



# General rogue wave solution to the discrete nonlinear Schrödinger equation

Yasuhiro Ohta<sup>a</sup>, Bao-Feng Feng<sup>b,\*</sup>

<sup>a</sup> Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

<sup>b</sup> School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, United States of America

## ARTICLE INFO

### Article history:

Received 7 January 2022

Received in revised form 21 May 2022

Accepted 31 May 2022

Available online 11 June 2022

### Keywords:

Fully discrete NLS equation

Breather

Rogue wave

KP–Toda reduction

## ABSTRACT

In the present paper, we are concerned with the rogue wave solution to an integrable discrete nonlinear Schrödinger (NLS) equation. First, through the KP–Toda reduction method, the general breather solution of the discrete NLS equation is deduced from two bilinear equations in discrete two-dimensional Toda-lattice hierarchy. Then, by taking an ingenious and successive limit to the breather solution, we derive the higher order rogue wave solution to the discrete NLS equation.

© 2022 Elsevier B.V. All rights reserved.

## 1. Introduction

Rogue waves (RWs) or freak waves are spontaneously excited local nonlinear waves with large amplitudes which appear from nowhere and disappear with no trace [1]. The first rogue wave recorded, known as “The Draupner Wave”, was measured in 1995 off the coast of Norway at 84 ft (25.6 m) with surrounding waves of approximately 40 ft (12 m), making the original rogue wave about twice the size of those around it [2]. Most recently, the record-breaking rogue wave was recorded off the coast of Vancouver Island in November 2020 measured at almost 58 ft (17.6 m) in comparison to surrounding waves of around 20 ft (6 m) [3]. The mathematical study of rogue waves starts with the rational solution of the nonlinear Schrödinger (NLS) equation, a generic model in nonlinear waves. The simplest form of such waves was firstly discovered by Peregrine in the NLS equation [4], and their higher order forms were found in 1986 in [5] and later by many authors (see, for example, [6–12]). Such extreme waves have been observed in various different contexts such as oceanography [13], hydrodynamic [14,15], Bose–Einstein condensate [16], plasma [17] and nonlinear optic [14,18,19].

Motivated by these physical applications, rogue wave solutions have been found in many other nonlinear wave equations such as the derivative Schrödinger (NLS) equation [20–24], the Manakov system [25–27], Davey–Stewartson I and II equation [28,29], the three-wave equation [30], the Boussinesq equation [31], the Yajima–Oikawa equation [32,33]. There are some developments in the study of RW most recently, Yang et al. found universal rogue wave patterns for RW behind many soliton equations [27]. Miller et al. developed a robust inverse scattering transform for the analysis of RW of infinite order [34].

On the other hand, in spite of potential applications in optical waveguides [35,36], the study of rogue waves in discrete integrable systems is much less. As far as we are aware, the rogue waves in semi-discrete NLS equation, or the so-called Ablowitz–Ladik equation [37–40] and in semi-discrete complex short pulse equation have been reported in the literature [41]. The inverse scattering transform for the AL equation was studied and discrete soliton solutions including the Kuznetsov–Ma, Akhmediev breather and Peregrine soliton solutions were obtained [42]. The vector discrete rogue waves in the coupled AL equations were investigated in [43,44]. It should be pointed out that the correspondence of breather and first order rogue wave solutions between the NLS and AL equations was found and has been extended to periodic solutions between Manakov system and coupled AL equation [43].

Since the monumental work by Ablowitz and Hirota [45–48], the study of discrete integrable systems has attracted much attention [49]. Discrete integrable systems become continuous integrable systems when an appropriate limiting procedure is applied. Discrete integrable systems have rich but complicated mathematical structures which go to ones of continuous integrable systems in appropriate continuous limits. What about the possible rogue wave solutions in fully discrete systems? This natural question motivates

\* Corresponding author.

E-mail address: [baofeng.feng@utrgv.edu](mailto:baofeng.feng@utrgv.edu) (B.-F. Feng).

our present work. In the present paper, we attempt to construct the rogue wave solutions to the fully discrete nonlinear Schrödinger (fd-NLS) equation, which was originally discovered by Ablowitz and Ladik [45,46] and rediscovered by Hirota and Ohta through Hirota's bilinear approach [50]. The bright soliton solutions and the reduction from the two-component Kadomtsev–Petviashvili hierarchy was discussed in [51,52].

The remainder of the present paper is organized as follows. In Section 2, we will construct general breather solution to the fd-NLS equation. We derive general rogue wave solutions to the fd-NLS equation by taking the limit to the general breather solution in Section 3. The paper is concluded in Section 4. In the Appendix, we give the derivation of two bilinear equations associated with fully discrete NLS equation from higher order discrete Kadomtsev–Petviashvili (KP) equation.

## 2. Breather solution for the fully discrete NLS equation

In order to derive breather solutions to the fd-NLS equation, we first give two bilinear equations in discrete KP–Toda hierarchy.

**Lemma 1.** *The following  $\tau$  function of discrete KP–Toda hierarchy*

$$\tau_n(k, K, l, L) = \det_{1 \leq i, j \leq N} \left( m_{ij}^{(n)}(k, K, l, L) \right), \quad (1)$$

satisfies the discrete bilinear equations

$$\begin{cases} (1 - aA)\tau_n(k+1, K+1, l, L)\tau_n(k, K, l, L) \\ - \tau_n(k+1, K, l, L)\tau_n(k, K+1, l, L) \\ + aA\tau_{n+1}(k+1, K, l, L)\tau_{n-1}(k, K+1, l, L) = 0, \end{cases} \quad (2)$$

$$\begin{cases} A(a-b)(1-aB)\tau_{n+1}(k+1, K, l+1, L+1)\tau_n(k, K+1, l, L) \\ - a(A-B)(1-Ab)\tau_{n+1}(k+1, K, l, L)\tau_n(k, K+1, l+1, L+1) \\ + Ab(1-aA)\tau_{n+1}(k, K, l+1, L+1)\tau_n(k+1, K+1, l, L) \\ - aB(1-aA)\tau_{n+1}(k+1, K+1, l, L)\tau_n(k, K, l+1, L+1) = 0, \end{cases} \quad (3)$$

if the matrix element  $m_{ij}^{(n)}$  satisfies

$$\begin{aligned} m_{ij}^{(n+1)}(k, K, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^{n+1} \phi_i^{(n)}(k, K, l, L) \psi_j^{(-n-1)}(k, K, l, L), \\ m_{ij}^{(n)}(k+1, K, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n a \phi_i^{(n)}(k+1, K, l, L) \psi_j^{(-n)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K+1, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n A \phi_i^{(n-1)}(k, K+1, l, L) \psi_j^{(-n-1)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K, l+1, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n b \phi_i^{(n)}(k, K, l+1, L) \psi_j^{(-n)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K, l, L+1) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n B \phi_i^{(n-1)}(k, K, l, L+1) \psi_j^{(-n-1)}(k, K, l, L), \end{aligned}$$

where  $\phi_i^{(n)}$  and  $\psi_j^{(n)}$  are arbitrary functions satisfying the linear dispersion relations,

$$\begin{aligned} \phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k-1, K, l, L) &= a \phi_i^{(n+1)}(k, K, l, L), \\ \phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K-1, l, L) &= A \phi_i^{(n-1)}(k, K, l, L), \\ \phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K, l-1, L) &= b \phi_i^{(n+1)}(k, K, l, L), \\ \phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K, l, L-1) &= B \phi_i^{(n-1)}(k, K, l, L), \\ \psi_j^{(n)}(k+1, K, l, L) - \psi_j^{(n)}(k, K, l, L) &= a \psi_j^{(n+1)}(k, K, l, L), \\ \psi_j^{(n)}(k, K+1, l, L) - \psi_j^{(n)}(k, K, l, L) &= A \psi_j^{(n-1)}(k, K, l, L), \\ \psi_j^{(n)}(k, K, l+1, L) - \psi_j^{(n)}(k, K, l, L) &= b \psi_j^{(n+1)}(k, K, l, L), \\ \psi_j^{(n)}(k, K, l, L+1) - \psi_j^{(n)}(k, K, l, L) &= B \psi_j^{(n-1)}(k, K, l, L). \end{aligned}$$

Here  $k, K, l, L$  are discrete independent variables and  $a, A, b, B$  are lattice constants.

The link of above two bilinear equations with higher order discrete KP hierarchy is given in the Appendix.

Based on the bilinear formulation, we are able to derive the breather solution of the fully discrete NLS equation, which is given by the following theorem.

**Theorem 1.** *The fully discrete NLS equation,*

$$\begin{cases} i(q_k^{t+1} - q_k^t) = (q_{k+1}^t + q_{k-1}^{t+1})(1 + \epsilon |q_k^t|^2) \Gamma_k^t, \\ \Gamma_{k+1}^t = \frac{1 + \epsilon |q_k^t|^2}{1 + \epsilon |q_k^{t+1}|^2} \Gamma_k^t, \end{cases} \quad (4)$$

where  $\epsilon = \pm 1$ , admits the  $N$ -breather solution,

$$q_k^t = \frac{g_k^t}{f_k^t} \frac{1-r}{2\sqrt{|r|}} e^{ik\theta} \left( \frac{1-c}{1-\bar{c}} \frac{1+r\bar{c}}{1+rc} \right)^t, \quad \Gamma_k^t = \frac{4rR}{(1+r)^2} \frac{f_{k-1}^{t+1} f_k^t}{f_k^{t+1} f_{k-1}^t}, \quad (5)$$

where  $\bar{\phantom{x}}$  means complex conjugate,  $r$  is a real constant whose sign coincides with  $\epsilon$ ,  $\epsilon = \text{sign } r$ ,  $c$  is a complex constant,  $R$  and  $\theta$  are determined by

$$\frac{1}{i} \frac{1+c}{1-\bar{c}} \frac{1-r\bar{c}}{1+rc} = Re^{i\theta}, \quad (6)$$

and  $f_k^t$  and  $g_k^t$  are given by

$$f_k^t = \tau_k^t(0), \quad g_k^t = \tau_k^t(1), \quad (7)$$

with

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} \left( A_{ij}^{(n)}(k, t) \right), \quad (8)$$

$$A_{ij}^{(n)} = \frac{1}{1 - r p_i \bar{p}_j} \left( a_i \varphi_n(p_i) \overline{a_j \varphi_{-n}(\bar{p}_j)} + b_i \varphi_n(-p_i) \overline{b_j \varphi_{-n}(-\bar{p}_j)} \right) - \frac{1}{1 + r p_i \bar{p}_j} \left( a_i \varphi_n(p_i) \overline{b_j \varphi_{-n}(-\bar{p}_j)} + b_i \varphi_n(-p_i) \overline{a_j \varphi_{-n}(\bar{p}_j)} \right), \quad (9)$$

$$\varphi_n(p) = \left( \frac{1-rp}{1+p} \right)^n \left( \frac{1+p}{1+rp} \right)^k \left( \frac{1+p}{1+rp} \frac{1+p/\bar{c}}{1+rcp} \right)^t, \quad (10)$$

where  $p_i$ ,  $a_i$ ,  $b_i$  are complex constants.

**Proof.** We set

$$\phi_i^{(n)}(k, K, l, L) = \sum_{v=1}^2 a_{iv} p_{iv}^n (1 - a p_{iv})^{-k} (1 - A/p_{iv})^{-K} (1 - b p_{iv})^{-l} (1 - B/p_{iv})^{-L}, \quad (11)$$

$$\psi_j^{(n)}(k, K, l, L) = \sum_{\mu=1}^2 b_{j\mu} q_{j\mu}^n (1 + a q_{j\mu})^k (1 + A/q_{j\mu})^K (1 + b q_{j\mu})^l (1 + B/q_{j\mu})^L, \quad (12)$$

where  $p_{iv}$ ,  $q_{j\mu}$ ,  $a_{iv}$ ,  $b_{j\mu}$  are arbitrary constants. Obviously, the  $\phi_i^{(n)}$  and  $\psi_j^{(n)}$  defined satisfy the above linear dispersion relations in [Lemma 1](#). Therefore the determinant  $\tau_n$  with element defined by

$$m_{ij}^{(n)}(k, K, l, L) \quad (13)$$

$$= \sum_{v=1}^2 \sum_{\mu=1}^2 \frac{a_{iv} b_{j\mu}}{p_{iv} + q_{j\mu}} \left( -\frac{p_{iv}}{q_{j\mu}} \right)^n \left( \frac{1 + a q_{j\mu}}{1 - a p_{iv}} \right)^k \left( \frac{1 + A/q_{j\mu}}{1 - A/p_{iv}} \right)^K \left( \frac{1 + b q_{j\mu}}{1 - b p_{iv}} \right)^l \left( \frac{1 + B/q_{j\mu}}{1 - B/p_{iv}} \right)^L \quad (14)$$

satisfies the bilinear equations (2)–(3).

Next, we proceed to the reduction procedure. By imposing

$$p_{i2} = -\frac{A}{a} \frac{1 - a p_{i1}}{p_{i1} - A}, \quad q_{j2} = -\frac{A}{a} \frac{1 + a q_{j1}}{q_{j1} + A},$$

we can show that  $\tau_n$  satisfies the following reduction condition,

$$\tau_n(k+1, K-1, l, L) = \tau_n(k, K, l, L) \prod_{i=1}^N \frac{q_{i1} q_{i2}}{p_{i1} p_{i2}}, \quad (15)$$

it follows that the bilinear equations (2)–(3) are reduced to

$$\begin{cases} (1 - aA) \sigma_{k+1}^t(n) \sigma_{k-1}^t(n) - \sigma_k^t(n) \sigma_k^t(n) + aA \sigma_k^t(n+1) \sigma_k^t(n-1) = 0, \\ A(a-b)(1-aB) \sigma_k^{t+1}(n+1) \sigma_k^t(n) - a(A-B)(1-Ab) \sigma_k^t(n+1) \sigma_k^{t+1}(n) \\ + Ab(1-aA) \sigma_{k-1}^{t+1}(n+1) \sigma_{k+1}^t(n) - aB(1-aA) \sigma_{k+1}^t(n+1) \sigma_{k-1}^{t+1}(n) = 0 \end{cases} \quad (16)$$

by taking  $\sigma_k^t(n) = \tau_n(k, 0, t, t)$ . Furthermore by taking

$$A = -\epsilon \bar{a}, \quad B = -\epsilon \bar{b}, \quad q_{j1} = \frac{\bar{a} \bar{p}_{j1} + \epsilon a}{a(1 - \bar{a} \bar{p}_{j1})}, \quad b_{j1} = \frac{\bar{a}_{j1}}{a(1 - \bar{a} \bar{p}_{j1})}, \quad b_{j2} = \frac{\bar{a}_{j2}}{a(1 - \bar{a} \bar{p}_{j2})}, \quad (17)$$

where  $\epsilon = +1$  or  $-1$ ,  $\sigma_k^t(n)$  satisfies the complex conjugate condition,

$$\sigma_k^t(-n)G = \overline{\sigma_k^t(n)G},$$

with a gauge factor  $G$  given below in Eq. (22). In order to simplify the final expression, let us parametrize by

$$|a|^2 = \epsilon \frac{r-2+1/r}{4}, \quad b = \frac{2a}{1-r} \frac{1-rc}{1+c}, \quad p_{i1} = \frac{1-1/r}{2a} \frac{1-rp_i}{1+p_i}, \quad (18)$$

$$a_{i1} = \epsilon \frac{r - 1/r}{4} a_i, \quad a_{i2} = -a(\bar{a} + \epsilon p_{i2}) b_i, \quad (19)$$

where  $r$  is a real constant,  $c$  is a complex constant,  $p_i$  represents the complex wave number of the  $i$ th breather,  $a_i$  and  $b_i$  stand for the complex phase constants of the  $i$ th breather. We comment that  $\epsilon$  is equal to the sign of  $r$  because of the positivity of  $|a|^2$ . Then we have

$$\begin{aligned} \tilde{m}_{ij}^n(k, K, l, L) &:= m_{ij}^n(k, K, l, L) \frac{-4}{(1-r)(1+p_i)(1+\bar{p}_j)} \left( -r \frac{1-\bar{p}_j^2}{1-r^2\bar{p}_j^2} \right)^n r^{K-k}(rc)^{-l} \left( r\bar{c} \frac{1-p_i^2/\bar{c}^2}{1-r^2p_i^2} \frac{1-\bar{p}_j^2}{1-r^2\bar{c}^2\bar{p}_j^2} \right)^L \\ &= \frac{a_i \bar{a}_j}{1-rp_i\bar{p}_j} \left( \frac{1-rp_i}{1+p_i} \frac{1+\bar{p}_j}{1-r\bar{p}_j} \right)^n \left( \frac{1+p_i}{1+rp_i} \frac{1+\bar{p}_j}{1+r\bar{p}_j} \right)^k \left( \frac{1-rp_i}{1-p_i} \frac{1-r\bar{p}_j}{1-\bar{p}_j} \right)^K \\ &\quad \times \left( \frac{1+p_i}{1+rcp_i} \frac{1+\bar{p}_j/c}{1+r\bar{p}_j} \right)^l \left( \frac{1+p_i/\bar{c}}{1+rp_i} \frac{1+\bar{p}_j}{1+r\bar{c}\bar{p}_j} \right)^L \\ &\quad - \frac{a_i \bar{b}_j}{1+rp_i\bar{p}_j} \left( \frac{1-rp_i}{1+p_i} \frac{1-\bar{p}_j}{1+r\bar{p}_j} \right)^n \left( \frac{1+p_i}{1+rp_i} \frac{1-\bar{p}_j}{1-r\bar{p}_j} \right)^k \left( \frac{1-rp_i}{1-p_i} \frac{1+r\bar{p}_j}{1+\bar{p}_j} \right)^K \\ &\quad \times \left( \frac{1+p_i}{1+rcp_i} \frac{1-\bar{p}_j/c}{1-r\bar{p}_j} \right)^l \left( \frac{1+p_i/\bar{c}}{1+rp_i} \frac{1-\bar{p}_j}{1-r\bar{c}\bar{p}_j} \right)^L \\ &\quad - \frac{b_i \bar{a}_j}{1+rp_i\bar{p}_j} \left( \frac{1+rp_i}{1-p_i} \frac{1+\bar{p}_j}{1-r\bar{p}_j} \right)^n \left( \frac{1-p_i}{1-rp_i} \frac{1+\bar{p}_j}{1+r\bar{p}_j} \right)^k \left( \frac{1+rp_i}{1+p_i} \frac{1-r\bar{p}_j}{1-\bar{p}_j} \right)^K \\ &\quad \times \left( \frac{1-p_i}{1-rp_i} \frac{1+\bar{p}_j/c}{1+r\bar{p}_j} \right)^l \left( \frac{1-p_i/\bar{c}}{1-rp_i} \frac{1+\bar{p}_j}{1+r\bar{c}\bar{p}_j} \right)^L \\ &\quad + \frac{b_i \bar{b}_j}{1-rp_i\bar{p}_j} \left( \frac{1+rp_i}{1-p_i} \frac{1-\bar{p}_j}{1+r\bar{p}_j} \right)^n \left( \frac{1-p_i}{1-rp_i} \frac{1-\bar{p}_j}{1-r\bar{p}_j} \right)^k \left( \frac{1+rp_i}{1+p_i} \frac{1+r\bar{p}_j}{1+\bar{p}_j} \right)^K \\ &\quad \times \left( \frac{1-p_i}{1-rp_i} \frac{1-\bar{p}_j/c}{1-r\bar{p}_j} \right)^l \left( \frac{1-p_i/\bar{c}}{1-rp_i} \frac{1-\bar{p}_j}{1-r\bar{c}\bar{p}_j} \right)^L, \end{aligned}$$

and the bilinear equations (16) are reduced to

$$\begin{cases} (1+r)^2 \tau_{k+1}^t(n) \tau_{k-1}^t(n) - 4r \tau_k^t(n) \tau_k^t(n) - (1-r)^2 \tau_k^t(n+1) \tau_k^t(n-1) = 0, \\ (1-c)(1+r\bar{c}) \tau_k^{t+1}(n+1) \tau_k^t(n) - (1-\bar{c})(1+rc) \tau_k^t(n+1) \tau_k^{t+1}(n) \\ -(1+\bar{c})(1-rc) \tau_{k+1}^{t+1}(n+1) \tau_{k+1}^t(n) + (1+c)(1-r\bar{c}) \tau_{k+1}^t(n+1) \tau_{k-1}^{t+1}(n) = 0, \end{cases} \quad (20)$$

where  $\tau_k^t(n)$  is given by

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} (\tilde{m}_{ij}^n(k, 0, t, t)) = \sigma_k^t(n) G, \quad (21)$$

with

$$G = \prod_{i=1}^N \frac{-4}{(1-r)(1+p_i)(1+\bar{p}_i)} \left( -r \frac{1-\bar{p}_i^2}{1-r^2\bar{p}_i^2} \right)^n r^{-k} \left( \frac{\bar{c}}{c} \frac{1-p_i^2/\bar{c}^2}{1-r^2p_i^2} \frac{1-\bar{p}_i^2}{1-r^2\bar{c}^2\bar{p}_i^2} \right)^t, \quad (22)$$

which is nothing but  $\tau_k^t(n)$  in Theorem 1. For  $f_k^t = \tau_k^t(0)$ ,  $g_k^t = \tau_k^t(1)$ ,  $\bar{g}_k^t = \tau_k^t(-1)$ , the bilinear equations are written as

$$\begin{cases} (1+r)^2 f_{k+1}^t f_{k-1}^t - 4r f_k^t f_k^t - (1-r)^2 g_k^t \bar{g}_k^t = 0, \\ (1-c)(1+r\bar{c}) g_k^{t+1} f_k^t - (1-\bar{c})(1+rc) g_k^t f_k^{t+1} \\ -(1+\bar{c})(1-rc) g_{k+1}^{t+1} f_{k+1}^t + (1+c)(1-r\bar{c}) g_{k+1}^t f_{k-1}^{t+1} = 0, \end{cases}$$

from which the fully discrete NLS equation (4) is straightforwardly derived through the variable transformation (5). This completes the proof of Theorem 1.  $\square$

In the case of  $N = 1$ , we have one-breather solution,

$$\tau_k^t(n) = \frac{1}{1-r|p|^2} (|a|^2 \varphi_n(p) \overline{\varphi_{-n}(\bar{p})} + |b|^2 \varphi_n(-p) \overline{\varphi_{-n}(-\bar{p})}) \quad (23)$$

$$- \frac{1}{1+r|p|^2} (a \bar{b} \varphi_n(p) \overline{\varphi_{-n}(-\bar{p})} + \bar{a} b \varphi_n(-p) \overline{\varphi_{-n}(p)}), \quad (24)$$

with (5)–(10) where  $p$ ,  $a$ ,  $b$  are complex constants (the index 1 of  $p_1$ ,  $a_1$ ,  $b_1$  are omitted for notational simplicity). The Akhmediev breather which is the breather solution localized in time  $t$  can be derived by taking the wave number  $p$  pure imaginary. If we require the regularity of the solution only on the lattice points, i.e.,  $f_k^t \neq 0$  for integers  $k$  and  $t$ , then  $r$  can be either positive or negative and the discrete NLS equations (4) of both focusing type and defocusing type admit regular breather solutions by locating the singularities (if exist) off the lattice points. On the other hand if we require the solution to be regular in whole real space, i.e.,  $f_k^t$  is non-zero for all

real numbers  $k$  and  $t$ , then  $r$  has to be positive and only the focusing discrete NLS admits the regular breathers for generic parameters. An example is shown in Fig. 1. Some exceptional regular solutions for  $r < 0$  can be derived from the above  $\tau_k^t(n)$  but usually they are not called breathers and we do not discuss about such solutions.

### 3. Rogue wave solution to fully discrete NLS equation

The rogue wave solution of rational function type can be derived as a limit of the breather solution. In Theorem 1, we take

$$a_i = \frac{1}{2} \left( 1 + \sum_{v=1}^i c_v p_i^{2v-1} \right), \quad b_i = \frac{1}{2} \left( 1 - \sum_{v=1}^i c_v p_i^{2v-1} \right), \quad (25)$$

and scale the  $\tau$  function by  $\tau_k^t(n) / \prod_{i=1}^N (p_i \bar{p}_i)^{2i-1}$  and finally take the limit  $p_i \rightarrow 0$  successively for  $i = 1, 2, \dots, N$ . Then the leading order of  $\tau_k^t(n)$  in  $p_i$ 's gives a polynomial of  $k$  and  $t$  which turns to be the rogue wave solution. This result is summarized in the following theorem.

**Theorem 2.** The  $N$ th order rogue wave solution for the fully discrete NLS equation (4) is given by the variable transformations (5)–(7) and

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} \left( B_{ij}^{(n)}(k, t) \right) = \det_{1 \leq i, j \leq N} \left( \sum_{v=1}^{2 \min(i, j)} r^{v-1} \Phi_{2i-v}^{(n)} \overline{\Phi_{2j-v}^{(-n)}} \right), \quad (26)$$

where

$$\Phi_i^{(n)} = S_i(x(n)) + \sum_{\mu=1}^{\left[ \frac{i+1}{2} \right]} c_\mu S_{i+1-2\mu}(x(n)). \quad (27)$$

Here  $x(n) = (x_1(n), x_2(n), \dots, x_h(n), \dots)$  with

$$x_h(n) = \frac{(-1)^h}{h} \left( (1 - (-r)^h)n - (1 - r^h)k - (1 - r^h + (1/\bar{c})^h - (rc)^h)t \right), \quad h = 1, 2, \dots, \quad (28)$$

where  $[ \ ]$  means the Gauss symbol and  $S_\mu(x)$  is the so-called elementary Schur function defined by  $\sum_{\mu=0}^{\infty} S_\mu(x) \lambda^\mu = \exp \sum_{h=1}^{\infty} x_h \lambda^h$ . This solution has  $N$  complex parameters  $c_\mu$ ,  $\mu = 1, 2, \dots, N$ .

**Proof.** Firstly  $\varphi_n(p)$  in (10) is written as  $\varphi_n(p) = \sum_{\mu=0}^{\infty} S_\mu(x(n)) p^\mu$ . We rewrite  $A_{ij}^{(n)}$  in Theorem 1 as

$$A_{ij}^{(n)} = \frac{1}{1 - (rp_i \bar{p}_j)^2} \left( a_i \varphi_n(p_i) - b_i \varphi_n(-p_i) \right) \left( \overline{a_j \varphi_{-n}(p_j) - b_j \varphi_{-n}(-p_j)} \right) \\ + \frac{rp_i \bar{p}_j}{1 - (rp_i \bar{p}_j)^2} \left( a_i \varphi_n(p_i) + b_i \varphi_n(-p_i) \right) \left( \overline{a_j \varphi_{-n}(p_j) + b_j \varphi_{-n}(-p_j)} \right).$$

Denoting  $a_i - b_i = p_i d_i$  and  $a_i + b_i = s_i$ , the four factors in the above expression are written as

$$a_i \varphi_n(p_i) - b_i \varphi_n(-p_i) = \sum_{\mu=0}^{\infty} S_\mu(x(n)) p_i^\mu (a_i - (-1)^\mu b_i) \\ = (p_i d_i, p_i s_i, p_i^3 d_i, p_i^3 s_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots), \\ \overline{a_j \varphi_{-n}(p_j) - b_j \varphi_{-n}(-p_j)} = \sum_{\mu=0}^{\infty} \overline{S_\mu(x(-n))} p_j^\mu (a_j - (-1)^\mu b_j) \\ = (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{p}_j \bar{d}_j, \bar{p}_j \bar{s}_j, \bar{p}_j^3 \bar{d}_j, \bar{p}_j^3 \bar{s}_j, \dots), \\ a_i \varphi_n(p_i) + b_i \varphi_n(-p_i) = \sum_{\mu=0}^{\infty} S_\mu(x(n)) p_i^\mu (a_i + (-1)^\mu b_i) \\ = (s_i, p_i^2 d_i, p_i^2 s_i, p_i^4 d_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots), \\ \overline{a_j \varphi_{-n}(p_j) + b_j \varphi_{-n}(-p_j)} = \sum_{\mu=0}^{\infty} \overline{S_\mu(x(-n))} p_j^\mu (a_j + (-1)^\mu b_j) \\ = (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{s}_j, \bar{p}_j^2 \bar{d}_j, \bar{p}_j^2 \bar{s}_j, \bar{p}_j^4 \bar{d}_j, \dots),$$

where  ${}^t v$  means transpose of  $v$ . Thus we obtain

$$A_{ij}^{(n)} = \sum_{\lambda=0}^{\infty} (rp_i \bar{p}_j)^{2\lambda} (p_i d_i, p_i s_i, p_i^3 d_i, p_i^3 s_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots) \\ \times (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{p}_j \bar{d}_j, \bar{p}_j \bar{s}_j, \bar{p}_j^3 \bar{d}_j, \bar{p}_j^3 \bar{s}_j, \dots)$$

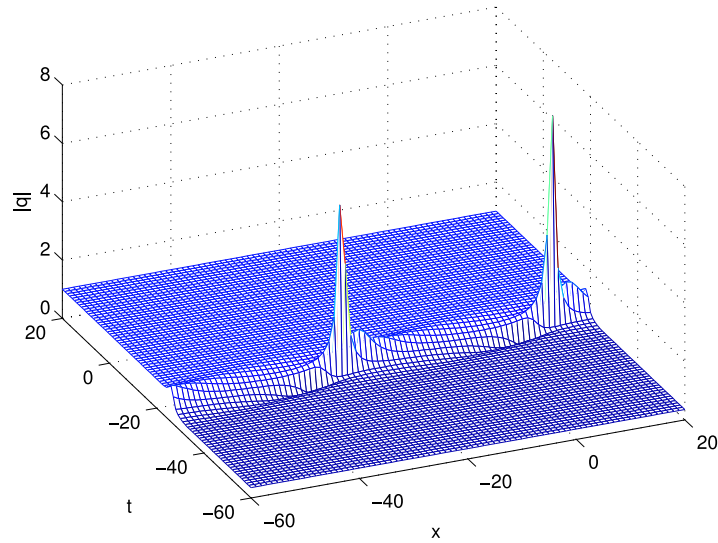


Fig. 1. One-breather solution for  $r = 0.5$ ,  $c = 3 + 2i$ ,  $p = 0.5i$ .

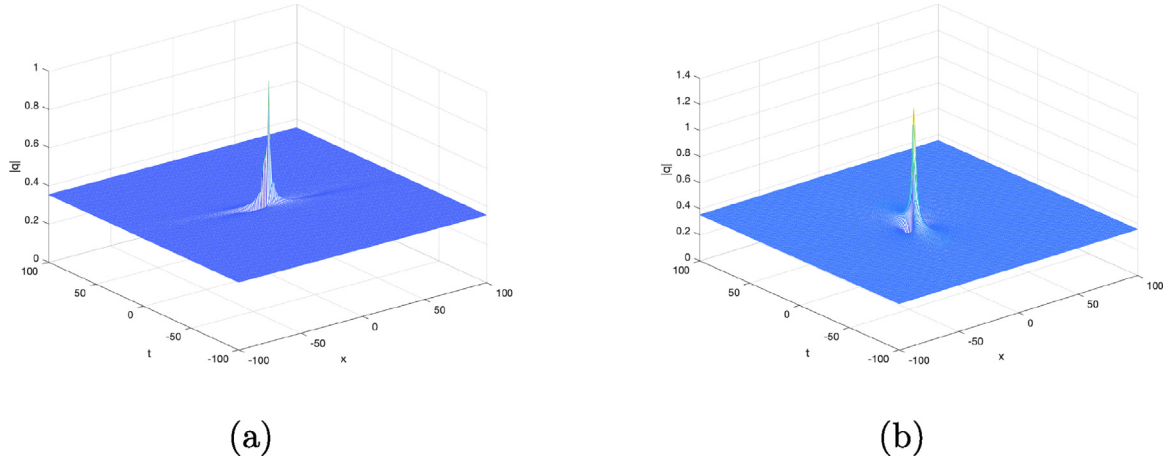
$$\begin{aligned}
 & + \sum_{\lambda=0}^{\infty} (rp_i \bar{p}_j)^{2\lambda+1} (s_i, p_i^2 d_i, p_i^2 s_i, p_i^4 d_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots) \\
 & \quad \times (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{s}_j, \bar{p}_j^2 \bar{d}_j, \bar{p}_j^2 \bar{s}_j, \bar{p}_j^4 \bar{d}_j, \dots) \\
 & = (p_i d_i \quad p_i s_i \quad p_i^3 d_i \quad p_i^3 s_i \quad \dots) \begin{pmatrix} S_0(x(n)) & & & & 0 \\ S_1(x(n)) & S_0(x(n)) & & & \\ S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & & \\ S_3(x(n)) & S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
 & \quad \times \begin{pmatrix} 1 & & & & 0 \\ & r & & & \\ & & r^2 & & \\ & & & r^3 & \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \dots \\ & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & \dots \\ & & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & \dots \\ & & & S_0(\overline{x(-n)}) & \dots \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} \bar{p}_j \bar{d}_j \\ \bar{p}_j \bar{s}_j \\ \bar{p}_j^3 \bar{d}_j \\ \bar{p}_j^3 \bar{s}_j \\ \vdots \end{pmatrix}.
 \end{aligned}$$

Therefore  $\tau_k^t(n)$  in Theorem 1 is given in the form of the following determinant,

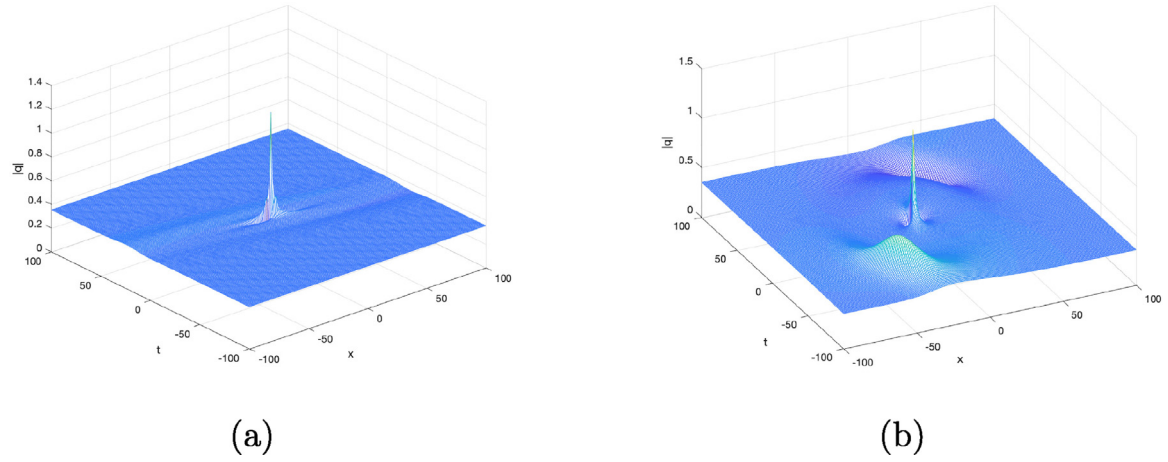
$$\begin{aligned}
 \tau_k^t(n) & = \begin{vmatrix} p_1 d_1 & p_1 s_1 & p_1^3 d_1 & p_1^3 s_1 & \dots \\ p_2 d_2 & p_2 s_2 & p_2^3 d_2 & p_2^3 s_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ p_N d_N & p_N s_N & p_N^3 d_N & p_N^3 s_N & \dots \end{vmatrix} \begin{pmatrix} S_0(x(n)) & & & & 0 \\ S_1(x(n)) & S_0(x(n)) & & & \\ S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & & \\ S_3(x(n)) & S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
 & \quad \times \begin{pmatrix} 1 & & & & 0 \\ & r & & & \\ & & r^2 & & \\ & & & r^3 & \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \dots \\ & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & \dots \\ & & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & \dots \\ & & & S_0(\overline{x(-n)}) & \dots \\ 0 & & & & \ddots \end{pmatrix} \\
 & \quad \times \begin{vmatrix} \bar{p}_1 \bar{d}_1 & \bar{p}_2 \bar{d}_2 & \dots & \bar{p}_N \bar{d}_N \\ \bar{p}_1 \bar{s}_1 & \bar{p}_2 \bar{s}_2 & \dots & \bar{p}_N \bar{s}_N \\ \bar{p}_1^3 \bar{d}_1 & \bar{p}_2^3 \bar{d}_2 & \dots & \bar{p}_N^3 \bar{d}_N \\ \bar{p}_1^3 \bar{s}_1 & \bar{p}_2^3 \bar{s}_2 & \dots & \bar{p}_N^3 \bar{s}_N \\ \vdots & \vdots & & \vdots \end{vmatrix}.
 \end{aligned}$$

let us take

$$a_i = \frac{1}{2} \left( 1 + \sum_{v=1}^i c_v p_i^{2v-1} \right), \quad b_i = \frac{1}{2} \left( 1 - \sum_{v=1}^i c_v p_i^{2v-1} \right).$$
[illegible]



**Fig. 2.** First order rogue wave solution (a)  $r = 2.0$ ,  $c = 1 + 2i$ ,  $c_1 = 2$  (b)  $r = 0.5$ ,  $c = 1 + 2i$ ,  $c_1 = 2$ .



**Fig. 3.** Second order rogue wave solution (a)  $r = 2.0$ ,  $c = 1 + 2i$ ,  $c_1 = 2.0$ ,  $c_2 = 2 + 2i$ ; (b)  $r = 0.5$ ,  $c = 2 + i$ ,  $c_1 = 2$ ,  $c_2 = 2 + 2i$ .

By calculating the matrix element, it is easy to see that the above determinant is equal to  $\det_{1 \leq i, j \leq N} (B_{ij}^{(n)}(k, t))$ . We complete the proof of Theorem 2.  $\square$

By taking  $N = 1$  we obtain the fully discrete Peregrine rogue wave solution,

$$\begin{aligned} \tau_k^t(n) = & -(1+r)n + (1-r)k + (1-r+1/\bar{c}-rc)t + c_1 \\ & \times ((1+r)n + (1-r)k + (1-r+1/c-r\bar{c})t + \bar{c}_1) + r, \end{aligned} \quad (29)$$

where  $c_1$  is a complex constant. Similarly to the breather solution, there are rogue wave solutions regular on the lattice for both focusing case ( $r > 0$ ) and defocusing case ( $r < 0$ ), since if there are zeros of  $f_k^t$  we can avoid explosion of solution by displacing the zeros off the lattice points. However for regularity of the solution on the real two dimensional space of  $(k, t)$ , we have to take  $r$  positive. First order and second-order rogue wave solutions are shown in Figs. 2 and 3, respectively.

There is an exceptional regular solution for  $r < 0$  which is obtained by taking  $c$  real and  $(\text{Im } c_1)^2 > -r$  in (29), but this is not a rogue wave solution but a traveling wave solution. An example is shown in Fig. 4.

#### 4. Concluding remarks

Even though the study of rogue waves has attracted much attention in more than one decade, the rogue wave solution in fully discrete integrable systems has not been reported yet. In this paper, we firstly constructed the general breather solution of the fully discrete NLS equation via the KP-Toda reduction method. Then we succeeded in constructing its general rogue wave solution by taking the limit of  $p_i \rightarrow 0$  successively for  $i = 1, \dots, N$ . More study of rogue wave solutions in other discrete systems including non-integrable ones is called for in the future.



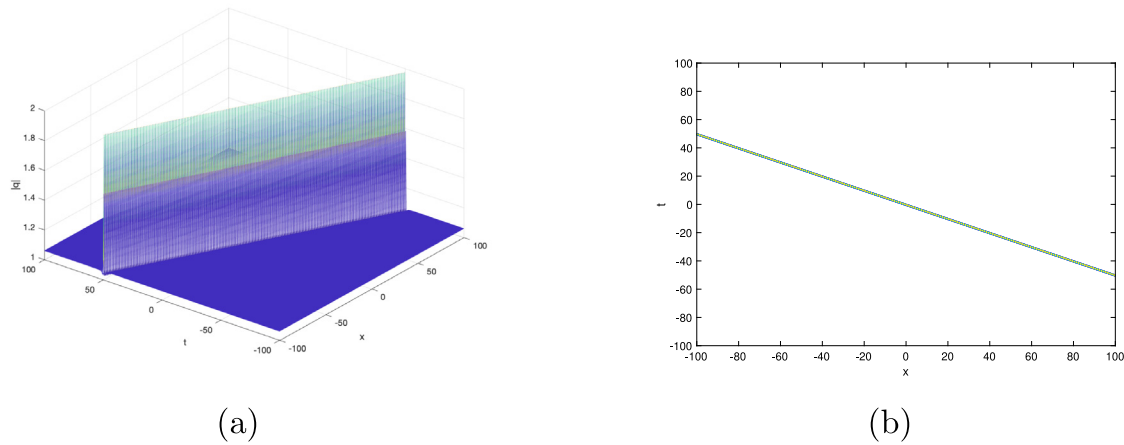


Fig. 4. Traveling wave solution with  $r = -0.5$ ,  $c = 1.0$ ,  $c_1 = 1 + i$ : (a) profile, (b) contour plot.

### CRediT authorship contribution statement

**Yasuhiro Ohta:** Concept, Method, Formulation, Revision. **Bao-Feng Feng:** Concept, Method, Writing.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgments

YO is partially supported by JSPS KAKENHI 18H01130. BF's work is partially supported by National Science Foundation (NSF) under Grant No. DMS-1715991 and U.S. Department of Defense (DoD), Air Force Office of Scientific Research (AFOSR) under grant No. W911NF2010276.

### Appendix

The discrete two-dimensional Toda lattice hierarchy in bilinear form can be written as

$$\begin{aligned}
 & \tau_n(k_1 + 1, \dots, k_v + 1, K_1 + 1) \tau_n(k_1, \dots, k_v, K_1) \\
 & - \frac{1}{\prod_{i=1}^v (1 - a_i A_1)} \tau_n(k_1 + 1, \dots, k_v + 1, K_1) \tau_n(k_1, \dots, k_v, K_1 + 1) \\
 & + A_1 \sum_{h=1}^v \frac{a_h^v}{(1 - a_h A_1) \prod_{\substack{i=1 \\ i \neq h}}^v (a_h - a_i)} \\
 & \times \tau_{n+1}(k_1, \dots, k_h + 1, \dots, k_v, K_1) \tau_{n-1}(k_1 + 1, \dots, k_h, \dots, k_v + 1, K_1 + 1) = 0,
 \end{aligned} \tag{30}$$

where  $n$  is the lattice number,  $k_1, \dots, k_v, K_1$  are discrete independent variables,  $a_1, \dots, a_v, A_1$  are corresponding difference intervals and  $v \geq 1$ .

Eq. (2) is derived from the above bilinear equation with  $v = 1$  and rewriting  $k_1, K_1, a_1, A_1$  to  $k, K, a, A$ , respectively. We also have the first modified Toda hierarchy,

$$\begin{aligned}
 & A_2 \tau_{n+1}(k_1 + 1, \dots, k_v + 1, K_1 + 1) \tau_n(k_{v+1} + 1, \dots, k_{2v} + 1, K_2 + 1) \\
 & - A_1 \prod_{i=1}^v \frac{a_{v+i}}{a_i} \tau_{n+1}(k_{v+1} + 1, \dots, k_{2v} + 1, K_2 + 1) \tau_n(k_1 + 1, \dots, k_v + 1, K_1 + 1) \\
 & + (A_1 - A_2) \prod_{i=1}^v \frac{1 - a_{v+i} A_1}{1 - a_i A_1} \tau_{n+1}(k_1 + 1, \dots, k_v + 1) \tau_n(k_{v+1} + 1, \dots, k_{2v} + 1, K_1 + 1, K_2 + 1)
 \end{aligned}$$

$$\begin{aligned}
& -A_1 \sum_{h=1}^v \frac{(1-a_h A_2) \prod_{i=1}^v (a_h - a_{v+i})}{a_h (1-a_h A_1) \prod_{\substack{i=1 \\ i \neq h}}^v (a_h - a_i)} \tau_{n+1}(k_h + 1, k_{v+1} + 1, \dots, k_{2v} + 1, K_2 + 1) \\
& \times \tau_n(k_1 + 1, \dots, k_{h-1} + 1, k_{h+1} + 1, \dots, k_v + 1, K_1 + 1) = 0,
\end{aligned} \tag{31}$$

where  $v \geq 0$  and  $a_h, A_h$  are difference intervals for discrete independent variables  $k_h, K_h$ , respectively. Here and hereafter we denote shifted independent variables only and omit unshifted independent variables for simplicity. Eq. (3) is derived from this bilinear equation by taking  $v = 1$  and rewriting  $k_1, k_2, K_1, K_2, a_1, a_2, A_1, A_2$  to  $k, l, K, L, a, b, A, B$ , respectively.

In principle, the above two bilinear equations can be obtained from the discrete KP hierarchy,

$$\sum_{h \in I} \frac{a_h^{|J|-|I|-2} \prod_{j \in J} (a_h - a_j)}{\prod_{i \in I \setminus \{h\}} (a_h - a_i)} \tau_{I \setminus \{h\}} \tau_{J \cup \{h\}} = 0, \tag{32}$$

where  $I, J$  are finite sets of integers satisfying  $|I| \geq |J| + 2$ ,  $a_h$  is the difference interval of  $k_h$ ,  $\tau_I$  means  $\tau$  function with shifts of  $k_i$  ( $i \in I$ ), i.e.,  $\tau_I = \tau(k_{i_1} + 1, k_{i_2} + 1, \dots, k_{i_v} + 1)$  for  $I = \{i_1, i_2, \dots, i_v\}$ . By taking  $I = \{-1, 0, 1, \dots, v\}$ ,  $J = \emptyset$ , we obtain

$$\sum_{h=-1}^v \frac{a_h^v}{\prod_{\substack{i=-1 \\ i \neq h}}^v (a_h - a_i)} \tau_{\{-1, 0, 1, \dots, v\} \setminus \{h\}} \tau_{\{h\}} = 0. \tag{33}$$

Taking the limit  $a_0 \rightarrow \infty$ , denoting  $a_{-1} = 1/A_1$  and applying variable transformations  $K_1 = k_{-1}$ ,  $n = -k_{-1} - k_0$ , the first bilinear equation (30) is derived. If we take  $I = \{-1, 0, 1, \dots, v, 2v + 1\}$  and  $J = \{-2, v + 1, \dots, 2v\}$ , we have

$$\sum_{h=-1}^v \frac{(a_h - a_{-2}) \prod_{j=v+1}^{2v} (a_h - a_j)}{(a_h - a_{2v+1}) \prod_{\substack{i=-1 \\ i \neq h}}^v (a_h - a_i)} \tau_{I \setminus \{h\}} \tau_{J \cup \{h\}} + \frac{(a_{2v+1} - a_{-2}) \prod_{j=v+1}^{2v} (a_{2v+1} - a_j)}{\prod_{i=-1}^v (a_{2v+1} - a_i)} \tau_{I \setminus \{2v+1\}} \tau_{J \cup \{2v+1\}} = 0.$$

Similarly by taking the limit  $a_0 \rightarrow \infty$ , denoting  $a_{-1} = 1/A_1$ ,  $a_{-2} = 1/A_2$ , taking  $a_{2v+1} = 0$  and demanding  $\tau$  independent of  $k_{2v+1}$ , the second bilinear equation (31) is recovered through variable transformations  $K_1 = k_{-1}$ ,  $K_2 = k_{-2}$ ,  $n = -k_{-2} - k_{-1} - k_0$ .

## References

- [1] N. Akhmediev, A. Ankiewicz, M. Taki, Waves that appear from nowhere and disappear without a trace, *Phys. Lett. A* 373 (2009) 675–678.
- [2] S. Haver, A possible freak wave event measured at the Draupner jacket January 1 1995, in: *Proceedings of Rogue Waves* (2004).
- [3] J. Gemmrich, L. Cicon, Generation mechanism and prediction of an observed extreme rogue wave, *Sci. Rep.* 12 (2022) 1718.
- [4] D.H. Peregrine, Water waves, Nonlinear Schrödinger equations and their solutions, *J. Aust. Math. Soc. B* 25 (1983) 16.
- [5] V.M. Eleonskii, I.M. Krichever, N.E. Kulagin, Rational multisoliton solutions of the nonlinear Schrödinger equation, *Sov. Phys. Dokl.* 31 (1986) 226–228.
- [6] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation, *Phys. Rev. E* 80 (2009) 026601.
- [7] D.J. Kedziora, A. Ankiewicz, N. Akhmediev, Second-order nonlinear Schrödinger equation breather solutions in the degenerate and rogue wave limits, *Phys. Rev. E* 85 (2012) 066601.
- [8] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spec. Top.* 185 (2010) 247.
- [9] P. Dubard, V.B. Matveev, Multi-rogue waves solutions to the focusing NLS equation and the KP-I equation, *Nat. Hazards Earth Syst. Sci.* 11 (2011) 667.
- [10] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A* 44 (2011) 435204.
- [11] B.L. Guo, L.M. Ling, Q.P. Liu, *Phys. Rev. E* 85 (2012) 026607.
- [12] Y. Ohta, J.K. Yang, *Proc. R. Soc. Lond. A* 468 (2012) 1716.
- [13] C. Kharif, E. Pelinovsky, A. Slunyaev, *Rogue Waves in the Ocean*, Springer, Heidelberg, 2009.
- [14] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, F.T. Arecchi, Rogue waves and their generating mechanisms in different physical contexts, *Phys. Rep.* 528 (2013) 47.
- [15] A. Chabchoub, N.P. Hoffmann, N. Akhmediev, Rogue wave observation in a water wave tank, *Phys. Rev. Lett.* 106 (2011) 204502.
- [16] Y.V. Bludov, V.V. Konotop, N. Akhmediev, Matter rogue waves, *Phys. Rev. A* 80 (2009) 033610.
- [17] H. Bailung, S.K. Sharma, Y. Nakamura, Observation of Peregrine solitons in a multicomponent plasma with negative ions, *Phys. Rev. Lett.* 107 (2011) 255005.
- [18] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Optical rogue waves, *Nature* 450 (2007) 1054.
- [19] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, J.M. Dudley, The peregrine soliton in nonlinear fibre optics, *Nat. Phys.* 6 (2010) 790.
- [20] S.W. Xu, J.S. He, L.H. Wang, The Darboux transformation of the derivative nonlinear Schrödinger equation, *J. Phys. A* 44 (2011) 305203.
- [21] B.L. Guo, L.M. Ling, Q.P. Liu, High-order solutions and generalized darbox transformations of derivative nonlinear Schrödinger equations, *Stud. Appl. Math.* 130 (2013) 317–344.
- [22] H.N. Chan, K.W. Chow, D.J. Kedziora, R.H.J. Grimshaw, E. Ding, Rogue wave modes for a derivative nonlinear Schrödinger model, *Phys. Rev. E* 89 (2014) 032914.
- [23] Y.S. Zhang, L.J. Guo, A. Chabchoub, J.S. He, Higher-order rogue wave dynamics for a derivative nonlinear Schrödinger equation, *Rom. J. Phys.* 62 (2017) 102.
- [24] B. Yang, J. Chen, J. Yang, Rogue waves in the generalized derivative nonlinear Schrödinger equations, *J. Nonlinear Sci.* 30 (2020) 3027–3056.
- [25] F. Baronio, A. Degasperis, M. Conforti, S. Wabnitz, Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves, *Phys. Rev. Lett.* 109 (2012) 044102.
- [26] F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, S. Wabnitz, Vector rogue waves and baseband modulation instability in the defocusing regime, *Phys. Rev. Lett.* 113 (2014) 034101.

- [27] B. Yang, J. Yang, Universal rogue wave patterns associated with the Yablonskii-Vorob'ev polynomial hierarchy, *Physica D* 425 (2021) 132958.
- [28] Y. Ohta, J. Yang, Rogue waves in the Davey–Stewartson I equation, *Phys. Rev. E* 86 (2012) 036604.
- [29] Y. Ohta, J. Yang, Dynamics of rogue waves in the Davey–Stewartson II equation, *J. Phys. A* 46 (2013) 105202.
- [30] B. Yang, J. Yang, General rogue waves in the three-wave resonant interaction systems, *IMA J. Appl. Math.* 86 (2021) 378–425.
- [31] B. Yang, J. Yang, General rogue waves in the Boussinesq equation, *J. Phys. Soc. Japan* 89 (2020) 024003.
- [32] J.C. Chen, Y. Chen, B.F. Feng, K. Maruno, Rational solutions to two-and one-dimensional multicomponent Yajima–Oikawa systems, *Phys. Lett. A* 379 (2015) 1510.
- [33] J.C. Chen, Y. Chen, B.F. Feng, K. Maruno, Y. Ohta, General high-order rogue wave of the (1+1)-dimensional Yajima–Oikawa system, *J. Phys. Soc. Japan* 87 (2018) 094007.
- [34] D. Bilman, P.D. Miller, Robust inverse scattering transform for the focusing nonlinear Schrödinger equation, *Comm. Pure Appl. Math.* 72 (2019) 1722–1805.
- [35] Y.V. Bludov, V.V. Konotop, N. Akhmediev, Rogue waves as spatial energy concentrators in arrays of nonlinear waveguides, *Opt. Lett.* 34 (2009) 3015–3017.
- [36] Y. Miyazawa, C. Chong, P.G. Kevrekidis, J. Yang, Rogue and solitary waves in coupled phononic crystals, *Phys. Rev. E* 105 (2022) 034202.
- [37] A. Ankiewicz, N. Akhmediev, J.M. Soto-Crespo, Discrete rogue waves of the Ablowitz–Ladik and Hirota equations, *Phys. Rev. E* 82 (2010) 026602.
- [38] A. Ankiewicz, N. Devine, M. Ünal, A. Chowdury, N. Akhmediev, A. Ankiewicz, N. Akhmediev, J.M. Soto-Crespo, Rogue waves and other solutions of single and coupled Ablowitz–Ladik and nonlinear Schrödinger equations, *J. Opt.* 15 (2013) 064008.
- [39] Y. Ohta, J.K. Yang, General rogue waves in the focusing and defocusing Ablowitz–Ladik equations, *J. Phys. A* 47 (2014) 255201.
- [40] X.-Y. Wen, Z. Yan, Modulational instability and dynamics of multi-rogue wave solutions for the discrete Ablowitz–Ladik equation, *J. Math. Phys.* 59 (2018) 073511.
- [41] B.-F. Feng, L. Ling, Z.N. Zhu, A focusing and defocusing semi-discrete complex short pulse equation and its various soliton solutions, *P. Roy. Soc. A-Math. Phys.* 477 (2021) 20200853.
- [42] B. Prinari, Discrete solitons of the focusing Ablowitz–Ladik equation with nonzero boundary conditions via inverse scattering, *J. Math. Phys.* 57 (2016) 083510.
- [43] A. Chowdury, A. Ankiewicz, N. Akhmediev, Solutions of the higher-order Manakov-type continuous and discrete equations, *Phys. Rev. E* 90 (2014) 012902.
- [44] X.-Y. Wen, Z. Yan, B.A. Malomed, Higher-order vector discrete rogue-wave states in the coupled Ablowitz–Ladik equations: Exact solutions and stability, *Chaos* 26 (2016) 123110.
- [45] M.J. Ablowitz, J.F. Ladik, A nonlinear difference scheme and inverse scattering, *Stud. Appl. Math.* 55 (1976) 213–229.
- [46] M.J. Ablowitz, J.F. Ladik, On the solution of a class of nonlinear partial difference equations, *Stud. Appl. Math.* 57 (1977) 1–12.
- [47] R. Hirota, Nonlinear partial difference equations, I. A difference analogue of the Korteweg–de Vries equation, *J. Phys. Soc. Japan* 43 (1977) 1427–1433.
- [48] R. Hirota, Nonlinear partial difference equations, II. Discrete-time Toda equation, *J. Phys. Soc. Japan* 43 (1977) 2074–2078.
- [49] J. Hietarinta, N. Joshi, F.W. Nijhoff, *Discrete Systems and Integrability*, Cambridge University Press, 2016.
- [50] R. Hirota, Y. Ohta, Discrete nonlinear Schrödinger equation, talk delivered at spring meeting of the physical society of Japan, 1991, Abstract available online: <http://ci.nii.ac.jp/naid/110001908012> (in Japanese).
- [51] S. Tsujimoto, Y. Ohta, R. Hirota, Difference scheme of nonlinear Schrödinger equation, in: *Proceedings of the Annual Meeting of Japan Society of Industrial and Applied Mathematics*, 1993, pp. 203–204, (in Japanese).
- [52] S. Tsujimoto, Discretization of integrable systems, in: Y. Nakamura (Ed.), *Applied Integrable Systems*, Shokabo, Tokyo, 2000, pp. 1–52, (in Japanese).