Hindawi
Journal of Function Spaces
Volume 2022, Article ID 9094502, 30 pages
https://doi.org/10.1155/2022/9094502

Research Article

On the Space of Locally Sobolev-Slobodeckij Functions

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Received 19 November 2021; Accepted 10 March 2022; Published 18 July 2022

Academic Editor: Guozhen Lu

The study of certain differential operators between Sobolev spaces of sections of vector bundles on compact manifolds equipped with rough metric is closely related to the study of locally Sobolev functions on domains in the Euclidean space. In this paper, we present a coherent rigorous study of some of the properties of locally Sobolev-Slobodeckij functions that are especially useful in the study of differential operators between sections of vector bundles on compact manifolds with rough metric. The results of this type in published literature generally can be found only for integer order Sobolev spaces $W^{m,p}$ or Bessel potential spaces H^s. Here, we have presented the relevant results and their detailed proofs for Sobolev-Slobodeckij spaces $W^{s,p}$ where s does not need to be an integer. We also develop a number of results needed in the study of differential operators on manifolds that do not appear to be in the literature.

1. Introduction

It is well-known that Sobolev spaces play a key role in the study of elliptic partial differential equations (PDEs) on domains in $\mathbb{R}^n$. There are many resources for properties of integer order Sobolev spaces of functions and their applications in PDEs (see, e.g., [1–3]). Also, there are variety of resources for properties of real order Sobolev spaces of functions and their applications (see, e.g., classical references such as [4–8] or more recent works such as [9–13]). Likewise, the study of elliptic PDEs on manifolds naturally leads to the study of Sobolev spaces of functions and more generally Sobolev spaces of sections of vector bundles on manifolds. As it turns out, the study of certain differential operators between Sobolev spaces of sections of vector bundles on manifolds equipped with rough metric and the study of low regularity geometry on Riemannian and semi-Riemannian manifolds are closely related to the study of spaces of locally Sobolev functions on domains in the Euclidean space (see, e.g., [14–16]).

In this paper, we focus on certain properties of spaces of locally Sobolev functions that are particularly useful in the study of differential operators on manifolds. Our work can be viewed as a continuation of the excellent work of Antonic and Burazin [17]; their work is mainly concerned with the properties of spaces of locally Sobolev functions with integer smoothness degree. In particular, they study the following fundamental questions for locally Sobolev spaces with integer smoothness degree:

(i) Topology and metrizability

(ii) Density of smooth functions

(iii) Reflexivity and the nature of the dual

(iv) Continuity of differentiation between certain spaces of locally Sobolev functions

Our main goal here is to provide a self-contained manuscript in which the known results are collected and stated in the general setting of Sobolev-Slobodeckij spaces and then develop certain other results that are useful in the study of differential operators on manifolds. In particular, we will discuss the following topics:

(i) General embedding results

(ii) Pointwise multiplication

(iii) Invariance under composition

The results of this type and other related results have been used in the literature—particularly in the study of Einstein constraint equations on manifolds equipped with rough metric—without complete proof. This paper should be viewed as a part of our efforts to fill some of the gaps. Interested readers can find other results in this direction in [13, 15, 16, 18]. Our hope is that the detailed presentation of this manuscript, along with these other four manuscripts, will help in better understanding the structure of the proofs and the properties of Sobolev-Slobodeckij spaces and locally Sobolev functions.

2. Notation and Conventions

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0$ denotes the set of nonnegative integers. For any nonnegative real number s, the integer part of s is denoted by $\lfloor s \rfloor$. The letter n is a positive integer and stands for the dimension of the space. For all $k \in \mathbb{N}$, $GL(k, \mathbb{R})$ is the set of all $k \times k$ invertible matrices with real entries.

Ω is a nonempty open set in $\mathbb{R}^n$. The collection of all compact subsets of Ω will be denoted by $\mathscr{K}(\Omega)$. If $\mathscr{F}(\Omega)$ is any function space on Ω and $K \in \mathscr{K}(\Omega)$, then $\mathscr{F}_K(\Omega)$ denotes the collection of elements in $\mathscr{F}(\Omega)$ whose support is inside K. Also,

$$\mathscr{F}_{\text{comp}}(\Omega) = \bigcup_{K \in \mathscr{K}(\Omega)} \mathscr{F}_K(\Omega). \tag{1}$$

If $\Omega' \subseteq \Omega$ and $f : \Omega' \longrightarrow \mathbb{R}$, we denote the extension by zero of f to the entire Ω by $\text{ext}^0_{\Omega',\Omega} f : \Omega \longrightarrow \mathbb{R}$, that is,

$$\text{ext}^0_{\Omega',\Omega} f(x) = \begin{cases} f(x), & \text{if } x \in \Omega', \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Lipschitz domain in $\mathbb{R}^n$ refers to a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. We say that a nonempty open set $\Omega \subseteq \mathbb{R}^n$ has the *interior Lipschitz property* provided that for each compact set $K \in \mathscr{K}(\Omega)$ there exists a bounded open set $\Omega' \subseteq \Omega$ with Lipschitz continuous boundary such that $K \subseteq \Omega'$.

Each element of $\mathbb{N}_0^n$ is called a multi-index. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, we let $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Also, for sufficiently smooth functions $u : \Omega \longrightarrow \mathbb{R}$ (or for any distribution u), we define the αth order partial derivative of u as follows:

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{3}$$

We use the notation $A \lesssim B$ to mean $A \leq cB$, where c is a positive constant that does not depend on the nonfixed parameters appearing in A and B. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

If X and Y are two topological spaces, we use the notation $X \hookrightarrow Y$ to mean $X \subseteq Y$, and the inclusion map is continuous.

3. Background Material

In this section, we collect some useful tools and facts we will need from topology and analysis. Statements without proof in this section are mainly taken from Rudin's functional analysis [19], Grubb's distributions and operators [20], excellent presentation of Reus [21], Treves' topological vector spaces [22] and the reference [16], or are direct consequences of statements in the aforementioned references.

3.1. Topological Vector Spaces

Definition 1. A *topological vector space* is a vector space X together with a topology τ with the following properties:

(i) For all $x \in X$, the singleton $\{x\}$ is a closed set

(ii) The maps

$$\begin{aligned} (x, y) &\mapsto x + y \quad (\text{from } X \times X \text{ into } X), \\ (\lambda, x) &\mapsto \lambda x \quad (\text{from } \mathbb{R} \times X \text{ into } X), \end{aligned} \tag{4}$$

are continuous where $X \times X$ and $\mathbb{R} \times X$ are equipped with the product topology

Definition 2. Suppose (X, τ) is a topological vector space and $Y \subseteq X$.

(i) Y is said to be *convex* if for all $y_1, y_2 \in Y$ and $t \in (0, 1)$ it is true that $ty_1 + (1-t)y_2 \in Y$

(ii) We say Y is *bounded* if for any neighborhood U of the origin (i.e., any open set containing the origin), there exits $t > 0$ such that $Y \subseteq tU$

Definition 3. Let (X, τ) be a topological vector space. X is said to be *metrizable* if there exists a metric $d : X \times X \longrightarrow [0, \infty)$ whose induced topology is τ. In this case, we say that the metric d is compatible with the topology τ.

Theorem 4 ([19, 20]). *Let (X, τ) be a topological vector space. The following are equivalent:*

(i) X is metrizable

(ii) There exists a translation invariant metric d on X whose collection of open sets is the same as τ. Translation invariant means

$$\forall x, y, a \in X \quad d(x + a, y + a) = d(x, y) \tag{5}$$

(iii) X has a countable local base at the origin

(Recall that a subcollection $\mathscr{B}$ of τ is said to be a local base at the origin if for any open set U containing the origin there is $B \in \mathscr{B}$ such that $0 \in B \subseteq U$.)

Remark 5. It can be shown that if d_1 and d_2 are two translation invariant metrics that induce the same topology on X, then the Cauchy sequences of (X, d_1) will be exactly the same as the Cauchy sequences of (X, d_2).

Definition 6. Let (X, τ) be a topological vector space. We say (X, τ) is *locally convex* if it has a convex local base at the origin.

Definition 7. Let (X, τ) be a metrizable locally convex topological vector space. Let d be any translation invariant metric on X that is compatible with τ. We say that X is *complete* if and only if the metric space (X, d) is a complete metric space. A complete metrizable locally convex topological vector space is called a *Frechet space*.

Definition 8. A *seminorm* on a vector space X is a real-valued function $p : X \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} &\forall x, y \in X \quad p(x+y) \leq p(x) + p(y), \\ &\forall x \in X \quad \forall \alpha \in \mathbb{R} \quad p(\alpha x) = |\alpha| p(x). \end{aligned} \tag{6}$$

If $\mathscr{P}$ is a family of seminorms on X, then we say $\mathscr{P}$ is *separating* provided that for all $x \neq 0$, there exists at least one $p \in \mathscr{P}$ such that $p(x) \neq 0$ (that is, if $p(x) = 0$ for all $p \in \mathscr{P}$, then $x = 0$). It easily follows from the definition that any seminorm is a nonnegative function.

Theorem 9. *Suppose that $(X, \|.\|_X)$ is a normed space. Let $p : X \longrightarrow \mathbb{R}$ be a seminorm on X. If p is continuous, then there exists a constant $C > 0$ such that*

$$\forall x \in X \quad p(x) \leq C\|x\|_X. \tag{7}$$

Proof. p is continuous at 0 so there exists $\delta > 0$ such that if $\|x\|_X \leq \delta$, then $|p(x)| < 1$. If $x \neq 0$, then $\delta(x/\|x\|_X)$ has norm equal to δ, and so for all $x \neq 0$, $p(\delta(x/\|x\|_X)) < 1$. Hence, for all $x \neq 0$, we have

$$p(x) \leq \frac{1}{\delta}\|x\|_X. \tag{8}$$

Since $p(0) = 0$, clearly the above inequality also holds for $x = 0$. □

Definition 10. Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. The *natural topology* induced by $\mathscr{P}$ is the smallest topology on X that is translation invariant and with respect to which every $p \in \mathscr{P}$ is a continuous function from X to $\mathbb{R}$. (Recall that *translation invariant* means if $U \subseteq X$ is open, then $U + x$ is open for every $x \in X$.)

Remark 11. Suppose that $\mathscr{P}$ and $\mathscr{P}'$ are two separating family of seminorms on a vector space X. Let τ and τ' be the corresponding natural topologies on X. It follows immediately from the definition that if (1) $p : (X, \tau') \longrightarrow \mathbb{R}$ is continuous for each $p \in \mathscr{P}$ and (2) $p' : (X, \tau) \longrightarrow \mathbb{R}$ is continuous for each $p' \in \mathscr{P}'$, then $\tau = \tau'$.

The following theorem can be viewed as an extension of Theorem 9.

Theorem 12 ([21], page 157). *Let X be a vector space and suppose $\mathscr{P}$ is a separating family of seminorms on X. Equip X with the corresponding natural topology. Then, a seminorm $q : X \longrightarrow \mathbb{R}$ is continuous if and only if there exist $C > 0$ and $p_1, \cdots, p_m \in \mathscr{P}$ such that for all $x \in X$*

$$q(x) \leq C(p_1(x) + \cdots + p_m(x)). \tag{9}$$

Theorem 13 ([19, 20]). *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X and τ is the corresponding natural topology on X. Then, (X, τ) is a locally convex topological vector space. Moreover, if $\mathscr{P} = \{p_k\}_{k \in \mathbb{N}}$ is countable, then the locally convex topological vector space (X, τ) is metrizable, and the following translation invariant metric on X is compatible with τ:*

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1 + p_k(x-y)}. \tag{10}$$

Corollary 14. *Suppose $\mathscr{P}$ is a countable separating family of seminorms on a vector space X and τ is the corresponding natural topology on X. Then, (X, τ) is a Frechet space if and only if it is complete.*

Theorem 15 ([23], Sections 6.4 and 6.5). *Let (X, τ) be a locally convex topological vector space. Then, there exists a separating family of seminorms on X whose corresponding natural topology is τ.*

Theorem 16 ([19], page 28). *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X and τ is the corresponding natural topology on X. Then, a set $E \subseteq X$ is bounded if and only if $p(E)$ is a bounded set in $\mathbb{R}$ for all $p \in \mathscr{P}$.*

Corollary 17. *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X and τ is the corresponding natural topology on X. It follows from Theorem 12 and Theorem 16 that if $E \subseteq X$ is bounded, then for any continuous seminorm $q : (X, \tau) \longrightarrow \mathbb{R}$, $q(E)$ is a bounded set in $\mathbb{R}$.*

Theorem 18 ([20], page 436, [23], Section 6.6). *Let (X, τ) be a topological vector space. Suppose $\mathbb{Q}$ is a separating family of seminorms on a vector space Y and τ' is the corresponding natural topology on Y. Then, a linear map $T : (X, \tau) \longrightarrow (Y, \tau')$ is continuous if and only if for each $q \in \mathbb{Q}$, $q \circ T$ is continuous on X.*

Theorem 19 ([20]). *Let X be a Frechet space and let Y be a topological vector space. When T is a linear map of X into Y, the following two properties are equivalent:*

(1) T is continuous

(2) $x_n \longrightarrow 0$ in $X \Rightarrow Tx_n \longrightarrow 0$ in Y

Theorem 20 ([19, 20]). *Let X and Y be two vector spaces and suppose $\mathcal{P}$ and $\mathcal{Q}$ are two separating families of seminorms on X and Y, respectively. Equip X and Y with the corresponding natural topologies. Then,*

(1) A sequence x_n converges to x in X if and only if for all $p \in \mathcal{P}$, $p(x_n - x) \longrightarrow 0$

(2) A linear operator $T : X \longrightarrow Y$ is continuous if and only if $\forall q \in \mathcal{Q} \exists c > 0, k \in \mathbb{N}, p_1, \cdots, p_k \in \mathcal{P}$ such that

$$\forall x \in X |q \circ T(x)| \leq c \max_{1 \leq i \leq k} p_i(x)$$

(3) A linear operator $T : X \longrightarrow \mathbb{R}$ is continuous if and only if $\exists c > 0, k \in \mathbb{N}, p_1, \cdots, p_k \in \mathcal{P}$ such that $\forall x \in X$ $|T(x)| \leq c \max_{1 \leq i \leq k} p_i(x)$

Definition 21. Let (X, τ) be a locally convex topological vector space.

(i) The *weak topology* on X is the natural topology induced by the separating family of seminorms $\{p_F\}_{F \in X^*}$ where

$$\forall F \in X^* \quad p_F : X \longrightarrow \mathbb{R}, \quad p_F(x) \coloneqq |F(x)|. \tag{11}$$

It can be shown that this topology is the smallest (weakest) topology with respect to which all the linear maps in $[(X, \tau)]^*$ are continuous. A sequence $\{x_m\}$ converges to x in X with respect to the weak topology if and only if $F(x_m) \longrightarrow F(x)$ in $\mathbb{R}$ for all $F \in X^*$. In this case, we may write $x_m \rightharpoonup x$. We denote the weak topology on X by $\sigma(X, X^*)$. It can be shown that $[(X, \tau)]^*$ is the same set as $[(X, \sigma(X, X^*))]^*$

(ii) The *weak * topology* on X^* is the natural topology induced by the separating family of seminorms $\{p_x\}_{x \in X}$ where

$$\forall x \in X \quad p_x : X^* \longrightarrow \mathbb{R}, \quad p_x(f) \coloneqq |f(x)|. \tag{12}$$

It can be shown that this topology is the weakest topology with respect to which all the linear maps $\{f \mapsto f(x)\}_{x \in X}$ (from X^* to $\mathbb{R}$) are continuous. A sequence $\{f_m\}$ converges to f in X^* with respect to the weak* topology if and only if $f_m(x) \longrightarrow f(x)$ in $\mathbb{R}$ for all $x \in X$. We denote the weak* topology on X^* by $\sigma(X^*, X)$

(iii) The *strong topology* on X^* is the natural topology induced by the separating family of seminorms $\{p_B\}_{B \subseteq X \text{bounded}}$ where for any bounded subset B of X,

$$p_B : X^* \longrightarrow \mathbb{R} \quad p_B(f) \coloneqq \sup \{|f(x)| : x \in B\}. \tag{13}$$

(It can be shown that for any bounded subset B of X and $f \in X^*$, $f(B)$ is a bounded subset of $\mathbb{R}$; see Theorem 16 and Theorem 28)

Remark 22.

(1) If X is a normed space, then the topology induced by the norm

$$\forall f \in X^* \quad \|f\|_{op} = \sup_{\|x\|_X = 1} |f(x)| \tag{14}$$

on X^* is the same as the strong topology on X^* ([22], page 198)

(2) In this manuscript, unless otherwise stated, we consider the topological dual of a locally convex topological vector space with the strong topology. Of course, it is worth mentioning that for many of the spaces that we will consider (including $X = \mathscr{E}(\Omega)$ or $X = D(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$), a sequence in X^* converges with respect to the weak* topology if and only if it converges with respect to the strong topology (for more details on this, see the definition and properties of *Montel spaces* in Section 34.4, page 356 of [22])

Theorem 23. *Let (X, τ) be a locally convex topological vector space. Then, the evaluation map*

$$J : (X, \tau) \longrightarrow X^{**} \coloneqq [(X^*, \textit{strong topology})]^*, \quad J(x)(F) \coloneqq F(x), \tag{15}$$

is a well-defined injective linear map. (X^{**} is called the bidual of X).

Definition 24. Let (X, τ) be a locally convex topological vector space. Let τ' denote the strong topology on X^{**} as the dual of $(X^*, \text{strong topology})$.

(i) If the evaluation map $J : (X, \tau) \longrightarrow (X^{**}, \tau')$ is bijective, then we say that (X, τ) is a *semireflexive space*

(ii) If the evaluation map $J : (X, \tau) \longrightarrow (X^{**}, \tau')$ is a linear topological isomorphism, then we say that (X, τ) is a *reflexive space*

Theorem 25 ([24], pages 16 and 17).

(i) Strong dual of a reflexive topological vector space is reflexive

(ii) Every semireflexive space whose topology is defined by the inductive limit of a sequence of Banach spaces is reflexive

(iii) Every semireflexive Frechet space is reflexive

Theorem 26. *Let (X, τ_X) and (Z, τ_Z) be two locally convex topological vector spaces. For all $x \in X$, let $l_x : X^* \longrightarrow \mathbb{R}$ be the linear map defined by $l_x(f) = f(x)$. Then,*

(1) a linear map $T : (Z, \tau_Z) \longrightarrow (X, \sigma(X, X^*))$ is continuous if and only if for all $F \in [(X, \tau_X)]^*$, the linear map $F \circ T : (Z, \tau_Z) \longrightarrow \mathbb{R}$ is continuous

(2) a linear map $T : (Z, \tau_Z) \longrightarrow (X^*, \sigma(X^*, X))$ is continuous if and only if for all $x \in X$, the linear map $l_x \circ T : (Z, \tau_Z) \longrightarrow \mathbb{R}$ is continuous

Theorem 27 ([21], page 163, [20], page 46). *Let X and Y be locally convex topological vector spaces and suppose $T : X \longrightarrow Y$ is a continuous linear map. Either equip both X^* and Y^* with the strong topology or equip both with the weak* topology. Then,*

(1) the map

$$T^* : Y^* \longrightarrow X^* \quad \langle T^* y, x \rangle_{X^* \times X} = \langle y, Tx \rangle_{Y^* \times Y} \tag{16}$$

is well-defined, linear, and continuous. (T^* is called the adjoint of T)

(2) If $T(X)$ is dense in Y, then $T^* : Y^* \longrightarrow X^*$ is injective

Theorem 28 ([19], page 70). *Let (X, τ) be a locally convex topological vector space. Then, a set $E \subseteq X$ is bounded with respect to τ if and only if it is bounded with respect to $\sigma(X, X^*)$.*

Corollary 29. *If (X, τ) is a locally convex topological vector space and $x_n \rightharpoonup x$ (i.e., x_n converges to x with respect to $\sigma(X, X^*)$), then $\{x_n\}$ is bounded with respect to both τ and $\sigma(X, X^*)$.*

Theorem 30. *Let (X, τ_X) and (Y, τ_Y) be two locally convex topological vector spaces. If $T : (X, \tau_X) \longrightarrow (Y^*, \sigma(Y^*, Y))$ is continuous, then $T : (X, \sigma(X, X^*)) \longrightarrow (Y^*, \sigma(Y^*, Y))$ is continuous. In particular, if $u_n \rightharpoonup u$ (i.e., u_n converges to u with respect to $\sigma(X, X^*)$), then $T(u_n) \longrightarrow T(u)$ in $(Y^*, \sigma(Y^*, Y))$.*

Proof. For all $y \in Y$, let $l_y : Y^* \longrightarrow \mathbb{R}$ be the map $l_y(F) = F(y)$. By Theorem 26, $T : (X, \sigma(X, X^*)) \longrightarrow (Y^*, \sigma(Y^*, Y))$ is continuous if $l_y \circ T : (X, \sigma(X, X^*)) \longrightarrow \mathbb{R}$ is continuous for all $y \in Y$. Let $y \in Y$.

(1) By definition of the weak* topology on Y^*, we know that the linear map $l_y : Y^* \longrightarrow \mathbb{R}$ is continuous

(2) By assumption $T : (X, \tau_X) \longrightarrow (Y^*, \sigma(Y^*, Y))$ is a continuous linear map

Therefore, $l_y \circ T$ belongs to $[(X, \tau_X)]^*$. Since $\sigma(X, X^*)$ is the weakest topology on X that makes all elements of $[(X, \tau_X)]^*$ continuous, we can conclude that $l_y \circ T : (X, \sigma(X, X^*)) \longrightarrow \mathbb{R}$ is continuous. □

Theorem 31 ([25], page 13). *Let (X, τ) be a Frechet space. Then, X is reflexive if and only if every bounded set E in X is relatively weakly compact (i.e., the closure of E with respect to $\sigma(X, X^*)$ is compact with respect to $\sigma(X, X^*)$).*

Theorem 32 ([26], page 167). *Let (X, τ) be a separable Frechet space. If $E \subseteq X$ is relatively weakly compact, then every infinite sequence in E has a subsequence that converges in $(X, \sigma(X, X^*))$.*

The next theorem is an immediate consequence of the previous theorems.

Theorem 33. *Suppose that (X, τ) is a separable reflexive Frechet space. Then, every bounded sequence in (X, τ) has a weakly convergent subsequence, that is, a subsequence that converges with respect to $\sigma(X, X^*)$.*

Theorem 34 ([27], page 61). *Let X and Y be two Banach spaces. Let $T : X \longrightarrow Y$ be a linear map. Then, T is continuous if and only if it is weak-weak continuous; that is, $T : (X, \|.\|_X) \longrightarrow (Y, \|.\|_Y)$ is continuous if and only if $T : (X, \sigma(X, X^*)) \longrightarrow (Y, \sigma(Y, Y^*))$ is continuous.*

Theorem 35. *Let X be a Banach space and Y be a closed subspace of X with the induced norm. Suppose that y_m is a sequence in Y and $y \in Y$. If $y_m \longrightarrow y$ in $(X, \sigma(X, X^*))$, then $y_m \longrightarrow y$ in $(Y, \sigma(Y, Y^*))$.*

Proof. This is a direct consequence of the fact that the following two topologies on the space Y are the same (see [27], page 70):

(1) The topology induced by $\sigma(X, X^*)$

(2) The topology $\sigma(Y, Y^*)$

□

Definition 36. Let X be a vector space and let $\{X_\alpha\}_{\alpha \in I}$ be a family of vector subspaces of X with the property that

(i) for each $\alpha \in I$, X_α is equipped with a topology that makes it a locally convex topological vector space, and

(ii) $\cup_{\alpha \in I} X_\alpha = X$

The *inductive limit topology* on X with respect to the family $\{X_\alpha\}_{\alpha \in I}$ is defined to be the largest topology with respect to which

(1) X is a locally convex topological vector space, and

(2) all the inclusions $X_\alpha \subseteq X$ are continuous

Theorem 37 ([21]). *Let X be a vector space equipped with the inductive limit topology with respect to $\{X_\alpha\}$ as described above. If Y is a locally convex vector space, then a linear map $T : X \longrightarrow Y$ is continuous if and only if $T|_{X_\alpha} : X_\alpha \longrightarrow Y$ is continuous for all $\alpha \in I$.*

Definition 38. Let X be a vector space and let $\{X_j\}_{j\in\mathbb{N}_0}$ be an increasing chain of subspaces of X:

$$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots. \tag{17}$$

Suppose that

(i) each X_j is equipped with a locally convex topology τ_j

(ii) for each j, the inclusion $(X_j, \tau_j) \hookrightarrow (X_{j+1}, \tau_{j+1})$ is a linear topological embedding with closed image

Then, the inductive limit topology on X with respect to the family $\{X_j\}_{j\in\mathbb{N}_0}$ is called a *strict inductive limit topology*.

Theorem 39 ([21]). *Suppose that X is equipped with the strict inductive limit topology with respect to the chain $\{X_j\}_{j\in\mathbb{N}_0}$. Then, a subset E of X is bounded if and only if there exists $m \in \mathbb{N}_0$ such that B is bounded in X_m.*

3.2. Function Spaces and Distributions

Definition 40. Let Ω be a nonempty open set in $\mathbb{R}^n$ and $m \in \mathbb{N}_0$.

$$\begin{aligned}
C(\Omega) &= \{f : \Omega \longrightarrow \mathbb{R} : f \text{ is continuous}\},\\
C^m(\Omega) &= \{f : \Omega \longrightarrow \mathbb{R} : \forall |\alpha| \le m\ \partial^\alpha f \in C(\Omega)\}\ \left(C^0(\Omega) = C(\Omega)\right),\\
BC(\Omega) &= \{f : \Omega \longrightarrow \mathbb{R} : f \text{ is continuous and bounded on } \Omega\},\\
BC^m(\Omega) &= \{f \in C^m(\Omega) : \forall |\alpha| \le m\ \partial^\alpha f \text{ is bounded on } \Omega\},\\
C^\infty(\Omega) &= \bigcap_{m\in\mathbb{N}_0} C^m(\Omega),\ BC^\infty(\Omega) = \bigcap_{m\in\mathbb{N}_0} BC^m(\Omega),\\
C_c^\infty(\Omega) &= \{f \in C^\infty(\Omega) : \text{support of } f \text{ is an element of } \mathscr{K}(\Omega)\}.
\end{aligned} \tag{18}$$

Let $0 < \lambda \le 1$. A function $F : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is called λ*-Holder continuous* if there exists a constant L such that

$$|F(x) - F(y)| \le L|x - y|^\lambda\ \forall x, y \in \Omega. \tag{19}$$

Clearly, a λ-Holder continuous function on Ω is uniformly continuous on Ω. 1-Holder continuous functions are also called *Lipschitz continuous* functions or simply Lipschitz functions. We define

$$\begin{aligned}
BC^{m,\lambda}(\Omega) &= \{f : \Omega \longrightarrow \mathbb{R} : \forall |\alpha| \le m\, \partial^\alpha f \text{ is } \lambda\\
&\qquad - \text{Holder continuous and bounded}\}\\
&= \{f \in BC^m(\Omega) : \forall |\alpha| \le m\, \partial^\alpha f \text{ is } \lambda\\
&\qquad - \text{Holder continuous}\},
\end{aligned}$$

$$BC^{\infty,\lambda}(\Omega) := \bigcap_{m\in\mathbb{N}_0} BC^{m,\lambda}(\Omega). \tag{20}$$

Theorem 41 [20]. *Let Ω be a nonempty open set in $\mathbb{R}^n$ and let $K \in \mathscr{K}(\Omega)$. There is a function $\psi \in C_c^\infty(\Omega)$ taking values in $[0, 1]$ such that $\psi = 1$ on a neighborhood containing K.*

Theorem 42 (exhaustion by compact sets) [20]. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. There exists a sequence of compact subsets $(K_j)_{j\in\mathbb{N}}$ such that $\cup_{j\in\mathbb{N}} \mathring{K}_j = \Omega$ and*

$$K_1 \subseteq \mathring{K}_2 \subseteq K_2 \subseteq \cdots \subseteq \mathring{K}_j \subseteq K_j \subseteq \cdots. \tag{21}$$

Moreover, as a direct consequence, if K is any compact subset of the open set Ω, then there exists an open set V such that $K \subseteq V \subseteq \bar{V} \subseteq \Omega$.

Theorem 43 [20]. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $\{K_j\}_{j\in\mathbb{N}}$ be an exhaustion of Ω by compact sets. Define*

$$V_0 = \mathring{K}_4, \forall j \in \mathbb{N} \quad V_j = \mathring{K}_{j+4} \setminus K_j. \tag{22}$$

Then,

(1) each V_j is an open bounded set and $\Omega = \cup_j V_j$

(2) the cover $\{V_j\}_{j\in\mathbb{N}_0}$ is locally finite in Ω; that is, each compact subset of Ω has nonempty intersection with only a finite number of the V_j's

(3) there is a family of functions $\psi_j \in C_c^\infty(\Omega)$ taking values in $[0, 1]$ such that supp $\psi_j \subseteq V_j$ *and*

$$\sum_{j\in\mathbb{N}_0} \psi_j(x) = 1 \text{ for all } x \in \Omega \tag{23}$$

Let Ω be a nonempty open set in $\mathbb{R}^n$. For all $\varphi \in C^\infty(\Omega)$, $j \in \mathbb{N}$, and $K \in \mathscr{K}(\Omega)$, we define

$$\|\varphi\|_{j,K} := \sup\{|\partial^\alpha \varphi(x)| : |\alpha| \le j, x \in K\}. \tag{24}$$

For all $j \in \mathbb{N}$ and $K \in \mathscr{K}(\Omega)$, $\|.\|_{j,K}$ is a seminorm on $C^\infty(\Omega)$. We define $\mathscr{E}(\Omega)$ to be $C^\infty(\Omega)$ equipped with the natural topology induced by the separating family of seminorms $\{\|.\|_{j,K}\}_{j\in\mathbb{N},K\in\mathscr{K}(\Omega)}$. It can be shown that $\mathscr{E}(\Omega)$ is a Frechet space.

For all $K \in \mathscr{K}(\Omega)$, we define $\mathscr{E}_K(\Omega)$ to be $C_K^\infty(\Omega)$ equipped with the subspace topology. Since $C_K^\infty(\Omega)$ is a closed subset of the Frechet space $\mathscr{E}(\Omega)$, $\mathscr{E}_K(\Omega)$ is also a Frechet space.

We define $D(\Omega) = \cup_{K\in\mathscr{K}(\Omega)} \mathscr{E}_K(\Omega)$ equipped with the inductive limit topology with respect to the family of vector subspaces $\{\mathscr{E}_K(\Omega)\}_{K\in\mathscr{K}(\Omega)}$. It can be shown that if $\{K_j\}_{j\in\mathbb{N}_0}$ is an exhaustion by compacts sets of Ω, then the inductive

limit topology on $D(\Omega)$ with respect to the family $\{\mathscr{E}_{K_j}\}_{j\in\mathbb{N}_0}$ is exactly the same as the inductive limit topology with respect to $\{\mathscr{E}_K(\Omega)\}_{K\in\mathscr{K}(\Omega)}$.

Remark 44. Suppose Y is a topological space and the mapping $T : Y \longrightarrow D(\Omega)$ is such that $T(Y) \subseteq \mathscr{E}_K(\Omega)$ for some $K \in \mathscr{K}(\Omega)$. Since $\mathscr{E}_K(\Omega) \hookrightarrow D(\Omega)$, if $T : Y \longrightarrow \mathscr{E}_K(\Omega)$ is continuous, then $T : Y \hookrightarrow D(\Omega)$ will be continuous.

Theorem 45 (convergence and continuity for $\mathscr{E}(\Omega)$). *Let Ω be a nonempty open set in $\mathbb{R}^n$. Let Y be a topological vector space whose topology is induced by a separating family of seminorms $\mathscr{Q}$.*

(1) A sequence $\{\varphi_m\}$ converges to φ in $\mathscr{E}(\Omega)$ if and only if $\|\varphi_m - \varphi\|_{j,K} \longrightarrow 0$ for all $j \in \mathbb{N}$ and $K \in \mathscr{K}(\Omega)$

(2) Suppose $T : \mathscr{E}(\Omega) \longrightarrow Y$ is a linear map. Then, the following are equivalent:

(i) T is continuous

(ii) For every $q \in \mathscr{Q}$, there exist $j \in \mathbb{N}$ and $K \in \mathscr{K}(\Omega)$, and $C > 0$ such that

$$\forall \varphi \in \mathscr{E}(\Omega) \quad q(T(\varphi)) \leq C\|\varphi\|_{j,K} \tag{25}$$

(iii) If $\varphi_m \longrightarrow 0$ in $\mathscr{E}(\Omega)$, then $T(\varphi_m) \longrightarrow 0$ in Y

(3) In particular, a linear map $T : \mathscr{E}(\Omega) \longrightarrow \mathbb{R}$ is continuous if and only if there exist $j \in \mathbb{N}$ and $K \in \mathscr{K}(\Omega)$, and $C > 0$ such that

$$\forall \varphi \in \mathscr{E}(\Omega) \quad |T(\varphi)| \leq C\|\varphi\|_{j,K} \tag{26}$$

(4) A linear map $T : Y \longrightarrow \mathscr{E}(\Omega)$ is continuous if and only if

$$\begin{aligned} &\forall j \in \mathbb{N}, \forall K \in \mathscr{K}(\Omega)\ \exists C > 0, k \in \mathbb{N}, q_1, \cdots, \\ &\quad q_k \in \mathscr{Q} \text{ such that } \forall y\ \|T(y)\|_{j,K} \\ &\quad \leq C \max_{1\leq i\leq k} q_i(y) \end{aligned} \tag{27}$$

Theorem 46 (convergence and continuity for $\mathscr{E}_K(\Omega)$). *Let Ω be a nonempty open set in $\mathbb{R}^n$ and $K \in \mathscr{K}(\Omega)$. Let Y be a topological vector space whose topology is induced by a separating family of seminorms $\mathscr{Q}$.*

(1) A sequence $\{\varphi_m\}$ converges to φ in $\mathscr{E}_K(\Omega)$ if and only if $\|\varphi_m - \varphi\|_{j,K} \longrightarrow 0$ for all $j \in \mathbb{N}$

(2) Suppose $T : \mathscr{E}_K(\Omega) \longrightarrow Y$ is a linear map. Then, the following are equivalent:

(i) T is continuous

(ii) For every $q \in \mathscr{Q}$, there exists $j \in \mathbb{N}$ and $C > 0$ such that

$$\forall \varphi \in \mathscr{E}_K(\Omega) \quad q(T(\varphi)) \leq C\|\varphi\|_{j,K} \tag{28}$$

(iii) If $\varphi_m \longrightarrow 0$ in $\mathscr{E}_K(\Omega)$, then $T(\varphi_m) \longrightarrow 0$ in Y

Theorem 47 (convergence and continuity for $D(\Omega)$). *Let Ω be a nonempty open set in $\mathbb{R}^n$. Let Y be a topological vector space whose topology is induced by a separating family of seminorms $\mathscr{Q}$.*

(1) A sequence $\{\varphi_m\}$ converges to φ in $D(\Omega)$ if and only if there is a $K \in \mathscr{K}(\Omega)$ such that $\operatorname{supp}\varphi_m \subseteq K$ and $\varphi_m \longrightarrow \varphi$ in $\mathscr{E}_K(\Omega)$

(2) Suppose $T : D(\Omega) \longrightarrow Y$ is a linear map. Then, the following are equivalent:

(i) T is continuous

(ii) For all $K \in \mathscr{K}(\Omega)$, $T : \mathscr{E}_K(\Omega) \longrightarrow Y$ is continuous

(iii) For every $q \in \mathscr{Q}$ and $K \in \mathscr{K}(\Omega)$, there exists $j \in \mathbb{N}$ and $C > 0$ such that

$$\forall \varphi \in \mathscr{E}_K(\Omega) \quad q(T(\varphi)) \leq C\|\varphi\|_{j,K} \tag{29}$$

(iv) If $\varphi_m \longrightarrow 0$ in $D(\Omega)$, then $T(\varphi_m) \longrightarrow 0$ in Y

(3) In particular, a linear map $T : D(\Omega) \longrightarrow \mathbb{R}$ is continuous if and only if for every $K \in \mathscr{K}(\Omega)$, there exists $j \in \mathbb{N}$ and $C > 0$ such that

$$\forall \varphi \in \mathscr{E}_K(\Omega) \quad |T(\varphi)| \leq C\|\varphi\|_{j,K} \tag{30}$$

Remark 48. Let Ω be a nonempty open set in $\mathbb{R}^n$. Here are two immediate consequences of the previous theorems and remark:

(1) The identity map

$$i_{D,\mathscr{E}} : D(\Omega) \longrightarrow \mathscr{E}(\Omega) \tag{31}$$

is continuous (that is, $D(\Omega) \hookrightarrow \mathscr{E}(\Omega)$)

(2) If $T : \mathscr{E}(\Omega) \longrightarrow \mathscr{E}(\Omega)$ is a continuous linear map such that $\operatorname{supp}(T\varphi) \subseteq \operatorname{supp}\varphi$ for all $\varphi \in \mathscr{E}(\Omega)$ (i.e., T is a *local* continuous linear map), then T restricts to a continuous linear map from $D(\Omega)$ to $D(\Omega)$. Indeed, the assumption $\operatorname{supp}(T\varphi) \subseteq \operatorname{supp}\varphi$ implies that $T(D(\Omega)) \subseteq D(\Omega)$. Moreover $T : D(\Omega) \longrightarrow D(\Omega)$ is continuous if and only if for $K \in \mathscr{K}(\Omega)$ $T : \mathscr{E}_K(\Omega) \longrightarrow D(\Omega)$ is continuous. Since $T(\mathscr{E}_K(\Omega))$

$\subseteq \mathscr{E}_K(\Omega)$, this map is continuous if and only if $T : \mathscr{E}_K(\Omega) \longrightarrow \mathscr{E}_K(\Omega)$ is continuous (see Remark 44). However, since the topology of $\mathscr{E}_K(\Omega)$ is the induced topology from $\mathscr{E}(\Omega)$, the continuity of the preceding map follows from the continuity of $T : \mathscr{E}(\Omega) \longrightarrow \mathscr{E}(\Omega)$

Theorem 49. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Then $D(\Omega)$ is separable.*

Definition 50. Let Ω be a nonempty open set in $\mathbb{R}^n$. The topological dual of $D(\Omega)$, denoted $D'(\Omega)$ $(D'(\Omega) = [D(\Omega)]^*)$, is called the *space of distributions* on Ω. Each element of $D'(\Omega)$ is called a *distribution* on Ω. The action of a distribution $u \in D'(\Omega)$ on a function $\varphi \in D(\Omega)$ is sometimes denoted by $\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}$ or simply $\langle u, \varphi \rangle$.

Remark 51. Every function $f \in L^1_{loc}(\Omega)$ defines a distribution $u_f \in D'(\Omega)$ as follows

$$\forall \varphi \in D(\Omega) \quad u_f(\varphi) \coloneqq \int_\Omega f\varphi dx. \tag{32}$$

In particular, every function $\varphi \in \mathscr{E}(\Omega)$ defines a distribution u_φ. It can be shown that the map $i : \mathscr{E}(\Omega) \longrightarrow D'(\Omega)$ which sends φ to u_φ is an injective linear continuous map ([21], page 11). Therefore, we can identify $\mathscr{E}(\Omega)$ with a subspace of $D'(\Omega)$; we sometimes refer to the map i as the "identity map."

Theorem 52 ([20], page 47). *Let Ω be a nonempty open set in $\mathbb{R}^n$. Equip $D'(\Omega)$ with the weak* topology. Then, under the above identification, $C_c^\infty(\Omega)$ is dense in $D'(\Omega)$.*

Theorem 53 ([22], page 302). *Let Ω be a nonempty open set in $\mathbb{R}^n$. Equip $D'(\Omega)$ with the strong topology. Then, under the identification described in Remark 51, $C_c^\infty(\Omega)$ is sequentially dense in $D'(\Omega)$.*

Remark 54.

(i) Clearly sequential density is a stronger notion than density. So $C_c^\infty(\Omega)$ is dense in $(D'(\Omega)$, strong topology)

(ii) Recall that, according to Remark 22, a sequence converges in $(D'(\Omega), \text{weak}^*)$ if and only if it converges in $(D'(\Omega)$, strong topology). This together with the fact that weak* topology is weaker than the strong topology implies that convergent sequences in both topologies converge to the same limit. Therefore, it follows from Theorem 53 that $C_c^\infty(\Omega)$ is sequentially dense in $(D'(\Omega), \text{weak}^*)$. Hence, Theorem 52 can be viewed as a corollary of Theorem 53

Theorem 55 ([21], page 9). *$D(\Omega)$ is reflexive. So $[(D'(\Omega), \textit{strong topology})]^*$ can be identified with the topological vector space $D(\Omega)$.*

Definition 56 (restriction of a distribution). Let Ω be an open subset of $\mathbb{R}^n$ and V be an open subset of Ω. We define the restriction map $res_{\Omega,V} : D'(\Omega) \longrightarrow D'(V)$ as follows:

$$\left\langle \text{res}_{\Omega,V} u, \varphi \right\rangle_{D'(V)\times D(V)} \coloneqq \left\langle u, \text{ext}^0_{V,\Omega}\varphi \right\rangle_{D'(\Omega)\times D(\Omega)}. \tag{33}$$

This is well-defined; indeed, $res_{\Omega,V} : D'(\Omega) \longrightarrow D'(V)$ is a continuous linear map as it is the adjoint of the continuous map $\text{ext}^0_{V,\Omega} : D(V) \longrightarrow D(\Omega)$. Given $u \in D'(\Omega)$, we sometimes write $u|_V$ instead of $res_{\Omega,V} u$.

Definition 57 (support of a distribution). Let Ω be a nonempty open set in $\mathbb{R}^n$. Let $u \in D'(\Omega)$.

(i) We say u is equal to zero on some open subset V of Ω if $u|_V = 0$

(ii) Let $\{V_i\}_{i\in I}$ be the collection of all open subsets of Ω such that u is equal to zero on V_i. Let $V = \cup_{i\in I} V_i$. The support of u is defined as follows:

$$\text{supp } u \coloneqq \Omega \setminus V \tag{34}$$

Note that supp u is closed in Ω but it is not necessarily closed in $\mathbb{R}^n$.

Theorem 58 ([21]). *Let Ω be a nonempty open set in $\mathbb{R}^n$ and let $u \in D'(\Omega)$. If $\varphi \in D(\Omega)$ vanishes on a neighborhood containing* supp *u, then $u(\varphi) = 0$.*

Theorem 59 ([21]). *Let $\{u_i\}$ be a sequence in $D'(\Omega)$, $u \in D(\Omega)$, and $K \in \mathscr{K}(\Omega)$ such that $u_i \longrightarrow u$ in $D'(\Omega)$ and* supp *$u_i \subseteq K$ for all i. Then, also* supp *$u \subseteq K$.*

Theorem 60 ([28], page 38). *Let Ω be a nonempty open set in $\mathbb{R}^n$. Suppose that $\{T_i\}$ is a sequence in $D'(\Omega)$ with the property that for all $\varphi \in D(\Omega)$, $\lim_{i\longrightarrow\infty} \langle T_i, \varphi \rangle_{D'(\Omega)\times D(\Omega)}$ exists. Then, there exists $T \in D'(\Omega)$ such that*

$$\forall \varphi \in D(\Omega) \quad \langle T, \varphi \rangle_{D'(\Omega)\times D(\Omega)} = \lim_{i\longrightarrow\infty} \langle T_i, \varphi \rangle_{D'(\Omega)\times D(\Omega)}. \tag{35}$$

Definition 61 (Sobolev-Slobodeckij spaces). Let Ω be a nonempty open set in $\mathbb{R}^n$. Let $s \in \mathbb{R}$ and $p \in (1,\infty)$.

(i) If $s = k \in \mathbb{N}_0$,

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{W^{k,p}(\Omega)} := \sum_{|\nu| \le k} \|\partial^\nu u\|_{L^p(\Omega)} < \infty \right\} \tag{36}$$

(ii) If $s = \theta \in (0,1)$

$$W^{\theta,p}(\Omega) = \left\{ u \in L^p(\Omega) : |u|_{W^{\theta,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\theta p}} dxdy \right)^{1/p} < \infty \right\} \tag{37}$$

(iii) If $s = k + \theta, k \in \mathbb{N}_0, \theta \in (0,1)$,

$$W^{s,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \|u\|_{W^{s,p}(\Omega)} := \|u\|_{W^{k,p}(\Omega)} + \sum_{|\nu|=k} |\partial^\nu u|_{W^{\theta,p}(\Omega)} < \infty \right\} \tag{38}$$

(iv) $W_0^{s,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$

(v) If $s < 0$,

$$W^{s,p}(\Omega) = \left(W_0^{-s,p'}(\Omega) \right)^* \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right) \tag{39}$$

(vi) For all compact sets $K \subset \Omega$, we define

$$W_K^{s,p}(\Omega) = \{ u \in W^{s,p}(\Omega) : \text{ supp } u \subseteq K \} \tag{40}$$

with $\|u\|_{W_K^{s,p}(\Omega)} := \|u\|_{W^{s,p}(\Omega)}$. Note that for $s < 0$, $W^{s,p}(\Omega)$ can be viewed as a subspace of $D'(\Omega)$ (see Theorem 68), and the support of $u \in W^{s,p}(\Omega)$ is interpreted as the support of a distribution.

(vii) $W_{\text{comp}}^{s,p}(\Omega) := \cup_{K\in\mathcal{K}(\Omega)} W_K^{s,p}(\Omega)$. $W_{comp}^{s,p}(\Omega)$ is equipped with the inductive limit topology with respect to the family of vector subspaces $\{W_K^{s,p}(\Omega)\}_{K\in\mathcal{K}(\Omega)}$. It can be shown that if $\{K_j\}_{j\in\mathbb{N}_0}$ is an exhaustion by compacts sets of Ω, then the inductive limit topology on $W_{comp}^{s,p}(\Omega)$ with respect to the family $\{W_{K_j}^{s,p}(\Omega)\}_{j\in\mathbb{N}_0}$ is exactly the same as the inductive limit topology with respect to $\{W_K^{s,p}(\Omega)\}_{K\in\mathcal{K}(\Omega)}$

Theorem 62. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \ge 1$, and $1 < p < \infty$. Then, $u \in W^{s,p}(\Omega)$ if and only if $u \in L^p(\Omega)$, and for all $1 \le i \le n$, $\partial u/\partial x^i \in W^{s-1,p}(\Omega)$.*

Proof. We consider two cases:

Case 1: $s = k \in \mathbb{N}$.

$$\begin{aligned} u \in W^{k,p}(\Omega) &\Longleftrightarrow u \in L^p(\Omega) \text{and } \partial^\alpha u \in L^p(\Omega) \quad \forall 1 \le |\alpha| \le k \\ &\Longleftrightarrow u \in L^p(\Omega) \text{ and } \partial^\beta \left[\frac{\partial u}{\partial x^i} \right] \in L^p(\Omega) \quad \forall 0 \le |\beta| \le k-1, 1 \le i \le n \\ &\Longleftrightarrow u \in L^p(\Omega) \text{ and } \frac{\partial u}{\partial x^i} \in W^{k-1,p}(\Omega) \quad \forall 1 \le i \le n \end{aligned} \tag{41}$$

Case 2: $s = k + \theta$, $k \in \mathbb{N}$, and $0 < \theta < 1$.

$$\begin{aligned} u \in W^{s,p}(\Omega) &\Longleftrightarrow u \in W^{k,p}(\Omega) \text{ and } \frac{\partial^\nu u(x) - \partial^\nu u(y)}{|x-y|^{n/p+\theta}} \in L^p(\Omega\times\Omega) \forall |\nu| = k \\ &\Longleftrightarrow u \in L^p(\Omega) \text{ and } \frac{\partial u}{\partial x^i} \in W^{k-1,p}(\Omega) \, \forall 1 \le i \\ &\le n \text{ and } \frac{\partial^\nu u(x) - \partial^\nu u(y)}{|x-y|^{n/p+\theta}} \in L^p(\Omega\times\Omega) \forall |\nu| = k \\ &\Longleftrightarrow u \in L^p(\Omega) \text{ and } \frac{\partial u}{\partial x^i} \in W^{k-1,p}(\Omega) \text{ and} \\ &\quad \frac{\partial^\beta \left(\partial u/\partial x^i\right)(x) - \partial^\beta \left(\partial u/\partial x^i\right)(y)}{|x-y|^{n/p+\theta}} \in L^p(\Omega\times\Omega) \forall |\beta| \\ &= k-1 \forall 1 \le i \le n \\ &\Longleftrightarrow u \in L^p(\Omega) \text{ and } \frac{\partial u}{\partial x^i} \in W^{s-1,p}(\Omega) \quad \forall 1 \le i \le n \end{aligned} \tag{42}$$

□

Remark 63. Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$, and $1 < p < \infty$. Clearly for $s \ge 0$, $C_c^\infty(\Omega) \subseteq W^{s,p}(\Omega)$. For $s < 0$, it is easy to see that for all $\varphi \in C_c^\infty(\Omega)$, the map $l_\varphi : W_0^{-s,p'}(\Omega) \longrightarrow \mathbb{R}$ which sends $u \in W_0^{-s,p'}(\Omega)$ to $\int_\Omega u\varphi \, dx$ belongs to $\left[W_0^{-s,p'}(\Omega)\right]^* = W^{s,p}(\Omega)$. The map $\varphi \mapsto l_\varphi$ is one-to-one and we can use it to identify $C_c^\infty(\Omega)$ with a subspace of $W^{s,p}(\Omega)$; we sometimes refer to the map that sends φ to l_φ as the "identity map." So we can talk about the identity map from $C_c^\infty(\Omega)$ to $W^{s,p}(\Omega)$ for all $s \in \mathbb{R}$.

Theorem 64 ([16]). *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \ge 0$, and $1 < p < \infty$. Then, $W^{s,p}(\Omega)$ is a reflexive Banach space.*

Corollary 65. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \ge 0$, and $1 < p < \infty$. A closed subspace of a reflexive space is reflexive, so $W_0^{s,p}(\Omega)$ is reflexive. Dual of a reflexive Banach space is a reflexive Banach space, so $W^{-s,p'}(\Omega)$ is a reflexive Banach space.*

Remark 66. Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \ge 0$, and $1 < p < \infty$. Since $W_0^{s,p}(\Omega)$ is reflexive, it can be identified with $\left[W_0^{s,p}(\Omega)\right]^{**}$ and we may write $\left[W^{-s,p'}(\Omega)\right]^* = W_0^{s,p}(\Omega)$ and talk about the duality pairing $\langle u, f \rangle_{W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)}$. To be more precise, we notice that the identification of $\left[W_0^{s,p}(\Omega)\right]^{**}$ and $W_0^{s,p}(\Omega)$ is done by the evaluation map

$$J : W_0^{s,p}(\Omega) \longrightarrow \left[W_0^{s,p}(\Omega)\right]^{**} \quad J(u)[f] = f(u). \tag{43}$$

Therefore, for all $u \in W_0^{s,p}(\Omega)$ and $f \in W^{-s,p'}(\Omega)$,

$$\begin{aligned}\langle u, f\rangle_{W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)} &= \langle J(u), f\rangle_{\left[W_0^{s,p}(\Omega)\right]^{**}\times W^{-s,p'}(\Omega)} \\ &= f(u) = \langle f, u\rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}.\end{aligned} \tag{44}$$

Theorem 67. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \geq 0$, and $1 < p < \infty$. Then, $C_c^\infty(\Omega)$ is dense in $W^{-s,p'}(\Omega)$. We may write this as $W_0^{-s,p'}(\Omega) = W^{-s,p'}(\Omega)$.*

Proof. Our proof will be based on a similar argument given in page 65 of [1]. Let $\varphi \mapsto l_\varphi$ be the mapping introduced in Remark 63. Our goal is to show that the set

$$V := \left\{l_\varphi : \varphi \in C_c^\infty(\Omega)\right\} \tag{45}$$

is dense in $W^{-s,p'}(\Omega)$. To this end, it is enough to show that if $F \in \left[W^{-s,p'}(\Omega)\right]^*$ is such that $F(l_\varphi) = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $F = 0$. Indeed, let F be such an element. By reflexivity of $W_0^{s,p}(\Omega)$, there exists $f \in W_0^{s,p}(\Omega)$ such that

$$\forall v \in W^{-s,p'}(\Omega) \quad F(v) = v(f). \tag{46}$$

Thus, for all $\varphi \in C_c^\infty(\Omega)$, we have

$$0 = F\left(l_\varphi\right) = l_\varphi(f) = \int_\Omega f(x)\varphi(x)\,dx. \tag{47}$$

So, by the fundamental lemma of the calculus of variations (see [27], page 110), we have $f = 0$ (as an element of $W^{s,p}(\Omega) \subseteq L^1_{loc}(\Omega)$) and therefore $F = 0$. □

Theorem 68. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$, and $1 < p < \infty$. Equip $D'(\Omega)$ with weak* topology or strong topology. Then*

$$D(\Omega) \hookrightarrow W^{s,p}(\Omega) \hookrightarrow D'(\Omega). \tag{48}$$

Proof. Recall that the convergent sequences in $D'(\Omega)$ equipped with strong topology are exactly the same as the convergent sequences of $D'(\Omega)$ equipped with the weak* topology (see Remark 22). This together with Theorem 19 implies that in the study of the continuity of the inclusion map from $W^{s,p}(\Omega)$ to $D'(\Omega)$, it does not matter whether we equip $D'(\Omega)$ with the strong topology or weak* topology. In the proof, as usual, we assume $D'(\Omega)$ is equipped with the strong topology. We consider two cases:

Case 1: $s \geq 0$. The continuity of the embedding $D(\Omega) \hookrightarrow W^{s,p}(\Omega)$ has been studied in [16]. Also clearly $W^{s,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow D'(\Omega)$. The former continuous embedding holds by the definition of $W^{s,p}(\Omega)$ and the latter embedding is continuous because if $u_m \longrightarrow 0$ in $L^p(\Omega)$, then for all $\varphi \in D(\Omega)$,

$$\left|\langle u_m, \varphi\rangle_{D'(\Omega)\times D(\Omega)} - 0\right| = \left|\int_\Omega u_m \varphi\,dx\right| \leq \|u_m\|_p \|\varphi\|_\infty \longrightarrow 0. \tag{49}$$

So, $u_m \longrightarrow 0$ in $D'(\Omega)$. This implies the continuity of the inclusion map from $L^p(\Omega)$ to $D'(\Omega)$ by Theorem 19

Case 2: $s < 0$. Since $W_0^{-s,p'}(\Omega) \hookrightarrow W^{-s,p'}(\Omega)$, it follows from previous case that $W_0^{-s,p'}(\Omega) \hookrightarrow D'(\Omega)$. Also since $D(\Omega) \subseteq W_0^{-s,p'}(\Omega)$ is dense in $D'(\Omega)$ (see Theorem 52, Theorem 53, and Remark 54), it follows that the inclusion map from $W_0^{-s,p'}(\Omega)$ to $D'(\Omega)$ is continuous with dense image. Thus, by Theorem 27, $D(\Omega) \hookrightarrow W^{s,p}(\Omega)$. Here, we used the facts that (1) the strong dual of the normed space $W_0^{-s,p'}(\Omega)$ is $W^{s,p}(\Omega)$ and that (2) the dual of $(D'(\Omega),$ strong topology) is $D(\Omega)$ (see Theorem 55). It remains to show that $W^{s,p}(\Omega) \hookrightarrow D'(\Omega)$. It follows from Case 1 that $D(\Omega) \hookrightarrow W_0^{-s,p'}(\Omega)$ and by definition $D(\Omega)$ is dense in $W_0^{-s,p'}(\Omega)$. So, by Theorem 27, $W^{s,p}(\Omega) \hookrightarrow D'(\Omega)$ □

Remark 69. Note that for $s \leq 0$, $W_0^{s,p}(\Omega)$ is the same as $W^{s,p}(\Omega)$. For $s > 0$, $W_0^{s,p}(\Omega)$ is a subspace of $W^{s,p}(\Omega)$ which contains $C_c^\infty(\Omega)$. So it follows from the previous theorem that

$$D(\Omega) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow D'(\Omega). \tag{50}$$

To be more precise, we should note that for $s < 0$, we identify $\varphi \in D(\Omega)$ with the corresponding distribution in $D'(\Omega)$. Under this identification, for all $s \in \mathbb{R}$ the "identity map" $i : D(\Omega) \longrightarrow W_0^{s,p}(\Omega)$ is continuous with dense image, and so its adjoint $i^* : \left[W_0^{s,p}(\Omega)\right]^* \longrightarrow D'(\Omega)$ will be an injective continuous map (Theorem 27), and we have

$$\begin{aligned}\langle i^* u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} &= \langle u, i\varphi\rangle_{\left[W_0^{s,p}(\Omega)\right]^*\times W_0^{s,p}(\Omega)} \\ &= \langle u, \varphi\rangle_{\left[W_0^{s,p}(\Omega)\right]^*\times W_0^{s,p}(\Omega)}.\end{aligned} \tag{51}$$

We usually identify $\left[W_0^{s,p}(\Omega)\right]^*$ with its image under i^* and view $\left[W_0^{s,p}(\Omega)\right]^*$ as a subspace of $D'(\Omega)$. So, under this identification, we can rewrite the above equality as follows:

$$\begin{aligned}\forall u \in \left[W_0^{s,p}(\Omega)\right]^* \quad \forall \varphi \in D(\Omega) \quad &\langle u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} \\ &= \langle u, \varphi\rangle_{\left[W_0^{s,p}(\Omega)\right]^*\times W_0^{s,p}(\Omega)}.\end{aligned} \tag{52}$$

Finally, noting that for all $s \in \mathbb{R}$ and $1 < p < \infty$, $\left[W_0^{s,p}(\Omega)\right]^* = W_0^{-s,p'}(\Omega)$ (see Definition 61, Theorem 67, and Corollary 65), we can write

$$\forall u \in W_0^{-s,p'}(\Omega) \quad \forall \varphi \in D(\Omega) \quad \langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi \rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}. \tag{53}$$

Theorem 70. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \geq 0$, and $1 < p < \infty$. Then,*

(1) the mapping $F \mapsto F|_{C_c^\infty(\Omega)}$ is an isometric isomorphism between $W^{-s,p'}(\Omega)$ and $[C_c^\infty(\Omega), \|.\|_{s,p}]^*$

(2) suppose $u \in D'(\Omega)$. If $u : (C_c^\infty(\Omega), \|.\|_{-s,p'}) \longrightarrow \mathbb{R}$ is continuous, then $u \in W_0^{s,p}(\Omega)$ (more precisely, there is a unique element in $W_0^{s,p}(\Omega)$ whose corresponding distribution is u). Moreover,

$$\|u\|_{W_0^{s,p}(\Omega)} = \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{-s,p'}(\Omega)}} \tag{54}$$

Proof. The first item has been studied in [16]. Here, we will prove the second item. Since $u : (C_c^\infty(\Omega), \|.\|_{-s,p'}) \longrightarrow \mathbb{R}$ is continuous, it can be extended to a continuous linear map $\tilde{u} : W^{-s,p'}(\Omega) \longrightarrow \mathbb{R}$. So $\tilde{u} \in [W^{-s,p'}(\Omega)]^*$. However, $W_0^{s,p}(\Omega)$ is reflexive, therefore there exists a unique $v \in W_0^{s,p}(\Omega)$ such that $\tilde{u} = J(v)$ where $J(v)$: $W^{-s,p'}(\Omega) \longrightarrow \mathbb{R}$ is the evaluation map defined by $J(v)(F) = \langle F, v \rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}$. To finish the proof, it is enough to show that $v = u$ as elements of $D'(\Omega)$. For all $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} \langle v, \varphi \rangle_{D'(\Omega)\times D(\Omega)} &= \int_\Omega v\varphi \, dx \overset{\text{Remark 63}}{=} \langle \varphi, v \rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \\ &= J(v)(\varphi) = \tilde{u}(\varphi) = u(\varphi) = \langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}. \end{aligned} \tag{55}$$

Also,

$$\begin{aligned} \|u\|_{W_0^{s,p}(\Omega)} &= \|v\|_{W_0^{s,p}(\Omega)} \\ &= \|J(v)\|_{\left[W^{-s,p'}(\Omega)\right]^*} \\ &= \|\tilde{u}\|_{\left[W^{-s,p'}(\Omega)\right]^*} \\ &= \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle \tilde{u}, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{-s,p'}(\Omega)}} \\ &= \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{-s,p'}(\Omega)}}. \end{aligned} \tag{56}$$

□

Corollary 71. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \geq 0$, and $1 < p < \infty$. Suppose that $u \in D'(\Omega)$. As a direct consequence of Theorem 70, we have the following:*

(i) If $\sup_{0\equiv\varphi\in C_c^\infty(\Omega)} (\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)} / \|\varphi\|_{W^{s,p}(\Omega)}) < \infty$, then $u \in W^{-s,p'}(\Omega)$ and

$$\|u\|_{W^{-s,p'}(\Omega)} = \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{s,p}(\Omega)}} \tag{57}$$

(ii) If $\sup_{0\equiv\varphi\in C_c^\infty(\Omega)} (\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)} / \|\varphi\|_{W^{-s,p'}(\Omega)}) < \infty$, then $u \in W_0^{s,p}(\Omega)$ and

$$\|u\|_{W^{s,p}(\Omega)} = \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{-s,p'}(\Omega)}} \tag{58}$$

That is, for any $e \in \mathbb{R}$ and $1 < q < \infty$, in order to show that $u \in D'(\Omega)$ belongs to $W_0^{e,q}(\Omega)$, it is enough to prove that

$$\sup_{0\equiv\varphi\in C_c^\infty(\Omega)} \frac{\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}}{\|\varphi\|_{W^{-e,q'}(\Omega)}} < \infty, \tag{59}$$

and in fact $\|u\|_{W^{e,q}(\Omega)} = \sup_{0\equiv\varphi\in C_c^\infty(\Omega)} (\langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)} / \|\varphi\|_{W^{-e,q'}(\Omega)})$.

Theorem 72. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$, and $1 < p < \infty$. Suppose that $K \in \mathcal{K}(\Omega)$. Then, $W_K^{s,p}(\Omega)$ is a closed subspace of $W^{s,p}(\Omega)$.*

Proof. It is enough to show that if $\{u_i\}$ is a sequence of elements in $W_K^{s,p}(\Omega)$ such that $u_i \longrightarrow u$ in $W^{s,p}(\Omega)$, then $u \in W_K^{s,p}(\Omega)$, i.e., supp $u \subseteq K$. By Theorem 68, we have $u_i \longrightarrow u$ in $D'(\Omega)$. Now it follows from Theorem 59 that supp $u \subseteq K$. Note that for any $s \geq 0$, we have $W^{s,p}(\Omega) \subseteq L^p(\Omega) \subseteq L^1_{loc}(\Omega)$; in this proof, we implicitly used the fact that for functions in $L^1_{loc}(\Omega)$, the usual definition of support agrees with the distributional definition of support. □

Next, we list several embedding theorems for Sobolev-Slobodeckij spaces.

Theorem 73 ([29], Section 2.8.1). *Suppose $1 < p \leq q < \infty$ and $-\infty < t \leq s < \infty$ satisfy $s - n/p \geq t - n/q$. Then, $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{t,q}(\mathbb{R}^n)$. In particular, $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{t,p}(\mathbb{R}^n)$.*

Theorem 74 ([30, 18]). *Let Ω be a nonempty bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary. Suppose $1 \leq p, q < \infty$ (p does NOT need to be less than or equal to q) and $0 \leq t \leq s$ satisfy $s - n/p \geq t - n/q$. If $s \in \mathbb{N}_0$, additionally assume that $s \neq t$. Then, $W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega)$. Furthermore, if $s > t$, then the embedding $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$ is compact.*

Theorem 75 ([16]). *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary nonempty open set.*

(1) Suppose $1 \leq p \leq q < \infty$ and $0 \leq t \leq s$ satisfy $s - n/p \geq t - n/q$. Then, $W_K^{s,p}(\Omega) \hookrightarrow W_K^{t,q}(\Omega)$ for all $K \in \mathcal{K}(\Omega)$

(2) *For all* $k_1, k_2 \in \mathbb{N}_0$ *with* $k_1 \leq k_2$ *and* $1 < p < \infty$, $W^{k_2,p}(\Omega) \hookrightarrow W^{k_1,p}(\Omega)$

(3) *If* $0 \leq t \leq s < 1$ *and* $1 < p < \infty$, *then* $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$

(4) *If* $0 \leq t \leq s < \infty$ *are such that* $\lfloor s \rfloor = \lfloor t \rfloor$ *and* $1 < p < \infty$, *then* $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$

(5) *If* $0 \leq t \leq s < \infty$, $t \in \mathbb{N}_0$, *and* $1 < p < \infty$, *then* $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$

Theorem 76 ([6]). *Let* Ω *be a nonempty bounded open subset of* $\mathbb{R}^n$ *with Lipschitz continuous boundary or* $\Omega = \mathbb{R}^n$. *If* $sp > n$, *then* $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^0(\Omega)$ *and* $W^{s,p}(\Omega)$ *is a Banach algebra.*

In the next several theorems, we will list certain multiplication properties of Sobolev spaces. Suppose $\varphi \in C^\infty(\Omega)$ and $u \in W^{s,p}(\Omega)$. If $s \geq 0$, then the product φu has a clear meaning. What if $s < 0$? In this case, $u|_{D(\Omega)}$ is a distribution and by the product φu we mean the distribution $(\varphi)(u|_{D(\Omega)})$; then φu is in $W^{s,p}(\Omega)$ if $(\varphi)(u|_{D(\Omega)})$: $(C_c^\infty(\Omega), \|.\|_{-s,p'}) \longrightarrow \mathbb{R}$ is continuous. Because then it possesses a unique extension to a continuous linear map from $W_0^{-s,p'}(\Omega)$ to $\mathbb{R}$ and so it can be viewed as an element of $[W_0^{-s,p'}(\Omega)]^* = W^{s,p}(\Omega)$. See Theorem 70 and Corollary 71. Also see Remark 89.

Theorem 77 (multiplication by smooth functions I, [31], page 203). *Let* $s \in \mathbb{R}$, $1 < p < \infty$, *and* $\varphi \in BC^\infty(\mathbb{R}^n)$. *Then, the linear map*

$$m_\varphi : W^{s,p}(\mathbb{R}^n) \longrightarrow W^{s,p}(\mathbb{R}^n), \quad u \mapsto \varphi u \tag{60}$$

is well-defined and bounded.

Theorem 78 (multiplication by smooth functions II, [16]). *Let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary.*

(1) *Let* $k \in \mathbb{N}_0$ *and* $1 < p < \infty$. *If* $\varphi \in BC^k(\Omega)$, *then the linear map* $W^{k,p}(\Omega) \longrightarrow W^{k,p}(\Omega)$ *defined by* $u \mapsto \varphi u$ *is well-defined and bounded*

(2) *Let* $s \in \mathbb{R}$ *and* $1 < p < \infty$. *If* $\varphi \in BC^\infty(\Omega)$, *then the linear map* $W^{s,p}(\Omega) \longrightarrow W^{s,p}(\Omega)$ *defined by* $u \mapsto \varphi u$ *is well-defined and bounded*

Theorem 79 (multiplication by smooth functions III, [16]). *Let* Ω *be any nonempty open set in* $\mathbb{R}^n$. *Let* $p \in (1,\infty)$.

(1) *If* $0 \leq s < 1$ *and* $\varphi \in BC^{0,1}(\Omega)$ *(that is,* $\varphi \in L^\infty(\Omega)$ *and* φ *is Lipschitz), then*

$$m_\varphi : W^{s,p}(\Omega) \longrightarrow W^{s,p}(\Omega), \quad u \mapsto \varphi u \tag{61}$$

is a well-defined bounded linear map

(2) *If* $k \in \mathbb{N}_0$ *and* $\varphi \in BC^k(\Omega)$, *then*

$$m_\varphi : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\Omega), \quad u \mapsto \varphi u \tag{62}$$

is a well-defined bounded linear map

(3) *If* $-1 < s < 0$ *and* $\varphi \in BC^{\infty,1}(\Omega)$ *or* $s \in \mathbb{Z}^-$ *and* $\varphi \in BC^\infty(\Omega)$, *then*

$$m_\varphi : W^{s,p}(\Omega) \longrightarrow W^{s,p}(\Omega), \quad u \mapsto \varphi u \tag{63}$$

is a well-defined bounded linear map

Theorem 80 (multiplication by smooth functions IV, [16]). *Let* Ω *be a nonempty open set in* $\mathbb{R}^n$, $K \in \mathcal{K}(\Omega)$, $p \in (1,\infty)$, *and* $-1 < s < 0$ *or* $s \in \mathbb{Z}^-$ *or* $s \in [0,\infty)$. *If* $\varphi \in C^\infty(\Omega)$, *then the linear map*

$$W_K^{s,p}(\Omega) \longrightarrow W_K^{s,p}(\Omega), \quad u \mapsto \varphi u \tag{64}$$

is well-defined and bounded.

Theorem 81 (multiplication by smooth functions V, [16]). *Let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary. Let* $K \in \mathcal{K}(\Omega)$. *Suppose* $s \in \mathbb{R}$ *and* $p \in (1,\infty)$. *If* $\varphi \in C^\infty(\Omega)$, *then the linear map* $W_K^{s,p}(\Omega) \longrightarrow W_K^{s,p}(\Omega)$ *defined by* $u \mapsto \varphi u$ *is well-defined and bounded.*

In the next definition, we introduce the notion of *smooth multiplication triple* which will play a key role in several theorems that will follow.

Definition 82 (smooth multiplication triple). Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$ and $1 < p < \infty$.

(i) We say that the triple (s, p, Ω) is a *smooth multiplication triple* if for all $\varphi \in C_c^\infty(\Omega)$, the map

$$m_\varphi : W^{s,p}(\Omega) \longrightarrow W^{s,p}(\Omega) \quad u \mapsto \varphi u \tag{65}$$

is well-defined and bounded

(ii) We say that the triple (s, p, Ω) is an *interior smooth multiplication triple* if for all $\varphi \in C_c^\infty(\Omega)$ and $K \in \mathcal{K}(\Omega)$, the map

$$m_\varphi : W_K^{s,p}(\Omega) \longrightarrow W_K^{s,p}(\Omega) \quad u \mapsto \varphi u \tag{66}$$

is well-defined and bounded

Remark 83.

(1) Every smooth multiplication triple is also an interior smooth multiplication triple

(2) It is a direct consequence of Theorems 77, 78, and 79 that

(i) if $\Omega = \mathbb{R}^n$ or Ω is bounded with Lipschitz continuous boundary, then for all $s \in \mathbb{R}$ and $1 < p < \infty$, (s, p, Ω) is a smooth multiplication triple

(ii) if Ω is any open set in $\mathbb{R}^n$, $1 < p < \infty$, and $s \in \mathbb{R}$ is not a noninteger with magnitude greater than 1, then (s, p, Ω) is a smooth multiplication triple

(3) It is a direct consequence of Theorem 80 and Theorem 81 that

(i) if $\Omega = \mathbb{R}^n$ or Ω is bounded with Lipschitz continuous boundary, then for all $s \in \mathbb{R}$ and $1 < p < \infty$, (s, p, Ω) is an interior smooth multiplication triple

(ii) if Ω is any open set in $\mathbb{R}^n$, $1 < p < \infty$, and $s \in \mathbb{R}$ is not a noninteger less than -1, then (s, p, Ω) is an interior smooth multiplication triple

(4) If (s, p, Ω) is a smooth multiplication triple and $K \in \mathcal{K}(\Omega)$, then $W_K^{s,p}(\Omega) \subseteq W_0^{s,p}(\Omega)$ (see the proof of Theorem 7.31 in [16]). Of course, if $s < 0$, then $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ and so $W_K^{s,p}(\Omega) \subseteq W_0^{s,p}(\Omega)$ holds for all $s < 0$, $1 < p < \infty$ and open sets $\Omega \subseteq \mathbb{R}^n$

Theorem 84. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \geq 0$, and $1 < p < \infty$. If (s, p, Ω) is a smooth multiplication triple so is $(-s, p', \Omega)$.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$. For all $u \in W^{-s,p'}(\Omega) = W_0^{-s,p'}(\Omega)$ and $\psi \in D(\Omega)$, we have

$$\begin{aligned} &\left|\langle \varphi u, \psi\rangle_{D'(\Omega)\times D(\Omega)}\right| \\ &= \left|\langle u, \varphi\psi\rangle_{D'(\Omega)\times D(\Omega)}\right| \overset{\text{Remark 69}}{=} \left|\langle u, \varphi\psi\rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}\right| \\ &\leq \|u\|_{W^{-s,p'}(\Omega)} \|\varphi\psi\|_{W^{s,p}(\Omega)} \lesssim \|u\|_{W^{-s,p'}(\Omega)} \|\psi\|_{W^{s,p}(\Omega)}. \end{aligned} \tag{67}$$

The last inequality holds because (s, p, Ω) is a smooth multiplication triple. It follows from Corollary 71 that $\varphi u \in W_0^{-s,p'}(\Omega)$ and $\|\varphi u\|_{W^{-s,p'}(\Omega)} \lesssim \|u\|_{W^{-s,p'}(\Omega)}$; that is, $m_\varphi : W^{-s,p'}(\Omega) \longrightarrow W^{-s,p'}(\Omega)$ is well-defined and continuous. □

Theorem 85. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$ and $1 < p < \infty$. If $s < 0$, further assume that $(-s, p', \Omega)$ is a smooth multiplication triple. Suppose that $\Omega' \subseteq \Omega$ and $K \in \mathcal{K}(\Omega')$. Then,*

(1) for all $u \in W_K^{s,p}(\Omega)$, $\|u\|_{W^{s,p}(\Omega)} \simeq \|u|_{\Omega'}\|_{W^{s,p}(\Omega')}$

(2) for all $u \in W_K^{s,p}(\Omega')$, $\|\text{ext}^0_{\Omega',\Omega} u\|_{W^{s,p}(\Omega)} \simeq \|u\|_{W^{s,p}(\Omega')}$

Proof. The claim follows from the argument presented in the proofs of Corollary 7.39 and Theorem 7.46 in [16]. □

Theorem 86 (([30], pages 598-605), ([6], Section 1.4)). *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\alpha \in \mathbb{N}_0^n$. Suppose Ω is a nonempty open set in $\mathbb{R}^n$. Then,*

(1) the linear operator $\partial^\alpha : W^{s,p}(\mathbb{R}^n) \longrightarrow W^{s-|\alpha|,p}(\mathbb{R}^n)$ is well-defined and bounded

(2) for $s < 0$, the linear operator $\partial^\alpha : W^{s,p}(\Omega) \longrightarrow W^{s-|\alpha|,p}(\Omega)$ is well-defined and bounded

(3) for $s \geq 0$ and $|\alpha| \leq s$, the linear operator $\partial^\alpha : W^{s,p}(\Omega) \longrightarrow W^{s-|\alpha|,p}(\Omega)$ is well-defined and bounded

(4) if Ω is bounded with Lipschitz continuous boundary, and if $s \geq 0$, $s - 1/p \neq$ integer (i.e., the fractional part of s is not equal to $1/p$), then the linear operator $\partial^\alpha : W^{s,p}(\Omega) \longrightarrow W^{s-|\alpha|,p}(\Omega)$ for $|\alpha| > s$ is well-defined and bounded

Theorem 87. *Assumptions:*

(i) $\Omega = \mathbb{R}^n$ or Ω is a bounded domain with Lipschitz continuous boundary

(ii) $s_i, s \in \mathbb{R}$, $s_i \geq s \geq 0$ for $i = 1, 2$

(iii) $1 < p_i \leq p < \infty$ for $i = 1, 2$

(iv) $s_i - s \geq n(1/p_i - 1/p)$

(v) $s_1 + s_2 - s > n(1/p_1 + 1/p_2 - 1/p)$

Claim: If $u \in W^{s_1,p_1}(\Omega)$ and $v \in W^{s_2,p_2}(\Omega)$, then $uv \in W^{s,p}(\Omega)$ and moreover the pointwise multiplication of functions is a continuous bilinear map

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \longrightarrow W^{s,p}(\Omega). \tag{68}$$

Remark 88. A number of other results concerning the sufficient conditions on the exponents s_i, p_i, s, p that guarantee the multiplication $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ is well-defined and continuous are discussed in detail in [18].

Remark 89. Suppose that (s, p, Ω) is a smooth multiplication triple with $s \geq 0$. $W^{-s,p'}(\Omega) = W_0^{-s,p'}(\Omega)$ is the dual of $W_0^{s,p}(\Omega)$ and $\langle u, f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}$ is the action of the functional u on the function f. As it was discussed before, if ψ is a function in $C_c^\infty(\Omega)$, $(\psi)(u|_{D(\Omega)})$ is defined as a product of a smooth function and a distribution. Since (s, p, Ω) is a smooth multiplication triple, $(-s, p', \Omega)$ will also be a smooth multiplication triple, and that means $(\psi)(u|_{D(\Omega)}) : (C_c^\infty(\Omega), \|.\|_{s,p}) \longrightarrow \mathbb{R}$ is continuous (see the note right after Theorem 76). We interpret ψu as an element of $W^{-s,p'}(\Omega) = [W_0^{s,p}(\Omega)]^*$ to be the unique continuous linear extension of $\psi(u|_{D(\Omega)})$ to the entire $W_0^{s,p}(\Omega)$. It is easy to

see that, this unique linear extension is given by

$$\langle \psi u, f\rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \coloneqq \langle u, \psi f\rangle_{W^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}, \tag{69}$$

that is, the above map is linear continuous and its restriction to $D(\Omega)$ is the same as $\psi(u|_{D(\Omega)})$. (Note that since (s,p,Ω) is a smooth multiplication triple, ψf is indeed an element of $W_0^{s,p}(\Omega)$.)

Theorem 90 ([32]). *Let $s \in [1,\infty)$ and $1 < p < \infty$, and let*

$$m = \begin{cases} s, & \text{if } s \text{ is an integer},\\ \lfloor s\rfloor + 1, & \text{otherwise}. \end{cases} \tag{70}$$

If $F \in C^m(\mathbb{R})$ is such that $F(0) = 0$ and $F, F', \cdots, F^{(m)} \in L^\infty(\mathbb{R})$ (in particular, note that every $F \in C_c^\infty(\mathbb{R})$ with $F(0) = 0$ satisfies these conditions), then the map $u \mapsto F(u)$ is well-defined and continuous from $W^{s,p}(\mathbb{R}^n) \cap W^{1,sp}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$.

Corollary 91. *Let s, p, and F be as in the previous theorem. Moreover, suppose $sp > n$. Then, the map $u \mapsto F(u)$ is well-defined and continuous from $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$. The reason is that when $sp > n$, we have $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{1,sp}(\mathbb{R}^n)$.*

In the remaining of this section, we will state certain useful properties of the topological vector space $W^{s,p}_{\text{comp}}$. The properties we will discuss here echo the ones stated in [24] for spaces H^s_{comp}.

Theorem 92. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$, and $1 < p < \infty$. Then, $D(\Omega)$ is continuously embedded in $W^{s,p}_{comp}(\Omega)$.*

Proof. For all $K \in \mathscr{K}(\Omega)$, we have

$$\mathscr{E}_K(\Omega) \hookrightarrow D(\Omega) \hookrightarrow W^{s,p}(\Omega). \tag{71}$$

This together with the fact that the image of $\mathscr{E}_K(\Omega)$ under the identity map is inside $W^{s,p}_K(\Omega)$ implies that

$$\mathscr{E}_K(\Omega) \hookrightarrow W^{s,p}_K(\Omega). \tag{72}$$

Also, by the definition of the inductive limit topology on $W^{s,p}_{\text{comp}}(\Omega)$, we have

$$W^{s,p}_K(\Omega) \hookrightarrow W^{s,p}_{\text{comp}}(\Omega). \tag{73}$$

It follows from (72) and (73) that for all $K \in \mathscr{K}(\Omega)$,

$$\mathscr{E}_K(\Omega) \hookrightarrow W^{s,p}_{\text{comp}}(\Omega), \tag{74}$$

which, by Theorem 37, implies that $D(\Omega) \hookrightarrow W^{s,p}_{\text{comp}}(\Omega)$. □

Theorem 93. *Let (s,p,Ω) be a smooth multiplication triple. Then, $C_c^\infty(\Omega)$ is dense in $W^{s,p}_{comp}(\Omega)$.*

Proof. We will follow the proof given in [24] for spaces H^s_{comp}. Let $u \in W^{s,p}_{\text{comp}}(\Omega)$. It is enough to show that there exists a sequence in $C_c^\infty(\Omega)$ that converges to u in $W^{s,p}_{\text{comp}}(\Omega)$ (this proves sequential density which implies density). By Meyers-Serrin theorem, there exists a sequence $\varphi_m \in C^\infty(\Omega) \cap W^{s,p}(\Omega)$ such that $\varphi_m \longrightarrow u$ in $W^{s,p}(\Omega)$. Let $\chi \in C_c^\infty(\Omega)$ be such that $\chi = 1$ on a neighborhood containing supp u (see Theorem 41). Let $K = \text{supp } \chi$. Since (s,p,Ω) is a smooth multiplication triple, multiplication by χ is a linear continuous map on $W^{s,p}(\Omega)$ and so $\chi\varphi_m \longrightarrow \chi u$ in $W^{s,p}(\Omega)$. Now, we note that $\chi u = u$ and for all m, $\chi\varphi_m$ are in $C_c^\infty(\Omega)$ with support inside K. Consequently, $\chi\varphi_m \longrightarrow u$ in $W^{s,p}_K(\Omega)$. Now, since $W^{s,p}_K(\Omega) \hookrightarrow W^{s,p}_{\text{comp}}(\Omega)$, we may conclude that $\chi\varphi_m$ is a sequence in $C_c^\infty(\Omega)$ that converges to u in $W^{s,p}_{\text{comp}}(\Omega)$. □

Remark 94. As a consequence, if (s,p,Ω) is a smooth multiplication triple, then $[W^{s,p}_{\text{comp}}(\Omega)]^*$ (equipped with the strong topology) is continuously embedded in $D'(\Omega)$. More precisely, the identity map $i : D(\Omega) \longrightarrow W^{s,p}_{\text{comp}}(\Omega)$ is continuous with dense image, and therefore, by Theorem 27, the adjoint $i^* : [W^{s,p}_{\text{comp}}(\Omega)]^* \longrightarrow D'(\Omega)$ is a continuous injective map. We have

$$\begin{aligned} \langle i^* u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} &= \langle u, i\varphi\rangle_{\left[W^{s,p}_{\text{comp}}(\Omega)\right]^*\times W^{s,p}_{\text{comp}}(\Omega)}\\ &= \langle u, \varphi\rangle_{\left[W^{s,p}_{\text{comp}}(\Omega)\right]^*\times W^{s,p}_{\text{comp}}(\Omega)}. \end{aligned} \tag{75}$$

We usually identify $[W^{s,p}_{\text{comp}}(\Omega)]^*$ with its image under i^* and view $[W^{s,p}_{comp}(\Omega)]^*$ as a subspace of $D'(\Omega)$. So, under this identification, we can rewrite the above equality as follows:

$$\forall u \in \left[W^{s,p}_{\text{comp}}(\Omega)\right]^* \quad \langle u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi\rangle_{\left[W^{s,p}_{\text{comp}}(\Omega)\right]^*\times W^{s,p}_{\text{comp}}(\Omega)}. \tag{76}$$

Next, we will prove that if (s,p,Ω) is a smooth multiplication triple, then $W^{s,p}_{\text{comp}}(\Omega)$ is separable. To this end, we need the following lemma.

Lemma 95. *Let (X,τ) and (Y,τ') be two topological spaces. Suppose that*

(1) A is dense in (X,τ)

(2) $T : (X,\tau) \longrightarrow (Y,\tau')$ is continuous

(3) $T(X)$ is dense in (Y,τ')

Then, $T(A)$ is dense in (Y,τ').

Proof. It is enough to show that $T(A)$ intersects every nonempty open set in (Y,τ'). So let $O \in \tau'$ be nonempty. Since $T(X)$ is dense in (Y,τ'), we have $O \cap T(X) \neq \varnothing$ and so $T^{-1}(O)$ is nonempty. Also, since T is continuous, $T^{-1}(O) \in \tau$. A is dense in (X,τ), so $A \cap T^{-1}(O) \neq \varnothing$. Therefore,

$$T(A) \cap O \supseteq T\left(A \cap T^{-1}(O)\right) \neq \varnothing. \tag{77}$$

□

Theorem 96. *Let (s, p, Ω) be a smooth multiplication triple. Then, $W^{s,p}_{comp}(\Omega)$ is separable.*

Proof. According to Theorems 92 and 93, $D(\Omega)$ is continuously embedded in $W^{s,p}_{\text{comp}}(\Omega)$ and it is dense in $W^{s,p}_{\text{comp}}(\Omega)$. Since $D(\Omega)$ is separable, it follows from Lemma 95 that $W^{s,p}_{\text{comp}}(\Omega)$ is separable. □

Theorem 97. *Let (s, p, Ω) be an interior smooth multiplication triple. Let $\{\psi_j\}_{j\in\mathbb{N}_0}$ be the partition of unity introduced in Theorem 43. Let S be the collection of all sequences whose terms are nonnegative integers. For all sequences $a = (a_0, a_1, \cdots) \in S$ define $q_{a,s,p} : W^{s,p}_{comp}(\Omega) \longrightarrow \mathbb{R}$ by*

$$q_{a,s,p}(u) = \sum_{j=0}^{\infty} a_j \left\| \psi_j u \right\|_{W^{s,p}(\Omega)}. \tag{78}$$

Then, $\{q_{a,s,p}\}_{a\in S}$ is a separating family of seminorms on $W^{s,p}_{comp}(\Omega)$ and the natural topology induced by this family on $W^{s,p}_{comp}(\Omega)$ is the same as the inductive limit topology on $W^{s,p}_{comp}(\Omega)$.

Proof. Note that support of every $u \in W^{s,p}_{\text{comp}}$ is compact, so for each u, only finitely many of $\psi_j u$'s are nonzero. Thus, the sum in the definition of $q_{a,s,p}$ is a finite sum. Now it is not hard to show that each $q_{a,s,p}$ is a seminorm and $\{q_{a,s,p}\}_{a\in S}$ is separating. Here, we will show that the topologies are the same. Let us denote the inductive limit topology on $W^{s,p}_{\text{comp}}(\Omega)$ by τ and the natural topology induced by the given family of seminorms τ'.

In what follows, we implicitly use the fact that both topologies are locally convex and translation invariant.

Step 1 ($\tau' \subseteq \tau$). We will prove that for each $K \in \mathscr{K}(\Omega)$, $W^{s,p}_K(\Omega) \hookrightarrow (W^{s,p}_{\text{comp}}(\Omega), \tau')$. This together with the definition of τ (the biggest topology with this property) implies that $\tau' \subseteq \tau$. Let $K \in \mathscr{K}(\Omega)$. By Theorem 18, it is enough to show that for all $a \in S$, $q_{a,s,p} \circ Id : W^{s,p}_K(\Omega) \longrightarrow \mathbb{R}$ is continuous. Since K is compact, there are only finitely may ψ_j's such that $K \cap \operatorname{supp}\ \psi_j \neq \varnothing$; let us call them $\psi_{j_1}, \cdots, \psi_{j_l}$. So, for all $u \in W^{s,p}_K(\Omega)$,

$$q_{a,s,p}(u) = a_{j_1} \left\| \psi_{j_1} u \right\|_{W^{s,p}(\Omega)} + \cdots + a_{j_l} \left\| \psi_{j_l} u \right\|_{W^{s,p}(\Omega)}. \tag{79}$$

By assumption, (s, p, Ω) is an interior smooth multiplication triple, so for each $j \in \{j_1, \cdots, j_l\}$, the mapping $u \mapsto \| \psi_j u \|_{W^{s,p}(\Omega)}$ from $W^{s,p}_K(\Omega) \longrightarrow \mathbb{R}$ is continuous. Hence, $q_{a,s,p} \circ Id : W^{s,p}_K(\Omega) \longrightarrow \mathbb{R}$ must be continuous.

Step 2 ($\tau \subseteq \tau'$). Since $(W^{s,p}_{\text{comp}}(\Omega), \tau)$ is a locally convex topological vector space, there exists a separating family of seminorms $\mathscr{P}$ whose corresponding natural topology is τ (see Theorem 15). We will prove that for all $\tilde{p} \in \mathscr{P}$, $\tilde{p} : (W^{s,p}_{\text{comp}}(\Omega), \tau') \longrightarrow \mathbb{R}$ is continuous. This together with the fact that τ is the smallest topology with this property shows that $\tau \subseteq \tau'$. Let $\tilde{p} \in \mathscr{P}$. By Theorem 12, it is enough to prove that there exists $a \in S$ such that

$$\forall u \in W^{s,p}_{\text{comp}}(\Omega) \quad \tilde{p}(u) \leq q_{a,s,p}(u). \tag{80}$$

For all $u \in W^{s,p}_{\text{comp}}(\Omega)$, we have $\tilde{p}(u) = \tilde{p}(\sum_j \psi_j u)$. Since u has compact support, only finitely many terms in the sum are nonzero, and so by the finite subadditivity of a seminorm, we get

$$\tilde{p}(u) = \tilde{p}\left(\sum_j \psi_j u\right) \leq \sum_j \tilde{p}\left(\psi_j u\right). \tag{81}$$

Now, note that $\psi_j u$ belongs to the normed space $W^{s,p}_{\text{supp}\ \psi_j}(\Omega)$. Since $\tilde{p} : (W^{s,p}_{\text{comp}}(\Omega), \tau) \longrightarrow \mathbb{R}$ is continuous and $W^{s,p}_{\text{supp}\ \psi_j}(\Omega) \hookrightarrow (W^{s,p}_{\text{comp}}(\Omega), \tau)$, we can conclude that $\tilde{p} : W^{s,p}_{\text{supp}\ \psi_j}(\Omega) \longrightarrow \mathbb{R}$ is continuous. Thus, by Theorem 9, there exists a positive integer a_j such that

$$\forall u \in W^{s,p}_{\text{comp}}(\Omega) \quad \tilde{p}\left(\psi_j u\right) \leq a_j \left\| \psi_j u \right\|_{W^{s,p}(\Omega)}. \tag{82}$$

It follows that for all $u \in W^{s,p}_{\text{comp}}(\Omega)$,

$$\tilde{p}(u) \leq \sum_j \tilde{p}\left(\psi_j u\right) \leq \sum_j a_j \left\| \psi_j u \right\|_{W^{s,p}(\Omega)} = q_{a,s,p}(u), \tag{83}$$

where $a = (a_0, a_1, \cdots)$ □

4. Spaces of Locally Sobolev Functions

Let $s \in \mathbb{R}$, $1 < p < \infty$. Let Ω be a nonempty open set in $\mathbb{R}^n$. We define

$$W^{s,p}_{loc}(\Omega) \coloneqq \left\{ u \in D'(\Omega) : \forall \varphi \in C^\infty_c(\Omega) \quad \varphi u \in W^{s,p}(\Omega) \right\}. \tag{84}$$

We equip $W^{s,p}_{loc}(\Omega)$ with the natural topology induced by the separating family of seminorms $\{|.|_{\varphi,s,p}\}_{\varphi\in C^\infty_c(\Omega)}$ (see Definition 10) where

$$\forall u \in W^{s,p}_{loc}(\Omega), \varphi \in C^\infty_c(\Omega) \quad |u|_{\varphi,s,p} \coloneqq \|\varphi u\|_{W^{s,p}(\Omega)}. \tag{85}$$

When s and p are clear from the context, we may just write $|u|_\varphi$ or $p_\varphi(u)$ instead of $|u|_{\varphi,s,p}$. It is easy to show that for all $\varphi \in C^\infty_c(\Omega)$, $|.|_{\varphi,s,p}$ is a seminorm on $W^{s,p}_{loc}(\Omega)$. The fact

that the family of seminorms $\{|.|_{\varphi,s,p}\}_{\varphi\in C_c^\infty(\Omega)}$ is separating will be proved in Theorem 103.

Remark 98. Note that, by item (1) of Theorem 20, $u_i \longrightarrow u$ in $W^{s,p}_{loc}(\Omega)$ if and only if $\varphi u_i \longrightarrow \varphi u$ in $W^{s,p}(\Omega)$ for all $\varphi \in C_c^\infty(\Omega)$.

Remark 99. Clearly if (s,p,Ω) is a smooth multiplication triple, then $W^{s,p}(\Omega) \subseteq W^{s,p}_{loc}(\Omega)$.

An equivalent description of locally Sobolev functions is described in the following theorem.

Theorem 100. *Suppose that (s,p,Ω) is a smooth multiplication triple. Then, $u \in D'(\Omega)$ is in $W^{s,p}_{loc}(\Omega)$ if and only if for every precompact open set V with $\bar{V} \subseteq \Omega$ there is $w \in W^{s,p}(\Omega)$ such that $w|_V = u|_V$.*

Proof. $\Rightarrow$: suppose $u \in W^{s,p}_{loc}(\Omega)$ and let V be a precompact open set such that $\bar{V} \subseteq \Omega$. Let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi = 1$ on a neighborhood containing $\bar{V}$. Let $w = \varphi u$. u is a locally Sobolev function, so $w \in W^{s,p}(\Omega)$; also clearly $w|_V = u|_V$.

$\Leftarrow$: suppose $u \in D'(\Omega)$ has the property that for every precompact open set V with $\bar{V} \subseteq \Omega$ there is $w \in W^{s,p}(\Omega)$ such that $w|_V = u|_V$. Let $\varphi \in C_c^\infty(\Omega)$. We need to show that $\varphi u \in W^{s,p}(\Omega)$. Note that supp φ is compact, so there exists a bounded open set V such that

$$\text{supp } \varphi \subseteq V \subseteq \bar{V} \subseteq \Omega. \tag{86}$$

By assumption, there exists $w \in W^{s,p}(\Omega)$ such that $w|_V = u|_V$. It follows from the hypothesis of the theorem that $\varphi w \in W^{s,p}(\Omega)$. Clearly $\varphi w = \varphi u$ on Ω. Therefore, $\varphi u \in W^{s,p}(\Omega)$. □

5. Overview of the Basic Properties

Material of this section is mainly an adaptation of the material presented in the excellent work of Antonic and Burazin [17], which is restricted to integer order Sobolev spaces, and Peterson [24], which is restricted to Hilbert spaces H^s. We have added certain details to the statements of the theorems and their proofs to ensure all the arguments are valid for both integer and noninteger order Sobolev-Slobodeckij spaces.

Definition 101. If A is a subset of $C_c^\infty(\Omega)$ with the property that,

$$\forall x \in \Omega\, \exists \varphi \in A \text{ such that } \varphi \geq 0 \text{ and } \varphi(x) \neq 0, \tag{87}$$

then we say A is an *admissible* family of functions.

Remark 102. Note that if A is an admissible family of functions, then for all $m \in \mathbb{N}$, the set $\{\varphi^m : \varphi \in A\}$ is also an admissible family of functions.

Theorem 103. *Let (s,p,Ω) be an interior smooth multiplication triple. If A is an admissible family of functions then*

(1) $W^{s,p}_{loc}(\Omega) = \{u \in D'(\Omega) : \forall \varphi \in A \quad \varphi u \in W^{s,p}(\Omega)\}$

(2) The collection $\{|.|_\varphi : \varphi \in A\}$ is a separating family of seminorms on $W^{s,p}_{loc}(\Omega)$

(3) The natural topology induced by the separating family of seminorms $\{|.|_\varphi : \varphi \in A\}$ is the same as the topology of $W^{s,p}_{loc}(\Omega)$

Proof.

(1) Let $u \in D'(\Omega)$ be such that $\varphi u \in W^{s,p}(\Omega)$ for all $\varphi \in A$. We need to show that if $\psi \in C_c^\infty(\Omega)$, then $\psi u \in W^{s,p}(\Omega)$. By the definition of A, for all $x \in$ supp ψ, there exists $\varphi_x \in A$ such that $\varphi_x(x) > 0$. Define

$$U_x := \{y \in \Omega : \varphi_x(y) > 0\}. \tag{88}$$

Clearly, $x \in U_x$, and since φ_x is continuous, U_x is an open set. $\{U_x\}_{x\in supp\psi}$ is an open cover of the compact set supp ψ. So there exist points $x_1, \cdots, x_k$ such that supp $\psi \subseteq U := U_{x_1} \cup \cdots \cup U_{x_k}$. If $y \in U$, then there exists $1 \leq i \leq k$ such that $y \in U_{x_i}$ and so $\varphi_{x_i}(y) > 0$. So the smooth function $\sum_{i=1}^k \varphi_{x_i}$ is nonzero on U. Thus, on U we have

$$\psi u = \frac{\psi}{\sum_{i=1}^k \varphi_{x_i}} \left(\sum_{i=1}^k \varphi_{x_i} u \right). \tag{89}$$

Indeed, if we define

$$\xi(z) = \begin{cases} \dfrac{\psi(z)}{\sum_{i=1}^k \varphi_{x_i}(z)}, & \text{if } z \in U, \\ 0, & \text{otherwise,} \end{cases} \tag{90}$$

then ξ is smooth with compact support in U and

$$\psi u = \xi \sum_{i=1}^k \varphi_{x_i} u \tag{91}$$

on the entire Ω. Now, note that for each i, $\varphi_{x_i} u$ is in $W^{s,p}(\Omega)$ (because by assumption $\varphi u \in W^{s,p}(\Omega)$ for all $\varphi \in A$). So $\sum_{i=1}^k \varphi_{x_i} u \in W^{s,p}(\Omega)$. Since $\xi \in C_c^\infty(\Omega)$ and $\sum_{i=1}^k \varphi_{x_i} u$ have compact support and (s,p,Ω) is an interior smooth multiplication triple, it follows that $\xi \sum_{i=1}^k \varphi_{x_i} u \in W^{s,p}(\Omega)$

(2) Now, we prove that $\{|.|_\varphi : \varphi \in A\}$ is a separating family of seminorms. We need to show that if $u \in$

$W^{s,p}_{loc}(\Omega)$ has the property that for all $\varphi \in A$ $|u|_\varphi = \|\varphi u\|_{W^{s,p}(\Omega)} = 0$, then $u = 0$. By definition of locally Sobolev functions, u is an element of $D'(\Omega)$. So, in order to show that $u = 0$, it is enough to prove that for all $\eta \in C^\infty_c(\Omega)$, $\langle u, \eta\rangle_{D'(\Omega)\times D(\Omega)} = 0$. We consider two cases:

Case 1. $A = C^\infty_c(\Omega)$. Let $\varphi \in A$ be such that $\varphi = 1$ on a neighborhood containing supp η. By assumption $\varphi u = 0$ in $W^{s,p}(\Omega)$ and so it is zero in $D'(\Omega)$. Now we have

$$\langle u, \eta\rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi\eta\rangle_{D'(\Omega)\times D(\Omega)} = \langle \varphi u, \eta\rangle_{D'(\Omega)\times D(\Omega)} = 0 \tag{92}$$

which is exactly what we wanted to prove

Case 2. $A \subset C^\infty_c(\Omega)$. We claim that if $\|\varphi u\|_{W^{s,p}(\Omega)} = 0$ for all $\varphi \in A$, then for any $\psi \in C^\infty_c(\Omega)$, $\|\psi u\|_{W^{s,p}(\Omega)} = 0$, and so this case reduces to the previous case. Indeed, if ψ is an arbitrary element of $C^\infty_c(\Omega)$, then by what was proved in item (1),

$$\psi u = \xi \sum_{i=1}^{k} \varphi_{x_i} u, \tag{93}$$

where by assumption, for each i, $\varphi_{x_i} u$ is zero as an element of $W^{s,p}(\Omega)$. Hence, $\psi u = 0$ in $W^{s,p}(\Omega)$

(3) Finally, we show that the natural topology $\tau_{\mathscr{P}}$ induced by $\mathscr{P} = \{|.|_\varphi : \varphi \in A\}$ is the same as the natural topology $\tau_{\mathscr{Q}}$ induced by $\mathscr{Q} = \{|.|_\varphi : \varphi \in C^\infty_c(\Omega)\}$. Obviously $\mathscr{P}$ is a subset of $\mathscr{Q}$, so it follows from the definition of natural topology induced by a family of seminorms (see Definition 10) that $\tau_{\mathscr{P}} \subseteq \tau_{\mathscr{Q}}$. In order to show that $\tau_{\mathscr{Q}} \subseteq \tau_{\mathscr{P}}$, it is enough to prove that for all $\psi \in C^\infty_c(\Omega)$, the map $|.|_\psi : (W^{s,p}_{loc}(\Omega), \tau_P) \longrightarrow \mathbb{R}$ is continuous. By what was shown in item (1), we can write

$$\forall u \in W^{s,p}_{loc}(\Omega)\ |u|_\psi = \|\psi u\|_{W^{s,p}(\Omega)} = \left\|\xi \sum_{i=1}^{k} \varphi_{x_i} u\right\|_{W^{s,p}(\Omega)} \preccurlyeq \sum_{i=1}^{k} \left\|\varphi_{x_i} u\right\|_{W^{s,p}(\Omega)} = \sum_{i=1}^{k} |u|_{\varphi_{x_i}}, \tag{94}$$

where the implicit constant does not depend on u. In the last inequality, we used the assumption that (s, p, Ω) is an interior smooth multiplication triple. Now, it follows from Theorem 20 that $|.|_\psi : (W^{s,p}_{loc}(\Omega), \tau_P) \longrightarrow \mathbb{R}$ is continuous

□

Lemma 104. *There exists an admissible family $A \subseteq C^\infty_c(\Omega)$ that has only countably many elements.*

Proof. Let $\{K_j\}_{j\in\mathbb{N}}$ be an exhaustion by compact sets for Ω. For each $j \in \mathbb{N}$, let $\varphi_j \in C^\infty_c(\Omega)$ be a nonnegative function such that $\varphi_j = 1$ on K_j and $\varphi_j = 0$ outside $\mathring{K}_{j+2}$. Clearly, $A = \{\varphi_j\}_{j\in\mathbb{N}}$ is a countable admissible family of functions. □

Corollary 105. *Let (s, p, Ω) be an interior smooth multiplication triple. Considering Theorem 13, it follows from the previous lemma and Theorem 103 that $W^{s,p}_{loc}(\Omega)$ is metrizable. Indeed, if $A = \{\varphi_j\}_{j=1}^\infty$ is a countable admissible family, then*

$$d(u, v) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|u - v|_{\varphi_j}}{1 + |u - v|_{\varphi_j}} \tag{95}$$

is a compatible translation invariant metric on $W^{s,p}_{loc}(\Omega)$.

Theorem 106. *Let (s, p, Ω) be an interior smooth multiplication triple. Then, $W^{s,p}_{loc}(\Omega)$ is a Frechet space.*

Proof. By Corollary 14, it is enough to show that $W^{s,p}_{loc}(\Omega)$ equipped with the metric in (95) is complete. Note that all admissible families result in equivalent topologies in $W^{s,p}_{loc}(\Omega)$. So we can choose the functions φ_j's in the definition of d to be the partition of unity introduced in Theorem 43. Now, suppose $\{u_m\}$ is a Cauchy sequence with respect to d. In what follows, we will prove that $\{u_m\}$ converges to a distribution u in $D'(\Omega)$. For now, let us assume this is true. We need to show that u is an element of $W^{s,p}_{loc}(\Omega)$; that is, we need to show that for all j, $\varphi_j u \in W^{s,p}(\Omega)$.

It follows from the definition of d that for each $j \in \mathbb{N}$, $\{\varphi_j u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $W^{s,p}(\Omega)$. Since $W^{s,p}(\Omega)$ is a Banach space, there exists f_j in $W^{s,p}(\Omega)$ such that $\varphi_j u_m \longrightarrow f_j$ in $W^{s,p}(\Omega)$. Note that $W^{s,p}(\Omega) \hookrightarrow D'(\Omega)$, so $\varphi_j u_m \longrightarrow f_j$ in $D'(\Omega)$, and thus, for all $\psi \in D(\Omega)$ we have

$$\begin{aligned} \left\langle f_j, \psi\right\rangle_{D'(\Omega)\times D(\Omega)} &= \lim_{m\longrightarrow\infty} \left\langle \varphi_j u_m, \psi\right\rangle_{D'(\Omega)\times D(\Omega)} \\ &= \lim_{m\longrightarrow\infty} \left\langle u_m, \varphi_j\psi\right\rangle_{D'(\Omega)\times D(\Omega)} \\ &= \left\langle u, \varphi_j\psi\right\rangle_{D'(\Omega)\times D(\Omega)} \\ &= \left\langle \varphi_j u, \psi\right\rangle_{D'(\Omega)\times D(\Omega)}. \end{aligned} \tag{96}$$

Hence, $\varphi_j u = f_j$ in $D'(\Omega)$. Since $f_j \in W^{s,p}(\Omega)$ we can conclude that $\varphi_j u \in W^{s,p}(\Omega)$.

It remains to show that $\{u_m\}$ converges in $D'(\Omega)$. To this end, it is enough to show that for all $\psi \in D(\Omega)$, the sequence $\{\langle u_m, \psi\rangle\}$ converges in $\mathbb{R}$ (see Theorem 60). Let $\psi \in D(\Omega)$. Since supp ψ is compact, there are only finitely many of φ_j's that are nonzero on the support of ψ (see

Theorem 43) which we denote by $\varphi_{j_1}, \cdots, \varphi_{j_l}$. So for each $x \in \text{supp } \psi$, $\varphi_{j_1}(x) + \cdots + \varphi_{j_l}(x) = 1$. We have

$$\begin{aligned}\langle u_m, \psi\rangle &= \left\langle u_m, \left(\varphi_{j_1}+\cdots+\varphi_{j_l}\right)\psi\right\rangle = \left\langle \left(\varphi_{j_1}+\cdots+\varphi_{j_l}\right)u_m, \psi\right\rangle \\ &= \left\langle \varphi_{j_1}u_m, \psi\right\rangle + \cdots + \left\langle \varphi_{j_l}u_m, \psi\right\rangle.\end{aligned} \tag{97}$$

$\lim_{m\longrightarrow\infty}\langle\varphi_{j_1}u_m, \psi\rangle, \cdots, \lim_{m\longrightarrow\infty}\langle\varphi_{j_l}u_m, \psi\rangle$ all exist (since $\varphi_{j_r}u_m$ is Cauchy in $W^{s,p}(\Omega)$, it is convergent in $W^{s,p}(\Omega)$, and so it is convergent in $D'(\Omega)$). Therefore, $\lim_{m\longrightarrow\infty}\langle u_m, \psi\rangle$ exists. □

Theorem 107. *Let (s, p, Ω) be a smooth multiplication triple (so we know that $W^{s,p}(\Omega) \subseteq W^{s,p}_{loc}(\Omega)$ and $W^{s,p}_{loc}(\Omega)$ is metrizable). Then, $W^{s,p}(\Omega) \hookrightarrow W^{s,p}_{loc}(\Omega)$.*

Proof. Since both spaces are metrizable, it suffices to show that if $u_i \longrightarrow u$ in $W^{s,p}(\Omega)$, then $u_i \longrightarrow u$ in $W^{s,p}_{loc}(\Omega)$. To this end, let φ be an arbitrary element of $C^\infty_c(\Omega)$. We need to show that if $u_i \longrightarrow u$ in $W^{s,p}(\Omega)$, then $\varphi u_i \longrightarrow \varphi u$ in $W^{s,p}(\Omega)$. But this is a consequence of the fact that (s, p, Ω) is a smooth multiplication triple. □

Theorem 108. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \in \mathbb{R}$, and $1 < p < \infty$. Then, $\mathscr{E}(\Omega)$ is continuously embedded in $W^{s,p}_{loc}(\Omega)$; i.e., the "identity map" from $\mathscr{E}(\Omega)$ to $W^{s,p}_{loc}(\Omega)$ is continuous.*

Proof. By Theorem 45, it is enough to show that if $\varphi_m \longrightarrow 0$ in $\mathscr{E}(\Omega)$, then $\varphi_m \longrightarrow 0$ in $W^{s,p}_{loc}(\Omega)$; that is, for all $\psi \in C^\infty_c(\Omega)$, $\psi\varphi_m \longrightarrow 0$ in $W^{s,p}(\Omega)$.

Let $\psi \in C^\infty_c(\Omega)$ and let m_ψ denote multiplication by ψ. Multiplication by smooth functions is a continuous linear operator on $\mathscr{E}(\Omega)$ ([21]). So $m_\psi : \mathscr{E}(\Omega) \longrightarrow \mathscr{E}(\Omega)$ is continuous. The range of this map is in the subspace $\mathscr{E}_{\text{supp }\psi}(\Omega)$. So $m_\psi : \mathscr{E}(\Omega) \longrightarrow \mathscr{E}_{\text{supp }\psi}(\Omega)$ is continuous. However, $\mathscr{E}_{\text{supp }\psi}(\Omega) \hookrightarrow D(\Omega)$. Hence, $m_\psi : \mathscr{E}(\Omega) \longrightarrow D(\Omega)$ is continuous. As a consequence, since $\varphi_m \longrightarrow 0$ in $\mathscr{E}(\Omega)$, $\psi\varphi_m \longrightarrow 0$ in $D(\Omega)$. Finally, since $D(\Omega) \hookrightarrow W^{s,p}(\Omega)$, we can conclude that $\psi\varphi_m \longrightarrow 0$ in $W^{s,p}(\Omega)$. □

Corollary 109. *Since $D(\Omega) \hookrightarrow \mathscr{E}(\Omega)$, it follows that under the hypotheses of Theorem 108, $D(\Omega)$ is continuously embedded in $W^{s,p}_{loc}(\Omega)$.*

Theorem 110. *Let (s, p, Ω) be a smooth multiplication triple. Then, $C^\infty_c(\Omega)$ is dense in $W^{s,p}_{loc}(\Omega)$.*

Proof. Let $u \in W^{s,p}_{loc}(\Omega)$. It is enough to show that there exists a sequence $\{\psi_j\}$ in $C^\infty_c(\Omega)$ such that $\psi_j \longrightarrow u$ in $W^{s,p}_{loc}(\Omega)$, i.e.,

$$\forall \xi \in C^\infty_c(\Omega) \quad \xi\psi_j \longrightarrow \xi u \quad \text{in} \quad W^{s,p}(\Omega). \tag{98}$$

First, note that, since (s, p, Ω) is a smooth multiplication triple, for all $\xi \in C^\infty_c(\Omega)$, there exists a constant $C_{\xi,s,p,\Omega}$ such that

$$\forall v \in W^{s,p}(\Omega) \quad \|\xi v\|_{W^{s,p}(\Omega)} \le C_{\xi,s,p,\Omega}\|v\|_{W^{s,p}(\Omega)}. \tag{99}$$

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be the admissible family introduced in the proof of Lemma 104. For each $\xi \in C^\infty_c(\Omega)$, there exists a number J_ξ such that for all $j \ge J_\xi$, $\varphi_j = 1$ on supp ξ. So,

$$\forall j \ge J_\xi \quad \varphi_j\xi = \xi. \tag{100}$$

Clearly, by definition of $W^{s,p}_{loc}(\Omega)$, for each j, $\varphi_j u \in W^{s,p}(\Omega)$, also $\varphi_j u$ has compact support, so $\varphi_j u \in W^{s,p}_0(\Omega)$ (see Remark 83). Hence, for each j, there exists $\psi_j \in C^\infty_c(\Omega)$ such that $\|\psi_j - \varphi_j u\| < 1/j$. We claim that $\xi\psi_j \longrightarrow \xi u$ in $W^{s,p}(\Omega)$. Indeed, given $\varepsilon > 0$ and $\xi \in C^\infty_c(\Omega)$, let $J > J_\xi$ be such that $1/J < \varepsilon/C_{\xi,s,p,\Omega}$. Then, for $j \ge J$, we have

$$\begin{aligned}\left\|\xi\psi_j - \xi u\right\|_{W^{s,p}(\Omega)} &= \left\|\xi\psi_j - \xi\varphi_j u\right\|_{W^{s,p}(\Omega)} \\ &= \left\|\xi\left(\psi_j - \varphi_j u\right)\right\|_{W^{s,p}(\Omega)} \\ &\le C_{\xi,s,p,\Omega}\left\|\psi_j - \varphi_j u\right\|_{W^{s,p}(\Omega)} \\ &< C_{\xi,s,p,\Omega}\frac{1}{J} < \varepsilon.\end{aligned} \tag{101}$$

□

Remark 111. As a consequence, if (s, p, Ω) is a smooth multiplication triple, then $[W^{s,p}_{loc}(\Omega)]^*$ (equipped with the strong topology) is continuously embedded in $D'(\Omega)$. More precisely, the identity map $i : D(\Omega) \longrightarrow W^{s,p}_{loc}(\Omega)$ is continuous with dense image, and therefore, by Theorem 27, the adjoint $i^* : [W^{s,p}_{loc}(\Omega)]^* \longrightarrow D'(\Omega)$ is a continuous injective map. We have

$$\begin{aligned}\langle i^*u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} &= \langle u, i\varphi\rangle_{\left[W^{s,p}_{loc}(\Omega)\right]^*\times W^{s,p}_{loc}(\Omega)} \\ &= \langle u, \varphi\rangle_{\left[W^{s,p}_{loc}(\Omega)\right]^*\times W^{s,p}_{loc}(\Omega)}.\end{aligned} \tag{102}$$

We usually identify $[W^{s,p}_{loc}(\Omega)]^*$ with its image under i^* and view $[W^{s,p}_{loc}(\Omega)]^*$ as a subspace of $D'(\Omega)$. So, under this identification, we can rewrite the above equality as follows:

$$\forall u \in \left[W^{s,p}_{loc}(\Omega)\right]^* \quad \langle u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi\rangle_{\left[W^{s,p}_{loc}(\Omega)\right]^*\times W^{s,p}_{loc}(\Omega)}. \tag{103}$$

Theorem 112. *Let (s, p, Ω) be a smooth multiplication triple. Then, $W^{s,p}_{loc}(\Omega)$ is separable.*

Proof. $D(\Omega)$ is continuously embedded in $W^{s,p}_{loc}(\Omega)$ and it is dense in $W^{s,p}_{loc}(\Omega)$. Since $D(\Omega)$ is separable, it follows from Lemma 95 that $W^{s,p}_{loc}(\Omega)$ is separable. □

As a direct consequence of the definitions, locally Sobolev functions and Sobolev functions with compact support are both subsets of the space of distributions. The next two theorems establish a duality connection between the two spaces. But first, we need to state a simple lemma.

Lemma 113. *Let X and Y be two topological spaces. Suppose that Y is Hausdorff. Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ be two continuous functions that agree on a dense subset A of X. Then, $f = g$ everywhere. (So, in particular, in order to show that two continuous mappings from X to Y are equal, we just need to show that they agree on some dense subset.)*

Proof. Suppose that there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Since Y is Hausdorff, there exist open neighborhoods U and V of $f(x_0)$ and $g(x_0)$, respectively, such that $U \cap V = \varnothing$. $f^{-1}(U) \cap g^{-1}(V)$ is a nonempty (x_0 is in it) open set in X so its intersection with A is nonempty. Let z be a point in the intersection of $f^{-1}(U) \cap g^{-1}(V)$ and A. Clearly, $f(z) \in U$ and $g(z) \in V$; but since $z \in A$, we have $f(z) = g(z)$. This contradicts the assumption that $U \cap V = \varnothing$. □

Theorem 114. *Suppose that (s, p, Ω) and $(-s, p', \Omega)$ are smooth multiplication triples. Define the mapping $T : W^{-s,p'}_{loc}(\Omega) \longrightarrow [W^{s,p}_{comp}(\Omega)]^*$ by*

$$\forall u \in W^{-s,p'}_{loc}(\Omega)\ \forall f \in W^{s,p}_{comp}(\Omega) \quad [T(u)](f) := \left\langle \psi_f u, f \right\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}, \tag{104}$$

where ψ_f is any function in $C^\infty_c(\Omega)$ that is equal to 1 on a neighborhood containing the support of f. Then,

(1) $[T(u)](f)$ does not depend on the choice of ψ_f

(2) For all $u \in W^{-s,p'}_{loc}(\Omega)$, $T(u)$ is indeed an element of $[W^{s,p}_{comp}(\Omega)]^*$

(3) $T : W^{-s,p'}_{loc}(\Omega) \longrightarrow [W^{s,p}_{comp}(\Omega)]^*$ is bijective

(4) Suppose $[W^{s,p}_{comp}(\Omega)]^*$ is equipped with the strong topology. Then, the bijective linear map $T : W^{-s,p'}_{loc}(\Omega) \longrightarrow [W^{s,p}_{comp}(\Omega)]^*$ is a topological isomorphism; i.e., it is continuous with continuous inverse. So $[W^{s,p}_{comp}(\Omega)]^*$ can be identified with $W^{-s,p'}_{loc}(\Omega)$ as topological vector spaces

Proof.

(1) For the first item, it is enough to show that if $\psi \in C^\infty_c(\Omega)$ is equal to zero on a neighborhood U containing supp f, then $\langle \psi u, f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} = 0$. Note that f is not necessarily in $C^\infty_c(\Omega)$, so we cannot directly apply the duality pairing identity stated in Remark 69. Let $\{f_m\}$ be sequence in $C^\infty_c(\Omega)$ such that $f_m \longrightarrow f$ in $W^{s,p}_0(\Omega)$. Let $\xi \in C^\infty_c(\Omega)$ be such that $\xi = 1$ on supp f and $\xi = 0$ outside U. By assumption, (s, p, Ω) is a smooth multiplication triple and so $\xi f_m \longrightarrow \xi f = f$ in $W^{s,p}_0(\Omega)$. Since elements of dual are continuous, we have

$$\begin{aligned}
&\langle \psi u, f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \\
&= \lim_{m\longrightarrow\infty} \langle \psi u, \xi f_m\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \overset{\text{Remark 69}}{=} \lim_{m\longrightarrow\infty} \langle \psi u, \xi f_m\rangle_{D'(\Omega)\times D(\Omega)} \\
&= \lim_{m\longrightarrow\infty} \langle u, \psi\xi f_m\rangle_{D'(\Omega)\times D(\Omega)} = \lim_{m\longrightarrow\infty} u(0) = 0.
\end{aligned} \tag{105}$$

Note that ξf_m is zero outside U and $\psi = 0$ in U, so $\psi\xi f_m = 0$ everywhere

(2) In order to show that $T(u)$ is an element of $[W^{s,p}_{\text{comp}}(\Omega)]^*$, we need to prove that $T(u)$: $W^{s,p}_{\text{comp}}(\Omega) \longrightarrow \mathbb{R}$ is linear and continuous. Linearity is obvious. In order to prove continuity, we need to show that for all $K \in \mathscr{K}(\Omega)$, $T(u)|_{W^{s,p}_K(\Omega)}$ is continuous (see Theorem 37). Let $K \in \mathscr{K}(\Omega)$ and fix a function $\psi \in C^\infty_c(\Omega)$ which is equal to 1 on a neighborhood containing K. For all $f \in W^{s,p}_K(\Omega)$ we have

$$\begin{aligned}
|[T(u)](f)| &= \left|\langle \psi u, f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}\right| \\
&\leq \|\psi u\|_{W^{-s,p'}(\Omega)} \times \|f\|_{W^{s,p}(\Omega)},
\end{aligned} \tag{106}$$

which proves the continuity of the linear map $T(u)$

(3) In order to prove that T is bijective, we give an explicit formula for the inverse. Recall that by definition, $W^{-s,p'}_{loc}(\Omega)$ is a subspace of $D'(\Omega)$ and by Remark 94, $[W^{s,p}_{\text{comp}}(\Omega)]^*$ can also be viewed as a subspace of $D'(\Omega)$. More precisely, if we let $i : D(\Omega) \longrightarrow W^{s,p}_{\text{comp}}(\Omega)$ be the "identity map" and $i^* : [W^{s,p}_{\text{comp}}(\Omega)]^* \longrightarrow D'(\Omega)$ be the adjoint of i, then i^* is a continuous injective linear map and

$$\begin{aligned}
\forall u \in \left[W^{s,p}_{\text{comp}}(\Omega)\right]^* \forall \varphi \in D(\Omega) \quad &\langle i^* u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} \\
&= \langle u, \varphi\rangle_{[W^{s,p}_{\text{comp}}(\Omega)]^*\times W^{s,p}_{\text{comp}}(\Omega)}.
\end{aligned} \tag{107}$$

Moreover, if $K \in \mathscr{K}(\Omega)$, then $W^{s,p}_K(\Omega) \hookrightarrow W^{s,p}_{\text{comp}}(\Omega)$, and therefore, if $u \in [W^{s,p}_{\text{comp}}(\Omega)]^*$, then $u|_{W^{s,p}_K(\Omega)} \in [W^{s,p}_K(\Omega)]^*$ and

$$\forall g \in W_K^{s,p}(\Omega) \quad \langle u, g\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)} = \left\langle u\Big|_{W_K^{s,p}(\Omega)}, g\right\rangle_{\left[W_K^{s,p}(\Omega)\right]^* \times W_K^{s,p}(\Omega)}. \tag{108}$$

Now, we claim that the image of i^* is in $W_{loc}^{-s,p'}(\Omega)$ and in fact i^* is the inverse of T. Let us first prove that the image of i^* is in $W_{loc}^{-s,p'}(\Omega)$. Let $u \in [W_{\text{comp}}^{s,p}(\Omega)]^*$. We need to show that for all $\varphi \in C_c^\infty(\Omega)$, $(\varphi)(i^*u) \in W^{-s,p'}(\Omega)$. To this end, we make use of Corollary 71. Let $\varphi \in C_c^\infty(\Omega)$ and let $K = \text{supp } \varphi$. For all $\psi \in D(\Omega)$, we have

$$\begin{aligned}
\left|\langle \varphi i^* u, \psi\rangle_{D'(\Omega)\times D(\Omega)}\right| &= \left|\langle i^* u, \varphi\psi\rangle_{D'(\Omega)\times D(\Omega)}\right| \\
&= \left|\langle u, \varphi\psi\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)}\right| \\
&= \left|\left\langle u\Big|_{W_K^{s,p}(\Omega)}, \varphi\psi\right\rangle_{\left[W_K^{s,p}(\Omega)\right]^* \times W_K^{s,p}(\Omega)}\right| \\
&\le \left\|u|_{W_K^{s,p}(\Omega)}\right\|_{\left[W_K^{s,p}(\Omega)\right]^*} \|\varphi\psi\|_{W_K^{s,p}(\Omega)} \\
&= \left\|u|_{W_K^{s,p}(\Omega)}\right\|_{\left[W_K^{s,p}(\Omega)\right]^*} \|\varphi\psi\|_{W^{s,p}(\Omega)} \\
&\preccurlyeq \left\|u|_{W_K^{s,p}(\Omega)}\right\|_{\left[W_K^{s,p}(\Omega)\right]^*} \|\psi\|_{W^{s,p}(\Omega)},
\end{aligned} \tag{109}$$

which, by Corollary 71, proves that $\varphi i^* u \in W^{-s,p'}(\Omega)$. Now, we prove i^* is the inverse of T. Note that for all $u \in W_{loc}^{-s,p'}(\Omega) \subseteq D'(\Omega)$ and $\varphi \in D(\Omega)$,

$$\begin{aligned}
\langle (i^* \circ T)(u), \varphi\rangle_{D'(\Omega)\times D(\Omega)} &\overset{\text{Equation (107)}}{=} \langle T(u), \varphi\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)} \\
&\overset{\text{Definition of } T}{=} \left\langle \psi_\varphi u, \varphi\right\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \overset{\text{Remark 69}}{=} \left\langle \psi_\varphi u, \varphi\right\rangle_{D'(\Omega)\times D(\Omega)} \\
&= \left\langle u, \psi_\varphi \varphi\right\rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi\rangle_{D'(\Omega)\times D(\Omega)}.
\end{aligned} \tag{110}$$

Therefore, $i^* \circ T$ is identity. Next, we show that for all $\nu \in [W_{\text{comp}}^{s,p}(\Omega)]^*$, $(T \circ i^*)(\nu) = \nu$. Note that $(T \circ i^*)(\nu)$ and ν both are in $[W_{\text{comp}}^{s,p}(\Omega)]^*$ and so they are continuous functions from $W_{\text{comp}}^{s,p}(\Omega)$ to $\mathbb{R}$. Since $D(\Omega)$ is dense in $W_{\text{comp}}^{s,p}(\Omega)$, according to Lemma 113, it is enough to show that for all $f \in D(\Omega)$, we have $[(T \circ i^*)(\nu)](f) = \nu(f)$.

$$\begin{aligned}
&\langle (T \circ i^*)(\nu), f\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)} \\
&\overset{\text{Definition of } T}{=} \left\langle \psi_f (i^*\nu), f\right\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \\
&\overset{\text{Remark 69}}{=} \left\langle \psi_f (i^*\nu), f\right\rangle_{D'(\Omega)\times D(\Omega)} \\
&= \left\langle i^*\nu, \psi_f f\right\rangle_{D'(\Omega)\times D(\Omega)} = \langle i^*\nu, f\rangle_{D'(\Omega)\times D(\Omega)} \\
&\overset{\text{Equation (107)}}{=} \langle \nu, f\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)}
\end{aligned} \tag{111}$$

(4) Let us denote the topology of $W_{loc}^{-s,p'}(\Omega)$ by τ and the strong topology on $[W_{\text{comp}}^{s,p}(\Omega)]^*$ by τ'. Our goal is to show that $T : (W_{loc}^{-s,p'}(\Omega), \tau) \longrightarrow ([W_{\text{comp}}^{s,p}(\Omega)]^*, \tau')$ and $T^{-1} = i^* : ([W_{\text{comp}}^{s,p}(\Omega)]^*, \tau') \longrightarrow (W_{loc}^{-s,p'}(\Omega), \tau)$ are both continuous maps. To this end, we make use of Theorem 20. Recall that τ is induced by the family of seminorms $\{p_\varphi : W_{loc}^{-s,p'}(\Omega) \longrightarrow \mathbb{R}\}_{\varphi \in C_c^\infty(\Omega)}$ where $p_\varphi(u) = \|\varphi u\|_{W^{-s,p'}(\Omega)}$. Also τ' is induced by the family of seminorms $\{p'_B : [W_{\text{comp}}^{s,p}(\Omega)]^* \longrightarrow \mathbb{R}\}$ where B varies over all bounded sets in $W_{\text{comp}}^{s,p}(\Omega)$ and $p'_B(u) = \sup_{f \in B}|u(f)|$

Step 1. Let B be a bounded subset of $W_{\text{comp}}^{s,p}(\Omega)$. Since B is bounded, there exists $K \in \mathcal{K}(\Omega)$ such that B is bounded in $W_K^{s,p}(\Omega)$ (see Theorem 39; note that the topology of $W_{\text{comp}}^{s,p}(\Omega)$ can be constructed as the inductive limit of $W_{K_j}^{s,p}(\Omega)$ where $\{K_j\}$ is an increasing chain of compact subsets of Ω). So there exists a constant C such that for all $f \in B$, $\|f\|_{W^{s,p}(\Omega)} \le C$. Let ψ be a function in $C_c^\infty(\Omega)$ which is equal to 1 on a neighborhood containing K. For all $u \in W_{loc}^{-s,p'}(\Omega)$, we have

$$\begin{aligned}
\left(p'_B \circ T\right)(u) &= \sup_{f \in B}|[T(u)](f)| \\
&\overset{\text{Definition of } T}{=} \sup_{f \in B}\left|\langle \psi u, f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}\right| \\
&\le \sup_{f \in B}\|\psi u\|_{W^{-s,p'}(\Omega)}\|f\|_{W^{s,p}(\Omega)} \le C p_\psi(u).
\end{aligned} \tag{112}$$

It follows from Theorem 20 that $T : (W_{loc}^{-s,p'}(\Omega), \tau) \longrightarrow ([W_{\text{comp}}^{s,p}(\Omega)]^*, \tau')$ is continuous

Step 2. Let $\varphi \in C_c^\infty(\Omega)$. Let K be a compact set whose interior contains supp φ. Since (s, p, Ω) is a smooth multiplication triple, there exists a constant $C_\varphi > 0$ such that for all $f \in W^{s,p}(\Omega)$, we have $\|\varphi f\|_{W^{s,p}(\Omega)} \le C_\varphi \|f\|_{W^{s,p}(\Omega)}$.

We have

$$\begin{aligned}
\left(p_\varphi \circ i^*\right)(u) &= \|\varphi i^* u\|_{W_0^{s,p}(\Omega)} = \|\varphi i^* u\|_{\left[W_0^{-s,p'}(\Omega)\right]^*} \\
&= \sup_{\xi \in C_c^\infty(\Omega), \|\xi\|_{W^{s,p}(\Omega)} \le 1}\left|\langle \varphi i^* u, \xi\rangle_{W_0^{s,p}(\Omega)\times W_0^{-s,p'}(\Omega)}\right| \\
&= \sup_{\xi \in C_c^\infty(\Omega), \|\xi\|_{W^{s,p}(\Omega)} \le 1}\left|\langle \varphi i^* u, \xi\rangle_{D'(\Omega)\times D(\Omega)}\right| \\
&= \sup_{\xi \in C_c^\infty(\Omega), \|\xi\|_{W^{s,p}(\Omega)} \le 1}\left|\langle i^* u, \varphi\xi\rangle_{D'(\Omega)\times D(\Omega)}\right| \\
&\le \sup_{\eta \in C^\infty_{\text{supp }\varphi}(\Omega), \|\eta\|_{W^{s,p}(\Omega)} \le C_\varphi}\left|\langle i^* u, \eta\rangle_{D'(\Omega)\times D(\Omega)}\right| \\
&\overset{\text{Equation (107)}}{=} \sup_{\eta \in C^\infty_{\text{supp }\varphi}(\Omega), \|\eta\|_{W^{s,p}(\Omega)} \le C_\varphi}\left|\langle u, \eta\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^* \times W_{\text{comp}}^{s,p}(\Omega)}\right|.
\end{aligned} \tag{113}$$

So, if we let B be the ball of radius $2C_\varphi$ centered at 0 in $W_K^{s,p}(\Omega)$ (clearly B is a bounded set in $W_{\text{comp}}^{s,p}(\Omega)$), we get

$$\left(p_\varphi \circ i^*\right)(u) \leq \sup_{f\in B}\left|\langle u, f\rangle_{\left[W_{\text{comp}}^{s,p}(\Omega)\right]^*\times W_{\text{comp}}^{s,p}(\Omega)}\right| = p_B'(u) \tag{114}$$

□

Corollary 115. *Suppose that (s,p,Ω) and $(-s,p',\Omega)$ are both smooth multiplication triples. By the previous theorem, $[W_{comp}^{-s,p'}(\Omega)]^*$ can be identified with $W_{loc}^{s,p}(\Omega)$. Also, by Remark 94, $[W_{comp}^{-s,p'}(\Omega)]^*$ is continuously embedded in $D'(\Omega)$. Therefore, $W_{loc}^{s,p}(\Omega)$ is continuously embedded in $D'(\Omega)$. Since $W_{loc}^{s,p}(\Omega)$ is a Frechet space, it follows from Theorem 19 and Remark 22 that the preceding statement remains true even if we consider $D'(\Omega)$ equipped with the weak* topology. So,*

$$\begin{aligned} W_{loc}^{s,p}(\Omega) &\hookrightarrow \left(D'(\Omega), \textit{strong topology}\right) \textit{ and } W_{loc}^{s,p}(\Omega) \\ &\hookrightarrow \left(D'(\Omega), \textit{weak}^* \textit{ topology}\right). \end{aligned} \tag{115}$$

Theorem 116. *Suppose that (s,p,Ω) and $(-s,p',\Omega)$ are smooth multiplication triples. Define the mapping $R: W_{comp}^{-s,p'}(\Omega) \longrightarrow [W_{loc}^{s,p}(\Omega)]^*$ by*

$$\forall u \in W_{comp}^{-s,p'}(\Omega)\ \forall f \in W_{loc}^{s,p}(\Omega) \quad [R(u)](f) \coloneqq \langle u, \psi_u f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}, \tag{116}$$

where ψ_u is any function in $C_c^\infty(\Omega)$ that is equal to 1 on a neighborhood containing the support of u. Then,

(1) $[R(u)](f)$ does not depend on the choice of ψ_u

(2) For all $u \in W_{comp}^{-s,p'}(\Omega)$, $R(u)$ is indeed an element of $[W_{loc}^{s,p}(\Omega)]^*$

(3) $R: W_{comp}^{-s,p'}(\Omega) \longrightarrow [W_{loc}^{s,p}(\Omega)]^*$ is bijective

(4) Suppose $[W_{loc}^{s,p}(\Omega)]^*$ is equipped with the strong topology. Then, the linear map R is bijective and continuous. In particular, $[W_{loc}^{s,p}(\Omega)]^*$ and $W_{comp}^{-s,p'}(\Omega)$ are isomorphic vector spaces

Proof.

(1) Note that since (s,p,Ω) is a smooth multiplication triple, $\psi_u f$ is in $W_0^{s,p}(\Omega)$. Also by assumption, $(-s, p', \Omega)$ is a smooth multiplication triple. Therefore, for each $K \in \mathcal{K}(\Omega)$, $W_K^{-s,p'}(\Omega) \hookrightarrow W_0^{-s,p'}(\Omega)$, and hence, $W_{\text{comp}}^{-s,p'}(\Omega) \hookrightarrow W_0^{-s,p'}(\Omega)$. So the pairing in the definition of $[R(u)](f)$ makes sense. The fact that the output is independent of the choice of ψ_u follows directly from Theorem 58

(2) Clearly, $R(u)$ is linear. Also $R(u)$ is continuous (so it is an element of $[W_{loc}^{s,p}(\Omega)]^*$). The reason is as follows: for all $f \in W_{loc}^{s,p}(\Omega)$, we have

$$\begin{aligned} |[R(u)](f)| &= \left|\langle u, \psi_u f\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}\right| \\ &\leq \|u\|_{W_0^{-s,p'}(\Omega)} \|\psi_u f\|_{W_0^{s,p}(\Omega)}. \end{aligned} \tag{117}$$

That is, for all $f \in W_{loc}^{s,p}(\Omega)$, we have $|[R(u)](f)| \lesssim \|\psi_u f\|_{W^{s,p}(\Omega)}$. It follows from Theorem 20 that $R(u) : W_{loc}^{s,p}(\Omega) \longrightarrow \mathbb{R}$ is continuous

(3) In order to prove that R is bijective we give an explicit formula for the inverse. Recall that by Remark 111, $[W_{loc}^{s,p}(\Omega)]^*$ can also be viewed as a subspace of $D'(\Omega)$. More precisely, if we let $i: D(\Omega) \longrightarrow W_{loc}^{s,p}(\Omega)$ be the "identity map" and $i^* : [W_{loc}^{s,p}(\Omega)]^* \longrightarrow D'(\Omega)$ be the adjoint of i, then i^* is a continuous injective linear map and

$$\begin{aligned} \forall u \in \left[W_{loc}^{s,p}(\Omega)\right]^* \forall \varphi \in D(\Omega) \quad & \langle i^* u, \varphi\rangle_{D'(\Omega)\times D(\Omega)} \\ &= \langle u, \varphi\rangle_{\left[W_{loc}^{s,p}(\Omega)\right]^*\times W_{loc}^{s,p}(\Omega)}. \end{aligned} \tag{118}$$

Now, we claim that the image of i^* is in $W_{\text{comp}}^{-s,p'}(\Omega)$ and in fact i^* is the inverse of R. Let us first prove that the image of i^* is in $W_{\text{comp}}^{-s,p'}(\Omega)$. $\mathscr{E}(\Omega)$ is continuously and densely embedded in $W_{loc}^{s,p}(\Omega)$ (continuity is proved in Theorem 108 and density is a direct consequence of the density of $C_c^\infty(\Omega)$ in $W_{loc}^{s,p}(\Omega)$). Therefore, $i^*([W_{loc}^{s,p}(\Omega)]^*)$ is indeed a subspace of $\mathscr{E}'(\Omega) \subseteq D'(\Omega)$ and so elements of $i^*([W_{loc}^{s,p}(\Omega)]^*)$ can be identified with distributions in $D'(\Omega)$ that have compact support. It remains to show that if $u \in [W_{loc}^{s,p}(\Omega)]^*$, then $i^* u \in W^{-s,p'}(\Omega)$. To this end, we make use of Corollary 71. For all $\varphi \in D(\Omega)$, we have

$$\begin{aligned} \left|\langle i^* u, \varphi\rangle_{D'(\Omega)\times D(\Omega)}\right| &\overset{\text{Equation (118)}}{=} \left|\langle u, \varphi\rangle_{\left[W_{loc}^{s,p}(\Omega)\right]^*\times W_{loc}^{s,p}(\Omega)}\right| \\ &= \left|\left\langle u|_{W^{s,p}(\Omega)}, \varphi\right\rangle_{\left[W_0^{s,p}(\Omega)\right]^*\times W_0^{s,p}(\Omega)}\right| \\ &\leq \left\|u|_{W_0^{s,p}(\Omega)}\right\|_{\left[W_0^{s,p}(\Omega)\right]^*} \|\varphi\|_{W^{s,p}(\Omega)}. \end{aligned} \tag{119}$$

So, by Corollary 71, we can conclude that $u \in$

$W^{-s,p'}(\Omega)$. In the above, we used the fact that $W_0^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega) \hookrightarrow W_{loc}^{s,p}(\Omega)$, and so for $u \in [W_{loc}^{s,p}(\Omega)]^*$, we have $u|_{W_0^{s,p}(\Omega)} \in [W_0^{s,p}(\Omega)]^*$.

Now, we prove that $i^* : [W_{loc}^{s,p}(\Omega)]^* \longrightarrow W_{\text{comp}}^{-s,p'}(\Omega)$ is the inverse of R. For all $u \in W_{\text{comp}}^{-s,p'}(\Omega)$ and $\varphi \in D(\Omega)$, we have

$$\begin{aligned}
&\langle (i^* \circ R)(u), \varphi \rangle_{D'(\Omega)\times D(\Omega)} \\
&\overset{\text{Equation (118)}}{=} \langle Ru, \varphi \rangle_{[W_{loc}^{s,p}(\Omega)]^* \times W_{loc}^{s,p}(\Omega)} \\
&\overset{\text{Definition of } R}{=} \langle u, \psi_u \varphi \rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \\
&\overset{\text{Remark 69}}{=} \langle u, \psi_u \varphi \rangle_{D'(\Omega)\times D(\Omega)} \\
&= \langle \psi_u u, \varphi \rangle_{D'(\Omega)\times D(\Omega)} = \langle u, \varphi \rangle_{D'(\Omega)\times D(\Omega)}.
\end{aligned} \tag{120}$$

Therefore, $(i^* \circ R)(u) = u$ for all $u \in W_{\text{comp}}^{-s,p'}(\Omega)$.

Now, we prove that $R \circ i^*$ is also the identity map. Considering Lemma 113, since $D(\Omega)$ is dense in $[W_{loc}^{s,p}(\Omega)]^*$, it is enough to show that for all $v \in [W_{loc}^{s,p}(\Omega)]^*$ and $f \in D(\Omega)$, $[R \circ i^*(v)](f) = v(f)$. We have

$$\begin{aligned}
&\langle (R \circ i^*)v, f \rangle_{[W_{loc}^{s,p}(\Omega)]^* \times W_{loc}^{s,p}(\Omega)} \overset{\text{Definition of } R}{=} \langle i^* v, \psi_{i^* v} f \rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)} \\
&\overset{\text{Remark 69}}{=} \langle i^* v, \psi_{i^* v} f \rangle_{D'(\Omega)\times D(\Omega)} = \langle \psi_{i^* v} i^* v, f \rangle_{D'(\Omega)\times D(\Omega)} \\
&= \langle i^* v, f \rangle_{D'(\Omega)\times D(\Omega)} \overset{\text{Equation (118)}}{=} \langle v, f \rangle_{[W_{loc}^{s,p}(\Omega)]^* \times W_{loc}^{s,p}(\Omega)},
\end{aligned} \tag{121}$$

which shows $R \circ i^*(v) = v$

(4) Let us denote the topology of $W_{\text{comp}}^{-s,p'}(\Omega)$ by τ and the strong topology on $[W_{loc}^{s,p}(\Omega)]^*$ by τ'. Our goal is to show that $R : (W_{\text{comp}}^{-s,p'}(\Omega), \tau) \longrightarrow ([W_{loc}^{s,p}(\Omega)]^*, \tau')$ is continuous. To this end, we make use of Theorem 20. Recall that τ is induced by the family of seminorms $\{q_{a,-s,p'} : W_{\text{comp}}^{-s,p'}(\Omega) \longrightarrow \mathbb{R}\}_{a\in S}$ where $q_{a,-s,p'}(u) = \sum_j a_j \|\psi_j u\|_{W^{-s,p'}(\Omega)}$ (here, we are using the notation introduced in Theorem 97). Also τ' is induced by the family of seminorms $\{p_B : [W_{loc}^{s,p}(\Omega)]^* \longrightarrow \mathbb{R}\}$ where B varies over all bounded sets in $W_{loc}^{s,p}(\Omega)$ and $p_B(u) = \sup_{f\in B}|u(f)|$.

Let B be a bounded subset of $W_{loc}^{s,p}(\Omega)$. Since B is bounded, for all $\varphi \in C_c^\infty(\Omega)$, the set $\{\|\varphi f\|_{W^{s,p}(\Omega)} : f \in B\}$ is bounded in $\mathbb{R}$ (see Theorem 16). Thus, for all $\varphi \in C_c^\infty(\Omega)$, there exists a positive integer a_φ such that for all $f \in B$, $\|\varphi f\|_{W^{s,p}(\Omega)} < a_\varphi$. Recall that $\{\psi_j\}$ in the definition of $q_{a,-s,p'}$ denotes a fixed partition of unity. For each j, let φ_j be a function in $C_c^\infty(\Omega)$ which is equal to 1 on a neighborhood containing the support of ψ_j. For all $u \in W_{\text{comp}}^{-s,p'}(\Omega)$, we have

$$\begin{aligned}
(p_B \circ R)(u) &= (p_B \circ R)\left(\sum_j \psi_j u\right) \le \sum_j (p_B \circ R)\left(\psi_j u\right) \\
&= \sup_{f\in B}\left|\left(R\left(\psi_j u\right)\right)(f)\right| \\
&\overset{\text{Definition of } R}{=} \sum_j \sup_{f\in B}\left|\left\langle \psi_j u, \varphi_j f \right\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}\right| \\
&\le \sum_j \sup_{f\in B}\left\|\psi_j u\right\|_{W^{-s,p'}(\Omega)}\left\|\varphi_j f\right\|_{W^{s,p}(\Omega)} \\
&\le \sum_j a_{\varphi_j}\left\|\psi_j u\right\|_{W^{-s,p'}(\Omega)} = q_{a,-s,p'}(u),
\end{aligned} \tag{122}$$

where $a = (a_{\varphi_1}, a_{\varphi_2}, \cdots)$. Note that the inequality $(p_B \circ R)(\sum_j \psi_j u) \le \sum_j (p_B \circ R)(\psi_j u)$ holds because u has compact support and so only finitely many terms in the sum are nonzero, so we can use the subadditivity of the seminorm and linearity of R.

It follows from Theorem 20 that $R : (W_{\text{comp}}^{-s,p'}(\Omega), \tau) \longrightarrow [W_{loc}^{s,p}(\Omega)]^*$ is continuous

□

Remark 117. According to the previous two theorems, we have the following:

(i) When $u \in W_{loc}^{-s,p'}(\Omega)$ is viewed as an element of $[W_{\text{comp}}^{s,p}(\Omega)]^*$, we have

$$\forall f \in W_{\text{comp}}^{s,p}(\Omega) \quad u(f) = \left\langle \psi_f u, f \right\rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}, \tag{123}$$

where ψ_f is any function in $C_c^\infty(\Omega)$ that is equal to 1 on a neighborhood containing supp f

(ii) When $u \in W_{\text{comp}}^{-s,p'}(\Omega)$ is viewed as an element of $[W_{loc}^{s,p}(\Omega)]^*$, we have

$$\forall f \in W_{loc}^{s,p}(\Omega) \quad u(f) = \langle u, \psi f \rangle_{W_0^{-s,p'}(\Omega)\times W_0^{s,p}(\Omega)}, \tag{124}$$

where ψ is any function in $C_c^\infty(\Omega)$ that is equal to 1 on a neighborhood containing supp u

Corollary 118. *Suppose that (s, p, Ω) and $(-s, p', \Omega)$ are both smooth multiplication triples. As a direct consequence of the previous theorems, the bidual of $W_{comp}^{s,p}(\Omega)$ is itself. So $W_{comp}^{s,p}(\Omega)$ is semireflexive. It follows from Theorem 25*

that $W^{s,p}_{comp}(\Omega)$ is reflexive and subsequently its dual $W^{-s,p'}_{loc}(\Omega)$ is reflexive.

Now, we put everything together to build general embedding theorems for spaces of locally Sobolev-Slobodeckij functions.

Theorem 119 (embedding theorem I). *Let Ω be a nonempty open set in $\mathbb{R}^n$. If $s_1, s_2 \in \mathbb{R}$ and $1 < p_1, p_2 < \infty$ are such that $W^{s_1,p_1}(\Omega) \hookrightarrow W^{s_2,p_2}(\Omega)$, then $W^{s_1,p_1}_{loc}(\Omega) \hookrightarrow W^{s_2,p_2}_{loc}(\Omega)$.*

Proof. We have

$$\begin{aligned} u \in W^{s_1,p_1}_{loc}(\Omega) &\Longleftrightarrow \forall \varphi \in C^\infty_c(\Omega), \quad \varphi u \in W^{s_1,p_1}(\Omega), \\ &\Rightarrow \forall \varphi \in C^\infty_c(\Omega), \quad \varphi u \in W^{s_2,p_2}(\Omega), \\ &\Longleftrightarrow u \in W^{s_2,p_2}_{loc}(\Omega). \end{aligned} \tag{125}$$

So, $W^{s_1,p_1}_{loc}(\Omega) \subseteq W^{s_2,p_2}_{loc}(\Omega)$. Now, note that for all $\varphi \in C^\infty_c(\Omega)$,

$$|u|_{\varphi,s_2,p_2} = \|\varphi u\|_{W^{s_2,p_2}(\Omega)} \preccurlyeq \|\varphi u\|_{W^{s_1,p_1}(\Omega)} = |u|_{\varphi,s_1,p_1}. \tag{126}$$

So, it follows from Theorem 20 that the inclusion map from $W^{s_1,p_1}_{loc}(\Omega)$ to $W^{s_2,p_2}_{loc}(\Omega)$ is continuous. □

Theorem 120 (embedding theorem II). *Let Ω be a nonempty open set in $\mathbb{R}^n$ that has the interior Lipschitz property. Suppose that $s_1, s_2 \in \mathbb{R}$ and $1 < p_1, p_2 < \infty$ are such that $W^{s_1,p_1}(U) \hookrightarrow W^{s_2,p_2}(U)$ for all bounded open sets U with Lipschitz continuous boundary. If $s_1 < 0$, further assume that $(-s_1, p_1', \Omega)$ is a smooth multiplication triple. If $s_2 < 0$, further assume that $(-s_2, p_2', \Omega)$ is a smooth multiplication triple. Then, $W^{s_1,p_1}_{loc}(\Omega) \hookrightarrow W^{s_2,p_2}_{loc}(\Omega)$.*

Proof. Suppose $u \in W^{s_1,p_1}_{loc}(\Omega)$ and $\varphi \in C^\infty_c(\Omega)$. Let Ω' be an open set in Ω that contains supp φ and has Lipschitz continuous boundary. We have

$$\begin{aligned} u \in W^{s_1,p_1}_{loc}(\Omega) &\Rightarrow \varphi u \in W^{s_1,p_1}(\Omega) \\ &\overset{\text{Theorem 85}}{\Rightarrow} (\varphi u)\big|_{\Omega'} \in W^{s_1,p_1}\left(\Omega'\right) \Rightarrow (\varphi u)\big|_{\Omega'} \\ &\cdot \in W^{s_2,p_2}\left(\Omega'\right) \overset{\text{Theorem 85}}{\Rightarrow} \varphi u \in W^{s_2,p_2}(\Omega). \end{aligned} \tag{127}$$

Since φ can be any element of $C^\infty_c(\Omega)$, we can conclude that if $u \in W^{s_1,p_1}_{loc}(\Omega)$, then $u \in W^{s_2,p_2}_{loc}(\Omega)$. In order to prove the continuity of the inclusion map, we can proceed as follows: let $\varphi \in C^\infty_c(\Omega)$ and choose Ω' as before.

$$\begin{aligned} |u|_{\varphi,s_2,p_2} = \|\varphi u\|_{W^{s_2,p_2}(\Omega)} &\overset{\text{Theorem 85}}{\simeq} \|\varphi u\|_{W^{s_2,p_2}(\Omega')} \\ &\preccurlyeq \|\varphi u\|_{W^{s_1,p_1}(\Omega')} \overset{\text{Theorem 85}}{\simeq} \|\varphi u\|_{W^{s_1,p_1}(\Omega)} = |u|_{\varphi,s_1,p_1}. \end{aligned} \tag{128}$$

So, it follows from Theorem 20 that the inclusion map from $W^{s_1,p_1}_{loc}(\Omega)$ to $W^{s_2,p_2}_{loc}(\Omega)$ is continuous. □

A version of compact embedding for spaces H^m_{loc} with integer smoothness degree has been studied in [17]. In what follows, we will state the corresponding theorem and its proof for spaces of locally Sobolev-Slobodeckij functions.

Lemma 121. *Suppose that (s, p, Ω) and $(-s, p', \Omega)$ are smooth multiplication triples. If u_m converges weakly to u in $W^{s,p}_{loc}(\Omega)$, then*

$$\forall \varphi \in C^\infty_c(\Omega) \quad \varphi u_m \rightharpoonup \varphi u \quad \text{in } W^{s,p}(\Omega). \tag{129}$$

Proof. The proof is based on the following well-known fact:

Fact 1. Let X be a topological space and suppose that x is a point in X. Let $\{x_m\}$ be a sequence in X. If every subsequence of $\{x_m\}$ contains a subsequence that converges to x, then $x_m \longrightarrow x$.

Let $\varphi \in C^\infty_c(\Omega)$. By Fact 1, it is enough to show that every subsequence of φu_m has a subsequence that converges weakly to φu in $W^{s,p}(\Omega)$. Let $\varphi u_{m'}$ be a subsequence of φu_m. We have

$$\begin{aligned} u_{m'} \rightharpoonup u \quad &\text{in } W^{s,p}_{loc}(\Omega) \overset{\text{Corollary 29}}{\Rightarrow} \{u_{m'}\} \text{ is bounded} \\ &\text{in } W^{s,p}_{loc}(\Omega) \overset{\text{Corollary 17}}{\Rightarrow} \{\varphi u_{m'}\} \text{ is bounded} \\ &\text{in } W^{s,p}(\Omega). \end{aligned} \tag{130}$$

Since $W^{s,p}(\Omega)$ is reflexive, there exists a subsequence $\varphi u_{m''}$ that converges weakly to some $F \in W^{s,p}(\Omega)$. To finish the proof, it is enough to show that $F = \varphi u$. We have

$$\begin{aligned} u_{m''} \rightharpoonup u \quad &\text{in } W^{s,p}_{loc}(\Omega) \Rightarrow u_{m''} \longrightarrow u \quad \text{in } \left(D'(\Omega), \text{weak}^*\right) \\ &\Rightarrow \varphi u_{m''} \longrightarrow \varphi u \quad \text{in } \left(D'(\Omega), \text{weak}^*\right). \end{aligned} \tag{131}$$

In the first line, we used Theorem 30 and the fact that $W^{s,p}_{loc}(\Omega) \hookrightarrow (D'(\Omega), \text{weak}^*)$ (see Corollary 115). In the second line, we used the fact that multiplication by smooth functions is a continuous operator on $(D'(\Omega), \text{weak}^*)$.

Similarly, since $W^{s,p}(\Omega) \hookrightarrow (D'(\Omega), \text{weak}^*)$, it follows from Theorem 30 that

$$\varphi u_{m''} \rightharpoonup F \text{ in } W^{s,p}(\Omega) \Rightarrow \varphi u_{m''} \longrightarrow F \text{ in } \left(D'(\Omega), \text{weak}^*\right). \tag{132}$$

Consequently, $\varphi u = F$ as elements of $D'(\Omega)$ and subsequently as elements of $W_0^{s,p}(\Omega)$. □

Theorem 122 (compact embedding). *Let Ω be a nonempty open set in $\mathbb{R}^n$ that has the interior Lipschitz property. Suppose that (s_1, p_1, Ω) and $(-s_1, p_1', \Omega)$ are smooth multiplication triples. If $s_2 < 0$, further assume that $(-s_2, p_2', \Omega)$ is a smooth multiplication triple. Moreover, suppose that s_1, s_2, p_1, and p_2 are such that $W^{s_1,p_1}(U)$ is compactly embedded in $W^{s_2,p_2}(U)$ for all bounded open sets U with Lipschitz continuous boundary. Then, every bounded sequence in $W_{loc}^{s_1,p_1}(\Omega)$ has a convergent subsequence in $W_{loc}^{s_2,p_2}(\Omega)$.*

Proof. The proof makes use of the following well-known fact:

Fact 2. Let X and Y be Banach spaces. Suppose that $T : X \longrightarrow Y$ is a linear compact operator. If the sequence x_m converges weakly (i.e., with respect to the weak topology) to x in X, then $T(x_n)$ converges to $T(x)$ (with respect to the norm of Y) in Y.

Let u_m be a bounded sequence in $W_{loc}^{s_1,p_1}(\Omega)$. By Theorem 33, since $W_{loc}^{s_1,p_1}(\Omega)$ is a separable reflexive Frechet space, there exists $u \in W_{loc}^{s_1,p_1}(\Omega)$ and a subsequence $\{u_{m'}\}$ such that $u_{m'} \rightharpoonup u$ in $W_{loc}^{s_1,p_1}(\Omega)$. We claim that $\{u_{m'}\}$ converges to u in $W_{loc}^{s_2,p_2}(\Omega)$, that is, for all $\varphi \in C_c^\infty(\Omega)$, $\varphi u_{m'} \longrightarrow \varphi u$ in $W_{loc}^{s_2,p_2}(\Omega)$. Suppose that $\varphi \in C_c^\infty(\Omega)$ and let $K \coloneqq \operatorname{supp} \varphi$. By Lemma 121, we have

$$\varphi u_{m'} \rightharpoonup \varphi u \text{ in } W^{s_1,p_1}(\Omega). \tag{133}$$

So, by Theorem 35,

$$\varphi u_{m'} \rightharpoonup \varphi u \text{ in } W_K^{s_1,p_1}(\Omega). \tag{134}$$

Let Ω' be an open bounded set in Ω with Lipschitz continuous boundary such that $K \subseteq \Omega'$. By Theorem 85, the restriction map from $W_K^{s_1,p_1}(\Omega)$ to $W^{s_1,p_1}(\Omega')$ is well-defined and continuous. It follows from Theorem 34 that this restriction map is weak-weak continuous. So $\varphi u_{m'} \rightharpoonup \varphi u$ in $W_K^{s_1,p_1}(\Omega)$ implies that $\varphi u_{m'} \rightharpoonup \varphi u$ in $W^{s_1,p_1}(\Omega')$. By assumption the identity map from $W^{s_1,p_1}(\Omega')$ to $W^{s_2,p_2}(\Omega')$ is compact, so it follows from Fact 2 that $\varphi u_{m'} \longrightarrow \varphi u$ in $W^{s_2,p_2}(\Omega')$ which subsequently implies $\varphi u_{m'} \longrightarrow \varphi u$ in $W^{s_2,p_2}(\Omega)$ by Theorem 85. □

6. Other Properties

The main results of this section do not appear to be in the literature in the generality appearing here, and they play a fundamental role in the study of the properties of differential operators between Sobolev spaces of sections of vector bundles on manifolds equipped with nonsmooth metric (see [15, 16]).

Theorem 123. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $s \geq 1$, and $1 < p < \infty$. Then, $u \in W_{loc}^{s,p}(\Omega)$ if and only if $u \in L_{loc}^p(\Omega)$ and for all $1 \leq i \leq n$, $\partial u/\partial x^i \in W_{loc}^{s-1,p}(\Omega)$.*

Proof.

$$u \in W_{loc}^{s,p}(\Omega) \Longleftrightarrow \forall \varphi \in C_c^\infty(\Omega) \quad \varphi u \in W^{s,p}(\Omega)$$

$$\overset{\text{Theorem 62}}{\Longleftrightarrow} \forall \varphi \in C_c^\infty(\Omega) \quad \varphi u \in L^p(\Omega) \text{ and for all } 1 \leq i \leq n, \frac{\partial(\varphi u)}{\partial x^i} \in W^{s-1,p}(\Omega). \tag{135}$$

Note that $\partial(\varphi u)/\partial x^i = (\partial\varphi/\partial x^i)u + \varphi(\partial u/\partial x^i)$. Since $(\partial\varphi/\partial x^i)u \in W^{s,p}(\Omega) \hookrightarrow W^{s-1,p}(\Omega)$, we have

$$\frac{\partial(\varphi u)}{\partial x^i} \in W^{s-1,p}(\Omega) \Longleftrightarrow \varphi \frac{\partial u}{\partial x^i} \in W^{s-1,p}(\Omega). \tag{136}$$

Therefore,

$$u \in W_{loc}^{s,p}(\Omega) \Longleftrightarrow \forall \varphi \in C_c^\infty(\Omega)\, \varphi u \in L^p(\Omega) \text{ and for all } 1 \leq i \leq n, \varphi \frac{\partial u}{\partial x^i} \in W^{s-1,p}(\Omega)$$

$$\Longleftrightarrow u \in L_{loc}^p(\Omega) \text{ and for all } 1 \leq i \leq n, \frac{\partial u}{\partial x^i} \in W_{loc}^{s-1,p}(\Omega). \tag{137}$$

□

Theorem 124. *Let Ω be a nonempty open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 < p < \infty$. Then, $u \in W_{loc}^{k,p}(\Omega)$ if and only if $\partial^\alpha u \in L_{loc}^p(\Omega)$ for all $|\alpha| \leq k$.*

Proof. We prove the claim by induction on k. For $k = 1$, we have

$$u \in W_{loc}^{1,p}(\Omega) \overset{\text{Theorem 123}}{\Longleftrightarrow} u \in L_{loc}^p(\Omega), \forall 1 \leq i \leq n \quad \frac{\partial u}{\partial x^i} \in L_{loc}^p(\Omega)$$

$$\Longleftrightarrow \forall |\alpha| \leq 1 \quad \partial^\alpha u \in L_{loc}^p(\Omega). \tag{138}$$

Now, suppose the claim holds for $k = m$. For $k = m + 1$, we have

$$u \in W_{loc}^{m+1,p}(\Omega) \overset{\text{Theorem 123}}{\Longleftrightarrow} u \in L_{loc}^p(\Omega), \forall 1 \leq i \leq n \ \frac{\partial u}{\partial x^i} \in W_{loc}^{m,p}(\Omega)$$

$$\overset{\text{induction hypothesis}}{\Longleftrightarrow} u \in L_{loc}^p(\Omega), \forall 1 \leq i \leq n \quad \forall 0 \leq |\alpha| \leq m\, \partial^\alpha \left[\frac{\partial u}{\partial x^i}\right] \in L_{loc}^p(\Omega)$$

$$\Longleftrightarrow u \in L_{loc}^p(\Omega), \forall 0 < |\beta| \leq m + 1\, \partial^\beta u \in L_{loc}^p(\Omega)$$

$$\Longleftrightarrow \forall |\beta| \leq m + 1\, \partial^\beta u \in L_{loc}^p(\Omega). \tag{139}$$

□

Theorem 125. *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\alpha \in \mathbb{N}_0^n$. Suppose Ω is a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. Then,*

(1) *the linear operator* $\partial^\alpha : W^{s,p}_{loc}(\mathbb{R}^n) \longrightarrow W^{s-|\alpha|,p}_{loc}(\mathbb{R}^n)$ *is well-defined and continuous*

(2) *for* $s < 0$, *the linear operator* $\partial^\alpha : W^{s,p}_{loc}(\Omega) \longrightarrow W^{s-|\alpha|,p}_{loc}(\Omega)$ *is well-defined and continuous*

(3) *for* $s \geq 0$ *and* $|\alpha| \leq s$, *the linear operator* $\partial^\alpha : W^{s,p}_{loc}(\Omega) \longrightarrow W^{s-|\alpha|,p}_{loc}(\Omega)$ *is well-defined and continuous*

(4) *if* $s \geq 0$, $s - 1/p \neq$ *integer (i.e., the fractional part of* s *is not equal to* $1/p$*), then the linear operator* $\partial^\alpha : W^{s,p}_{loc}(\Omega) \longrightarrow W^{s-|\alpha|,p}_{loc}(\Omega)$ *for* $|\alpha| > s$ *is well-defined and continuous*

Proof. This is the counterpart of Theorem 86 for locally Sobolev functions. Here, we will prove the first item. The remaining items can be proved using a similar technique.

Step 1. First we prove by induction on $|\alpha|$ that if $u \in W^{s,p}_{loc}(\mathbb{R}^n)$, then $\partial^\alpha u \in W^{s-|\alpha|,p}_{loc}(\mathbb{R}^n)$. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$; we need to show that $\varphi\partial^\alpha u \in W^{s-|\alpha|,p}(\mathbb{R}^n)$. If $|\alpha| = 0$, there is nothing to prove. If $|\alpha| = 1$, there exists $1 \leq i \leq n$ such that $\partial^\alpha = \partial/\partial x^i$. We have

$$\varphi\partial^\alpha u = \varphi\frac{\partial u}{\partial x^i} = \frac{\partial(\varphi u)}{\partial x^i} - \frac{\partial\varphi}{\partial x^i}u. \tag{140}$$

By assumption, $\varphi u \in W^{s,p}(\mathbb{R}^n)$, and so it follows from Theorem 86 that the first term on the right hand side is in $W^{s-1,p}(\mathbb{R}^n)$. Also, since $u \in W^{s,p}_{loc}(\mathbb{R}^n)$, the second term on the right hand side is in $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s-1,p}(\mathbb{R}^n)$. Hence, $\varphi\partial^\alpha u \in W^{s-1,p}(\mathbb{R}^n)$. Now, suppose the claim holds for all $|\alpha| \leq k$. Suppose α is a multi-index such that $|\alpha| = k + 1$. Clearly, there exists $1 \leq i \leq n$ such that $\partial^\alpha = (\partial/(\partial x^i))\partial^\beta$ where β is a multi-index with $|\beta| = k$. By the induction hypothesis, $\partial^\beta u \in W^{s-|\beta|,p}_{loc}(\mathbb{R}^n)$ and so by the argument that was presented for the base case, we have $(\partial/(\partial x^i))\partial^\beta u \in W^{s-|\beta|-1,p}_{loc}(\mathbb{R}^n) = W^{s-|\alpha|,p}_{loc}(\mathbb{R}^n)$

Step 2. In this step, we prove the continuity. Again, we use induction on $|\alpha|$. Let $|\alpha| = 1$. Choose i as in the previous step. For every $\varphi \in C^\infty_c(\mathbb{R}^n)$, we have

$$\left\|\varphi\frac{\partial}{\partial x^i}u\right\|_{s-1,p} = \left\|\frac{\partial(\varphi u)}{\partial x^i} - \frac{\partial\varphi}{\partial x^i}u\right\|_{s-1,p} \leq \left\|\frac{\partial(\varphi u)}{\partial x^i}\right\|_{s-1,p} + \left\|\frac{\partial\varphi}{\partial x^i}u\right\|_{s-1,p} \preccurlyeq \|\varphi u\|_{s,p} + \left\|\frac{\partial\varphi}{\partial x^i}u\right\|_{s,p}. \tag{141}$$

On the right hand side, we have sum of two of the seminorms that define the topology of $W^{s,p}_{loc}(\mathbb{R}^n)$. It follows from item (2) of Theorem 20 that $\partial^\alpha : W^{s,p}_{loc}(\mathbb{R}^n) \longrightarrow W^{s-1,p}_{loc}(\mathbb{R}^n)$ is continuous. Now, suppose the claim holds for all $|\alpha| \leq k$. Suppose α is a multi-index such that $|\alpha| = k + 1$. Clearly, there exists $1 \leq i \leq n$ such that $\partial^\alpha = (\partial/(\partial x^i))\partial^\beta$ where β is a multi-index with $|\beta| = k$. We have

$$\begin{aligned}
\|\varphi\partial^\alpha u\|_{s-|\alpha|,p} = \left\|\varphi\frac{\partial}{\partial x^i}\left(\partial^\beta u\right)\right\|_{s-|\alpha|,p} &\overset{\text{argument of the base case}}{\preccurlyeq} \left\|\varphi\partial^\beta u\right\|_{s-|\alpha|+1,p} \\
&+ \left\|\frac{\partial\varphi}{\partial x^i}\partial^\beta u\right\|_{s-|\alpha|+1,p} \preccurlyeq \left\|\varphi\partial^\beta u\right\|_{s-|\beta|,p} + \left\|\frac{\partial\varphi}{\partial x^i}\partial^\beta u\right\|_{s-|\beta|,p} \\
&\overset{\text{induction hypothesis;Theorem 20}}{\preccurlyeq} \max\left(\|\varphi_1 u\|_{s,p},\cdots,\|\varphi_k u\|_{s,p}\right) \\
&+ \left\|\frac{\partial\varphi}{\partial x^i}\partial^\beta u\right\|_{s-|\beta|,p} \overset{\text{induction hypothesis;Theorem 20}}{\preccurlyeq} \max \\
&\cdot \left(\|\varphi_1 u\|_{s,p},\cdots,\|\varphi_k u\|_{s,p}\right) + \max\left(\|\psi_1 u\|_{s,p},\cdots,\|\psi_l u\|_{s,p}\right) \\
&\preccurlyeq \max\left(\|\varphi_1 u\|_{s,p},\cdots,\|\varphi_k u\|_{s,p}, \|\psi_1 u\|_{s,p},\cdots,\|\psi_l u\|_{s,p}\right)
\end{aligned} \tag{142}$$

for some $\varphi_1, \cdots, \varphi_k$ and $\psi_1, \cdots, \psi_l$ in $C^\infty_c(\mathbb{R}^n)$. It follows from item (2) of Theorem 20 that $\partial^\alpha : W^{s,p}_{loc}(\mathbb{R}^n) \longrightarrow W^{s-|\alpha|,p}_{loc}(\mathbb{R}^n)$ is continuous □

Next, we want to establish a counterpart of Theorem 76 for locally Sobolev-Slobodeckij spaces. To this end, first we state and prove a simple lemma.

Lemma 126. *Let* Ω *be a nonempty open subset of* $\mathbb{R}^n$. *Suppose* $u : \Omega \longrightarrow \mathbb{R}$ *and* $\tilde{u} : \Omega \longrightarrow \mathbb{R}$ *are such that* $u = \tilde{u}$ *a.e. If* $\tilde{u}$ *is continuous, then* $\operatorname{supp}\tilde{u} \subseteq \operatorname{supp} u$.

Proof by Contradiction. Suppose $x \in \operatorname{supp}\tilde{u} \setminus \operatorname{supp} u$. Since x belongs to the complement of supp u, which is an open set, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \Omega$ and $B_\varepsilon(x) \cap \operatorname{supp} u = \varnothing$. Since $x \in \operatorname{supp}\tilde{u}$, there exists $y \in B_{\varepsilon/4}(x)$ such that $\tilde{u}(y) \neq 0$. $\tilde{u}$ is continuous, therefore there exists $0 < \delta < \varepsilon/4$ such that $\tilde{u}(z) \neq 0$ for all $z \in B_\delta(y) \subseteq B_\varepsilon(x)$. But $u = 0$ a.e. on $B_\varepsilon(x)$. This contradicts the fact that $u = \tilde{u}$ *a.e.* □

Theorem 127. *Let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *with Lipschitz continuous boundary or* $\Omega = \mathbb{R}^n$. *Suppose* $u \in W^{s,p}_{loc}(\Omega)$ *where* $sp > n$. *Then,* u *has a continuous version.*

Proof. Let $\{V_j\}_{j\in\mathbb{N}_0}$ and $\{\psi_j\}_{j\in\mathbb{N}_0}$ be as in Theorem 43. Note that $u = \sum_j \psi_j u$. For all j, $\psi_j u \in W^{s,p}(\Omega)$ so by Theorem 76, there exists $\tilde{u}_j \in C(\Omega)$ such that $\psi_j u = \tilde{u}_j$ on $\Omega \setminus A_j$ where A_j is a set of measure zero. Also by Lemma 126 $\operatorname{supp}\tilde{u}_j \subseteq \operatorname{supp}\psi_j$. Therefore for any $x \in \Omega$, only finitely many of $\tilde{u}_j(x)$'s are nonzero. So we may define $\tilde{u} : \Omega \longrightarrow \mathbb{R}$ by $\tilde{u} = \sum_j \tilde{u}_j$. Clearly, $\tilde{u} = u$ on $\Omega \setminus A$ where $A = \cup A_j$ (so A is a set of measure zero). Consequently $\tilde{u} = u$ *a.e.* It remains to show that $\tilde{u} : \Omega \longrightarrow \mathbb{R}$ is indeed continuous. To this end, suppose $a_m \longrightarrow a$ in Ω. We need to prove that $\tilde{u}(a_m) \longrightarrow \tilde{u}(a)$. Let $\varepsilon > 0$ be such that $\overline{B_\varepsilon(a)} \subseteq \Omega$. So $B_\varepsilon(a)$ intersects only finitely many of supp $\tilde{u}_j$'s; let us denote them by $\tilde{u}_{r_1}, \cdots, \tilde{u}_{r_l}$. Also since $a_m \longrightarrow a$, there

exists M such that for all $m \geq M$, $a_m \in B_\varepsilon(a)$. Hence,

$$\begin{aligned} \tilde{u}(a) &= \sum_j \tilde{u}_j(a) = \tilde{u}_{r_1}(a) + \cdots + \tilde{u}_{r_l}(a), \\ \forall m \geq M \quad \tilde{u}(a_m) &= \tilde{u}_{r_1}(a_m) + \cdots + \tilde{u}_{r_l}(a_m). \end{aligned} \tag{143}$$

Recall that $\tilde{u}_{r_1} + \cdots + \tilde{u}_{r_l}$ is a finite sum of continuous functions and so it is continuous. Thus,

$$\lim_{m \longrightarrow \infty} \tilde{u}(a_m) = \lim_{m \longrightarrow \infty} \left(\tilde{u}_{r_1} + \cdots + \tilde{u}_{r_l}\right)(a_m) = \tilde{u}_{r_1}(a) + \cdots + \tilde{u}_{r_l}(a) = \tilde{u}(a). \tag{144}$$

□

Remark 128. In the above proof, the only place we used the assumption of Ω being Lipschitz was in applying Theorem 76. We can replace this assumption by the weaker assumption that Ω has the interior Lipschitz property. Then, since supp $(\psi_j u)$ is compact, there exists Ω' with Lipschitz boundary that contains supp $(\psi_j u)$. Then, by Theorem 85, $\psi_j u \in W^{s,p}(\Omega')$ and so it has a continuous version $\widehat{u}_j \in C(\Omega')$. Since $\psi_j u = \widehat{u}_j$ almost everywhere on Ω' and $\psi_j u = 0$ outside of the compact set supp ψ_j, we can conclude that $\text{ext}^0_{\Omega',\Omega}\widehat{u}_j$ is in $C(\Omega)$ and it is almost everywhere equal to $\psi_j u$. We set $\tilde{u}_j = \text{ext}^0_{\Omega',\Omega}\widehat{u}_j$. The rest of the proof will be exactly the same as before.

Theorem 129. *Let $\Omega = \mathbb{R}^n$ or Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. Suppose $s_1, s_2, s \in \mathbb{R}$ and $1 < p_1, p_2, p < \infty$ are such that*

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega). \tag{145}$$

Then,

(1) $W^{s_1,p_1}_{loc}(\Omega) \times W^{s_2,p_2}_{loc}(\Omega) \hookrightarrow W^{s,p}_{loc}(\Omega)$

(2) for all $K \in \mathscr{K}(\Omega)$, $W^{s_1,p_1}_{loc}(\Omega) \times W^{s_2,p_2}_{K}(\Omega) \hookrightarrow W^{s,p}(\Omega)$. *In particular, if* $f \in W^{s_1,p_1}_{loc}(\Omega)$, *then the mapping* $u \mapsto fu$ *is a well-defined continuous linear map from* $W^{s_2,p_2}_K(\Omega)$ *to* $W^{s,p}(\Omega)$

Remark 130. In the above theorem, since the locally Sobolev spaces on Ω are metrizable, the continuity of the mapping

$$W^{s_1,p_1}_{loc}(\Omega) \times W^{s_2,p_2}_{loc}(\Omega) \longrightarrow W^{s,p}_{loc}(\Omega), \quad (u, v) \mapsto uv \tag{146}$$

can be interpreted as follows: if $u_i \longrightarrow u$ in $W^{s_1,p_1}_{loc}(\Omega)$ and $v_i \longrightarrow v$ in $W^{s_2,p_2}_{loc}(\Omega)$, then $u_i v_i \longrightarrow uv$ in $W^{s,p}_{loc}(\Omega)$. Also since $W^{s_2,p_2}_K(\Omega)$ is considered as a normed subspace of $W^{s_2,p_2}(\Omega)$, we have a similar interpretation of the continuity of the mapping in item (2).

Proof.

(1) Suppose $u \in W^{s_1,p_1}_{loc}(\Omega)$ and $v \in W^{s_2,p_2}_{loc}(\Omega)$. First, we show that uv is in $W^{s,p}_{loc}(\Omega)$. Clearly, the set $A = \{\varphi^2 : \varphi \in C^\infty_c(\Omega)\}$ is an admissible family of test functions. So in order to show that $uv \in W^{s,p}_{loc}(\Omega)$, it is enough to show that for all $\varphi \in C^\infty_c(\Omega)$, $\varphi^2 uv = (\varphi u)(\varphi v)$ is in $W^{s,p}(\Omega)$. This is clearly true because $\varphi u \in W^{s_1,p_1}(\Omega)$, $\varphi v \in W^{s_2,p_2}(\Omega)$, and by assumption

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega) \tag{147}$$

In order to prove the continuity of the map $(u, v) \mapsto uv$, suppose $u_i \longrightarrow u$ in $W^{s_1,p_1}_{loc}(\Omega)$ and $v_i \longrightarrow v$ in $W^{s_2,p_2}_{loc}(\Omega)$. We need to show that $u_i v_i \longrightarrow uv$ in $W^{s,p}_{loc}(\Omega)$. That is, we need to prove that for all $\varphi \in C^\infty_c(\Omega)$,

$$\varphi^2 u_i v_i \longrightarrow \varphi^2 uv \quad \text{in } W^{s,p}(\Omega). \tag{148}$$

We have

$$\begin{aligned} &u_i \longrightarrow u \text{ in } W^{s_1,p_1}_{loc}(\Omega) \Rightarrow \varphi u_i \longrightarrow \varphi u \text{ in } W^{s_1,p_1}(\Omega), \\ &v_i \longrightarrow v \text{ in } W^{s_2,p_2}_{loc}(\Omega) \Rightarrow \varphi v_i \longrightarrow \varphi v \text{ in } W^{s_2,p_2}(\Omega). \end{aligned} \tag{149}$$

By assumption, $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, so

$$(\varphi u_i)(\varphi v_i) \longrightarrow (\varphi u)(\varphi v) \quad \text{in } W^{s,p}(\Omega). \tag{150}$$

(2) Suppose $u \in W^{s_1,p_1}_{loc}(\Omega)$ and $v \in W^{s_2,p_2}_{K}(\Omega)$. First, we show that uv is in $W^{s,p}(\Omega)$. To this end, let $\varphi \in C^\infty_c(\Omega)$ be such that $\varphi = 1$ on a neighborhood containing K. We have

$$uv = u(\varphi v) = \underbrace{(\varphi u)}_{\in W^{s_1,p_1}(\Omega)} \underbrace{v}_{\in W^{s_2,p_2}(\Omega)} \in W^{s,p}(\Omega) \tag{151}$$

Now, we prove the continuity. Suppose $u_i \longrightarrow u$ in $W^{s_1,p_1}_{loc}(\Omega)$ and $v_i \longrightarrow v$ in $W^{s_2,p_2}_{K}(\Omega)$. Let φ be as before. We have

$$\begin{aligned} u_i \longrightarrow u \quad \text{in } W^{s_1,p_1}_{loc}(\Omega) \Rightarrow \varphi u_i \longrightarrow \varphi u \quad \text{in } W^{s_1,p_1}(\Omega), \\ v_i \longrightarrow v \quad \text{in } W^{s_2,p_2}(\Omega). \end{aligned} \tag{152}$$

This together with the assumption that $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ implies $\varphi u_i v_i \longrightarrow \varphi uv$ in $W^{s,p}(\Omega)$. Since $\varphi v = v$ and $\varphi v_i = v_i$, we conclude that $u_i v_i \longrightarrow uv$ in $W^{s,p}(\Omega)$.

□

Remark 131. In the above theorem, the assumption that Ω is Lipschitz or $\mathbb{R}^n$ was used only to ensure that we can apply Theorem 103 and to make sure that the locally Sobolev spaces involved are metrizable. For item (1), we can use the weaker assumption that (s_1, p_1, Ω), (s_2, p_2, Ω), and (s, p, Ω) are interior smooth multiplication triples. For item (2), we just need to assume that (s_1, p_1, Ω) is an interior smooth multiplication triple.

Corollary 132. *Let Ω be the same as the previous theorem. If $sp > n$, then $W^{s,p}_{loc}(\Omega)$ is closed under multiplication. Moreover, if*

$$(f_1)_m \longrightarrow f_1 \text{ in } W^{s,p}_{loc}(\Omega), \cdots, (f_l)_m \longrightarrow f_l \text{ in } W^{s,p}_{loc}(\Omega), \quad (153)$$

then

$$(f_1)_m \cdots (f_l)_m \longrightarrow f_1 \cdots f_l \text{ in } W^{s,p}_{loc}(\Omega). \quad (154)$$

The next theorem plays a key role in the study of differential operators on manifolds equipped with nonsmooth metrics (see [15]).

Theorem 133. *Let $\Omega = \mathbb{R}^n$ or let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. Let $s \in \mathbb{R}$ and $p \in (1,\infty)$ be such that $sp > n$. Let $B : \Omega \longrightarrow GL(k, \mathbb{R})$. Suppose for all $x \in \Omega$ and $1 \le i, j \le k$, $B_{ij}(x) \in W^{s,p}_{loc}(\Omega)$. Then,*

(1) $\det B \in W^{s,p}_{loc}(\Omega)$

(2) moreover if for each $m \in \mathbb{N}$ $B_m : \Omega \longrightarrow GL(k, \mathbb{R})$ and for all $1 \le i, j \le k$ $(B_m)_{ij} \longrightarrow B_{ij}$ in $W^{s,p}_{loc}(\Omega)$, then $\det B_m \longrightarrow \det B$ *in* $W^{s,p}_{loc}(\Omega)$

Proof.

(1) By the Leibniz formula, we have

$$\det B(x) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) B_{\sigma(1),1} \cdots B_{\sigma(k),k}. \quad (155)$$

By assumption, for all $1 \le i \le k$, $B_{\sigma(i),i}$ is in $W^{s,p}_{loc}(\Omega)$. Since $sp > n$, it follows from Corollary 132 that $\det B \in W^{s,p}_{loc}(\Omega)$

(2) Since $(B_m)_{ij} \longrightarrow B_{ij}$ in $W^{s,p}_{loc}(\Omega)$, it again follows from Corollary 132 that for all $\sigma \in S_n$,

$$(B_m)_{\sigma(1),1} \cdots (B_m)_{\sigma(k),k} \longrightarrow B_{\sigma(1),1} \cdots B_{\sigma(k),k} \quad \text{in } W^{s,p}_{loc}(\Omega). \quad (156)$$

Thus, $\det B_m \longrightarrow \det B$ in $W^{s,p}_{loc}(\Omega)$

□

Theorem 134. *Let $\Omega = \mathbb{R}^n$ or let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with the Lipschitz continuous boundary. Let $s \ge 1$ and $p \in (1,\infty)$ be such that $sp > n$.*

(1) Suppose that $u \in W^{s,p}_{loc}(\Omega)$ and that $u(x) \in I$ for all $x \in \Omega$ where I is some interval in $\mathbb{R}$. If $F : I \longrightarrow \mathbb{R}$ is a smooth function, then $F(u) \in W^{s,p}_{loc}(\Omega)$

(2) Suppose that $u_m \longrightarrow u$ in $W^{s,p}_{loc}(\Omega)$ and that for all $m \ge 1$ and $x \in \Omega$, $u_m(x), u(x) \in I$ where I is some open interval in $\mathbb{R}$. If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function, then $F(u_m) \longrightarrow F(u)$ in $W^{s,p}_{loc}(\Omega)$

(3) If $F : \longrightarrow \mathbb{R}$ is a smooth function, then the map taking u to $F(u)$ is continuous from $W^{s,p}_{loc}(\Omega)$ to $W^{s,p}_{loc}(\Omega)$

Proof. The proof of part (1) generalizes the argument given in [33]. Let $k = \lfloor s \rfloor$. First, we show that $F(u) \in W^{k,p}_{loc}(\Omega)$. To this end, we fix a multi-index $|\alpha| = m \le k$ and we show that $\partial^\alpha(F(u)) \in L^p_{loc}(\Omega)$ (see Theorem 124).

It follows from the chain rule (and induction) that $\partial^\alpha(F(u))$ is a sum of the terms of the form

$$F^{(l)}(u) \partial^{\beta_1} u \cdots \partial^{\beta_r} u, \quad (157)$$

where $l \in \mathbb{N}$ and $\sum_{i=1}^r |\beta_i| = m$. It is a consequence of Theorem 129 that if $s_1, s_2 \ge s_3 \ge 0$ and $s_1 + s_2 - s_3 > n/p$, then $W^{s_1,p}_{loc}(\Omega) \times W^{s_2,p}_{loc}(\Omega) \hookrightarrow W^{s_3,p}_{loc}(\Omega)$. As a consequence,

$$\begin{gathered} W^{s-|\beta_1|,p}_{loc}(\Omega) \times W^{s-|\beta_2|,p}_{loc}(\Omega) \hookrightarrow W^{s-|\beta_1|-|\beta_2|,p}_{loc}(\Omega), \\ W^{s-|\beta_1|-|\beta_2|,p}_{loc}(\Omega) \times W^{s-|\beta_3|,p}_{loc}(\Omega) \hookrightarrow W^{s-|\beta_1|-|\beta_2|-|\beta_3|,p}_{loc}(\Omega), \\ \vdots \\ W^{s-|\beta_1|-\cdots-|\beta_{r-1}|,p}_{loc}(\Omega) \times W^{s-|\beta_r|,p}_{loc}(\Omega) \hookrightarrow W^{s-|\beta_1|-\cdots-|\beta_r|,p}_{loc}(\Omega) = W^{s-m,p}_{loc}(\Omega). \end{gathered} \quad (158)$$

Considering this and the fact that $\partial^{\beta_i} u \in W^{s-|\beta_i|,p}_{loc}(\Omega)$, we have

$$\partial^{\beta_1} u \cdots \partial^{\beta_r} u \in W^{t,p}_{loc}(\Omega) \quad (159)$$

for all $0 \le t \le s - m$. In particular, $\partial^{\beta_1} u \cdots \partial^{\beta_r} u \in W^{0,p}_{loc}(\Omega) = L^p_{loc}(\Omega)$. Also, since F is smooth and u is continuous, $F^{(l)}(u) \in L^\infty_{loc}(\Omega)$. Therefore,

$$F^{(l)}(u) \partial^{\beta_1} u \cdots \partial^{\beta_r} u \in L^p_{loc}(\Omega). \quad (160)$$

So, $F(u) \in W^{k,p}_{loc}(\Omega)$ where $k = \lfloor s \rfloor$. Now, for noninteger s, we use a bootstrapping argument to show that $F(u)$ in fact belongs to $W^{s,p}_{loc}(\Omega)$.

F' is smooth; therefore, $F'(u) \in W^{k,p}_{loc}(\Omega)$. Also $\partial u / \partial x^i \in W^{s-1,p}_{loc}(\Omega)$ (note that $s - 1 \ge 0$). By Theorem 129, we have

$$W^{k,p}_{loc}(\Omega) \times W^{s-1,p}_{loc}(\Omega) \hookrightarrow W^{t-1,p}_{loc}(\Omega) \quad (161)$$

provided that

$$k \geq t-1 \geq 0,\ s-1 \geq t-1 \geq 0,\ k+(s-1)-(t-1) > \frac{n}{p}. \tag{162}$$

Therefore, $\partial/\partial x^i(F(u)) = F'(u)\partial u/\partial x^i \in W^{t-1,p}_{loc}(\Omega)$ for all $1 \leq t \leq s$ such that $t < k + (s - n/p)$. Consequently, $F(u) \in W^{t,p}_{loc}(\Omega)$ for all $1 \leq t \leq s$ such that $t < k + (s - n/p)$ (see Theorem 123). Now, we can repeat this argument by starting with "F' is smooth, therefore $F'(u) \in W^{t,p}_{loc}(\Omega)$ for all $1 \leq t \leq s$ such that $t < k + (s - n/p)$." This results in $F(u) \in W^{t,p}_{loc}(\Omega)$ for all $1 \leq t \leq s$ such that $t < k + 2(s - n/p)$. Repeating this, a finite number of times shows that $F(u) \in W^{s,p}_{loc}(\Omega)$.

Our next goal is to prove items (2) and (3). First, we note that if $0 \in I$ then without loss of generality (WLOG) we may assume that $F(0) = 0$. Indeed, the constant function $F(0)$ is an element of $W^{s,p}_{loc}(\Omega)$. So,

$$F(u_m) \longrightarrow F(u) \text{ in } W^{s,p}_{loc}(\Omega) \Longleftrightarrow \tilde{F}(u_m) \longrightarrow \tilde{F}(u) \text{ in } W^{s,p}_{loc}(\Omega), \tag{163}$$

where $\tilde{F}(t) = F(t) - F(0)$. Thus WLOG we may assume that $F(0) = 0$.

Let $\{K_j\}_{j\in\mathbb{N}_0}$, $\{V_j\}_{j\in\mathbb{N}_0}$, and $\{\psi_j\}_{j\in\mathbb{N}_0}$ be as in Theorem 43. Clearly, $\{\psi_j\}$ is an admissible family of functions. Therefore, in order to show that $F(u_m) \longrightarrow F(u)$ in $W^{s,p}_{loc}(\Omega)$, it is enough to prove that

$$\forall r \in \mathbb{N}_0 \quad \psi_r(F(u_m) - F(u)) \longrightarrow 0 \quad \text{in } W^{s,p}(\Omega) as\, m \longrightarrow \infty. \tag{164}$$

Let $\psi_{r_1}, \cdots, \psi_{r_k}$ be those admissible test functions whose support intersects the support of ψ_r. So,

$$\forall x \in \operatorname{supp} \psi_r \quad \sum_{j\in\mathbb{N}_0} \psi_j u = \psi_{r_1} u + \cdots + \psi_{r_k} u. \tag{165}$$

Consequently,

$$\begin{aligned} \psi_r(F(u_m) - F(u)) = \psi_r F\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right) \\ - \psi_r F\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right). \end{aligned} \tag{166}$$

Since $u_m \longrightarrow u$ in $W^{s,p}_{loc}(\Omega)$, for all $1 \leq i \leq k$, we have

$$\psi_{r_i} u_m \longrightarrow \psi_{r_i} u \quad \text{in } W^{s,p}(\Omega), \tag{167}$$

and so,

$$\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m \longrightarrow \psi_{r_1} u + \cdots + \psi_{r_k} u \quad \text{in } W^{s,p}(\Omega). \tag{168}$$

Since $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we have

$$\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m \longrightarrow \psi_{r_1} u + \cdots + \psi_{r_k} u \quad \text{in } L^\infty(\Omega). \tag{169}$$

Consequently, for the continuous representatives of $\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m$ and $\psi_{r_1} u + \cdots + \psi_{r_k} u$, we have uniform convergence. From this point, we work with these continuous versions. The continuous function $\psi_{r_1} u + \cdots + \psi_{r_k} u$ attains its max and min on the compact set $\operatorname{supp} \psi_r$ which we denote by $A_{\max}$ and $A_{\min}$, respectively. Note that

$$\forall x \in \operatorname{supp} \psi_r \quad \left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right)(x) = u(x) \in I. \tag{170}$$

So, $A_{\max}$ and $A_{\min}$ are in I (that is, $[A_{\min}, A_{\max}] \subseteq I$). Let $\varepsilon > 0$ be such that $[A_{\min} - 2\varepsilon, A_{\max} + 2\varepsilon] \subseteq I$. By (169) there exists M such that

$$\begin{aligned} &\forall m \geq M, \forall x \in \operatorname{supp} \psi_r \quad \left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right)(x) \\ &\quad \in [A_{\min} - \varepsilon, A_{\max} + \varepsilon] \subseteq I. \end{aligned} \tag{171}$$

Let $\xi \in C^\infty_c(\mathbb{R})$ be such that $\xi = 1$ on $[A_{\min} - \varepsilon, A_{\max} + \varepsilon]$ and $\xi = 0$ outside of $[A_{\min} - 2\varepsilon, A_{\max} + 2\varepsilon] \subseteq I$. Define $\widehat{F} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\widehat{F}(t) = \begin{cases} \xi(t)F(t), & \text{if } t \in I, \\ 0, & \text{if } t \in I. \end{cases} \tag{172}$$

Clearly, $\widehat{F} : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function and $\widehat{F}(0) = 0$. Moreover, $\widehat{F} = F$ on $[A_{\min} - \varepsilon, A_{\max} + \varepsilon]$. Also, for all $x \in \Omega$ and $m \geq M$, we have

$$\begin{aligned} &\psi_r(F(u_m) - F(u)) \\ &\quad = \psi_r F\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right) - \psi_r F\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right) \\ &\quad = \psi_r \widehat{F}\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right) - \psi_r \widehat{F}\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right). \end{aligned} \tag{173}$$

Indeed, if $x \in \operatorname{supp} \psi_r$, then both sides are equal to zero. If $x \in \operatorname{supp} \psi_r$, then

$$\begin{aligned} &\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right)(x) \in [A_{\min}, A_{\max}], \\ &\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right)(x) \in [A_{\min} - \varepsilon, A_{\max} + \varepsilon], \end{aligned} \tag{174}$$

and so,

$$\begin{aligned} F\left(\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right)(x)\right) &= \widehat{F}\left(\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right)(x)\right), \\ F\left(\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right)(x)\right) &= \widehat{F}\left(\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right)(x)\right). \end{aligned} \tag{175}$$

$\widehat{F}$ is a smooth function and its value at 0 is 0. Also, by

assumption, $sp > n$. Therefore, the mapping $v \longrightarrow \psi_r \widehat{F}(v)$ from $W^{s,p}(\Omega)$ to $W^{s,p}(\Omega)$ is continuous. Hence,

$$\psi_r \widehat{F}\left(\psi_{r_1} u_m + \cdots + \psi_{r_k} u_m\right) \longrightarrow \psi_r \widehat{F}\left(\psi_{r_1} u + \cdots + \psi_{r_k} u\right) \quad \text{in } W^{s,p}(\Omega). \tag{176}$$

That is,

$$\psi_r(F(u_m) - F(u)) \longrightarrow 0 \quad \text{in } W^{s,p}(\Omega). \tag{177}$$

So, we proved item (2). Finally, we note that $W^{s,p}_{loc}(\Omega)$ is metrizable. So continuity of the mapping $u \longrightarrow F(u)$ is equivalent to sequential continuity which was proved in item (2). □

7. Conclusion

Sobolev-Slobodeckij spaces play a key role in the study of elliptic differential operators in nonsmooth setting. The study of certain differential operators between Sobolev spaces of sections of vector bundles on compact manifolds equipped with rough metric is closely related to the study of locally Sobolev functions on domains in the Euclidean space. In the present paper, we provided a self-contained rigorous study of certain fundamental properties of locally Sobolev-Slobodeckij spaces. In particular, by introducing notions such as "smooth multiplication triple" and "interior smooth multiplication triple," we rigorously studied completeness, separability, nature of the dual space, general embedding results, continuity of differentiation, and invariance under composition by smooth functions for locally Sobolev-Slobodeckij spaces.

Data Availability

All results are obtained without any software and found by manual computations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

An earlier draft of this manuscript was posted to arXiv: https://arxiv.org/abs/1806.02188. MH was supported in part by NSF (Awards 1262982, 1620366, and 2012857). AB was supported by NSF (Award 1262982).

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