



Inverse scattering transform for the complex short-pulse equation by a Riemann–Hilbert approach

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Abstract In this paper, we develop the inverse scattering transform (IST) for the complex short-pulse equation (CSP) on the line with zero boundary conditions at space infinity. The work extends to the complex case the Riemann–Hilbert approach to the IST for the real short-pulse equation proposed by A. Boutet de Monvel, D. Shepelsky and L. Zielinski in 2017. As a byproduct of the IST, soliton solutions are also obtained. Unlike the real SPE, in the complex case discrete eigenvalues are not necessarily restricted to the imaginary axis, and, as consequence, smooth 1-soliton solutions exist for any choice of discrete eigenvalue $k_1 \in \mathbb{C}$ with $\text{Im } k_1 < |\text{Re } k_1|$. The 2-soliton solution is obtained for arbitrary eigenvalues k_1, k_2 , providing also the breather solution of the real SPE in the special case $k_2 = -k_1^*$.

1 Introduction

The short-pulse equation (SPE):

$$v_{xt} = v + \frac{1}{6}(v^3)_{xx} \equiv v + \frac{1}{2}(v^2 v_x)_x \quad (1)$$

was originally introduced as an equation for pseudospherical surfaces [1], and later derived as a model for the propagation of ultra-short optical pulses in nonlinear media, with $v = v(x, t)$ a real-valued function representing the magnitude of the electric field [2–4], and, more recently, as model for short pulses in metamaterials [5]. The real SPE has been the subject of many investigations, and various aspects of the dynamics of solutions have been elucidated: solitons and periodic solutions [6–11], well-posedness of the initial-value problem (IVP) [12, 13], wave breaking conditions [14], as well as integrable hierarchy and bi-Hamiltonian structure [15, 16], and a multicomponent generalization [18]. The real SPE was also recently studied by a Riemann–Hilbert approach. Analysis of the long-time asymptotics provided the basis for the formulation, in spectral terms, of a sufficient condition for wave breaking [19].

In 2015, a complex-valued version of the SPE, i.e., a complex short pulse (CSP) equation, was derived from Maxwell’s equations and studied by one of the authors [20] of this work.

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This CSP:

$$u_{xt} = u + \frac{1}{2}(|u|^2 u_x)_x, \quad (2)$$

which reduces to the SPE when $u(x, t)$ is a real-valued function, i.e., $u(x, t) = u^*(x, t)$, is also a completely integrable system, with a Lax pair

$$\Phi_x = U \Phi = \begin{pmatrix} -ik & ku_x \\ -ku_x^* & ik \end{pmatrix} \Phi, \quad (3a)$$

$$\Phi_t = V \Phi = \begin{pmatrix} \frac{i}{4k} - \frac{i}{2}k|u|^2 & -\frac{i}{2}u + \frac{1}{2}k|u|^2 u_x \\ -\frac{i}{2}u^* - \frac{1}{2}k|u|^2 u_x^* & -\frac{i}{4k} + \frac{i}{2}k|u|^2 \end{pmatrix} \Phi, \quad (3b)$$

such that the zero-curvature condition $U_t - V_x + [U, V] = 0$ (where $[\cdot, \cdot]$ denotes the standard matrix commutator) is satisfied iff u is a solution of the CSP equation (2). Spectral/scattering problems of the form (3a) in the real case were first introduced in [21] for the Wadati–Konno–Ichikawa equation, and considered further in [22–29]. In particular, these latter articles are the guide used to derive soliton solutions of the real SPE equation via Gelfand–Levitan–Marchenko equations in [30].

For future convenience, we write

$$U = -ikU_\infty = -ik\sigma_3 + U_1, \quad U_1 = k \begin{pmatrix} 0 & u_x \\ -u_x^* & 0 \end{pmatrix}, \quad (4)$$

$$V = \frac{i}{4k}\sigma_3 + V_o, \quad V_o = -\frac{i}{2}k|u|^2\sigma_3 + \frac{1}{2}|u|^2 U_1 - \frac{i}{2} \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}. \quad (5)$$

In addition to the CSP equation [20], which pertains to a focusing dispersion regime, there also exists a defocusing CSP equation [31], and both can be viewed as analogues of the focusing/defocusing nonlinear Schrödinger (NLS) equation when dealing with ultra-short pulses, i.e., pulses whose time duration is of the order of a picosecond or less. For the focusing CSP equation, its multi-bright soliton solution was found in pfaffian form in [20], and in determinant form in [32] by combining Hirota's bilinear method and the Kadomtsev–Petviashvili (KP) hierarchy reduction method. The multi-breather and the higher order rogue wave solutions were constructed via Darboux transformations [33]. For the defocusing CSP equation, its multi-dark soliton solution was obtained by the KP hierarchy reduction method [34] and the generalized Darboux transformation method [31], respectively.

Like its real counterpart, in addition to standard soliton solutions, the CSP equation admits loop solitons, which are not single-valued, as well as solutions that “breathe” between single-valued and multi-valued states. Also, the interaction of single-valued solitons can result in a multi-valued solution. The shock (in the spatial derivative) implied by transition from a single-valued to a multi-valued solution is not dependent on a scaling, hence, the “signature” of the shock in the spectral data should be very different than that observed for KdV and NLS. In this work, we extend the Riemann–Hilbert approach to the inverse scattering transform (IST) for the real short-pulse equation recently developed in [19] to the CSP equation. Most of the construction valid for the real SPE can be adapted to the complex case, with suitable modifications. The plan of this work is as follows. In Sect. 2, we formulate the direct problem: The Jost eigenfunctions are defined using gauge transformations which are the analog of those introduced in [19] for the real SPE; time-independent scattering data are introduced in the usual way in terms of the eigenfunctions, and their symmetries are presented. Note that unlike the real SPE, in the complex case the discrete eigenvalues are not necessarily symmetric with respect to the imaginary axis. In Sect. 3, the inverse problem for the eigenfunctions is

formulated as a Riemann–Hilbert problem (RHP), and the reconstruction of the solution of the complex SPE in terms of the eigenfunctions is addressed. As an example, in Sect. 4, the one-soliton solution for the complex SPE is obtained and its properties are discussed. While the one-soliton solution of the real SPE is always a multi-valued function, the complex SPE, for which discrete eigenvalues are not necessarily confined to the imaginary axis, admits smooth envelope 1-soliton solutions for any choice of the discrete eigenvalue $k_1 \in \mathbb{C}$ with $\text{Im } k_1 < |\text{Re } k_1|$. The 2-soliton solution is also obtained for arbitrary eigenvalues k_1, k_2 , which reduces to the breather solution of the real SPE when $k_2 = -k_1^*$. We also analyze the long-time asymptotics of the two-soliton solution in the direction of each soliton, showing that it reduces, as expected, to single soliton solutions, and compute the shift in position and phase due to the soliton interactions in terms of the soliton eigenvalues. Section 5 is devoted to some concluding remarks.

2 Direct problem

2.1 Lax pairs and eigenfunctions

The matrices U and V in the Lax pair (3) are traceless, which implies that the determinant of a square matrix solution to (3) is independent of x and t . Also, since the Lax matrices U, V have singularities in the complex k -plane at $k = 0$ and $k = \infty$, in order to control the behavior of the eigenfunctions at these singular points it is convenient to perform suitable gauge transformations, like the ones introduced for the real SPE in [19], which have also successfully being used for other systems, including the Camassa–Holm equation, the Degasperis–Procesi equation, and the short-wave equation [35–41].

First gauge transformation We consider a transformation

$$\hat{\Phi}(x, t, k) = P(x, t) \Phi(x, t, k), \quad (6)$$

which converts the Lax pair (3) into the form:

$$\hat{\Phi}_x + Q_x \hat{\Phi} = \hat{U} \hat{\Phi}, \quad (7a)$$

$$\hat{\Phi}_t + Q_t \hat{\Phi} = \hat{V} \hat{\Phi}. \quad (7b)$$

As in [19], we choose $P(x, t)$ so that (i) Q is diagonal while (ii) $\hat{U}$ and $\hat{V}$ vanish as $x \rightarrow \pm\infty$, and (iii) remain bounded as the spectral parameter $k \rightarrow \infty$. In order to accomplish this, we choose P to be the eigenvector matrix which diagonalizes U_∞ in (4).

The eigenvalues of U_∞ are $\pm q = \pm\sqrt{1 + |u_x|^2}$, and one can define the eigenvector matrix P such that

$$P = p \begin{pmatrix} 1 & -\alpha \\ \alpha^* & 1 \end{pmatrix}, \quad P^{-1} = p \begin{pmatrix} 1 & \alpha \\ -\alpha^* & 1 \end{pmatrix}, \quad (8)$$

with

$$p = \sqrt{\frac{1+q}{2q}}, \quad q = \sqrt{1 + |u_x|^2}, \quad \alpha = \frac{-iu_x}{1+q},$$

satisfies

$$P U_\infty P^{-1} = q \sigma_3.$$

Then, with $\hat{\Phi} = P \Phi$ (as in (6)), and P as defined in (8), Eq. (3a) transforms to (7a), with

$$\hat{U} = P_x P^{-1}, \quad \text{and} \quad Q_x = ikq\sigma_3. \quad (9)$$

Moreover,

$$\hat{U} = \frac{i}{2q^2(1+q)} \begin{pmatrix} q \operatorname{Im}(u_{xx}u_x^*) & q(q+1)u_{xx} - u_x \operatorname{Re}(u_{xx}u_x^*) \\ q(q+1)u_{xx}^* - u_x^* \operatorname{Re}(u_{xx}u_x^*) & -q \operatorname{Im}(u_{xx}u_x^*) \end{pmatrix}, \quad (10)$$

which is traceless.

Letting

$$\hat{\Phi}_t = \check{V} \hat{\Phi}, \quad \check{V} = PV P^{-1} + P_t P^{-1} = \hat{V} - Q_t,$$

and

$$Q_t = \left(\frac{1}{4ik} + \frac{i}{2} k |u|^2 q \right) \sigma_3, \quad (11)$$

we obtain (7b) with

$$\hat{V} = \frac{i}{4kq} \begin{pmatrix} 1-q & -iu_x \\ iu_x^* & q-1 \end{pmatrix} + \frac{i}{4q(1+q)} \begin{pmatrix} |u|^2 \operatorname{Im}(u_{xx}u_x^*) + 2q \operatorname{Im}(u_x u^*) & \frac{|u|^2}{2q} \left((q+1)^2 u_{xx} - u_{xx}^* u_x^2 \right) \\ \frac{|u|^2}{2q} \left((q+1)^2 u_{xx}^* - u_{xx} (u_x^*)^2 \right) & -|u|^2 \operatorname{Im}(u_{xx}u_x^*) - 2q \operatorname{Im}(u_x u^*) \end{pmatrix}, \quad (12)$$

which is also traceless.

From (9) and (11), we then define

$$Q = i\theta\sigma_3, \quad \theta = k\xi - \frac{t}{4k}, \quad (13)$$

where

$$\xi = x - \int_x^\infty \left(\sqrt{1 + |u_y|^2} - 1 \right) dy. \quad (14)$$

The definition (13) guarantees that $(Q_x)_t = (Q_t)_x$ iff the complex SPE is satisfied. The Lax-pair Eqs. (7) are then satisfied for $\hat{U}$, $\hat{V}$ as defined, respectively, by (10) and (12).

The travel-time parameter ξ , defined in (14), was used in the development of the scattering theory for the Schrödinger equation of the form: $-u_{xx} + Qu = k^2 Hu$, where $H(x)$ is a positive function, possibly with jump discontinuities, and positive limits as $x \rightarrow \pm\infty$ (cf. [42]). The travel-time parameter was also employed in the development of the scattering theory for the space-derivative operator in the Lax pair $[\varphi_x = U\varphi]$ for the Camassa–Holm equation [35]. The map (14) that defines the travel-time parameter is well-defined and single-valued as long as u is differentiable with respect to x and $|u_x| \rightarrow 0$ sufficiently rapidly as $x \rightarrow \pm\infty$.

Modified eigenfunctions In the following analysis, it is convenient to consider eigenfunctions of the Lax pair with constant boundary conditions as $x \rightarrow \pm\infty$. Hence, we introduce:

$$M(x, t, k) = \hat{\Phi}(x, t, k) G(x, t, k), \quad \text{where} \quad G(x, t, k) = e^{Q(x, t, k)}. \quad (15)$$

Because Q is traceless, we have $[\det G]_x = [\det G]_t = 0$, and hence $\det G = 1$. Then

$$M_x + [Q_x, M] = \hat{U}M, \quad (16a)$$

$$M_t + [Q_t, M] = \hat{V}M. \quad (16b)$$

Now define the *modified Jost solutions* $M_{\pm}(x, t, k)$ as the unique solutions of the Volterra integral equations

$$M_{\pm}(x, t, k) = I_2 \mp \int_{\pm\infty}^x dy e^{ik \left(\int_x^y dz \sqrt{1+|u_z|^2} \right) \sigma_3} \hat{U}(y, t) M_{\pm}(y, t, k) e^{-ik \left(\int_x^y dz \sqrt{1+|u_z|^2} \right) \sigma_3}. \quad (17)$$

Then, $M_{\pm}$ are solutions of (16a) such that $\lim_{x \rightarrow \pm\infty} M_{\pm}(x, t, k) = I_2$.

Assuming that, for each $t \in \mathbb{R}$, $u_x, u_{xx} \in L^1(\mathbb{R})$, and denoting the columns of the solution $M_{\pm}$ as $M_{\pm,j}$ for $j = 1, 2$, and the open upper-half (lower-half) complex plane by $\mathbb{C}^+$ ($\mathbb{C}^-$), one can prove the following:

1. $\det M_{\pm}(x, t, k) = 1$;
2. $M_{-,1}(x, t, k)$ and $M_{+,2}(x, t, k)$ are continuous for $k \in \mathbb{C}^+ \cup \mathbb{R}$, analytic for $k \in \mathbb{C}^+$, and tend to I_2 as $k \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$;
3. $M_{+,1}(x, t, k)$ and $M_{-,2}(x, t, k)$ are continuous for $k \in \mathbb{C}^- \cup \mathbb{R}$, analytic for $k \in \mathbb{C}^-$, and tend to I_2 as $k \rightarrow \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$.

First, note that the above assumptions on the potential imply that $\|\hat{U}(\cdot, t)\| \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$. Also note that $\hat{U}$ does not depend on k . The proof of statements 1–3 then follows by applying Gronwall's inequality to the integral Eq. (17) for the columns of the Jost functions, provided $\frac{dk}{dx} > 0$.

2.2 Scattering data and symmetries

Scattering matrix We introduce the *scattering matrix*

$$S(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}$$

as the solution of

$$M_{-}(x, t, k) e^{-i\theta(x,t,k)\sigma_3} = M_{+}(x, t, k) e^{-i\theta(x,t,k)\sigma_3} S(k). \quad (18)$$

Then, for $k \in \mathbb{R}$,

$$a(k) = \det(M_{-,1}(x, t, k), M_{+,2}(x, t, k)), \quad (19a)$$

$$\bar{a}(k) = \det(M_{+,1}(x, t, k), M_{-,2}(x, t, k)), \quad (19b)$$

$$b(k) = e^{-2i\theta(x,t,k)} \det(M_{+,1}(x, t, k), M_{-,1}(x, t, k)), \quad (19c)$$

$$\bar{b}(k) = e^{2i\theta(x,t,k)} \det(M_{+,2}(x, t, k), M_{-,2}(x, t, k)), \quad (19d)$$

where $b(k)$ and $\bar{b}(k)$ are continuous for $k \in \mathbb{R}$ and vanish as $k \rightarrow \pm\infty$. From the above representation it follows that the scattering coefficient $a(k)$ can be extended continuously to $k \in \mathbb{C}^+ \cup \mathbb{R}$, it is analytic for $k \in \mathbb{C}^+$, and tends to 1 as $k \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$.

Analogously, $\bar{a}(k)$ can be extended continuously to $k \in \mathbb{C}^- \cup \mathbb{R}$, it is analytic for $k \in \mathbb{C}^-$, and tends to 1 as $k \rightarrow \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$.

Symmetry Using the integral Eqs. (17) one can verify that the modified Jost solutions satisfy the symmetry relation:

$$M_{\pm}^*(x, t, k^*) = \sigma_2 M_{\pm}(x, t, k) \sigma_2 \quad k \in \mathbb{R}. \quad (20)$$

[In the case of the real SPE, in addition to (20), the modified Jost solutions are also such that:

$$M_{\pm}^*(x, t, k^*) = M_{\pm}(x, t, -k).$$

However, this second symmetry does not necessarily hold for the complex SPE, and this has far-reaching consequences regarding the spectrum of the scattering operator and the corresponding envelope soliton solutions, as elucidated below.]

As a consequence of the symmetry relation (20) for the modified Jost solutions, the scattering matrix satisfies:

$$S^*(k) = \sigma_2 S(k) \sigma_2, \quad k \in \mathbb{R},$$

which implies that

$$\bar{a}^*(k^*) = a(k), \quad k \in \mathbb{C}^+ \cup \mathbb{R}, \quad (21a)$$

$$\bar{b}(k) = -b(k), \quad k \in \mathbb{R}, \quad (21b)$$

where we have extended the first symmetry relation to $\mathbb{C}^+$ by virtue of Schwarz reflection principle.

Reflection coefficients and norming constants In this work, we make a genericity assumption that: (i) the discrete eigenvalues are finite in number and algebraically simple; and (ii) there are no spectral singularities. In other words, $a(k)$ is assumed to have only simple zeros in $\mathbb{C}^+$, and no real zeros. By virtue of the conjugation symmetry, $\bar{a}(k)$ then has only simple zeros in $\mathbb{C}^-$, and does not have any real zeros. We denote the zeros of $a(k)$ in $\mathbb{C}^+$ by $k_1, \dots, k_N$, and the zeros of $\bar{a}(k)$ in $\mathbb{C}^-$ by $k_1^*, \dots, k_N^*$. Under the same conditions, we define the *reflection coefficients*:

$$\rho(k) = b(k)/a(k), \quad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k),$$

which are continuous functions of $k \in \mathbb{R}$ that vanish as $k \rightarrow \pm\infty$, and (by (21)) satisfy the symmetry

$$\bar{\rho}^*(k) = -\rho(k), \quad k \in \mathbb{R}.$$

Let us denote by τ_j the residue of $1/a(k)$ at the simple pole k_j , and by $\bar{\tau}_j$ the residue of $1/\bar{a}(k)$ at k_j^* . Then, the proportionality of $e^{-i\theta(x,t,k_j)} M_{-,1}(x, t, k_j)$ and $e^{i\theta(x,t,k_j)} M_{+,2}(x, t, k_j)$ allows us to define the *norming constants* as the solutions of

$$\tau_j e^{-i\theta(x,t,k_j)} M_{-,1}(x, t, k_j) = C_j e^{i\theta(x,t,k_j)} M_{+,2}(x, t, k_j). \quad (22a)$$

Similarly, the proportionality of $e^{i\theta(x,t,k_j^*)} M_{-,2}(x, t, k_j^*)$ and $e^{-i\theta(x,t,k_j^*)} M_{+,1}(x, t, k_j^*)$ allows us to define $\bar{C}_j$ as the solutions of

$$\bar{\tau}_j e^{i\theta(x,t,k_j^*)} M_{-,2}(x, t, k_j^*) = \bar{C}_j e^{-i\theta(x,t,k_j^*)} M_{+,1}(x, t, k_j^*). \quad (22b)$$

From the symmetry relations (20) and (21) it follows that:

$$\bar{\tau}_j = \tau_j^*, \quad \bar{C}_j = -C_j^* \quad j = 1, \dots, N. \quad (23)$$

3 Inverse problem

3.1 Riemann–Hilbert problem formulation

The RHP formulation for the solution of integrable systems makes use of the fact that solutions of the linear equations that constitute the Lax pair are sectionally analytic (or sectionally meromorphic) functions of the spectral parameter considered as a complex variable. First, we show the necessity of parameterization by the travel-time parameter for solution of the RHP at times other than the time for which the solution is given (e.g., for $t > 0$ with given initial data at $t = 0$). Then, we show how to solve the problem with the travel-time parameter. Finally, we show how to recover the solution $u(x, t)$ from the time-dependent solution of the RHP.

3.1.1 RHP with the original variables (x, t)

The analyticity properties of the modified Jost functions $M_{\pm}(x, t, k)$ and of the scattering coefficients $a(k)$, $\bar{a}(k)$ allow us to rewrite the scattering relation (18) as a jump condition for the piecewise, meromorphic matrix function of k defined as:

$$\mu(x, t, k) = \begin{cases} (M_{-,1}(x, t, k)/a(k) \ M_{+,2}(x, t, k)) & k \in \mathbb{C}^+ \\ (M_{+,1}(x, t, k) \ M_{-,2}(x, t, k)/\bar{a}(k)) & k \in \mathbb{C}^- \end{cases}. \quad (24)$$

In terms of the limiting values of $\mu(x, t, k)$ as k approaches the real axis from $\mathbb{C}^{\pm}$, the scattering relation (18) can be written as

$$\mu_+(x, t, k) = \mu_-(x, t, k) \left[I_2 - e^{-i\theta(x,t,k)\sigma_3} J(k) e^{i\theta(x,t,k)\sigma_3} \right], \quad (25)$$

where the “jump matrix”, $J(k)$, defined for $k \in \mathbb{R}$, is given by

$$J(k) = \begin{pmatrix} \rho(k)\bar{\rho}(k) & \bar{\rho}(k) \\ -\rho(k) & 0 \end{pmatrix}. \quad (26)$$

Our assumptions about the meromorphic structure of the scattering data are equivalent to

$$1/a(k) = A(k) + \sum_{j=1}^{\mathcal{N}} \frac{\tau_j}{k - k_j}, \quad 1/\bar{a}(k) = \bar{A}(k) + \sum_{j=1}^{\mathcal{N}} \frac{\bar{\tau}_j}{k - k_j^*},$$

where: $A(k)$ is continuous for $k \in \mathbb{C}^+ \cup \mathbb{R}$, analytic for $k \in \mathbb{C}^+$, and tends to 1 as $k \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$; similarly, $\bar{A}(k)$ is continuous for $k \in \mathbb{C}^- \cup \mathbb{R}$, analytic for $k \in \mathbb{C}^-$, and tends to 1 as $k \rightarrow \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$. Then (18) implies that, for $k \in \mathbb{R}$,

$$\begin{aligned} M_{+,1}(x, t, k) &= M_{-,1}(x, t, k)A(k) + \sum_{j=1}^{\mathcal{N}} \frac{M_{-,1}(x, t, k) - M_{-,1}(x, t, k_j)}{k - k_j} \\ &\quad + \sum_{j=1}^{\mathcal{N}} \frac{C_j e^{2i\theta(x,t,k_j)} M_{+,2}(x, t, k_j)}{k - k_j} - e^{2i\theta(x,t,k)} M_{+,2}(x, t, k) \rho(k), \end{aligned} \quad (27a)$$

and

$$M_{+,2}(x, t, k) = M_{-,2}(x, t, k) \bar{A}(k) + \sum_{j=1}^{\mathcal{N}} \frac{M_{-,2}(x, t, k) - M_{-,2}(x, t, k_j^*)}{k - k_j^*} \\ + \sum_{j=1}^{\mathcal{N}} \frac{\bar{C}_j e^{-2i\theta(x, t, k_j^*)} M_{+,1}(x, t, k_j^*)}{k - k_j^*} - e^{-2i\theta(x, t, k)} M_{+,1}(x, t, k) \bar{\rho}(k), \quad (27b)$$

where we have used the residue conditions (22).

To extend (27) to the complex half-planes, we employ the Cauchy projectors

$$\mathcal{P}_{\pm}[f](k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - (k \pm i0)} d\zeta, \quad (28)$$

where the notation $k \pm i0$ indicates that, when $k \in \mathbb{R}$, the limit is taken from within $\mathbb{C}^{\pm}$. Plemelj's formulas describe the action of a Cauchy projector on functions analytic in a half plane: if $f_{\pm}$ are, respectively, analytic in $\mathbb{C}^{\pm}$, and decay $O(1/k)$ as $k \rightarrow \infty$, then $\mathcal{P}_{\pm}[f_{\pm}] = \pm f_{\pm}$; conversely, $\mathcal{P}_{+}[f_{-}] = 0$, and $\mathcal{P}_{-}[f_{+}] = 0$. Applying $\mathcal{P}_{+}$ to the jump condition (27a) and $\mathcal{P}_{-}$ to (27b), we obtain

$$M_{+,1}(x, t, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{\mathcal{N}} \frac{e^{2i\theta(x, t, k_j)} M_{+,2}(x, t, k_j) C_j}{k - k_j} \\ + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \frac{e^{2i\theta(x, t, \zeta)} M_{+,2}(x, t, \zeta) \rho(\zeta)}{\zeta - (k - i0^{+})}, \quad (29a)$$

$$M_{+,2}(x, t, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\mathcal{N}} \frac{e^{-2i\theta(x, t, k_j^*)} M_{+,1}(x, t, k_j^*) \bar{C}_j}{k - k_j^*} \\ - \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \frac{e^{-2i\theta(x, t, \zeta)} M_{+,1}(x, t, \zeta) \bar{\rho}(\zeta)}{\zeta - (k + i0^{-})}. \quad (29b)$$

To close the system (29), one would substitute $k = k_{\ell}^*$ in (29a) and $k = k_{\ell}$ in (29b) for $\ell = 1, 2, \dots, \mathcal{N}$ to obtain a $2\mathcal{N} \times 2\mathcal{N}$ linear algebraic-integral system. However, this system depends on θ , which, in turn, depends on $u(x, t)$ (cf. (13), (14)). Hence, in this formulation, the Riemann–Hilbert problem cannot be used to reconstruct the solution, $u(x, t)$ for some $t > 0$ based only on scattering data derived from $u(x, 0)$. To resolve the circular dependence, we reformulate the RHP in terms of the travel-time parameter.

3.1.2 RHP with the travel-time parameter

In this section, we show that a modified RHP which depends parametrically on ξ , instead of x , is properly posed. Introducing

$$\hat{\mu}(\xi, t, k) = \mu(x(\xi, t), t, k), \quad (30)$$

we rewrite the jump condition (25) as

$$\hat{\mu}_{+}(\xi, t, k) = \hat{\mu}_{-}(\xi, t, k) [I_2 - J(\xi, t, k)] \quad k \in \mathbb{R}, \quad (31)$$

with jump matrix

$$\hat{J}(\xi, t, k) = e^{-i\hat{\theta}(\xi, t, k)\sigma_3} J(k) e^{i\hat{\theta}(\xi, t, k)\sigma_3}, \quad (32)$$

where $J(k)$ is given by (26) and

$$\hat{\theta}(\xi, t, k) = k\xi - \frac{t}{4k}. \quad (33)$$

The difference between (33) and (13) is that, in (33), we do not assume a mapping between ξ and x (in either direction). Similarly, the sectionally meromorphic function $\hat{\mu}(\xi, t, k)$ depends explicitly on ξ without any assumption on a mapping to or from x .

Applying the Cauchy projectors (28) to the jump condition (31), with the assumption that all poles are simple, we find:

$$\begin{aligned} \hat{\mu}(\xi, t, k) = I_2 + \sum_{j=1}^{\mathcal{N}} \frac{\text{Res}_{k=k_j} \hat{\mu}_+(\xi, t, k)}{k - k_j} + \sum_{j=1}^{\mathcal{N}} \frac{\text{Res}_{k=k_j^*} \hat{\mu}_-(\xi, t, k)}{k - k_j^*} \\ + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{\mu}_-(\xi, t, \zeta) J(\xi, t, \zeta)}{\zeta - k} d\zeta, \end{aligned}$$

for all $k \in \mathbb{C}^\pm$. The expressions for $\hat{\mu}_+(\xi, t, k)$ and $\hat{\mu}_-(\xi, t, k)$ are formally identical, except for the fact that the integral appearing on the right-hand side is a $\mathcal{P}_+$ or a $\mathcal{P}_-$ projector, respectively. This formal solution of the RHP of course assumes that the integral in the RHS is well-defined. The system is then closed using the residue conditions (22) for the modified eigenfunctions, which can be written in terms of the meromorphic matrix $\hat{\mu}(\xi, t, k)$ as follows:

$$\text{Res}_{k=k_j} \hat{\mu}_1(\xi, t, k) = e^{2i\hat{\theta}(\xi, t, k_j)} \hat{\mu}_2(\xi, t, k_j) C_j \quad j = 1, \dots, \mathcal{N}, \quad (34a)$$

$$\text{Res}_{k=k_j^*} \hat{\mu}_2(\xi, t, k) = e^{-2i\hat{\theta}(\xi, t, k_j^*)} \hat{\mu}_1(\xi, t, k_j^*) \bar{C}_j \quad j = 1, \dots, \mathcal{N}, \quad (34b)$$

where the subscript $j = 1, 2$, in $\hat{\mu}(\xi, t, k)$ identifies its j -th column. Finally, note that the symmetries of eigenfunctions and scattering data (20) and (21) imply

$$\hat{\mu}^*(\xi, t, k^*) = \sigma_2 \hat{\mu}(\xi, t, k) \sigma_2. \quad (35)$$

3.2 From the solution of the RHP to the solution of the complex SPE

The last step in the inverse problem is to recover the solution of the complex SPE from the solution $\hat{\mu}(\xi, t, k)$ of the RHP, and specifically from its value at $k = 0$. In order to establish the behavior of $\hat{\mu}(\xi, t, k)$ as $k \rightarrow 0$, it is convenient to write the Lax pair (3) as

$$\Phi_x + ik\sigma_3\Phi = U_1\Phi, \quad \Phi_t + \frac{1}{4ik}\sigma_3\Phi = V_o\Phi, \quad (36)$$

where U_1 and V_o are as in (4), and hence vanish as $|x| \rightarrow \infty$, while $U_1(x, t, 0)$ is the zero matrix.

We introduce

$$\begin{aligned} \theta_o(x, t, k) = kx - \frac{t}{4k}, \quad Q_o(x, t, k) = i\theta_o(x, t, k)\sigma_3, \\ M_o(x, t, k) = \Phi(x, t, k) e^{Q_o(x, t, k)}. \end{aligned} \quad (37)$$

Then (36) becomes

$$\partial_x M_o(x, t, k) + [\partial_x Q_o, M_o] = U_1 M_o, \quad \partial_t M_o(x, t, k) + [\partial_t Q_o, M_o] = V_o M_o.$$

The modified eigenfunctions $M_{o\pm}(x, t, k)$ of the above Lax pair are solutions of the associated Volterra integral equations:

$$M_{o\pm}(x, t, k) = I_2 + \int_{\pm\infty}^x e^{ik(y-x)\sigma_3} U_1(y, t, k) M_{o\pm}(y, t, k) e^{ik(x-y)\sigma_3} dy.$$

Because $U_1(x, t, 0)$ is the zero matrix, we have

$$M_{o\pm}(x, t, 0) \equiv I_2 \quad \forall x, t \in \mathbb{R}.$$

Moreover,

$$M_{o\pm}(x, t, k) = I_2 + k \begin{pmatrix} 0 & u(x, t) \\ -u^*(x, t) & 0 \end{pmatrix} + O(k^2), \quad (38)$$

as $k \rightarrow 0$.

Because $M_{o\pm}$ and $M_{\pm}$ are, respectively, the fundamental solutions of a system of ODEs that are related one another by gauge transformations [cf. Eqs. (6), (15) and (37)], there exist matrices $B_{\pm}(k)$, independent of x, t such that:

$$M_{\pm}(x, t, k) = P(x, t) M_{o\pm}(x, t, k) e^{-Q_o(x, t, k)} B_{\pm}(k) e^{Q(x, t, k)}. \quad (39)$$

Evaluating these relationships in the limit $x \rightarrow \pm\infty$ we obtain

$$B_+(k) = I_2, \quad B_-(k) = e^{ik\beta} \sigma_3, \quad \beta = \int_{-\infty}^{+\infty} (\sqrt{1 + |u_x|^2} - 1) dx. \quad (40)$$

Note that β is a constant of the motion for any solution u of the complex SPE. Taking into account (38) and (40), from (39) one then obtains the following asymptotics for $M_{\pm}(x, t, k)$ as $k \rightarrow 0$:

$$M_+(x, t, k) = P(x, t) \left(I_2 + k \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} - ik \int_x^{+\infty} \left(\sqrt{1 + |u_y|^2} - 1 \right) \sigma_3 dy + O(k^2) \right), \quad (41a)$$

$$M_-(x, t, k) = P(x, t) \left(I_2 + k \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} + ik \int_{-\infty}^x \left(\sqrt{1 + |u_y|^2} - 1 \right) \sigma_3 dy + O(k^2) \right). \quad (41b)$$

In turn, the above asymptotics for the eigenfunctions imply that

$$\begin{aligned} S(k) &= e^{Q(x, t, k)} M_+^{-1}(x, t, k) M_-(x, t, k) e^{-Q(x, t, k)} \\ &= e^{Q(x, t, k)} (I_2 + ik\beta\sigma_3 + O(k^2)) e^{-Q(x, t, k)} = I_2 + ik\beta\sigma_3 + O(k^2), \end{aligned}$$

and, consequently,

$$a(k) = 1 + ik\beta + O(k^2), \quad b(k) = O(k^2), \quad (42)$$

as $k \rightarrow 0$. Finally, substituting (41a) and (42) into (24) and taking into account (14), we obtain

$$\mu(x, t, k) = P(x, t) \left(I_2 - ik \begin{pmatrix} x - \xi & iu \\ -iu^* & \xi - x \end{pmatrix} + O(k^2) \right).$$

Now notice that $\mu(x, t, 0) = P(x, t)$, and $\hat{\mu}(\xi, t, k) = \mu(x(\xi), t, k)$ so that

$$\lim_{k \rightarrow 0} \frac{i}{k} [\hat{\mu}^{-1}(\xi, t, 0) \hat{\mu}(\xi, t, k) - I_2] = \begin{pmatrix} x - \xi & i\hat{u}(\xi, t) \\ -i\hat{u}^*(\xi, t) & \xi - x \end{pmatrix}. \quad (43)$$

As a consequence, from the solution of the RHP for $\hat{\mu}(\xi, t, k)$ one obtains a parametric representation of the solution $u(x, t)$ of the Cauchy problem for the complex SPE given by:

$$u = \lim_{k \rightarrow 0} \frac{1}{k} (\hat{\mu}^{-1}(\xi, t, 0) \hat{\mu}(\xi, t, k))_{1,2}. \quad (44a)$$

$$x = \xi + \lim_{k \rightarrow 0} \frac{i}{k} [(\hat{\mu}^{-1}(\xi, t, 0) \hat{\mu}(\xi, t, k))_{1,1} - 1], \quad (44b)$$

where we have made use of the identification $u(x, t) = \hat{u}(\xi, t)$. Note that u is not necessarily single-valued as a function of x as this depends on (44b) being one-to-one.

4 Soliton solutions

In this section, we compute the one-soliton and two-soliton solutions of the complex SPE from the RHP. As usual, solving the RHP amounts to solving a linear system of algebraic integral equations for the eigenfunctions corresponding to an assigned set of scattering data (reflection coefficient $\rho(k)$, discrete eigenvalues $\{k_j\}_{j=1}^{\mathcal{N}}$ and associated norming constants $\{C_j\}_{j=1}^{\mathcal{N}}$) determined by the initial datum $u(x, 0)$ of the complex SPE. The solution $u(x, t)$ at any time $t > 0$ is then recovered in terms of the value of the eigenfunctions at $k = 0$. Solitons are reflectionless solutions, and they can be obtained from the inverse problem by imposing that $\rho(k) \equiv 0$ for all $k \in \mathbb{R}$. In this case, the solution of the RHP reduces to solving a system of $\mathcal{N}$ algebraic equations, with $\mathcal{N}$ being the number of solitons.

4.1 One-soliton solution

As an example, in a straightforward way, we compute the one-soliton solution of the complex SPE from the RHP (31). We assume the spectrum comprises one discrete eigenvalue $k_1 = \eta_1 + i\nu_1 \in \mathbb{C}^+$ and its complex conjugate $k_1^* \in \mathbb{C}^-$, with associated norming constants $C_1 \in \mathbb{C}$ and $\bar{C}_1 = -C_1^*$ (cf. (23)) and no continuous spectrum ($\rho(k) \equiv 0$ for all $k \in \mathbb{R}$, which implies that the jump matrix J in (32) is identically zero). Hence, the solution of the RHP has the form

$$\hat{\mu}(\xi, t, k) = \begin{pmatrix} \frac{k-b_{11}}{k-k_1} & \frac{b_{21}}{k-k_1^*} \\ \frac{b_{12}}{k-k_1} & \frac{k-b_{22}}{k-k_1^*} \end{pmatrix},$$

where b_{ij} are functions of ξ, t . The symmetry (35) implies that

$$b_{11} = b_{22}^*, \quad b_{21} = -b_{12}^* \quad (45)$$

and, therefore,

$$\hat{\mu}(\xi, t, k) = \begin{pmatrix} \frac{b_{11}}{k_1} & \frac{b_{12}^*}{k_1^*} \\ -\frac{b_{12}}{k_1} & \frac{b_{11}}{k_1^*} \end{pmatrix} + k \begin{pmatrix} \frac{b_{11}-k_1}{k_1^2} & \frac{b_{12}^*}{(k_1^*)^2} \\ -\frac{b_{12}}{k_1^2} & \frac{b_{11}-k_1^*}{(k_1^*)^2} \end{pmatrix} + O(k^2),$$

which leads to

$$\hat{\mu}^{-1}(\xi, t, 0)\hat{\mu}(\xi, t, k) = I_2 + k \begin{pmatrix} \frac{1}{k_1} - \frac{b_{11}^*}{|b_{11}|^2 + |b_{12}|^2} & \frac{k_1}{k_1^*} \frac{b_{12}^*}{|b_{11}|^2 + |b_{12}|^2} \\ -\frac{k_1^*}{|b_{11}|^2 + |b_{12}|^2} & \frac{1}{k_1^*} - \frac{b_{11}}{|b_{11}|^2 + |b_{12}|^2} \end{pmatrix} + O(k^2). \quad (46)$$

To recover the solution of the CSP, we employ (44), which yields a solution in terms of b_{11} and b_{12} .

To determine the explicit dependence of b_{ij} on ξ, t we use the residue conditions (34) (with (45)) to obtain the relations

$$b_{11} = k_1 + e^{2i\hat{\theta}_1} \frac{b_{12}^*}{2iv_1} C_1, \quad b_{12} = e^{2i\hat{\theta}_1} \frac{k_1 - b_{11}^*}{2iv_1} C_1, \quad (47)$$

where $\hat{\theta}_1 = \hat{\theta}(\xi, t, k_1)$ [cf. (33)]. Hence,

$$b_{11} = \frac{4v_1^2 k_1 + |C_1|^2 k_1^* e^{-2v_1 \zeta_1}}{4v_1^2 + |C_1|^2 e^{-2v_1 \zeta_1}}, \quad b_{12} = \frac{4v_1^2 C_1 e^{-v_1 \zeta_1}}{4v_1^2 + |C_1|^2 e^{-2v_1 \zeta_1}} e^{i\phi_1}, \quad (48)$$

where

$$\zeta_1 = 2\xi + t/2|k_1|^2, \quad \phi_1 = \eta_1(2\xi - t/2|k_1|^2).$$

Combining (48) with (46), and employing (44), we obtain the following parametric form for the one-soliton solution:

$$u = \frac{v_1}{|k_1|^2} e^{-i(\phi_1 - \bar{\phi}_1)} \operatorname{sech}[\nu_1(\zeta_1 - x_1)], \quad (49a)$$

$$x = \xi + \frac{2v_1}{|k_1|^2} \frac{1}{1 + e^{2v_1(\zeta_1 - x_1)}}, \quad (49b)$$

where the phase variables, ζ_1 and ϕ_1 , are as above, and

$$x_1 = \frac{1}{v_1} \log(|C_1|/2v_1), \quad \text{and} \quad \bar{\phi}_1 = 2 \arg k_1 - \arg C_1.$$

The parameterization of x (49b) satisfies

$$\frac{\partial x}{\partial \xi} = 1 - \frac{2v_1^2}{|k_1|^2} \operatorname{sech}^2[\nu_1(2\xi + t/(2|k_1|^2) - x_1)]. \quad (50)$$

Therefore, we have three types of solitons:

- *smooth*: If $v_1 < |\eta_1|$, then $\partial x / \partial \xi$ is always positive, which leads to a smooth envelope soliton;
- *loop*: If $v_1 > |\eta_1|$, then $\partial x / \partial \xi$ has two zeros, which leads to a loop in the envelope of u ;
- *cuspon*: If $v_1 = |\eta_1|$, then $\partial x / \partial \xi$ has only one zero. In x, t variables, we obtain a soliton with a cusp in the envelope.

4.2 Two-soliton solution

In this subsection, we consider the case of a reflectionless solution corresponding to 2 discrete eigenvalues, $k_j = \eta_j + i\nu_j$, $j = 1, 2$ in the UHP, with associated norming constants, C_j ,

$j = 1, 2$. The meromorphic function is of the form

$$\begin{aligned}\hat{\mu}(\xi, t, k) &= \begin{pmatrix} 1 + \frac{B_{11}}{k-k_1} + \frac{B_{22}}{k-k_2} & -\frac{B_{12}^*}{k-k_1^*} - \frac{B_{21}^*}{k-k_2^*} \\ \frac{B_{12}}{k-k_1} + \frac{B_{21}}{k-k_2} & 1 + \frac{B_{11}^*}{k-k_1^*} + \frac{B_{22}^*}{k-k_2^*} \end{pmatrix} \\ &= I_2 + \begin{pmatrix} -\frac{B_{11}}{k_1} - \frac{B_{22}}{k_2} & \frac{B_{12}^*}{k_1^*} + \frac{B_{21}^*}{k_2^*} \\ -\frac{B_{12}}{k_1} - \frac{B_{21}}{k_2} & -\frac{B_{11}^*}{k_1^*} - \frac{B_{22}^*}{k_2^*} \end{pmatrix} \\ &\quad + k \begin{pmatrix} -\frac{B_{11}}{k_1^2} - \frac{B_{22}}{k_2^2} & \frac{B_{12}^*}{(k_1^*)^2} + \frac{B_{21}^*}{(k_2^*)^2} \\ -\frac{B_{12}}{k_1^2} - \frac{B_{21}}{k_2^2} & -\frac{B_{11}^*}{(k_1^*)^2} - \frac{B_{22}^*}{(k_2^*)^2} \end{pmatrix} + O(k^2),\end{aligned}\quad (51)$$

where B_{ij} depend on ξ, t and the symmetry $\hat{\mu}^*(k^*) = \sigma_2 \hat{\mu}(k) \sigma_2$ is taken into account.

To determine the B_{ij} , we use the residue conditions (34) at $k = k_1$ and $k = k_2$ to generate the system of linear equations

$$\begin{aligned}B_{11} &= -e^{2i\hat{\theta}_1} \frac{B_{12}^*}{k_1 - k_1^*} C_1 - e^{2i\hat{\theta}_1} \frac{B_{21}^*}{k_1 - k_2^*} C_1, \\ B_{12} &= e^{2i\hat{\theta}_1} C_1 + e^{2i\hat{\theta}_1} \frac{B_{11}^*}{k_1 - k_1^*} C_1 + e^{2i\hat{\theta}_1} \frac{B_{22}^*}{k_1 - k_2^*} C_1, \\ B_{22} &= -e^{2i\hat{\theta}_2} \frac{B_{12}^*}{k_2 - k_1^*} C_2 - e^{2i\hat{\theta}_2} \frac{B_{21}^*}{k_2 - k_2^*} C_2, \\ B_{21} &= e^{2i\hat{\theta}_2} C_2 + e^{2i\hat{\theta}_2} \frac{B_{11}^*}{k_2 - k_1^*} C_2 + e^{2i\hat{\theta}_2} \frac{B_{22}^*}{k_2 - k_2^*} C_2,\end{aligned}$$

where, as above, $\hat{\theta}_j = k_j \xi - \frac{t}{4k_j}$ for $j = 1, 2$. Solving the linear system, we obtain

$$B_{11} = \frac{1}{\Delta} \left[\frac{|C_1|^2}{k_1^* - k_1} e_1^2 - \frac{|C_1|^2 |C_2|^2 |k_1 - k_2|^2 (k_1^* - k_2^*)}{8i v_1 v_2^2 |k_1 - k_2^*|^2 (k_1^* - k_2)} e_1^2 e_2^2 - \frac{C_1 C_2^*}{k_1 - k_2^*} e^{2i(\hat{\theta}_1 - \hat{\theta}_2^*)} \right], \quad (52a)$$

$$B_{22} = \frac{1}{\Delta} \left[\frac{|C_2|^2}{k_2^* - k_2} e_2^2 - \frac{|C_1|^2 |C_2|^2 |k_1 - k_2|^2 (k_2^* - k_1^*)}{8i v_2 v_1^2 |k_1 - k_2^*|^2 (k_2^* - k_1)} e_1^2 e_2^2 - \frac{C_1^* C_2}{k_2 - k_1^*} e^{2i(\hat{\theta}_2 - \hat{\theta}_1^*)} \right], \quad (52b)$$

$$B_{12} = \frac{1}{\Delta} \left[C_1 e^{2i\hat{\theta}_1} + \frac{|C_1|^2 C_2 (k_2 - k_1)}{2i v_1 (k_1^* - k_2)^2} e_1^2 e^{2i\hat{\theta}_2} + \frac{|C_2|^2 C_1 (k_1 - k_2)}{4v_2^2 (k_1 - k_2^*)} e_2^2 e^{2i\hat{\theta}_1} \right], \quad (52c)$$

$$B_{21} = \frac{1}{\Delta} \left[C_2 e^{2i\hat{\theta}_2} + \frac{|C_2|^2 C_1 (k_1 - k_2)}{2i v_2 (k_2^* - k_1)^2} e_2^2 e^{2i\hat{\theta}_1} + \frac{|C_1|^2 C_2 (k_2 - k_1)}{4v_1^2 (k_2 - k_1^*)} e_1^2 e^{2i\hat{\theta}_2} \right], \quad (52d)$$

where

$$\begin{aligned}\Delta &= 1 + \frac{|C_1|^2}{4v_1^2} e_1^2 + \frac{|C_2|^2}{4v_2^2} e_2^2 + \frac{|k_1 - k_2|^4}{16v_1^2 v_2^2 |k_1 - k_2^*|^4} |C_1|^2 |C_2|^2 e_1^2 e_2^2 \\ &\quad - 2 \operatorname{Re} \left[\frac{C_1 C_2^*}{(k_1 - k_2^*)^2} e^{2i(\hat{\theta}_1 - \hat{\theta}_2^*)} \right],\end{aligned}\quad (53)$$

and

$$e_j^2 = e^{2i(\hat{\theta}_j - \hat{\theta}_j^*)} \equiv e^{-2v_j \zeta_j} \quad j = 1, 2, \quad (54)$$

where $\zeta_j = 2\xi + t/(2|k_j|^2)$ for $j = 1, 2$.

Inserting (51) into (44a) and (44b), we obtain

$$\begin{aligned} x - \xi &= i \left[\left(1 - \frac{B_{11}}{k_1} - \frac{B_{22}}{k_2} \right) \left(-\frac{B_{11}^*}{(k_1^*)^2} - \frac{B_{22}^*}{(k_2^*)^2} \right) + \left(\frac{B_{12}}{k_1} + \frac{B_{21}}{k_2} \right) \left(\frac{B_{12}^*}{(k_1^*)^2} + \frac{B_{21}^*}{(k_2^*)^2} \right) \right] \\ &= -\frac{i}{k_1^2} B_{11} - \frac{i}{k_2^2} B_{22} + \frac{i}{k_1 |k_1|^2} (|B_{11}|^2 + |B_{12}|^2) + \frac{i}{k_2 |k_2|^2} (|B_{21}|^2 + |B_{22}|^2) \\ &\quad + \frac{i}{k_1^* k_2^2} (B_{11}^* B_{22} + B_{21} B_{12}^*) + \frac{i}{k_2^* k_1^2} (B_{11} B_{22}^* + B_{12} B_{21}^*), \end{aligned} \quad (55a)$$

$$\hat{u}^* = \frac{B_{12}(B_{22}(k_1 - k_2) + k_2^2) + B_{21}(B_{11}(k_2 - k_1) + k_1^2)}{k_1^2 k_2^2}, \quad (55b)$$

where we used $\det \hat{\mu}(k) = 1$ for all k , which follows from (24) and (30) taking (19) into account.

Finally, using the identity

$$\begin{aligned} 0 = \det \hat{\mu}(0) - 1 &= \frac{B_{11}}{k_1} + \frac{B_{22}}{k_2} + \frac{B_{11}^*}{k_1^*} + \frac{B_{22}^*}{k_2^*} - \frac{|B_{11}|^2 + |B_{12}|^2}{|k_1|^2} - \frac{|B_{22}|^2 + |B_{21}|^2}{|k_2|^2} \\ &\quad - \frac{B_{11}^* B_{22} + B_{12}^* B_{21}}{k_1^* k_2} - \frac{B_{11} B_{22}^* + B_{12} B_{21}^*}{k_1 k_2^*}, \end{aligned}$$

and Eq. (52), we obtain the following parametric form for the two-soliton solution:

$$\begin{aligned} \hat{u}^*(\xi, t) &= \frac{1}{\Delta} \left[\frac{e^{2i\hat{\theta}_1} C_1}{k_1^2} - \frac{|C_1|^2 C_2 e_1^2 e^{2i\hat{\theta}_2} (k_2 - k_1)^2 (k_1^*)^2}{(k_1 - k_1^*)^2 (k_2 - k_1^*)^2 k_1^2 k_2^2} \right. \\ &\quad \left. + \frac{e^{2i\hat{\theta}_2} C_2}{k_2^2} - \frac{|C_2|^2 C_1 e_2^2 e^{2i\hat{\theta}_1} (k_2 - k_1)^2 (k_2^*)^2}{(k_1 - k_2^*)^2 (k_2 - k_2^*)^2 k_1^2 k_2^2} \right], \end{aligned} \quad (56)$$

$$\begin{aligned} x - \xi &= \frac{1}{\Delta} \left[\frac{|C_1|^2 e_1^2}{2v_1 |k_1|^2} + \frac{|C_2|^2 e_2^2}{2v_2 |k_2|^2} + \left(\frac{v_1}{|k_1|^2} + \frac{v_2}{|k_2|^2} \right) \right. \\ &\quad \left. \frac{|k_1 - k_2|^4 |C_1|^2 |C_2|^2 e_1^2 e_2^2}{8v_1^2 v_2^2 |k_1 - k_2^*|^4} - 2 \operatorname{Re} \left(\frac{i C_1 C_2^*}{k_1 k_2^* (k_1 - k_2^*)} e^{2i(\hat{\theta}_1 - \hat{\theta}_2^*)} \right) \right]. \end{aligned} \quad (57)$$

If $|k_1| = |k_2|$ then the solution is a generalization of the breathing pulse solution of the real SPE, with the real pulse the special case when $k_2 = -k_1^*$ and $C_2 = -C_1^*$.

On the other hand, generically, the magnitudes of the eigenvalues are not equal and the solitons separate at long times. Moreover, the two solitons will pass through one another as we pass from the limit $t \rightarrow -\infty$ to the limit $t \rightarrow +\infty$. Below we compute the long-time asymptotics of the 2-soliton solution along the direction of each soliton.

4.3 Long-time asymptotics of the 2-soliton solution

Without loss of generality, we posit that $|k_1| < |k_2|$, so that the velocities of the solitons, $v_j = 1/(4|k_j|^2)$ for $j = 1, 2$, satisfy $v_1 > v_2$. We also introduce $\zeta_j = 2\xi + t/(2|k_j|^2)$

for $j = 1, 2$ to denote coordinates in the respective moving reference frame of each soliton. Now note that

$$e_2^2 = \exp[-2v_2\zeta_1 + v_2(|k_1|^{-2} - |k_2|^{-2})t]$$

is exponentially decaying as $t \rightarrow -\infty$, and exponentially growing as $t \rightarrow +\infty$ when ζ_1 is fixed. Similarly,

$$e^{2i\hat{\theta}_j} = e^{i\eta_j(2\xi - t/(2|k_j|^2)) - v_j\zeta_j}, \quad e^{2i\hat{\theta}_j^*} = e^{i\eta_j(2\xi - t/(2|k_j|^2)) + v_j\zeta_j},$$

so $e^{2i\hat{\theta}_j}$ has the same asymptotic behavior as e_j , and $e^{2i\hat{\theta}_j^*}$ behaves like $1/e_j$ as $|t| \rightarrow \infty$. Then, one can easily compute the limit of the 2-soliton solution (56) as $t \rightarrow -\infty$ with ζ_1 fixed, which yields

$$\hat{u}(\xi, t) \sim \frac{v_1}{(k_1^*)^2} e^{-i\eta_1(2\xi - t/(2|k_1|^2)) - i \arg C_1} \operatorname{sech}[v_1(2\xi + t/(2|k_1|^2) - x_1^-)]$$

i.e., a 1-soliton solution (49) centered at $x_1^- = v_1^{-1} \log(|C_1|/(2v_1))$. On the other hand, the limit $t \rightarrow +\infty$ with ζ_1 fixed yields

$$\begin{aligned} \hat{u}(\xi, t) &\sim \left(\frac{k_2}{k_2^*}\right)^2 \frac{(k_1 - k_2^*)(k_1^* - k_2^*)}{(k_1^* - k_2)(k_1 - k_2)} \\ &\frac{v_1}{(k_1^*)^2} e^{-i\eta_1(2\xi - t/(2|k_1|^2)) - i \arg C_1} \operatorname{sech}[v_1(2\xi + t/(2|k_1|^2) - x_1^+)] \end{aligned}$$

where

$$v_1 x_1^+ = \log\left(\frac{|C_1|}{2v_1} \frac{|k_1 - k_2|^2}{|k_1 - k_2^*|^2}\right),$$

so still a 1-soliton solution, but with a shift in the overall phase, and a shift in the center of the soliton:

$$v_1(x_1^+ - x_1^-) = \log(|k_1 - k_2^*|^2/|k_1 - k_2|^2).$$

The corresponding expressions for the asymptotic behavior along the direction of the second soliton are obtained by interchanging indices 1 and 2 and by switching the limits $t \rightarrow +\infty$ and $t \rightarrow -\infty$. As a consequence, one also has $v_2(x_2^+ - x_2^-) = -v_1(x_1^+ - x_1^-)$.

As to the long-time asymptotics of $x - \xi$, one can easily check that when $t \rightarrow -\infty$ with ζ_1 fixed, (57) gives the same result as (49), with x_1 replaced by x_1^- above, i.e.

$$x - \xi \sim \frac{2v_1}{|k_1|^2} \frac{1}{1 + \exp[2v_1(2\xi + t/(2|k_1|^2) - x_1^-)]}. \quad (58)$$

When $t \rightarrow +\infty$, the limit of (57) for ζ_1 fixed yields:

$$x - \xi \sim \frac{2v_2}{|k_2|^2} \frac{1 + \left(1 + \frac{v_1|k_2|^2}{v_2|k_1|^2}\right) \exp[-2v_1(2\xi + t/(2|k_1|^2) - x_1^+)]}{1 + \exp[-2v_1(2\xi + t/(2|k_1|^2) - x_1^+)]}, \quad (59)$$

again with x_1^+ given above. Note that this implies

$$\frac{\partial x}{\partial \xi} \sim 1 - \frac{2v_1^2}{|k_1|^2} \operatorname{sech}^2[v_1(2\xi + t/(2|k_1|^2) - x_1^+)],$$

and therefore the condition $v_1 < |\eta_1|$ ensures the smoothness of the fast soliton in both limits as $t \rightarrow \pm\infty$ along the soliton direction. The same results can obviously be obtained for the second soliton, by simply switching the indices.

5 Concluding remarks

The Riemann–Hilbert formulation of the IST for the complex SPE elucidates the spectral “signature” that reveals whether its soliton solutions are single-valued, or evolve into envelope loop solitons. Moreover, the time-travel variable reflects the solution of the RHP that can be obtained from the spectral data (specifically, eigenvalues/poles and norming constant). On the other hand, the traveling pulse of the real SPE is a bound-state “breather” composed of two complex solitons which are comingled so that the complex-valued parts of the constituent solitons cancel one another. This can be seen directly from the two-soliton solution of the complex SPE by the Riemann–Hilbert formulation. Although this also occurs when comparing soliton solutions of the modified KdV or sine-Gordon equations and their complexified versions (and, in fact, also NLS restricted to real potentials), we stress that unlike KdV and sine-Gordon, and similarly to NLS, the complex SPE was derived from first principles as the relevant physical model to describe ultra-short pulse propagation in optical fibers.

On a different note, it is worth mentioning that the parametric representation of the solution solves initial-value problem for the complex SPE for all times $0 \leq t < T$ for which the map (44b) $\xi \rightarrow x$ is one-to-one. While the solution on the variables ξ, t always exists globally, if the bijectivity of the map $\xi \rightarrow x$ is broken, then wave-breaking occurs and (44b) and (44a) provide a continuation of the solution of the initial-value problem for the complex SPE after wave breaking.

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