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On Maximally Mixed Equilibria of Two-Dimensional Perfect Fluids

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Abstract

The vorticity of a two-dimensional perfect (incompressible and inviscid) fluid is transported by its area preserving flow. Given an initial vorticity distribution ω_0, predicting the long time behavior which can persist is an issue of fundamental importance. In the infinite time limit, some irreversible mixing of ω_0 can occur. Since kinetic energy E is conserved, not all the mixed states are relevant and it is natural to consider only the ones with energy E_0 corresponding to ω_0. The set of said vorticity fields, denoted by $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, contains all the possible end states of the fluid motion. A. Shnirelman introduced the concept of *maximally mixed states* (any further mixing would necessarily change their energy), and proved they are perfect fluid equilibria. We offer a new perspective on this theory by showing that any minimizer of any strictly convex Casimir in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ is maximally mixed, as well as discuss its relation to classical statistical hydrodynamics theories. Thus, (weak) convergence to equilibrium cannot be excluded solely on the grounds of vorticity transport and conservation of kinetic energy. On the other hand, on domains with symmetry (for example straight channel or annulus), we exploit all the conserved quantities and the characterizations of $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ to give examples of open sets of initial data which can be arbitrarily close to any shear or radial flow in L^1 of vorticity but do not weakly converge to them in the long time limit.

1. Introduction

Let $M \subset \mathbb{R}^2$ be a bounded domain possibly with boundary ∂M having exterior unit normal $\hat{n}$, for example, the flat two-torus $\mathbb{T}^2$, the periodic channel $\mathbb{T} \times [0, 1]$ or the disk $\mathbb{D}$. The Euler equations governing the motion of a fluid which is perfect (inviscid and incompressible) and confined to M read [28] as

$$\partial_t u + u \cdot \nabla u = -\nabla p, \qquad \text{in} \quad M, \tag{1.1}$$

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$$\nabla \cdot u = 0, \qquad \text{in} \quad M, \tag{1.2}$$
$$u|_{t=0} = u_0, \qquad \text{in} \quad M, \tag{1.3}$$
$$u \cdot \hat{n} = 0, \qquad \text{on } \partial M. \tag{1.4}$$

In terms of the vorticity $\omega := \nabla^\perp \cdot u$ where $\nabla^\perp := (-\partial_2, \partial_1)$, the system above can be reformulated as

$$\partial_t \omega + u \cdot \nabla \omega = 0 \qquad \text{in} \quad M, \tag{1.5}$$
$$\omega|_{t=0} = \omega_0, \qquad \text{in} \quad M, \tag{1.6}$$

where $u = K_M[\omega] = \nabla^\perp \Delta^{-1} \omega$ is recovered by the Biot-Savart law. Equations (1.5)–(1.6) say that the vorticity is transported by particle trajectories, namely the solution admits the representation

$$\omega(t) = \omega_0 \circ \Phi_t^{-1}, \tag{1.7}$$

where
$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t = u(\Phi_t, t), \qquad \Phi_0 = \mathrm{id} \tag{1.8}$$

is the Lagrangian flowmap.

To formalize the study of the dynamics of two-dimensional fluids, let X be a ball in L^∞

$$X := \{\omega \in L^\infty(M) \, : \, \|\omega\|_{L^\infty(M)} \leqq 1\}. \tag{1.9}$$

Yudovich [45] proved that X is a good phase space for the Euler equations in that (1.5)–(1.6) forms an infinite dimensional, time reversible, dynamical system on X for all finite times. We call the time $t \in \mathbb{R}$ solution operator $S_t : X \circlearrowleft$. Our interest is the long time behavior of this dynamical system. Since $\omega(t) = S_t(\omega_0)$ satisfies $\|\omega(t)\|_{L^\infty(M)} = \|\omega_0\|_{L^\infty(M)} \leqq 1$, we have

$$\omega(t_i) \overset{*}{\rightharpoonup} \overline{\omega} \quad \text{along subsequences} \quad t_i \to \infty$$

where, we recall that weak-$*$ convergence is defined for $f_n \in L^\infty(M)$ by

$$\lim_{n\to\infty} \int_M \varphi(x) f_n(x) \mathrm{d}x = \int_M \varphi(x) \overline{f}(x) \mathrm{d}x, \qquad \forall \varphi \in L^1(M). \tag{1.10}$$

If $\|f_n\|_{L^\infty}$ is uniformly bounded (as is the case for $\|\omega(t_n)\|_{L^\infty(M)}$), then this notion of weak convergence agrees with others such as weak convergence in L^2. Denoting the weak-$*$ closure in $L^\infty(M)$ by $\overline{(\cdot)}^*$, we introduce the Omega limit set

$$\Omega_+(\omega_0) := \bigcap_{s \geqq 0} \overline{\{S_t(\omega_0), t \geqq s\}}^*, \tag{1.11}$$

which is the collection of all such weak-$*$ limits as $t \to \infty$ along the solution $\omega(t) = S_t(\omega_0)$ passing through $\omega_0 \in X$ at time 0. The set $\Omega_+(\omega_0)$ represents all possible 'coarsened' persistent motions launched by ω_0.

Our interest is understanding what can be ruled out *kinematically* in $\Omega_+(\omega_0)$, based solely on the transport structure of the vorticity equation and accounting for the conserved quantities. Recall that the conservation laws for the Euler equation, which hold on general planar domains (possibly multiply connected), are

$$\begin{aligned}
&\text{energy:}\quad \mathsf{E}(\omega(t)) = \mathsf{E}(\omega_0),\\
&\mathsf{E}(\omega) := \frac{1}{2}\int_M |K_M[\omega](x)|^2 \mathrm{d}x = \frac{1}{2}\int_M |u(x)|^2 \mathrm{d}x,\\
&\text{Casimirs:}\quad \mathsf{I}_f(\omega(t)) = \mathsf{I}_f(\omega_0),\\
&\mathsf{I}_f(\omega) := \int_M f(\omega(x))\mathrm{d}x, \qquad \text{for any continuous } f : X \to \mathbb{R},\\
&\text{circulation:}\quad \mathsf{K}_i(\omega(t)) = \mathsf{K}_i(\omega_0),\\
&\mathsf{K}_i(\omega) := \int_{\Gamma_i} u \cdot \mathrm{d}\ell, \qquad \text{for connected components } \Gamma_i \text{ of } \partial M.
\end{aligned}$$

If the domain has additional symmetries there can be additional invariants such as linear momentum on the torus and channel[1] and angular momentum on the disk:

$$\begin{aligned}
&\text{linear momentum on } M = \mathbb{T}\times[0,1]: \quad \mathsf{M}(\omega(t)) = \mathsf{M}(\omega_0),\\
&\mathsf{M}(\omega) := \int_M e_1 \cdot u(x)\mathrm{d}x = \int_M (x_2\omega(x) + u_1(x_1, 1))\mathrm{d}x,\\
&\text{angular momentum on } M = \mathbb{D}: \quad \mathsf{A}(\omega(t)) = \mathsf{A}(\omega_0),\\
&\mathsf{A}(\omega) := -\frac{1}{2}\int_M (1-|x|^2)\omega(x)\mathrm{d}x = \int_M x^\perp \cdot u(x)\mathrm{d}x.
\end{aligned}$$

However, for domains without Euclidean symmetries, linear and angular momentum conservation are lost due to pressure effects. Casimirs and circulations are the *only* invariants for general area preserving transformations of the vorticity (see Izosimov and Khesin [25,26]). Together with energy, they are the only *known* conservation laws (first integrals) for perfect fluids in 2D which hold for all data and on arbitrary domains. For simplicity of presentation, we will primarily work on simply connected domains where the circulation does not impose any additional constraints beyond the constancy of the domain-averaged vorticity.

From (1.7), we know that the vorticity function is, at every instant, an area preserving rearrangement of its initial datum. Thus, let $\mathscr{D}_\mu(M)$ denote the group of area preserving diffeomorphisms on M and denote the orbit of $\omega_0 \in X$ in $\mathscr{D}_\mu(M)$ by

$$\mathcal{O}_{\omega_0} := \{\omega_0 \circ \varphi \ : \ \varphi \in \mathscr{D}_\mu(M)\}, \tag{1.12}$$

where we understand φ to be in the component of the identity. Since φ are area preserving, the Casimirs I_f are constant along orbits $\mathcal{O}_{\omega_0}$, just as they are for Euler according to the representation (1.7). To get closer to the Euler dynamics, we consider the intersection of this orbit with constant energy fields

$$\mathcal{O}_{\omega_0,\mathsf{E}_0} := \mathcal{O}_{\omega_0} \cap \{\mathsf{E} = \mathsf{E}_0\}. \tag{1.13}$$

[1] These can be related to conservation of circulation of the harmonic component of u around fixed non-contractible loops.

In fact, we have that $\omega(t) = S_t(\omega_0) \in \mathcal{O}_{\omega_0,\mathsf{E}_0}$ for all $t \in \mathbb{R}$. In the coarse-grained infinite-time picture captured by weak-$*$ limits, there is a marked difference between the energy (also circulations and, on domains with symmetry, momentum) and the Casimirs: the energy is weak-$*$ continuous whereas the non-linear Casimirs are not. In fact, if $\omega(t_i) \overset{*}{\rightharpoonup} \overline{\omega}$ we can only deduce by lower-semicontinuity that

$$\mathsf{I}_f(\overline{\omega}) \leqq \liminf_{i\to\infty} \mathsf{I}_f(\omega(t_i)) = \mathsf{I}_f(\omega_0) \quad \text{for any convex } f. \tag{1.14}$$

Consequently, we have the following containments

$$\Omega_+(\omega_0) \subset \overline{\mathcal{O}_{\omega_0,\mathsf{E}_0}}^* \subset \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\} \tag{1.15}$$

where the last containment is a consequence of energy being weak-$*$ continuous. Loss of enstrophy on weak limits, namely $\|\bar{\omega}\|_{L^2}^2 < \|\omega_0\|_{L^2}^2$ (or, more generally speaking, with a strict inequality in (1.14) for any convex Casimir) is associated to fine-scale *mixing*. This behavior is often observed in the long time limit of the Euler evolution and is conjectured to be typical [43]. Due to mixing, on $\overline{\mathcal{O}_{\omega_0}}^*$ the Casimirs are no longer constant but convex Casimirs do not increase in view of (1.14). Thanks to this, we have a partial ordering structure (a "mixing order") on $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$:

Definition 1.1. Given $\omega_1, \omega_2 \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, we say that $\omega_1 \preceq \omega_2$ if there exists a strictly convex function f such that $\mathsf{I}_f(\omega_1) \leqq \mathsf{I}_f(\omega_2)$.

Such a partial order was first introduced by Shnirelman in [41] using an equivalent characterization (see Lemma 2.3 below). In view of this ordering, it is natural to introduce the notion of a *minimal element*:

Definition 1.2. An $\omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ is *minimal* if for all ω such that $\omega \preceq \omega^*$ then $\omega^* \preceq \omega$.

Namely, a minimal element (termed *minimal flow*) ω^* has the property that if $\omega \preceq \omega^*$ then $\mathsf{I}_f(\omega) = \mathsf{I}_f(\omega^*)$ for all convex f. We can therefore think that minimal elements are *maximally mixed* versions of ω_0 at a given fixed energy. Shnirelman [41] establishes the existence of a minimal flows in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ for any $\omega_0 \in X$ as an application of the Zorn's Lemma. See also the discussion by Arnold and Khesin [1].

One of the main purposes of this paper is to offer a different perspective of certain minimal flows, specifically to those that naturally arise from Definition 1.2. In §5 we prove the following:

Theorem 1. *Let $M \subset \mathbb{R}^2$ be a bounded planar domain with smooth boundary and let $f : X \to \mathbb{R}$ be a strictly convex function. Given any $\omega_0 \in X$ with energy E_0, there exists a minimizer $\omega^* \in X$ such that*

$$\mathsf{I}_f(\omega^*) = \min_{\omega \,\in\, \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E}=\mathsf{E}_0\}} \mathsf{I}_f(\omega). \tag{1.16}$$

Any such minimizer ω_* is a minimal flow in the sense of Shnirelman and enjoys the following properties:

(i) ω_* *is a stationary solution of the Euler equation having the property that there exists a bounded monotone function* $F : \mathbb{R} \to \mathbb{R}$ *such that* $\omega_* = F(\psi_*)$,
(ii) there exists a continuous convex function Φ *and scalars* $\alpha, \beta, \gamma \in \mathbb{R}$ *with* $\alpha^2 + \beta^2 \neq 0$ *such that* ω_* *is a minimizer on X (the unconstrained space) of the functional*

$$J_\Phi(\omega) = \mathsf{I}_{\Phi+\alpha f}(\omega) + \beta(\mathsf{E}(\omega) - \mathsf{E}_0) + \gamma \int_M (\omega - \omega_0)\mathrm{d}x. \tag{1.17}$$

The theorem above gives a method to produce stationary and minimal solutions of the Euler equations by solving a variational problem on $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. Certain characterizations of this set are available, see §2. Using these, in Appendix A, we give a concrete and explicit instance of Shnirelman's maximal mixing theory as it applies to vortex patches with a finite number of regions. In this case, (1.16) can be seen as an optimization problem with a *finite* number of inequality constraints. Point (ii) of the Theorem 1 gives the natural extension of such characterization for a general ω_0, inspired by work of Rakotoson and Serre [35]. In fact, the main purpose of Point (ii) is to get minimal flows by solving an *unconstrained* variational problem, which can give more information about minimal flows in some cases. For instance, if $\alpha \neq 0$, then $\Phi + \alpha f$ is strictly convex and thus the minimizer of (1.17) is unique and satisfies $\omega = F(\psi)$ where $F(z) := (\Phi' + \alpha f')^{-1}(-\beta z - \gamma)$. In this case, F is Lipschitz and strictly increasing or decreasing.

Remark 1.3. [Minimal flows as minimizers] The minimal flows obtained by Shnirelman through the Zorn's lemma need not to be same as the one given in Theorem 1. However, a remarkable property of minimal flows established by Shnirelman (see also Lemma 3.2 herein) is that they are *all* stationary solutions of the Euler equation having the property $\omega^* = F(\psi^*)$ for some bounded monotone function F. When F is *strictly* monotone increasing, then ω^* is in fact a minimizer of the strictly convex functional $\mathsf{I}_{F^{-1}}$, with F^{-1} being the inverse of F. This follows by the standard Lagrange multiplier rule in the set $X \cap \{\mathsf{E} = \mathsf{E}_0\}$. It remains an open issue to say that any minimal flow in the set $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ (particularly those having regions of constant vorticity, see Appendix B) are minimizers of some strictly convex functional.

Remark 1.4. (Examples of minimal and non-minimal flows) A privileged family of minimal flows are the so called *Arnold stable* states. They satisfy $\omega = F(\psi)$ for a Lipschitz F satisfying either of the following two conditions

$$-\lambda_1 < F'(\psi) < 0, \qquad \text{or} \qquad 0 < F'(\psi) < \infty, \tag{1.18}$$

where $\lambda_1 := \lambda_1(\Omega) > 0$ is the smallest eigenvalue of $-\Delta$ in M. These flows are Lyapunov stable in the L^2 topology of vorticity under the 2D Euler evolution. Any Arnold stable steady state is a minimal flow, since any mixing of them necessarily results in a change of energy. For an area preserving rearrangement, this follows by the fact that they are local maximizers or minimizers of the energy on their isovortical sheet $\mathcal{O}_{\omega_*}$, see for example [1,19]. On the other hand, any shear flow (on the channel) or circular flow (on the disk) having an inflection point cannot be a minimal flow. This is implied by [41] and Lemma 3.2 herein.

Remark 1.5. (Non-uniqueness and regularity of minimal flows) Given $\omega_0 \in X$ with energy E_0, there is no reason for the minimal flow in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ to be unique—different convex functions f in the variational problem (1.16) can give rise to distinct minimal flows. See Remark A.4 for a datum $\omega_0 \in X$ with infinitely many minimal flows on its orbit. This example also shows that minimal flows in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ can have better regularity than the datum $\omega_0 \in X$.

Remark 1.6. (Connections to Statistical Hydrodynamics) Understanding the coarse-scale features of the possible end-states of the fluid flow has been the subject of *statistical hydrodynamics theories*, laid out by Onsager in his foundational paper [34]. Many of these theories [4,18,30] (see [5,37] for a review). lead to a variational problem to minimize a given strictly convex Casimir in $X \cap \{\mathsf{E} = \mathsf{E}_0\}$ (compare to (1.16)), where in particular the minimizer will also be a stationary state of the form $\omega = F(\psi)$. However, these motions may not be accessible dynamically since they do not account for all known constraints on the structure of the solution given initial data. Notable exceptions are the theories of Miller, Robert, Sommeria [31,32,36–38], where the variational problem is for a probability distribution whose mean represents the coarse-grained vorticity. This point of view is based on the representation of weak-$*$ limits through Young's measures and it also lead to vorticities satisfying $\omega = F(\psi)$. In some particular case, for example $F' > 0$, the MRS variational approach is equivalent to solving (1.16) [5]. However, in general it is not possible to deduce similar properties a priori.

Studying properties of certain states in the set $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ sheds light on some questions concerning relaxation to equilibrium. A consequence of Theorem 1 is that, for any initial data with bounded vorticity, there *always exist* stationary solutions (with bounded vorticity, but not necessarily smooth) in the set $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. We thus deduce that accounting for all known kinematic constraints on the solution is not enough to rule out relaxation to equilibrium via Euler evolution for any initial datum, at least in a weak-$*$ sense. In a similar direction, Choffrut and Šverák [9] gave a full characterization of the steady states nearby certain Arnold stable ones on annular domains. Izosimov and Khesin [25] gave necessary conditions on ω_0 to have smooth steady Euler solution in its orbit $\mathcal{O}_{\omega_0}$. In the other direction, Ginzburg and Khesin [20,21] showed that if M is a simply connected planar domain and ω_0 is Morse, positive and has both a local maximum and minimum in the interior, then $\mathcal{O}_{\omega_0}$ contains no smooth Euler steady state.

Thus, convergence to equilibrium at long time can occur, although theorems (and likely scenarios) are very rare. The results [2,23,24,29] are the only to fully characterize the Omega limit sets for Euler, albeit for very smooth perturbations of special equilibria. For instance, if $\overline{\omega}$ is the vorticity of a (class of) strictly monotone shear flow on $\mathbb{T} \times [0,1]$ or $\mathbb{T} \times \mathbb{R}$, then for any ω_0 in a Gevrey-2 neighborhood of $\overline{\omega}$, one has $\Omega_+(\omega_0) = \{\overline{\omega}_{\omega_0}\}$, where $\overline{\omega}_{\omega_0}$ is the vorticity of a (slightly modified) shear flow nearby $\overline{\omega}$. The convergence happens weakly, not strongly, in L^2 so some amount of mixing definitively occurs. These remarkable results—termed *inviscid damping*—show that certain full neighborhoods in the (Gevrey) phase space relax to equilibrium at long time, a feature consistent with Theorem 1 and the theories of Statistical hydrodynamics mentioned in Remark 1.6. The fact that these stable equilibria are symmetric is no accident. On domains with symmetry, also the Arnold stable steady states must inherit the symmetry of the domain they occupy [12]; all such on the channel are shears, while on the annulus they are circular. It is unclear if the flows $\overline{\omega}_{\omega_0}$ are minimal. On the other hand, convergence to symmetric equilibria (even in this weak sense) seems to be the exception rather than the rule more generally. For instance, Lin and Zeng [27] discovered non-shear Catseye steady states nearby (at low regularity) to the Couette shear flow,and we refer to [8,15,33] for related results. These results provide an obstruction to inviscid damping back to a shear flow for general perturbations nearby certain shear flows of a given structure. However, they do not rule out convergence to shear flow for some perturbations.

In a similar spirit, we show here that there exist open sets of small, sufficiently coarse, perturbations of any shear flow on the periodic channel (actually, of any bounded vorticity field on the channel) that cannot possibly damp back to a shear flow. Unlike those previous works, we do this not by finding other nearby steady states, but rather by excluding shear flows directly from a set containing the Omega limit set.

Theorem 2. *Let $M = \mathbb{T} \times [0, 1]$ and $\omega_b \in L^\infty(M)$. For any $\delta > 0$, there exists $\xi \in C^\infty(M)$ such that*

$$\|\xi - \omega_b\|_{L^1} \lesssim \delta \tag{1.19}$$

and for which the set $\overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$ contains no shear flows.

The idea behind our construction, carried out in §6, is to insert a large perturbation at small spatial scales in the form of regularized point vorticies of width ε, see Fig. 1. Namely, the field ξ is comprised of highly peaked vortices embedded in the background ω_b, that is there is $0 < \varepsilon := \varepsilon(\|\omega_b\|_{L^\infty}) < \delta$ so that

$$\|\xi - \omega_b\|_{L^\infty} \approx \varepsilon^{-2}, \quad |\mathrm{supp}(\xi - \omega_b)| \lesssim \varepsilon^2, \quad \mathsf{E}(\xi) - \mathsf{E}_b \approx \delta^2 |\log(\varepsilon)|,$$
$$|\mathsf{M}(\xi) - \mathsf{M}_b| \lesssim \delta.$$

In view of the Biot-Savart law, from which the velocity is recovered from the vorticity by $u = \nabla^\perp \Delta^{-1} \omega$, these perturbations exploit the (logarithmic) singularity

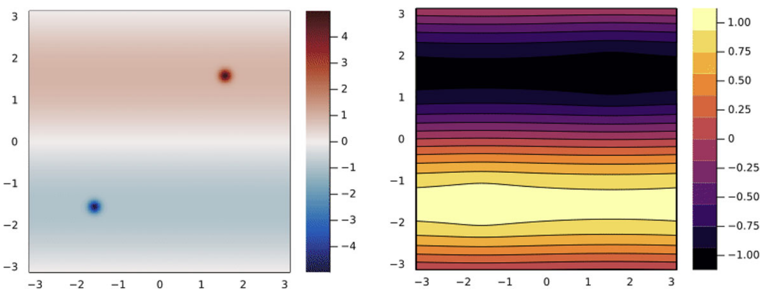

Fig. 1. Example of a datum ω_0 from Theorem 2—a perturbation (at the level of the stream-function) of the Kolmogorov flow $\omega_S = \sin(y)$ by two equal and opposite approximate point vortices. Vorticity colormap (left) and streamfunction contour plot (right)

of the Green's function of the Laplacian in two-dimensions and thus have energy $|\log \varepsilon|$. We show that for small ε, one cannot rearrange such a configuration into a shear flow while conserving the energy. This is because shear flows are fundamentally 1D objects in the sense that the Biot-Savart kernel is non-singular acting on functions of one variable. Similarly, radial flows can be excluded on the annulus by exploiting conservation of angular momentum.

In view of the containment (1.15), Theorem 2 implies that the Euler solution starting from this data cannot weakly converge to a shear flow. In fact, Theorem 2 holds for fields $\tilde{\xi}$ in an open neighborhood of ξ in L^∞. These results show that the Euler dynamics cannot totally "shear out" highly peaked coherent vortices, but they do not rule out damping to some asymmetric equilibria. However, numerical simulations (see Fig. 2) suggest that it is more likely that the Euler solutions relax to some time dependent (but recurrent) states. For additional discussion, see [17,42].

Remark 1.7. (Asymmetry of minimal flows) By including the constraint on the momentum in (1.16), we can combine Theorem 1 and Theorem 2 to see that all the minimal flows obtained as minimizers of strictly convex functionals in the set $\overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$ cannot be shear flows, thus providing examples of minimal flows which do not conform to the symmetries of the domain.

Remark 1.8. (Perturbations of shear flows) The ω_b of Theorem 2 can be any shear flow $u_b(x_1, x_2) := (v(x_2), 0)$ with bounded vorticity $\omega_b(x_1, x_2) := -v'(x_2)$. Our result shows that not only is the regularity important for convergence back to a

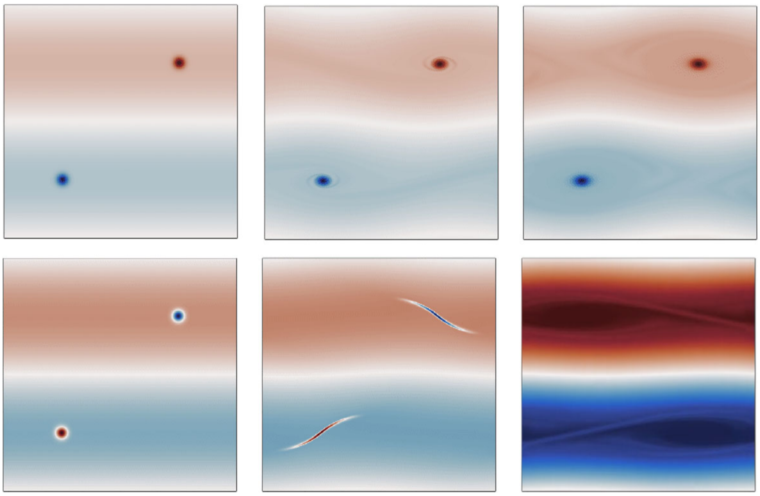

Fig. 2. Direct numerical simulations [13,14] of the time evolution (from left to right) of initial data with localized vortices rotating with and against the background Kolmogorov shear flow under Navier–Stokes evolution with Reynolds number Re $\approx 10^3$. The long time behavior exhibits no tendency to return to shear. In the case of the co-rotating vortices, it appears possible that the evolution weakly damps to a non-shear equilibrium, possibly a minimal flow. However, in both case, periodic or quasi-period structures appear to be present at small scales it is unclear whether those can disappear in the long time limit

shear flow, as highligthed by the results [8,15,27,33], but also the proximity must be measured in a quite strong sense. In fact, our perturbation is extremely large in any L^p (on vorticity) with $p > 1$ and also in L^2 velocity.

2. The weak-$*$ closure of the orbit

To understand the maximal mixing theory, it is important to study the structure of $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. In fact, this problem was considered in different mathematical contexts [7,10,11,39,41,43]. Here, we present a self-contained description of the different characterizations which we will exploit. Let us recall here a particular characterization of the weak-$*$ closure of the orbit of a scalar function in the group of area preserving diffeomorphisms $\mathscr{D}_\mu(M)$, used in [7,11,41]. Denote the collection of evaluation maps along area preserving diffeomorphisms by

$$\mathscr{E}_\mu(M) := \{i_\varphi \,:\, \varphi \in \mathscr{D}_\mu(M)\}, \tag{2.1}$$

where i_φ is the evaluation map, that is if $f : M \to \mathbb{R}$ then $(i_\varphi f)(x) = \int_M f(y)\delta(y-\varphi(x))\mathrm{d}y = f(\varphi(x))$. We associate to i_φ the positive measure $\delta(y-\varphi(x))\mathrm{d}y$. The following is established in [6,7]:

Proposition 2.1. *We have*

$$\overline{\mathscr{E}_\mu(M)}^* = \mathscr{K}(M), \tag{2.2}$$

where $\mathscr{K}(M)$ is the convex space of polymorphisms or bistochastic operators

$$\mathscr{K}(M) := \Big\{K : M \times M \to \mathbb{R} \;\; such\ that \;\; K \geqq 0, \\ \int_M K(x,\cdot)\mathrm{d}x = \int_M K(\cdot, y)\mathrm{d}y = 1\Big\}. \tag{2.3}$$

Remark 2.2. (Examples of polymorphisms) Bistochastic operators are the infinite dimensional extension of bistochastic matrices. Few important examples are the following:

1) Let $\varphi \in \mathscr{D}_\mu(M)$. Then the insertion operator $K_\varphi = i_\varphi$ is bistochastic and $(K_\varphi\omega)(x) = \omega(\varphi(x))$
2) The *complete mixing* operator K_{mix} given by

$$(K_{\mathsf{mix}}\omega)(x) = \frac{1}{|M|}\int_M \omega(y)\mathrm{d}y \tag{2.4}$$

is bistochastic. On $M = \mathbb{T}^2$, this operator is the projection onto the zero Fourier mode.
3) On $M = \mathbb{T}^2$, some frequency cut-offs define bistochastic operators, for example the Fejér kernel:

$$F_N(x) = \frac{1}{(2\pi)^2}\sum_{k_1,k_2=-N}^{N}\left(1-\frac{|k_1|}{N}\right)\left(1-\frac{|k_2|}{N}\right)e^{i(k_1x_1+k_2x_2)}. \tag{2.5}$$

In view of $\int_{\mathbb{T}^2} F_N(x)\mathrm{d}x = \widehat{F}(0) = 1$, this kernel has the following properties:

a) $F_N(x) \geqq 0$,

b) $\widehat{F_N}(k) = \begin{cases} \left(1 - \frac{|k_1|}{N}\right)\left(1 - \frac{|k_2|}{N}\right) & 1 \leqq |k_1|, |k_2| \leqq N \\ 0 & \max\{|k_1|, |k_2|\} > N \end{cases},$

c) $\int_{\mathbb{T}^2} F_N(x)\mathrm{d}x = 1$.

Given $\omega \in L^2$, define a frequency cut-off as follows

$$(K_N\omega)(x) = \int_{\mathbb{T}^2} F_N(x-y)\omega(y)\mathrm{d}y = (F_N * \omega)(x). \tag{2.6}$$

Thanks to the properties of F_N, it can be verified that K_N is a bistochastic operator (see [44, §3.1]).

The set of polymorphisms is relevant to the weak-$*$ closure of the orbit since (see Proposition 2.4 in §2), given any $\omega_0 \in X$, we have

$$\overline{\mathcal{O}_{\omega_0}}^* = \{\omega \in X : \ \omega = K\omega_0 \ \text{ for } K \in \mathscr{K}\}. \tag{2.7}$$

Shnirelman uses the characterization (2.7) of $\overline{\mathcal{O}_{\omega_0}}^*$ to impart a partial ordering in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$; that is given $\omega_1, \omega_2 \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, we say that $\omega_1 \preceq \omega_2$ if there exists a polymorphism $K \in \mathscr{K}$ such that $\omega_1 = K\omega_2$. Minimal elements (flows) are defined in the same way as Definition 1.2, only now using this new partial order. However, these two definitions are entirely equivalent in view of the following Lemma (which we prove in §3):

Lemma 2.3. *Given $\omega \in X$ and $K_1 \in \mathscr{K}$, let $\omega_1 = K_1\omega$. There exists $\widetilde{K} \in \mathscr{K}$ such that $\omega = \widetilde{K}\omega_1$ if and only if $\mathsf{I}_f(\omega_1) = \mathsf{I}_f(\omega)$ for any strictly convex $f : \mathbb{R} \to \mathbb{R}$.*

The intuition that minimal flows are maximally mixed, quantified by the conservation of all Casimirs in their weak-$*$ closure with our definition, can be rephrased also with bistochastic operators. The application of a bistochastic operator K could mix ω^*, but mixing is an irreversible process that prevent us from recovering ω^* from $K\omega^*$. On the other hand, for a minimal flow we can always go back. We are therefore excluding any mixing of ω^*. Thus, the class of available transformations of a minimal flow is a subset of area preserving maps (not necessarily a diffeomorphism). In fact, Lemma 2.3 shows that if $\omega = K\omega^*$ and $\omega^* = \widetilde{K}\omega$ then ω and ω^* are equimeasurable.

Here we prove the following characterizations of the weak-$*$ closure of the orbit:

Proposition 2.4. *Consider X, $\mathscr{K}$ as in (1.9) and (2.3) respectively. Given any $\omega_0 \in X$, we have*

$$\overline{\mathcal{O}_{\omega_0}}^* = \{\omega \in X : \ \omega = K\omega_0 \ \textit{for } K \in \mathscr{K}\}, \tag{2.8}$$

$$= \left\{\omega \in X \ : \ \int_M \omega \,\mathrm{d}x = \int_M \omega_0 \,\mathrm{d}x, \ \textit{and} \ \ \int_M (\omega - c)_+ \,\mathrm{d}x \leqq \int_M (\omega_0 - c)_+ \,\mathrm{d}x \ \ \textit{for all} \ \ c \in \mathbb{R}\right\}, \tag{2.9}$$

$$= \left\{ \omega \in X \,:\, \int_M \omega \, \mathrm{d}x = \int_M \omega_0 \, \mathrm{d}x, \; \textit{and} \; \int_M f(\omega) \, \mathrm{d}x \leqq \int_M f(\omega_0) \, \mathrm{d}x \;\; \textit{for any convex } f \right\}. \quad (2.10)$$

The description of the set $\overline{\mathcal{O}_{\omega_0}}^*$ has been a classical topic in rearrangement inequalities theory and the characterizations (2.8)-(2.10) can be found for example in [10,11,39]. The (2.8) has been used by Shnirelman [41], while the characterization (2.9) also appears in the lecture notes of Šverák [43]. In view of $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\} = \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, see (1.15), Proposition 2.4 completes our characterization of $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. In the following, we present a self contained proof of these characterizations.

Proof. We divide the proof in several steps.
$\diamond$ Step I: ($\overline{\mathcal{O}_{\omega_0}}^* =$ (2.8)) This characterization is a direct consequence of Proposition (2.1), whose proof can be found for example in [7] or [6, Sec 1.4]. We review the main arguments of the proof since in the sequel we need to exploit some technical lemma used for it.

As observed in Remark 2.2, we know $\mathscr{E}_\mu(M) \subset \mathscr{K}(M)$. Since $\mathscr{K}(M)$ is weak-$*$ closed, we infer $\overline{\mathscr{E}_\mu(M)}^* \subseteq \mathscr{K}(M)$. Thus, to prove (2.2) we only have to show that for every $K \in \mathscr{K}$ there exists a sequence $\{\phi_n\} \subset \mathscr{E}_\mu(M)$ such that

$$\lim_{n\to+\infty} \int_M f(x, \phi_n(x)) \mathrm{d}x = \iint_{M\times M} f(x, y) K(x, y) \mathrm{d}x\mathrm{d}y, \quad \text{for all } f \in C(M \times M). \quad (2.11)$$

Indeed, choosing $f(x, y) = g(x)\omega_0(y)$, we see that any element in $\overline{\mathcal{O}_{\omega_0}}^*$ is of the form $K\omega_0$, meaning that the characterization (2.8) is proved. The proof of (2.11) relies on the following key lemma, which we prove below.

Lemma 2.5. *Let $Q_1, Q_2 \subset M$ be two squares with centers x_1, x_2 respectively and $|Q_1| = |Q_2|$. Let $p : M \to M$ be a permutation of these two squares, namely*

$$p(x) = \begin{cases} x - x_1 + x_2 & \textit{if } x \in Q_1, \\ x - x_2 + x_1 & \textit{if } x \in Q_2, \\ x & \textit{otherwise}. \end{cases} \quad (2.12)$$

Then, there exists $\{\varphi_n\} \in \mathscr{E}_\mu(M)$ such that $\varphi_n \to p$ in $L^2(M)$.

In particular, permutations of squares are in $\overline{\mathcal{O}_{\omega_0}}^{L_2}$. The main idea is to discretize the problem (2.11) and use permutations of squares as building blocks to construct the approximating sequence ϕ_n. This is analogous to the decomposition of a doubly stochastic matrix in terms of permutation matrices, which is the classical Birkhoff's theorem. More precisely, given m sufficiently large, we can cover the interior of M with $N_m < +\infty$ squares $\{Q_i^m\}_{i=1}^{N_m}$ of area 4^{-m} up to an error $O(2^{-m})$. Then, approximate the measure

$$\mu_K(x, y) = K(x, y)\mathrm{d}x\mathrm{d}y$$

by

$$\gamma_m = \sum_{i,j} \mu_K (Q_i^m \times Q_j^m) \delta_{(x_i^m, x_j^m)}, \tag{2.13}$$

where x_i^m is the center of the cube Q_i^m. The measure γ_m is discrete and can be identified with a matrix $A = (a_{ij})$ where $a_{ij} = 4^m \mu_K (Q_i^m \times Q_j^m)$. Since K is bistochastic, the matrix A is also bistochastic, that is $\sum_i a_{ij} = \sum_j a_{ij} = 1$. We can therefore apply the Birkhoff's theorem to rewrite the matrix as a convex combination of permutation matrices, namely

$$a_{ij} = \sum_{k=1}^{K} \theta_k \delta_{\sigma_k(i), j}, \qquad \sum_{k=1}^{K} \theta_k = 1, \tag{2.14}$$

where $K \leqq N_m^2$ and σ is a permutation of $\{1, \dots N_m\}$. A permutation of squares can be approximated with a permutation matrix. Indeed, if p_σ is the permutation of the squares Q_i^m, $Q_{\sigma(i)}^m$, then

$$\sum_i \int_{Q_i^m} f(x, p_\sigma(x)) \mathrm{d}x = 4^{-m} \sum_i f(x_i^m, x_{\sigma(i)}^m) + C\eta(2^{-m}), \tag{2.15}$$

where η is the modulus of continuity of f. We are associating the discrete measure $4^{-m}\delta_{(x_i^m, x_{\sigma(i)}^m)}$ to p_σ up to a small error. Therefore, the proof of (2.11) is a standard approximation argument combined with the Birkhoff theorem and Lemma 2.5. We refer to [6, Sec 1.4] for a detailed proof of the approximation argument. Instead, let us show the proof of Lemma 2.5, see [7, Lemma 1.2], which we are going to use also in the proof of Theorem 1.

Proof of Lemma 2.5. First observe that if $\varphi_n^1, \varphi_n^2 \in \mathscr{E}_\mu(M)$ and $\varphi_n^1 \to h_1, \varphi_n^2 \to h_2$ in $L^2(M)$ then $\varphi_n^1 \circ \varphi_n^2 \to h_1 \circ h_2$ in $L^2(M)$. Hence, it is enough to prove that we can exchange two adjacent squares, since any permutation of squares can be written as a combination of exchanges between adjacent squares (refining further the grid covering M if necessary). To exchange adjacent squares, it is enough to approximate the central symmetry with respect to squares and rectangles.[2] For instance, given $Q = [-a, a]^2$, we need to approximate the map $c(x) = -x$ if $x \in Q$ and $c(x) = x$ otherwise. Notice that Q can be written as the union of the level sets for the function

$$g(x) = \max\{|x_1|, |x_2|\}, \quad \text{so that} \quad Q = \{x | \, g(x) \leqq a\}.$$

The idea is now to use the function g to construct a velocity field which moves the particles along the streamlines, where the velocity can be tuned in order to reach the point $-x$ at time $t = 1$ (a rigid rotation), see Fig. 3.

[2] Equivalently, we could also exchange adjacent triangles. This can be useful to extend the proof to smooth compact manifolds.

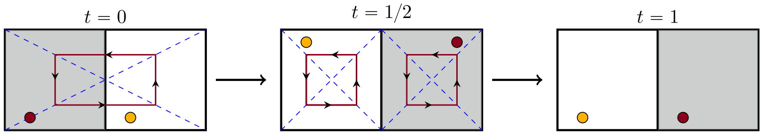

Fig. 3. First we act with the central symmetry for the rectangle. Then we use the central symmetry in each squares

In this case, since g is not differentiable everywhere, we cannot directly use $\nabla^{\perp} g$. However, it is enough to approximate g on a smaller domain. In polar coordinates one has

$$g(r, \theta) = r^2 \max\{\cos^2(\theta), \sin^2(\theta)\} = r^2(1 + |\cos(2\theta)|) := r^2 f(\theta), \quad (2.16)$$

so that a possible approximation of g is given by

$$r^2 f_\varepsilon(\theta) := r^2(1 + \sqrt{\varepsilon^2 + \cos^2(2\theta)}). \quad (2.17)$$

Also at the origin we may have problems, but since we are looking for an approximation up to zero Lebesgue measure sets, it is enough to prove that we approximate the central symmetry on the set $Q_\varepsilon = \{\varepsilon < r^2 f_\varepsilon(\theta) \leqq 2 - \varepsilon\}$. This can be proved by defining the streamfunction

$$\psi_\varepsilon(r, \theta) = \frac{1}{2}\lambda_\varepsilon r^2 f_\varepsilon(\theta), \qquad \lambda_\varepsilon = \int_0^\pi \frac{\mathrm{d}s}{f(s)} > 0, \quad (2.18)$$

with associated velocity field $\boldsymbol{v}_\varepsilon = \nabla^{\perp}\psi_\varepsilon$, for which it is not difficult to show that $\boldsymbol{v}$ moves a particle x to $-x$ in time $t = 1$, see [7]. The flow generated by $\partial_t \phi_\varepsilon = \boldsymbol{v}_\varepsilon(\phi_\varepsilon)$ is such that $\phi_\varepsilon(1, x) = -x$ on Q_ε. Once this is is done, we can choose $\varepsilon = 2^{-n}$ and define $c_n = \mathrm{id}$ on $M \backslash Q$, $c_n(x) = \phi_\varepsilon(1, x)$ on Q_ε and any smooth approximation between id and ϕ_ε on $Q \backslash Q_\varepsilon$. Then, c and c_n are equal up to a set of measure $O(2^{-n})$ and thus, being clearly uniformly bounded, $c_n \to c$ in $L^2(M)$.

For the central symmetry with respect to a rectangle $R = [-a, a] \times [-b, b]$, just notice that $R = \{x | \max\{|x_1|/a, |x_2|/b\} \leqq 1\}$, so we can repeat the construction above modifying the function g. □

◇ Step 2: ((2.8) ⇒ (2.9)) Since K is bistochastic and $(\cdot)_+$ is convex, by Jensen's inequality (see (3.1)) it follows that

$$\begin{aligned}
\mathscr{K}_{\omega_0} &:= \{\omega \in X : \ \omega = K\omega_0 \ \text{ for } K \in \mathscr{K}\} \\
&\subseteq \mathscr{S}_{\omega_0} := \left\{\omega \in X \ : \ \int_M \omega \,\mathrm{d}x = \int_M \omega_0 \,\mathrm{d}x, \quad \int_M (\omega - c)_+ \,\mathrm{d}x \right. \\
&\left. \leqq \int_M (\omega_0 - c)_+ \,\mathrm{d}x \quad \text{for all} \quad c \in \mathbb{R} \right\}. \quad (2.19)
\end{aligned}$$

It thus remain to prove that given an element $\omega \in \mathscr{S}_{\omega_0}$, there exists $K \in \mathscr{K}_{\omega_0}$ such that $\omega = K\omega_0$. This is indeed a classical result in *rearrangement inequalities* [11] which we prove below.

For any set $A \in \mathbb{R}^2$ define $A^\#$ as the ball centered at the origin such that $|A| = |A^\#|$. Given a function f, its distribution function is given by

$$d_f(t) = |\{x \in M : f(x) > t\}| \qquad \text{for any } t \in \mathbb{R}. \tag{2.20}$$

The *Hardy-Littlewood-Polya decreasing rearrangement* [22] is defined as

$$f^*(s) = \sup\{\tau \in \mathbb{R} : d_f(\tau) > s\} \qquad \text{for } s \in [0, |M|), \tag{2.21}$$

and the *Schwarz spherical decreasing rearrangement* is given by

$$f^\#(x) = f^*(4\pi|x|), \qquad \text{for } x \in B_R(0), \quad R = |M|. \tag{2.22}$$

The function $f^\#$ is obtained by rearranging the level sets of f in a symmetric and radially decreasing way. The functions f, f^* are equimeasurable, and hence also $f^\#$. This imply that

$$\{f > t\}^\# = \{f^\# > t\}, \tag{2.23}$$

since both sets are balls centered at the origin with the same volume. We are also going to use the *Hardy-Littlewood-Polya inequality* which read as

$$\int_M fg \, \mathrm{d}x \leqq \int_{B_R} f^\# g^\# \, \mathrm{d}x = \int_{[0,R]} f^* g^* \, \mathrm{d}s. \tag{2.24}$$

This can be easily proved through the layer cake decomposition.

To prove $\mathscr{S}_{\omega_0} = \mathscr{K}_{\omega_0 t}$, we first observe that the following conditions are equivalent:

(i) $f \in \mathscr{S}_g$
(ii) $\int_0^r f^* \leqq \int_0^r g^*$ for any $r \in [0, R]$ and $\int_0^R f^* = \int_0^R g^*$
(iii) $\int_{B_r} f^\# \leqq \int_{B_r} g^\#$ for any $r \in [0, R]$ and $\int_{B_R} f^\# = \int_{B_R} g^\#$.

The equivalence between (ii) and (iii) is straightforward, while (i) $\iff$ (ii) is proved in [10, Theorem 1.6].

If (ii) holds, for any $u \geqq 0$ we have

$$\int_M f u \, \mathrm{d}x \leqq \int_{B_R} f^\# u^\# \, \mathrm{d}x \leqq \int_{B_R} g^\# u^\# \, \mathrm{d}x \tag{2.25}$$

where the first inequality is (2.24). To prove the last inequality above, let $u = \sum_{i=0}^N a_i \chi_{A_i}$ with $a_i > a_{i+1} \geqq 0$. Then $u^* = \sum_{i=0}^N a_i \chi_{[r_i, r_{i+1}]}$ with

$$r_0 = 0, \quad r_{N+1} = R, \quad r_{i+1} - r_i = |A_i|.$$

Defining $F(r) = \int_0^r f^*$, $G(r) = \int_0^r g^*$, observe that

$$\int_{B_R} (f^\# - g^\#) u^\# \, \mathrm{d}x = \int_0^R u^* \frac{\mathrm{d}}{\mathrm{d}r}(F - G)\mathrm{d}r = \sum_{i=0}^N a_i((F(r_{i+1}) - G(r_{i+1})) - (F(r_i) - G(r_i))). \tag{2.26}$$

Then, since $F(0) = G(0) = 0$ and $F(R) = G(R)$ by the conservation of the mean, we deduce that

$$\sum_{i=1}^{N} a_i((F(r_{i+1}) - G(r_{i+1})) - (F(r_i) - G(r_i)) \\ = \sum_{i=1}^{N-1} (a_{i-1} - a_i)(F(r_i) - G(r_i)) \leqq 0 \tag{2.27}$$

and the last inequality follows by $a_{i-1} \geqq a_i$ and the fact that $F(r_i) \leqq G(r_i)$ in account of (ii) above. The general case is recovered by a standard approximation argument.

Finally, given $\omega \in \mathscr{S}_{\omega_0}$, assume that $\omega \notin \mathscr{K}_{\omega_0}$. We now follow the arguments in [16,39]. Since $\mathscr{K}_{\omega_0}$ is convex and weakly closed in L^1, if $\omega \in L^\infty(M)\backslash\mathscr{K}_{\omega_0} \subset L^1(M)\backslash\mathscr{K}_{\omega_0}$, by the Hahn-Banach theorem there exists $g \in L^\infty$ such that

$$\int_M gK\omega_0 \, \mathrm{d}x < \int_M g\omega \, \mathrm{d}x, \qquad \text{for any } K \in \mathscr{K}. \tag{2.28}$$

Since $\int K\omega_0 = \int \omega_0 = \int \omega$, we can assume that $g \geqq 0$. Then, for each $f \in L^1$ there exists a measure preserving map $\sigma_f : M \to B_R$ such that $f = f^\# \circ \sigma_f$ [16]. Hence, let $g = g^\# \circ \sigma_g$ and $\omega_0 = \omega_0^\# \circ \sigma_{\omega_0}$. Now, if σ_g and σ_{ω_0} are one-to-one it is enough to choose $K(x, y) = \delta(y - (\sigma_{\omega_0}^{-1} \circ \sigma_g)(x))$ to get

$$\int_M gK\omega_0 \, \mathrm{d}x = \int_M (g^\#\omega_0^\#) \circ \sigma_g \, \mathrm{d}x = \int_{B_R} g^\#\omega_0^\# \, \mathrm{d}x \geqq \int_{B_R} g^\#\omega^\# \, \mathrm{d}x \geqq \int_M g\omega \, \mathrm{d}x \tag{2.29}$$

where the last two bounds follows by (2.25), but this is a contradiction and hence $\omega \in \mathscr{K}_{\omega_0}$. When σ_g and σ_{ω_0} are not one-to-one, we need to define bistochastic operators $\widetilde{K} : L^1(B_R) \to L^1(M)$ with adjoint $\widetilde{K}^* : L^\infty(M) \to L^\infty(B_R)$ where

$$\int_M f\widetilde{K}g \, \mathrm{d}x = \int_{B_R} g\widetilde{K}^* f \, \mathrm{d}x. \tag{2.30}$$

The operators $\widetilde{K}$ are the weak-$*$ closure of area preserving diffeomorphisms from B_R to M. If $\widetilde{K}$ is associated to an area preserving map then

$$\widetilde{K}^*\widetilde{K} = \mathrm{id}.$$

This extension is necessary since if $\widetilde{K}(x, y) = \delta(y - \sigma(x))$ for $\sigma : B_R \to M$ area preserving map then $\widetilde{K}^*$ is not in general associated to an area preserving map [39]. Anyway, we know that $g = g^\# \circ \sigma_g := \widetilde{K}_1 g^\#$ and $\omega_0 = \omega_0^\# \circ \sigma_{\omega_0} := \widetilde{K}_2\omega_0^\#$. Choosing $K = \widetilde{K_1}\widetilde{K_2}^* : L^\infty(M) \to L^\infty(M)$, with $K \in \mathscr{K}$, we conclude

$$\int_M gK\omega_0 \, \mathrm{d}x = \int_M (\widetilde{K}_1 g^\#)\widetilde{K}_1(\widetilde{K_2}^*\widetilde{K}_2\omega_0^\#) \, \mathrm{d}x = \int_{B_R} g^\#\omega_0^\# \, \mathrm{d}x \\ \geqq \int_{B_R} g^\#\omega^\# \, \mathrm{d}x \geqq \int_M g\omega \, \mathrm{d}x, \tag{2.31}$$

which is a contradiction with (2.28), meaning that we must have $\omega \in \mathscr{K}_{\omega_0}$.

⋄ STEP 3: ((2.9) = (2.10)) This was proved in [10, Theorem 2.5] and also used in [3,44]. The inclusion (2.9)⊆ (2.10) is obvious. Let us show a short proof of the remaining inclusion. We first observe that in (2.9) it is enough to consider

$$c \in [\min\{\omega_0\}, \max\{\omega_0\}] := I_0.$$

Indeed, if $c \geqq \max\{\omega_0\}$ the inequality is trivial. Then, from the characterization (2.8) we know that $\omega \in I_0$. Thus, for all $c < \min\{\omega_0\}$ we have

$$\int_M (\omega - c)_+ \,\mathrm{d}x = \int_M (\omega - c)\,\mathrm{d}x = \int_M (\omega_0 - c)\,\mathrm{d}x = \int_M (\omega_0 - c)_+ \,\mathrm{d}x, \tag{2.32}$$

where the identity in the middle follows by the conservation of the mean. Then, to prove that (2.10)⊂ (2.9) let us first consider $f \in C^2$. Given $s \in I_0$, integrating by parts we have

$$\begin{aligned}\int_{I_0} (s-c)_+ f''(c)\mathrm{d}c &= f'(c)(s-c)_+\Big|_{\min\{\omega_0\}}^{\max\{\omega_0\}} - \int_s^{\max\{\omega_0\}} f'(c)\mathrm{d}c \\ &= f(s) + f'(\min\{\omega_0\})(s - \min\{\omega_0\}) - f(\max\{\omega_0\}).\end{aligned} \tag{2.33}$$

Using conservation of the mean again, since $f'' \geqq 0$ by convexity, combining (2.33) with (2.9) we get

$$\begin{aligned}\int_M f(\omega) - f(\omega_0)\mathrm{d}x &= \int_M \mathrm{d}x \int_{I_0} ((\omega - c)_+ - (\omega_0 - c)_+) f''(c)\mathrm{d}c \\ &= \int_{I_0} f''(c)\mathrm{d}c \int_M ((\omega - c)_+ - (\omega_0 - c)_+)\mathrm{d}x \leqq 0.\end{aligned} \tag{2.34}$$

For any convex function f, the representation (2.33) is given by

$$f(s) = \alpha_0 + \alpha_1 s + \int (s-c)_+ \mathrm{d}\alpha(c), \tag{2.35}$$

where α_0, α_1 are constants and $\alpha(c)$ is a positive measure. This again follows by approximation. □

3. Characterization of the minimizers

In this section, we aim at proving Theorem 1 in different steps. We first prove the existence of a minimizer. Then we show *(i)* and *(ii)*.

3.1. Existence

We exploit the characterization (2.8). Define

$$\inf_{\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E}=\mathsf{E}_0\}} \mathsf{I}_f(\omega) = \alpha.$$

Let $\omega_n \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ be a minimizing sequence, namely $\mathsf{I}_f(\omega_n) \to \alpha$. Thanks to Proposition 2.4, we know that $\omega_n = K_n\omega_0$ for $K_n \in \mathscr{K}$. Since $\mathscr{K}$ is weakly compact in L^2, there exists a converging subsequence $K_{n_j} \rightharpoonup K^*$ so that $\omega_{n_j} \rightharpoonup \omega^* := K^*\omega_0$ in L^2. Since $\mathsf{E}(\omega_{n_j}) = \mathsf{E}_0$, by compactness we have $\mathsf{E}(\omega^*) = \mathsf{E}_0$. Hence, $\omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. In account of the lower semicontinuity, we get $\mathsf{I}_f(\omega^*) \leqq \liminf \mathsf{I}_f(\omega_{n_j}) = \alpha$, so that $\mathsf{I}_f(\omega^*) = \alpha$. Therefore, the minimum is attained in $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$.

3.2. Minimizers are minimal

To argue that the minimizers are minimal, we must use Lemma 2.3 which we now prove.

Proof of Lemma 2.3. We first show that $\mathsf{I}_f(K\omega) \leqq \mathsf{I}_f(\omega)$ for any $K \in \mathscr{K}$. Since K is bistochastic, we know that $\mu_x(\mathrm{d}y) = K(x,y)\mathrm{d}y$ is a probability measure. By Jensen's inequality, we have

$$\mathsf{I}_f(K\omega) = \int_M f\left(\int_M \omega(y)\mu_x(\mathrm{d}y)\right)\mathrm{d}x \leqq \iint_{M\times M} f(\omega(y))\mu_x(\mathrm{d}y)\mathrm{d}x = \mathsf{I}_f(\omega), \tag{3.1}$$

where the last identity follows by $\int K(x,y)\mathrm{d}x = 1$. Therefore, we know that $\mathsf{I}_f(\omega_1) \leqq \mathsf{I}_f(\omega)$ and if $\omega = \widetilde{K}\omega_1$ also $\mathsf{I}_f(\omega) \leqq \mathsf{I}_f(\omega_1)$ meaning that $\mathsf{I}_f(\omega) = \mathsf{I}_f(\omega_1)$.

It thus remain to prove that if $\mathsf{I}_f(\omega_1) = \mathsf{I}_f(\omega)$ then there exists $\widetilde{K} \in \mathscr{K}$ such that $\omega = \widetilde{K}\omega_1$. Since the bound (3.1) is obtained pointwise for the integrand, when equality holds we have that for a.a. $x \in M$

$$f\left(\int_M \omega(y)\mu_x(\mathrm{d}y)\right) = \int_M f(\omega(y))\mu_x(\mathrm{d}y). \tag{3.2}$$

Since f is strictly convex, the equality case in the Jensen's inequality holds if and only if $\omega(y) = c_x\mu_x$-almost everywhere for c_x constant in y. Given a function $g : M \to \mathbb{R}$, define the set

$$S^g_{a,b} = \{y \in M : \ a < g(y) < b\}. \tag{3.3}$$

Since ω is μ_x-almost everywhere constant, observe that

$$\mu_x(S^\omega_{a,b}) = \begin{cases} 1 & a < c_x < b \\ 0 & \text{otherwise} \end{cases}. \tag{3.4}$$

In addition, since $\omega_1 = K_1\omega = \int \omega(y)\mu_x(\mathrm{d}y)$ and $\omega(y)$ is μ_x-almost everywhere constant, we infer

$$|S^{\omega_1}_{a,b}| = |\{x \in M : a < \omega_1(x) < b\}| = |\{x \in M : a < \int \omega(y)\mu_x(\mathrm{d}y) < b\}|$$
$$= |\{x \in M : a < c_x < b\}| = |\{x \in M : \mu_x(S^{\omega}_{a,b}) = 1\}|. \tag{3.5}$$

Since K_1 is bistochastic we also have

$$\begin{aligned}|S^{\omega}_{a,b}| &= \int_{\{y\in M:\, a<\omega(y)<b\}} \mathrm{d}y = \iint_{M\times\{y\in M:\, a<\omega(y)<b\}} K_1(x,y)\mathrm{d}y\mathrm{d}x \\ &= \iint_{M\times\{y\in M:\, a<\omega(y)<b\}} \mu_x(\mathrm{d}y)\mathrm{d}x \\ &= \int_M \mu_x(S^{\omega}_{a,b})\mathrm{d}x = |\{x \in M : \mu_x(S^{\omega}_{a,b}) = 1\}| = |S^{\omega_1}_{a,b}|,\end{aligned}$$

meaning that ω and ω_1 are equimeasurable. Indeed, choosing $a = -\infty$, $b = -t$ and $a = t$, $b = +\infty$, we get

$$|\{x \in M : |\omega_1(x)| > t\}| = |\{x \in M : |\omega(x)| > t\}|. \tag{3.6}$$

Through the layer-cake representation, this implies that

$$\|\omega_1\|_{L^p} = \|\omega\|_{L^p}, \qquad \text{for any } 1 \leqq p < \infty. \tag{3.7}$$

We now have to "invert" K_1. From (2.11), since $\omega_1 = K_1\omega$ we know that there exists a sequence of permutations p_n such that $\omega \circ p_n \rightharpoonup \omega_1$ in L^2. Combining the weak convergence with (3.7) and the fact that p_n is area preserving, notice that

$$\|\omega \circ p_n - \omega_1\|^2_{L^2} = \|\omega \circ p_n\|^2_{L^2} + \|\omega_1\|^2_{L^2} - 2\int_M \omega_1(\omega \circ p_n)\,\mathrm{d}x \overset{n\to\infty}{\longrightarrow} \|\omega\|^2_{L^2}$$
$$+ \|\omega_1\|^2_{L^2} - 2\,\|\omega_1\|^2_{L^2} = 0. \tag{3.8}$$

Namely $\omega \circ p_n \to \omega_1$ in L^2. Since p_n is area preserving, we also get $\omega_1 \circ p_n^{-1} \to \omega$ in L^2. We can then define $\widetilde{K} \in \mathscr{K}$ as the operator obtained in the weak limit of $i_{p_n^{-1}}$ (see Step 1 in §4), that is

$$\lim_{n_j\to\infty} \int_M g(x)(\omega_1 \circ p_{n_j}^{-1})(x)\mathrm{d}x = \int_M g(x)\int_M \omega_1(y)\widetilde{K}(x,y)\mathrm{d}y\mathrm{d}x. \tag{3.9}$$

Since $\omega_1 \circ p_n^{-1} \to \omega$ in L^2, we get $\omega = \widetilde{K}\omega_1$, whence the lemma is proved. □

As a consequence of Lemma 2.3 we have

Corollary 3.1. *Any minimizer ω^* of the variation problem (1.16) is a minimal flow.*

Proof of Corollary 3.1. Let ω^* be a minimizer of I_f with f strictly convex, see (1.16). Consider $\omega_1 \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ and let K be such that $\omega_1 = K\omega^*$. As shown in the proof of Lemma 2.3, we have $\mathsf{I}_f(\omega_1) \leqq \mathsf{I}_f(\omega^*)$. However, being ω^* a minimizer we must have $\mathsf{I}_f(\omega_1) = \mathsf{I}_f(\omega^*)$. Thus, ω^* is minimal in the sense of Definition 1.2. □

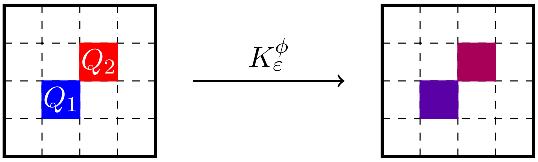

Fig. 4. The operator K_ε^ϕ is a proper mixing if we are exchanging squares where ω has different values

3.3. Minimal flows are stationary

Having at hand a minimal flow as a solution to the problem (1.16), it remains to show that is indeed a stationary solution.

Lemma 3.2. (Minimal flows are Euler steady states) *For any minimal flow $\omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, there exists a bounded monotone function $F : \mathbb{R} \to \mathbb{R}$ such that $\omega^* = F(\psi^*)$ where $\Delta\psi^* = \omega^*$.*

The lemma above was proved by Shnirelman in [41] with an alternative (but equivalent in light of Lemma 2.3) definition of minimal flows. We show this can be directly proved directly using Definition 1.2.

Proof of Lemma 3.2. We adapt the variational argument used by Shnirelman in [41, Theorem 2] (and also by Segre and Kida in [40, Sec A.2]) to our situation. Let ϕ be a permutation of two arbitrary squares Q_1, Q_2 in M. By Lemma 2.5, we know that we can associate a bistochastic operator to ϕ. Then, by convexity of $\mathscr{K}$, notice that the operator K_ε defined as

$$K_\varepsilon^\phi \omega = (1-\varepsilon)\omega + \varepsilon(\omega \circ \phi) \tag{3.10}$$

is bistochastic. See Fig. 4 for a visualization of such operator.

By the choice of ϕ, computing the first variation of the energy we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathsf{E}(K_\varepsilon^\phi\omega^*)|_{\varepsilon=0} &= \int_{Q_1\cup Q_2}(\omega^*(x)-\omega^*(\phi(x)))\psi^*(x)\mathrm{d}x\\ &= \int_{Q_1}(\omega^*(x)-\omega^*(\phi(x)))(\psi^*(x)-\psi^*(\phi(x)))\mathrm{d}x. \end{aligned}\tag{3.11}$$

We claim that the last integral in (3.11) cannot change sign. Otherwise, there are $K_\varepsilon^{\phi_1}$ and $K_\varepsilon^{\phi_2}$, exchanging squares Q_1^ℓ, Q_2^ℓ with $\ell = 1, 2$, such that

$$\mathsf{E}(K_\varepsilon^{\phi_1}\omega^*) > \mathsf{E}(\omega^*), \qquad \mathsf{E}(K_\varepsilon^{\phi_2}\omega^*) < \mathsf{E}(\omega^*). \tag{3.12}$$

If the inequalities above hold, there exists $0 < \lambda < 1$ such that energy is preserved:

$$\tilde{K}_\varepsilon\omega := (\lambda K_\varepsilon^{\phi_1}\omega + (1-\lambda)K_\varepsilon^{\phi_2}\omega) = (1-\varepsilon)\omega + \varepsilon(\lambda\omega\circ\phi_1 + (1-\lambda)\omega\circ\phi_2), \tag{3.13}$$

$$\mathsf{E}(\tilde{K}_\varepsilon\omega^*) = \mathsf{E}(\omega^*) = \mathsf{E}_0. \tag{3.14}$$

This means that $\tilde{K}_\varepsilon \omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. Notice that the condition (3.12) implies that $\omega \circ \phi_i$ is not equal to ω almost everywhere (namely, we are not exchanging squares where the vorticity is a constant). Then, since f is strictly convex and $\omega \circ \phi_i$ is not equal to ω almost everywhere, notice that

$$\begin{aligned}
&\mathsf{I}_f(\tilde{K}_\varepsilon \omega^*) - \mathsf{I}_f(\omega^*) \\
&\quad = \int_M \big(f\left((1-\varepsilon)\omega^* + \varepsilon(\lambda\omega^* \circ \phi_1 + (1-\lambda)\omega^* \circ \phi_2)\right) - f(\omega^*)\big)\, \mathrm{d}x \\
&\quad < \varepsilon \int_M \big(\lambda f(\omega^* \circ \phi_1) + (1-\lambda) f(\omega^* \circ \phi_2) - f(\omega^*)\big)\, \mathrm{d}x = 0, \qquad (3.15)
\end{aligned}$$

where the last identity follows since ϕ_i are area preserving maps. Therefore $\mathsf{I}_f(\tilde{K}_\varepsilon \omega^*) < \mathsf{I}_f(\omega^*)$. Since $\tilde{K}_\varepsilon \omega^* \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$, this contradicts ω^* being minimal according to Definition 1.2. Hence, this implies

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathsf{E}(K_\varepsilon^\phi \omega^*)|_{\varepsilon=0} = \int_{Q_1} (\omega^*(x) - \omega^*(\phi(x)))(\psi^*(x) - \psi^*(\phi(x)))\mathrm{d}x \geqq 0 \text{ (or } \leqq 0). \qquad (3.16)$$

Since the choice of Q_1 is arbitrary, almost everywhere in M we get

$$\begin{aligned}
&(\omega^*(x) - \omega^*(y))(\psi^*(x) - \psi^*(y)) \\
&\quad \geqq 0, \quad \text{or} \quad (\omega^*(x) - \omega^*(y))(\psi^*(x) - \psi^*(y)) \leqq 0. \qquad (3.17)
\end{aligned}$$

One can now directly apply [41, Lemma 1], whose proof is recalled here for convenience of the reader. Since $\Delta\psi^* = \omega^*$ and $\omega^* \in L^\infty$, one has $\psi^* \in C^{1,\beta}(\overline{M})$ for any $0 < \beta < 1$. Thus, $\psi^*(M)$ is the segment $[\min\psi^*, \max\psi^*]$. For each $x \in M$, consider $(\psi^*(x), \omega^*(x))$ on the ψ^*-ω^* plane. We know $\bigcup_{x\in M}\psi^*(x) = [\min\psi, \max\psi]$ while $\bigcup_{x\in M}\omega^*(x) \subseteq [\min\omega^*, \max\omega^*]$. We can therefore write the relation $\omega = F(\psi)$ with F bounded, but in general can be multivalued. Thanks to (3.17), F must be a monotone function, non-decreasing when $\geqq 0$, non-increasing for the case $\leqq 0$ in (3.17). To check that F is also single-valued, one needs to study what happens when ψ is constant. Let $M_j = \{x \in M : \psi(x) \equiv \psi_j\}$. If $|M_j| > 0$ then there exists a ball $B_j \subset M_j$, so that $0 \equiv \Delta\psi|_{B_j} = F(\psi_j)$. Since $F(\psi)$ is monotone, there is at most one value such that $F(\psi_\ell^-) \leqq 0 \leqq F(\psi_\ell^+)$, but here we can define $F(\psi_\ell) \equiv 0$, so that $\omega = F(\psi)$ is a relation satisfying the properties required in *(i)*. □

Since ω^*, the minimizer of I_f, is minimal, applying Lemma 3.2 concludes the proof of (i) in Thm 1.

3.4. Unconstrained characterization

We exploit the abstract optimality theorem given by Rakotoson and Serre in [35, Theorem 2], which reads as follows:

Theorem 3. *Let* X, Y *be two normed real vector spaces whose dual spaces are respectively* $\mathsf{X}^*, \mathsf{Y}^*$*. Let* $\mathsf{C} \subset \mathsf{Y}$ *be a convex cone*[3] *with non-empty interior. Let* g_0 *be an optimal solution of the problem*

$$J(g_0) = \inf\{J(g) : g \in \mathsf{X}, \quad Sg \in -\mathsf{C}\}, \tag{3.18}$$

where $J : \mathsf{X} \to \mathbb{R}$ *and* $S : \mathsf{X} \to \mathsf{Y}$*. Suppose that*

(H1) For all $h \in X$*, the first variations of* J, S *along* h *are well defined at* g_0*, that is*

$$\lim_{\varepsilon \to 0^+} \frac{J(g_0 + \varepsilon h) - J(g_0)}{\varepsilon} = J'(g_0; h),$$
$$\lim_{\varepsilon \to 0^+} \frac{S(g_0 + \varepsilon h) - S(g_0)}{\varepsilon} = S'(g_0; h). \tag{3.19}$$

(H2) The map $h \mapsto J'(g_0; h) \in \mathbb{R}$ *is convex. The map* $h \mapsto S'(g_0; h) \in \mathsf{Y}$ *is convex in the following sense: for all* $\lambda \in [0, 1]$ *and* $h_1, h_2 \in \mathsf{X}$ *one has*

$$S'(g_0; \lambda h_1 + (1 - \lambda)h_2) - \lambda S'(g_0; h) - (1 - \lambda)S'(g_0; h) \in -\mathsf{C}. \tag{3.20}$$

Then, there exists $c_0 \geqq 0$ *and* $\lambda^* \in \mathsf{C}^* = \{L \in Y^* : \text{ for all } f \in \mathsf{C},\ \langle L, f\rangle \geqq 0\}$*, such that the following holds true: for all* $h \in \mathsf{X}$

$$c_0 J'(g_0; h) + \langle \lambda^*, S'(g_0; h)\rangle \geqq 0, \tag{3.21}$$
$$\langle \lambda^*, Sg_0\rangle = 0. \tag{3.22}$$

with $(c_0, \lambda^*) \neq (0, 0)$*.*

Remark 3.3. Theorem 3 is a natural generalization of the Karush-Kuhn-Tucker theory [46] to the case with an infinite number of inequality constraints, see also Appendix A.

We aim at applying Theorem (3) in the following setting: let C be the convex cone

$$\begin{aligned} \mathsf{C} = \{f \in L^\infty(\mathbb{R}) \ : \ f(x) \geqq 0, \ \ \text{for almost everywhere } x \in \mathbb{R}\} \times [0, \infty) \times \{0\} \\ =: \mathsf{C}_1 \times [0, \infty) \times \{0\}. \end{aligned} \tag{3.23}$$

Observe that

$$\mathsf{C}^* = \mathsf{C}_1^* \times [0, +\infty) \times \mathbb{R}, \tag{3.24}$$

where C_1^* consists of non-negative, bounded and finitely additive measures that are absolutely continuous with respect to the Lebesgue measure. For any $\omega \in X$, with X given in (1.9), we define the functional $S : X \to L^\infty(\mathbb{R}) \times \mathbb{R}^2$ as

$$S\omega := (S_1\omega, S_2\omega, S_3\omega), \tag{3.25}$$

[3] C is a convex cone if for each $k \in \mathsf{C}$ and $\alpha \in \mathbb{R}_+$ then $\alpha k \in \mathsf{C}$ and $\mathsf{C} + \mathsf{C} \subseteq \mathsf{C}$.

$$(S_1\omega)(c) = \int_M ((\omega - c)_+ - (\omega_0 - c)_+)\,\mathrm{d}x, \qquad \text{for } c \in \mathbb{R}, \tag{3.26}$$

$$S_2\omega = \int_M (\omega_0 - \omega)\,\mathrm{d}x, \tag{3.27}$$

$$S_3\omega = \mathsf{E}(\omega_0) - \mathsf{E}(\omega). \tag{3.28}$$

In account of the characterization (2.9), imposing $S\omega \in -\mathsf{C}$ is equivalent to ask that $\omega \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$. In fact, we just need to check that the mean is conserved. Take $c^* = \min\{\min\{\omega_0 - 1\}, \min\{\omega - 1\}\}$. Then

$$0 \geqq S_1(\omega)(c^*) = \int_M (\omega - c^* - (\omega_0 - c^*))\,\mathrm{d}x = \int_M \omega\,\mathrm{d}x - \int_M \omega_0\,\mathrm{d}x. \tag{3.29}$$

Combining the inequality above with $S_2\omega \leqq 0$ we recover the conservation of the mean. The first variation of S_1 is

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_M ((\omega^* + \varepsilon h - c)_+ - (\omega^* - c)_+)\right) \\ &\quad = \int_M (\chi_{\{\omega^*>c\}} h + \chi_{\{\omega^*=c\}} h_+), \end{aligned} \tag{3.30}$$

so we get that

$$\begin{aligned} S'(\omega, h)(c) &= (S_1'(\omega, h)(c), S_2'h, S_3'(\omega, h)) \\ &= \left(\int_M (\chi_{\{\omega>c\}} h + \chi_{\{\omega=c\}} h_+),\ -\int_M h,\ -\int_M \psi h\right), \end{aligned} \tag{3.31}$$

where $\Delta\psi = \omega$. From the identity above, we deduce that (3.20) holds true. Notice that the linearity with respect to h of S_3' is crucial.

Thanks to this construction, we can rewrite the variational problem (1.16) as

$$\min_{\omega \in \overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E}=\mathsf{E}_0\}} \mathsf{I}_f(\omega) = \min\{\mathsf{I}_f(\omega) : \omega \in X, \quad S\omega \in -\mathsf{C}\}, \tag{3.32}$$

where S, C are respectively defined in (3.25), (3.23) and X in (1.9).

Since I_f, S satisfy the hypotheses *(H1)-(H2)* in Theorem 3, we obtain the following as a consequence of Theorem 3:

Proposition 3.4. *Let $f \in C^1(\mathbb{R})$ be a convex function. Let ω^* be an optimal solution to (3.32). There exists a non-negative measure measure $\lambda^* \in \mathsf{C}_1^*$, $\lambda_f, \lambda_{\mathsf{E}} \in \mathbb{R}$, $\lambda_f^2 + \lambda_{\mathsf{E}}^2 \neq 0$, $\lambda_m \geqq 0$ such that for all $h \in X$*

$$\int_M (\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* - \lambda_m) h\,\mathrm{d}x + \int_{\mathbb{R}}\left(\int_M (\chi_{\{\omega^*>c\}} h + \chi_{\{\omega^*=c\}} h_+)\,\mathrm{d}x\right)\mathrm{d}\lambda^*(c) \geqq 0 \tag{3.33}$$

$$\int_{\mathbb{R}}\left(\int_M ((\omega^* - c)_+ - (\omega_0 - c)_+)\,\mathrm{d}x\right)\mathrm{d}\lambda^*(c) = 0, \tag{3.34}$$

Moreover, defining the Plateau set

$$P(\omega^*) = \{c \in [\operatorname{essinf}\omega^*, \operatorname{esssup}\omega^*] : |\omega^* = c| > 0\}, \tag{3.35}$$

in the sense of distribution we have

$$\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* \in [-\Gamma_2, -\Gamma_1], \tag{3.36}$$

$$\Gamma_1 = \int_{\mathbb{R}} \chi_{\{\omega^*(x)>c\}} \mathrm{d}\lambda^*(c) - \lambda_m, \qquad \Gamma_2 = \Gamma_1 + \chi_{P(\omega^*)}(x) \int_{P(\omega^*)} \mathrm{d}\lambda^*(c). \tag{3.37}$$

This imply that there exists a convex function Φ such that an optimal solution to ([3.32]) is a minimizer in X of the unconstrained functional

$$J_\Phi(\omega) = \lambda_f \mathsf{I}_f(\omega) + \mathsf{I}_\Phi(\omega) + \lambda_{\mathsf{E}}(\mathsf{E}(\omega) - \mathsf{E}_0). \tag{3.38}$$

Remark 3.5. At this level of generality, we are not able to exclude the case $\lambda_f = 0$. This degenerate scenario might include stationary states with constant vorticity. See Appendix B for an example of a minimal flow having this property in a region. We also stress that the conservation of the energy and the mean for ω^* follows by the fact that $S\omega^* \in -\mathsf{C}$.

Remark 3.6. The fact that one can rewrite a constrained minimization problem as an unconstrained one as (3.38), it is standard with a finite number of inequality constraints (the Lagrange multiplier rule or the more general Karush-Kuhn-Tucker theory). That the same happens also with infinite number of constraints was observed, for instance, by Rakotoson and Serre in [35, Remark after Theorem 1]. Our Proposition 3.4 is different to the result obtained in [35] because we use the characterization (2.9) instead of the one with the symmetric decreasing rearrangement; see Step 2 in §2.

Proof. We can apply Theorem 3 to the problem (3.32) to obtain that

$$\int_M (\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* - \lambda_m) h \,\mathrm{d}x + \int_{\mathbb{R}} \left(\int_M (\chi_{\{\omega^*>c\}} h + \chi_{\{\omega^*=c\}} h_+) \,\mathrm{d}x \right) \mathrm{d}\lambda^*(c) \geqq 0, \tag{3.39}$$

with

$$(\lambda_f, \lambda_{\mathsf{E}}, \lambda_m, \lambda^*) \neq (0, 0, 0, 0). \tag{3.40}$$

The equality (3.34) is the orthogonality condition (3.22) for S_1.

We then have to prove that $\lambda_f^2 + \lambda_{\mathsf{E}}^2 \neq 0$. Assume by contradiction that $\lambda_f = \lambda_{\mathsf{E}} = 0$. We can assume $\lambda_m,\ \lambda^* \neq 0$, since if one of the two is zero, also the other must be zero by the arbitrariness of h, whence contradicting (3.40). By a slight abuse of notation we can set $\lambda_m = 1$. From (3.39), we have

$$\int_M h \,\mathrm{d}x \leqq \int_{\mathbb{R}} \left(\int_M (\chi_{\{\omega^*>c\}} h + \chi_{\{\omega^*=c\}} h_+) \,\mathrm{d}x \right) \mathrm{d}\lambda^*(c). \tag{3.41}$$

If $|\omega^* = \text{ess sup}\,\omega^*| > 0$, take $h = \chi_{\{\omega^* = \text{ess sup}\,\omega^*\}}$. On the left hand side of the inequality above, we have something strictly positive. Therefore, λ^* must be a

Dirac mass at $c = \text{ess sup}\,\omega^*$, but this contradicts $\lambda^* \in \mathsf{C}_1^*$ (the Dirac mass is not absolutely continuous with respect to the Lebesgue measure for instance).

Otherwise, recall the definition of the Plateau set given in (3.35). Taking $h = g\chi_{M\backslash P(\omega^*)}$ or $h = -g\chi_{M\backslash P(\omega^*)}$, we see that the inequality (3.42) become the identity

$$\int_{\{M\backslash P(\omega^*)\}} g\,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\{M\backslash P(\omega^*)\}} \chi_{\{\omega^*>c\}} g\,\mathrm{d}x \right) \mathrm{d}\lambda^*(c). \tag{3.42}$$

By Fubini's theorem, we rewrite the above as

$$\int_{\{M\backslash P(\omega^*)\}} \left(1 - \int_{\mathbb{R}} \chi_{\{\omega^*>c\}} \mathrm{d}\lambda^*(c)\right) g\,\mathrm{d}x = 0. \tag{3.43}$$

Taking g to be the term inside brackets, we get

$$1 = \int_{-\infty}^{\text{ess sup}\,\omega^*} \chi_{\{\omega^*(x)>c\}} \mathrm{d}\lambda^*(c), \qquad \text{for a.a. } x \in \{M\backslash P(\omega^*)\}. \tag{3.44}$$

Since we are considering the case $|\omega^* = \text{ess sup}\,\omega^*| = 0$ and we are taking the essential supremum, we can find at least one point $\tilde{x} \in \{M\backslash P(\omega^*)\}$ such that $\omega^*(\tilde{x}) = \text{ess sup}\,\omega^*$ can be taken in (3.44). But (3.44) would imply that λ^* is a Dirac mass concentrated in ess sup ω^*, whence contradicting the continuity w.r.t. the Lebesgue measure of λ^*. Therefore, $\lambda_f^2 + \lambda_{\mathsf{E}}^2 = 0$ is not possible.

To prove (3.36), considering $h \leqq 0$ in (3.33) and using Fubini's theorem as in (3.43), by the definition of Γ_1 in (3.37) we get

$$\int_M (\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* + \Gamma_1)(-h)\,\mathrm{d}x \leqq 0. \tag{3.45}$$

This means $\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* + \Gamma_1 \leqq 0$ in the sense of distribution since we are testing against the non-negative function $-h$. The lower bound with $-\Gamma_2$ follows analogously by testing against $h \geqq 0$ in (3.33).

Finally, to prove (3.38), it is enough to observe that Γ_1 and Γ_2 are of the form $g_1 \circ \omega^*$ and $g_2 \circ \omega^*$ with

$$g_1(s) = \int_{\mathbb{R}} \chi_{\{s>c\}} \mathrm{d}\lambda^*(c) - \lambda_m, \qquad g_2(s) = g_1(s) + \chi_{P(s)} \int_{P(s)} \mathrm{d}\lambda^*(c). \tag{3.46}$$

The second term in g_2 is interpreted as $P(s) = 1$ if $s = c$ for some $c \in [\text{ess inf}\,\omega^*, \text{ess sup}\,\omega^*]$. We can now argue as in [35]. Namely, since $g_1 \leqq g_2$ and are both decreasing, it means that there exists a convex function Φ whose sub-differential, denoted by $\partial\Phi$,[4] at s is the interval $[g_1(s), g_2(s)]$. Thus

$$\lambda_f f'(\omega^*) - \lambda_{\mathsf{E}}\psi^* \in \partial\Phi(\omega^*), \tag{3.47}$$

which, by the definition of the subdifferential, is equivalent to be a minimizer of (3.38) [46]. Note that since Φ is convex, it has at most finitely many discontinuities in its derivative and hence it can be chosen Lipschitz. □

[4] For instance, $(\partial|x|)(0) = [-1, 1]$.

4. Excluding shear flows at infinite times

We now turn our attention to the proof of Theorem 2. In the sequel, we denote $x = (x_1, x_2)$ with $x_1 \in \mathbb{T}$ and $x_2 \in [0, 1]$. We exploit the periodicity in x_1 by taking the Fourier transform on the horizontal variable. For any $f \in L^2(M)$, let

$$f(x_1, x_2) = \sum_{k\in\mathbb{Z}} \hat{f}_k(x_2)e^{ikx_1}, \qquad \hat{f}_k(x_2) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx_1} f(x_1, x_2)\mathrm{d}x_1. \tag{4.1}$$

Given a vorticity ω, the associated streamfunction ψ satisfy

$$\begin{cases} (\partial_{x_2}^2 - k^2)\widehat{\psi}_k = \widehat{\omega}_k, \\ k\widehat{\psi}_k(0) = k\widehat{\psi}_k(1) = 0. \end{cases} \tag{4.2}$$

When $k = 0$, we set $\widehat{\psi}_0(0) = 0$ and, in principle, we could choose $\widehat{\psi}_0(1)$ as we wish. We fix the value of this constant exploiting the conservation of the linear momentum M. Namely, we set

$$\widehat{\psi}_0(1) = \frac{\mathsf{M}}{2\pi}. \tag{4.3}$$

With this choice, the Green's function in the periodic channel with Dirichlet boundary conditions is

$$G_k(x_2, z) = -\begin{cases} x_2(1-z)\mathbb{1}_{k=0} \\ +\frac{1}{k\sinh(k)}\sinh(kx_2)\sinh(k(1-z)), & \text{for } 0 \leqq x_2 \leqq z \leqq 1, \\ (1-x_2)z\mathbb{1}_{k=0} \\ +\frac{1}{k\sinh(k)}\sinh(k(1-x_2))\sinh(kz), & \text{for } 0 \leqq z \leqq x_2 \leqq 1, \end{cases} \tag{4.4}$$

and $\widehat{\psi}_k$ is given by

$$\widehat{\psi}_k(x_2) = \int_0^1 G_k(x_2, z)\widehat{\omega}_k(z)\mathrm{d}z. \tag{4.5}$$

Notice that $\widehat{\omega}_0(x_2) = -\partial_{x_2}\widehat{u}_{1;0}(x_2)$, from which we deduce

$$\widehat{\psi}_0(x_2) = -\int_0^{x_2} \widehat{u}_{1;0}(z)\mathrm{d}z + 2x_2\int_0^1 \widehat{u}_{1;0}(z)\mathrm{d}z, \tag{4.6}$$

and that (4.3) is satisfied.

Recall that ω_b is the background vorticity with energy E_b, momentum M_b and let $\delta > 0$ be a given constant. Let $0 < \varepsilon < \delta$ be a small parameter. We define ξ as a small spatial scales perturbation of ω_b:

$$\xi = \omega_b + \delta\varepsilon^{-2}\mathbb{1}_{\mathsf{B}_\varepsilon}(x_1)\mathbb{1}_{\mathsf{A}_\epsilon}(x_2),$$

$$\mathsf{A}_\varepsilon = [1/2-\varepsilon, 1/2+\varepsilon], \ \mathsf{B}_\varepsilon = [\pi-\varepsilon, \pi+\varepsilon]. \tag{4.7}$$

$$\omega = \xi - \omega_b. \tag{4.8}$$

Notice that ω is an L^∞ approximation of a point vortex (Dirac point mass vorticity). The construction can easily be smoothed to make the approximations C^∞, since we measure closeness to the background only in an integral sense. By the definition of ω, one has

$$\|\xi - \omega_b\|_{L^1} = \|\omega\|_{L^1} = \delta\varepsilon^{-2} \int_{\mathsf{A}_\varepsilon \times \mathsf{B}_\varepsilon} \mathrm{d}x_2 \mathrm{d}x_1 = 4\delta. \tag{4.9}$$

Since the linear momentum is a linear functional, we have $\mathsf{M}(\xi) = \mathsf{M}_b + \mathsf{M}(\omega)$. Thus as $y \in [0, 1]$, a bound analogous to (4.9) readily give us that $|\mathsf{M}(\xi) - \mathsf{M}_b| \lesssim \delta$.

We now turn our attention to the energy. It is natural to expect that $\mathsf{E}(\omega)$ is of order $|\log(\varepsilon)|$ and, for our perturbation, we can compute this explicitly. The Fourier transform of ω is

$$\widehat{\omega}_k(x_2) = \delta \frac{\varepsilon^{-1}}{\pi} \mathbb{1}_{\mathsf{A}_\varepsilon}(x_2) \mathbb{1}_{k=0} + \delta\varepsilon^{-2} \mathbb{1}_{\mathsf{A}_\varepsilon}(x_2) \frac{e^{-ik\pi}}{\pi} \frac{\sin(k\varepsilon)}{k}. \tag{4.10}$$

By Plancherel's theorem, the energy is given by

$$\begin{aligned} \mathsf{E}(\omega) = -\frac{1}{2} \int_M \omega\psi \mathrm{d}x &= -\delta \frac{\varepsilon^{-1}}{2\pi} \int_{1/2-\varepsilon}^{1/2+\varepsilon} \widehat{\psi}_0(x_2) \mathrm{d}x_2 \\ &\quad - \sum_{k \neq 0} \delta \frac{\varepsilon^{-2}}{2\pi k} e^{ik\pi} \sin(k\varepsilon) \int_{1/2-\varepsilon}^{1/2+\varepsilon} \widehat{\psi}_k(x_2) \mathrm{d}x_2 \\ &:= \frac{\delta}{2\pi} \big(\mathcal{E}_0 + \sum_{k\neq 0} \mathcal{E}_k\big). \end{aligned} \tag{4.11}$$

For the $k = 0$ part, since $\partial_{x_2 x_2} \widehat{\psi}_0 = \widehat{\omega}_0$, by Taylor's theorem we get

$$\widehat{\psi}_0(x_2) = \widehat{\psi}_0(1/2) + \frac{\partial_{x_2} \widehat{\psi}_0(1/2)}{2} (x_2 - 1/2) + \frac{\widehat{\omega}_0(\tilde{x}_2)}{6} (x_2 - 1/2)^2 \tag{4.12}$$

with $\tilde{x}_2$ between x_2 and $1/2$. Therefore

$$\mathcal{E}_0 = -2\widehat{\psi}_0(1/2) - \delta \frac{\varepsilon^{-2}}{6\pi} \int_{1/2-\varepsilon}^{1/2+\varepsilon} (x_2 - 1/2)^2 \mathrm{d}x_2 = -2\widehat{\psi}_0(1/2) - \delta \frac{\varepsilon}{18\pi}. \tag{4.13}$$

Using (4.5) and (4.10), by the continuity of the Green's function, we infer that

$$\widehat{\psi}_0(1/2) = \delta \frac{2}{\pi} G_0(1/2, 1/2) + \delta O(\varepsilon) = -\delta \frac{1}{2\pi} + \delta O(\varepsilon), \tag{4.14}$$

so that

$$\mathcal{E}_0 = \frac{\delta}{\pi} + \delta O(\varepsilon). \tag{4.15}$$

To compute $\mathcal{E}_k$, we need to know the stream function for $x_2 \in [1/2 - \varepsilon, 1/2 + \varepsilon]$. By (4.5), when $x_2 \in [1/2 - \varepsilon, 1/2 + \varepsilon]$ and $k \neq 0$ we have

$$
\begin{aligned}
e^{ik\pi}\widehat{\psi}_k(x_2) = & -\delta\varepsilon^{-2}\frac{\sin(k\varepsilon)}{\pi k}\frac{1}{k\sinh(k)}\Big(\sinh(k(1-x_2))\int_{1/2-\varepsilon}^{x_2}\sinh(kz)\mathrm{d}z \\
& + \sinh(kx_2)\int_{x_2}^{1/2+\varepsilon}\sinh(k(1-z))\mathrm{d}z\Big) \\
= & -\delta\varepsilon^{-2}\frac{\sin(k\varepsilon)}{\pi k}\frac{\sinh(k(1-x_2))}{k^2\sinh(k)}(\cosh(kx_2)-\cosh(k(1/2-\varepsilon))) \\
& -\delta\varepsilon^{-2}\frac{\sin(k\varepsilon)}{\pi k}\frac{\sinh(kx_2)}{k^2\sinh(k)}(\cosh(k(1-x_2))-\cosh(k(1/2-\varepsilon))).
\end{aligned}
\tag{4.16}
$$

Since $\sinh(a+b) = \sinh(a)\cosh(b) + \sinh(b)\cosh(a)$ we rewrite the identity above as

$$
e^{ik\pi}\widehat{\psi}_k(x_2) = \delta\varepsilon^{-2}\frac{\sin(k\varepsilon)}{\pi k^3}\left(\frac{\cosh(k(1/2-\varepsilon))}{\sinh(k)}(\sinh(k(1-x_2))+\sinh(kx_2))-1\right).
\tag{4.17}
$$

Computing the integral and using $\cosh(a+b)-\cosh(a-b) = 2\sinh(a)\sinh(b)$, we get

$$
\begin{aligned}
& e^{ik\pi}\int_{1/2-\varepsilon}^{1/2+\varepsilon}\widehat{\psi}_k(x_2)\mathrm{d}x_2 \\
& = 2\delta\varepsilon^{-1}\frac{\sin(k\varepsilon)}{\pi k^3}\left(\frac{\cosh(k/2-k\varepsilon)}{(k\varepsilon)\sinh(k)}(\cosh(k/2+k\varepsilon)-\cosh(k/2-k\varepsilon))-1\right) \\
& = 2\delta\varepsilon^{-1}\frac{\sin(k\varepsilon)}{\pi k^3}\left(2\frac{\cosh(k/2-k\varepsilon)}{\sinh(k)}\sinh(k/2)\frac{\sinh(k\varepsilon)}{k\varepsilon}-1\right).
\end{aligned}
\tag{4.18}
$$

From standard properties of the hyperbolic functions, notice that

$$
\begin{aligned}
2\frac{\cosh(k(1/2-\varepsilon))}{\sinh(k)}\sinh(k/2) &= \frac{\cosh(k(1/2-\varepsilon))}{\cosh(k/2)} \\
&= e^{-|k|\varepsilon} + e^{-|k|(1+\varepsilon)}\left(\frac{e^{2|k|\varepsilon}-1}{1+e^{-|k|}}\right).
\end{aligned}
\tag{4.19}
$$

If $|k\varepsilon| < 1/10$, combining (4.18) with (4.19), using Taylor's formula we infer that

$$
\begin{aligned}
\mathcal{E}_k = & \frac{2\delta/\varepsilon^3}{\pi}\frac{(\sin(k\varepsilon))^2}{k^4}\left((e^{-|k|\varepsilon}-1) + e^{-|k|(1+\varepsilon)}\frac{\sinh(k\varepsilon)}{k\varepsilon}\left(\frac{e^{2|k|\varepsilon}-1}{1+e^{-|k|}}\right)\right. \\
& \left. + e^{-|k|\varepsilon}\left(\frac{\sinh(k\varepsilon)}{k\varepsilon}-1\right)\right) \\
\approx & -\frac{\delta}{|k|} + O(\delta)e^{-|k\varepsilon|/4}.
\end{aligned}
\tag{4.20}
$$

From the identity above for $\mathcal{E}_k$ we deduce the following (rough) bound at large frequencies

$$
|\mathcal{E}_k| \lesssim \frac{\varepsilon}{(\varepsilon k)^4} \qquad \text{for} \qquad |k\varepsilon| \geqq \tfrac{1}{10}.
\tag{4.21}
$$

Putting together (4.15), (4.20) and (4.21) we obtain

$$\begin{aligned}\mathsf{E}(\omega) &\approx \delta^2\Bigg(1 + O(\varepsilon) + \sum_{|k|<(10\varepsilon)^{-1}} \left(\frac{1}{|k|} + O(1)e^{-|k\varepsilon|/4}\right) + \sum_{|k|\geqq(10\varepsilon)^{-1}} \frac{O(\varepsilon)}{(\varepsilon k)^4}\Bigg) \\ &\approx \delta^2\big(|\log(\varepsilon)| + 1 + O(\varepsilon)\big). \end{aligned} \tag{4.22}$$

Taking ε sufficiently small, we finally get $\mathsf{E}(\omega) \approx \delta^2|\log(\varepsilon)|$. Moreover, the main contribution to the energy of ξ is given by ω. Indeed,

$$\mathsf{E}(\xi) = \mathsf{E}(\omega) + \mathsf{E}_b + 2\int_M \psi_b \omega \mathrm{d}x \tag{4.23}$$

Since $\|\psi_b\|_{L^\infty} \lesssim \|\omega_b\|_{L^\infty}$, we have $\left|\int_M \psi_b\omega \mathrm{d}x\right| \lesssim \|\omega_b\|_{L^\infty} \|\omega\|_{L^1} \lesssim \delta \|\omega_b\|_{L^\infty}$. Taking ε sufficiently small so that $\mathsf{E}(\omega) \approx \delta^2|\log(\varepsilon)| \gg \mathsf{E}_b + \delta \|\omega_b\|_{L^\infty}$ we have

$$\mathsf{E}(\xi) \approx \delta^2|\log(\varepsilon)|. \tag{4.24}$$

We are now ready to prove that ξ cannot be rearranged into a shear flow in the set $\overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$. Assume by contradiction that $\widetilde{\omega}_{\mathsf{S}} \in \overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$ is a shear flow, namely $\widetilde{\omega}_{\mathsf{S}} \equiv \widetilde{\omega}_{\mathsf{S}}(x_2)$. By the characterization given in (2.8), we know that

$$\|\tilde{\omega}_{\mathsf{S}}\|_{L^\infty} \lesssim \varepsilon^{-2} \tag{4.25}$$

Moreover, to obtain a shear flow from ξ there are two possibilities:

(1) rearrange the value ε^{-2} in horizontal strips whose total size is ε^2,
(2) do a proper mixing of ω and ω_b and rearrange everything to get a function depending only on x_2.

This last procedure creates a shear flow whose L^∞ norm is smaller with respect to a rearrangement but the resulting shear flow could be big in a larger set. In particular, the worst case scenario is to create a shear flow of the form

$$\widetilde{\omega}_{\mathsf{S}}(x_2) = \begin{cases} O(\mu^{-p}) & \text{on } \widetilde{A}_{\mu^2},\ |\widetilde{A}_{\mu^2}| \leqq \mu^2, \\ O(1) & \text{on } [0,1]\backslash A_{\mu^2}, \end{cases} \tag{4.26}$$

where $0 < \mu \ll 1$ and $p > 0$ are numbers that need to be controlled with the constraints imposed on ω_{S} to belong to $\overline{\mathcal{O}_\xi}^*$. For instance, having $\mu = 1/|\log(\varepsilon)|$ and p too large will give rise to an energy even larger than the one of ξ. However, thanks to the characterization (2.9), if $\widetilde{\omega}_{\mathsf{S}}(x_2) \in \overline{\mathcal{O}_\xi}^*$ one has

$$\int_M |\widetilde{\omega}_{\mathsf{S}}(x_2)| \mathrm{d}x \leqq \int_M |\xi| \mathrm{d}x \leqq \|\omega_b\|_{L^1} + 4\delta, \tag{4.27}$$

which imply

$$\mu^{-p} \lesssim \mu^{-2}(\|\omega_b\|_{L^1} + 4\delta + 1) = O(\mu^{-2}) \tag{4.28}$$

Therefore, the worst case scenario in $\overline{\mathcal{O}_\xi}^*$ is (4.26) with $p = 2$. Split now the vorticity into the large and $O(1)$ part as

$$\widetilde{\omega}_{\mathsf{S}}(x_2) = \widetilde{\omega}_{\mathsf{S}}(x_2)(\mathbb{1}_{A_{\mu^2}} + \mathbb{1}_{[0,1]\setminus A_{\mu^2}})(x_2) := \widetilde{\omega}_{\mathsf{S}}^L(x_2) + \widetilde{\omega}_{\mathsf{S}}^1(x_2). \tag{4.29}$$

We rewrite the energy as

$$\begin{aligned}\mathsf{E}(\widetilde{\omega}_{\mathsf{S}}) &= -\frac{1}{2}\int_0^1 \widetilde{\omega}_{\mathsf{S}}(x_2)\left(\int_0^1 G_0(y,z)\widetilde{\omega}_{\mathsf{S}}(z)\mathrm{d}z\right)\mathrm{d}x_2 \\ &= -\frac{1}{2}\int_0^1 (\widetilde{\omega}_{\mathsf{S}}^L(x_2) + \widetilde{\omega}_{\mathsf{S}}^1(x_2))\left(\int_0^1 G_0(y,z)(\widetilde{\omega}_{\mathsf{S}}^L(z) + \widetilde{\omega}_{\mathsf{S}}^1(z))\mathrm{d}z\right)\mathrm{d}x_2 \\ &:= I_{L,L} + I_{L,1} + I_{1,L} + I_{1,1},\end{aligned} \tag{4.30}$$

where $I_{L,L}$ is the integral containing two large vorticities and so on. Using the boundedness of G_0, we control each term as follows:

$$|I_{L,L}| \lesssim \mu^{-4}\int_{A_{\mu^2}\times A_{\mu^2}} \mathrm{d}x_2\mathrm{d}z = O(1), \tag{4.31}$$

$$|I_{L,1}| + |I_{1,L}| \lesssim \mu^{-2}\int_{A_{\mu^2}\times([0,1]\setminus A_{\mu^2})} \mathrm{d}x_2\mathrm{d}z = O(1), \tag{4.32}$$

$$|I_{1,1}| \lesssim \int_{([0,1]\setminus A_{\mu^2})\times([0,1]\setminus A_{\mu^2})} \mathrm{d}x_2\mathrm{d}z = O(1). \tag{4.33}$$

Therefore, the energy of shear flow $\widetilde{\omega}_{\mathsf{S}}$ obtained through a rearrangement of ξ would satisfy

$$\mathsf{E}(\widetilde{\omega}_{\mathsf{S}}) \lesssim 1. \tag{4.34}$$

In view of (4.24), there is a large energy gap $\mathsf{E}(\xi) \gg \mathsf{E}(\widetilde{\omega}_{\mathsf{S}})$. Thus for any $\widetilde{\omega}_{\mathsf{S}} \in \overline{\mathcal{O}_\xi}^* \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$, we have $\widetilde{\omega}_{\mathsf{S}} \notin \overline{\mathcal{O}_\xi}^* \cap \{\mathsf{E} = \mathsf{E}(\xi)\} \cap \{\mathsf{M} = \mathsf{M}(\xi)\}$. This completes the proof. □

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Data availability statement The code used to produce the data used to visualize the findings of this study is freely available at https://github.com/navidcy/2D-Euler. See also [13,14].

the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A: Maximal mixing theory prediction for vortex patches

Let us now give some examples of the predictions of Shnirelman's maximal mixing theory (Theorem 1 herein). Consider the vorticity to be a finite number of vortex patches, namely $\omega_0 \in X$ given by

$$\omega_0 = \sum_{i=1}^{N} a_i \chi_{A_i}, \tag{A.1}$$

with $a_i \in \mathbb{R}$, $a_i \neq a_j$ and $\cup_i A_i = M$. Without loss of generality, assume that $a_1 \leqq a_2 \leqq \cdots \leqq a_N$. In this case, the characterization (2.9) of the weak-$*$ closure of the orbit of ω_0 can be refined as follows:

Proposition A.1. *Given any $\omega_0 \in X$ of the form (A.1), we have*

$$\overline{\mathcal{O}_{\omega_0}}^* = \left\{\omega \in X \ : \ \int_M \omega = \int_M \omega_0, \int_M (\omega - a_i)_+ \quad \leqq \int_M (\omega_0 - a_i)_+ \ \ \textit{for all} \quad i = 1, \ldots, N\right\}. \tag{A.2}$$

Remark A.2. If $\omega_0 = a_1\chi_{A_1} + a_2\chi_{A_2}$ is comprised of two patches of equal magnitude but opposite strength (for example $a_1 = -a_2 = 1$) occupying equal areas ($|A_1| = |A_2| = \frac{1}{2}|M|$), Prop. A.1 gives

$$\overline{\mathcal{O}_{\omega_0}}^* = \left\{\omega \in X \ : \ \int_M \omega = \int_M \omega_0\right\}. \tag{A.3}$$

Indeed, considering $a_1 < a_2$, from $\int_M(\omega - a_2)_+ \leqq \int_M(\omega_0 - a_2)_+ = 0$ we find $\omega \leqq a_2$. To get the lower bound, let $\varepsilon > 0$. Then

$$\begin{aligned} \int_M (\omega - (a_1+\varepsilon))_+ \leqq \int_M (\omega_0 - (a_1+\varepsilon))_+ &= \int_M (\omega_0 - (a_1+\varepsilon)) \\ &= \int_M (\omega - (a_1+\varepsilon)), \end{aligned} \tag{A.4}$$

where in the last identity we used the conservation of the mean. Recalling the definition of the positive and negative part, that is $(f)_+ = (f+|f|)/2$ and $(f)_- = (|f| - f)/2$, from (A.4) we get $\int_M(\omega - (a_1+\varepsilon))_- \leqq 0$. This imply $\omega \geqq a_1 + \varepsilon$. Sending $\varepsilon \to 0$, we obtain $a_1 \leqq \omega \leqq a_2$. Since we do not have any other constraints, we deduce that any $\omega \in X$ (assuming $a_1 = -a_2$) can be taken.

The main point of (A.2) is that we have a *finite* number of inequality constraints. This is extremely useful since we can characterize the minimizer of (1.16) through the Karush-Kuhn-Tucker (KKT) theory [46].[5] We thus obtain

[5] The extension of the Lagrange multiplier rule when we have inequality constraints.

Proposition A.3. *Let $f \in C^1$ be a convex function. Consider ω_0 as in (A.1) with energy E_0. Then, there exists $\mu_0, \mu_1, \{\lambda_i\}_{i=1}^N \in \mathbb{R}$ such that $\omega^* \in X$ solving*

$$\int_M \left(f'(\omega^*) + \mu_0 \psi^* + \mu_1 + \sum_{i=i}^{N} \lambda_i (\chi_{\omega^* > a_i} + \chi_{\omega^* = a_i} \chi_{w>0}) \right) w \geqq 0, \quad \text{for all } w \in X \tag{A.5}$$

is a minimizer of (1.16). Moreover, for $i = 1, \ldots, N$

$$\lambda_i \geqq 0, \qquad \lambda_i \int_M \left((\omega^* - a_i)_+ - (\omega_0 - a_i)_+ \right) = 0,$$
$$\int_M (\omega^* - a_i)_+ \leqq \int_M (\omega_0 - a_i)_+. \tag{A.6}$$

Remark A.4. In the case of equal patches with opposite strength, that is $\omega_0 = a_1 \chi_{A_1} + a_2 \chi_{A_2}$ with $a_1 = -a_2 = 1$ and $|A_1| = |A_2| = |M|/2$, we can even obtain a stronger characterization. Indeed, from (A.3) we know that it is enough to just impose the conservation of the mean and that $\omega \in X$. In this special case, a standard trick [43,46] in variational problems is to first modify the convex function f as

$$F(\omega) = \begin{cases} f(\omega), & \text{if } |\omega| \leqq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{A.7}$$

This modified convex function will automatically impose that the minimizer ω^* belongs to X. Hence, thanks to (A.3) and the standard Lagrange multiplier rule, any minimizer for the problem (1.16) satisfies $F'(\omega^*) + \mu_0 \psi^* + \mu_1 = 0$, where μ_0, μ_1 are chosen to guarantee the conservation of the energy and the mean respectively. We note that different convex functions f correspond to different minimal flows, thereby showing they need not be unique in the set $\overline{\mathcal{O}_{\omega_0}}^* \cap \{\mathsf{E} = \mathsf{E}_0\}$ for certain ω_0.

Proof of Proposition. A.1. We just have to show that

$$\overline{\mathcal{O}_{\omega_0}}^* \supseteq S_{\omega_0} := \left\{ \omega \in X \ \int_M \omega = \int_M \omega_0, \quad \int_M (\omega - a_i)_+ \leqq \int_M (\omega_0 - a_i)_+ \quad \text{for all} \quad i = 1, \ldots, N \right\},$$

since the reverse inclusion directly follows from (2.9). We are going to exploit the characterization (2.9) of $\overline{\mathcal{O}_{\omega_0}}^*$. We first observe that for $\omega \in \overline{\mathcal{O}_{\omega_0}}^*$ it is enough to consider $c \in [a_1, a_N]$ (one can argue as in Remark A.2). Hence, it is enough to prove that for any $\omega \in S_{\omega_0}$ one has

$$\int_M (\omega - c)_+ \leqq \int_M (\omega_0 - c)_+ \qquad \text{for all } c \in [a_1, a_N]. \tag{A.8}$$

When $c = a_i$, $i = 1, \ldots, N$ there is nothing to prove. Assume that $a_\ell < c < a_{\ell+1}$ for some $\ell \in \{1, \ldots, N\}$. Let $\lambda > 0$ be such that $c = \lambda a_\ell + (1 - \lambda) a_{\ell+1}$. By the convexity of the positive part function, we deduce that

$$\int_M (\omega - c)_+ = \int_M (\lambda(\omega - a_\ell) + (1 - \lambda)(\omega - a_{\ell+1}))_+$$

$$\leqq \lambda \int_M (\omega_0 - a_\ell)_+ + (1-\lambda) \int_M (\omega_0 - a_{\ell+1})_+, \qquad \text{(A.9)}$$

where in the last inequality we used $\omega \in S_{\omega_0}$. Since ω_0 is of the form (A.1) and $a_\ell < c < a_{\ell+1}$, we have

$$\int_M (\omega_0 - c)_+ = \sum_{i \geqq \ell+1} (a_i - c)|A_i|, \qquad \int_M (\omega_0 - a_\ell)_+ = \sum_{i \geqq \ell+1} (a_i - a_\ell)|A_i|, \qquad \text{(A.10)}$$

$$\int_M (\omega_0 - a_{\ell+1})_+ = \sum_{i \geqq \ell+1} (a_i - a_{\ell+1})|A_i|. \qquad \text{(A.11)}$$

Therefore

$$\lambda \int_M (\omega_0 - a_\ell)_+ + (1-\lambda) \int_M (\omega_0 - a_{\ell+1})_+ = \sum_{i \geqq \ell+1} (a_i - c)|A_i| = \int_M (\omega_0 - c)_+.$$

Combining the identities above with (A.9) we prove (A.8), so that $S_{\omega_0} = \overline{\mathcal{O}_{\omega_0}}^*$. □

We now turn our attention to the proof of Proposition A.3.

Proof of Proposition A.3. Thanks to the characterization (A.2), solving (1.16) corresponds to solve a minimum problem with a *finite* number of inequality constraints. Thus we construct the Lagrange function

$$\begin{aligned} L(\omega, \boldsymbol{\lambda}) = \lambda^* I_f(\omega) + \mu_0 (E(\omega) - E(\omega_0)) + \mu_1 \left(\int_M (\omega - \omega_0) \right) \\ + \sum_{i=1}^N \lambda_i \int_M \left((\omega - a_i)_+ - (\omega_0 - a_i)_+ \right), \end{aligned} \qquad \text{(A.12)}$$

with $\boldsymbol{\lambda} = (\lambda^*, \mu_0, \mu_1, \lambda_1, \dots, \lambda_N) \in \mathbb{R}^{N+3}$. Appealing to the KKT theory [46, Theorem 47.E], we know that solving (1.16) is equivalent to solve $\min_{\omega \in X} L(\omega, \boldsymbol{\lambda})$, for a fixed $\boldsymbol{\lambda}$ such that for all $i = 1, \dots, N$ the conditions (A.6) hold with (in our present notation) ω_* replaced by ω. In addition, the so-called Slater condition, that is there exists $\widetilde{\omega} \in X$ such that $\int_M (\widetilde{\omega} - a_i)_+ < \int_M (\omega_0 - a_i)_+$, guarantees that $\lambda^* \neq 0$. This is clearly satisfied in our case, and therefore we consider $\lambda^* = 1$. The coefficients μ_0, μ_1 are chosen to guarantee the conservation of the energy and the mean, respectively.
Then, let ω^* be a minimizer of (1.16). Defining $\omega_\varepsilon = \omega^* + \varepsilon w$, with $w \in X$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\omega_\varepsilon, \boldsymbol{\lambda})|_{\varepsilon=0} \\
&\quad = \int_M (f'(\omega^*) + \mu_0\psi^* + \mu_1) w + \sum_{i=1}^N \lambda_i \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \\
&\qquad \times \left(\int_M ((\omega^* + \varepsilon w - a_i)_+ - (\omega^* - a_i)_+) \right) \\
&\quad = \int_M \left(f'(\omega^*) + \mu_0\psi^* + \mu_1 + \sum_{i=i}^N \lambda_i (\chi_{\omega^*>a_i} + \chi_{\omega^*=a_i}\chi_{w>0}) \right) w. \quad \text{(A.13)}
\end{aligned}
$$

If the term on the right hand side of (A.13) is negative, we would have found $L(\omega_\varepsilon, \boldsymbol{\lambda}) < L(\omega^*, \boldsymbol{\lambda})$, whence contradicting the optimality of ω^*. Therefore, (A.5) is proved. □

Appendix B: An example of minimal flow with piecewise constant vorticity

Consider a shear flow $u(x_1, x_2) = (U(x_2), 0)$ in a periodic channel $M = \mathbb{T} \times [-1, 1]$ with a convex profile $U(x_2)$. It is minimal, even if the function $U(x_2)$ is not strictly convex, and has a flat piece (for example $U(x_2) = \text{const.}$ for $a \leqq x_2 \leqq b$). Let the streamfunction ψ and vorticity $\omega = -U'(x_2)$ defined by

$$
\psi(x_2) = \begin{cases} \frac{1}{2}x_2 + \frac{1}{8} & x_2 \in [-1, -\frac{1}{2}], \\ -\frac{1}{2}x_2^2 & x_2 \in [-\frac{1}{2}, \frac{1}{2}], \\ -\frac{1}{2}x_2 + \frac{1}{8} & x_2 \in [\frac{1}{2}, 1]. \end{cases} \qquad U(x_2) = \begin{cases} -\frac{1}{2} & x_2 \in [-1, -\frac{1}{2}], \\ x_2 & x_2 \in [-\frac{1}{2}, \frac{1}{2}], \\ \frac{1}{2} & x_2 \in [\frac{1}{2}, 1]. \end{cases}
$$

$$
\omega(x_2) = \begin{cases} -1 & x_2 \in (-\frac{1}{2}, \frac{1}{2}), \\ 0 & \text{otherwise}. \end{cases}
$$

We now define squares where the vorticity is 0 and -1. For instance

$$
Q_0 = \begin{cases} [-\frac{\delta}{2}, \frac{\delta}{2}] \times [x_{2,0}, x_{2,0} + \delta] & \frac{1}{2} < x_{2,0} < 1 - \delta, \\ [-\frac{\delta}{2}, \frac{\delta}{2}] \times [x_{2,0} - \delta, x_{2,0}] & -1 + \delta < x_{2,0} < -\frac{1}{2}. \end{cases} \tag{B.1}
$$

$$
Q_{-1} = \begin{cases} [-\frac{\delta}{2}, \frac{\delta}{2}] \times [x_{2,0}, x_{2,0} + \delta] & 0 < x_{2,0} < \frac{1}{2} - \delta, \\ [-\frac{\delta}{2}, \frac{\delta}{2}] \times [x_{2,0} - \delta, x_{2,0}] & -\frac{1}{2} + \delta < x_{2,0} \leqq 0. \end{cases} \tag{B.2}
$$

Notice that $Q_0 \subset \mathbb{T} \times ([1/2, 1] \cup [-1, -1/2])$ and $Q_{-1} \subset \mathbb{T} \times [-1/2, 1/2]$. Let Φ be the area preserving map that exchange Q_0 with Q_{-1} and define $K_\varepsilon\omega = ((1-\varepsilon)\mathrm{id} + \varepsilon\Phi)(\omega)$. Then, by definition we know that

$$
\omega|_{Q_0} = 0, \qquad \omega \circ \Phi|_{Q_0} = \omega|_{Q_{-1}} = -1
$$

$$
\psi|_{Q_0} = \begin{cases} \frac{1}{2}y + \frac{1}{8} & x_{2,0} < -\frac{1}{2}, \\ -\frac{1}{2}y + \frac{1}{8} & x_{2,0} > \frac{1}{2}, \end{cases} \qquad \psi \circ \Phi|_{Q_0} = \psi|_{Q_{-1}} = -\frac{1}{2}x_2^2.
$$

The first variation of the energy is $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E(K_\varepsilon\omega) = \int_{Q_0}(\omega - \omega\circ\phi)(\psi - \psi\circ\phi)$. Consequently, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E(K_\varepsilon\omega) &= \pi\delta\int_{x_{2,0}}^{x_{2,0}+\delta}(x_2^2 - x_2 + \tfrac{1}{4})\mathrm{d}x_2 = \pi\delta\int_{x_{2,0}}^{x_{2,0}+\delta}(x_2 - \tfrac{1}{2})^2\mathrm{d}x_2 > 0,\\
&\qquad \text{for} \quad x_{2,0} > \tfrac{1}{2},\\
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E(K_\varepsilon\omega) &= \pi\delta\int_{x_{2,0}-\delta}^{x_{2,0}}(x_2^2 + x_2 + \tfrac{1}{4})\mathrm{d}x_2 = \pi\delta\int_{x_{2,0}-\delta}^{x_{2,0}}(x_2 + \tfrac{1}{2})^2\mathrm{d}x_2 > 0,\\
&\qquad \text{for} \quad x_{2,0} < -\tfrac{1}{2}.
\end{aligned}$$

Therefore, for any proper mixing we increase the energy—it is an “energy deficient" minimal flow [41]. Another example is the circular flow inside a disk $\mathbb{D}$ given by $u(r,\theta) = V(r)e_\theta$ where $V(r) = 0$ for $0 \leqq r \leqq a$ and convex $V(r)$ which grows $a \leqq r \leqq 1$.

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