau functions for the breather I solution are written as:

$$f = \frac{1}{4\omega_1^2} + \frac{1}{\left(\Omega_1 + \Omega_1^*\right)^2} - \frac{\mathrm{e}^{-\zeta_1}}{2\omega_1 A_1} - \frac{\mathrm{e}^{-\zeta_1^*}}{2\omega_1 B_1} + \frac{\mathrm{e}^{-\zeta_1 - \zeta_1^*}}{A_1 B_1}, \tag{157}$$

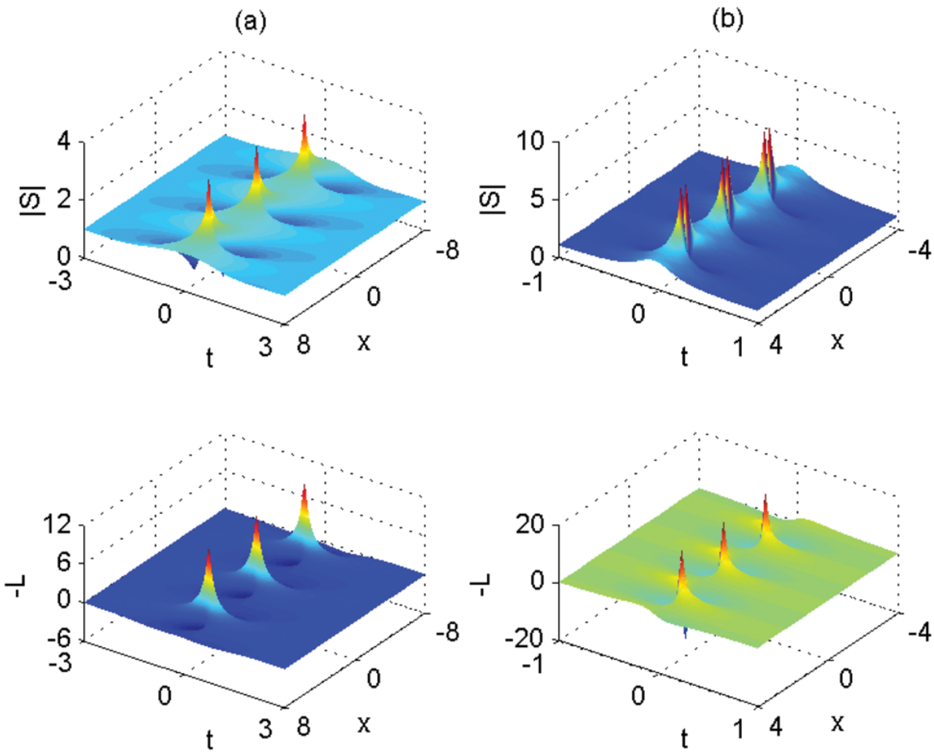

Figure 8. The breather I solution with the parameter $\sigma = \rho = 1$ and $\zeta_{1,0} = 0$: (a) the general breather: $\alpha = -2$, $\omega_1 = \frac{\sqrt{3}}{3}$, and $\Omega_1 = \frac{\sqrt{15}}{6} + \frac{1}{2}\mathrm{i}$; (b) the homoclinic orbit: $\alpha = \frac{3}{2}$, $\omega_1 = \frac{\sqrt{7}}{2}$, and $\Omega_1 = \frac{\sqrt{2}}{2}$.

$$g = \frac{1}{4\omega_1^2} + \frac{1}{\left(\Omega_1 + \Omega_1^*\right)^2} - \frac{P_1 \mathrm{e}^{-\zeta_1}}{2\omega_1 A_1} - \frac{Q_1 \mathrm{e}^{-\zeta_1^*}}{2\omega_1 B_1} + \frac{P_1 Q_1 \mathrm{e}^{-\zeta_1 - \zeta_1^*}}{A_1 B_1}, \tag{158}$$

with

$$A_1 = \mathrm{i}\omega_1 + \Omega_1, \qquad B_1 = \mathrm{i}\omega_1 + \Omega_1^*,$$
$$P_1 = -\frac{\mathrm{i}\omega_1 + \left(\Omega_1^* - \mathrm{i}a\right)}{\mathrm{i}\omega_1 - \left(\Omega_1^* - \mathrm{i}a\right)}, \qquad Q_1 = -\frac{\omega_1 + (a - \mathrm{i}\Omega_1)}{\omega_1 - (a - \mathrm{i}\Omega_1)},$$

and $\zeta_1 = 2\mathrm{i}\omega_1 x - \frac{4\mathrm{i}\sigma\alpha\rho^2\omega_1}{(\mathrm{i}\Omega_1 - \alpha)^2 - \omega_1^2} t + \zeta_{1,0}$, in which the parameters need to satisfy (151) for $k = 1$. In particular, if we take the imaginary part of the coefficient of t be zero, one breather solution reduces to the homoclinic orbit solution. Such two kinds of solutions are exhibited in Fig. 8 by choosing the different parameters. More specifically, Fig. 8(a) represents the general breather solution and Fig. 8(b) indicates the homoclinic orbit

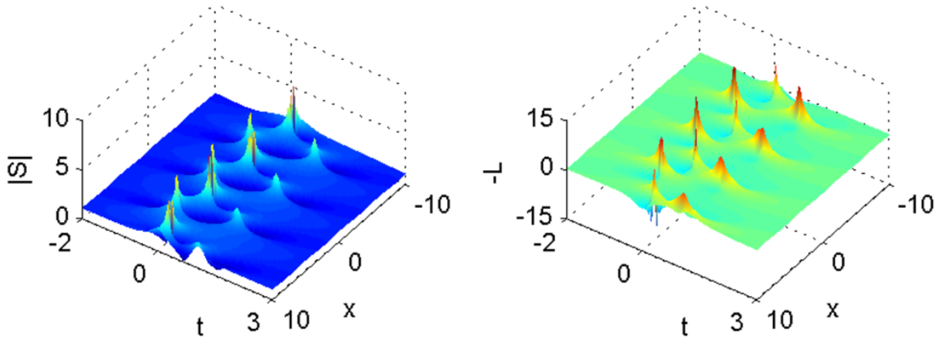

Figure 9. The breather I solution for $M = 2$ with the parameter $\sigma = \rho = 1$, $\alpha = -2$, $\omega_1 = \frac{\sqrt{3}}{3}$, $\Omega_1 = \frac{\sqrt{15}}{6} + \frac{1}{2}\mathrm{i}$, $\omega_2 = \frac{\sqrt{265}}{15}$, $\Omega_1 = \frac{\sqrt{110}}{15} + \frac{1}{3}\mathrm{i}$, and $\zeta_{1,0} = \zeta_{2,0} = 0$.

solution with the parameters satisfying the condition Im $[\frac{4\mathrm{i}\sigma\alpha\rho^2\omega_1}{(\mathrm{i}\Omega_1-\alpha)^2-\omega_1^2}] = 0$. For $M = 2$, the expression of the breather solution is too complicated to list and we only illustrate it in Fig. 9.

4.3. *The breather II solution to the (1+1)-dimensional DYO system*

The breather II solution for the the (1+1)-dimensional DYO system is given in the following theorem:

THEOREM 4. *The tau functions satisfying bilinear equations* (69)–(71) *are given by the determinants* $f = |F_{k,l}|$ *and* $g = |G_{k,l}|$, *where the matrix elements are defined by*

$$F_{k,k} = \begin{pmatrix} -\frac{\mathrm{i}}{(\mathrm{i}\omega_k+\Omega_k)e^{\zeta_k}} + \frac{\mathrm{i}}{2\omega_k} & \frac{\mathrm{i}}{2\omega_k+\mathrm{i}(\Omega_k-\Omega_k^*)} \\ \frac{\mathrm{i}}{-2\omega_k+\mathrm{i}(\Omega_k-\Omega_k^*)} & \frac{\mathrm{i}}{(\mathrm{i}\omega_k+\Omega_k^*)e^{\zeta_k^*}} - \frac{\mathrm{i}}{2\omega_k} \end{pmatrix}, \tag{159}$$

$$G_{k,k} = \begin{pmatrix} -\frac{\mathrm{i}}{(\mathrm{i}\omega_k+\Omega_k)e^{\zeta_k}}\left(\frac{\mathrm{i}a-\mathrm{i}\Omega_k+\omega_k}{\mathrm{i}a-\mathrm{i}\Omega_k-\omega_k}\right) + \frac{\mathrm{i}}{2\omega_k} & \frac{\mathrm{i}}{2\omega_k+\mathrm{i}(\Omega_k-\Omega_k^*)} \\ \frac{\mathrm{i}}{-2\omega_k+\mathrm{i}(\Omega_k-\Omega_k^*)} & \frac{\mathrm{i}}{(\mathrm{i}\omega_k+\Omega_k^*)e^{\zeta_k^*}}\left(\frac{\mathrm{i}\Omega_k^*-\mathrm{i}a-\omega_k}{\mathrm{i}\Omega_k^*-\mathrm{i}a+\omega_k}\right) - \frac{\mathrm{i}}{2\omega_k} \end{pmatrix}, \tag{160}$$

$$F_{k,l} = G_{k,l} = \begin{pmatrix} \frac{\mathrm{i}}{(\omega_k+\omega_l)+\mathrm{i}(\Omega_k-\Omega_l)} & \frac{\mathrm{i}}{(\omega_k+\omega_l)+\mathrm{i}(\Omega_k-\Omega_l^*)} \\ -\frac{\mathrm{i}}{(\omega_k+\omega_l)+\mathrm{i}(\Omega_k^*-\Omega_l)} & -\frac{\mathrm{i}}{(\omega_k+\omega_l)+\mathrm{i}(\Omega_k^*-\Omega_l^*)} \end{pmatrix}, \tag{161}$$

with $\zeta_k = 2\omega_k x - \frac{4\sigma\alpha\rho^2\omega_k}{(\Omega_k-\alpha)^2+\omega_k^2}t + \zeta_{k,0}$. *Here,* $\Omega_k, \zeta_{k,0}$ *are complex parameters and* ω_k *are real parameters for* $k = 1, 2 \ldots, M$, *and these parameters satisfy* *the constraint condition:*

$$\frac{\sigma\alpha\rho^2}{(\Omega_k-\alpha)^2+\omega_k^2}-\Omega_k=0. \tag{162}$$

Proof. Indeed, the breather II solution can be directly obtained from the breather I solution by taking $p_{2k-1}\to -\mathrm{i}p_{2k-1}$, $p_{2k}\to \mathrm{i}p_{2k}$, $\bar{p}_{2k-1}\to -\mathrm{i}\bar{p}_{2k-1}$, and $\bar{p}_{2k}\to \mathrm{i}\bar{p}_{2k}$. More specifically, by setting

$$p_{2k-1}=\omega_k+\mathrm{i}\Omega_k, \qquad p_{2k}=\omega_k+\mathrm{i}\Omega_k^*, \qquad \bar{p}_{2k-1}=\omega_k-\mathrm{i}\Omega_k,$$
$$\bar{p}_{2k}=\omega_k-\mathrm{i}\Omega_k^*, \tag{163}$$

and $a=\mathrm{i}\alpha$, $\xi'_{2k-1,0}=\xi'^*_{2k,0}\equiv\xi_{k,0}$, $\bar{\xi}'_{2k-1,0}=\bar{\xi}'^*_{2k,0}\equiv\eta_{k,0}$, $\zeta_{k,0}=\xi_{k,0}+\eta_{k,0}$, where $\Omega_k,\xi_{k,0},\eta_{k,0},\zeta_{k,0}$ are complex parameters and ω_k are real parameters for $k=1,2\ldots,M$, one has

$$\xi_{2k-1}+\bar{\xi}_{2k-1}=\xi_{2k}^*+\bar{\xi}_{2k}^*=2\omega_k x+4\omega_k\left[\frac{\sigma\alpha\rho^2}{(\Omega_k-\alpha)^2+\omega_k^2}-\Omega_k\right]y$$
$$-\frac{4\sigma\alpha\rho^2\omega_k}{(\Omega_k-\alpha)^2+\omega_k^2}t+\zeta_{k,0}, \tag{164}$$

and Δ_0 is real, $\bar{\Delta}_1=\Delta_1^*$, and $\Delta_2^{-1}=\Delta_2^*$.

Let

$$\tilde{F}=H_1^TF^TH_1, \qquad \tilde{G}=H_1^TG^TH_1, \tag{165}$$

we can verify

$$\tilde{F}_{ij}^*=\hat{F}_{ij}, \qquad \tilde{G}_{ij}^*=\hat{G}_{ij}, \tag{166}$$

and

$$|\tilde{F}|=\left|H_1^TF^TH_1\right|=|F|, \qquad |\tilde{G}|=\left|H_1^TG^TH_1\right|=|G|. \tag{167}$$

Similar to the breather I solution, due to the gauge freedom and $|\Delta_2|=1$, we redefine $f=|F|$, $\hat{f}=|\hat{F}|$, $g=|G|$, and $\hat{g}=|\hat{G}|$, and then the complex conjugate condition (137) is satisfied and the breather solution for the $(2+1)$-dimensional DYO bilinear equations (138)–(140) is obtained. Further, inserting the dimensional reduction condition (162) gives the bilinear form of the $(1+1)$-dimensional DYO system (69)–(71), and the second kind of breather solution is derived in Theorem 4. ■

4.3.1. The breather II solution for $M=1$ and $M=2$. By taking $M=1$, the tau functions for the breather II solution read

$$f=\frac{1}{4\omega_1\left[1+4\frac{\omega_1^2}{(\Omega_1-\Omega_1^*)^2}\right]}-\frac{\mathrm{e}^{-\zeta_1}}{2\omega_1A_1}-\frac{\mathrm{e}^{-\zeta_1^*}}{2\omega_1B_1}+\frac{\mathrm{e}^{-\zeta_1-\zeta_1^*}}{A_1B_1}, \tag{168}$$

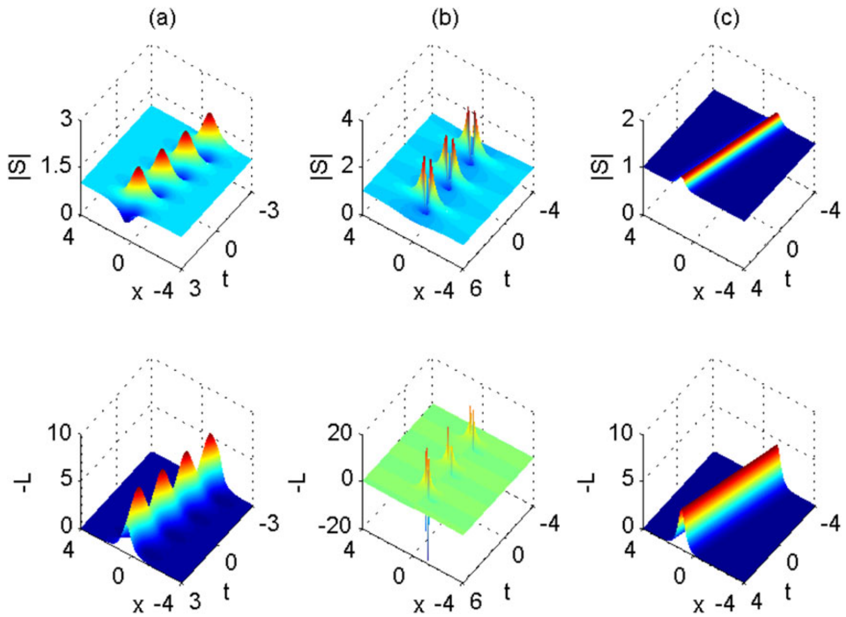

Figure 10. The breather II solution with the parameters: (a) the general breather: $\sigma = \rho = \omega_1 = 1$, $\alpha = \frac{1}{4}$, $\Omega_1 = \frac{1}{8} + \frac{3\sqrt{7}}{8}\mathrm{i}$; (b) the Kuznetsov–Ma breather: $\sigma = 1$, $\rho = \sqrt{2}$, $\alpha = \frac{\sqrt{3}}{2}$, $\omega_1 = \frac{1}{2}$, $\Omega = \mathrm{i}$; and (c) the anti-dark soliton: $\sigma = \rho = 1$, $\alpha = \frac{1}{2}$, $\omega_1 = 2$, $\Omega_1 = \frac{1}{3} + \frac{47}{6r_0} - \frac{1}{6}r_0$, and $r_0 = (91 + 18\sqrt{346})^{\frac{1}{3}}$.

$$g = \frac{1}{4\omega_1\left[1 + 4\frac{\omega_1^2}{(\Omega_1 - \Omega_1^*)^2}\right]} - \frac{P_1 \mathrm{e}^{-\zeta_1}}{2\omega_1 A_1} - \frac{Q_1 \mathrm{e}^{-\zeta_1^*}}{2\omega_1 B_1} + \frac{P_1 Q_1 \mathrm{e}^{-\zeta_1 - \zeta_1^*}}{A_1 B_1}, \quad (169)$$

with

$$A_1 = \mathrm{i}\omega_1 + \Omega_1, \qquad B_1 = \mathrm{i}\omega_1 + \Omega_1^*,$$
$$P_1 = -\frac{\mathrm{i}\omega_1 - (a - \Omega_1)}{\mathrm{i}\omega_1 + (a - \Omega_1)}, \qquad Q_1 = -\frac{\mathrm{i}\omega_1 - \left(a - \Omega_1^*\right)}{\mathrm{i}\omega_1 + \left(a - \Omega_1^*\right)},$$

and $\zeta_1 = 2\omega_1 x - \frac{4\sigma\alpha\rho^2\omega_1}{(\Omega_1-\alpha)^2+\omega_1^2}t + \zeta_{1,0}$, in which the parameters need to satisfy (162) for $k = 1$. Particularly, if the real part of the coefficient of t is taken as zero, one breather solution reduces to the Kuznetsov–Ma breather solution, whereas if the imaginary part of the coefficient of t be taken as zero, the one breather solution degenerates to one dark soliton solution. These kinds of solutions are displayed in Fig. 10 by choosing the different parameters. For $M = 2$, the expression of the breather solution is too complicated to be written here and we only illustrate it in Fig. 11.

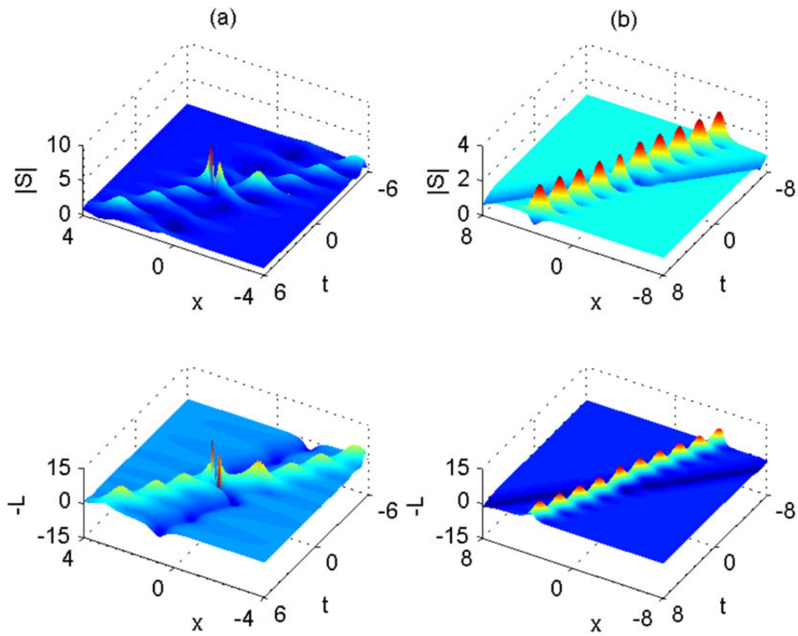

Figure 11. The breather II solution for $M = 2$ with the parameter $\sigma = \rho = 1$, $\alpha = \frac{1}{2}$, and $\zeta_{1,0} = \zeta_{2,0} = 0$: (a) the breather-breather: $\omega_1 = -1$, $\Omega_1 = \frac{1}{4} + \frac{\sqrt{15}}{4}\mathrm{i}$, $\omega_2 = -0.8241$, and $\Omega_2 = 0.1517 + 0.8335\mathrm{i}$; (b) the breather-dark soliton: $\omega_1 = -1$, $\Omega_1 = \frac{1}{4} + \frac{\sqrt{15}}{4}\mathrm{i}$, $\omega_2 = -1$, and $\Omega_2 = \frac{1}{2}$.

5. Summary and conclusion

In the present paper, we give a thorough study for the DYO system which describes an LW–SW interaction model. We first show that such a system can be classified into three types same as the DNLS equation. Then, the N-bright and N-dark soliton solutions in terms of Gram determinants are constructed via the KP-hierarchy reduction method. Based on these soliton solutions, the properties of soliton propagation and collision have been discussed in details. Particularly, it is found that when the SW takes dark soliton solution, it allows the nonzero background anti-dark soliton under certain parameters' condition. The asymptotic analysis of two-soliton solutions reveals that for both kinds of soliton only elastic collision exists and each soliton suffers the phase shifts in the LW and SW.

We also propose a new (2+1)-dimensional DYO system and provide its soliton and breather solutions. Moreover, by considering different reductions, two types of breather solutions to (1+1)-dimensional DYO systems are obtained. These two types of breather solutions are related by $p_{2k-1} \to -\mathrm{i}p_{2k-1}$, $p_{2k} \to \mathrm{i}p_{2k}$, $\bar{p}_{2k-1} \to -\mathrm{i}\bar{p}_{2k-1}$, and $\bar{p}_{2k} \to \mathrm{i}\bar{p}_{2k}$, in which the homoclinic orbit and Kuznetsov–Ma breather solutions are two special cases, respectively.

Finally, it should be pointed out that similar to the multicomponent YO system [21, 22], we can extend the present study to obtain multisoliton and breather solutions of the multicomponent DYO system which is composed of multi-SWs and one LW. In addition, in parallel to the investigation of the integrable YO system [23], the integrable semidiscrete analogue of the DYO system is worth to be expected. We will report the relevant results in the future work.

Acknowledgments

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Appendix A

In this appendix, we give the proof of Lemma 1. By using the differential formula of determinant

$$\partial_x \det_{1\le i,j\le N}(a_{ij}) = \sum_{i,j=1}^{N} \Delta_{ij}\partial_x a_{ij}, \tag{A1}$$

and the expansion formula of bordered determinant

$$\det\begin{pmatrix} a_{ij} & b_i \\ c_j & d \end{pmatrix} = -\sum_{i,j}^{N} \Delta_{ij} b_i c_j + d\,\det(a_{ij}), \tag{A2}$$

with Δ_{ij} being the (i, j)-cofactor of the matrix (a_{ij}), one can check that the derivatives and shifts of tau functions are expressed by the bordered determinants as follows:

$$\partial_{x_1}\tau_{0,0}^{0,0} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix}, \tag{A3}$$

$$\partial_{x_1}^2 \tau_{0,0}^{0,0} = \begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1x_1} & \mathbf{0} & 0 \end{vmatrix}, \tag{A4}$$

$$\partial_{x_2} \tau_{0,0}^{0,0} = \begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1x_1} & \mathbf{0} & 0 \end{vmatrix}, \tag{A5}$$

$$\partial_{x_1} \tau_{1,-1}^{0,0} = \begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}, \tag{A6}$$

$$\partial_{x_1}^2 \tau_{1,-1}^{0,0} = \begin{vmatrix} A & I & \Phi_{x_1x_1}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \Phi^T & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 & 0 \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix}, \tag{A7}$$

$$\partial_{x_2} \tau_{1,-1}^{0,0} = \begin{vmatrix} A & I & \Phi_{x_1x_1}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \Phi^T & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 & 0 \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix}. \tag{A8}$$

On the other hand, tau function $\tau_{1,-1}^{0,0}$ can be rewritten as:

$$\tau_{0,0}^{0,-1} = \begin{vmatrix} A' & I \\ -I & B \end{vmatrix} = \begin{vmatrix} a_{ij} - e^{\xi_i} e^{\bar{\xi}_j} & I \\ -I & B \end{vmatrix} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 1 \end{vmatrix}, \tag{A9}$$

then

$$\tau_{0,0}^{0,-1} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix} + \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}. \tag{A10}$$

Similarly, one can get another expression of tau function $\tau_{-1,1}^{0,-1}$

$$\tau_{-1,1}^{0,-1} = \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}. \tag{A11}$$

Furthermore, we have

$$\partial_{x_1} \tau_{0,0}^{0,-1} = \begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}, \tag{A12}$$

$$\partial_{x_1}^2 \tau_{0,0}^{0,-1} = \begin{vmatrix} A & I & \Phi^T_{x_1x_1} \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \Phi & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 & 0 \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix}, \tag{A13}$$

$$\partial_{x_2} \tau_{0,0}^{0,-1} = \begin{vmatrix} A & I & \Phi^T_{x_1x_1} \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \Phi & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 & 0 \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix}, \tag{A14}$$

$$\partial_y \tau_{0,0}^{0,0} = \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}, \tag{A15}$$

$$\partial_y \tau_{0,0}^{0,-1} = \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \mathbf{0}^T & \Phi^T \\ -I & B & \Psi & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 & 0 \\ -\bar{\Phi} & \mathbf{0} & 0 & 0 \end{vmatrix}. \tag{A16}$$

By using above formulae, the bilinear equations (32)–(34) become

$$\begin{aligned} &(\partial_{x_2} - \partial_{x_1x_1})\tau_{1,-1}^{0,0} \times \tau_{0,0}^{0,0} - (\partial_{x_2} + \partial_{x_1x_1})\tau_{0,0}^{0,0} \times \tau_{1,-1}^{0,0} + 2\partial_{x_1}\tau_{0,0}^{0,0} \times \partial_{x_1}\tau_{1,-1}^{0,0} \\ &= 2\begin{vmatrix} A & I & \Phi^T & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 & 0 \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix} \begin{vmatrix} A & I \\ -I & B \end{vmatrix} - 2\begin{vmatrix} A & I & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix} \\ &\quad +2\begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} \begin{vmatrix} A & I & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}, \end{aligned} \tag{A17}$$

$$\begin{aligned} &(\partial_{x_1x_1} - \partial_{x_2})\tau_{0,0}^{0,-1} \times \tau_{0,0}^{0,0} + (\partial_{x_1x_1} + \partial_{x_2})\tau_{0,0}^{0,0} \times \tau_{0,0}^{0,-1} - 2\partial_{x_1}\tau_{0,0}^{0,0} \times \partial_{x_1}\tau_{0,0}^{0,-1} \\ &= 2\left(\begin{vmatrix} A & I & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix} + \begin{vmatrix} A & I & \Phi & \Phi^T_{x_1} \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 & 0 \\ \bar{\Phi}_{x_1} & \mathbf{0} & 0 & 0 \end{vmatrix} \right) \begin{vmatrix} A & I \\ -I & B \end{vmatrix} \end{aligned}$$

$$+2\begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix}\left(\begin{vmatrix} A & I \\ -I & B \end{vmatrix}+\begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}\right)$$

$$-2\begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ -\bar{\Phi}_{x_1} & \mathbf{0} & 0 \end{vmatrix}\begin{vmatrix} A & I & \Phi_{x_1}^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}, \tag{A18}$$

$$\partial_y \tau_{0,0}^{0,0} \times \tau_{0,0}^{0,-1} - \partial_y \tau_{0,0}^{0,-1} \times \tau_{0,0}^{0,0} + \tau_{1,-1}^{0,0}\tau_{-1,1}^{0,-1}$$

$$= \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}\left(\begin{vmatrix} A & I \\ -I & B \end{vmatrix}+\begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}\right)$$

$$-\left(\begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix} - \begin{vmatrix} A & I & \mathbf{0}^T & \Phi^T \\ -I & B & \Psi^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 & 0 \\ -\bar{\Phi} & \mathbf{0} & 0 & 0 \end{vmatrix}\right)\begin{vmatrix} A & I \\ -I & B \end{vmatrix}$$

$$+\begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}\begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}, \tag{A19}$$

which are nothing but Jacobi's identities.

Appendix B

To prove Lemma 2, we first define

$$\theta_i(n,k) = \mathrm{i} p_i p_i^n (p_i - a)^k \mathrm{e}^{\xi_i}, \tag{B1}$$

$$\omega_i(n,k) = (-\bar{p}_i)^{-n}[-(\bar{p}_i + a)]^{-k}\mathrm{e}^{\bar{\xi}_i}, \tag{B2}$$

which satisfy the differential and difference rules:

$$\partial_{x_2}\theta_i(n,k) = \partial_{x_1}^2\theta_i(n,k), \qquad \partial_{x_2}\omega_i(n,k) = -\partial_{x_1}^2\omega_i(n,k), \tag{B3}$$

$$\partial_{x_{-1}}\theta_i(n,k) = \theta_i(n,k-1), \qquad \partial_{x_{-1}}\omega_i(n,k) = -\omega_i(n,k+1), \tag{B4}$$

$$(\partial_{x_1} - a)\theta_i(n,k) = \theta_i(n,k+1), \qquad (\partial_{x_1} + a)\omega_i(n,k) = -\omega_i(n,k-1). \tag{B5}$$

Then, one can easily verify that

$$\partial_{x_1} m_{ij}^{n,k} = \theta_i(n,k)\omega_j(n,k), \qquad \partial_{x_{-1}} m_{ij}^{n,k} = -\theta_i(n,k-1)\omega_j(n,k+1), \tag{B6}$$

$$\partial_{x_2} m_{ij}^{n,k} = [\partial_{x_1}\theta_i(n,k)]\omega_j(n,k) - \theta_i(n,k)[\partial_{x_1}\omega_j(n,k)], \tag{B7}$$

$$\begin{aligned} m_{ij}^{n,k+1} &= m_{ij}^{n,k} + \theta_i(n,k)\omega_j(n,k+1), \\ m_{ij}^{n+1,k} &= m_{ij}^{n,k} + \theta_i(n,k)\omega_j(n+1,k). \end{aligned} \tag{B8}$$

Define

$$\mathbf{m} = \begin{pmatrix} m_{11}^{n,k} & m_{12}^{n,k} & \cdots & m_{1N}^{n,k} \\ m_{21}^{n,k} & m_{22}^{n,k} & \cdots & m_{2N}^{n,k} \\ \vdots & \vdots & \vdots & \vdots \\ m_{N1}^{n,k} & m_{N2}^{n,k} & \cdots & m_{NN}^{n,k} \end{pmatrix},$$

and

$$\Theta(n,k) = (\theta_1(n,k), \ldots, \theta_N(n,k))^T, \qquad \Omega(n,k) = (\omega_1(n,k), \ldots, \omega_N(n,k)),$$

then the following formulae are derived:

$$\partial_{x_1}\tau_{n,k} = \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix}, \tag{B9}$$

$$\partial_{x_1}^2\tau_{n,k} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\partial_{x_1}\Omega(n,k) & 0 \end{vmatrix}, \tag{B10}$$

$$\partial_{x_2}\tau_{n,k} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\partial_{x_1}\Omega(n,k) & 0 \end{vmatrix}, \tag{B11}$$

$$a\partial_{x_{-1}}\tau_{n,k} = \begin{vmatrix} \mathbf{m} & a\Theta(n,k-1) \\ \Omega(n,k+1) & 0 \end{vmatrix}, \tag{B12}$$

$$\tau_{n,k+1} = \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k+1) & 1 \end{vmatrix}, \tag{B13}$$

$$\tau_{n,k-1} = \begin{vmatrix} \mathbf{m} & \Theta(n,k-1) \\ \Omega(n,k) & 1 \end{vmatrix}, \tag{B14}$$

$$(\partial_{x_1} + a)\tau_{n,k+1} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k+1) & a \end{vmatrix}, \tag{B15}$$

$$\begin{aligned} (\partial_{x_1} + a)^2\tau_{n,k+1} &= \begin{vmatrix} \mathbf{m} & \partial_{x_1}^2\Theta(n,k) \\ -\Omega(n,k+1) & a^2 \end{vmatrix} \\ &\quad + \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n,k+1) & a & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix}, \end{aligned} \tag{B16}$$

$$(\partial_{x_2} + a^2)\tau_{n,k+1} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}^2 \Theta(n,k) \\ -\Omega(n,k+1) & a^2 \end{vmatrix}$$

$$- \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n,k+1) & a & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix}. \quad \text{(B17)}$$

Similarly,

$$\tau_{n+1,k} = \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n+1,k) & 1 \end{vmatrix}, \quad \text{(B18)}$$

$$\tau_{n+1,k-1} = \begin{vmatrix} \mathbf{m} & a\Theta(n,k-1) \\ \Omega(n+1,k) & 1 \end{vmatrix}, \quad \text{(B19)}$$

$$\partial_{x_1}\tau_{n+1,k} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n+1,k) & 0 \end{vmatrix}, \quad \text{(B20)}$$

$$(a\partial_{x_{-1}} - 1)\tau_{n+1,k} = \begin{vmatrix} \mathbf{m} & \Theta(n,k) & a\Theta(n,k-1) \\ -\Omega(n+1,k) & 1 & -1 \\ \Omega(n,k+1) & -1 & 0 \end{vmatrix}, \quad \text{(B21)}$$

$$\partial_{x_1}^2\tau_{n+1,k} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}^2\Theta(n,k) \\ -\Omega(n+1,k) & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n+1,k) & 0 & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix}, \quad \text{(B22)}$$

$$\partial_{x_2}\tau_{n+1,k} = \begin{vmatrix} \mathbf{m} & \partial_{x_1}^2\Theta(n,k) \\ -\Omega(n+1,k) & 0 \end{vmatrix}$$

$$- \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n+1,k) & 0 & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix}. \quad \text{(B23)}$$

Thus, the bilinear equations (76)–(78) become

$$\left[\left(\partial_{x_2} + a^2\right) - \left(\partial_{x_1} + a\right)^2\right]\tau_{n,k+1} \times \tau_{n,k} - (\partial_{x_2} + \partial_{x_1x_1})\tau_{n,k} \times \tau_{n,k+1}$$

$$+2(\partial_{x_1} + a)\tau_{n,k+1} \times \partial_{x_1}\tau_{n,k}$$

$$= -2 \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n,k+1) & a & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix} |\mathbf{m}|$$

$$-2 \begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix} \begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k+1) & 1 \end{vmatrix}$$

$$+2\begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k+1) & a \end{vmatrix}\begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix}, \tag{B24}$$

$$\begin{aligned}
&(\partial_{x_2}+\partial_{x_1x_1})\tau_{n,k}\times\tau_{n+1,k}(\partial_{x_2}-\partial_{x_1x_1})\tau_{n+1,k}\times\tau_{n,k}-2\partial_{x_1}\tau_{n,k}\times\partial_{x_1}\tau_{n+1,k}\\
&\quad=2\begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix}\begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n+1,k) & 1 \end{vmatrix}\\
&\quad-2\begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) & \Theta(n,k) \\ -\Omega(n+1,k) & 0 & 1 \\ -\Omega(n,k) & 0 & 0 \end{vmatrix}|\mathbf{m}|\\
&\quad-2\begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k) & 0 \end{vmatrix}\begin{vmatrix} \mathbf{m} & \partial_{x_1}\Theta(n,k) \\ -\Omega(n+1,k) & 0 \end{vmatrix},
\end{aligned} \tag{B25}$$

$$\begin{aligned}
&a\partial_{x_1}\tau_{n,k}\times\tau_{n+1,k}-(a\partial_{x_1}-1)\tau_{n+1,k}\times\tau_{n,k}-\tau_{n,k+1}\tau_{n+1,k-1}\\
&=\begin{vmatrix} \mathbf{m} & a\Theta(n,k-1) \\ \Omega(n,k+1) & 0 \end{vmatrix}\begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n+1,k) & 1 \end{vmatrix}\\
&-\begin{vmatrix} \mathbf{m} & \Theta(n,k) & a\Theta(n,k-1) \\ -\Omega(n+1,k) & 1 & -1 \\ \Omega(n,k+1) & -1 & 0 \end{vmatrix}|\mathbf{m}|\\
&-\begin{vmatrix} \mathbf{m} & \Theta(n,k) \\ -\Omega(n,k+1) & 1 \end{vmatrix}\begin{vmatrix} \mathbf{m} & a\Theta(n,k-1) \\ \Omega(n+1,k) & 1 \end{vmatrix},
\end{aligned} \tag{B26}$$

which are satisfied by Jacobi's identities.

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