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# The robust inverse scattering method for focusing Ablowitz–Ladik equation on the non-vanishing background

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ABSTRACT

In this paper, we consider the robust inverse scattering method for the Ablowitz–Ladik (AL) equation on the non-vanishing background, which can be used to deal with arbitrary-order poles on the branch points and spectral singularities in a unified way. The Darboux matrix is constructed with the aid of loop group method and considered within the framework of robust inverse scattering transform. Various soliton solutions are constructed without using the limit technique. These solutions include general soliton, breathers, as well as high order rogue wave solutions.



## 1. Introduction

The nonlinear Schrödinger equation (NLSE)

$$iq_\tau = q_{xx} + 2\sigma|q|^2 q, \quad \sigma = \pm 1 \tag{1}$$

is a universal model for weakly nonlinear dispersive waves, both in the focusing case $\sigma = 1$ and defocusing case $\sigma = -1$. It appears in many different physical contexts, for instance: the deep water waves, nonlinear optics, Bose–Einstein condensates [1,2]. The integrable semi-discrete analogue for NLSE (1)

$$iu_{n,t} = u_{n+1} - 2u_n + u_{n+1} + \sigma|u_n|^2(u_{n+1} + u_{n-1}), \quad u_n = hq_n, \quad t = \tau/h^2, \tag{2}$$

was discovered by Ablowitz and Ladik in 1970s [3,4], therefore it is often called the Ablowitz–Ladik (AL) equation. Besides being used as a difference numerical schemes for its continuous counterpart, the AL equation also possesses numerous physical applications, such as the dynamics of anharmonic lattices [5], self-trapping on a dimer [6], Heisenberg spin chains [7,8] and so on [9].

Recently, there are quite a lot of work for the study of the AL equation (2) on the non-vanishing background using different approaches, such as the inverse scattering method [10–13], Hirota's bilinear method [14], Darboux transformation [15,16], algebraic geometry method [17–19]. For the traditional scattering analysis on the rogue wave solution, it shares the same scattering data with the background. Thus, it cannot be used to analyze the rogue wave solutions, as well as high order ones. Very recently a robust inverse scattering method was proposed by Bilman and Miller so as to deal with the rogue wave solutions [20]. The key point is to use the normalization method to reconstruct the meromorphic matrix function. By applying the Darboux transformation method within the frame of robust inverse scattering method, the general high order rogue waves could be obtained without taking the limit on the spectral parameter.

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In this work, we use the robust inverse scattering method [20–22] to analyze the focusing AL equation (2) on the non-vanishing background:

$$\lim_{n\to\pm\infty} u_n = u_n^{\pm} \equiv \rho e^{i\left[\theta(n+\frac{1}{2})+2\omega t+\theta_{\pm}\right]}, \quad \omega = 1-(1+\rho^2)\cos\theta, \tag{3}$$

where parameters $\theta$, $\rho$ and $\theta_{\pm}$ are real constants. Comparing with the previous studies on the non-vanishing background, the boundary condition (3) involves the parameter $\theta$, which can be used to regulate the modulational instability [23]. We will show the fact in the Appendix: if $\theta \neq \frac{\pi}{2}+n\pi$ ($n\in\mathbb{Z}$), the background is modulational unstable; otherwise, it is modulational stable. In the modulational unstable region, there exist Akhmediev breather, Kuznetz–Ma breather, Tajiri–Watanabe breather and rogue wave solutions; whereas, with the modulational stable background the corresponding solutions will turn to either periodic solutions or (rational) soliton solutions of W-shape. These phenomena are different from the focusing NLSE (1), whose non-vanishing background is always modulational unstable. With the aid of the Darboux transformation, the general soliton solutions, which include multi-breather solution, the high order lattice rogue wave and so on, are constructed by the Bäcklund transformation. In the original work of robust inverse scattering transform [20], the authors used a Darboux matrix with the unit determinant. Instead, we use the Darboux matrix of the loop group version [24,25], which could provide a compact formula for soliton solutions. Some special propositions for the exact solutions could be analyzed by the elementary Bäcklund transformation.

The rest of the paper is arranged as follows: In Section 2, the robust inverse scattering transform for the AL equation (2) on the non-vanishing background is constructed by following the method proposed in [20]. A key step to construct robust inverse scattering transform is to seek a new Riemann–Hilbert problem to capture the solutions with spectral singularity. Comparing with the robust inverse scattering method for NLSE (1), in which the Riemann–Hilbert problem is normalized in the neighborhood of $\infty$ with the identity matrix, here for the AL equation the Riemann–Hilbert problem of (2) is normalized with a diagonal matrix depending on $n$ and $t$ in the neighborhood of $\infty$ and 0. So we need more careful analysis on this procedure to obtain the robust inverse scattering transform. As a by-product, the conservation laws are obtained by performing the expansions in the neighborhood of $\infty$ and 0 respectively, which are given in Appendix B. In Section 3, we construct the elementary Darboux matrix by the loop group method. Through the robust inverse scattering transform, the general Darboux matrix can possess the Riemann–Hilbert representation. Then the compact solitonic formulas are obtained. In Section 4, various exact solutions including breathers, (rational) solitons of W-shape and rogue wave solutions are constructed and analyzed. The interaction between breathers and solitons are analyzed by performing the asymptotic analysis. The highest peak value for the high order rogue wave solution is obtained by the Bäcklund transformation. Section 5 is devoted to some conclusions and discussions.

## 2. The robust inverse scattering transform to the AL equation on the non-vanishing background

The Lax pair for the AL equation (2) is given by [3,4]:

$$\mathbf{v}_{n+1} = \mathbf{X}_n\mathbf{v}_n, \quad \mathbf{X}_n = \begin{bmatrix} z & u_n \\ -\bar{u}_n & z^{-1}\end{bmatrix}, \tag{4a}$$

$$\mathbf{v}_{n,t} = \mathbf{T}_n\mathbf{v}_n, \quad \mathbf{T}_n = \begin{bmatrix} -iu_n\bar{u}_{n-1}-\frac{i}{2}(z-z^{-1})^2 & -iu_nz+iu_{n-1}z^{-1} \\ -i\bar{u}_nz^{-1}+i\bar{u}_{n-1}z & iu_{n-1}\bar{u}_n+\frac{i}{2}(z-z^{-1})^2\end{bmatrix}, \tag{4b}$$

where $\mathbf{v}_n$ is a two-component vector, $z\in\mathbb{C}$ is the spectral parameter and $u_n(t)$ is the potential function, the overbar represents the complex conjugation. The compatibility condition $\mathbf{X}_{n,t}+\mathbf{X}_n\mathbf{T}_n-\mathbf{T}_{n+1}\mathbf{X}_n=0$ gives the AL equation (2).

To study the above Cauchy problem conveniently, we set the following gauge transformation

$$u_n = w_n e^{i\left[\theta(n+\frac{1}{2})+2\omega t\right]}, \quad \mathbf{f}_n = e^{-\frac{i}{2}[\theta n+2\omega t]\boldsymbol{\sigma}_3}\mathbf{v}_n, \quad \lambda = ze^{-\frac{i\theta}{2}},$$

where $\boldsymbol{\sigma}_3$ is the third Pauli matrix. It follows that Eq. (2) becomes

$$iw_{n,t} = (1+|w_n|^2)(w_{n+1}e^{i\theta}+w_{n-1}e^{-i\theta})-2\cos\theta(1+\rho^2)w_n, \tag{5}$$

and the Lax pair (4) turns to

$$\mathbf{f}_{n+1} = \mathbf{L}_n\mathbf{f}_n, \quad \mathbf{L}_n \equiv \lambda\mathbf{E}_+ + \mathbf{Q}_n + \mathbf{E}_-\lambda^{-1}, \tag{6a}$$

$$\mathbf{f}_{n,t} = \mathbf{M}_n\mathbf{f}_n, \tag{6b}$$

where

$$\begin{gathered}\mathbf{E}_+ = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad \mathbf{E}_- = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad \mathbf{Q}_n = \begin{bmatrix}0 & w_n\\ -\bar{w}_n & 0\end{bmatrix},\\ \mathbf{M}_n = \begin{bmatrix} -iw_n\bar{w}_{n-1}e^{i\theta}-\frac{i}{2}(\lambda e^{\frac{i\theta}{2}}-\lambda^{-1}e^{-\frac{i\theta}{2}})^2-i\omega & -iw_ne^{i\theta}\lambda+iw_{n-1}e^{-i\theta}\lambda^{-1}\\ -i\bar{w}_ne^{-i\theta}\lambda^{-1}+i\bar{w}_{n-1}e^{i\theta}\lambda & iw_{n-1}\bar{w}_ne^{-i\theta}+\frac{i}{2}(\lambda e^{\frac{i\theta}{2}}-\lambda^{-1}e^{-\frac{i\theta}{2}})^2+i\omega\end{bmatrix}.\end{gathered} \tag{7}$$

Thus the Cauchy problem for (2) is equivalent to the Cauchy problem for (5) with boundary condition

$$\lim_{n\to\pm\infty} w_n = w^{\pm} \equiv \rho e^{i\theta_{\pm}}. \tag{8}$$

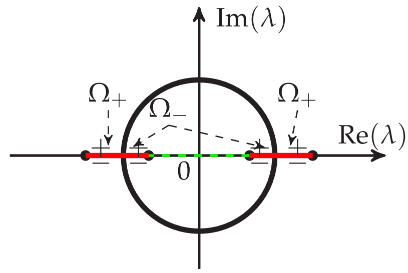


**Fig. 1.** The two red segments denote the branch cuts. The four black points denote the branch points. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

### 2.1. Scattering analysis

For direct scattering problem, we merely need to analyze the spectral problem (6a). To analyze the spectral problem, we introduce the following transformation:

$$\mathbf{f}_n^{\pm} = \exp\left(\frac{\mathrm{i}}{2}\theta_{\pm}\boldsymbol{\sigma_3}\right)\boldsymbol{\chi}_n^{\pm}, \tag{9}$$

under which the spectral problem can be reduced to

$$\boldsymbol{\chi}_{n+1}^{\pm} = [\boldsymbol{\Lambda} + \mathbf{Q}_n^{\pm}]\boldsymbol{\chi}_n^{\pm},$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & \rho \\ -\rho & \lambda^{-1} \end{bmatrix}, \quad \mathbf{Q}_n^{\pm} = \begin{bmatrix} 0 & w_n \mathrm{e}^{-\mathrm{i}\theta_{\pm}} - \rho \\ -(\bar{w}_n \mathrm{e}^{\mathrm{i}\theta_{\pm}} - \rho) & 0 \end{bmatrix}.$$

We diagonalize the matrix

$$\boldsymbol{\Lambda} = r\mathbf{M}\zeta^{\boldsymbol{\sigma_3}}\mathbf{M}^{-1}, \quad r = \sqrt{1+\rho^2}, \quad \zeta^{\boldsymbol{\sigma_3}} = \mathrm{diag}\left(\zeta, \zeta^{-1}\right)$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix}, \quad \xi = \frac{1-\lambda^2+\sqrt{(1-\lambda^2)^2-4\rho^2\lambda^2}}{2\rho\lambda},$$

and $\zeta$ satisfies the following equation

$$r(\zeta+\zeta^{-1}) = \lambda+\lambda^{-1}, \quad \text{i.e. } \zeta = \frac{1+\lambda^2+\sqrt{(1+\lambda^2)^2-4r^2\lambda^2}}{2r\lambda}, \tag{10}$$

where $\zeta(\lambda)$ defines a two sheet Riemann surface $\Gamma$ with genus one, which is pieced together by two pieces $\Gamma_+$ and $\Gamma_-$ slit along the line segments $\Omega = [-(r+\rho), -(r-\rho)] \cup [(r-\rho), (r+\rho)]$ as shown in Fig. 1. For the functions $\zeta(\lambda)$ and $\xi(\lambda)$ on the branch cut $\Omega$, we have the jump conditions $\zeta_- = \zeta_+\zeta_-^2$ and $\xi_- = \xi_+\xi_-^2$, where $\zeta_\pm = \zeta(\lambda \pm \mathrm{i}0^+)$ and $\xi_\pm = \xi(\lambda \pm \mathrm{i}0^+)$ are the nontangential limits from the "$\pm$" side of branch cut $\Omega$ respectively (see Fig. 1). The branch cut $\Omega$ can be decomposed into two pieces $\Omega = \Omega_+ \cup \Omega_-$ (see Fig. 1), where

$$\Omega_+ = [1, (r+\rho)] \cup [-(r+\rho), -1]$$

and

$$\Omega_- = [(r-\rho), 1] \cup [-1, -(r-\rho)].$$

We can verify that $|\zeta(\lambda)| = 1$ when $\lambda \in \{\lambda : |\lambda| = 1\} \cup \Omega$, and $|\xi(\lambda)| = 1$ for $\lambda \in \Omega$. In the piece $\Gamma_+$, the functions $\xi(\lambda)$ and $\zeta(\lambda)$ are the meromorphic function in the whole complex plane with the first order pole at $\lambda = 0$ and the removable singularity at $\lambda = \infty$. In the piece $\Gamma_-$, the functions $\xi(\lambda)$ and $\zeta(\lambda)$ are the analytic function in the whole complex plane with the first order pole at $\lambda = \infty$ and the removable singularity at $\lambda = 0$. In what follows, we just consider the analysis on the piece $\Gamma_-$ since we can apply the similar analysis for the piece $\Gamma_+$. Since the functions $\xi(\lambda)$ and $\zeta(\lambda)$ are analytic in the region $\lambda \in S_{\mathrm{in}}/\Omega_-$ and the norm of them equals to 1 on the boundary $\Omega_-$, we have $|\xi(\lambda)| \leq \max\{1, \max_{|\lambda|=1}|\xi(\lambda)|\} = (1+r)/\rho$ and $|\zeta(\lambda)| \leq 1$ by the maximum modulus principle, where $S_{\mathrm{in}} := \{\lambda : |\lambda| < 1\}$. Both equations $\xi^2 = 1$ and $\zeta^2 = 1$ have the same four simple roots $\lambda = r \pm \rho$ and $\lambda = -r \pm \rho$.

Then we introduce the gauge transformation $\boldsymbol{\tau}_n^+ = \left(\prod_{k=n}^{+\infty} \frac{1+|w_k|^2}{r^2}\right)\mathbf{M}^{-1}\boldsymbol{\chi}_n^+$ and $\boldsymbol{\tau}_n^- = \mathbf{M}^{-1}\boldsymbol{\chi}_n^-$. It follows that

$$\boldsymbol{\tau}_n^+ = [r^{-1}\zeta^{-\boldsymbol{\sigma_3}} - r^{-2}\mathbf{M}^{-1}\mathbf{Q}_n^+\mathbf{M}]\boldsymbol{\tau}_{n+1}^+, \qquad \boldsymbol{\tau}_{n+1}^- = [r\zeta^{\boldsymbol{\sigma_3}} + \mathbf{M}^{-1}\mathbf{Q}_n^-\mathbf{M}]\boldsymbol{\tau}_n^-. \tag{11}$$

Based on above equation (11) and boundary condition (8), we can rewrite them as the summation equation

$$\mathbb{I} - (r\zeta^{\boldsymbol{\sigma_3}})^{-n}\boldsymbol{\tau}_n^+ = r^{-2}\sum_{k=n}^{+\infty}(r\zeta^{\boldsymbol{\sigma_3}})^{-k}\mathbf{M}^{-1}\mathbf{Q}_k^+\mathbf{M}\boldsymbol{\tau}_{k+1}^+, \tag{12}$$

$$(r\zeta^{\boldsymbol{\sigma_3}})^{-n}\boldsymbol{\tau}_n^- - \mathbb{I} = \sum_{k=-\infty}^{n-1}(r\zeta^{\boldsymbol{\sigma_3}})^{-(k+1)}\mathbf{M}^{-1}\mathbf{Q}_k^-\mathbf{M}\tau_k^-. \tag{13}$$

From Eq. (12), multiply $\mathbf{M}(r\zeta^{\sigma_3})^n$ from the left and $(r\zeta^{\sigma_3})^{-n}$ from the right at both sides, we can obtain the following equation

$$\boldsymbol{\mu}_n^+ = \mathbf{M} - r^{-2}\sum_{k=n}^{+\infty}\mathbf{M}(r\zeta^{\sigma_3})^{n-k}\mathbf{M}^{-1}\mathbf{Q}_k^+\boldsymbol{\mu}_{k+1}^+(r\zeta^{\sigma_3})^{-n+k+1}, \tag{14}$$

$$\boldsymbol{\mu}_n^- = \mathbf{M} + \sum_{k=-\infty}^{n-1}\mathbf{M}(r\zeta^{\sigma_3})^{n-(k+1)}\mathbf{M}^{-1}\mathbf{Q}_k^-\boldsymbol{\mu}_k^-(r\zeta^{\sigma_3})^{-n+k}, \tag{15}$$

where $\boldsymbol{\mu}_n^\pm = \mathbf{M}\boldsymbol{\tau}_n^\pm(r\zeta^{\sigma_3})^{-n} = \boldsymbol{\chi}_n^\pm(r\zeta^{\sigma_3})^{-n}$.

Next, we consider the analytic property for solution $\boldsymbol{\mu}_n^\pm = [\boldsymbol{\mu}_{n,1}^\pm, \boldsymbol{\mu}_{n,2}^\pm]$. Then Eqs. (14)–(15) can be rewritten as the following form:

$$\boldsymbol{\mu}_{n,1}^+ = \begin{bmatrix}1\\ \xi\end{bmatrix} - \zeta\sum_{k=n}^{+\infty}\mathbf{G}_1(k,n;\lambda)\frac{\mathbf{Q}_k^+}{r}\boldsymbol{\mu}_{k+1,1}^+, \quad \boldsymbol{\mu}_{n,1}^- = \begin{bmatrix}1\\ \xi\end{bmatrix} + \sum_{k=-\infty}^{n-1}\mathbf{G}_1(k+1,n;\lambda)\frac{\mathbf{Q}_k^-}{r\zeta}\boldsymbol{\mu}_{k,1}^-, \tag{16}$$

$$\boldsymbol{\mu}_{n,2}^+ = \begin{bmatrix}\xi\\ 1\end{bmatrix} - \sum_{k=n}^{+\infty}\mathbf{G}_2(k,n;\lambda)\frac{\mathbf{Q}_k^+}{r\zeta}\boldsymbol{\mu}_{k+1,2}^+, \quad \boldsymbol{\mu}_{n,2}^- = \begin{bmatrix}\xi\\ 1\end{bmatrix} + \zeta\sum_{k=-\infty}^{n-1}\mathbf{G}_2(k+1,n;\lambda)\frac{\mathbf{Q}_k^-}{r}\boldsymbol{\mu}_{k,2}^-, \tag{17}$$

where

$$\mathbf{G}_1(k,n;\lambda) = \mathbf{M}\begin{bmatrix}1 & 0\\ 0 & \zeta^{2(k-n)}\end{bmatrix}\mathbf{M}^{-1} = \mathbb{I} + \frac{\zeta^{2(k-n)}-1}{1-\xi^2}\begin{bmatrix}-\xi^2 & \xi\\ -\xi & 1\end{bmatrix},$$
$$\mathbf{G}_2(k,n;\lambda) = \mathbf{M}\begin{bmatrix}\zeta^{2(n-k)} & 0\\ 0 & 1\end{bmatrix}\mathbf{M}^{-1} = \mathbb{I} + \frac{\zeta^{2(n-k)}-1}{1-\xi^2}\begin{bmatrix}1 & -\xi\\ \xi & -\xi^2\end{bmatrix}.$$

The existence and analytic properties of solution $\boldsymbol{\mu}_n^\pm$ can be summarized as the following lemmas. The proofs of two following lemmas are followed by the method of Ref. [26], which are given in Appendix A.

**Lemma 1.** *If $\sum_{n=n_0}^{\infty}|w_n e^{-i\theta_+} - \rho| < \infty$ for any finite $n_0 \in \mathbb{Z}$ and take arbitrary $0 < \epsilon < \min(r-\rho, 1-(r-\rho))$, then the solution $\boldsymbol{\mu}_{n,1}^+$ is analytic in the set $S_{\rm in}\backslash\Omega_-$ and uniformly bounded to $S_{\rm in}\backslash(\Omega_-\cup B_\epsilon(-(r-\rho))\cup B_\epsilon(r-\rho))$, where $B_\epsilon(z_0) = \{z\in\mathbb{C} : |z-z_0| < \epsilon/2\}$; the solution $\boldsymbol{\mu}_{n,2}^+$ is analytic in the set $S_{\rm out}\setminus\Omega_+$ and uniformly bounded for $S_{\rm out}\setminus(\Omega_+\cup B_\epsilon(-(r+\rho))\cup B_\epsilon(r+\rho))$, where $S_{\rm out} := \{\lambda : 1 < |\lambda| < \infty\}$. In a similar way, if $\sum_{n=-\infty}^{n_0}|w_n e^{-i\theta_-} - \rho| < \infty$ for any finite $n_0\in\mathbb{Z}$, then solution $\boldsymbol{\mu}_{n,1}^-$ is analytic in the set $S_{\rm out}\setminus\Omega_+$ and uniformly bounded for $S_{\rm out}\backslash(\Omega_+\cup B_\epsilon(-(r+\rho))\cup B_\epsilon(r+\rho))$; solution $\boldsymbol{\mu}_{n,2}^-$ is analytic in the set $S_{\rm in}\backslash\Omega_-$ and uniformly bounded for $S_{\rm in}\backslash(\Omega_-\cup B_\epsilon(-(r-\rho))\cup B_\epsilon(r-\rho))$. Moreover, the solutions of above summation equations are unique in the set $S_{\rm out}\setminus\Omega_+$ or $S_{\rm in}\setminus\Omega_-$ under the space of bounded functions.*

**Lemma 2.** *If $\sum_{k=n}^{+\infty}(1+|k|)\|\mathbf{Q}_k^+\| < +\infty$, there exists a positive constant $\epsilon > 0$, such that solution $\boldsymbol{\mu}_{n,1}^+$ can be uniformly bounded to the region $(B_{2\epsilon}(r-\rho)\cup B_{2\epsilon}(-(r-\rho)))\setminus\Omega_-$; the solution $\boldsymbol{\mu}_{n,2}^+$ can be uniformly bounded to the region $(B_{2\epsilon}(r+\rho)\cup B_{2\epsilon}(-(r+\rho)))\setminus\Omega_+$.*

*If $\sum_{-\infty}^{k=n-1}(1+|k|)\|\mathbf{Q}_k^-\| < +\infty$, there exists a positive constant $\epsilon > 0$, such that solution $\boldsymbol{\mu}_{n,1}^-$ can be uniformly bounded to the region $(B_{2\epsilon}(r+\rho)\cup B_{2\epsilon}(-(r+\rho)))\setminus\Omega_+$; the solution $\boldsymbol{\mu}_{n,2}^-$ can be uniformly bounded to the region $(B_{2\epsilon}(r-\rho)\cup B_{2\epsilon}(-(r-\rho)))\setminus\Omega_-$.*

Together with the above two lemmas on the Jost functions, we conclude that

**Theorem 1.** *If $\sum_{k=n_0}^{\infty}(1+|k|)|w_k e^{-i\theta_+} - \rho| < \infty$ for any finite $n_0\in\mathbb{Z}$, then the solution $\boldsymbol{\mu}_{n,1}^+$ is analytic in the region $S_{\rm in}\setminus\Omega_-$ and is continuous to its boundary; the solution $\boldsymbol{\mu}_{n,2}^+$ is analytic in the region $S_{\rm out}\setminus\Omega_+$ and is continuous to its boundary.*

*If $\sum_{k=-\infty}^{n_0}(1+|k|)|w_k e^{-i\theta_-} - \rho| < \infty$ for any finite $n_0\in\mathbb{Z}$, then solution $\boldsymbol{\mu}_{n,1}^-$ is analytic in the region $S_{\rm out}\setminus\Omega_+$ and is continuous to its boundary; solution $\boldsymbol{\mu}_{n,2}^-$ is analytic in the region $S_{\rm in}\setminus\Omega_-$ and is continuous to its boundary.*

### 2.2. Scattering matrix

The Jost solutions $\mathbf{J}_\pm(n;\lambda) = \mathbf{f}_n^\pm(\lambda)(\mathbf{E}(\lambda))^{-n}$ are linear dependent, which can be related by

$$\mathbf{J}_-(n;\lambda) = \mathbf{J}_+(n;\lambda)\mathbf{E}(\lambda)^n\mathbf{S}(\lambda)\mathbf{E}(\lambda)^{-n}, \ \ \lambda\in\{\lambda : |\lambda| = 1\}, \tag{18}$$

where $\mathbf{S}(\lambda)$ is the scattering matrix,

$$\mathbf{S}(\lambda) = \begin{bmatrix}a(\lambda) & c(\lambda)\\ b(\lambda) & d(\lambda)\end{bmatrix}, \ \ \mathbf{E}(\lambda) = r\begin{bmatrix}\zeta(\lambda) & 0\\ 0 & \zeta(\lambda)^{-1}\end{bmatrix}.$$

By above substitution and Eqs. (6), we have

$$\mathbf{J}_\pm(n+1;\lambda) = \mathbf{L}_n(\lambda)\mathbf{J}_\pm(n;\lambda)\mathbf{E}(\lambda)^{-1}. \tag{19}$$

Furthermore, we can obtain that

$$\mathbf{J}_\pm^\dagger(n+1;\lambda^*) = [\mathbf{E}(\lambda^*)^{-1}]^\dagger\mathbf{J}_\pm^\dagger(n;\lambda^*)\mathbf{L}_n^\dagger(\lambda^*) = \frac{1}{r^2}\mathbf{E}(\lambda)\mathbf{J}_\pm^\dagger(n;\lambda^*)\mathbf{L}_n^\dagger(\lambda^*),$$

here $\lambda^* \equiv \bar{\lambda}^{-1}$, and $^\dagger$ represents the Hermite conjugate. Together with the symmetry property of $\mathbf{L}_n(\lambda)$, i.e. $\mathbf{L}_n^\dagger(\lambda^*)\mathbf{L}_n(\lambda) = (1+|\omega_n|^2)\mathbb{I}_2$, we arrive at

$$\mathbf{J}_\pm^\dagger(n+1;\lambda^*)\mathbf{J}_\pm(n+1;\lambda) = \rho_n \mathbf{E}(\lambda)\mathbf{J}_\pm^\dagger(n;\lambda^*)\mathbf{J}_\pm(n;\lambda)\mathbf{E}(\lambda)^{-1},$$

where $\rho_n = \frac{1}{r^2}(1+|\omega_n|^2)$.

On the other hand, boundary conditions for $\mathbf{J}_\pm(n;\lambda)$ are

$$\mathbf{J}_\pm(\pm\infty;\lambda) = \exp\left(\frac{\mathrm{i}}{2}\theta_\pm \boldsymbol{\sigma_3}\right)\begin{bmatrix} 1 & \rho^{-1}(r\zeta-\lambda) \\ \rho^{-1}(r\zeta-\lambda) & 1 \end{bmatrix}.$$

It follows that

$$\mathbf{J}_\pm^\dagger(\pm\infty;\lambda^*) = \begin{bmatrix} 1 & \rho^{-1}(r\zeta^{-1}-\lambda^{-1}) \\ \rho^{-1}(r\zeta^{-1}-\lambda^{-1}) & 1 \end{bmatrix}\exp\left(-\frac{\mathrm{i}}{2}\theta_\pm \boldsymbol{\sigma_3}\right).$$

Then we can obtain the following proposition:

**Proposition 1.** *The Jost solutions $\mathbf{J}_\pm$ possesses the following symmetry relation:*

$$\begin{aligned} \mathbf{J}_+^\dagger(n;\lambda^*)\mathbf{J}_+(n;\lambda) &= [1-\xi^2]\Delta_n^+, \quad \Delta_n^+ = \prod_{l=n}^{+\infty}\rho_l^{-1}, \\ \mathbf{J}_-^\dagger(n;\lambda^*)\mathbf{J}_-(n;\lambda) &= [1-\xi^2]\Delta_n^-, \quad \Delta_n^- = \prod_{l=-\infty}^{n-1}\rho_l. \end{aligned} \tag{20}$$

On the other hand, we have the following relation for determinants $\mathbf{J}_\pm$,

$$\begin{aligned} \det(\mathbf{J}_+(n;\lambda)) &= [1-\xi^2]\prod_{l=n}^{+\infty}\rho_l^{-1}, \\ \det(\mathbf{J}_-(n;\lambda)) &= [1-\xi^2]\prod_{l=-\infty}^{n-1}\rho_l. \end{aligned} \tag{21}$$

It follows that

$$\det(\mathbf{S}(\lambda)) = \prod_{l=-\infty}^{+\infty}\rho_l \equiv v.$$

Through above relations, we can obtain that

$$\mathbf{J}_-^\dagger(n;\lambda^*) = \mathbf{E}(\lambda)^n\mathbf{S}^\dagger(\lambda^*)\mathbf{E}(\lambda)^{-n}\mathbf{J}_+^\dagger(n;\lambda^*)$$

and

$$\mathbf{J}_-^{-1}(n;\lambda) = \mathbf{E}(\lambda)^n\mathbf{S}^{-1}(\lambda)\mathbf{E}(\lambda)^{-n}\mathbf{J}_+^{-1}(n;\lambda).$$

With the aid of Proposition 1, we can obtain that the following proposition.

**Proposition 2.** *The scattering matrix $\mathbf{S}(\lambda)$ possesses the following symmetry properties:*

$$\mathbf{S}^\dagger(\lambda^*) = v\mathbf{S}(\lambda)^{-1}.$$

Based on Proposition 2, we can rewrite $\mathbf{S}(\lambda)$ as

$$\mathbf{S}(\lambda) = \begin{bmatrix} a(\lambda) & -\bar{b}(\lambda^*) \\ b(\lambda) & \bar{a}(\lambda^*) \end{bmatrix}. \tag{22}$$

There exists another symmetry relation for Jost solution $\mathbf{J}_\pm(n;\lambda)$:

$$\mathbf{J}_\pm(n;\lambda) = \boldsymbol{\sigma_3}\mathbf{J}_\pm(n;-\lambda)\boldsymbol{\sigma_3}. \tag{23}$$

It follows that the scattering matrix possesses the following symmetry relation:

$$\mathbf{S}(\lambda) = \boldsymbol{\sigma_3}\mathbf{S}(-\lambda)\boldsymbol{\sigma_3}. \tag{24}$$

Based on the analytic properties of Jost functions, we define the following analytic matrices:

$$\Phi^+(n;\lambda) = \left[\mathbf{J}_{-,1}(n;\lambda), \mathbf{J}_{+,2}(n;\lambda)\right], \quad \Phi^-(n;\lambda) = \left[\mathbf{J}_{+,1}(n;\lambda), \mathbf{J}_{-,2}(n;\lambda)\right]$$

which are analytic in the region $S_{\mathrm{in}}/\Omega_-$ and $S_{\mathrm{out}}/\Omega_+$ respectively.

**Proposition 3.** *The analytic matrices $\Phi^{\pm}(n;\lambda)$ possess the following asymptotic behavior:*

$$\begin{aligned}\Phi^{+}(n;\lambda)&=\begin{bmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_-} & 0\\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_+}\Delta_n^{+}\end{bmatrix}+\mathcal{O}(\lambda^{-1}),\quad \text{as } \lambda\to\infty\\ \Phi^{-}(n;\lambda)&=\begin{bmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_+}\Delta_n^{-} & 0\\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_-}\end{bmatrix}+\mathcal{O}(\lambda),\quad \text{as } \lambda\to 0\end{aligned} \tag{25}$$

**Proof.** When $\lambda\to\infty$, the matrices $\mathbf{E}(\lambda)$ and $\Phi^{+}(n;\lambda)$ have the asymptotic expansion

$$\begin{aligned}\mathbf{E}(\lambda)&=\mathbf{E}_{+}\lambda+\mathbf{E}_0^{[\infty]}\lambda^{-1}+\mathcal{O}(\lambda^{-3}),\\ \Phi^{+}(n;\lambda)&=\Phi_0^{+}(n)+\Phi_1^{+}(n)\lambda^{-1}+\mathcal{O}(\lambda^{-2}),\end{aligned} \tag{26}$$

where

$$\mathbf{E}_0^{[\infty]}=\begin{bmatrix}1-r^2 & 0\\ 0 & r^2\end{bmatrix}.$$

Inserting the above expansions into (19) and comparing the coefficients, we have

$$\begin{aligned}\Phi_0^{+}(n+1)\mathbf{E}_{+}&=\mathbf{E}_{+}\Phi_0^{+}(n),\\ \Phi_1^{+}(n+1)\mathbf{E}_{+}&=\mathbf{Q}_n\Phi_0^{+}(n)+\mathbf{E}_{+}\Phi_1^{+}(n),\end{aligned} \tag{27}$$

which implies that

$$\Phi_0^{+}(n)=\begin{bmatrix}c^{+} & 0\\ 0 & \alpha_n^{+}\end{bmatrix},\quad \Phi_1^{+}(n)=\begin{bmatrix}\beta^{+} & -\alpha_n^{+}w_n\\ -\overline{w}_{n-1} & \gamma_n^{+}\end{bmatrix},$$

where $c^{+}$, $\beta^{+}$ are constants which are independent of $n$. On the other hand, we know the boundary conditions

$$\mathbf{J}_{-,1}(n;\lambda)=\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_-\boldsymbol{\sigma}_3}\begin{bmatrix}1\\ \xi\end{bmatrix},\quad \text{as } n\to-\infty,$$

and

$$\mathbf{J}_{+,2}(n;\lambda)=\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_+\boldsymbol{\sigma}_3}\begin{bmatrix}\xi\\ 1\end{bmatrix},\quad \text{as } n\to+\infty,$$

which deduce that $c^{+}=\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_-}$, $\beta^{+}=0$ and $\alpha_n^{+}\to\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_+}$ as $n\to+\infty$. Meanwhile, the determinant has the following relation

$$\det(\Phi^{+}(n+1;\lambda))=\frac{1+|w_n|^2}{r^2}\det(\Phi^{+}(n;\lambda))$$

which infers that $\alpha_{n+1}^{+}=\frac{1+|w_n|^2}{r^2}\alpha_n^{+}$. Together with the boundary condition, we obtain $\alpha_n^{+}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_+}\Delta_n$.

While $\lambda\to 0$, the matrices $\mathbf{E}(\lambda)$ and $\Phi_{-}(n)$ have the asymptotic expansion

$$\begin{aligned}\mathbf{E}(\lambda)&=\mathbf{E}_{-}\lambda^{-1}+\mathbf{E}_0^{[0]}\lambda+\mathcal{O}(\lambda^{3}),\\ \Phi^{-}(n;\lambda)&=\Phi_0^{-}(n)+\Phi_1^{-}(n)\lambda+\mathcal{O}(\lambda^{2}),\end{aligned} \tag{28}$$

where

$$\mathbf{E}_0^{[0]}=\begin{bmatrix}r^2 & 0\\ 0 & 1-r^2\end{bmatrix}.$$

Inserting the above expansions into (19) and comparing the coefficients of $\lambda$, we have

$$\begin{aligned}\Phi_0^{-}(n+1)\mathbf{E}_{-}&=\mathbf{E}_{-}\Phi_0^{-}(n),\\ \Phi_1^{-}(n+1)\mathbf{E}_{-}&=\mathbf{Q}_n\Phi_0^{-}(n)+\mathbf{E}_{-}\Phi_1^{-}(n),\end{aligned} \tag{29}$$

which implies that

$$\Phi_0^{-}(n)=\begin{bmatrix}\alpha_n^{-} & 0\\ 0 & c^{-}\end{bmatrix},\quad \Phi_1^{-}(n)=\begin{bmatrix}\gamma_n^{-} & w_{n-1}\\ \alpha_n^{-}\overline{w}_n & \beta^{-}\end{bmatrix}.$$

Similar as above, by the boundary conditions, we have $c^{-}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_-}$, $\beta^{-}=0$ and $\alpha_n^{-}=\Delta_n^{-}\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_+}$. This completes the proof. □

Through the definition of scattering matrix $\mathbf{S}(\lambda)$ (18) and (22), we know that

$$\begin{aligned}a(\lambda)&=\frac{\det(\Phi^{+}(n;\lambda))}{(1-\xi^2)\Delta_n^{+}},\quad b(\lambda)=\frac{\det\left(\left[\mathbf{J}_{+,1}(n;\lambda),\mathbf{J}_{-,1}(n;\lambda)\right]\right)}{(1-\xi^2)\Delta_n^{+}}\zeta^{2n},\\ \bar{a}(\lambda^*)&=\frac{\det(\Phi^{-}(n;\lambda))}{(1-\xi^2)\Delta_n^{+}},\quad \bar{b}(\lambda^*)=-\frac{\det\left(\left[\mathbf{J}_{-,2}(n;\lambda),\mathbf{J}_{+,2}(n;\lambda)\right]\right)}{(1-\xi^2)\Delta_n^{+}}\zeta^{-2n}.\end{aligned} \tag{30}$$

In virtue of Proposition 3, the $a(\lambda)$ function has the following asymptotic expression:

$$a(\lambda)=\mathrm{e}^{\frac{\mathrm{i}}{2}(\theta_--\theta_+)}+\mathcal{O}(\lambda^{-1}),\quad \text{as } \lambda\to\infty. \tag{31}$$

The function $a(\lambda)$ can be analytically extended to the region $\lambda \in S_{\mathrm{out}}/\Omega_+$. For convenience, assuming that it only has finitely many zeros and has no zeros on the boundary $\partial(S_{\mathrm{out}}/\Omega_+)$, then $a(\lambda)$ can be represented in the form

$$a(\lambda)=\hat{a}(\lambda)\prod_{i=1}^{k}\left(\frac{\lambda^2-\lambda_i^2}{\lambda^2-(\lambda_i^*)^2}\right)^{m_i} \tag{32}$$

where $\hat{a}(\lambda)$ is an analytic function without zeros in the region $\lambda \in S_{\mathrm{out}}/\Omega_+$, and $m_i \in \mathbb{Z}^+$ is the order of zeros. Denote $\Phi_\pm(n;\lambda)=\Phi^\pm(n;\lambda)\zeta^{n\sigma_3}$. The kernel of $\Phi_+(n;\lambda)$ and high order information at $\lambda=\lambda_i$ are determined by

$$\begin{bmatrix} \Phi_+^{[0]}(n;\lambda_i) & 0 & \cdots & 0\\ \Phi_+^{[1]}(n;\lambda_i) & \Phi_+^{[0]}(n;\lambda_i) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_+^{[m_i-1]}(n;\lambda_i) & \Phi_+^{[m_i-2]}(n;\lambda_i) & \cdots & \Phi_+^{[0]}(n;\lambda_i)\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_0(\lambda_i)\\ \boldsymbol{\gamma}_1(\lambda_i)\\ \vdots\\ \boldsymbol{\gamma}_{m_i-1}(\lambda_i)\end{bmatrix}=0, \tag{33}$$

where $\Phi_+(n;\lambda)=\sum_{j=0}^{\infty}\Phi_+^{[j]}(n;\lambda_i)\,(\lambda-\lambda_i)^j$. It is easy to verify the symmetry relation:

$$\overline{\Phi}_+(n;\lambda)=\boldsymbol{\sigma_2}\Phi_-(n;\lambda^*)\boldsymbol{\sigma_2},\quad \boldsymbol{\sigma_2}=\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}, \tag{34}$$

which infers that

$$\begin{bmatrix} \Phi_-^{[0]}(n;\lambda_i^*) & 0 & \cdots & 0\\ \Phi_-^{[1]}(n;\lambda_i^*) & \Phi_-^{[0]}(n;\lambda_i^*) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_-^{[m_i-1]}(n;\lambda_i^*) & \Phi_-^{[m_i-2]}(n;\lambda_i^*) & \cdots & \Phi_-^{[0]}(n;\lambda_i^*)\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_0(\lambda_i)}\\ \boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_1(\lambda_i)}\\ \vdots\\ \boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_{m_i-1}(\lambda_i)}\end{bmatrix}=0, \tag{35}$$

where $\Phi_-(n;\lambda)=\sum_{j=0}^{\infty}\Phi_-^{[j]}(n;\lambda_i^*)\left(\lambda-\lambda_i^*\right)^j$.

From the symmetry relation (23), we know that $\mathrm{Ker}(\Phi_+(n;\lambda))$ and high order information at $\lambda=-\lambda_i$ are

$$\begin{bmatrix} \Phi_+^{[0]}(n;-\lambda_i) & 0 & \cdots & 0\\ \Phi_+^{[1]}(n;-\lambda_i) & \Phi_+^{[0]}(n;-\lambda_i) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_+^{[m_i-1]}(n;-\lambda_i) & \Phi_+^{[m_i-2]}(n;-\lambda_i) & \cdots & \Phi_+^{[0]}(n;-\lambda_i)\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma_3}\boldsymbol{\gamma}_0(\lambda_i)\\ \boldsymbol{\sigma_3}\boldsymbol{\gamma}_1(\lambda_i)\\ \vdots\\ \boldsymbol{\sigma_3}\boldsymbol{\gamma}_{m_{i-1}}(\lambda_i)\end{bmatrix}=0, \tag{36}$$

where $\Phi_+(n;\lambda)=\sum_{j=0}^{\infty}\Phi_+^{[j]}(n;-\lambda_i)\,(\lambda+\lambda_i)^j$, and $\mathrm{Ker}(\Phi_-(n;\lambda))$ at $\lambda=-\lambda_i^*$ is

$$\begin{bmatrix} \Phi_-^{[0]}(n;-\lambda_i^*) & 0 & \cdots & 0\\ \Phi_-^{[1]}(n;-\lambda_i^*) & \Phi_-^{[0]}(n;-\lambda_i^*) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_-^{[m_i-1]}(n;-\lambda_i^*) & \Phi_-^{[m_i-2]}(n;-\lambda_i^*) & \cdots & \Phi_-^{[0]}(n;-\lambda_i^*)\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma_3}\boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_0(\lambda_i)}\\ \boldsymbol{\sigma_3}\boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_1(\lambda_i)}\\ \vdots\\ \boldsymbol{\sigma_3}\boldsymbol{\sigma_2}\overline{\boldsymbol{\gamma}_{m_i-1}(\lambda_i)}\end{bmatrix}=0, \tag{37}$$

where $\Phi_-(n;\lambda)=\sum_{j=0}^{\infty}\Phi_-^{[j]}(n;-\lambda_i^*)\left(\lambda+\lambda_i^*\right)^j$. These kernels and high order information conditions (33), (35), (36), (37) completely determine the degenerate property of meromorphic function $\Phi^\pm(n;\lambda)$ in the neighborhood of $\pm\lambda_i$ and $\pm\lambda_i^*$, which can be used to construct the solitonic solutions.

The conservation laws for AL-equation (5) on the non-vanishing background can be established based on the expansion of above analytic Jost functions, which is shown in Appendix B.

### *2.3. Riemann–Hilbert Problem*

In this subsection, we construct the corresponding Riemann–Hilbert problem. Firstly, we define the following sectional meromorphic functions:

$$\mathbf{M}(n;\lambda)=\begin{cases}\mathbf{M}^+(n;\lambda)=\Phi^+(n;\lambda)\mathrm{diag}\left(\frac{1}{a(\lambda)},1\right), & \lambda\in S_{out}\setminus\Omega_+,\\[2mm] \mathbf{M}^-(n;\lambda)=\Phi^-(n;\lambda)\mathrm{diag}\left(1,\frac{1}{\bar{a}(\lambda^*)}\right), & \lambda\in S_{in}\setminus\Omega_-,\end{cases} \tag{38}$$

Rewriting the matrix function $\mathbf{M}^\pm(n;\lambda)$ in a uniform form:

$$\mathbf{M}^+(n;\lambda)=\mathbf{J}_+(n;\lambda)\zeta^{n\sigma_3}\begin{bmatrix}1 & 0\\ r(\lambda) & 1\end{bmatrix}\zeta^{-n\sigma_3},\quad r(\lambda)=\frac{b(\lambda)}{a(\lambda)},$$
$$\mathbf{M}^-(n;\lambda)=\mathbf{J}_+(n;\lambda)\zeta^{n\sigma_3}\begin{bmatrix}1 & -\bar{r}(\lambda^*)\\ 0 & 1\end{bmatrix}\zeta^{-n\sigma_3},$$

we deduce the jump condition between $\mathbf{M}^+(n;\lambda)$ and $\mathbf{M}^-(n;\lambda)$ on $S=\{\lambda:|\lambda|=1\}$

$$\mathbf{M}^+(n;\lambda)=\mathbf{M}^-(n;\lambda)\mathbf{V}_1,\quad \mathbf{V}_1=\zeta^{n\sigma_3}\begin{bmatrix}1+r(\lambda)\bar{r}(\lambda^*) & \bar{r}(\lambda^*)\\ r(\lambda) & 1\end{bmatrix}\zeta^{-n\sigma_3},\quad \lambda\in S. \tag{39}$$

Then we consider the jump condition on the cut $\Omega$. We know the following boundary conditions for $\mathbf{J}_{\pm}(n;\lambda)$

$$\mathbf{J}_{\pm}(\pm\infty;\lambda_{\pm}) = \exp\left(\frac{\mathrm{i}}{2}\theta_{\pm}\boldsymbol{\sigma}_3\right)\begin{bmatrix}1 & \xi_{\pm}\\ \xi_{\pm} & 1\end{bmatrix},$$

where $\lambda_{\pm} = \lambda \pm \mathrm{i}\epsilon$, $\epsilon \to 0^{+}$, and $\xi_{\pm} = \xi(\lambda_{\pm})$. It follows that

$$\mathbf{J}_{\pm}(\pm\infty;\lambda_{+}) = \mathbf{J}_{\pm}(\pm\infty;\lambda_{-})\mathbf{S}_b, \quad \lambda \in \Omega,$$

where

$$\mathbf{S}_b = \begin{bmatrix}0 & \xi_{+}\\ \xi_{+} & 0\end{bmatrix}.$$

Therefore, by the uniqueness of solutions to difference equations, we obtain that

$$\mathbf{J}_{\pm}(n;\lambda_{+}) = \mathbf{J}_{\pm}(n;\lambda_{-})\mathbf{S}_b, \ \ \lambda \in \Omega. \tag{40}$$

Then we know that

$$\begin{aligned}
a(\lambda_{+}) &= \frac{\det\left(\left[\mathbf{J}_{-,1}(n;\lambda_{+}),\mathbf{J}_{+,2}(n;\lambda_{+})\right]\right)}{\Delta_n(1-\xi_{+}^2)}\\
&= \frac{\xi_{+}^2(1-\xi_{-}^2)}{(1-\xi_{+}^2)}\frac{\det\left(\left[\mathbf{J}_{-,2}(n;\lambda_{-}),\mathbf{J}_{+,1}(n;\lambda_{-})\right]\right)}{\Delta_n(1-\xi_{-}^2)}\\
&= \bar{a}(\lambda_{-}^{*})
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
b(\lambda_{+}) &= \frac{\det\left(\left[\mathbf{J}_{+,1}(n;\lambda_{+}),\mathbf{J}_{-,1}(n;\lambda_{+})\right]\right)}{\Delta_n(1-\xi_{+}^2)}\zeta_{+}^{2n}\\
&= \frac{\xi_{+}^2(1-\xi_{-}^2)}{(1-\xi_{+}^2)}\frac{\det\left(\left[\mathbf{J}_{+,2}(n;\lambda_{-}),\mathbf{J}_{-,2}(n;\lambda_{-})\right]\right)}{\Delta_n(1-\xi_{-}^2)}\zeta_{+}^{2n}\\
&= -\bar{b}(\lambda_{-}^{*})
\end{aligned} \tag{42}$$

which deduces that $r(\lambda_{+}) = -\bar{r}(\lambda_{-}^{*})$. Furthermore, we obtain the jump condition on $\Omega_{\pm}$. If $\lambda \in \Omega_{+}$, then we have

$$\begin{aligned}
\mathbf{M}^{+}(n;\lambda_{+}) =& \mathbf{J}_{+}(n;\lambda_{+})\zeta_{+}^{n\boldsymbol{\sigma}_3}\begin{bmatrix}1 & 0\\ r(\lambda_{+}) & 1\end{bmatrix}\zeta_{+}^{-n\boldsymbol{\sigma}_3}\\
=& \mathbf{M}^{+}(n;\lambda_{-})\frac{1}{\xi_{-}}\begin{bmatrix}1 & 0\\ r(\lambda_{-})\zeta_{-}^{-2n} & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ r(\lambda_{+})\zeta_{+}^{-2n} & 1\end{bmatrix}\\
=& \mathbf{M}^{+}(n;\lambda_{-})\mathbf{V}_{2,+}, \quad \mathbf{V}_{2,+} = \frac{1}{\xi_{-}}\begin{bmatrix}-\bar{r}(\lambda_{-}^{*})\zeta_{-}^{2n} & 1\\ 1+r(\lambda_{-})\bar{r}(\lambda_{-}^{*}) & -r(\lambda_{-})\zeta_{-}^{-2n}\end{bmatrix}.
\end{aligned} \tag{43}$$

Similarly, if $\lambda \in \Omega_{-}$, we have

$$\mathbf{M}^{+}(n;\lambda_{+}) = \mathbf{M}^{+}(n;\lambda_{-})\mathbf{V}_{2,-}, \quad \mathbf{V}_{2,-} = \frac{1}{\xi_{-}}\begin{bmatrix}\bar{r}(\lambda_{-}^{*})\zeta_{-}^{2n} & 1+r(\lambda_{-})\bar{r}(\lambda_{-}^{*})\\ 1 & r(\lambda_{-})\zeta_{-}^{-2n}\end{bmatrix}. \tag{44}$$

In summary, we can define the following Riemann–Hilbert problem:

**Riemann–Hilbert Problem 1.**

- *The sectionally analytic matrix-valued function* $\mathbf{M}(n;\lambda)$ *in* $\mathbb{C}/\{S\cup\Omega\}$*;*
- *The jump conditions:*

$$\begin{aligned}
\mathbf{M}^{+}(n;\lambda) &= \mathbf{M}^{-}(n;\lambda)\mathbf{V}_1, \quad \lambda \in S,\\
\mathbf{M}^{+}(n;\lambda) &= \mathbf{M}^{-}(n;\lambda)\mathbf{V}_{2,\pm}, \quad \lambda \in \Omega^{\pm},
\end{aligned} \tag{45}$$

  *where* $V_1$ *and* $V_{2,\pm}$ *are given by* (39), (43) and (44).
- *The principal part of* $\mathbf{M}(n;\lambda)$ *is given by*

$$\mathbf{M}(n;\lambda) = \left[\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left(\frac{\mathbf{M}_{1,j}^{[i]}(n)}{(\lambda-\lambda_i)^j} + \frac{\mathbf{M}_{1,j}^{[-i]}(n)}{(\lambda+\lambda_i)^j}\right), \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(\frac{\mathbf{M}_{2,j}^{[i]}(n)}{(\lambda-\lambda_i^{*})^j} + \frac{\mathbf{M}_{2,j}^{[-i]}(n)}{(\lambda+\lambda_i^{*})^j}\right)\right], \tag{46}$$

  *where the* $\mathbf{M}_{k,j}^{[s]}$*'s are the column vectors of the principal part, which will be determined by the analytic part of* $\mathbf{M}(n;\lambda)$ *by the conditions* (33), (35), (36) and (37).

- *The normalization condition:*

$$
\begin{aligned}
\mathbf{M}(n;\lambda) &= \begin{bmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_-} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_+}\Delta_n^+ \end{bmatrix} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty, \\
\mathbf{M}(n;\lambda) &= \begin{bmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_+}\Delta_n^- & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_-} \end{bmatrix} + \mathcal{O}(\lambda), \quad \textit{as } \lambda \to 0.
\end{aligned} \tag{47}
$$

Following the way in [20], the existence and uniqueness of solutions of the above Riemann–Hilbert Problem 1 for all $n \in \mathbb{Z}$ follows by means of Zhou's vanishing lemma argument [27] after replacing the poles by jumps along small circular contours and the Schwartz reflection about the unit circle $\{\lambda : |\lambda| = 1\}$.

### 2.4. Evolution of scattering data

**Lemma 3.** *Let $w_n(t)$ be a solution of Eq. (5) which decays to $\rho \mathrm{e}^{\mathrm{i}\theta_\pm}$ as $n \to \pm\infty$, and $\mathbf{f}_n^\pm(t;\lambda)$ be the Jost solutions for each $t \in [0, +\infty)$. Then $\mathbf{W}_\pm(n, t; \lambda) = \mathbf{f}_n^\pm(t;\lambda)\mathbf{C}^\pm(t;\lambda)$ solves the Lax pair simultaneously, where*

$$
\mathbf{C}^\pm(t;\lambda) = \mathrm{e}^{\rho^2 \sin(\theta)t + \mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda \mathrm{e}^{\mathrm{i}\theta})\gamma \boldsymbol{\sigma_3} t}, \quad \gamma = \frac{\sqrt{(1+\lambda^2)^2 - 4r^2\lambda^2}}{2\lambda}. \tag{48}
$$

**Proof.** Since the function $\mathbf{f}_n^\pm(t;\lambda)$ solves the $n$-part of Lax pair for the fixed $t$, by the fundamental solution theory of difference equation there exists a solution $\mathbf{W}_\pm(n, t; \lambda) = \mathbf{f}_n^\pm(t;\lambda)\mathbf{C}^\pm(t;\lambda)$ solves the Lax pair (5) simultaneously. Inserting the ansatz $\mathbf{W}_\pm(n, t; \lambda) = \mathbf{f}_n^\pm(t;\lambda)\mathbf{C}^\pm(t;\lambda)$ into the $t$-part of Lax pair, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{f}_n^\pm(t;\lambda)\right)\mathbf{C}^\pm(t;\lambda) + \mathbf{f}_n^\pm(t;\lambda)\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{C}^\pm(t;\lambda)\right) = \mathbf{M}_n \mathbf{f}_n^\pm(t;\lambda)\mathbf{C}^\pm(t;\lambda). \tag{49}
$$

Through the boundary conditions (8), we know that

$$
\mathbf{f}_n^\pm(t;\lambda) \to \exp\left(\frac{\mathrm{i}}{2}\theta_\pm \boldsymbol{\sigma_3}\right)\begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} r^n \zeta^{n\boldsymbol{\sigma_3}} \tag{50}
$$

and

$$
\mathbf{M}_n \to \mathbf{M}_n^\pm = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_\pm \boldsymbol{\sigma_3}}\left(\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\begin{bmatrix} \lambda & \rho \\ -\rho & \lambda^{-1} \end{bmatrix} + \delta \mathbb{I}_2\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_\pm \boldsymbol{\sigma_3}} \tag{51}
$$

where $\delta = \frac{\mathrm{i}}{2}(\mathrm{e}^{\mathrm{i}\theta}\lambda^2 - \mathrm{e}^{-\mathrm{i}\theta}\lambda^{-2}) + \mathrm{i}(r^2\cos(\theta) - \mathrm{e}^{\mathrm{i}\theta}\rho^2 - \mathrm{e}^{-\mathrm{i}\theta})$. Solving the differential equation for $\mathbf{C}^\pm(t;\lambda)$ with the initial data $\mathbf{C}^\pm(0;\lambda) = \mathbb{I}_2$, we obtain the solution (48), which completes the proof.

Through the evolution of Jost solutions, the evolution of scattering data and kernel $\mathrm{Ker}(\Phi_\pm(n;\lambda))$ can be stated as

$$
\mathbf{W}_+(n, t; \lambda) = \mathbf{W}_-(n, t; \lambda)\mathbf{S}(\lambda), \quad \mathbf{W}_\pm(n, t; \lambda) = \Phi_\pm(n, t; \lambda)\mathrm{e}^{\rho^2 \sin(\theta)t + \mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t} \tag{52}
$$

and

$$
\begin{bmatrix}
\mathbf{W}_+^{[0]}(n,t;\lambda_i) & 0 & \cdots & 0 \\
\mathbf{W}_+^{[1]}(n,t;\lambda_i) & \mathbf{W}_+^{[0]}(n,t;\lambda_i) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{W}_+^{[m_i-1]}(n,t;\lambda_i) & \mathbf{W}_+^{[m_i-2]}(n,t;\lambda_i) & \cdots & \mathbf{W}_+^{[0]}(n,t;\lambda_i)
\end{bmatrix}
\begin{bmatrix} \boldsymbol{\gamma_0}(\lambda_i) \\ \boldsymbol{\gamma_1}(\lambda_i) \\ \vdots \\ \boldsymbol{\gamma_{m_i-1}}(\lambda_i) \end{bmatrix} = 0, \tag{53}
$$

where $i = 1, 2, \ldots, k$,

$$
\mathbf{W}_+(n, t; \lambda) = \sum_{j=0}^{\infty} \mathbf{W}_+^{[j]}(n, t; \lambda_i)(\lambda - \lambda_i)^j.
$$

The evolution of kernel $\mathrm{Ker}(\Phi_\pm(n;\lambda))$ at the other points $-\lambda_i, \pm\lambda_i^*$ can be stated as above by the symmetric relations (23), (34).

Then the evolution of jump matrices $\mathbf{V}_1(t)$ and $\mathbf{V}_{2,\pm}(t)$ are given by

$$
\mathbf{V}_1(\lambda; t) = \mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}\mathbf{V}_1(\lambda)\mathrm{e}^{-\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}, \tag{54}
$$

and

$$
\mathbf{V}_{2,\pm}(\lambda; t) = \mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma_-\boldsymbol{\sigma_3} t}\mathbf{V}_{2,\pm}(\lambda)\mathrm{e}^{-\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma_+\boldsymbol{\sigma_3} t}, \quad \gamma_\pm = \gamma(\lambda_\pm). \tag{55}
$$

The evolution of kernel and high order information at $\lambda = \lambda_i$ can be represented as

$$
\begin{bmatrix} \boldsymbol{\gamma_0}(\lambda_i; t) \\ \boldsymbol{\gamma_1}(\lambda_i; t) \\ \vdots \\ \boldsymbol{\gamma_{m_i-1}}(\lambda_i; t) \end{bmatrix}
=
\begin{bmatrix}
\mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t} & 0 & \cdots & 0 \\
\frac{\mathrm{d}\,\mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}}{\mathrm{d}\lambda} & \mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\mathrm{d}^{m_i-1}\mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}}{\mathrm{d}\lambda^{m_i-1}} & \frac{\mathrm{d}^{m_i-2}\mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}}{\mathrm{d}\lambda^{m_i-2}} & \cdots & \mathrm{e}^{\mathrm{i}(\lambda^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \lambda\mathrm{e}^{\mathrm{i}\theta})\gamma\boldsymbol{\sigma_3} t}
\end{bmatrix}\Bigg|_{\lambda=\lambda_i}
\begin{bmatrix} \boldsymbol{\gamma_0}(\lambda_i) \\ \boldsymbol{\gamma_1}(\lambda_i) \\ \vdots \\ \boldsymbol{\gamma_{m_i-1}}(\lambda_i) \end{bmatrix}, \tag{56}
$$

and $i = 1, 2, \ldots, k$.

### 2.5. Robust inverse scattering method

The above procedure is the classical inverse scattering method. However, under the classical one, the rogue waves and high order rogue wave cannot be captured. To solve this problem, we apply the robust inverse scattering method [20]. The key point of the robust inverse scattering method is to construct a new analytic function instead of original Jost function. The new analytic function is constructed by the following proposition:

**Proposition 4.** *Suppose that $w_n(t)$ is a bounded classic solution for Eq. (5). For arbitrary $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, there exists a unique analytic solution $\mathbf{U}(n,t;\lambda)$ on the torus region $\Gamma = \{\lambda : R^{-1} < |\lambda| \leq R, R > r+\rho\}$ which solves the Lax pair (6) simultaneously with the initial condition $\mathbf{U}(0,0;\lambda) = \mathbb{I}_2$.*

**Proof.** Firstly, we define the shift operator $E\mathbf{f}_n = \mathbf{f}_{n+1}$. Since $w_n(t)$ satisfies Eq. (5), the compatibility condition $\frac{\mathrm{d}}{\mathrm{d}t}(E\mathbf{f}_n(t)) = E(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}_n(t))$, i.e

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_n(t;\lambda) + \mathbf{L}_n(t;\lambda)\mathbf{M}_n(t;\lambda) - (E\mathbf{M}_n(t;\lambda))\mathbf{L}_n(t;\lambda) = 0 \tag{57}$$

is valid automatically. The compatibility condition guarantees that the value of $\mathbf{f}_n(t)$ from the origin point $(n,t) = (0,0)$ by the difference–differential equation (6) is independent of the path. Actually, we can show the iteration on the basic path from $(n,0)$ to $(n+1,t)$:

$$\begin{aligned}\mathbf{f}_{n+1}(t;\lambda) =& E(\mathbf{f}_n(0;\lambda)) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}(E\mathbf{f}_n(t';\lambda))\mathrm{d}t' \\ =& E\left(\mathbf{f}_n(0;\lambda) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}\mathbf{f}_n(t';\lambda)\mathrm{d}t'\right)\end{aligned} \tag{58}$$

where the first equation is along the path $(n,0) \to (n+1,0) \to (n+1,t)$ and the second equation is along the path $(n,0) \to (n,t) \to (n+1,t)$. The equivalence between the first and second equality is from the compatibility condition (57). Fixed a path, the solution $\mathbf{U}(n,t;\lambda)$ with the initial data $\mathbf{U}(0,0;\lambda) = \mathbb{I}_2$ can be represented as the following integral form:

$$\begin{aligned}&\mathbf{U}(n,t;\lambda) = \mathbf{L}_{n-1}(0;\lambda)\mathbf{L}_{n-2}(0;\lambda)\cdots\mathbf{L}_0(0;\lambda) + \int_0^t \mathbf{M}_n(t';\lambda)\mathbf{U}(n,t';\lambda)\mathrm{d}t', \quad n \in \mathbb{Z}^+, \\ &\mathbf{U}(n,t;\lambda) = (\mathbf{L}_n(0;\lambda))^{-1}(\mathbf{L}_{n+1}(0;\lambda))^{-1}\cdots(\mathbf{L}_{-1}(0;\lambda))^{-1} + \int_0^t \mathbf{M}_n(t';\lambda)\mathbf{U}(n,t';\lambda)\mathrm{d}t', \quad n \in \mathbb{Z}^-,\end{aligned} \tag{59}$$

The standard Picard iteration and the assumption that $w_n(t)$ is a bounded classical solution of (5) guarantee uniform convergence of iterating series. Then, the first expression at the right hand side of Eq. (59) and $\mathbf{M}_n(t;\lambda)$ are analytic in the region $\Gamma$, so is $\mathbf{U}(n,t;\lambda)$.

In view of above proposition, the existence and uniqueness theorem of ordinary differential equation, the analytic solution on the region $\Gamma$ can be constructed:

$$\begin{aligned}\mathbf{U}^{\mathrm{in}}(n,t;\lambda) &= \mathbf{U}_B(n,t;\lambda)[\mathbf{U}_B(0,0;\lambda)]^{-1} \\ &= \mathbf{M}^{\pm}(n,t;\lambda)r^n e^{\sin(\theta)\rho^2 t} e^{[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\gamma t]\boldsymbol{\sigma}_3}[\mathbf{M}^{\pm}(0,0;\lambda)]^{-1},\end{aligned} \tag{60}$$

where the matrix function $\mathbf{U}_B(n,t;\lambda)$ is the fundamental solutions for the Lax pair (6).

We now define the sectionally analytic matrix function:

$$\mathbf{M}(n,t;\lambda) = \begin{cases} \mathbf{U}^{\mathrm{in}}(n,t;\lambda)r^{-n}e^{-\sin(\theta)\rho^2 t}e^{-[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\gamma t]\boldsymbol{\sigma}_3}, & \lambda \in \Gamma, \\ \mathbf{M}^{\pm}(n,t;\lambda), & \lambda \in \Gamma_{\pm}. \end{cases} \tag{61}$$

where the regions $\Gamma_{\pm}$ and $\Gamma$ are shown in Fig. 2, which solves the following Riemann–Hilbert problem:

**Riemann–Hilbert Problem 2.** *The $2 \times 2$ matrix function $\mathbf{M}(n,t;\lambda)$ that has the following properties:*

**Analyticity** *$\mathbf{M}(n,t;\lambda)$ is analytic in $\lambda \in \mathbb{C} \setminus \{\partial\Gamma \cup \Omega\}$.*

**Jump condition** *$\mathbf{M}(n,t;\lambda)$ takes continuous boundary values $\mathbf{M}_{\pm}(n,t;\lambda)$ on $\partial\Gamma \cup \Omega$, and they are related by the jump conditions of the form $\mathbf{M}_+(n,t;\lambda) = \mathbf{M}_-(n,t;\lambda)\mathbf{V}(n,t;\lambda)$ on $\partial\Gamma \cup \Omega$, where*

$$\begin{aligned}\mathbf{V}(n,t;\lambda) =& e^{-2[n\ln(\zeta_+)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\delta_+ t]\boldsymbol{\sigma}_3}, \quad \lambda \in \Omega, \\ \mathbf{V}(n,t;\lambda) =& e^{[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\delta t]\boldsymbol{\sigma}_3}[\mathbf{M}^{+}(0,0;\lambda)]e^{-[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\gamma t]\boldsymbol{\sigma}_3}, \quad \lambda \in \partial\Gamma_+, \\ \mathbf{V}(n,t;\lambda) =& e^{[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\delta t]\boldsymbol{\sigma}_3}[\mathbf{M}^{-}(0,0;\lambda)]^{-1}e^{-[n\ln(\zeta)+\mathrm{i}(\lambda^{-1}e^{-\mathrm{i}\theta}-\lambda e^{\mathrm{i}\theta})\gamma t]\boldsymbol{\sigma}_3}, \quad \lambda \in \partial\Gamma_-,\end{aligned} \tag{62}$$

**Normalization**

$$\begin{aligned}\mathbf{M}(n,t;\lambda) &= \begin{bmatrix} e^{\frac{\mathrm{i}}{2}\theta_-} & 0 \\ 0 & e^{-\frac{\mathrm{i}}{2}\theta_+}\Delta_n^+(t)\end{bmatrix} + \mathcal{O}(\lambda^{-1}), \quad \text{as } \lambda \to \infty, \\ \mathbf{M}(n,t;\lambda) &= \begin{bmatrix} e^{\frac{\mathrm{i}}{2}\theta_+}\Delta_n^-(t) & 0 \\ 0 & e^{-\frac{\mathrm{i}}{2}\theta_-}\end{bmatrix} + \mathcal{O}(\lambda), \quad \text{as } \lambda \to 0.\end{aligned} \tag{63}$$

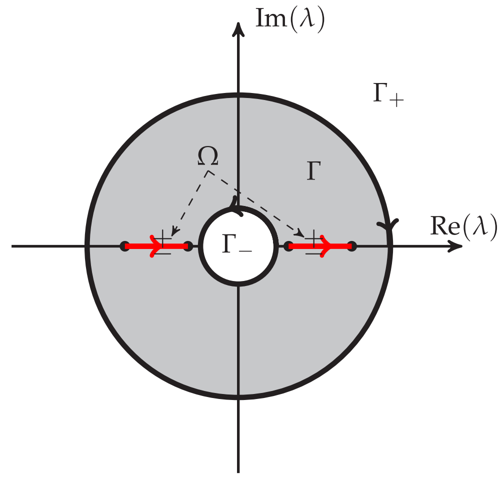


**Fig. 2.** The two red segments denote the branch cuts. The four black points denote the branch points. Definition of the regions $\Gamma$, $\Gamma_\pm$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Remark 1.** In the assumption of function $a(\lambda)$: (32), it only has finitely many zeros. Then we can choose the radii of annulus $\Gamma$ to include all zeros of $a(\lambda)$ and $a^*(\lambda)$.

The existence and uniqueness of Riemann–Hilbert Problem 2 can be proved by mimicking the proof of Theorem 2.4 in Ref. [20] but replacing the Schwartz symmetry on the line with that on the contour $\{\lambda : |\lambda| = 1\}$. Finally, the potential function $w_n$ can be recovered from the following formula:

$$w_n = -\lim_{\lambda\to\infty} \lambda \frac{\mathbf{M}_{1,2}(n,t;\lambda)}{\mathbf{M}_{2,2}(n,t;\lambda)}. \tag{64}$$

## 3. Darboux transformation in the frame of robust inverse scattering

In this section, we firstly construct the Darboux matrix by the loop group method [28]. Then by the robust inverse scattering transform, the Darboux matrix can be inserted into a proper Riemann–Hilbert problem, which can be used to reconstruct the potential functions.

### 3.1. Darboux transformation

In this subsection, we construct the elementary Darboux matrix $\mathbf{T}_1(n,t;\lambda)$ for the spectral problem of AL equation (5). Assume that we have an analytic solution $\Phi(n,t;\lambda)$ in the region $\mathbb{C}\setminus\{0,\infty\}$ that solves the Lax pair (6). The loop group construction infers that the Darboux matrix $\mathbf{T}_1(n,t;\lambda)$ is linear fractional transformation of matrix. The conditions of Darboux matrix can be summarized as

**(1) The ansatz of Darboux matrix**

$$\mathbf{T}_1(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & a_1(n,t) \end{bmatrix}\left(\mathbb{I} + \frac{|x_1(n,t)\rangle\langle y_1(n,t)|}{\lambda-\lambda_1^*} - \frac{\boldsymbol{\sigma_3}|x_1(n,t)\rangle\langle y_1(n,t)|\boldsymbol{\sigma_3}}{\lambda+\lambda_1^*}\right) \tag{65}$$

**(2) The kernel conditions and residue conditions**

$$\mathrm{Ker}(\mathbf{T}_1(n,t;\lambda_1)) = \Phi(n,t;\lambda_1)\mathbf{c}_1, \quad \mathrm{Ker}(\mathbf{T}_1(n,t;-\lambda_1)) = \boldsymbol{\sigma_3}\Phi(n,t;\lambda_1)\mathbf{c}_1,$$

and

$$\operatorname*{Res}_{\lambda=\lambda_1^*}(\mathbf{T}_1(n,t;\lambda))\sigma_2\overline{\Phi}(n,t;\lambda_1)\bar{\mathbf{c}}_1 = 0, \quad \operatorname*{Res}_{\lambda=-\lambda_1^*}(\mathbf{T}_1(n,t;\lambda))\sigma_2\boldsymbol{\sigma_3}\overline{\Phi}(n,t;\lambda_1)\bar{\mathbf{c}}_1 = 0,$$

where $\mathbf{c}_1$ is a column vector.

**(3) The normalization conditions**

$$\begin{aligned} &\mathbf{T}_1(n,t;\lambda) \to \begin{bmatrix} b_1(n,t) & 0 \\ 0 & \beta_1 \end{bmatrix} + \mathcal{O}(\lambda), \quad \text{as } \lambda\to 0, \\ &\mathbf{T}_1(n,t;\lambda) \to \begin{bmatrix} 1 & 0 \\ 0 & c_1(n,t) \end{bmatrix} + \mathcal{O}(\lambda^{-1}), \quad \text{as } \lambda\to\infty, \end{aligned} \tag{66}$$

where $\beta_1$ is an undetermined constant independent with $n$ and $t$, $b_1(n,t)$ and $c_1(n,t)$ are the undetermined functions.

**(4) The symmetry for the potential functions**

$$\mathbf{L}_n^{[1]}(t) = \lambda\mathbf{E}_+ + \lambda^{-1}\mathbf{E}_- + \mathbf{Q}_n^{[1]}(t) \equiv \mathbf{T}_1(n+1,t;\lambda)\mathbf{L}_n(t)[\mathbf{T}_1(n,t;\lambda)]^{-1}, \tag{67}$$

where

$$\mathbf{Q}_n^{[1]} = \begin{bmatrix} 0 & w_n^{[1]} \\ -\bar{w}_n^{[1]} & 0 \end{bmatrix}. \tag{68}$$

These above four conditions determine the elementary Darboux matrix with the form:

$$\mathbf{T}_1(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{|\lambda_1|^2}{1+2|\varphi_1|^2\beta^{-1}} \end{bmatrix} \left[ \mathbb{I} - \left( \frac{\lambda_1^* \mathbf{K}^{-1}|y_1\rangle\langle y_1|}{\lambda - \lambda_1^*} - \frac{\lambda_1^* \boldsymbol{\sigma}_3 \mathbf{K}^{-1}|y_1\rangle\langle y_1|\boldsymbol{\sigma}_3}{\lambda + \lambda_1^*} \right) \right] \tag{69}$$

where $\mathbf{K} = \mathrm{diag}\,(\alpha, \beta)$, $\alpha = \frac{\langle y_1|y_1\rangle}{|\lambda_1|^2-1} - \frac{\langle y_1|\boldsymbol{\sigma}_3|y_1\rangle}{|\lambda_1|^2+1}$, $\beta = \frac{\langle y_1|y_1\rangle}{|\lambda_1|^2-1} + \frac{\langle y_1|\boldsymbol{\sigma}_3|y_1\rangle}{|\lambda_1|^2+1}$ and $|y_1\rangle = (\psi_1, \varphi_1)^{\mathrm{T}} = \Phi(n,t;\lambda_1)\mathbf{c}_1$, and $\langle y_1| = (|y_1\rangle)^\dagger$ uses the Dirac bra–ket notation. Note that $\mathbf{T}_1$ depends only on span$(|y_1\rangle)$, i.e. it is invariant under any rescaling $|y_1\rangle \mapsto k|y_1\rangle$, $k \in \mathbb{C}$. The corresponding Bäcklund transformation can be obtained by performing the expansion of Eq. (67) in the neighborhood of $\infty$ together with Eq. (69):

$$w_n^{[1]} = \frac{\left(|\psi_1|^2 + |\lambda_1|^2|\varphi_1|^2\right) w_n + \lambda_1^*(|\lambda_1|^4 - 1)\psi_1\bar{\varphi}_1}{|\lambda_1|^2|\psi_1|^2 + |\varphi_1|^2}. \tag{70}$$

The new analytic matrix solution can be constructed with the form:

$$\Phi_{[1]}(n,t;\lambda) = \mathbf{T}_1(n,t;\lambda)\Phi(n,t;\lambda)(\mathbf{T}_1(0,0;\lambda))^{-1}, \tag{71}$$

which satisfies the Lax pair (6) with the new potential function (70).

The Darboux transformation can be iterated to yield the multi ones or high order ones. To represent the solution by the determinant form, we rewrite the multi-fold Darboux matrices with the following theorems:

**Theorem 2.** *Suppose we have $N$ different solutions $|y_i\rangle$ for (6) at $\lambda = \lambda_i$, then the Darboux matrix for analytic solution matrix $\Phi(n,t;\lambda)$ is*

$$\mathbf{T}_N(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & a_N(n,t) \end{bmatrix} \left[ \mathbb{I} + \sum_{i=1}^{N} \left( \frac{|x_i\rangle\langle y_i|}{\lambda - \lambda_i^*} - \frac{\boldsymbol{\sigma}_3|x_i\rangle\langle y_i|\boldsymbol{\sigma}_3}{\lambda + \lambda_i^*} \right) \right] \tag{72}$$

*where $\langle x_i| = |x_i\rangle^\dagger$, $\langle y_i| = |y_i\rangle^\dagger$, $|x_i\rangle$ can be determined by the following linear equations:*

$$\mathbf{X}_1 = -\mathbf{Y}_1\mathbf{A}^{-1}, \ \mathbf{X}_2 = -\mathbf{Y}_2\mathbf{B}^{-1},$$
$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = [|x_1\rangle, |x_2\rangle, \ldots, |x_N\rangle], \quad \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = [|y_1\rangle, |y_2\rangle, \ldots, |y_N\rangle],$$

*here*

$$\mathbf{A} = \left( \frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*} - \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j + \lambda_i^*} \right)_{1\le i,j\le N}, \quad \mathbf{B} = \left( \frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*} + \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j + \lambda_i^*} \right)_{1\le i,j\le N}$$

*and*

$$a_N(n,t) = \frac{\prod_{i=1}^N |\lambda_i|^2}{1 + 2\mathbf{Y}_2\mathbf{B}^{-1}\mathbf{D}\mathbf{Y}_2^\dagger}, \quad \mathbf{D} = \mathrm{diag}\left(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_N\right).$$

Based on the Darboux matrix (72), we obtain the Bäcklund transformation between new potential functions and old ones:

$$w_n^{[N]} = \frac{w_n - b_N}{a_N},$$

where

$$b_N = 2\sum_{i=1}^{N} (|x_i\rangle\langle y_i|)_{1,2} = -2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{Y}_1^\dagger.$$

It follows that

$$w_n^{[N]} = \frac{\left(w_n + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{Y}_2^\dagger\right)\left(1 + 2\mathbf{Y}_2\mathbf{B}^{-1}\mathbf{D}\mathbf{Y}_2^\dagger\right)}{\prod_{i=1}^N |\lambda_i|^2}. \tag{73}$$

We take the seed solution with $w_n = \rho$, then the general solution formula can be represented by the following theorem:

**Theorem 3.** *The general solitonic solution formula (73) with $w_n = \rho$ can be represented as*

$$w_n^{[N]} = \rho\frac{\det(\mathbf{H})}{\det(\mathbf{M})}, \tag{74}$$

*where*

$$\mathbf{M} = \left[ \frac{\bar{\lambda}_i\lambda_j\bar{\psi}_i\psi_j + \bar{\varphi}_i\varphi_j}{\bar{\lambda}_i^2\lambda_j^2 - 1} \right]_{1\le i,j\le N},$$

$$\mathbf{H} = \left[ \frac{\bar{\psi}_i\psi_j + \bar{\lambda}_i\lambda_j\bar{\varphi}_i\varphi_j}{\bar{\lambda}_i^2\lambda_j^2 - 1} + \frac{\bar{\varphi}_i\psi_j}{\rho\bar{\lambda}_i} \right]_{1\le i,j\le N}.$$

**Proof.** Firstly, from the Darboux matrix, we know that

$$\left(1 + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{D}\mathbf{Y}_1^\dagger\right)\left(1 + 2\mathbf{Y}_2\mathbf{B}^{-1}\mathbf{D}\mathbf{Y}_2^\dagger\right) = \prod_{i=1}^{N}|\lambda_i|^4.$$

Then the solution formula becomes

$$w_n^{[N]} = \left(\prod_{i=1}^{N}|\lambda_i|^2\right)\frac{\rho + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{Y}_2^\dagger}{1 + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{D}\mathbf{Y}_1^\dagger}.$$

Denote

$$|y_j\rangle = \begin{bmatrix}\psi_j\\ \varphi_j\end{bmatrix},\quad \langle y_j| = \begin{bmatrix}\bar{\psi}_j & \bar{\varphi}_j\end{bmatrix}.$$

It can be easily shown that

$$1 + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{D}\mathbf{Y}_1^\dagger = \frac{\det(\mathbf{A} + 2\mathbf{D}\mathbf{Y}_1^\dagger\mathbf{Y}_1)}{\det(\mathbf{A})},$$

and

$$\begin{aligned}\rho + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{Y}_2^\dagger &= \rho\left(1 + \frac{2}{\rho}\psi\mathbf{A}^{-1}\varphi^\dagger\right)\\ &= \rho\frac{\det(\mathbf{A} + \frac{2}{\rho}\varphi^\dagger\psi)}{\det(\mathbf{A})},\end{aligned}$$

where

$$\begin{aligned}\psi &= \begin{bmatrix}\psi_1, & \psi_2, & \cdots, & \psi_N\end{bmatrix},\\ \varphi &= \begin{bmatrix}\varphi_1, & \varphi_2, & \cdots, & \varphi_N\end{bmatrix}.\end{aligned}$$

Moreover, the solution formula can be represented as

$$w_n^{[N]} = \rho\prod_{i=1}^{N}|\lambda_i|^2\frac{\det\left(\frac{1}{2}\mathbf{A} + \frac{1}{\rho}\varphi^\dagger\psi\right)}{\det(\frac{1}{2}\mathbf{A} + \mathbf{D}\psi^\dagger\psi)}.$$

In what follows, we calculate the explicit elements for matrices

$$\begin{aligned}\left(\frac{1}{2}\mathbf{A} + \frac{1}{\rho}\varphi^\dagger\psi\right)_{i,j} &= \frac{1}{2}\left(\frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*} - \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j + \lambda_i^*}\right) + \frac{1}{\rho}\bar{\varphi}_i\psi_j\\ &= \frac{\bar{\lambda}_i}{\bar{\lambda}_i^2\lambda_j^2 - 1}\left(\bar{\psi}_i\psi_j + \bar{\lambda}_i\lambda_j\bar{\varphi}_i\varphi_j\right) + \frac{1}{\rho}\bar{\varphi}_i\psi_j,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{2}\mathbf{A} + \mathbf{D}\psi^\dagger\psi\right)_{i,j} &= \frac{1}{2}\left(\frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*} - \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j + \lambda_i^*}\right) + \bar{\lambda}_i\bar{\psi}_i\psi_j\\ &= \frac{\bar{\lambda}_i}{\bar{\lambda}_i^2\lambda_j^2 - 1}\left(\bar{\psi}_i\psi_j + \bar{\lambda}_i\lambda_j\bar{\varphi}_i\varphi_j\right) + \bar{\lambda}_i\bar{\psi}_i\psi_j\\ &= \frac{\bar{\lambda}_i}{\bar{\lambda}_i^2\lambda_j^2 - 1}\left(\bar{\lambda}_i\lambda_j\bar{\psi}_i\psi_j + \bar{\varphi}_i\varphi_j\right)\bar{\lambda}_i\lambda_j.\end{aligned}$$

Finally, we obtain the solution formula (74). □

### *3.2. High order darboux matrix*

In Theorem 2, we assume that the spectral parameters $\lambda_i$'s are different. It is natural to ask what happens if we have the same spectral parameter. Actually, this case corresponds exactly the generalized Darboux transformation. As presented in the literature, the generalized Darboux transformation comes from the elementary Darboux transformation. The key step to yield the generalized Darboux matrix is to construct the fundamental solution at $\lambda = \lambda_1$. However, it will fail if we apply the Darboux matrix directly. In this work, the normalization method [20] (Eq. (71)) is used to deal with this problem.

Following the construction from the appendix of [22], we have the following theorem:

**Theorem 4.** *Suppose we have the vector functions* $|y_1^{[j]}\rangle = \Phi^{[j]}(n,t;\lambda_1)\mathbf{c}_1$ *at* $\lambda = \lambda_1$, *where*

$$\Phi(n,t;\lambda) = \sum_{j=1}^{\infty} \Phi^{[j]}(n,t;\lambda_1)(\lambda-\lambda_1)^j, \tag{75}$$

*then the high order Darboux matrix for solution matrix* $\Phi(n,t;\lambda)$ *is*

$$\mathbf{T}_N(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & a_N(n,t) \end{bmatrix} \left[\mathbb{I} + \mathbf{X}\mathbf{L}(\lambda,\bar{\lambda}_1)\mathbf{Y}^\dagger + \boldsymbol{\sigma}_3\mathbf{X}\mathbf{L}(-\lambda,\bar{\lambda}_1)\mathbf{Y}^\dagger\boldsymbol{\sigma}_3\right] \tag{76}$$

*where* $\mathbf{X}$ *can be determined by the following linear equations:*

$$\mathbf{X}_1 = -\mathbf{Y}_1\mathbf{A}^{-1},\ \ \mathbf{X}_2 = -\mathbf{Y}_2\mathbf{B}^{-1},\ \ \mathbf{X} \equiv \begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix},\quad \mathbf{Y} \equiv \begin{bmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{bmatrix} = \left[\left|y_1^{[0]}\right\rangle, \left|y_1^{[1]}\right\rangle, \dots, \left|y_1^{[N-1]}\right\rangle\right],$$

$$\mathbf{L}(\lambda,\bar{\lambda}_1) = \frac{\bar{\lambda}_1\mathbb{I}}{\lambda\bar{\lambda}_1 - 1} + \sum_{i=1}^{N-1} \frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(\frac{x}{x\lambda-1}\right)\Big|_{x=\bar{\lambda}_1}(\mathbf{L}_0)^i,\quad \mathbf{L}_0 = \left(\delta_{i,j-1}\right)_{1\le i,j\le N},$$

*and* $\delta_{i,j-1}$ *is the standard Christoffel symbols* $\delta_{i,j} = \begin{cases} 1, & i=j, \\ 0, & i\neq j, \end{cases}$ *here*

$$\mathbf{A} = \left(\mathbf{J}_1^\dagger\mathbf{C}_-\mathbf{J}_1 + \mathbf{K}_1^\dagger\mathbf{C}_-\mathbf{K}_1\right) - \left(\mathbf{J}_1^\dagger\mathbf{C}_+\mathbf{J}_1 - \mathbf{K}_1^\dagger\mathbf{C}_+\mathbf{K}_1\right),$$
$$\mathbf{B} = \left(\mathbf{J}_1^\dagger\mathbf{C}_-\mathbf{J}_1 + \mathbf{K}_1^\dagger\mathbf{C}_-\mathbf{K}_1\right) + \left(\mathbf{J}_1^\dagger\mathbf{C}_+\mathbf{J}_1 - \mathbf{K}_1^\dagger\mathbf{C}_+\mathbf{K}_1\right)$$

*and*

$$\begin{aligned}
\mathbf{C}_\pm &= \left(\frac{1}{(i-1)!(j-1)!}\frac{\mathrm{d}^{i+j-2}}{\mathrm{d}x^{i-1}\mathrm{d}y^{j-1}}\left(\frac{x}{xy\pm 1}\right)\Big|_{x=\bar{\lambda}_1,y=\lambda_1}\right)_{1\le i,j\le N}, \\
\mathbf{J}_1 &= \psi_1^{[0]}\mathbb{I}_N + \sum_{j=1}^{N-1}\psi_1^{[j]}\mathbf{E}^j,\quad \mathbf{E} = \left(\delta_{i,j+1}\right)_{1\le i,j\le N}, \\
\mathbf{K}_1 &= \varphi_1^{[0]}\mathbb{I}_N + \sum_{j=1}^{N-1}\varphi_1^{[j]}\mathbf{E}^j,\quad \left|y_1^{[j]}\right\rangle = \left[\psi_1^{[j]}, \varphi_1^{[j]}\right]^T,
\end{aligned} \tag{77}$$

*and*

$$a_N(n,t) = \frac{|\lambda_1|^{2N}}{1 + 2\bar{\lambda}_1\mathbf{Y}_2\mathbf{B}^{-1}\mathbf{Y}_2^\dagger}.$$

*The Bäcklund transformation between old potential functions and new ones is*

$$w_n^{[N]} = \frac{\left(w_n + 2\mathbf{Y}_1\mathbf{A}^{-1}\mathbf{Y}_2^\dagger\right)\left(1 + 2\bar{\lambda}_1\mathbf{Y}_2\mathbf{B}^{-1}\mathbf{Y}_2^\dagger\right)}{|\lambda_1|^{2N}}. \tag{78}$$

The above Darboux matrix has one higher order pole, it can extend to the general case with lots of different higher order poles. By a similar calculation, we get the similar formula for high order solutions as in Theorem 3, which will be given in the subsequent section.

### 3.3. Riemann–Hilbert Problem for the Darboux matrix

We reconsider the Darboux matrix in the frame of robust inverse scattering transform. Following the steps in [22], we define the following sectional analytic matrix function:

$$\mathbf{N}(n,t;\lambda) = \begin{cases} \mathbf{N}^+(n,t;\lambda) = \mathbf{T}_N(n,t;\lambda), & \lambda \in \{\lambda||\lambda| > R\} \cup \{\lambda||\lambda| < R^{-1}\} \\ \mathbf{N}^-(n,t;\lambda) = \mathbf{T}_N(n,t;\lambda)\Phi(n,t;\lambda)\mathbf{T}_N^{-1}(0,0;\lambda)\Phi^{-1}(n,t;\lambda), & \lambda \in \{\lambda|R^{-1} < |\lambda| < R\}, \end{cases} \tag{79}$$

where $R > \max|\lambda_i|$, which solves the following Riemann–Hilbert problem:

**Riemann–Hilbert Problem 3.**

**Analyticity** $\mathbf{N}(n,t;\lambda)$ *is analytic in* $\lambda \in \mathbb{C} \setminus \left(\{|\lambda| = R^{-1}\} \cup \{|\lambda| = R\}\right)$.

**Jump condition** $\mathbf{N}(n,t;\lambda)$ *takes continuous boundary values* $\mathbf{N}^\pm(n,t;\lambda)$ *on* $\{\lambda||\lambda| = R\} \cup \{\lambda||\lambda| = R^{-1}\}$, *and they are related by the jump conditions of the form* $\mathbf{N}^+(n,t;\lambda) = \mathbf{N}^-(n,t;\lambda)\mathbf{V}(n,t;\lambda)$ *on* $\{\lambda||\lambda| = R\} \cup \{\lambda||\lambda| = R^{-1}\}$, *where*

$$\mathbf{V}(n,t;\lambda) = \Phi(n,t;\lambda)\mathbf{T}_N(0,0;\lambda)\Phi^{-1}(n,t;\lambda) \tag{80}$$

**Normalization**

$$\mathbf{N}(n,t;\lambda)\to\begin{bmatrix}1 & 0\\ 0 & a_N(n,t)\end{bmatrix},\quad \lambda\to\infty, \tag{81}$$

*and*

$$\mathbf{N}(n,t;\lambda)\to\prod_{i=1}^{N}|\lambda_i|^2\begin{bmatrix}[a_N(n,t)]^{-1} & 0\\ 0 & 1\end{bmatrix},\quad \lambda\to 0. \tag{82}$$

Through the recovering formula (64) and above Riemann–Hilbert Problem 3, the potential function is given by

$$w_n^{[N]}(t)=\lim_{\lambda\to\infty}\frac{\rho-\lambda\mathbf{N}_{12}(n,t;\lambda)}{\mathbf{N}_{22}(n,t;\lambda)}. \tag{83}$$

The jump matrix $\mathbf{V}(n,t;\lambda)$ of Riemann–Hilbert Problem 3 depends on the Darboux matrix at $(x,t)=(0,0)$ and a fundamental matrix solution of Lax pair $\Phi(n,t;\lambda)$ with a trivial seed solution $w_n(t)$. As has been shown elsewhere [22], this Riemann–Hilbert representation of the Darboux matrix is an effective method for analyzing the large order rogue waves or solitons for the integrable equations.

## 4. Soliton, breather and rogue wave solution

Through the formulas of Bäcklund transformation (70) or (83), we can construct various exact solutions. To this end, we first solve the linear system with $w_n=\rho$ and $\lambda=\lambda_i$.

If $\lambda_i\neq r\pm\rho,-r\pm\rho$, inserting the seed solution $w_n=\rho$ into Lax pair (6), we have

$$\begin{aligned}\mathbf{f}_{n+1}&=\mathbf{U}_i\mathbf{f}_n,\\ \mathbf{f}_{n,t}&=\left[\beta_i\mathbf{U}_i+\delta_i\mathbb{I}\right]\mathbf{f}_n,\end{aligned} \tag{84}$$

where

$$\mathbf{U}_i=\begin{bmatrix}\lambda_i & \rho\\ -\rho & \lambda_i^{-1}\end{bmatrix},\quad \delta_i=\delta(\lambda_i),\quad \beta_i=\mathrm{i}\left(\lambda_i^{-1}\mathrm{e}^{-\mathrm{i}\theta}-\lambda_i\mathrm{e}^{\mathrm{i}\theta}\right).$$

Then we diagonalize the matrix $\mathbf{U}_i$:

$$\mathbf{U}_i=r\mathbf{V}_i\zeta_i^{\boldsymbol{\sigma}_3}\mathbf{V}_i^{-1},\quad \zeta_i=\zeta(\lambda_i),$$

where

$$\mathbf{V}_i=\begin{bmatrix}1 & \xi_i\\ \xi_i & 1\end{bmatrix},\quad \xi_i=\xi(\lambda_i).$$

By the above diagonalization, the fundamental solution for linear system (84) can be solved simultaneously:

$$\mathbf{f}_n(t;\lambda_i)=r^n\mathrm{e}^{\sin(\theta)\rho^2 t}\mathbf{V}_i\mathrm{e}^{(n\ln\zeta_i+\beta_i\gamma_i t)\boldsymbol{\sigma}_3},\quad \gamma_i=\gamma(\lambda_i). \tag{85}$$

Inserting into formulas (74) the special vector solutions

$$\begin{bmatrix}\psi_i\\ \varphi_i\end{bmatrix}=\frac{\mathbf{f}_n(t;\lambda_i)}{r^n\mathrm{e}^{\sin(\theta)\rho^2 t}}\begin{bmatrix}\frac{1}{2}\mathrm{e}^{c_i+\mathrm{i}\omega_i/2}\\ -\frac{1}{2}\mathrm{e}^{-c_i+\mathrm{i}\omega_i/2}\end{bmatrix}=\begin{bmatrix}\sinh\left(\alpha_i+\frac{\mathrm{i}}{2}\omega_i\right)\\ -\sinh\left(\alpha_i-\frac{\mathrm{i}}{2}\omega_i\right)\end{bmatrix},\quad \omega_i=\arccos\left(\frac{\lambda_i-\lambda_i^{-1}}{2\rho}\right) \tag{86}$$

where

$$\alpha_i=n\ln\zeta_i+\beta_i\gamma_i t+c_i,\qquad \mathrm{e}^{-\mathrm{i}\omega_i}=-\xi_i=\frac{\lambda_i-\lambda_i^{-1}}{2\rho}-\mathrm{i}\sqrt{1-\left(\frac{\lambda_i-\lambda_i^{-1}}{2\rho}\right)^2},$$

and $c_i$ is a complex constant, we can obtain various solitonic solutions.

If $\lambda_i=r\pm\rho,-r\pm\rho$, we will use the normalization method to obtain the fundamental solution for the Lax pair (84), which will be shown later. With these fundamental solutions, we will obtain the lattice rational solutions of AL equation (5).

### 4.1. Single soliton solution

In this subsection, we construct a single soliton solution and analyze its dynamic behaviors. Plugging the solution (86) into the solution formula, we obtain

$$w_n^{[1]}=\rho\left[\frac{|\psi_1|^2+|\lambda_1|^2|\varphi_1|^2+(\rho\bar{\lambda}_1)^{-1}(|\lambda_1|^4-1)\bar{\varphi}_1\psi_1}{|\lambda_1|^2|\psi_1|^2+|\varphi_1|^2}\right], \tag{87}$$

To analyze the properties of soliton solution, we need to simplify the expression (87). By the following identities

$$\sinh(X)\sinh(Y)=\frac{1}{2}\left(\cosh(X+Y)-\cosh(X-Y)\right),$$

$$\begin{aligned}&A_1\cosh(X+Y_1)+A_2\cosh(X+Y_2)+A_3\cosh(X+Y_3)\\ &=\left[(A_1\mathrm{e}^{Y_1}+A_2\mathrm{e}^{Y_2}+A_3\mathrm{e}^{Y_3})(A_1\mathrm{e}^{-Y_1}+A_2\mathrm{e}^{-Y_2}+A_3\mathrm{e}^{-Y_3})\right]^{1/2}\cosh(X+\omega),\end{aligned}$$

where

$$\omega=\frac{1}{2}\ln\left(\frac{A_1\mathrm{e}^{Y_1}+A_2\mathrm{e}^{Y_2}+A_3\mathrm{e}^{Y_3}}{A_1\mathrm{e}^{-Y_1}+A_2\mathrm{e}^{-Y_2}+A_3\mathrm{e}^{-Y_3}}\right),$$

we can obtain the following relations:

$$\begin{aligned}|\lambda_1|^2|\psi_1|^2+|\varphi_1|^2&=r_1\cosh(\alpha_1+\bar{\alpha}_1+\theta_1)-r_2\cosh(\alpha_1-\bar{\alpha}_1+\theta_2),\\ |\psi_1|^2+|\lambda_1|^2|\varphi_1|^2+(\rho\bar{\lambda}_1)^{-1}(|\lambda_1|^4-1)\bar{\varphi}_1\psi_1&=r_3\cosh(\alpha_1+\bar{\alpha}_1+\theta_3)-r_4\cosh(\alpha_1-\bar{\alpha}_1+\theta_4)\end{aligned}\tag{88}$$

where

$$\theta_i=\frac{1}{2}\ln\left(\frac{p_i}{q_i}\right),\ \ r_i=(p_iq_i)^{1/2},\ \ i=1,2,3,4,$$

and

$$\begin{aligned}p_1=&|\lambda_1|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_1-\bar{\omega}_1)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_1-\bar{\omega}_1)},\ \ q_1=|\lambda_1|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_1-\bar{\omega}_1)}+\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_1-\bar{\omega}_1)},\\ p_2=&|\lambda_1|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_1+\bar{\omega}_1)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_1+\bar{\omega}_1)},\ \ q_2=|\lambda_1|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_1+\bar{\omega}_1)}+\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_1+\bar{\omega}_1)},\\ p_3=&-\frac{|\lambda_1|^2\mathrm{e}^{\mathrm{i}\omega_1}+\mathrm{e}^{\mathrm{i}\bar{\omega}_1}}{|\lambda_1|^2\mathrm{e}^{-\mathrm{i}\bar{\omega}_1}+\mathrm{e}^{-\mathrm{i}\omega_1}}p_1,\quad q_3=-\frac{|\lambda_1|^2\mathrm{e}^{-\mathrm{i}\omega_1}+\mathrm{e}^{-\mathrm{i}\bar{\omega}_1}}{|\lambda_1|^2\mathrm{e}^{\mathrm{i}\bar{\omega}_1}+\mathrm{e}^{\mathrm{i}\omega_1}}q_1,\\ p_4=&-\frac{|\lambda_1|^2\mathrm{e}^{-\mathrm{i}\omega_1}+\mathrm{e}^{\mathrm{i}\bar{\omega}_1}}{|\lambda_1|^2\mathrm{e}^{\mathrm{i}\omega_1}+\mathrm{e}^{-\mathrm{i}\bar{\omega}_1}}p_2,\quad q_4=-\frac{|\lambda_1|^2\mathrm{e}^{\mathrm{i}\omega_1}+\mathrm{e}^{-\mathrm{i}\bar{\omega}_1}}{|\lambda_1|^2\mathrm{e}^{-\mathrm{i}\omega_1}+\mathrm{e}^{\mathrm{i}\bar{\omega}_1}}q_2.\end{aligned}$$

Then it is easy to verify that $r_1=r_3$, $r_2=r_4$, and

$$\theta_3=\theta_1+\mathrm{i}(\omega_1+\bar{\omega}_1),\ \ \theta_4=\theta_2+\mathrm{i}(\omega_1-\bar{\omega}_1).$$

Finally, we obtain a compact expression for $w_n^{[1]}$:

$$w_n^{[1]}=\rho\left[\frac{\cosh(\chi+\mathrm{i}(\omega_1+\bar{\omega}_1))-G\cosh(\varpi+\mathrm{i}(\omega_1-\bar{\omega}_1))}{\cosh(\chi)-G\cosh(\varpi)}\right],\tag{89}$$

where $\chi=\alpha_1+\bar{\alpha}_1+\theta_1$, $\varpi=\alpha_1-\bar{\alpha}_1+\theta_2$, $G=r_2/r_1$. From (89), we can see that the velocity of soliton is $-\frac{\mathrm{Re}(\beta_1\gamma_1)}{\mathrm{Re}(\ln(\zeta_1))}$. If soliton solutions exhibit the breather behavior, it will oscillate along lines which are perpendicular to the line $2\mathrm{Im}(\ln(\zeta_1))\left(n+\frac{\mathrm{Im}(\beta_1\gamma_1)}{\mathrm{Im}(\ln(\zeta_1))}t\right)=$ const. Moreover, we can obtain the asymptotic behavior for the single soliton solution:

$$\begin{aligned}w_n^{[1]}\to&\rho\mathrm{e}^{\mathrm{i}(\omega_1+\bar{\omega}_1)},\quad \chi\to+\infty,\\ w_n^{[1]}\to&\rho\mathrm{e}^{-\mathrm{i}(\omega_1+\bar{\omega}_1)},\quad \chi\to-\infty.\end{aligned}\tag{90}$$

Thus, if $2(\omega_1+\bar{\omega}_1) \mod (2\pi)\neq 0$, there is a non-trivial phase difference. Through the reduced formula (89), we obtain the phase difference and the velocity of the localized lattice wave solution. However it is hard to analyze its maximum value. To answer this problem, we give the following proposition:

**Proposition 5.** *If $|\lambda_1|>1$ and $c_1=\frac{1}{2}\ln\left(\frac{1+(r+\rho)\lambda_1\mathrm{e}^{-\mathrm{i}\omega_1}}{(r+\rho)\lambda_1+\mathrm{e}^{-\mathrm{i}\omega_1}}\right)$, then the modulus of solution (87) attains the maximum $\frac{1}{2}\left((r+\rho)|\lambda_1|^2-(r-\rho)|\lambda_1|^{-2}\right)$ at $(n,t)=(0,0)$.*

**Proof.** Rewriting the formula (87) as the following form:

$$w_n^{[1]}=\rho\left[\frac{1+|\lambda_1|^2|\varphi_1/\psi_1|^2+(\rho\bar{\lambda}_1)^{-1}(|\lambda_1|^4-1)\bar{\varphi}_1/\bar{\psi}_1}{|\lambda_1|^2+|\varphi_1/\psi_1|^2}\right],\tag{91}$$

for arbitrary values on $\varphi_1/\psi_1$, we can find by choosing $\psi_1=1$ and $\varphi_1=(r+\rho)\lambda_1$ the maximum value of $|w_n^{[1]}|$ for the solution (87) is $\frac{1}{2}\left((r+\rho)|\lambda_1|^2-(r-\rho)|\lambda_1|^{-2}\right)$. So if $(\psi_1(0,0),\varphi_1(0,0))=k(1,(r+\rho)\lambda_1)$, where $k\in\mathbb{C}$ is a complex constant, the modulus of solution $|w_n^{[1]}(t)|$ will attain the maximum value at $(n,t)=(0,0)$. Actually, by choosing $c_1=\frac{1}{2}\ln\left(\frac{1+(r+\rho)\lambda_1\mathrm{e}^{-\mathrm{i}\omega_1}}{(r+\rho)\lambda_1+\mathrm{e}^{-\mathrm{i}\omega_1}}\right)$ we can achieve this aim. □

Consequently, the parameter $c_1$ in Proposition 5 can be used to generate the solutions with the maximum value at $(n,t)=(0,0)$.

**Remark 2.** Similar as the proof of Proposition 5, we can establish the maximum value between both old and new functions. Rewriting the formula of elementary Bäcklund transformation (70):

$$w_n^{[1]}=w_n\left[\frac{\left(1+|\lambda_1|^2|\varphi_1/\psi_1|^2\right)+(w_n\bar{\lambda}_1)^{-1}(|\lambda_1|^4-1)\bar{\varphi}_1/\bar{\psi}_1}{|\lambda_1|^2+|\varphi_1/\psi_1|^2}\right],$$

the maximum value of $|w_n^{[1]}|$: $\max|w_n^{[1]}|=\frac{1}{2}\left((\sqrt{1+m^2}+m)|\lambda_1|^2-(\sqrt{1+m^2}-m)|\lambda_1|^{-2}\right)$ will attain by choosing $\psi_1(0,0)=1$, $\varphi_1(0,0)=(m+\sqrt{1+m^2})\lambda_1$ and $m=\max|w_n|=|w_n(t)|_{(n,t)=(0,0)}$. The high order or multi-fold Darboux transformation is the recursive iteration of elementary one. Thus the above proposition can be considered as the general rule for the solution generating by the Darboux transformations.

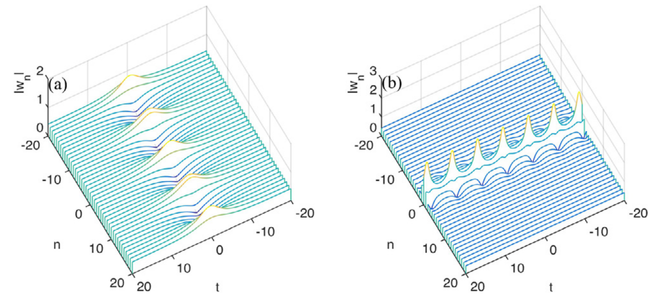


**Fig. 3.** $\rho = \frac{5}{12}$, $\theta = 0$, (a): Akhmediev lattice breathers: $\lambda_1 = \frac{5}{4}$, $\max|w_n| = 0.96$. (b): Kuznetsov–Ma lattice breather: $\lambda_1 = \frac{7}{4}$, $\max|w_n| = 2.19$. The constant $c_1$ is given by Proposition 5.

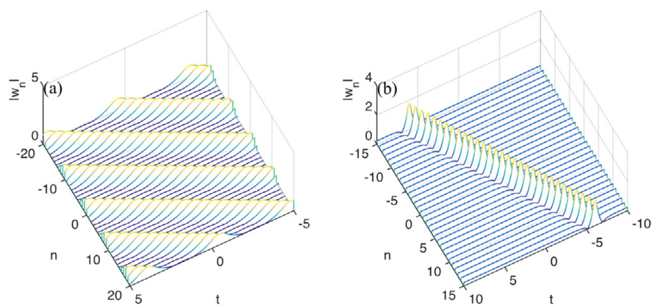


**Fig. 4.** $\rho = \frac{5}{12}$, $\theta = \pi/2$, (a): periodic lattice solution: $\lambda_1 = \frac{5}{4}$, $\max|w_n| = 0.96$, (b): lattice soliton of W-shape: $\lambda_1 = \frac{7}{4}$, $\max|w_n| = 2.19$. The constant $c_1$ is given by Proposition 5.

Now we turn to analyze the dynamic behavior of above soliton solution, which is closely related with the properties of modulational instability to the background solution. The modulational instability analysis is given in Appendix A. We find that the modulational instability for the focusing AL equation (5) is different from the focusing NLSE (1). The plane wave solution of focusing NLSE (1) is always modulational unstable. However, for the focusing AL equation (5), if $\theta \neq \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$), the plane wave solution is modulational unstable; while if $\theta = \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$), it is modulational stable.

For the modulational unstable background $\theta \neq \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$), the Akhmediev lattice breather (localized in time and periodic in $n$, see Fig. 3(a)) is obtained by taking the parameters $\lambda_1 \in (1, r+\rho) \cup (-r-\rho, -1)$. If $\theta = 0$, the Kuznetsov–Ma lattice breather (localized in $n$ and periodic in time, see Fig. 3(b)) can be obtained by taking the parameters $\lambda_1 \in (r+\rho, \infty) \cup (-\infty, -r-\rho)$. For other choices of parameters, we obtain the Tajiri–Watanabe lattice breather [29].

On the other hand, for the modulational stable background $\theta = \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$), if $\lambda_1 \in (1, r+\rho) \cup (-r-\rho, -1)$, a periodic lattice solution occurs instead of Akhmediev lattice breather (see Fig. 4(a)). If $\lambda_1 \in (r+\rho, \infty) \cup (-\infty, -r-\rho)$, the soliton solution of W-shape appears (see Fig. 4(b)). For other choices of parameters $\lambda_1 \in S_{out}/\mathbb{R}$, we also obtain the Tajiri–Watanabe lattice breather solution.

### 4.2. Multi-solitonic solution

In this subsection, we consider the interaction law for the multi-solitonic solutions with the parameters $\lambda_i$ and $c_i$, ($|\lambda_i| > 1$), $\mathrm{Re}(\ln(\zeta_i)) > 0$, $i = 1, 2, \ldots, N$. The velocity parameters are arranged with the order $s_1 < s_2 < \cdots < s_N$, where $s_i = -\frac{\mathrm{Re}(\beta_i\gamma_i)}{\mathrm{Re}(\ln(\zeta_i))}$. We analyze the asymptotic behavior of the $k$th localized lattice wave solution. To this end, we decompose the $N$-fold Darboux matrix into two Darboux matrices:

$$\mathbf{T}_N(n,t;\lambda) = \mathbf{T}^{[k]}(n,t;\lambda)\mathbf{T}_{(k)}(n,t;\lambda) \tag{92}$$

where

$$\mathbf{T}^{[k]}(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{|\lambda_k|^2}{1+2|\hat{\varphi}_k|^2\beta_k^{-1}} \end{bmatrix} \left[ \mathbb{I} - \left( \frac{\lambda_k^* \mathbf{K}^{-1}|\hat{y}_k\rangle\langle\hat{y}_k|}{\lambda - \lambda_k^*} - \frac{\lambda_k^* \boldsymbol{\sigma_3}\mathbf{K}^{-1}|\hat{y}_k\rangle\langle\hat{y}_k|\boldsymbol{\sigma_3}}{\lambda + \lambda_k^*} \right) \right] \tag{93}$$

with $\mathbf{K} = \mathrm{diag}\,(\alpha_k, \beta_k)$, $\alpha_k = \frac{\langle\hat{y}_k|\hat{y}_k\rangle}{|\lambda_k|^2-1} - \frac{\langle\hat{y}_k|\boldsymbol{\sigma_3}|\hat{y}_k\rangle}{|\lambda_k|^2+1}$, $\beta_k = \frac{\langle\hat{y}_k|\hat{y}_k\rangle}{|\lambda_k|^2-1} + \frac{\langle\hat{y}_k|\boldsymbol{\sigma_3}|\hat{y}_k\rangle}{|\lambda_k|^2+1}$, $|\hat{y}_k\rangle = (\hat{\psi}_k, \hat{\varphi}_k)^{\mathrm{T}} = \mathbf{T}_{(k)}(n,t;\lambda_k)|y_k\rangle$

$$\mathbf{T}_{(k)}(n,t;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & a_{(k)}(n,t) \end{bmatrix} \left[ \mathbb{I} + \sum_{i=1, i\neq k}^{N} \left( \frac{|x_i^{(k)}\rangle\langle y_i|}{\lambda - \lambda_i^*} - \frac{\boldsymbol{\sigma_3}|x_i^{(k)}\rangle\langle y_i|\boldsymbol{\sigma_3}}{\lambda + \lambda_i^*} \right) \right] \tag{94}$$

and

$$\begin{aligned}
&\mathbf{X}_1^{(k)} = -\mathbf{Y}_1^{(k)}\mathbf{A}_{(k)}^{-1},\ \mathbf{X}_2^{(k)} = -\mathbf{Y}_2^{(k)}\mathbf{B}_{(k)}^{-1},\\
&\begin{bmatrix}\mathbf{X}_1^{(k)}\\ \mathbf{X}_2^{(k)}\end{bmatrix} = \left[|x_1^{(k)}\rangle, |x_2^{(k)}\rangle, \ldots, |x_{k-1}^{(k)}\rangle, |x_{k+1}^{(k)}\rangle, \ldots, |x_N^{(k)}\rangle\right],\\
&\begin{bmatrix}\mathbf{Y}_1^{(k)}\\ \mathbf{Y}_2^{(k)}\end{bmatrix} = \left[|y_1\rangle, |y_2\rangle, \ldots, |y_{k-1}\rangle, |y_{k+1}\rangle, \ldots, |y_N\rangle\right],
\end{aligned}$$

$$\mathbf{A}_{(k)} = \left(\frac{\langle y_i|y_j\rangle}{\lambda_j-\lambda_i^*} - \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j+\lambda_i^*}\right)_{1\le i,j\le N; i,j\neq k},\ \ \mathbf{B}_{(k)} = \left(\frac{\langle y_i|y_j\rangle}{\lambda_j-\lambda_i^*} + \frac{\langle y_i|\boldsymbol{\sigma}_3|y_j\rangle}{\lambda_j+\lambda_i^*}\right)_{1\le i,j\le N; i,j\neq k}$$

$$a_{(k)}(n,t) = \frac{\prod_{i=1; i\neq k}^N |\lambda_i|^2}{1+2\mathbf{Y}_2^{(k)}\mathbf{B}_{(k)}^{-1}\mathbf{D}_{(k)}(\mathbf{Y}_2^{(k)})^\dagger},\quad \mathbf{D}_{(k)} = \operatorname{diag}\left(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{k-1}, \bar{\lambda}_{k+1}, \ldots, \bar{\lambda}_N\right).$$

Along the line $\mathrm{Re}(\ln(\zeta_k))(n-s_kt) = \mathrm{const}$, and $\mathrm{Re}(\alpha_i) = \mathrm{Re}(\ln(\zeta_i))[n-s_kt+(s_k-s_i)t]$, if $i>k$, $\mathrm{Re}(\alpha_i)\to\mp\infty$ as $t\to\pm\infty$; if $i<k$, then $\mathrm{Re}(\alpha_i)\to\pm\infty$ as $t\to\pm\infty$. Up to a scalar function[1], we obtain the asymptotic expression for $|y_i\rangle$ $(i\neq k)$ as $t\to+\infty$

$$\begin{aligned}
&|y_i\rangle \propto |y_i^+\rangle = \begin{bmatrix}1\\ \xi_i\end{bmatrix} + \mathcal{O}(\mathrm{e}^{-c_i^{(k)}|t|}),\quad i<k,\\
&|y_i\rangle \propto |y_i^+\rangle = \begin{bmatrix}\xi_i\\ 1\end{bmatrix} + \mathcal{O}(\mathrm{e}^{-c_i^{(k)}|t|}),\quad i>k,
\end{aligned}\tag{95}$$

where $c_i^{(k)} = 4\mathrm{Re}(\ln(\zeta_i))|s_i-s_k|$; as $t\to-\infty$

$$\begin{aligned}
&|y_i\rangle \propto |y_i^-\rangle = \begin{bmatrix}\xi_i\\ 1\end{bmatrix} + \mathcal{O}(\mathrm{e}^{-c_i^{(k)}|t|}),\quad i<k,\\
&|y_i\rangle \propto |y_i^-\rangle = \begin{bmatrix}1\\ \xi_i\end{bmatrix} + \mathcal{O}(\mathrm{e}^{-c_i^{(k)}|t|}),\quad i>k.
\end{aligned}\tag{96}$$

It follows that

$$\begin{aligned}
&\mathbf{T}_{(k)}(n,t;\lambda) = \mathbf{T}_{(k)}^{\pm}(\lambda) + \mathcal{O}(\mathrm{e}^{-c^{(k)}|t|}),\\
&\mathbf{T}_{(k)}^{\pm}(\lambda) = \begin{bmatrix}1 & 0\\ 0 & a_{(k)}^{\pm}\end{bmatrix}\left[\mathbb{I} + \sum_{i=1, i\neq k}^{N}\left(\frac{|x_i^{(k)\pm}\rangle\langle y_i^{\pm}|}{\lambda-\lambda_i^*} - \frac{\boldsymbol{\sigma}_3|x_i^{(k)\pm}\rangle\langle y_i^{\pm}|\boldsymbol{\sigma}_3}{\lambda+\lambda_i^*}\right)\right]
\end{aligned}\tag{97}$$

as $t\to\pm\infty$, where $c^{(k)} = 4\min\limits_{i\neq k}(\mathrm{Re}(\ln(\zeta_i))|s_i-s_k|)$,

$$\begin{aligned}
&\mathbf{X}_1^{(k)\pm} = -\mathbf{Y}_1^{(k)\pm}(\mathbf{A}_{\pm}^{(k)})^{-1},\ \mathbf{X}_2^{(k)\pm} = -\mathbf{Y}_2^{(k)\pm}(\mathbf{B}_{\pm}^{(k)})^{-1},\\
&\begin{bmatrix}\mathbf{X}_1^{(k)\pm}\\ \mathbf{X}_2^{(k)\pm}\end{bmatrix} = \left[|x_1^{(k)\pm}\rangle, |x_2^{(k)\pm}\rangle, \ldots, |x_{k-1}^{(k)\pm}\rangle, |x_{k+1}^{(k)\pm}\rangle, \ldots, |x_N^{(k)\pm}\rangle\right],\\
&\begin{bmatrix}\mathbf{Y}_1^{(k)\pm}\\ \mathbf{Y}_2^{(k)\pm}\end{bmatrix} = \left[|y_1^{\pm}\rangle, |y_2^{\pm}\rangle, \ldots, |y_{k-1}^{\pm}\rangle, |y_{k+1}^{\pm}\rangle, \ldots, |y_N^{\pm}\rangle\right],
\end{aligned}$$

and

$$\mathbf{A}_{\pm}^{(k)} = \left(\frac{\langle y_i^{\pm}|y_j^{\pm}\rangle}{\lambda_j-\lambda_i^*} - \frac{\langle y_i^{\pm}|\boldsymbol{\sigma}_3|y_j^{\pm}\rangle}{\lambda_j+\lambda_i^*}\right)_{1\le i,j\le N; i,j\neq k},\ \ \mathbf{B}_{\pm}^{(k)} = \left(\frac{\langle y_i^{\pm}|y_j^{\pm}\rangle}{\lambda_j-\lambda_i^*} + \frac{\langle y_i^{\pm}|\boldsymbol{\sigma}_3|y_j^{\pm}\rangle}{\lambda_j+\lambda_i^*}\right)_{1\le i,j\le N; i,j\neq k}$$

$$a_{(k)}^{\pm} = \frac{\prod_{i=1, i\neq k}^N |\lambda_i|^2}{1+2\mathbf{Y}_2^{(k)\pm}\left(\mathbf{B}_{\pm}^{(k)}\right)^{-1}\mathbf{D}_{(k)}(\mathbf{Y}_2^{(k)\pm})^\dagger}.$$

Furthermore, we have the following asymptotic behavior for $|\hat{y}_k\rangle$:

$$|\hat{y}_k\rangle = \mathbf{T}_{(k)}(n,t;\lambda_k)|y_k\rangle = \mathbf{T}_{(k)}^{\pm}(\lambda_k)|y_k\rangle + \mathcal{O}(\mathrm{e}^{-c^{(k)}|t|}).\tag{98}$$

1 The Darboux matrices are invariant under the rescaling of the vector $|y_i\rangle \mapsto d_i|y_i\rangle$, $d_i\in\mathbb{C}$.

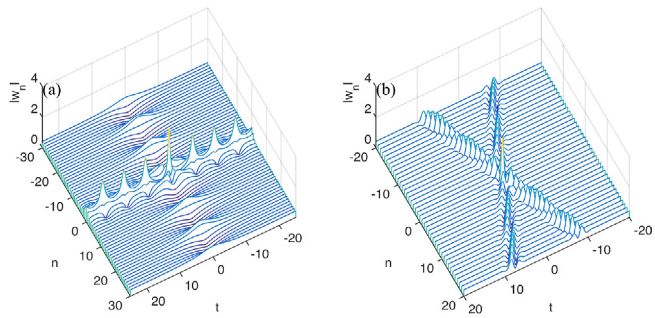


**Fig. 5.** $\rho = \frac{5}{12}$, $\theta = 0$ (a): The interaction between Akhmediev breather and Kuznetsov–Ma breather. $\lambda_1 = \frac{5}{4}$, $\lambda_2 = \frac{7}{4}$, $\max|w_n| = 3.52$, (b): The interaction of two Tajiri–Watanabe breather. $\lambda_1 = 1 - \mathrm{i}$, $\lambda_2 = 1 + \mathrm{i}$, $\max|w_n| = 2.92$. The constants $c_1$, $c_2$ are given by Proposition 5.

To give the asymptotic behavior of the $k$th soliton solution, we calculate the exact form of $\mathbf{T}^{\pm}_{(k)}(\lambda_k)|y_k\rangle$:

$$\begin{aligned}\mathbf{T}^{\pm}_{(k)}(\lambda_k)|y_k\rangle &= \mathbf{T}^{\pm}_{(k)}(\lambda_k)\begin{bmatrix}1 & \xi_k\\ \xi_k & 1\end{bmatrix}\begin{bmatrix}\mathrm{e}^{[n\ln(\zeta_k)+\beta_k\gamma_k t+c_k]}\\ \mathrm{e}^{-[n\ln(\zeta_k)+\beta_k\gamma_k t+c_k]}\end{bmatrix}\\ &= \begin{bmatrix}\gamma_k^{\pm} & \delta_k^{\pm}\xi_k\mathrm{e}^{\mathrm{i}\Delta_k^{\pm}}\\ \gamma_k^{\pm}\xi_k\mathrm{e}^{-\mathrm{i}\Delta_k^{\pm}} & \delta_k^{\pm}\end{bmatrix}\begin{bmatrix}\mathrm{e}^{[n\ln(\zeta_k)+\beta_k\gamma_k t+c_k]}\\ \mathrm{e}^{-[n\ln(\zeta_k)+\beta_k\gamma_k t+c_k]}\end{bmatrix}\end{aligned} \tag{99}$$

where

$$\begin{aligned}\gamma_k^{\pm} &= 1 - \mathbf{Y}_1^{(k)\pm}(\mathbf{A}_{\pm}^{(k)})^{-1}\mathbf{C}_1^{(k)\pm}, \quad \delta_k^{\pm} = a_{(k)}^{\pm}\left(1 - \mathbf{Y}_2^{(k)\pm}(\mathbf{B}_{\pm}^{(k)})^{-1}\mathbf{C}_2^{(k)\pm}\right),\\ \mathbf{C}_1^{(k)\pm} &= \left[\frac{\langle y_i^{\pm}|y_k^{(1)}\rangle}{\lambda_k - \lambda_i^*} - \frac{\langle y_i^{\pm}|\boldsymbol{\sigma}_3|y_k^{(1)}\rangle}{\lambda_k + \lambda_i^*}\right]^{\mathrm{T}}_{1\le i\le N; i\ne k}, \quad \mathbf{C}_2^{(k)\pm} = \left[\frac{\langle y_i^{\pm}|y_k^{(2)}\rangle}{\lambda_k - \lambda_i^*} - \frac{\langle y_i^{\pm}|\boldsymbol{\sigma}_3|y_k^{(2)}\rangle}{\lambda_k + \lambda_i^*}\right]^{\mathrm{T}}_{1\le i\le N; i\ne k},\\ \Delta_k^{\pm} &= \pm 2\left(\sum_{i=1}^{k-1}\mathrm{Re}(\omega_i) - \sum_{i=k+1}^{N}\mathrm{Re}(\omega_i)\right), \quad |y_k^{(1)}\rangle = \begin{bmatrix}1\\ \xi_k\end{bmatrix}, \quad |y_k^{(2)}\rangle = \begin{bmatrix}\xi_k\\ 1\end{bmatrix}.\end{aligned} \tag{100}$$

Thus, through the above analysis we obtain the following proposition:

**Proposition 6.** *The $N$-soliton solution with $N$ distinct velocities, $s_k$, $k = 1, 2, \dots, N$, has the following asymptotic behavior along the line $n - s_k t = \mathrm{const}$ as $t \to \pm\infty$:*

$$w_n^{[N]} = \rho\left[\frac{\cosh(\chi_k^{\pm} + \mathrm{i}(\omega_k + \bar{\omega}_k)) - G_k\cosh(\varpi_k^{\pm} + \mathrm{i}(\omega_k - \bar{\omega}_k))}{\cosh(\chi_k^{\pm}) - G_k\cosh(\varpi_k^{\pm})}\right]\mathrm{e}^{\mathrm{i}\Delta_k^{\pm}} + \mathcal{O}(\mathrm{e}^{-c^{(k)}|t|}), \tag{101}$$

*where* $\chi_k^{\pm} = 2\mathrm{Re}\left(\alpha_k^{\pm}\right) + \theta_1^{(k)}$, $\varpi_k^{\pm} = 2\mathrm{i}\,\mathrm{Im}\left(\alpha_k^{\pm}\right) + \theta_2^{(k)}$, $G_k = \frac{r_2^{(k)}}{r_1^{(k)}}$, $\alpha_k^{\pm} = n\ln(\zeta_k) + \beta_k\gamma_k t + c_k + \frac{1}{2}\ln(\gamma_k^{\pm}/\delta_k^{\pm})$,

$$\begin{aligned}\theta_1^{(k)} &= \frac{1}{2}\ln\left(\frac{|\lambda_k|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)}}{|\lambda_k|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)} + \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)}}\right), \quad \theta_2^{(k)} = \frac{1}{2}\ln\left(\frac{|\lambda_k|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)}}{|\lambda_k|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)} + \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)}}\right),\\ r_1^{(k)} &= \left[\left(|\lambda_k|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)}\right)\left(|\lambda_k|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)} + \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k-\bar{\omega}_k)}\right)\right]^{1/2},\\ r_2^{(k)} &= \left[\left(|\lambda_k|^2\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)}\right)\left(|\lambda_k|^2\mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)} + \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega_k+\bar{\omega}_k)}\right)\right]^{1/2}.\end{aligned}$$

Through the above proposition, we know that the interaction between different types of localized lattice waves is elastic. To visualize the result of the above proposition, we will exhibit some numeric examples to illustrate their dynamic behaviors.

We give some examples to exhibit interactions between breathers or solitons. In the modulational unstable background, we show the interaction between Akhmediev lattice breather and Kuznetsov–Ma lattice breather in Fig. 5(a). It is seen that two breathers are perpendicular to each other. Fig. 5(b) shows the interaction of two Tajiri–Watanabe lattice breathers.

For the modulational stable background, the types of soliton solutions are richer than the modulational unstable one. Fig. 6(a) illustrates the interaction between a Tajiri–Watanabe lattice breather and a lattice soliton of W-shape which is elastic. Fig. 6(b) shows the interaction of two solitons of W-shape which is also elastic. Fig. 7(a) shows the interaction of periodic lattice solution and a Tajiri–Watanabe lattice breather.

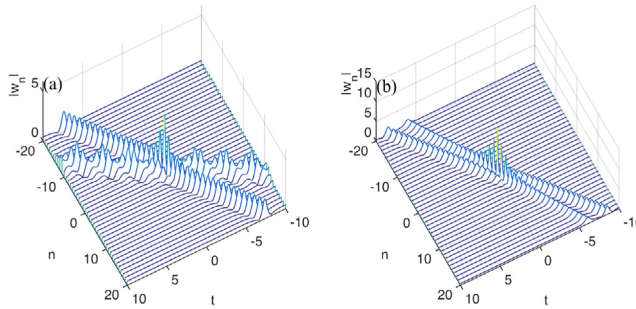


**Fig. 6.** $\rho=\frac{5}{12}$, $\theta=\frac{\pi}{2}$ (a): The interaction between a Tajiri–Watanabe breather and a soliton of W-shape. $\lambda_1=\frac{7}{4}$, $\lambda_2=\frac{7}{4}+\mathrm{i}$, $\max|w_n|=9.3$, (b): The interaction for two solitons of W-shape. $\lambda_1=\frac{7}{4}$, $\lambda_2=\frac{9}{4}$, $\max|w_n|=11.6$. The constants $c_1$, $c_2$ are given by Proposition 5.

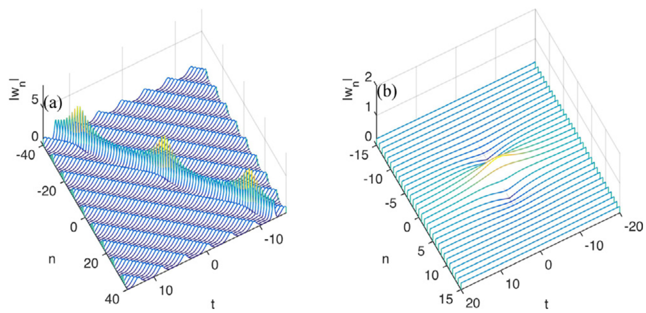


**Fig. 7.** $\rho=\frac{5}{12}$, $\theta=\frac{\pi}{2}$ (a): The interaction between a Tajiri–Watanabe breather and a periodic wave. $\lambda_1=\frac{5}{4}$, $\lambda_2=\frac{9}{4}$, $\max|w_n|=5.89$, (b): Rogue wave solution. $\theta=0$, $\rho=\frac{9}{40}$, $\lambda_1=\frac{5}{4}$, $n_0=t_0=0$, $\max|w_n|=\frac{11529}{16000}$. The constants $c_1$, $c_2$ are given by Proposition 5.

### 4.3. Rational and rogue wave solutions

To obtain the rational solution for the AL equation (5), we normalize the fundamental matrix solution of the Lax pair (6)

$$\begin{aligned}\Phi(n,t;\lambda)&=r^{n-n_0}\mathrm{e}^{\sin(\theta)\rho^2(t-t_0)}\mathbf{V}\mathrm{e}^{\eta\boldsymbol{\sigma}_3}\mathbf{V}^{-1}\\&=r^{n-n_0}\mathrm{e}^{\sin(\theta)\rho^2(t-t_0)}\left(\cosh(\eta)\mathbb{I}_2-\frac{\rho\sinh(\eta)}{\gamma}\begin{bmatrix}\frac{1-\lambda^2}{2\rho\lambda}&-1\\1&-\frac{1-\lambda^2}{2\rho\lambda}\end{bmatrix}\right)\end{aligned}\tag{102}$$

with $\Phi(n_0,t_0;\lambda)=\mathbb{I}_2$, where $\eta=(n-n_0)\ln(\zeta)+\beta\gamma(t-t_0)+c$, which is analytic in the region $\mathbb{C}\setminus\{0,\infty\}$ and has the removable singularity at the branch points $\lambda=r\pm\rho$ or $-r\pm\rho$. Now we choose the vector solutions

$$\begin{bmatrix}\psi(n,t;\lambda)\\\varphi(n,t;\lambda)\end{bmatrix}=\Phi(n,t;\lambda)\begin{bmatrix}1\\(r+\rho)\lambda\end{bmatrix}\tag{103}$$

which can be expanded in the deleted neighborhood of $\lambda=\lambda_1\equiv r+\rho$:

$$\begin{aligned}\psi(n,t;\lambda)&=\sum_{i=0}^{\infty}\psi_1^{[i]}(\lambda-\lambda_1)^i,\\\varphi(n,t;\lambda)&=\sum_{i=0}^{\infty}\varphi_1^{[i]}(\lambda-\lambda_1)^i\end{aligned}\tag{104}$$

for the fixed $n$ and $t$, where $\psi_1^{[i]}=\psi_1^{[i]}(n,t)$, $\varphi_1^{[i]}=\varphi_1^{[i]}(n,t)$ and the constant $c$ can be choosing as the form $c=\sum_{i=1}^{\infty}c_i(\lambda-\lambda_1)^i$. The constant vector $(1,(r+\rho)\lambda)^{\mathrm{T}}$ in Eq. (103) is chosen by the fixed form to obtain the solutions with maximum peak by Proposition 5. Even though Proposition 5 just shows the maximum occurs at $(n,t)=(0,0)$ for the single soliton solutions, actually it still work for the high order soliton or multi-solitons due to the Darboux–Bäcklund transformation can be iterated recursively.

Combining Theorems 3 and 4, the $N$th order rational solution for (5) can be represented by the following determinant:

$$w_n^{[N]}=\rho\frac{\det(\mathbf{H})}{\det(\mathbf{M})},\tag{105}$$

where

$$\mathbf{M}=\mathbf{K}_1^{\dagger}\mathbf{C}\mathbf{K}_1+\mathbf{J}_2^{\dagger}\mathbf{C}\mathbf{J}_2,\quad\mathbf{H}=\mathbf{K}_2^{\dagger}\mathbf{C}\mathbf{K}_2+\mathbf{J}_1^{\dagger}\mathbf{C}\mathbf{J}_1+\mathbf{F}^{\dagger}\mathbf{K}_{1,1}^{\dagger}\mathbf{J}_{1,1},$$

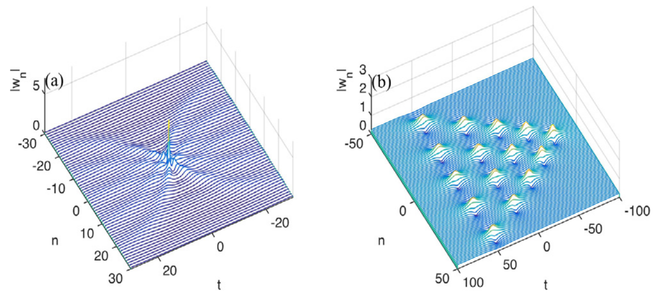


**Fig. 8.** Parameters $N=5$, $\lambda_1=\frac{5}{4}$, $\rho=\frac{9}{40}$, $\theta=0$, $n_0=t_0=0$. (a): 5th order fundamental rogue wave. $c=0$, $\max|w_n|=5.78$, (b): 5th order rogue wave with triangle shape. $c=-300\mathrm{i}\epsilon\ln(\zeta_1(\epsilon))$.

with

$$\mathbf{C}=\left(\frac{1}{(i-1)!(j-1)!}\frac{\mathrm{d}^{i+j-2}}{\mathrm{d}x^{i-1}\mathrm{d}y^{j-1}}\left(\frac{1}{x^2y^2-1}\right)\Big|_{x=\lambda_1,y=\bar{\lambda}_1}\right)_{1\le i,j\le N} \tag{106}$$

$$\mathbf{F}=\frac{\mathbb{I}_N}{\bar{\lambda}_1}-\frac{\mathbf{E}}{(\bar{\lambda}_1)^2}+\frac{\mathbf{E}^2}{(\bar{\lambda}_1)^3}+\cdots+\frac{(-1)^{N-1}\mathbf{E}^{N-1}}{(\bar{\lambda}_1)^N},\quad \mathbf{E}=\left(\delta_{i,j+1}\right)_{1\le i,j\le N}, \tag{107}$$

$$\begin{aligned}\mathbf{J}_1&=\psi_1^{[0]}\mathbb{I}_N+\psi_1^{[1]}\mathbf{E}+\psi_1^{[2]}\mathbf{E}^2+\cdots+\psi_1^{[N-1]}\mathbf{E}^{N-1},\quad \mathbf{J}_2=\lambda_1\mathbf{J}_1+\mathbf{J}_1\mathbf{E},\\ \mathbf{K}_1&=\varphi_1^{[0]}\mathbb{I}_N+\varphi_1^{[1]}\mathbf{E}+\varphi_1^{[2]}\mathbf{E}^2+\cdots+\varphi_1^{[N-1]}\mathbf{E}^{N-1},\quad \mathbf{K}_2=\lambda_1\mathbf{K}_1+\mathbf{K}_1\mathbf{E},\end{aligned} \tag{108}$$

and $\mathbf{K}_{1,1}$ and $\mathbf{J}_{1,1}$ represents the first row of matrices $\mathbf{K}_1$ and $\mathbf{J}_1$ respectively. Then we discuss the dynamic behavior for the solutions (105).

*4.3.1. The first order rational solution*

Based on the formula (105), we have the first order rational lattice solution for the AL equation (5) by setting $N=1$. We take $\rho=\frac{1}{2}\left(p-\frac{1}{p}\right)$, $r=\frac{1}{2}\left(p+\frac{1}{p}\right)$ and $c_0=0$, where $t_0$ and $n_0$ are real constants, $p$ is a real constant. With the aid of Eq. (102), we can obtain the special vector functions

$$\begin{aligned}\psi_1^{[0]}&=(p^2-1)(n-n_0)+\frac{\mathrm{i}}{2p^2}(p^4-1)(\mathrm{e}^{-\mathrm{i}\theta}-p^2\mathrm{e}^{\mathrm{i}\theta})(t-t_0)+1,\\ \varphi_1^{[0]}&=-\left((p^2-1)(n-n_0)+\frac{\mathrm{i}}{2p^2}(p^4-1)(\mathrm{e}^{-\mathrm{i}\theta}-p^2\mathrm{e}^{\mathrm{i}\theta})(t-t_0)-p^2\right).\end{aligned} \tag{109}$$

It follows that the first order rational lattice solution is

$$w_n^{[1]}=\rho\left[-1+\frac{4p^2(p^2+1)^2[p^2-\mathrm{i}(t-t_0)(p^2-1)^2\cos(\theta)]}{4p^6+[(2p^2(n-n_0)+(p^2+1)^2\sin(\theta)(t-t_0))^2+(t-t_0)^2\cos^2(\theta)(p^4-1)^2](p^2-1)^2}\right]. \tag{110}$$

There are two kinds of different dynamical behavior for the first order rational lattice solution:

**Rational lattice soliton of W-shape** If $\theta=\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$, i.e. the modulational stable background, the rational solution (110) is soliton of W-shape (see Fig. 9(a)).

**Rogue wave** If $\theta\neq\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$, i.e. the modulational unstable background, the rational solution (110) is the rogue wave solution (see Fig. 7(b)).

The rational solution (110) for the AL equation (5) is derived by bilinear method [14] and Darboux transformation [16], and their dynamics behavior is also studied. Thus we omit the details to discuss the dynamics behavior.

*4.3.2. The high order rational soliton and rogue wave solutions*

If we take $N>1$, we can obtain the high order rational solution. Under the modulational unstable background, it is rogue wave solution. While under the modulational stable background, it is soliton of W-shape. By choosing special parameters, we exhibit different dynamics in Figs. 8 and 9(b).

The high order rational solutions are found in [14] by Hirota's bilinear method and the highest peak value is obtained from the solution formula directly. We use Remark 2 of Proposition 5 to determine the rational solutions with the highest peak value:

**Proposition 7.** *The maximum peak value $m_i$ of the fundamental ith order rational solution (105) with $c=0$ and $n_0=t_0=0$ has the following recursion relationship:*

$$\begin{aligned}m_1&=\frac{1}{2}\left((r+\rho)^3-(r-\rho)^3\right),\\ m_i&=\frac{1}{2}\left[(r+\rho)^2(\sqrt{1+m_{i-1}^2}+m_{i-1})-(r-\rho)^2(\sqrt{1+m_{i-1}^2}-m_{i-1})\right],\quad i\in\mathbb{Z},\quad i\ge 2.\end{aligned} \tag{111}$$

*The functions $|w_n^{[i]}|$ attain the maximum value at point $(n,t)=(0,0)$.*

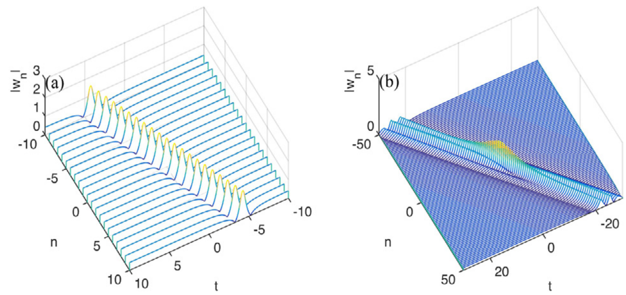


**Fig. 9.** Parameters $\lambda_1 = \frac{5}{4}$, $\rho = \frac{9}{40}$, $\theta = \frac{\pi}{2}$, $n_0 = t_0 = 0$. (a): Rational lattice soliton of W-shape. $N = 1$, $\max|w_n| = 1.78$, (b) Third order lattice soliton of W-shape. $N = 3$, $\max|w_n| = 2.28$.

Through Proposition 7, we find that the peak value of rogue wave to the AL equation (5) is different from the NLSE (1) with $m_i = 2i + 1$. The increasing rate of the peak value to the AL equation (5) with the order $m_i/m_{i-1} \approx (r + \rho)^2$ is much faster than the NLSE (1).

The left panel of Fig. 8 shows the profile of the fifth order fundamental rogue waves. By Proposition 7, we find that the maximum value of lattice rogue wave is 5.78. Meanwhile, this proposition is also satisfied for the high order lattice soliton of W-shape. The right panel in Fig. 9 is the third order lattice soliton of W-shape with the maximum value 2.28 at $(n, t) = (0, 0)$ by Proposition 7.

## 5. Conclusion and discussion

In this paper, we perform the robust inverse scattering analysis for the AL equation (5) on the non-vanishing background. Based on the loop group method, the Darboux transformation is constructed within the framework of robust inverse scattering method. The multi-solitonic solution and high order rogue waves solution are derived by the Bäcklund transformation. Their dynamic behaviors are clarified by the asymptotic analysis.

The classical inverse scattering method for the AL equation, along with exact solutions, (5) are constructed several years ago. Comparing with the previous studies, the formulas of exact solutions obtained in this work are more compact. The interaction between different solitonic solutions are analyzed by asymptotic analysis. The maximum amplitude for peak of solitonic solutions are derived by the Bäcklund transformation. What is more important, under the frame of robust inverse scattering method, we can analyze the infinite order rogue waves for the AL equation similar to the NLSE [22]. We expect to report the results in the near future.

## CRediT authorship contribution statement

**Yiren Chen:** Methodology, Validation, Formal analysis, Funding acquisition. **Bao-Feng Feng:** Conceptualization, Methodology, Validation, Writing - original draft, Writing - review & editing, Funding acquisition. **Liming Ling:** Conceptualization, Methodology, Software, Validation, Formal analysis, Writing - original draft, Writing - review & editing, Funding acquisition.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Proof of Lemmas 1 and 2

**Proof of Lemma 1.** Actually, we merely need to prove the solution $\boldsymbol{\mu}_{n,1}^{+}$ is analytic in $S_{\rm in} \setminus \Omega_{-}$, since other solutions can be proved in a similar manner. Firstly, we introduce the matrix norm

$$\|\mathbf{A}\| = \max(|a_{i,j}|), \ \ \mathbf{A} = (a_{i,j})_{1\leq i\leq m, 1\leq j\leq n}.$$

Using the standard iteration method, a solution for the first equation of summation equation (16) can be written in the form of Neumann series

$$\boldsymbol{\mu}_{n,1}^{+} = \sum_{k=0}^{\infty} \mathbf{g}_n^{(k)},$$

where

$$\mathbf{g}_n^{(0)} = \begin{bmatrix} 1 \\ \xi \end{bmatrix}, \quad \mathbf{g}_n^{(k)} = -\zeta \sum_{l=n}^{+\infty} \mathbf{G}_1(k, n; \lambda) \frac{\mathbf{Q}_k^+}{r} \mathbf{g}_{l+1}^{(k-1)}.$$

We then consider the convergence of above series. We can establish the following estimation:

$$\|\mathbf{G}_1(k, n; \lambda)\| \le C(\epsilon), \quad C(\epsilon) \equiv \frac{(1+r)^2}{\rho^2} \left| \frac{1}{1-\xi^2} \right|.$$

By the lemma A.2 in Ref. [2], we can obtain the following estimation

$$\|\mathbf{g}_n^{(m)}\| \le \frac{1+r}{\rho} \frac{1}{m!} \left( \frac{C(\epsilon)}{r} \sum_{k=n}^{+\infty} \|\mathbf{Q}_k^+\| \right)^m = \frac{1+r}{\rho} \frac{1}{m!} \left( \frac{C(\epsilon)}{r} \sum_{k=n}^{+\infty} \left| w_n \mathrm{e}^{-\mathrm{i}\theta_+} - \rho \right| \right)^m .$$

By above estimation, we deduce that the series is uniformly bounded in $S_{\text{in}} \setminus (\Omega_- \cup B_\epsilon(-(r-\rho)) \cup B_\epsilon(r-\rho))$ and internally closed uniformly convergent in the region $S_{\text{in}} \setminus \Omega_-$. It follows that the solution $\boldsymbol{\mu}_{n,1}^+$ is analytic in the region $S_{\text{in}} \setminus \Omega_-$.

We proceed to the proof for the uniqueness property of solution $\boldsymbol{\mu}_{n,1}^+$ . Suppose we have another solution $\hat{\boldsymbol{\mu}}_{n,1}^+$, then

$$\boldsymbol{\mu}_{n,1}^+ - \hat{\boldsymbol{\mu}}_{n,1}^+ = -\zeta \sum_{k=n}^{+\infty} \mathbf{G}_1(k, n; \lambda) \frac{\mathbf{Q}_k^+}{r} (\boldsymbol{\mu}_{k+1,1}^+ - \hat{\boldsymbol{\mu}}_{k+1,1}^+).$$

It follows that

$$\|\boldsymbol{\mu}_{n,1}^+ - \hat{\boldsymbol{\mu}}_{n,1}^+\| \le \sum_{k=n}^{+\infty} \frac{\|\mathbf{G}_1(k, n; \lambda)\mathbf{Q}_k^+\|}{r} \|\boldsymbol{\mu}_{k+1,1}^+ - \hat{\boldsymbol{\mu}}_{k+1,1}^+\|.$$

Iterating above inequality as above, we establish the following estimate

$$\|\boldsymbol{\mu}_{n,1}^+ - \hat{\boldsymbol{\mu}}_{n,1}^+\| \le \frac{1}{m!} \left( \frac{C(\epsilon)}{r} \sum_{k=n}^{+\infty} \left| w_n \mathrm{e}^{-\mathrm{i}\theta_+} - \rho \right| \right)^m .$$

When $m \to +\infty$, we can deduce that $\|\boldsymbol{\mu}_{n,1}^+ - \hat{\boldsymbol{\mu}}_{n,1}^+\| = 0$. Thus the uniqueness property is proved. □

**Proof of Lemma 2.** Here we merely need to obtain the sharp estimation for $\boldsymbol{\mu}_{n,1}^+$, the other Jost solutions can be proved in a similar way. Firstly, by the following limit

$$\lim_{z \to r-\rho} \frac{\zeta^{2(k-n)} - 1}{1-\xi^2} = -\frac{\rho}{r}(k-n).$$

It follows that there exists a positive $\epsilon$, such that in the region $B_{2\epsilon}(r-\rho)/\Omega_-$ the following estimates hold

$$\left| \xi^2 \frac{\zeta^{2(k-n)} - 1}{1-\xi^2} \right| \le k-n, \quad \left| \xi \frac{\zeta^{2(k-n)} - 1}{1-\xi^2} \right| \le k-n, \quad \left| \frac{\zeta^{2(k-n)} - 1}{1-\xi^2} \right| \le k-n.$$

Then we can obtain the following estimation

$$\left\| \zeta \mathbf{G}_1(k, n; \lambda) \frac{\mathbf{Q}_k^+}{r} \right\| \le \frac{(1+(k-n))}{r} \|\mathbf{Q}_k^+\|.$$

It follows from the iteration of standard Neumann series that

$$\|\boldsymbol{\mu}_{n,1}^+\| \le 1 + \frac{1}{r} \sum_{k=n}^{+\infty} (1+(k-n)) \, \|\mathbf{Q}_k^+\| \|\boldsymbol{\mu}_{k+1,1}^+\|.$$

If $n \ge 0$, we obtain the estimation

$$\|\boldsymbol{\mu}_{n,1}^+\| \le 1 + \frac{1}{r} \sum_{k=n}^{+\infty} (1+k) \|\mathbf{Q}_k^+\| \|\boldsymbol{\mu}_{k+1,1}^+\|.$$

Iterating the above inequality, by the estimation

$$\sum_{k=n}^{+\infty} (1+|k|) \|\mathbf{Q}_k^+\| < +\infty,$$

we obtain that

$$\|\boldsymbol{\mu}_{n,1}^+\| \le \exp(R(n)) < +\infty, \; R(n) = \frac{1}{r} \sum_{k=n}^{+\infty} (1+k) \|\mathbf{Q}_k^+\|. \tag{112}$$

If $n < 0$, then

$$\begin{aligned}\|\boldsymbol{\mu}_{n,1}^{+}\| &\leq 1+\sum_{k=n}^{+\infty}\frac{k}{r}\|\mathbf{Q}_k^{+}\|\|\boldsymbol{\mu}_{k+1,1}^{+}\|+\frac{1}{r}\sum_{k=n}^{+\infty}(1-n)\,\|\mathbf{Q}_k^{+}\|\|\boldsymbol{\mu}_{k+1,1}^{+}\|\\ &\leq 1+\sum_{k=1}^{+\infty}\frac{k}{r}\|\mathbf{Q}_k^{+}\|\|\boldsymbol{\mu}_{k+1,1}^{+}\|+\frac{1}{r}\sum_{k=n}^{+\infty}(1-n)\,\|\mathbf{Q}_k^{+}\|\|\boldsymbol{\mu}_{k+1,1}^{+}\|.\end{aligned}$$

On the other hand, through the estimate (112) in above case $n \geq 0$, we have

$$\|\boldsymbol{\mu}_{k,1}^{+}\| \leq \exp(R(k)).$$

Thus

$$\|\boldsymbol{\mu}_{n,1}^{+}\| \leq K_1+\frac{(1-n)}{r}\sum_{k=n}^{+\infty}\|\mathbf{Q}_k^{+}\|\|\boldsymbol{\mu}_{k+1,1}^{+}\|$$

where

$$K_1 = 1+\frac{\exp(R(1))}{r}\sum_{k=1}^{+\infty}k\|\mathbf{Q}_k^{+}\|.$$

Furthermore, we have

$$\frac{\|\boldsymbol{\mu}_{n,1}^{+}\|}{K_1(1+|n|)} \leq \frac{1}{(1+|n|)}+\sum_{k=n}^{+\infty}\frac{(1+|k+1|)}{r}\|\mathbf{Q}_k^{+}\|\frac{\|\boldsymbol{\mu}_{k+1,1}^{+}\|}{K_1(1+|k+1|)} \leq 1+\sum_{k=n}^{+\infty}\frac{(1+|k+1|)}{r}\|\mathbf{Q}_k^{+}\|\frac{\|\boldsymbol{\mu}_{k+1,1}^{+}\|}{K_1(1+|k+1|)}.$$

Finally, we obtain

$$\|\boldsymbol{\mu}_{n,1}^{+}\| \leq K_1(1+|n|)\exp\left[\frac{1}{r}\sum_{k=n}^{+\infty}(1+|k+1|)\,\|\mathbf{Q}_k^{+}\|\right] < +\infty.$$

Thus the uniformly bounded property for $\boldsymbol{\mu}_{n,1}^{+}$ in the region $B_{2\epsilon}(r-\rho)/\Omega_-$ is proved. Since in the region $B_{2\epsilon}(-r+\rho)\setminus\Omega_-$ the functions $\xi(\lambda)$ and $\zeta(\lambda)$ have the similar structure as $B_{2\epsilon}(r-\rho)\setminus\Omega_-$, the similar estimate can be established. □

## Appendix B. Conservation laws

We use the difference Riccati equation to derive the conservation laws. The conserved quantities for the defocusing AL equation under non-vanishing background was derived by Ablowitz et al. [11] by expanding the analytic function in the neighborhood of $\infty$. Here we expand the analytic function both in the neighborhood of $\infty$ and 0, then the whole list of conserved quantities are obtained.

Firstly, we rewrite the linear spectral problem in the form:

$$\begin{bmatrix}\psi_{n+1}\\ \varphi_{n+1}\end{bmatrix}=\begin{bmatrix}\lambda & w_n\\ -\bar{w}_n & \lambda^{-1}\end{bmatrix}\begin{bmatrix}\psi_n\\ \varphi_n\end{bmatrix},$$

then we have

$$\frac{\varphi_{n+1}}{\psi_{n+1}}=\frac{-\bar{w}_n\psi_n+\lambda^{-1}\varphi_n}{\lambda\psi_n+w_n\varphi_n}.$$

Introducing the notation $A_n=\varphi_n/\psi_n$, then we obtain the difference Riccati equation:

$$A_{n+1}(\lambda+w_nA_n)=-\bar{w}_n+\lambda^{-1}A_n.$$

Moreover, denote $\Pi_n=w_nA_n$, it follows that

$$w_n\Pi_{n+1}(\lambda+\Pi_n)=-w_{n+1}|w_n|^2+\lambda^{-1}w_{n+1}\Pi_n. \tag{113}$$

Assuming

$$\begin{bmatrix}\psi_n(\lambda)\\ \varphi_n(\lambda)\end{bmatrix}=\exp\left(\frac{\mathrm{i}}{2}\theta_-\boldsymbol{\sigma_3}\right)(r\zeta)^n\boldsymbol{\mu}_{n,1}^{-}(\lambda) \tag{114}$$

together with Proposition 3 implies the ansatz

$$\Pi_n=\sum_{i=1}^{+\infty}I_i(n)\lambda^{-2i+1}. \tag{115}$$

Plugging the above ansatz (115) into Eq. (113), comparing the coefficient of $\lambda$, we have

$$\begin{aligned}&I_1(n+1)=-w_{n+1}\bar{w}_n,\\ &I_2(n+1)=-w_{n+1}\bar{w}_{n-1}(1+|w_n|^2),\\ &I_k(n+1)=\frac{w_{n+1}}{w_n}I_{k-1}(n)-\sum_{j=1}^{k-1}I_j(n+1)I_{k-j}(n),\;\; k=3,4,\ldots.\end{aligned}$$

Through Eq. (114), we have

$$\frac{\psi_{n+1}}{\psi_n}=r\zeta e^{g(n+1;\lambda)-g(n;\lambda)}=\lambda+\Pi_n,\quad \psi_n=e^{\frac{i}{2}\theta_-+g(n;\lambda)}(r\zeta)^n,\quad \lim_{n\to-\infty}g(n;\lambda)=0,$$

which induces that

$$g(n+1;\lambda)-g(n;\lambda)=\ln\left(\frac{\lambda+\Pi_n}{r\zeta}\right).$$

Notice that the parameter $\zeta$ can be expanded at infinity:

$$\zeta^{-1}=r\lambda^{-1}+\frac{(\rho^2+r^2)^2-1}{4r}\lambda^{-3}+\mathcal{O}(\lambda^{-5}),\quad \lambda\to\infty \tag{116}$$

which deduces that

$$\ln\left(\frac{\lambda+\Pi_n}{r\zeta}\right)=(I_1(n)+\rho^2)\lambda^{-2}+\left(I_2(n)-\frac{1}{2}I_1^2(n)+\frac{1}{2}\rho^2\left(2r^2+\rho^2\right)\right)\lambda^{-4}+\mathcal{O}(\lambda^{-6}),\quad \lambda\to\infty. \tag{117}$$

As a result, we obtain the conversation laws:

$$\begin{aligned}\lim_{n\to+\infty}g(n;\lambda)&=\sum_{n=-\infty}^{+\infty}[g(n+1;\lambda)-g(n;\lambda)]\\&=\sum_{k=1}^{+\infty}C_k\lambda^{-2k},\quad \lambda\to\infty\end{aligned}$$

where

$$\begin{aligned}C_1&=\sum_{n=-\infty}^{\infty}\left(\rho^2-w_n\bar{w}_{n-1}\right),\\C_2&=\sum_{n=-\infty}^{\infty}\left[-w_n\bar{w}_{n-2}(1+|w_{n-1}|^2)-\frac{1}{2}w_n^2\bar{w}_{n-1}^2+\frac{1}{2}\rho^2\left(2r^2+\rho^2\right)\right],\ \ldots.\end{aligned}$$

Similarly, the assumption

$$\begin{bmatrix}\psi_n\\ \varphi_n\end{bmatrix}=\exp\left(\frac{i}{2}\theta_+\boldsymbol{\sigma_3}\right)(r\zeta)^n\boldsymbol{\mu}^+_{n,1}(\lambda), \tag{118}$$

implies the ansatz

$$\Pi_n=\sum_{i=1}^{+\infty}J_i(n)\lambda^{2i-1}. \tag{119}$$

Inserting the ansatz (119) into Eq. (113), comparing the coefficient of $\lambda$, it follows that

$$\begin{aligned}&J_1(n)=|w_n|^2,\\&J_2(n)=w_n\bar{w}_{n+1}(1+|w_n|^2),\\&J_k(n)=\frac{w_n}{w_{n+1}}\left(J_{k-1}(n+1)+\sum_{j=1}^{k-1}J_j(n+1)J_{k-j}(n)\right),\ k=3,4,\ldots.\end{aligned}$$

In view of Eq. (118), we arrive at

$$\frac{\psi_{n+1}}{\psi_n}=r\zeta e^{h(n+1;\lambda)-h(n;\lambda)}=\lambda+\Pi_n,\quad \psi_n=e^{\frac{i}{2}\theta_++h(n;\lambda)}(r\zeta)^n,\quad \lim_{n\to+\infty}h(n;\zeta)=0,$$

which induces that

$$h(n+1;\lambda)-h(n;\lambda)=\ln\left(\frac{\lambda+\Pi_n}{r\zeta}\right).$$

Together with the expansion at origin

$$\zeta^{-1}=\frac{\lambda^{-1}}{r}-\frac{\rho^2}{r}\lambda+\mathcal{O}(\lambda^3),\quad \lambda\to 0, \tag{120}$$

we have

$$\ln\left(\frac{\lambda+\Pi_n}{r\zeta}\right)=\ln\left(\frac{1+J_1(n)}{r^2}\right)+\left(\frac{J_2(n)}{1+J_1(n)}-\rho^2\right)\lambda^2+\mathcal{O}(\lambda^4),\quad \lambda\to 0. \tag{121}$$

Thus we obtain another sequence of conservation laws

$$\lim_{n\to-\infty} h(n;\lambda) = -\sum_{n=-\infty}^{+\infty}[h(n+1;\lambda)-h(n;\lambda)] = \sum_{k=0}^{+\infty} D_k\lambda^{2k}, \quad \lambda\to 0.$$

Here

$$D_0 = -\sum_{n=-\infty}^{\infty} \ln\left(\frac{1+|w_n|^2}{r^2}\right), \quad D_1 = \sum_{n=-\infty}^{\infty}\left(\rho^2 - w_n\bar{w}_{n+1}\right), \ \dots. \tag{122}$$

The conservation laws are derived by expanding the Jost function in the neighborhood of 0 instead of taking determinant as [11].

## Appendix C. Modulation instability analysis

The AL equation (5) is linearized about the plane wave solution by the substitution of

$$w_n = \rho + Q_n$$

where $Q_n$ is small perturbations which nonlinear effect can be neglected. The linearized equation is

$$\mathrm{i}Q_{n,t} = (1+\rho^2)(Q_{n+1}\mathrm{e}^{\mathrm{i}\theta} + Q_{n-1}\mathrm{e}^{-\mathrm{i}\theta}) - 2(1+\rho^2)\cos(\theta)Q_n + 2\rho^2\cos(\theta)(Q_n+\bar{Q}_n). \tag{123}$$

The stability of the plane wave solution can be determined by the complete basis for the solutions of the linearized equation (123). Solutions to linear equation (123) can be decomposed into a summation of various normal Fourier modes of the form

$$Q_n = f\mathrm{e}^{\mathrm{i}(\beta n-\lambda t)} + \bar{g}\mathrm{e}^{-\mathrm{i}(\beta n-\bar{\lambda}t)}.$$

Substituting the Fourier modes into linearized equation (123), the following linear system is obtained:

$$\begin{bmatrix} \Omega_1 & 2\rho^2\cos(\theta) \\ -2\rho^2\cos(\theta) & \Omega_2 \end{bmatrix}\begin{bmatrix} f \\ g\end{bmatrix} = 0,$$

where

$$\Omega_1 = 2(1+\rho^2)\cos(\theta+\beta) - 2\cos(\theta) - \lambda,$$
$$\Omega_2 = 2\cos(\theta) - 2(1+\rho^2)\cos(\theta-\beta) - \lambda.$$

The linearized dispersion relation is obtained by setting the determinant of above matrix to zero:

$$[\lambda + 2(1+\rho^2)\sin(\theta)\sin(\beta)]^2 + 4\cos^2(\theta)\left[\rho^4 - \left[\left(1+\rho^2\right)\cos(\beta)-1\right]^2\right] = 0.$$

Unstable Fourier modes occur if and only if $\lambda$ is nonreal. Indeed, we merely need to analyze the discriminant of above quadratic equation:

$$\Delta = -16\cos^2(\theta)\left[\rho^4 - \left[\left(1+\rho^2\right)\cos(\beta)-1\right]^2\right].$$

There are two cases: For the first case: $\theta \neq \frac{\pi}{2} + k\pi$, $k\in\mathbb{Z}$, we know that

$$1 - \frac{2}{\rho^2+1} = \frac{\rho^2-1}{\rho^2+1} \leq \cos\beta \leq 1,$$

which infers the modulational instability for arbitrary $\rho$.

For the second case: $\theta = \frac{\pi}{2} + k\pi$, $k\in\mathbb{Z}$, since $\Delta = 0$, the background is modulational stable.

On the other hand, we consider the case of $\beta = 0$ with the limit $\lambda = \beta\mu$ when $\beta\to 0$. It follows that $\Delta = -16\rho(1+\rho^2)\cos^2(\theta) < 0$, thus the background is modulational unstable for the case $\theta \neq \frac{\pi}{2} + k\pi$, $k\in\mathbb{Z}$.

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