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General High-order Rogue Waves of the (1+1)-Dimensional Yajima–Oikawa System

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General high-order rogue wave solutions for the (1+1)-dimensional Yajima–Oikawa (YO) system are derived by using the Hirota's bilinear method and the KP hierarchy reduction method. These rogue wave solutions are presented in terms of determinants in which the elements are algebraic expressions. The dynamics of first-order and higher-order rogue wave are investigated in details for different values of the free parameters. It is shown that the fundamental (first-order) rogue waves can be classified into three different patterns: bright, intermediate and dark ones. The higher-order rogue waves correspond to the superposition of fundamental rogue waves. Especially, compared with the nonlinear Schrödinger equation, there exists an essential parameter α to control the pattern of rogue wave for both first-order and higher-order rogue waves since the YO system does not possess the Galilean invariance.

1. Introduction

Rogue waves, which are initially used for the vivid description of the spontaneous and monstrous ocean surface waves,[1] have recently attracted considerable attention both experimentally and theoretically. Rogue waves have been observed in a variety of different fields, including optical systems,[2–4] Bose–Einstein condensates,[5,6] superfluids,[7,8] plasma,[9,10] capillary waves[11] and even finance.[12] Compared with the stable solitons, rogue waves are the localized structures with the instability and unpredictability.[13,14] A typical model for characterizing the rogue wave is the celebrated nonlinear Schrödinger (NLS) equation. The most fundamental rogue wave of the NLS equation is described by Peregrine soliton,[15] which is the first-order rogue wave and expressed in a simple rational form including the polynomials up to second order. This rational solution has localized behavior in both space and time, and its maximum amplitude attains three times the constant background. The Peregrine soliton can be obtained from a breather solution when the period is taken to infinity. More recently, significant progress on higher-order rogue waves has been made[16–41] since a few special higher-order rogue waves from first-order to fourth-order were provided theoretically by Akhmediev et al.[16] via the Darboux transformation. The higher-order rogue waves were also excited experimentally in a water wave tank,[42,43] which guarantees that such nonlinear complicated waves are meaningful physically. In fact, higher-order rogue waves can be treated as the nonlinear superposition of fundamental rogue wave and they are usually expressed in terms of complicated higher-order rational polynomials. These higher-order waves were also localized in both coordinates and could exhibit higher peak amplitudes or multiple intensity peaks.

Another major development of importance is the study of rogue waves in multicomponent coupled systems, as a lot of complex physical systems usually contain interacting wave components with different modes and frequencies.[44–59] As stated in Ref. 44, the cross-phase modulation term in coupled systems leads to the varying instability regime characters. Due to the additional degrees of freedom, there exist more abundant pattern structures and dynamics characters for rogue waves in coupled systems. For instance, in the scalar NLS equation, because of the existence of Galilean invariance, the velocity of the background field does not influence the pattern of rogue waves. However, for the coupled NLS system, the relative velocity between different component fields has real physical effects, and cannot be removed by any trivial transformation. This fact brings some novel patterns for rogue waves such as dark rogue waves,[45] the interaction between rogue waves and other nonlinear waves,[45,46] a four-petaled flower structure[47] and so on. In particular, those more various higher-order rogue waves in coupled nonlinear models enrich the realization and understanding of the mechanisms underlying the complex dynamics of rogue waves.

Among coupled wave dynamics systems, the long wave-short wave resonance interaction (LSRI) is a fascinating physical process in which a resonant interaction takes place between a weakly dispersive long-wave (LW) and a short-wave (SW) packet when the phase velocity of the former exactly or almost matches the group velocity of the latter.[60,61] The theoretical investigation of this LSRI was first done by Zakharov[62] on Langmuir waves in plasma. In the case of long wave propagating in one direction, the general Zakharov system was reduced to the LSRI system which is often called the one-dimensional (1D) Yajima–Oikawa (YO) system.[63] This phenomenon has been predicted in diverse areas such as plasma physics,[62,63] hydrodynamics[64–68] and nonlinear optics.[69,70] For instance, this resonance interaction can occur between the long gravity wave and the capillary-gravity one,[64] between long and short internal waves,[65] and between a long internal wave and a short surface wave in a two layer fluid.[66] In a second-order nonlinear negative refractive index medium, it can be achieved when the short wave lies on the negative index branch while the long wave resides in the positive index branch.[70] The 1D YO system (1D LSRI system) can be written in a dimensionless form:

$$\mathrm{i}S_t - S_{xx} + SL = 0, \tag{1}$$

$$L_t = -4(|S|^2)_x, \tag{2}$$

where S and L represent the short wave and long wave component, respectively. The 1D YO system was shown to be integrable with a Lax pair, and was solved by the inverse scattering transform method.[63] It admits both bright and dark soliton solutions.[67,68] In Refs. 71 and 72, it is shown that the 1D YO system can be derived from the so-called k-constrained KP hierarchy with $k = 2$ while the NLS equation with $k = 1$. Very recently, the first-order rogue wave solutions to the 1D YO system have been derived by using the Hirota's bilinear method[73] and the Darboux transformation.[74,75] These vector parametric solutions indicate interesting structures such that the long wave field always keeps a single hump structure, whereas the short-wave field can be manifested as bright, intermediate and dark rogue wave. In our previous paper,[30] rational solutions including lump solutions and rogue wave solutions of the 2D multi-component YO system were presented and some rogue wave solutions of the 1D multi-component YO system through reduction were discussed. Nevertheless, as far as we know, there is no report about high-order rogue wave solutions for the 1D YO system. Therefore, the objective of present paper is to study high-order rogue wave solutions of the 1D YO system (1)–(2) by using the bilinear method in the framework of KP hierarchy reduction. As will be shown in the subsequent section, a general rogue wave solutions in the form of Gram determinant is derived based on Hirota's bilinear method and the KP hierarchy reduction method. This determinant solution can generate rogue waves of any order without singularity.

The remainder of this paper is organized as follows. In Sect. 2, we start with a set of bilinear equations satisfied by the τ functions in Gram determinant of the KP hierarchy, and reduce them to bilinear equations satisfied by the 1D YO system (1)–(2). The reductions include mainly dimension reduction and complex conjugate reduction. We should emphasize here that the most crucial and difficult issue is to find a general algebraic expression for the element of determinant such that the dimension reduction can be realized. In Sect. 3, the dynamical behaviors of fundamental and higher-order rogue wave solutions are illustrated for different choices of free parameters. The paper is concluded in Sect. 4 by a brief summary and discussion.

2. Derivation of General Rogue Wave Solutions

This section is the core of the present paper, in which an explicit expression for general rogue wave solutions of the 1D YO system (1)–(2) will be derived by Hirota's bilinear method. To this end, let us first introduce dependent variable transformations

$$S = e^{\mathrm{i}[\alpha x + (h+\alpha^2)t]}\,\frac{g}{f}, \quad L = h - 2\frac{\partial^2}{\partial x^2}\log f, \tag{3}$$

where f is a real-valued function, g is a complex-valued function and α and h are real constants. Then the 1D YO system (1)–(2) is converted into the following bilinear equations

$$(D_x^2 + 2\mathrm{i}\alpha D_x - \mathrm{i}D_t)g\cdot f = 0, \tag{4}$$

$$(D_x D_t + 4)f\cdot f = 4gg^*, \tag{5}$$

where * denotes the complex conjugation hereafter and the D is Hirota's bilinear differential operator defined by

$$D_x^n D_t^m (a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')|_{x=x',t=t'}.$$

Prior to the tedious process in deriving the polynomial solutions of the functions f and g, we highlight the main steps of the detailed derivation, as shown in the subsequent subsections.

Firstly, we start from the following bilinear equations of the KP hierarchy:

$$(D_{x_1}^2 + 2aD_{x_1} - D_{x_2})\tau_{n+1}\cdot\tau_n = 0, \tag{6}$$

$$\left(\frac{1}{2}D_{x_1}D_{t_a} - 1\right)\tau_n\cdot\tau_n = -\tau_{n+1}\tau_{n-1}, \tag{7}$$

which admit a wide class of solutions in terms of Gram or Wronski determinant. Among these determinant solutions, we need to look for algebraic solutions to satisfy the reduction condition:

$$(\partial_{x_2} + 2\mathrm{i}\partial_{t_a})\tau_n = c\tau_n, \tag{8}$$

such that these algebraic solutions satisfy the (1+1)-dimensional bilinear equations:

$$(D_{x_1}^2 + 2aD_{x_1} - D_{x_2})\tau_{n+1}\cdot\tau_n = 0, \tag{9}$$

$$(\mathrm{i}D_{x_1}D_{x_2} - 4)\tau_n\cdot\tau_n = -4\tau_{n+1}\tau_{n-1}. \tag{10}$$

Furthermore, by introducing the variable transformations:

$$x_1 = x, \quad x_2 = -\mathrm{i}t, \tag{11}$$

and taking $f = \tau_0$, $g = \tau_1$, $h = \tau_{-1}$, and $a = \mathrm{i}\alpha$, the above bilinear equations (9)–(10) become

$$(D_x^2 + 2\mathrm{i}\alpha D_x - \mathrm{i}D_t)g\cdot f = 0, \tag{12}$$

$$(D_x D_t + 4)f\cdot f = 4gh. \tag{13}$$

Lastly, by requiring the real and complex conjugation condition:

$$f = \tau_0 : \text{real}, \quad g = \tau_1, \quad h = \tau_{-1} = g^*, \tag{14}$$

in the algebraic solutions, then the bilinear equations (12)–(13) are reduced to the bilinear equations (4)–(5), hence the general higher-order rogue wave solutions are obtained through the reductions.

2.1 Gram determinant solution for the bilinear equations in the KP hierarchy

In this subsection, through the Lemma below, we present and prove that a pair of bilinear equations are satisfied by the τ functions of the KP hierarchy.

Lemma 2.1. *Let $m_{ij}^{(n)}$, depending on $\varphi_i^{(n)}$ and $\psi_j^{(n)}$, be function of the variables x_1, x_2, and t_a, and satisfy the following differential and difference relations:*

$$\partial_{x_1} m_{ij}^{(n)} = \varphi_i^{(n)}\psi_j^{(n)},$$

$$\partial_{x_2} m_{ij}^{(n)} = [\partial_{x_1}\varphi_i^{(n)}]\psi_j^{(n)} - \varphi_i^{(n)}[\partial_{x_1}\psi_j^{(n)}],$$

$$\partial_{t_a} m_{ij}^{(n)} = -\varphi_i^{(n-1)}\psi_j^{(n+1)},$$

$$m_{ij}^{(n+1)} = m_{ij}^{(n)} + \varphi_i^{(n)}\psi_j^{(n+1)}, \tag{15}$$

where $\varphi_i^{(n)}$ *and* $\psi_j^{(n)}$ *are functions satisfying*

$$\partial_{x_2}\varphi_i^{(n)} = \partial_{x_1}^2\varphi_i^{(n)}, \quad \varphi_i^{(n+1)} = (\partial_{x_1} - a)\varphi_i^{(n)},$$
$$\partial_{x_2}\psi_j^{(n)} = -\partial_{x_1}^2\psi_j^{(n)}, \quad \psi_i^{(n-1)} = -(\partial_{x_1} + a)\psi_j^{(n)}. \tag{16}$$

Then the τ *functions of the following determinant form*

$$\tau_n = \det_{1\le i,j\le N}(m_{ij}^{(n)}), \tag{17}$$

satisfy the following bilinear equations (6) and (7) in KP hierarchy:

$$(D_{x_1}^2 + 2aD_{x_1} - D_{x_2})\tau_{n+1}\cdot\tau_n = 0, \tag{18}$$
$$\left(\frac{1}{2}D_{x_1}D_{t_a} - 1\right)\tau_n\cdot\tau_n = -\tau_{n+1}\tau_{n-1}. \tag{19}$$

Proof. By using the differential formula of determinant

$$\partial_x \det_{1\le i,j\le N}(a_{ij}) = \sum_{i,j=1}^{N}\Delta_{ij}\partial_x a_{ij}, \tag{20}$$

and the expansion formula of bordered determinant

$$\det\begin{pmatrix} a_{ij} & b_i \\ c_j & d\end{pmatrix} = -\sum_{i,j}^{N}\Delta_{ij}b_ic_j + d\det(a_{ij}), \tag{21}$$

with Δ_{ij} being the (i,j)-cofactor of the matrix (a_{ij}), one can check that the derivatives and shifts of the τ function are expressed by the bordered determinants as follows:

$$\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix}, \quad \partial_{x_1}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\partial_{x_1}\psi_j^{(n)} & 0\end{vmatrix},$$

$$\partial_{x_2}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\partial_{x_1}\psi_j^{(n)} & 0\end{vmatrix}, \quad \partial_{t_a}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n+1)} & 0\end{vmatrix},$$

$$(\partial_{x_1}\partial_{t_a} - 1)\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} & \varphi_i^{(n)} \\ \psi_j^{(n+1)} & 0 & -1 \\ -\psi_j^{(n)} & -1 & 0\end{vmatrix},$$

$$\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & 1\end{vmatrix}, \quad \tau_{n-1} = \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n)} & 1\end{vmatrix}, \quad (\partial_{x_1} + a)\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a\end{vmatrix},$$

$$(\partial_{x_1} + a)^2\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}^2\varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a^2\end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a & 1 \\ -\psi_j^{(n)} & 0 & 0\end{vmatrix},$$

$$(\partial_{x_2} + a^2)\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}^2\varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a^2\end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a & 1 \\ -\psi_j^{(n)} & 0 & 0\end{vmatrix}.$$

With the help of these relations, one has

$$\begin{aligned}
&(\partial_{x_1}\partial_{t_a} - 1)\tau_n\times\tau_n - \partial_{x_1}\tau_n\times\partial_{t_a}\tau_n + (-\tau_{n-1})(-\tau_{n+1}) \\
&= \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} & \varphi_i^{(n)} \\ \psi_j^{(n+1)} & 0 & -1 \\ -\psi_j^{(n)} & -1 & 0\end{vmatrix}\times|m_{ij}^{(n)}| - \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix}\times\begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ \psi_j^{(n+1)} & 0\end{vmatrix} \\
&\quad + \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n-1)} \\ -\psi_j^{(n)} & -1\end{vmatrix}\times\begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n+1)} & -1\end{vmatrix},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{1}{2}(\partial_{x_1}^2 + 2a\partial_{x_1} - \partial_{x_2})\tau_{n+1}\times\tau_n - (\partial_{x_1} + a)\tau_{n+1}\times\partial_{x_1}\tau_n + \tau_{n+1}\times\frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2})\tau_n \\
&= \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a & 1 \\ -\psi_j^{(n)} & 0 & 0\end{vmatrix}\times|m_{ij}^{(n)}| - \begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix}\times\begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} \\ -\psi_j^{(n+1)} & a\end{vmatrix} \\
&\quad + \begin{vmatrix} m_{ij}^{(n)} & \partial_{x_1}\varphi_i^{(n)} \\ -\psi_j^{(n)} & 0\end{vmatrix}\times\begin{vmatrix} m_{ij}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & 1\end{vmatrix}.
\end{aligned} \tag{23}$$

The r.h.s of both (22) and (23) are identically zero because of the Jacobi identity and hence the τ functions (17) satisfy the bilinear equations (18) and (19). This completes the proof. □

2.2 Algebraic solutions for the (1+1)-dimensional YO system

This subsection is crucial in the KP hierarchy reduction method. We will construct an algebraic expression for the elements of τ function of preceding subsection so that the dimension reduction condition (8) is satisfied. The main result is given by the following Lemma.

Lemma 2.2. *Suppose the entries $m_{kl}^{(\nu\mu n)}$ of the matrix m are*

$$m_{kl}^{(\nu\mu n)} = (A_k^{(\nu)} B_l^{(\mu)} m^{(n)})|_{p=\zeta, q=\zeta^*}, \tag{24}$$

where

$$m^{(n)} = \frac{1}{p+q}\left(-\frac{p-a}{q+a}\right)^n e^{\xi+\eta}, \quad \xi = px_1 + p^2 x_2,$$
$$\eta = qx_1 - q^2 x_2,$$
$$\zeta = \zeta_r + \mathrm{i}\zeta_i, \quad \zeta_r = \pm\frac{\sqrt{3}}{12}\frac{K^2 - 4\alpha^2}{K},$$
$$\zeta_i = \frac{1}{12}K + \frac{1}{3}\frac{\alpha^2}{K} + \frac{2}{3}\alpha,$$
$$K = (8\alpha^3 + 108 + 12\sqrt{12\alpha^3 + 81})^{1/3}, \quad \left(\alpha > -\frac{3}{2}2^{1/3}\right),$$

and $A_k^{(\nu)}$ and $B_l^{(\mu)}$ are differential operators with respect to p and q, respectively, given by

$$A_n^{(\nu)} = \sum_{k=0}^{n} a_k^{(\nu)} \frac{[(p-a)\partial_p]^{n-k}}{(n-k)!}, \quad n \geq 0, \tag{25}$$

$$B_n^{(\nu)} = \sum_{k=0}^{n} b_k^{(\nu)} \frac{[(q+a)\partial_q]^{n-k}}{(n-k)!}, \quad n \geq 0, \tag{26}$$

where $a_k^{(\nu)}$ and $b_k^{(\nu)}$ are constants satisfying the iterated relations

$$a_k^{(\nu+1)} = \sum_{j=0}^{k} \frac{2^{j+2}(p-a)^2 + (-1)^j \dfrac{2\mathrm{i}}{p-a} + 2a(p-a)}{(j+2)!} a_{k-j}^{(\nu)},$$
$$\nu = 0, 1, 2, \ldots, \tag{27}$$

$$b_k^{(\nu+1)} = \sum_{j=0}^{k} \frac{2^{j+2}(q+a)^2 - (-1)^j \dfrac{2\mathrm{i}}{q+a} - 2a(q+a)}{(j+2)!} b_{k-j}^{(\nu)},$$
$$\nu = 0, 1, 2, \ldots, \tag{28}$$

then the determinant

$$\tau_n = \det_{1\leq i,j\leq N}(m_{2i-1,2j-1}^{(N-i,N-j,n)})$$
$$= \begin{vmatrix} m_{11}^{(N-1,N-1,n)} & m_{13}^{(N-1,N-2,n)} & \cdots & m_{1,2N-1}^{(N-1,0,n)} \\ m_{31}^{(N-2,N-1,n)} & m_{33}^{(N-2,N-2,n)} & \cdots & m_{3,2N-1}^{(N-2,0,n)} \\ \vdots & \vdots & & \vdots \\ m_{2N-1,1}^{(0,N-1,n)} & m_{2N-1,3}^{(0,N-2,n)} & \cdots & m_{2N-1,2N-1}^{(0,0,n)} \end{vmatrix}, \tag{29}$$

satisfies the bilinear equations

$$(D_{x_1}^2 + 2aD_{x_1} - D_{x_2})\tau_{n+1}\cdot\tau_n = 0, \tag{30}$$

$$(\mathrm{i}D_{x_1}D_{x_2} - 4)\tau_n\cdot\tau_n = -4\tau_{n+1}\tau_{n-1}. \tag{31}$$

Proof. Firstly, we introduce the functions $\tilde{m}^{(n)}$, $\tilde{\varphi}^{(n)}$, and $\tilde{\psi}^{(n)}$ of the form

$$\tilde{m}^{(n)} = \frac{1}{p+q}\left(-\frac{p-a}{q+a}\right)^n e^{\tilde{\xi}+\tilde{\eta}}, \quad \tilde{\varphi}^{(n)} = (p-a)^n e^{\tilde{\xi}},$$
$$\tilde{\psi}^{(n)} = \left(-\frac{1}{q+a}\right)^n e^{\tilde{\eta}},$$

where

$$\tilde{\xi} = \frac{1}{p-a}t_a + px_1 + p^2 x_2, \quad \tilde{\eta} = \frac{1}{q+a}t_a + qx_1 - q^2 x_2.$$

These functions satisfy the differential and difference rules:

$$\partial_{x_1}\tilde{m}^{(n)} = \tilde{\varphi}^{(n)}\tilde{\psi}^{(n)},$$
$$\partial_{x_2}\tilde{m}^{(n)} = [\partial_{x_1}\tilde{\varphi}^{(n)}]\tilde{\psi}^{(n)} - \tilde{\varphi}^{(n)}[\partial_{x_1}\tilde{\psi}^{(n)}],$$
$$\partial_{t_a}\tilde{m}^{(n)} = -\tilde{\varphi}^{(n-1)}\tilde{\psi}^{(n+1)},$$
$$\tilde{m}^{(n+1)} = \tilde{m}^{(n)} + \tilde{\varphi}^{(n)}\tilde{\psi}^{(n+1)},$$

and

$$\partial_{x_2}\tilde{\varphi}^{(n)} = \partial_{x_1}^2\tilde{\varphi}^{(n)}, \quad \tilde{\varphi}^{(n+1)} = (\partial_{x_1} - a)\tilde{\varphi}^{(n)},$$
$$\partial_{x_2}\tilde{\psi}^{(n)} = -\partial_{x_1}^2\tilde{\psi}^{(n)}, \quad \tilde{\psi}^{(n-1)} = -(\partial_{x_1} + a)\tilde{\psi}^{(n)}.$$

We then define

$$\tilde{m}_{ij}^{(\nu\mu n)} = A_i^{(\nu)} B_j^{(\mu)}\tilde{m}^{(n)}, \quad \tilde{\varphi}_i^{(\nu n)} = A_i^{(\nu)}\tilde{\varphi}^{(n)},$$
$$\tilde{\psi}_j^{(\mu n)} = B_j^{(\mu)}\tilde{\psi}^{(n)}. \tag{32}$$

Since the operators $A_i^{(\nu)}$ and $B_j^{(\mu)}$ commute with differential operators ∂_{x_1}, ∂_{x_2}, and ∂_{t_a}, these functions $\tilde{m}_{ij}^{(\nu\mu n)}$, $\tilde{\varphi}_i^{(\nu n)}$, and $\tilde{\psi}_j^{(\mu n)}$ obey the differential and difference relations as well (15)–(16). From Lemma 2.1, we know that for an arbitrary sequence of indices $(i_1, i_2, \ldots, i_N; \nu_1, \nu_2, \ldots, \nu_N; j_1, j_2, \ldots, j_N; \mu_1, \mu_2, \ldots, \mu_N)$, the determinant

$$\tilde{\tau}_n = \det_{1\leq k,l\leq N}(\tilde{m}_{i_k,j_l}^{(\nu_k,\mu_l,n)})$$

satisfies the bilinear equations (18) and (19), for instance, the bilinear equations (18) and (19) hold for $\tilde{\tau}_n = \det_{1\leq i,j\leq N}(\tilde{m}_{2i-1,2j-1}^{(N-i,N-j,n)})$ with arbitrary parameters p and q.

Based on the Leibniz rule, one has,

$$[(p-a)\partial_p]^m\left(p^2 + \frac{2\mathrm{i}}{p-a}\right)$$
$$= \sum_{l=0}^{m}\binom{m}{l}\left[2^l(p-a)^2 + (-1)^l\frac{2\mathrm{i}}{p-a} + 2a(p-a)\right]$$
$$\times[(p-a)\partial_p]^{m-l} + a^2[(p-a)\partial_p]^m, \tag{33}$$

and

$$[(q+a)\partial_q]^m\left(q^2 - \frac{2\mathrm{i}}{q+a}\right)$$
$$= \sum_{l=0}^{m}\binom{m}{l}\left[2^l(q+a)^2 - (-1)^l\frac{2\mathrm{i}}{q+a} - 2a(q+a)\right]$$
$$\times[(q+a)\partial_q]^{m-l} + a^2[(q+a)\partial_q]^m. \tag{34}$$

Furthermore, one can derive

$$\left[A_n^{(\nu)}, p^2 + \frac{2\mathrm{i}}{p-a}\right]$$
$$= \sum_{k=0}^{n-1}\frac{a_k^{(\nu)}}{(n-k)!}\left[((p-a)\partial_p)^{n-k}, p^2 + \frac{2\mathrm{i}}{p-a}\right]$$
$$= \sum_{k=0}^{n-1}\frac{a_k^{(\nu)}}{(n-k)!}\sum_{l=1}^{n-k}\binom{n-k}{l}$$

$$\times\left(2^l(p-a)^2+(-1)^l\frac{2\mathrm{i}}{p-a}+2a(p-a)\right)$$
$$\times((p-a)\partial_p)^{n-k-l},$$

where [,] is the commutator defined by $[X,Y]=XY-YX$.

Let ζ be the solution of the cubic equation

$$2(p-a)^2-\frac{2\mathrm{i}}{p-a}+2a(p-a)=0,$$

with $a=\mathrm{i}\alpha$, then ζ has an explicit expression given previously. Hence we have

$$\left[A_n^{(\nu)},p^2+\frac{2\mathrm{i}}{p-a}\right]\bigg|_{p=\zeta}=0,$$

for $n=0,1$ and

$$\begin{aligned}
&\left[A_n^{(\nu)},p^2+\frac{2\mathrm{i}}{p-a}\right]\bigg|_{p=\zeta}\\
&=\sum_{k=0}^{n-2}\frac{a_k^{(\nu)}}{(n-k)!}\sum_{l=2}^{n-k}\binom{n-k}{l}\left\{2^l(p-a)^2+(-1)^l\frac{2\mathrm{i}}{p-a}+2a(p-a)\right\}[(p-a)\partial_p]^{n-k-l}|_{p=\zeta}\\
&=\sum_{k=0}^{n-2}\sum_{j=0}^{n-k-2}\frac{a_k^{(\nu)}}{(j+2)!(n-k-j-2)!}\left\{2^{j+2}(p-a)^2+(-1)^j\frac{2\mathrm{i}}{p-a}+2a(p-a)\right\}\\
&\quad\times[(p-a)\partial_p]^{n-k-j-2}|_{p=\zeta}\\
&=\sum_{\hat{k}=0}^{n-2}\left(\sum_{\hat{j}=0}^{\hat{k}}\frac{2^{\hat{j}+2}(p-a)^2+(-1)^{\hat{j}}\frac{2\mathrm{i}}{p-a}+2a(p-a)}{(\hat{j}+2)!}a_{\hat{k}-\hat{j}}^{(\nu)}\right)\frac{((p-a)\partial_p)^{n-2-\hat{k}}}{(n-2-\hat{k})!}\Bigg|_{p=\zeta}\\
&=\sum_{\hat{k}=0}^{n-2}a_{\hat{k}}^{(\nu+1)}\frac{((p-a)\partial_p)^{n-2-\hat{k}}}{(n-2-\hat{k})!}\Bigg|_{p=\zeta}\\
&=A_{n-2}^{(\nu+1)}|_{p=\zeta},
\end{aligned}$$

for $n\geq 2$. Thus the differential operator $A_n^{(\nu)}$ satisfies the following relation

$$\left[A_n^{(\nu)},p^2+\frac{2\mathrm{i}}{p-a}\right]\bigg|_{p=\zeta}=A_{n-2}^{(\nu+1)}|_{p=\zeta},\tag{35}$$

where we define $A_n^{(\nu)}=0$ for $n<0$.

Similarly, it is shown that the differential operator $B_n^{(\nu)}$ satisfies

$$\left[B_n^{(\nu)},q^2-\frac{2\mathrm{i}}{q+a}\right]\bigg|_{q=\zeta^*}=B_{n-2}^{(\nu+1)}|_{q=\zeta^*},\tag{36}$$

where we define $B_n^{(\nu)}=0$ for $n<0$.

Consequently, by referring to above two relations, we have

$$\begin{aligned}
&(\partial_{x_2}+2\mathrm{i}\partial_{t_a})\tilde{m}_{kl}^{(\nu\mu n)}|_{p=\zeta,q=\zeta^*}\\
&=[A_k^{(\nu)}B_l^{(\mu)}(\partial_{x_2}+2\mathrm{i}\partial_{t_a})\tilde{m}^{(n)}]|_{p=\zeta,q=\zeta^*}\\
&=\left(A_k^{(\nu)}B_l^{(\mu)}\left(p^2-q^2+2\mathrm{i}\left(\frac{1}{p-a}+\frac{1}{q+a}\right)\right)\tilde{m}^{(n)}\right)\bigg|_{p=\zeta,q=\zeta^*}\\
&=\left(A_k^{(\nu)}\left(p^2+\frac{2\mathrm{i}}{p-a}\right)B_l^{(\mu)}\tilde{m}^{(n)}\right)\bigg|_{p=\zeta,q=\zeta^*}-\left(A_k^{(\nu)}B_l^{(\mu)}\left(q^2-\frac{2\mathrm{i}}{q+a}\right)\tilde{m}^{(n)}\right)\bigg|_{p=\zeta,q=\zeta^*}\\
&=\left\{\left(\left(p^2+\frac{2\mathrm{i}}{p-a}\right)A_k^{(\nu)}+A_{k-2}^{(\nu+1)}\right)B_l^{(\mu)}\tilde{m}^{(n)}\right\}\bigg|_{p=\zeta,q=\zeta^*}\\
&\quad-\left\{A_k^{(\nu)}\left(\left(q^2-\frac{2\mathrm{i}}{q+a}\right)B_l^{(\mu)}+B_{l-2}^{(\mu+1)}\right)\tilde{m}^{(n)}\right\}\bigg|_{p=\zeta,q=\zeta^*}\\
&=\left(\zeta^2+\frac{2\mathrm{i}}{\zeta-\mathrm{i}\alpha}\right)\tilde{m}_{kl}^{(\nu\mu n)}|_{p=\zeta,q=\zeta^*}+\tilde{m}_{k-2,l}^{(\nu+1,\mu,n)}|_{p=\zeta,q=\zeta^*}\\
&\quad-\left(\zeta^{*2}-\frac{2\mathrm{i}}{\zeta^*+\mathrm{i}\alpha}\right)\tilde{m}_{kl}^{(\nu\mu n)}|_{p=\zeta,q=\zeta^*}-\tilde{m}_{k,l-2}^{(\nu,\mu+1,n)}|_{p=\zeta,q=\zeta^*}.
\end{aligned}$$

By using the formula (20) and the above relation, the differential of the following determinant

$$\tilde{\tilde{\tau}}_n = \det_{1\le i,j\le N}(\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*})$$

can be calculated as

$$\begin{aligned}
&(\partial_{x_2} + 2\mathrm{i}\partial_{t_a})\tilde{\tilde{\tau}}_n \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}(\partial_{x_2} + 2\mathrm{i}\partial_{t_a})(\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*}) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}\Bigg[\left(\zeta^2 + \frac{2\mathrm{i}}{\zeta - \mathrm{i}\alpha}\right)\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*} + m^{(N-i+1,N-j,n)}_{2i-3,2j-1}|_{p=\zeta,q=\zeta^*} \\
&\quad - \left(\zeta^{*2} - \frac{2\mathrm{i}}{\zeta^* + \mathrm{i}\alpha}\right)\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*} - m^{(N-i,N-j+1,n)}_{2i-1,2j-3}|_{p=\zeta,q=\zeta^*}\Bigg] \\
&= \left(\zeta^2 + \frac{2\mathrm{i}}{\zeta - \mathrm{i}\alpha}\right)N\tilde{\tilde{\tau}}_n + \sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}\tilde{m}^{(N-i+1,N-j,n)}_{2i-3,2j-1}|_{p=\zeta,q=\zeta^*} \\
&\quad - \left(\zeta^{*2} - \frac{2\mathrm{i}}{\zeta^* + \mathrm{i}\alpha}\right)N\tilde{\tilde{\tau}}_n - \sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}\tilde{m}^{(N-i,N-j+1,n)}_{2i-1,2j-3}|_{p=\zeta,q=\zeta^*},
\end{aligned}$$

where Δ_{ij} is the (i,j)-cofactor of the matrix $(\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*})_{1\le i,j\le N}$. For the term $\sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}\tilde{m}^{(N-i+1,N-j,n)}_{2i-3,2j-1}|_{p=\zeta,q=\zeta^*}$, it vanishes, since for $i = 1$ this summation is a determinant with the elements in first row being zero and for $i = 2, 3, \ldots$ this summation is a determinant with two identical rows. Similarly, the term $\sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_{ij}\tilde{m}^{(N-i,N-j+1,n)}_{2i-1,2j-3}|_{p=\zeta,q=\zeta^*}$ vanishes. Therefore, $\tilde{\tilde{\tau}}_n$ satisfies the reduction condition

$$(\partial_{x_2} + 2\mathrm{i}\partial_{t_a})\tilde{\tilde{\tau}}_n = \left(\zeta^2 - \zeta^{*2} + \frac{2\mathrm{i}}{\zeta - \mathrm{i}\alpha} + \frac{2\mathrm{i}}{\zeta^* + \mathrm{i}\alpha}\right)N\tilde{\tilde{\tau}}_n. \quad (37)$$

Since $\tilde{\tilde{\tau}}_n$ is a special case of $\tilde{\tau}_n$, it also satisfies the bilinear equations (18) and (19) with τ_n replaced by $\tilde{\tilde{\tau}}_n$. From (18), (19), and (37), it is obvious that $\tilde{\tilde{\tau}}_n$ satisfies the (1+1)-dimensional bilinear equations

$$(D_{x_1}^2 + 2aD_{x_1} - D_{x_2})\tilde{\tilde{\tau}}_{n+1}\cdot\tilde{\tilde{\tau}}_n = 0, \quad (38)$$

$$(\mathrm{i}D_{x_1}D_{x_2} - 4)\tilde{\tilde{\tau}}_n\cdot\tilde{\tilde{\tau}}_n = -4\tilde{\tilde{\tau}}_{n+1}\tilde{\tilde{\tau}}_{n-1}. \quad (39)$$

Due to the reduction condition (37), t_a becomes a dummy variable which can be taken as zero. Thus $\tilde{m}^{(N-i,N-j,n)}_{2i-1,2j-1}|_{p=\zeta,q=\zeta^*}$ and $\tilde{\tilde{\tau}}_n$ reduce to $m^{(N-i,N-j,n)}_{2i-1,2j-1}$ and τ_n (29) in Lemma 2.2. Therefore τ_n satisfies the bilinear equations (30) and (31) and the proof is complete. □

2.3 Complex conjugate condition and regularity

From Lemma 2.2, by taking the independent variable transformations:

$$x_1 = x, \quad x_2 = -\mathrm{i}t, \quad (40)$$

it is found that $f = \tau_0$, $g = \tau_1$ and $h = \tau_{-1}$ satisfy the (1+1)-dimensional bilinear equations:

$$(D_x^2 + 2\mathrm{i}\alpha D_x - \mathrm{i}D_t)g\cdot f = 0, \quad (41)$$

$$(D_xD_t + 4)f\cdot f = 4gh. \quad (42)$$

Next we consider the complex conjugate condition and the regularity (non-singularity) of solutions. The complex conjugate condition requires

$$\tau_0 : \text{real}, \quad \tau_{-1} = \tau_1^*. \quad (43)$$

Since $x_1 = x$ is real and $x_2 = -\mathrm{i}t$ is pure imaginary, the complex conjugate condition can be easily satisfied by taking the parameters $a_k^{(0)}$ and $b_k^{(0)}$ to be complex conjugate to each other. It then follows

$$b_k^{(\nu)}|_{q=\zeta^*} = (a_k^{(\nu)}|_{p=\zeta})^*, \quad (44)$$

for $\nu = 0, 1, 2, \ldots$. Then, by referring to (44), we have

$$m_{ij}^{(\nu,\mu,n)*} = m_{ij}^{(\nu,\mu,n)}|_{a_k^{(\nu)}\leftrightarrow b_k^{(\nu)}, x_2\leftrightarrow -x_2, a\leftrightarrow -a, \zeta\leftrightarrow\zeta^*} = m_{ji}^{(\mu,\nu,-n)}, \quad (45)$$

which implies

$$\tau_n^* = \tau_{-n}. \quad (46)$$

On the other hand, under the condition (44), we can show that τ_0 is nonzero for all (x, t). Note that $f = \tau_0$ is the determinant of a Hermitian matrix $M = (m^{(N-i,N-j,0)}_{2i-1,2j-1})_{1\le i,j\le N}$. From the appendix in Ref. 56, it is known that when the real part of the parameter ζ is positive, the element of the Hermitian matrix M can be written as an integral

$$m^{(N-i,N-j,0)}_{2i-1,2j-1} = \int_{-\infty}^{x} A^{(N-i)}_{2i-1}B^{(N-j)}_{2j-1}e^{\xi+\eta}\,dx|_{p=\zeta,q=\zeta^*}. \quad (47)$$

For any non-zero column vector $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_N)^T$ and $\upsilon^\dagger$ being its complex transpose, one can obtain

$$\begin{aligned}
\upsilon^\dagger M\upsilon &= \sum_{i,j=1}^{N}\upsilon_i^\dagger m^{(N-i,N-j,0)}_{2i-1,2j-1}\upsilon_j \\
&= \int_{-\infty}^{x}\left|\sum_{i=1}^{N}\upsilon_i^\dagger A^{(N-i)}_{2i-1}e^{\xi}|_{p=\zeta}\right|^2 dx > 0,
\end{aligned}$$

which shows that the Hermitian matrix M is positive definite, hence the denominator $f = \det M > 0$.

When the real part of the parameter p is negative, the element of the Hermitian matrix M can be cast into

$$m^{(N-i,N-j,0)}_{2i-1,2j-1} = -\int_{x}^{+\infty} A^{(N-i)}_{2i-1}B^{(N-j)}_{2j-1}e^{\xi+\eta}\,dx|_{p=\zeta,q=\zeta^*}. \quad (48)$$

Then one obtains

$$\upsilon^\dagger M\upsilon = \sum_{i,j=1}^{N}\upsilon_i^\dagger m^{(N-i,N-j,0)}_{2i-1,2j-1}\upsilon_j$$

$$= -\int_x^{+\infty}\left|\sum_{i=1}^{N} v_i^{\dagger} A_{2i-1}^{(N-i)} e^{\xi}|_{p=\zeta}\right|^2 dx < 0,$$

which proves that the Hermitian matrix M is negative definite, hence the denominator $f = \det M < 0$. Therefore, for either positive or negative of the parameter ζ, the rogue wave solution of the short wave and long wave components is always nonsingular.

To summarize the results, we have the following theorem for the general higher order rogue wave solutions of the 1D YO system (1)–(2):

Theorem 2.3. *The 1D YO system (1)–(2) has the non-singular rational solutions*

$$S = e^{\mathrm{i}[\alpha x+(h+\alpha^2)t]}\frac{\tau_1}{\tau_0}, \quad L = h - 2\frac{\partial^2}{\partial x^2}\log\tau_0, \tag{49}$$

with

$$\tau_n = \det_{1\le i,j\le N}(m_{2i-1,2j-1}^{(N-i,N-j,n)}) = \begin{vmatrix} m_{11}^{(N-1,N-1,n)} & m_{13}^{(N-1,N-2,n)} & \cdots & m_{1,2N-1}^{(N-1,0,n)} \\ m_{31}^{(N-2,N-1,n)} & m_{33}^{(N-2,N-2,n)} & \cdots & m_{3,2N-1}^{(N-2,0,n)} \\ \vdots & \vdots & & \vdots \\ m_{2N-1,1}^{(0,N-1,n)} & m_{2N-1,3}^{(0,N-2,n)} & \cdots & m_{2N-1,2N-1}^{(0,0,n)} \end{vmatrix}, \tag{50}$$

where $g = \tau_1$, $f = \tau_0$, and $h = \tau_{-1}$, and the elements in determinant τ_n are defined by

$$m_{i,j}^{(\nu,\mu,n)} = \sum_{k=0}^{i}\sum_{l=0}^{j}\frac{a_k^{(\nu)}}{(i-k)!}\frac{a_l^{(\mu)*}}{(j-l)!} \times[(p-\mathrm{i}\alpha)\partial_p]^{i-k}[(q+\mathrm{i}\alpha)\partial_q]^{j-l} \times\frac{1}{p+q}\left(-\frac{p-\mathrm{i}\alpha}{q+\mathrm{i}\alpha}\right)^n e^{(p+q)x-(p^2-q^2)\mathrm{i}t}|_{p=\zeta,q=\zeta^*}, \tag{51}$$

and

$$\zeta = \zeta_r + \mathrm{i}\zeta_i, \quad \zeta_r = \pm\frac{\sqrt{3}}{12}\frac{K^2-4\alpha^2}{K},$$
$$\zeta_i = \frac{1}{12}K + \frac{1}{3}\frac{\alpha^2}{K} + \frac{2}{3}\alpha,$$

with

$$K = (8\alpha^3 + 108 + 12\sqrt{12\alpha^3+81})^{1/3}, \quad \left(\alpha > -\frac{3}{2}2^{1/3}\right),$$

where $a_k^{(\nu)}$ are complex constants and need to satisfy the relations:

$$a_k^{(\nu+1)} = \sum_{j=0}^{k}\frac{2^{j+2}(p-\mathrm{i}\alpha)^2 + (-1)^j\frac{2\mathrm{i}}{p-\mathrm{i}\alpha} + 2\mathrm{i}\alpha(p-\mathrm{i}\alpha)}{(j+2)!} \times a_{k-j}^{(\nu)}, \quad \nu = 0,1,2,\ldots. \tag{52}$$

3. Dynamics of Rogue Wave Solutions

In this section, we analyze the dynamics of rogue wave solutions to the 1D YO system in detail. To this end, we fix the parameter $\zeta_r = \frac{\sqrt{3}}{12}\frac{K^2-4\alpha^2}{K}$ without loss of generality. Meanwhile, due to the fact that the long wave L is a real-valued function and its rogue wave structure is always bright,[73–75] in what follows, we omit the discussion of the long wave component and only consider the dynamical properties of the complex short wave component S.

3.1 *Fundamental rogue wave*

According to Theorem 2.3, in order to obtain the first-order rogue wave, we need to take $N = 1$ in Eqs. (49)–(52). For simplicity, we set $a_0^{(0)} = b_0^{(0)} = 1$, $a_1^{(0)} = b_1^{(0)} = 0$, then the functions f and g take the form

$$f = \frac{1}{\zeta_r}e^{2\zeta_r(x+2\zeta_i t)}(\theta\theta^* + \theta_0), \tag{53}$$

$$g = \frac{1}{\zeta_r}e^{2\zeta_r(x+2\zeta_i t)}\left[-\frac{\zeta_r+\mathrm{i}(\zeta_i-\alpha)}{\zeta_r-\mathrm{i}(\zeta_i-\alpha)}\right] \times\left[\left(\theta-\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)\left(\theta^*+\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)+\theta_0\right], \tag{54}$$

with

$$\theta = (k_1+\mathrm{i}k_2)x + (h_1+\mathrm{i}h_2)t + (l_1+\mathrm{i}l_2),$$
$$k_1 = \frac{1}{2}(\alpha+\zeta_r-\zeta_i), \quad k_2 = \frac{1}{2}(\alpha-\zeta_r-\zeta_i),$$
$$h_1 = \alpha(\zeta_i-\zeta_r) + \zeta_r^2 + 2\zeta_r\zeta_i - \zeta_i^2,$$
$$h_2 = \alpha(\zeta_i+\zeta_r) + \zeta_r^2 - 2\zeta_r\zeta_i - \zeta_i^2,$$
$$l_1 = \frac{\zeta_i-\alpha}{4\zeta_r} - \frac{1}{4}, \quad l_2 = \frac{\zeta_i-\alpha}{4\zeta_r} + \frac{1}{4},$$
$$\theta_0 = \frac{(\zeta_i-\alpha)^2}{8\zeta_r^2} + \frac{1}{8}.$$

Thus, the fundamental rogue wave solution is given by

$$S = e^{-\mathrm{i}[\alpha x+(h+\alpha^2)t]}\left[-\frac{\zeta_r+\mathrm{i}(\zeta_i-\alpha)}{\zeta_r-\mathrm{i}(\zeta_i-\alpha)}\right]\left[1+\frac{\mathrm{i}(L_1+L_2)-1/2}{L_1^2+L_2^2+\theta_0}\right], \tag{55}$$

$$L = h + 4\frac{(k_1L_1+k_2L_2)^2 - (k_2L_1-k_1L_2)^2 - (k_1^2+k_2^2)\theta_0}{(L_1^2+L_2^2+\theta_0)^2}, \tag{56}$$

where $L_1 = k_1x + h_1t + l_1$ and $L_2 = k_2x + h_2t + l_2$.

It is found that the modular square of the short-wave component $|S|^2$ possesses critical points

$$(x_1,t_1) = \left(\frac{1}{2\zeta_r}, 0\right), \tag{57}$$

$$(x_2,t_2) = \left(\frac{1}{2}\frac{\mu_1(\alpha-2\zeta_i)}{\zeta_r\Delta} + \frac{1}{2\zeta_r}, \frac{1}{4}\frac{\mu_1}{\zeta_r\Delta}\right), \tag{58}$$

$$(x_3,t_3) = \left(-\frac{1}{2}\frac{\mu_2(\alpha\zeta_i+\zeta_r^2-\zeta_i^2)}{\zeta_r^2\Delta} + \frac{1}{2\zeta_r}, \frac{1}{4}\frac{\mu_2(\alpha-\zeta_i)}{\zeta_r^2\Delta}\right), \tag{59}$$

with

$$\mu_1 = \pm\sqrt{3(\alpha-\zeta_i)^2 - \zeta_r^2},$$
$$\mu_2 = \pm\sqrt{3\zeta_r^2 - (\alpha-\zeta_i)^2},$$
$$\Delta = (\alpha-\zeta_i)^2 + \zeta_r^2.$$

Note that (x_3,t_3) are also two characteristic points, at which the values of the amplitude are zero.

At these points, the local quadratic forms are

$$H(\tilde{x},\tilde{t}) = \left[\frac{\partial^2|S|^2}{\partial x^2}\frac{\partial^2|S|^2}{\partial t^2} - \left(\frac{\partial^2|S|^2}{\partial x\partial t}\right)^2\right]\Bigg|_{(\tilde{x},\tilde{t})},$$

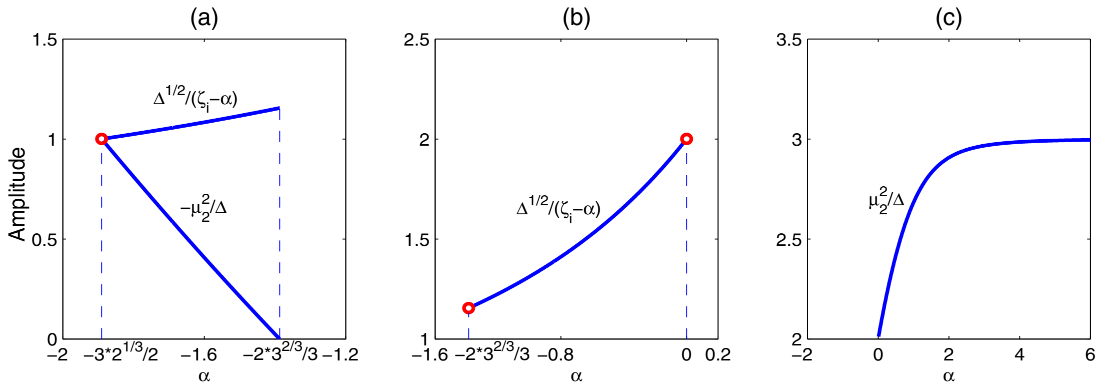

Fig. 1. (Color online) The extreme values of $|S|$ with the parameter α: (a) dark state; (b) intermediate state; and (c) bright state.

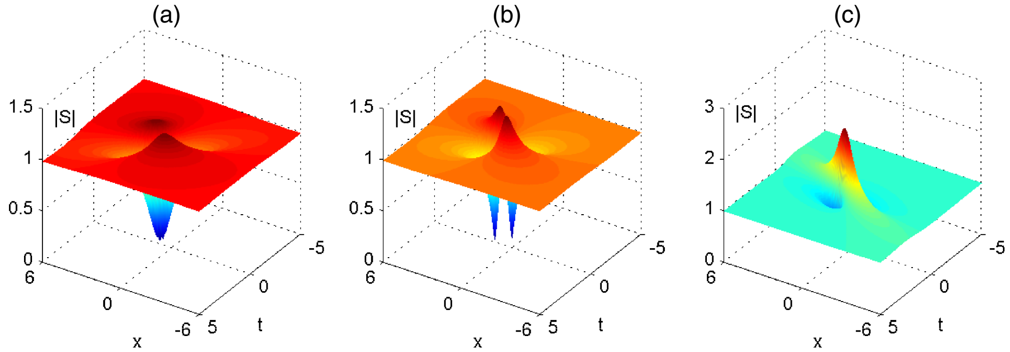

Fig. 2. (Color online) First-order rogue wave of the YO system: (a) dark state $\alpha = -\frac{7}{5}$; (b) intermediate state $\alpha = -1$; and (c) bright state $\alpha = \frac{1}{3}$.

$$H(x_1,t_1) = -\frac{16384\zeta_r^{10}\mu_1^2\mu_2^2}{\Delta^4},$$
$$H(x_2,t_2) = \frac{64\zeta_r^{10}\mu_1^2\Delta^2}{(\alpha-\zeta_i)^{10}}, \quad H(x_3,t_3) = 64\mu_2^2\Delta^2, \tag{60}$$

and the second derivatives are

$$H_1(\tilde{x},\tilde{t}) = \left.\frac{\partial^2|S|^2}{\partial x^2}\right|_{(\tilde{x},\tilde{t})},$$
$$H_1(x_1,t_1) = \frac{192\zeta_r^4[(\alpha-\zeta_i)^2-\zeta_r^2]}{\Delta^2},$$
$$H_2(x_2,t_2) = -\frac{6\zeta_r^4\Delta}{(\alpha-\zeta_i)^4}, \quad H_3(x_3,t_3) = 6\Delta. \tag{61}$$

Based on the above analysis, the fundamental rogue wave can be classified into three patterns:

(a) Dark state ($-\frac{3}{2}2^{1/3} < \alpha \leqslant -\frac{2}{3}3^{2/3}$): two local maxima at (x_2,t_2) with the $|S|$'s amplitude $\frac{\sqrt{\Delta}}{\zeta_i-\alpha}$, one local minimum at (x_1,t_1) with the $|S|$'s amplitude $-\frac{\mu_2^2}{\Delta}$. Especially, when $\alpha = -\frac{2}{3}3^{2/3}$, the local minimum is located at the characteristic point $(x_1,t_1) = (x_3,t_3) = (\frac{1}{3}3^{5/6},0)$.

(b) Intermediate state ($-\frac{2}{3}3^{2/3} < \alpha < 0$): two local maxima at (x_2,t_2) with the $|S|$'s amplitude $\frac{\sqrt{\Delta}}{\zeta_i-\alpha}$, two local minima at two characteristic points (x_3,t_3).

(c) Bright state ($\alpha \geqslant 0$): two local minima at two characteristic points (x_3,t_3), one local maxima at (x_1,t_1) with the $|S|$'s amplitude $\frac{\mu_2^2}{\Delta}$. Particularly, $\alpha = 0$, the local maximum is located at $(x_1,t_1) = (x_2,t_2) = (\frac{1}{3}3^{1/2},0)$.

At the extreme points, the evolution of the amplitudes for the short wave with the parameter α is exhibited in Fig. 1. It can be clearly seen that, for the dark state, as α changes from $-\frac{3}{2}2^{1/3}$ to $-\frac{2}{3}3^{2/3}$, the maximal amplitudes increase from 1 to $\frac{2}{3}\sqrt{3}$ while the minimal one decreases from 1 to 0; for the intermediate state, as α changes from $-\frac{2}{3}3^{2/3}$ to 0, the maximal amplitudes increase from $(\frac{2}{3}\sqrt{3}$ to 2 while the minimal amplitude is always zero; for the bright state, as $\alpha \geq 0$, the maximal amplitude is changed from 2 to its asymptotic value of 3 while the minimal value is always zero.

Figure 2 displays three patterns of fundamental rogue wave for short wave component. In three cases, the amplitudes of the short wave uniformly approach to the background 1 as (x,t) goes to infinity. Figure 2(a) exhibits a dark rogue wave, in which it has one hole falling to 0.0247 at $(0.8420,0)$ and two humps with the height 1.1450 at $(0.1531,0.4981)$ and $(1.5309,-0.4981)$. For Fig. 2(b), as an example of the intermediate state of rogue wave, it attains its maxima 1.3118 at $(0.1645,0.3356)$ and $(1.1780,-0.3356)$, and minima 0 at $(1.0975,0.2823)$ and $(0.2451,-0.2823)$. In Fig. 2(c), the amplitude of the short wave features a bright rogue wave, which possesses the two zero-amplitudes points $(1.0715,0.2168)$ and $(0.0966,-0.2168)$ and acquires a maximum of 2.2965 at $(0.5840,0)$. This bright rogue wave is similar to the Peregrine soliton, but its structure possesses the moving zero-amplitude points and the varying peak height owing to the arbitrary parameter α.

From Eqs. (55)–(56), it is known that the family of first-order rogue solutions contains two free parameters α and h. The latter one h is merely a constant determining the background of the long wave component. Therefore, based on the previous discussion, it is found that the feature of

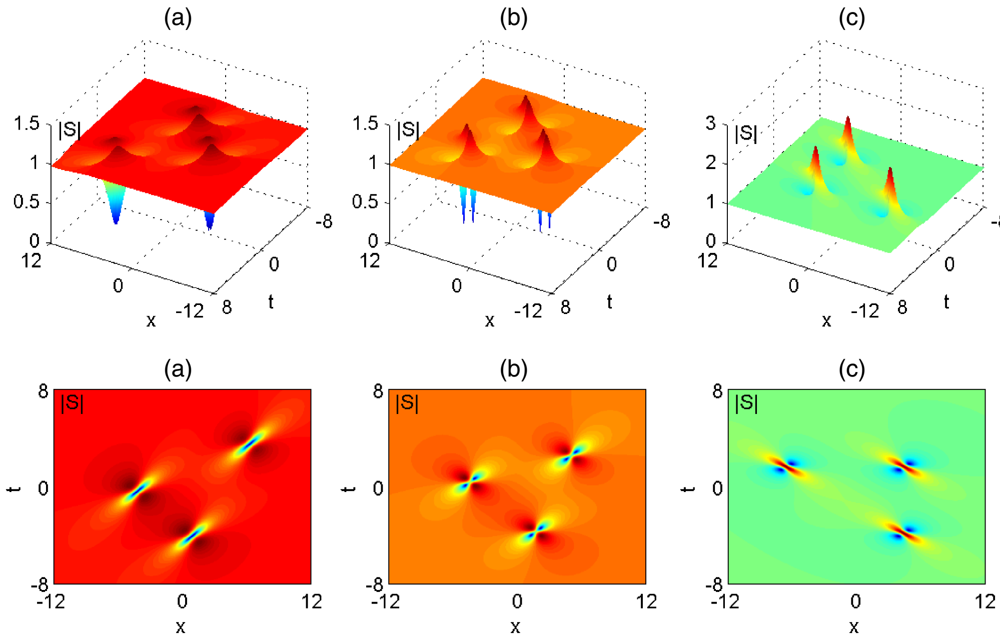

Fig. 3. (Color online) Second-order rogue wave of the YO system $a_3^{(0)} = 85$: (a) dark state $\alpha = -\frac{7}{5}$; (b) intermediate state $\alpha = -1$; and (c) bright state $\alpha = \frac{1}{3}$.

rogue waves for the short wave component depends on the parameter α. The choice of the parameter α determines these local waves patterns such as the number, the position, the height and the type of extrema. We comment here that the same parameter is also introduced in the construction of dark–dark soliton solution for the coupled NLS system[76] and the coupled YO system,[77] in which this treatment results in the generation of non-degenerate dark–dark soliton solution. As interpreted in Ref. 76, this parameter can be formally removed by the Galilean transformation in the scalar NLS equation, while the same copies cannot be removed simultaneously in the coupled NLS system. For the YO system, it contains the long wave and short wave coupling and is not Galilean invariant, so the introduction of the parameter α is necessary and essential for the construction of the general rogue wave solutions including intermediate and dark rogue wave ones.

3.2 Higher-order rogue wave

The second-order rogue wave solution is obtained from Eqs. (49)–(52) with $N = 2$. In this case, setting $a_0^{(0)} = b_0^{(0)} = 1$, $a_1^{(0)} = b_1^{(0)} = 0$, $a_2^{(0)} = b_2^{(0)} = 0$, we obtain the functions f and g as follows

$$f = \begin{vmatrix} m_{11}^{(1,1,0)} & m_{13}^{(1,0,0)} \\ m_{31}^{(0,1,0)} & m_{33}^{(0,0,0)} \end{vmatrix}, \quad g = \begin{vmatrix} m_{11}^{(1,1,1)} & m_{13}^{(1,0,1)} \\ m_{31}^{(0,1,1)} & m_{33}^{(0,0,1)} \end{vmatrix}, \tag{62}$$

where the elements are determined by

$$m_{11}^{(1,1,n)} = A_1^{(1)} B_1^{(1)} \tilde{m}^{(n)}|_{p=\zeta,q=\zeta^*},$$
$$m_{13}^{(1,0,n)} = A_1^{(1)} B_3^{(0)} \tilde{m}^{(n)}|_{p=\zeta,q=\zeta^*},$$
$$m_{31}^{(0,1,n)} = A_3^{(0)} B_1^{(1)} \tilde{m}^{(n)}|_{p=\zeta,q=\zeta^*},$$
$$m_{33}^{(0,0,n)} = A_3^{(0)} B_3^{(0)} \tilde{m}^{(n)}|_{p=\zeta,q=\zeta^*},$$

with the differential operators

$$A_1^{(1)} = a_0^{(1)}(p - \mathrm{i}\alpha)\partial_p + a_1^{(1)},$$
$$B_1^{(1)} = a_0^{(1)*}(q + \mathrm{i}\alpha)\partial_q + a_1^{(1)*},$$
$$A_3^{(0)} = \frac{1}{6}[(p - \mathrm{i}\alpha)\partial_p]^3 + a_3^{(0)},$$
$$B_3^{(0)} = \frac{1}{6}[(q + \mathrm{i}\alpha)\partial_q]^3 + a_3^{(0)*},$$

and

$$\tilde{m}^{(n)} = \frac{1}{p+q}\left(-\frac{p - \mathrm{i}\alpha}{q + \mathrm{i}\alpha}\right)^n e^{(p+q)x-(p^2-q^2)\mathrm{i}t}.$$

In addition

$$a_0^{(1)} = 2(p - \mathrm{i}\alpha)^2 + \frac{\mathrm{i}}{p - \mathrm{i}\alpha} + \mathrm{i}\alpha(p - \mathrm{i}\alpha),$$
$$a_1^{(1)} = \frac{1}{3}\left[4(p - \mathrm{i}\alpha)^2 - \frac{\mathrm{i}}{p - \mathrm{i}\alpha} + \mathrm{i}\alpha(p - \mathrm{i}\alpha)\right].$$

Three groups of second-order rogue wave solutions with different values of the parameter $a_3^{(0)}$ are displayed in Figs. 3 and 4. As shown in these figures, the second-order rogue waves can be viewed as the superposition of three fundamental rogue waves, and they have different dynamical behaviors for different values of the parameters α and $a_3^{(0)}$. Here we first observe that the structure of the second-order rogue wave exhibits the triangle arrays of three elementary patterns occurring in the first-order rogue waves, i.e., the fundamental dark, intermediate and bright ones, respectively, when the value of the parameter α is chosen the same as in Figs. 2(a)–2(c). Then it is easy to see that the dynamical behavior for the second-order rogue waves depends on the values of $a_3^{(0)}$ when the value of α is fixed. For example, two groups of rogue wave solutions with $a_3^{(0)} = 50\mathrm{i}$ and $-50\mathrm{i}$ are shown in Fig. 4, the triangles for the arrays of elementary rogue waves are symmetric.

For the construction of third and higher-order rogue waves, which represent the superposition of more fundamental ones, one need to take larger N in (49)–(52). The expressions is too complicated to illustrate here. However we can show the dynamical structures of rogue waves graphically. For $N = 3$,

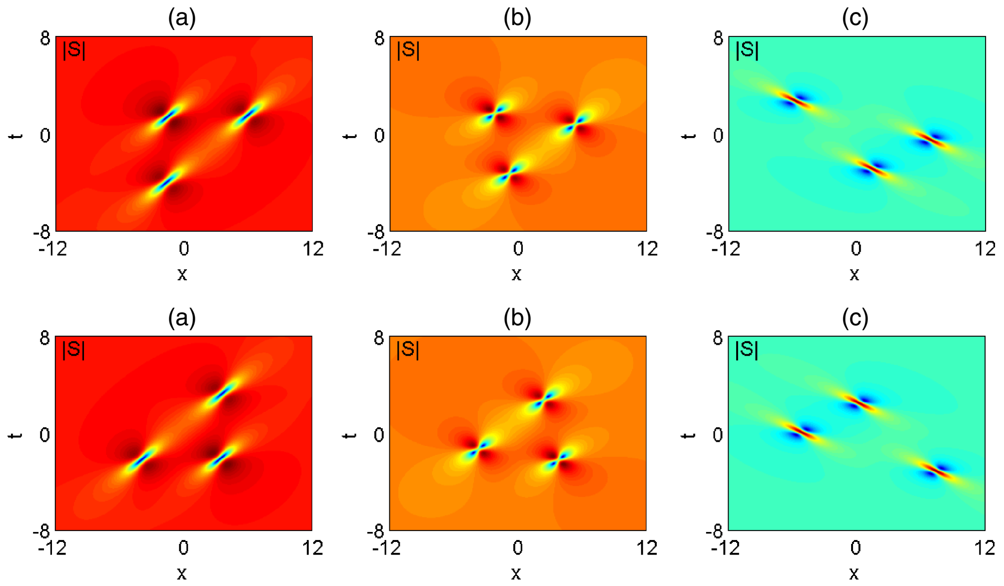

Fig. 4. (Color online) Second-order rogue wave of the YO system $a_3^{(0)} = 50\mathrm{i}$ (upper), $a_3^{(0)} = -50\mathrm{i}$ (bottom): (a) dark state $\alpha = -\frac{7}{5}$; (b) intermediate state $\alpha = -1$; and (c) bright state $\alpha = \frac{1}{3}$.

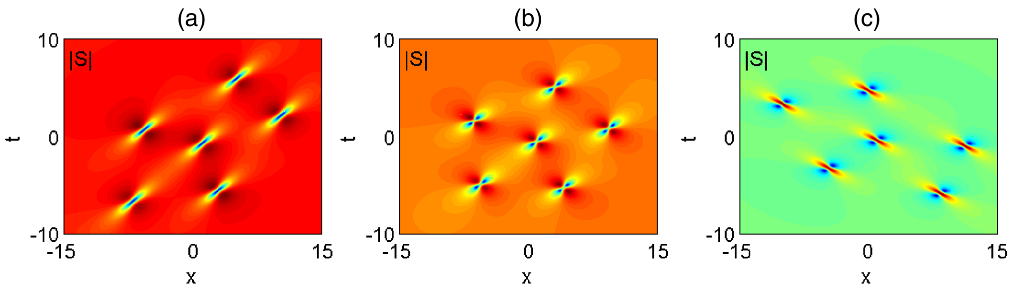

Fig. 5. (Color online) Third-order rogue wave of the YO system $a_5^{(0)} = 2000$: (a) dark state $\alpha = -\frac{7}{5}$; (b) intermediate state $\alpha = -1$; and (c) bright state $\alpha = 0$.

we choose $a_0^{(0)} = 1$, $a_2^{(0)} = a_3^{(0)} = a_4^{(0)} = 0$, $a_5^{(0)} = 2000$ in Eqs. (49)–(52), and plot the third-order rogue wave solution in Fig. 5. It can be seen that this third-order rogue waves exhibit the superposition of six fundamental rogue waves and they constitute a shape of pentagon.

Finally, we remark that the fundamental pattern occurring in the higher-order rogue wave completely depends on the parameter α (see Figs. 2–5). In other words, three types of fundamental rogue waves and their higher-order superposition appear at three different intervals of α, i.e., $(-\frac{3}{2}2^{1/3}, -\frac{2}{3}3^{2/3}]$ for dark state, $(-\frac{2}{3}3^{2/3}, 0)$ for intermediate state and $[0, +\infty)$ for bright state. Therefore, there is no pattern of superposition among different types of fundamental rogue waves, for example, between bright ones and dark ones. The underlying reason is due to the fact that only a single parameter α is introduced in the 1D YO system. In constructing general solutions to the coupled YO system with multi-short wave components,[30] the multiple copies of α_i can be introduced which allows the superposition of different types of fundamental rogue waves by taking appropriate values of the parameters. In addition, we comment that the 1D YO system with one short wave and one long wave coupling is different from the vector NLS equation representing two short wave coupling. As reported in Refs. 49 and 50, two copies of α can be imposed and different fundamental rogue wave's superpositions can be exhibited. The comparison reveals that the degree of freedom in the 1D YO system is less than the one in the two-component NLS system.

Theoretically speaking, three types of rouge wave can be observed in the experiments modeled by the YO system, which can describe the long wave-short wave resonance interaction in nonlinear optics[69,70] and other physical fields. Specifically, as reported in Ref. 46 regarding bright-dark rouge waves of the vector NLS equations, nonlinear optics is a fertile ground to develop the phenomenon of vector rogue waves. Thus, the certain experimental conditions in nonlinear optics can give rise to three patterns of rogue wave in first- and higher-order cases. Furthermore, in the relevant numerical simulation, each types of rouge wave may be characterized from the initial seeds with the different frequencies spectrum associated with the modulation instability analysis.[46,74]

4. Summary and Discussions

In this paper, we have derived general high-order rogue wave solutions for the 1D YO system by virtue of the Hirota's bilinear method. These rogue wave solutions are obtained by the KP hierarchy reduction method and are expressed in terms of determinants whose elements are algebraic formulae. By choosing different values of parameters in the rogue wave solutions, we have analytically and graphically studied the dynamics of first-order, second-order, and third-order rogue wave solutions. As a result, the fundamental (first-order) rogue waves have been classified into three different patterns: bright, intermediate and dark states. The higher-order rogue waves correspond to the superposition of fundamental rogue waves. In particular, we should mention here that, in compared with the NLS equation, there exists an essential parameter α to control the pattern of rogue wave for both first- and higher-order rogue waves since the YO system does not possess the Galilean invariance.

Apart from rogue waves appearing in the continuous models, the behavior of rogue waves in discrete systems have recently drawn much attention.[31–33] Paralleling to the novel patterns of rogue waves such as dark and intermediate ones in the continuous coupled systems with multiple waves, the discrete counterparts of such rogue waves can be attained in discrete systems. We have recently proposed an integrable semi-discrete analogue of the 1D YO system.[78] Thus the semi-discrete rogue wave, especially the semi-discrete dark and intermediate ones are worthy to be expected. We will report the results on this topic in the future.

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