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A Note on the Bilinearization of the Generalized Derivative Nonlinear Schrödinger Equation

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The bilinearization of the generalized derivative nonlinear Schrödinger (GDNLS) equation is investigated systematically. It is known recently that the GDNLS equation can be decomposed into two different bilinear systems under the vanishing and nonvanishing boundary condition, respectively. However, it remains a question of how these two systems are related. In this letter, we show that all the bilinear equations can be derived uniformly from the KP hierarchy through appropriate reductions. Bright and dark soliton solutions in terms of Gram-type determinants are presented.

To explore the effects of higher order perturbations, various extensions of the nonlinear Schrödinger (NLS) equation have been proposed and studied. Among them, one class of the integrable models is called the derivative nonlinear Schrödinger (DNLS) equation due to the existence of derivative-type nonlinearities. Generally speaking, there are mainly three types of the DNLS equation: the first one is the Kaup–Newell (KN) equation[1)]

$$\mathrm{i}u_t + u_{xx} + 2\mathrm{i}(|u|^2u)_x = 0, \tag{1}$$

the second one is called the Chen–Lee–Liu (CLL) equation[2)]

$$\mathrm{i}u_t + u_{xx} + 2\mathrm{i}|u|^2u_x = 0, \tag{2}$$

while the third one named the Gerdjikov–Ivanov (GI) equation[3)]

$$\mathrm{i}u_t + u_{xx} - 2\mathrm{i}u^2u_x^* + 2|u|^4u = 0, \tag{3}$$

here the superscript denotes complex conjugation. These three integrable equations are linked each other through gauge transformations. Indeed, this kind of gauge-equivalent connection was firstly found between the KN equation and the CLL equation by Wadati.[4)] Following the idea of gauge transformation, Kundu proposed a generalized derivative nonlinear Schrödinger (GDNLS) equation, which can be written as the normalized form[5,6)]

$$\begin{aligned}\mathrm{i}u_t + u_{xx} + 2\mathrm{i}\gamma|u|^2u_x + 2\mathrm{i}(\gamma-1)u^2u_x^*\\ + (\gamma-1)(\gamma-2)|u|^4u = 0,\end{aligned} \tag{4}$$

where γ is an arbitrary real constant. It is obvious that the GDNLS equation reduces to the KN equation for $\gamma = 2$, the CLL equation for $\gamma = 1$ and the GI equation for $\gamma = 0$. Clarkson and Cosgrove[7)] have performed the Painlevé integrability test for the GDNLS equation (4). These DNLS equations (1)–(4) have been studied in the framework of the bilinear formalism. Nakamura and Chen bilinearized the CLL equation and constructed bright soliton solution.[8)] The bright soliton solution expressed by Wronski-type determinants for the GDNLS equation was derived by Kakei et al. in which two auxiliary independent variables are introduced.[6)] Under the nonvanishing boundary condition, dark soliton solutions to the KN and the CLL equations were presented in Refs. 9 and 10. Through the Kadomtsev–Petviashvili (KP) hierarchy reduction method, general rogue waves for the GDNLS equation (4) were recently constructed by decomposing it into two bilinear systems.[11)]

In this letter, we will study the bilinearization of the GDNLS equation (4) systematically under both the vanishing and nonvanishing boundary conditions, and derive soliton solutions via the KP hierarchy reductions. The main results show that even though the GDNLS equation (4) can be decomposed into two bilinear systems either under the vanishing or nonvanishing boundary condition, all these bilinear members can be obtained uniformly from the KP hierarchy through appropriate reductions.

A. The bright soliton solution. For the vanishing boundary condition, we introduce the following dependent variable transformation

$$u = \frac{f^{*\gamma-1}g}{f^\gamma}, \tag{5}$$

where g and f are complex functions. Then the GDNLS equation (4) can be decoupled into a set of bilinear equations

$$B_1 = 0, \quad B_3 = 0, \quad B_4 = 0, \quad B_5 = 0, \tag{6}$$

or another set of bilinear equations

$$B_2 = 0, \quad B_3 = 0, \quad B_4 = 0, \quad B_5 = 0, \tag{7}$$

where B_i $(i = 1\cdots5)$ are defined by

$$B_1 \equiv (\mathrm{i}D_t + D_x^2)g\cdot f, \tag{8}$$

$$B_2 \equiv (\mathrm{i}D_t + D_x^2)g\cdot f^*, \tag{9}$$

$$B_3 \equiv (\mathrm{i}D_t + D_x^2)f\cdot f^*, \tag{10}$$

$$B_4 \equiv D_xf\cdot f^* - \mathrm{i}|g|^2, \tag{11}$$

$$B_5 \equiv D_x^2f\cdot f^* - \mathrm{i}D_xg\cdot g^*, \tag{12}$$

with D_x and D_t being the Hirota's bilinear operators

$$D_x^nD_t^m(a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')|_{x=x',t=t'}.$$

For the arbitrary value of γ, either the bilinear system (6) or (7) gives the GDNLS equation (4). But such bilinear equations are different from the ones given by Kakei et al.[6)] in which two auxiliary dependent variables need to be introduced. The connection of two bilinear systems is given by

$$fB_2 + g\partial_xB_4 = f^*B_1 + g(B_3 - B_5) + 2g_xB_4,$$

upon using the following trilinear identities

$$fD_t g\cdot f^* = f^* D_t g\cdot f + gD_t f\cdot f^*, \tag{13}$$

$$fD_x^2 g\cdot f^* + g\partial_x(D_x f\cdot f^*) = f^* D_x^2 g\cdot f + 2g_x D_x f\cdot f^*. \tag{14}$$

Subsequently, we show how the bilinear equations (8)–(12) are derived uniformly from the KP hierarchy. To this end, we list the following bilinear equations in the two-component hierarchy

$$(D_{x_2} - D_{x_1}^2)G\cdot F = 0, \tag{15}$$

$$(D_{x_2} - D_{x_1}^2)G\cdot \bar F = 0, \tag{16}$$

$$(D_{y_2} - D_{y_1}^2)F\cdot \bar F = 0, \tag{17}$$

$$D_{x_1} F\cdot \bar F - \mu G\bar G = 0, \tag{18}$$

$$D_{x_2} F\cdot \bar F - \mu D_{x_1} G\cdot \bar G = 0, \tag{19}$$

which admit the following Gram-type determinant solutions

$$F = |A|, \quad \bar F = |A'|, \tag{20}$$

$$G = \begin{bmatrix} A & \Phi^T \\ -\bar\Psi & 0 \end{bmatrix}, \quad \bar G = \begin{bmatrix} A' & \Psi^T \\ -\bar\Phi & 0 \end{bmatrix}, \tag{21}$$

where A and A' are $N\times N$ matrices, and Φ, Ψ, $\bar\Phi$, $\bar\Psi$ are N-component row vectors, whose elements are defined by

$$a_{ij} = \frac{1}{p_i+\bar p_j}e^{\xi_i+\bar\xi_j} - \frac{\mu q_i}{q_i+\bar q_j}e^{\eta_i+\bar\eta_j},$$

$$a'_{ij} = \frac{1}{p_i+\bar p_j}e^{\xi_i+\bar\xi_j} + \frac{\mu \bar q_j}{q_i+\bar q_j}e^{\eta_i+\bar\eta_j},$$

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad \Psi = (e^{\eta_1}, e^{\eta_2}, \dots, e^{\eta_N}),$$

$$\bar\Phi = (e^{\bar\xi_1}, e^{\bar\xi_2}, \dots, e^{\bar\xi_N}), \quad \bar\Psi = (e^{\bar\eta_1}, e^{\bar\eta_2}, \dots, e^{\bar\eta_N}),$$

$$\xi_i = p_i x_1 + p_i^2 x_2 + \xi_{i,0}, \quad \eta_i = q_i y_1 + q_i^2 y_2 + \eta_{i,0},$$

$$\bar\xi_i = \bar p_i x_1 - \bar p_i^2 x_2 + \bar\xi_{i,0}, \quad \bar\eta_i = \bar q_i y_1 - \bar q_i^2 y_2 + \bar\eta_{i,0}.$$

Here p_i, $\bar p_i$, q_i, $\bar q_i$, $\xi_{i,0}$, $\bar\xi_{i,0}$, $\eta_{i,0}$, $\bar\eta_{i,0}$, and μ are arbitrary constants.

In order to perform the dimension reduction, we rewrite tau functions F and $\bar F$ as

$$F = e^{\sum_{i=1}^N \xi_i+\bar\xi_i}\left|\frac{1}{p_i+\bar p_j} - \frac{\mu q_i}{q_i+\bar q_j}e^{\eta_i+\bar\eta_j-\xi_i-\bar\xi_j}\right|$$

$$= e^{\sum_{i=1}^N \xi_i+\bar\xi_i}|b_{ij}|,$$

$$\bar F = e^{\sum_{i=1}^N \xi_i+\bar\xi_i}\left|\frac{1}{p_i+\bar p_j} + \frac{\mu \bar q_j}{q_i+\bar q_j}e^{\eta_i+\bar\eta_j-\xi_i-\bar\xi_j}\right|$$

$$= e^{\sum_{i=1}^N \xi_i+\bar\xi_i}|b'_{ij}|.$$

By imposing the constraint condition $q_i = p_i$ and $\bar q_i = \bar p_i$, we have the following relations

$$(\partial_{x_i} + \partial_{y_i})b_{ij} = 0, \quad (\partial_{x_i} + \partial_{y_i})b'_{ij} = 0, \quad (i = 1, 2), \tag{22}$$

which give rise to the dimensional reduction relations

$$(\partial_{x_i} + \partial_{y_i})F = c_i F, \quad (\partial_{x_i} + \partial_{y_i})\bar F = c_i \bar F, \quad (i = 1, 2), \tag{23}$$

with $c_1 = \sum_{i=1}^N (p_i + \bar p_i)$ and $c_2 = \sum_{i=1}^N (p_i^2 - \bar p_i^2)$. This implies $D_{y_1}^2 F\cdot\bar F = D_{x_1}^2 F\cdot \bar F$, $D_{y_2}F\cdot\bar F = -D_{x_2}F\cdot\bar F$. Thus, Eq. (17) reduces to

$$(D_{x_2} + D_{x_1}^2)F\cdot\bar F = 0. \tag{24}$$

Furthermore, we set $\mu = -\mathrm{i}$, $x_1 = x$, $x_2 = \mathrm{i}t$, $\eta_{i,0} = \bar\eta_{i,0} = 0$ and require $\bar p_i = p_i^*$, $\bar\xi_{i,0} = \xi_{i,0}^*$, then the following complex conjugate relations hold

$$a_{ij}^* = a'_{ji}, \quad F^* = \bar F, \quad G^* = \bar G. \tag{25}$$

Thus, by defining

$$F = f^*, \quad \bar F = f, \quad G = g, \quad \bar G = g^*, \tag{26}$$

the bilinear equations (15)–(16), (24), and (18)–(19) are reduced to the bilinear equations (8)–(12). As a byproduct, the bright soliton solution of the GDNLS equation (4) is given by the following theorem. Here we comment that upon dimension reduction, y_1 and y_2 become dummy variables, which can be taken as zero values. Thus, η_i and $\bar\eta_i$ can also be treated as zero values.

Theorem 1. *The GDNLS equation (4) possesses the bright soliton solution (5) with the Gram-type determinants*

$$f = |\tilde A|, \quad g = \begin{bmatrix} \tilde A & \Phi^T \\ -\mathbf{I} & 0 \end{bmatrix}, \tag{27}$$

where $\tilde A$ is a $N\times N$ matrix, and Φ, $\mathbf{I}$ are N-component row vectors, whose elements are given by

$$\tilde a_{ij} = \frac{1}{p_i + p_j^*}e^{\xi_i+\xi_j^*} - \frac{\mathrm{i}p_j^*}{p_i+p_j^*},$$

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad \mathbf{I} = (1, 1, \dots, 1).$$

Here $\xi_i = p_i x + \mathrm{i}p_i^2 t + \xi_{i,0}$, p_i and $\xi_{i,0}$ are complex constants.

B. The dark soliton solution. For the nonvanishing boundary condition, we consider the dependent variable transformation with the background

$$u = \rho e^{\mathrm{i}\phi}\frac{f^{*\gamma-1}g}{f^\gamma}, \tag{28}$$

where $\phi = [\kappa - (\gamma-1)\rho^2]x - [(\gamma-1)(\kappa+\rho^2)^2 - \gamma\kappa^2 - 2\kappa\rho^2]t$, and ρ, κ are real constants. Then the GDNLS equation (4) can be decomposed into one set of bilinear equations

$$D_1 = 0, \quad D_3 = 0, \quad D_4 = 0, \quad D_5 = 0, \tag{29}$$

or another set of bilinear equations

$$D_2 = 0, \quad D_3 = 0, \quad D_4 = 0, \quad D_5 = 0, \tag{30}$$

where D_i $(i = 1\cdots 5)$ are defined by

$$D_1 \equiv \mathrm{i}D_t g\cdot f + D_x^2 g\cdot f + 2\mathrm{i}\kappa D_x g\cdot f, \tag{31}$$

$$D_2 \equiv \mathrm{i}D_t g\cdot f^* + D_x^2 g\cdot f^* + 2\mathrm{i}(\kappa+\rho^2)D_x g\cdot f^*, \tag{32}$$

$$D_3 \equiv \mathrm{i}D_t f\cdot f^* + D_x^2 f\cdot f^* + 2\mathrm{i}\rho^2 D_x f\cdot f^*, \tag{33}$$

$$D_4 \equiv D_x f\cdot f^* - \mathrm{i}\rho^2(|g|^2 - |f|^2), \tag{34}$$

$$D_5 \equiv D_x^2 f\cdot f^* - \mathrm{i}\rho^2 D_x g\cdot g^* + \rho^2(2\kappa-\rho^2)(|g|^2 - |f|^2). \tag{35}$$

Two bilinear systems are linked by

$$fD_2 + g\partial_x D_4$$

$$= f^* D_1 + g(D_3 - D_5) + [2g_x + \mathrm{i}(2\kappa-\rho^2)g]D_4,$$

upon using (13), (14) and the following trilinear identities

$$fD_x g\cdot f^* = f^* D_x g\cdot f + gD_x f\cdot f^*, \tag{36}$$

$$2fD_x g\cdot f^* = gD_x f\cdot f^* - g\partial_x(|f|^2) + 2g_x|f|^2, \quad (f\leftrightarrow f^*). \tag{37}$$

Next we show how the bilinear equations (31)–(35) are reduced uniformly from the KP hierarchy. To this end, let us start with the bilinear equations in the extended KP hierarchy

$$(D_{x_2} - D_{x_1}^2 - 2aD_{x_1})\tau_{n+1,k,l} \cdot \tau_{n,k,l} = 0, \tag{38}$$

$$(D_{x_2} - D_{x_1}^2 - 2bD_{x_1})\tau_{n,k+1,l-1} \cdot \tau_{n,k,l-1} = 0, \tag{39}$$

$$(D_{x_2} + D_{x_1}^2 - 2cD_{x_1})\tau_{n,k,l-1} \cdot \tau_{n,k,l} = 0, \tag{40}$$

$$[(c-b)D_{x_{-1}} + 1]\tau_{n,k,l} \cdot \tau_{n,k,l-1} = \tau_{n,k-1,l}\tau_{n,k+1,l-1}, \tag{41}$$

$$\begin{aligned}&[(c-b)D_{x_1}D_{x_{-1}} + D_{x_1} - 2(c-b)]\tau_{n,k,l} \cdot \tau_{n,k,l-1}\\ &\quad + [D_{x_1} + 2(c-b)]\tau_{n,k-1,l} \cdot \tau_{n,k+1,l-1} = 0,\end{aligned} \tag{42}$$

which have the Gram-type determinant solutions

$$\tau_{n,k,l} = |m_{ij}^{n,k,l}|_{1\le i,j\le N}, \tag{43}$$

where the entries of the determinant are given by

$$\begin{aligned}m_{ij}^{n,k,l} = \delta_{ij} + \frac{\mathrm{i}(p_i - c)}{p_i + \bar{p}_j}\left(-\frac{p_i - a}{\bar{p}_j + a}\right)^n\left(-\frac{p_i - b}{\bar{p}_j + b}\right)^k\\ \times\left(-\frac{p_i - c}{\bar{p}_j + c}\right)^l \mathrm{e}^{\xi_i + \bar{\xi}_j},\end{aligned} \tag{44}$$

with

$$\xi_i = \frac{1}{p_i - b}x_{-1} + p_i x_1 + p_i^2 x_2 + \xi_{i,0},$$

$$\bar{\xi}_j = \frac{1}{\bar{p}_j + b}x_{-1} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_{j,0},$$

where p_i, $\bar{p}_j$, $\xi_{i,0}$, $\bar{\xi}_{j,0}$, a, b, and c are arbitrary constants. By imposing the reduction condition

$$(c-b)\left(\frac{1}{p_i - b} + \frac{1}{\bar{p}_i + b}\right) = \frac{1}{c}(p_i + \bar{p}_i), \quad a + c = b, \tag{45}$$

or

$$(p_i - b)(\bar{p}_i + b) = c(c-b), \quad a + c = b, \tag{46}$$

one can check that $\tau_{n,k,l}$ satisfies

$$(c-b)\partial_{x_{-1}}\tau_{n,k,l} = \frac{1}{c}\partial_{x_1}\tau_{n,k,l}, \quad \tau_{n,k,l} = \tau_{n-1,k+1,l-1}. \tag{47}$$

It then follows that bilinear equations (41) and (42) become

$$(D_{x_1} + c)\tau_{n,k,l} \cdot \tau_{n,k,l-1} = c\tau_{n-1,k,l-1}\tau_{n+1,k,l}, \tag{48}$$

$$\begin{aligned}&(D_{x_1}^2 + cD_{x_1} + 2ac)\tau_{n,k,l} \cdot \tau_{n,k,l-1}\\ &\quad + (cD_{x_1} - 2ac)\tau_{n-1,k,l-1} \cdot \tau_{n+1,k,l} = 0.\end{aligned} \tag{49}$$

Furthermore, a substitution of (48) into (49) leads to

$$\begin{aligned}&D_{x_1}^2\tau_{n,k,l} \cdot \tau_{n,k,l-1} + cD_{x_1}\tau_{n-1,k,l-1} \cdot \tau_{n+1,k,l}\\ &\quad + (c^2 - 2ac)(\tau_{n-1,k,l-1}\tau_{n+1,k,l} - \tau_{n,k,l}\tau_{n,k,l-1}) = 0.\end{aligned} \tag{50}$$

Next, let us take $a = \mathrm{i}\kappa$, $c = \mathrm{i}\rho^2$, $x_1 = x$, $x_2 = \mathrm{i}t$, and require $\bar{p}_i = p_i^*$, $\bar{\xi}_{i,0} = \xi_{i,0}^*$, then the following complex conjugate relations hold

$$\tau_{0,0,0}^* = \tau_{0,0,-1}, \quad \tau_{1,0,0}^* = \tau_{-1,0,-1}. \tag{51}$$

Therefore, by defining

$$\tau_{0,0,0} = f, \quad \tau_{0,0,-1} = f^*, \quad \tau_{1,0,0} = g, \quad \tau_{-1,0,-1} = g^*, \tag{52}$$

the bilinear equations (38)–(40), (48), and (50) coincide exactly with the bilinear equations (31)–(35). Meanwhile, we obtain the dark soliton solution of the GDNLS equation (4) given by the following theorem.

Theorem 2. *The GDNLS equation (4) admits the dark soliton solution (28) with the Gram-type determinants*

$$f = \left|\delta_{ij} + \frac{\mathrm{i}(p_i - \mathrm{i}\rho^2)}{p_i + \bar{p}_j}\mathrm{e}^{\xi_i + \bar{\xi}_j}\right|_{1\le i,j\le N}, \tag{53}$$

$$g = \left|\delta_{ij} + \frac{\mathrm{i}(p_i - \mathrm{i}\rho^2)}{p_i + \bar{p}_j}\left(-\frac{p_i - \mathrm{i}\kappa}{\bar{p}_j + \mathrm{i}\kappa}\right)\mathrm{e}^{\xi_i + \bar{\xi}_j}\right|_{1\le i,j\le N}, \tag{54}$$

where $\xi_i = p_i x + \mathrm{i}p_i^2 t + \xi_{i,0}$, *and* p_i, $\xi_{i,0}$ *are complex constants,* κ, ρ *are real constants, which satisfy the constraint condition*

$$(p_i - \mathrm{i}(\kappa + \rho^2))(p_i^* + \mathrm{i}(\kappa + \rho^2)) = \kappa\rho^2. \tag{55}$$

In summary, we have investigated the bilinearization of the GDNLS equation under both the vanishing and nonvanishing boundary conditions, and we have derived soliton solutions via the KP hierarchy reduction. It is shown that the GDNLS equation can be derived from two different bilinear systems. We have shown that, for either the vanishing or nonvanishing boundary condition, these bilinear members can be uniformly reduced from the KP hierarchy through appropriate reductions. As a byproduct, we have constructed bright and dark soliton solutions to the GDNLS equation expressed by Gram-type determinants.

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