

# A SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR AN INFINITE DIMENSIONAL SYSTEM WITH STATE-DEPENDENT DAMPING

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We study the validity of a Smoluchowski–Kramers approximation for a class of wave equations in a bounded domain of  $\mathbb{R}^n$  subject to a state-dependent damping and perturbed by a multiplicative noise. We prove that in the small mass limit the solution converges to the solution of a stochastic quasilinear parabolic equation where a noise-induced extra drift is created.

**1. Introduction.** In this article we study the following class of stochastic wave equations with state-dependent damping:

$$(1.1) \quad \begin{cases} \mu \partial_t^2 u_\mu = \Delta u_\mu - \gamma(u_\mu) \partial_t u_\mu + f(u_\mu) + \sigma(u_\mu) \partial_t w^\mathcal{Q}, & t > 0, x \in \mathcal{O}, \\ u_\mu(0) = u_0, \quad \partial_t u_\mu(0) = v_0, & u_\mu|_{\partial\mathcal{O}} = 0, \end{cases}$$

and their small mass limit as  $\mu \rightarrow 0$ . Here,  $\mathcal{O}$  is a bounded domain on  $\mathbb{R}^n$ , and  $w^\mathcal{Q}(t, x)$  is a cylindrical Wiener process which is white in time and colored in space. The friction coefficient  $\gamma$  is a strictly positive, bounded and continuously differentiable function, and  $f$  and  $\sigma$  are Lipschitz continuous functions.

By Newton’s second law of motion, the solution  $u_\mu(t, x)$  of equation (1.1) can be interpreted as the displacement field of the particles in a continuum body occupying domain  $\mathcal{O}$ , subject to a random external force field  $\sigma(u_\mu) \partial_t w^\mathcal{Q}$  and a state-dependent damping force  $\gamma(u_\mu) \partial_t u_\mu$  which is proportional to the velocity field. In addition, the particles are subject to the interaction forces between neighboring particles, represented by the Laplace operator  $\Delta$ , and the nonlinear reaction, represented by  $f$ . Here,  $\mu$  represents the constant density of the particles, and we are interested in the regime when  $\mu \rightarrow 0$  which is the so-called Smoluchowski–Kramers approximation limit (ref. [23] and [31]).

A series of papers (ref. [3, 4] and [29]) studied the limiting behavior of  $u_\mu$ , for a large class of reaction terms  $f$  and for both additive and multiplicative noise. In all those papers the friction coefficient  $\gamma$  is assumed to be constant and a perturbative limit is obtained. Namely, it is proved that, in the small mass limit,  $u_\mu$  converges to the solution of the following parabolic problem:

$$(1.2) \quad \begin{cases} \gamma \partial_t u = \Delta u + f(u) + \sigma(u) \partial_t w^\mathcal{Q}, \\ u(0) = u_0, \quad u|_{\partial\mathcal{O}} = 0. \end{cases}$$

More precisely, it is shown that, for every  $T > 0$  and  $\eta > 0$ ,

$$(1.3) \quad \lim_{\mu \rightarrow 0} \mathbb{P} \left( \sup_{t \in [0, T]} \|u_\mu(t) - u(t)\|_{L^2(\mathcal{O})} > \eta \right) = 0.$$

In fact, in [10] it is proved that when  $f$  is Lipschitz continuous, the following stronger convergence holds:

$$(1.4) \quad \lim_{\mu \rightarrow 0} \mathbb{E} \sup_{t \in [0, T]} \|u_\mu(t) - u(t)\|_{L^2(\mathcal{O})}^p = 0$$

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for every  $p \geq 1$ . Note that several problems related to this type of limit have been addressed in a variety of finite and infinite dimensional contexts (see, e.g., [14] and [32], for the finite dimensional case, and [3, 4, 6, 25–28] and [29], for the infinite dimensional case).

Having proved the validity of the small mass limit in any fixed time interval, it is important to understand how stable this limit is for long times. To this purpose, [7] studies the convergence of the statistically invariant states for a class of semilinear wave equations with linear damping, that is, equation (1.1) with constant friction coefficient  $\gamma$ , both with Lipschitz and with polynomial nonlinearity  $f$ . A similar problem is studied in [3] when the two systems are of gradient type. In that case the Boltzmann distribution for the solution of the second order equation is explicitly given in terms of a Gibbs measure. It turns out that the first marginal of the Boltzmann distribution does not depend on  $\mu$  and coincides with the invariant measure of the limiting first order equation. In the case studied in [7], there is no explicit expression for the invariant distributions of (1.1). Nevertheless, it is shown that the first marginals of any sequence of invariant measures for (1.1) converge in a suitable Wasserstein metric to the unique invariant measure of equation (1.2). In the same spirit, [8] and [9] studied the convergence of the quasi-potentials  $V_\mu(u, v)$  which describe the asymptotics of the exit times and the large deviation principle for the invariant measures to equation (1.1). In [8], gradient systems are considered so that  $V_\mu$  is explicitly computed, and it is shown that  $V^\mu(u)$ , the infimum of  $V_\mu(u, v)$  over all  $v \in H^{-1}(\mathcal{O})$ , coincides for every  $\mu > 0$  with  $V(u)$ , the quasi-potential associated with equation (1.2). In [9], the nongradient case is studied, and it is shown that  $V^\mu(u)$  converges pointwise to  $V(u)$ , as  $\mu$  goes to zero.

In all the aforementioned papers, the case of a constant friction coefficient  $\gamma$  is considered, and the limiting equation (1.2) is formally obtained by taking  $\mu = 0$  in (1.1). However, there are relevant situations in which this is not true. This happens, for example, in the case when the constant friction is replaced by a magnetic field. As a matter of fact, even in the case of a constant magnetic field and finite dimension, the small mass limit does not yield the solution of the first order equation (ref. [5, 11, 16] and [24] for the finite dimensional case and [10] for the infinite dimensional case). In this case a possible strategy consists in regularizing the problem by adding a small friction or by smoothing the noise in time, and in the double limit it is possible to give a meaning to the Smoluchowski–Kramers approximation. Notice that in [10] the limiting equation is an SPDE of hyperbolic type.

In the present paper we are dealing with another situation when the small mass limit does not give a perturbative result. As mentioned at the beginning of this Introduction, we consider a wave equation perturbed by a multiplicative noise, having a friction term whose intensity is state dependent. This problem has been extensively studied in finite dimension in a series of papers (see [21] and references therein, and also [2, 15, 19, 22]). In these papers it is shown how the interplay between the nonconstant friction coefficient and the noise creates an additional drift in the limiting first order equation, when the mass  $\mu$  goes to zero. More precisely, the following system is studied:

$$\begin{cases} dx_\mu(t) = v_\mu(t) dt, & x_\mu(0) = x \in \mathbb{R}^d, \\ \mu dv_\mu(t) = [b(x_\mu(t)) - \gamma(x_\mu(t))v_\mu(t)] dt + \sigma(x_\mu(t)) dW(t), & v_\mu(0) = v \in \mathbb{R}^d, \end{cases}$$

where  $\gamma$  is a matrix valued function defined on  $\mathbb{R}^d$  such that, for some positive  $\gamma_0$ ,

$$\inf_{x \in \mathcal{O}} \xi^T \gamma(x) \xi \geq \gamma_0 |\xi|^2, \quad \xi \in \mathbb{R}^d.$$

It is proved that, as  $\mu$  goes to zero,  $x_\mu$  converges in  $L^2$ , with respect to the uniform norm in  $C([0, T]; \mathbb{R}^d)$ , to the solution of the first order equation,

$$dx(t) = \left( \frac{b(x(t))}{\gamma(x(t))} + S(x(t)) \right) dt + \frac{\sigma(x(t))}{\gamma(x(t))} dW(t), \quad x(0) = x,$$

where the noise induced drift  $S(x)$  is given by

$$S_i(x) = \frac{\partial}{\partial x_l} [(\gamma^{-1})_{ij}(x)] J_{jl}(x),$$

and the matrix valued function  $J$  is the solution of the Lyapunov equation

$$J(x)\gamma^*(x) + \gamma(x)J(x) = \sigma(x)\sigma^*(x).$$

Our purpose here is to understand if something similar also happens in the case of infinite dimensional systems. In fact, in what follows we will prove that, for every initial condition  $(u_0, v_0) \in H^1(\mathcal{O}) \times L^2(\mathcal{O})$  and for every  $\delta > 0$  and  $p < \infty$ ,

$$(1.5) \quad \lim_{\mu \rightarrow 0} \mathbb{P}(\|u_\mu - u\|_{C([0,T]; H^{-\delta}(\mathcal{O}))} + \|u_\mu - u\|_{L^p(\mathcal{O})} > \eta) = 0, \quad \eta > 0,$$

where  $u$  is the unique solution of the quasilinear stochastic parabolic equation

$$(1.6) \quad \begin{cases} \partial_t u = \frac{1}{\gamma(u)} \Delta u + \frac{f(u)}{\gamma(u)} - \frac{\gamma'(u)}{2\gamma^3(u)} \sum_{i=1}^{\infty} (\sigma(u) Q e_i)^2 + \frac{\sigma(u)}{\gamma(u)} \partial_t w^Q, & t > 0, x \in \mathcal{O}, \\ u(0) = u_0, & u|_{\partial\mathcal{O}} = 0. \end{cases}$$

It is important to notice that if  $\sigma$  is constant, then the noise induced term,

$$H(u) := -\frac{\gamma'(u)}{2\gamma^3(u)} \sum_{i=1}^{\infty} (\sigma(u) Q e_i)^2,$$

coincides with the Stratonovich-to-Itô correction so that equation (1.6) can be written as

$$\partial_t u = \frac{1}{\gamma(u)} \Delta u + \frac{f(u)}{\gamma(u)} + \frac{\sigma}{\gamma(u)} \circ \partial_t w^Q.$$

However, if  $G(u)$  denotes the Itô-to-Stratonovich correction,

$$G(u) = -\frac{1}{2} \sum_{i=1}^{\infty} \partial_u \left( \frac{\sigma(u) Q e_i}{\gamma(u)} \right) \left( \frac{\sigma(u) Q e_i}{\gamma(u)} \right),$$

then

$$H(u) + G(u) = -\frac{1}{2\gamma^2(u)} \sum_{i=1}^{\infty} (\sigma(u) Q e_i) \partial_u (\sigma(u) Q e_i),$$

and this is manifestly nontrivial, in general, when  $\sigma$  is not constant. This means that, in the case of an arbitrary state-dependent diffusion coefficient  $\sigma$ , the small mass limit does not lead to the perturbative parabolic quasilinear equation, obtained by taking  $\mu = 0$  and replacing the Itô's with the Stratonovich's integral.

We would also like to point out that, unlike in finite dimension, here we are not handling systems of equations. This means, in particular, that  $\gamma$  is a scalar function, and, for every function  $u : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ , we can write

$$(1.7) \quad \gamma(u(t, x)) \partial_t u(t, x) = \partial_t [g(u(t, x))], \quad t \in [0, T], x \in \mathcal{O},$$

where  $g' = \gamma$ . The case of systems and of matrix valued friction coefficients requires a different analysis and will be investigated in a forthcoming paper.

Our first step in the proof of (1.5) is proving that, for every fixed  $\mu > 0$ , equation (1.1) has a unique adapted solution  $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0, T]; H^1(\mathcal{O}) \times L^2(\mathcal{O})))$ . Because of the nonconstant friction, it is convenient to reformulate equation (1.1) in terms of the new variables  $(u_\mu, g(u_\mu)/\mu + \partial_t u_\mu)$ . However, due to the presence of the nonlinear term  $g(u)$ ,

using the theory of linear semigroups, as done in the previous papers [3, 4] and [10], turns out to be the wrong path to follow. Instead, here it is more appropriate to use the theory of monotone nonlinear operators (see [1]).

Having proved the well-posedness of (1.1), next we prove the uniform bounds of the solutions  $(u_\mu, \partial_t u_\mu)$  which are required to obtain tightness. This is one of the most delicate parts of the paper. Actually, even when using the Itô formula for a nicely chosen energy functional, the more classical arguments that work in finite dimension fail. Nevertheless, we are able to prove that  $u_\mu$  is bounded with respect to  $\mu$  in  $L^2(\Omega; C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H^1(\mathcal{O})))$ . Even more delicate are the bounds for the velocity  $\partial_t u_\mu$ . Of course, we know that we cannot have any uniform bounds with respect to  $\mu$ . However, we expected to have

$$\sup_{\mu \in (0, 1)} \mu^\alpha \mathbb{E} \sup_{t \in [0, T]} \|\partial_t u_\mu(t)\|_{L^2(\mathcal{O})}^2 < \infty$$

for  $\alpha = 1$ . As a matter of fact, by using an argument by contradiction we can prove the bound above only for  $\alpha = 3/2$ , but this is enough to obtain the fundamental limit

$$(1.8) \quad \lim_{\mu \rightarrow 0} \mu \mathbb{E} \sup_{t \in [0, T]} \|\partial_t u_\mu(t)\|_{L^2(\mathcal{O})} = 0.$$

After defining  $\rho_\mu = g(u_\mu)$ , these uniform bounds are fundamental to prove the tightness of the family  $(\rho_\mu)_{\mu > 0}$  in  $L^p(0, T; L^2(\mathcal{O})) \cap C([0, T]; H^{-\delta}(\mathcal{O}))$  for every  $p < \infty$  and  $\delta > 0$ . We show that, for every  $\mu > 0$ , the function  $\rho_\mu$  solves the equation

$$\begin{aligned} \rho_\mu(t) + \mu \partial_t u_\mu(t) &= g(u_0) + \mu v_0 + \int_0^t \operatorname{div}[b(\rho_\mu(s)) \nabla \rho_\mu(s)] ds \\ &\quad + \int_0^t F(\rho_\mu(s)) ds + \int_0^t \sigma_g(\rho_\mu(s)) dw^\mathcal{Q}(s), \end{aligned}$$

where  $b = 1/\gamma \circ g^{-1}$ ,  $F = f \circ g^{-1}$  and  $\sigma_g(h) = \sigma(g^{-1} \circ h)$ . Working with this equation, instead of (1.1), makes the use of the a priori bounds and of limit (1.8) more direct.

Once tightness is proved, we have the weak convergence of the sequence  $(\rho_\mu)_{\mu > 0}$  to some  $\rho$  that solves the quasilinear parabolic SPDE,

$$(1.9) \quad \begin{cases} \partial_t \rho = \operatorname{div}[b(\rho) \nabla \rho] + F(\rho) + \sigma_g(\rho) dw^\mathcal{Q}(t), & t > 0, x \in \mathcal{O}, \\ \rho(0, x) = g(u_0), & \rho(t, x) = 0, & x \in \partial \mathcal{O}. \end{cases}$$

Then, since we can prove pathwise uniqueness for equation (1.9), from weak convergence we get the convergence in probability. Finally, a generalized Itô formula stated in the [Appendix](#) allows us to get the convergence of  $u_\mu$  to the solution of equation (1.6).

We would like to remind that equations like (1.9) have attracted a lot of attention in recent years, and several papers have studied their well-posedness in  $C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H^1(\mathcal{O}))$ , in case of periodic boundary conditions, under considerably more general assumptions on the coefficients  $b$  that can be matrix valued and even degenerate (see [13] and [20]). Our  $b$  here is scalar valued and nondegenerate, but this allows us to have, at least in the additive case, weaker assumptions on the regularity of the noise than in [13] and [20]. In particular, as a byproduct of our small mass limit, we get the well-posedness of equation (1.9) for a noise that, for example, in the case of constant  $\sigma$  is only assumed to live in  $L^2(\mathcal{O})$  which seems to be a new result.

*Organization of the paper.* In Section 2 we introduce the notations, and we describe the assumptions we make on the coefficients and on the noise in equation (1.1). In Section 3 we study the well-posedness of equation (1.1) in space  $L^2(\Omega; C([0, T]; H^1(\mathcal{O}) \times L^2(\mathcal{O})))$  for every  $T > 0$  and every fixed  $\mu > 0$ . In Section 4 we prove some uniform bounds with respect

to  $\mu$  for the solutions  $((u_\mu, \partial_t u_\mu))_{\mu>0}$  in suitable functional spaces. In Section 5 these bounds allow us to prove the tightness of the distributions of  $g(u_\mu)$  for  $\mu > 0$  sufficiently small. In Section 6 we prove the validity of pathwise uniqueness for equation (1.9). In Section 7 we give the proof of the convergence in probability of  $u_\mu$ , as  $\mu$  goes to zero, and we identify the limit  $u$  as the solution of the first order equation (1.6).

**2. Preliminaries.** Throughout the present paper  $\mathcal{O}$  is a bounded domain in  $\mathbb{R}^n$  with  $n \geq 1$ , and it has a boundary of class  $C^3$ . We denote by  $H$  the Hilbert space  $L^2(\mathcal{O})$  and by  $\langle \cdot, \cdot \rangle_H$  the corresponding inner product.  $H^1$  is the completion of  $C_0^\infty(\mathcal{O})$  with respect to norm

$$\|u\|_{H^1}^2 := \|\nabla u\|_H^2 = \int_{\mathcal{O}} |\nabla u(x)|^2 dx,$$

and  $H^{-1}$  is the dual space to  $H^1$ . Then,  $H^1$ ,  $H$  and  $H^{-1}$  are all complete separable metric spaces, and the following relation between them holds

$$(2.1) \quad H^1 \subset H \subset H^{-1},$$

where both inclusions are compact embeddings. In what follows, we shall denote

$$\mathcal{H} = H \times H^{-1}, \quad \mathcal{H}_1 = H^1 \times H.$$

Given the domain  $\mathcal{O}$ , we denote by  $(e_i)_{i \in \mathbb{N}} \subset H^1$  the complete orthonormal basis of  $H$ , which diagonalizes the Laplacian  $\Delta$ , endowed with Dirichlet boundary conditions on  $\partial\mathcal{O}$ . Moreover, we denote by  $(-\alpha_i)_{i \in \mathbb{N}}$  the corresponding sequence of eigenvalues, that is,

$$\Delta e_i = -\alpha_i e_i, \quad i \in \mathbb{N}.$$

Given

$$u = \sum_{i=1}^{\infty} b_i e_i, \quad v = \sum_{i=1}^{\infty} c_i e_i,$$

for some sequences of real numbers  $(b_i)_{i \in \mathbb{N}}$  and  $(c_i)_{i \in \mathbb{N}}$ , we have

$$(2.2) \quad \langle u, v \rangle_{H^1} = \sum_{i=1}^{\infty} \alpha_i b_i c_i, \quad \langle u, v \rangle_H = \sum_{i=1}^{\infty} b_i c_i, \quad \langle u, v \rangle_{H^{-1}} = \sum_{i=1}^{\infty} \frac{1}{\alpha_i} b_i c_i.$$

From (2.2) we can derive the Poincaré inequality

$$(2.3) \quad \|u\|_H \leq \frac{1}{\sqrt{\alpha_1}} \|u\|_{H^1}, \quad u \in H^1, \quad \|u\|_{H^{-1}} \leq \frac{1}{\sqrt{\alpha_1}} \|u\|_H, \quad u \in H.$$

As for the stochastic perturbation, we assume that  $w^Q(t)$  is a cylindrical  $Q$ -Wiener process, defined on a complete stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . This means that  $w^Q(t)$  can be formally written as

$$w^Q(t) = \sum_{i=1}^{\infty} Q e_i \beta_i(t),$$

where  $(\beta_i)_{i \in \mathbb{N}}$  is a sequence of independent standard Brownian motions on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ,  $Q : H \rightarrow H$  is a bounded linear operator and  $(e_i)_{i \in \mathbb{N}}$  is the complete orthonormal system, introduced above, that diagonalizes the Laplace operator, endowed with Dirichlet boundary conditions.

In what follows, we shall denote by  $H_Q$  the set  $Q(H)$ .  $H_Q$  is the reproducing kernel of the noise  $w^Q$  and is a Hilbert space, endowed with the inner product

$$\langle Qh, Qk \rangle_{H_Q} = \langle h, k \rangle_H, \quad h, k \in H.$$

Notice that the sequence  $(Qe_i)_{i \in \mathbb{N}}$  is a complete orthonormal system in  $H_Q$ . Moreover, if  $U$  is any Hilbert space containing  $H_Q$  such that the embedding of  $H_Q$  into  $U$  is Hilbert–Schmidt, we have that

$$(2.4) \quad w^Q \in C([0, T]; U).$$

Next, we recall that, for every two separable Hilbert spaces  $E$  and  $F$ ,  $\mathcal{L}_2(E, F)$  denotes the space of Hilbert–Schmidt operators from  $E$  into  $F$ .  $\mathcal{L}_2(E, F)$  is a Hilbert space, endowed with the inner product

$$\langle A, B \rangle_{\mathcal{L}_2(E, F)} = \text{Tr}_E[A^*B] = \text{Tr}_F[BA^*].$$

Throughout this article we will always assume that the three hypotheses below are true.

ASSUMPTION 1. The mapping  $\sigma : H \rightarrow \mathcal{L}_2(H_Q, H)$  is defined by

$$[\sigma(h)Qe_i](x) = \sigma_i(x, h(x)), \quad x \in \mathcal{O},$$

for every  $h \in H$  and  $i \in \mathbb{N}$ , for some mapping  $\sigma_i : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ . We assume  $\sigma$  is bounded, that is,

$$\sigma_\infty := \sup_{h \in H} \|\sigma(h)\|_{\mathcal{L}_2(H_Q, H)} < \infty,$$

and

$$(2.5) \quad \sup_{x \in \mathcal{O}} \sum_{i=1}^{\infty} |\sigma_i(x, y_1) - \sigma_i(x, y_2)|^2 \leq L|y_1 - y_2|^2, \quad y_1, y_2 \in \mathbb{R}.$$

Notice that (2.5) implies  $\sigma$  is Lipschitz continuous in the sense that, for any  $h_1, h_2 \in H$ ,

$$\|\sigma(h_1) - \sigma(h_2)\|_{\mathcal{L}_2(H_Q, H)}^2 \leq L\|h_1 - h_2\|_H^2.$$

REMARK 2.1. If  $\sigma$  is constant, then Assumption 1 means that  $\sigma Q$  is a Hilbert–Schmidt operator in  $H$ . Equivalently, in case  $\sigma$  is the identity operator, this means that the noise  $w^Q$  lives in  $H$  so that we can take  $U = H$ .

If  $\sigma$  is not constant, then Assumption 1 is satisfied if, for example,

$$[\sigma(h)Qk](x) = \lambda(h(x))Qk(x), \quad x \in \mathcal{O}, h, k \in H$$

for some  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$  bounded and Lipschitz continuous and for some  $Q \in \mathcal{L}(H)$  such that

$$\sum_{i=1}^{\infty} \|Qe_i\|_{L^\infty(\mathcal{O})}^2 < \infty.$$

In case  $Q$  is diagonalizable with respect the basis  $(e_i)_{i \in \mathbb{N}}$ , with  $Qe_i = \lambda_i e_i$ , the condition above reads

$$(2.6) \quad \sum_{i=1}^{\infty} \lambda_i^2 \|e_i\|_{L^\infty(\mathcal{O})}^2 < \infty.$$

In general (see [17]), it holds that

$$\|e_i\|_{L^\infty(\mathcal{O})} \leq ci^\alpha$$

for some  $\alpha > 0$ , and (2.6) becomes

$$\sum_{i=1}^{\infty} \lambda_i^2 i^{2\alpha} < \infty.$$

In particular, when  $n = 1$  or the domain is a hyperrectangle in higher dimension, the eigenfunctions  $(e_i)_{i \in \mathbb{N}}$  are equibounded, and (2.6) becomes  $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$ .

ASSUMPTION 2. The mapping  $\gamma$  belongs to  $C_b^1(\mathbb{R})$ , and there exist  $\gamma_0$  and  $\gamma_1$  such that

$$(2.7) \quad 0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \quad r \in \mathbb{R}.$$

In what follows, we shall define

$$g(r) = \int_0^r \gamma(\sigma) d\sigma, \quad r \in \mathbb{R}.$$

Clearly,  $g(0) = 0$  and  $g'(r) = \gamma(r)$ . In particular, due to (2.7),  $g$  is uniformly Lipschitz continuous on  $\mathbb{R}$ . Moreover,  $g$  is strictly increasing and

$$(2.8) \quad (g(r_1) - g(r_2))(r_1 - r_2) \geq \gamma_0 |r_1 - r_2|^2, \quad r_1, r_2 \in \mathbb{R}.$$

ASSUMPTION 3. The mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous. Moreover, there exist  $\lambda < \alpha_1$ ,  $\delta < 1$  and  $c \geq 0$  such that

$$(2.9) \quad f(r)r \leq \lambda r^2 + c(1 + |r|^{1+\delta}), \quad r \in \mathbb{R}.$$

Any Lipschitz continuous function  $f$ , having sublinear growth, satisfies (2.9). Condition (2.9) also allows linear growth for  $f$ , but in this case we need

$$\sup_{r_1, r_2 \in \mathbb{R}} \frac{f(r_1) - f(r_2)}{r_1 - r_2} < \alpha_1.$$

In particular, (2.9) is satisfied if

$$\|f\|_{\text{Lip}} < \alpha_1.$$

**3. Well posedness of equation (1.1).** In this section we study the existence and uniqueness of solutions to the nonlinear stochastic wave equations (1.1) with initial data  $(u_0, v_0) \in \mathcal{H}_1$  for every fixed  $\mu > 0$ . Notice that the second order equations (1.1) can be written as the following system:

$$(3.1) \quad \begin{cases} du_\mu(t) = v_\mu(t) dt, & u_\mu(0) = u_0, \\ dv_\mu(t) = \frac{1}{\mu} [\Delta u_\mu(t) - \gamma(u_\mu(t))v_\mu(t) + f(u_\mu(t))] dt \\ \quad + \frac{1}{\mu} \sigma(u_\mu(t)) dw^\mathcal{Q}(t), & v_\mu(0) = v_0, \\ u_\mu(t)|_{\partial\mathcal{O}} = 0, & t > 0. \end{cases}$$

Now, if we define

$$(3.2) \quad \eta := \partial_t u + \frac{g(u)}{\mu},$$

and  $z = (u, \eta)$ , system (3.1) can be rewritten as

$$(3.3) \quad dz_\mu(t) = A_\mu(z_\mu(t)) dt + \Sigma_\mu(z_\mu(t)) dw^\mathcal{Q}(t), \quad z_\mu(0) = \left(u_0, v_0 + \frac{g(u_0)}{\mu}\right),$$

where we denoted

$$\Sigma_\mu(u, \eta) = \frac{1}{\mu} (0, \sigma(u)), \quad (u, \eta) \in \mathcal{H},$$

and

$$A_\mu(u, \eta) = \left( \frac{-g(u)}{\mu} + \eta, \frac{1}{\mu} [\Delta u + f(u)] \right), \quad (u, \eta) \in D(A_\mu) = \mathcal{H}_1.$$

This means that the adapted  $\mathcal{H}_1$ -valued process  $z_\mu(t) = (u_\mu(t), \eta_\mu(t))$  is the unique solution of the equation

$$(3.4) \quad z_\mu(t) = (u_0, g(u_0)/\mu + v_0) + \int_0^t A_\mu(z_\mu(s)) ds + \int_0^t \Sigma_\mu(z_\mu(s)) dw^\mathcal{Q}(s),$$

if and only if the adapted  $\mathcal{H}_1$ -valued process  $(u_\mu(t), v_\mu(t)) := (u_\mu(t), -g(u_\mu(t))/\mu + \eta_\mu(t))$  is the unique solution of the system

$$(3.5) \quad \begin{cases} u_\mu(t) = u_0 + \int_0^t v_\mu(s) ds \\ \mu v_\mu(t) = \mu v_0 + \int_0^t [\Delta u_\mu(s) - \gamma(u_\mu(s))v_\mu(s) + f(u_\mu(s))] ds \\ \quad + \int_0^t \sigma(u_\mu(s)) dw^\mathcal{Q}(s). \end{cases}$$

In this section we are interested in the well-posedness of equation (3.1) (and, equivalently, of (3.3)) and not on the dependence of its solution on  $\mu$ . Thus, without any loss of generality we will only consider the case when  $\mu = 1$  and, for simplicity of notation, we will denote  $A_1$  and  $\Sigma_1$  by  $A$  and  $\Sigma$ , respectively.

We start our study of equation (3.3) by analyzing the nonlinear operator  $A$ . To this purpose, it is immediate to check that

$$(3.6) \quad \|A(z)\|_{\mathcal{H}} \leq c(1 + \|z\|_{\mathcal{H}_1}), \quad z \in D(A),$$

because  $f$  and  $g$  are both Lipschitz continuous. In the next lemma we prove that the nonlinear operator  $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$  is *quasi- $m$ -dissipative*. For all the details on the definitions and the basic results about maximal monotone nonlinear operators that we are using below, we refer to [1], Chapters 2 and 3.

**LEMMA 3.1.** *Under Assumptions 2 and 3, there exists  $\kappa \geq 0$  such that, for every  $z_1, z_2 \in D(A)$ ,*

$$(3.7) \quad \langle A(z_1) - A(z_2), z_1 - z_2 \rangle_{\mathcal{H}} \leq \kappa \|z_1 - z_2\|_{\mathcal{H}}^2.$$

Moreover, there exists  $\lambda_0 > 0$  such that

$$(3.8) \quad \text{Range}(I - \lambda A) = \mathcal{H}, \quad \lambda \in (0, \lambda_0).$$

**PROOF.** For every  $z_1 = (u_1, \eta_1)$  and  $z_2 = (u_2, \eta_2)$  in  $D(A)$ , we have

$$\begin{aligned} \langle A(z_1) - A(z_2), z_1 - z_2 \rangle_{\mathcal{H}} &= -\langle g(u_1) - g(u_2), u_1 - u_2 \rangle_H + \langle \eta_1 - \eta_2, u_1 - u_2 \rangle_H \\ &\quad + \langle \Delta u_1 - \Delta u_2, \eta_1 - \eta_2 \rangle_{H^{-1}} + \langle f(u_1) - f(u_2), \eta_1 - \eta_2 \rangle_{H^{-1}} \\ &\leq -\gamma_0 \|u_1 - u_2\|_H^2 + \|f(u_1) - f(u_2)\|_{H^{-1}} \|\eta_1 - \eta_2\|_{H^{-1}} \\ &\leq -\gamma_0 \|u_1 - u_2\|_H^2 + c(\|u_1 - u_2\|_H^2 + \|\eta_1 - \eta_2\|_{H^{-1}}^2), \end{aligned}$$

where the first inequality follows from (2.8) and the second inequality follows from the Lipschitz continuity of  $f$  and the Poincaré inequality (2.3). In particular, there exists some  $\kappa \geq 0$  such that (3.7) holds.

Next, in order to prove (3.8), we need to show that, if  $\lambda$  is sufficiently small, then, for every  $h = (h_1, h_2) \in \mathcal{H}$ , there exists  $z = (u, \eta) \in \mathcal{H}_1$  such that

$$z - \lambda A(z) = h,$$

or, equivalently, there exists some  $u \in H^1$  such that

$$(3.9) \quad u - \lambda^2 \Delta u = -\lambda g(u) + \lambda^2 f(u) + (h_1 + \lambda h_2).$$



In particular, if we define

$$\Gamma_\lambda(u) = (I - \lambda^2 \Delta)^{-1} [-\lambda g(u) + \lambda^2 f(u) + (h_1 + \lambda h_2)],$$

we need to prove that there exists some  $\lambda_0 > 0$  such that  $\Gamma_\lambda : H \rightarrow H$  is a contraction for every  $\lambda \in (0, \lambda_0)$ . By

$$\|(I - \lambda^2 \Delta)^{-1}\|_{\mathcal{L}(H)} \leq 1, \quad \lambda > 0,$$

we have

$$\begin{aligned} \|\Gamma_\lambda(u_1) - \Gamma_\lambda(u_2)\|_H &\leq c\lambda(\|g(u_1) - g(u_2)\|_H + \lambda\|f(u_1) - f(u_2)\|_H) \\ &\leq c\lambda(1 + \lambda)\|u_1 - u_2\|_H \end{aligned}$$

which implies that  $\Gamma_\lambda$  is a contraction for small enough  $\lambda$ .  $\square$

Now, if we define  $\bar{\lambda} := \lambda_0 \wedge \kappa^{-1}$ , due to Lemma 3.1 we have that the operator

$$J_\lambda(z) := (I - \lambda A)^{-1}(z), \quad z \in \mathcal{H}, \lambda \in (0, \bar{\lambda}),$$

is well defined and is Lipschitz continuous from  $\mathcal{H}$  into  $\mathcal{H}$  with Lipschitz constant  $(1 - \lambda\kappa)^{-1}$  (see [1], Proposition 3.2). Thus, for every  $\lambda \in (0, \bar{\lambda})$ , we can introduce the *Yosida approximation* of  $A$ , defined as

$$(3.10) \quad A^\lambda(z) := \frac{1}{\lambda} [J_\lambda(z) - z] = A(J_\lambda(z)), \quad z \in \mathcal{H}.$$

By the Lipschitz continuity of  $J_\lambda$ , it is easy to check that

$$\|A^\lambda(z_1) - A^\lambda(z_2)\|_{\mathcal{H}} \leq \frac{2}{\lambda(1 - \lambda\kappa)} \|z_1 - z_2\|_{\mathcal{H}}, \quad z_1, z_2 \in \mathcal{H}.$$

Moreover,  $A^\lambda$  is quasi-dissipative in  $\mathcal{H}$ . Actually, by (3.7) and the definition of  $A^\lambda$  in (3.10), we have

$$\begin{aligned} \langle A^\lambda(z_1) - A^\lambda(z_2), z_1 - z_2 \rangle_{\mathcal{H}} &= -\langle A^\lambda(z_1) - A^\lambda(z_2), (J_\lambda(z_1) - z_1) - (J_\lambda(z_2) - z_2) \rangle_{\mathcal{H}} \\ &\quad + \langle A^\lambda(z_1) - A^\lambda(z_2), J_\lambda(z_1) - J_\lambda(z_2) \rangle_{\mathcal{H}} \\ (3.11) \quad &\leq \kappa \|J_\lambda(z_1) - J_\lambda(z_2)\|_{\mathcal{H}}^2 \\ &\leq \frac{\kappa}{1 - \lambda\kappa} \|z_1 - z_2\|_{\mathcal{H}}^2 \end{aligned}$$

for any  $z_1, z_2 \in \mathcal{H}$ . Moreover, as shown in [1], Proposition 3.2, for every  $z \in D(A)$  we have

$$(3.12) \quad \|A^\lambda(z)\|_{\mathcal{H}} \leq \frac{1}{1 - \lambda\kappa} \|A(z)\|_{\mathcal{H}},$$

and then

$$(3.13) \quad \|J_\lambda(z) - z\|_{\mathcal{H}} = \lambda \|A^\lambda(z)\|_{\mathcal{H}} \leq \frac{\lambda}{1 - \lambda\kappa} \|A(z)\|_{\mathcal{H}}.$$

Finally, as shown in [1], Proposition 3.5, we have

$$(3.14) \quad \lim_{\lambda \rightarrow 0} \|A^\lambda(z) - A(z)\|_{\mathcal{H}} = 0, \quad z \in D(A).$$

Now, we are ready to prove the main result of this section.

**THEOREM 3.2.** *Under Assumptions 1, 2 and 3, for every  $(u_0, v_0) \in \mathcal{H}_1$  and every  $T > 0$  and  $\mu > 0$ , there exists a unique adapted process  $(u_\mu, v_\mu) \in L^2(\Omega, C([0, T], \mathcal{H}_1))$  which solves equation (3.5).*

PROOF. Without loss of generality, we only consider  $\mu = 1$  here. As we have seen, the well-posedness of equation (3.5) is equivalent to the well-posedness of equation (3.3). Therefore, here we deal with equation (3.3).

For every  $\lambda \in (0, \bar{\lambda})$ , we introduce the approximating problem

$$(3.15) \quad dz_\lambda(t) = A^\lambda(z_\lambda(t)) dt + \Sigma(z_\lambda(t)) dw^Q(t), \quad z_\lambda(0) = (u_0, v_0 + g(u_0)).$$

By Assumption 1 the mapping  $\Sigma : \mathcal{H} \rightarrow \mathcal{L}_2(H_Q, \mathcal{H}_1)$  is bounded and Lipschitz continuous since

$$(3.16) \quad \|\Sigma(z)\|_{\mathcal{L}_2(H_Q, \mathcal{H}_1)} = \|\sigma(u)\|_{\mathcal{L}_2(H_Q, H)}$$

for all  $z = (u, \eta) \in \mathcal{H}$ . Then, since  $A^\lambda$  is Lipschitz continuous in  $\mathcal{H}$ , there exists a unique solution

$$z_\lambda = (u_\lambda, \eta_\lambda) \in L^p(\Omega, C([0, T]; \mathcal{H}))$$

to equation (3.15) for every  $T > 0$  and  $p \geq 1$ , using the classical fixed point theorem for contractions. Moreover, thanks to (3.11), we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 &= 2\mathbb{E} \langle A^\lambda(z_\lambda(t)), z_\lambda(t) \rangle_{\mathcal{H}} + \mathbb{E} \|\Sigma(z_\lambda(t))\|_{\mathcal{L}_2(H_Q, \mathcal{H})}^2 \\ &= 2\mathbb{E} \langle A^\lambda(z_\lambda(t)) - A^\lambda(0), z_\lambda(t) \rangle_{\mathcal{H}} + 2\mathbb{E} \langle A^\lambda(0), z_\lambda(t) \rangle_{\mathcal{H}} \\ &\quad + \mathbb{E} \|\Sigma(z_\lambda(t))\|_{\mathcal{L}_2(H_Q, \mathcal{H})}^2 \\ &\leq \frac{2\kappa}{1 - \lambda\kappa} \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 + \|A^\lambda(0)\|_{\mathcal{H}}^2 + \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 + c. \end{aligned}$$

Moreover, due to (3.12),

$$\|A^\lambda(0)\|_{\mathcal{H}}^2 \leq \frac{1}{(1 - \lambda\kappa)^2} \|A(0)\|_{\mathcal{H}}^2 = \frac{1}{(1 - \lambda\kappa)^2} \|f(0)\|_{H^{-1}}^2.$$

Therefore, there exists a constant  $c$ , independent of  $\lambda \leq \bar{\lambda}/2$ , such that

$$\frac{d}{dt} \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 \leq c(\mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 + 1).$$

By Grönwall's inequality this implies

$$(3.17) \quad \sup_{\lambda \in (0, \bar{\lambda}/2)} \sup_{t \in [0, T]} \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}}^2 < \infty.$$

In the rest of the proof, for an arbitrary  $z = (u, \eta) \in \mathcal{H}$ , we denote  $z_1 = u$  and  $z_2 = \eta$ :

*Step 1.* There exists  $c > 0$  such that

$$(3.18) \quad \mathbb{E} \|z_\lambda(t)\|_{\mathcal{H}_1}^2 + \gamma_0 \int_0^t \mathbb{E} \|J_\lambda(z_\lambda(s))\|_{H^1}^2 ds \leq \|z_0\|_{\mathcal{H}_1}^2 + c \int_0^t \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}}^2 ds + ct$$

for all  $\lambda \in (0, \hat{\lambda})$ , where

$$\hat{\lambda} := \frac{\bar{\lambda}}{2} \wedge \frac{\gamma_0}{8\|f\|_{\text{Lip}}}.$$

*Proof of Step 1.* We apply the Itô formula to

$$K(z) = \|z\|_{\mathcal{H}_1}^2 = \|u\|_{H^1}^2 + \|\eta\|_H^2,$$

and we get

$$\begin{aligned}
 dK(z_\lambda(t)) &= \langle A^\lambda(z_\lambda(t)), DK(z_\lambda(t)) \rangle_{\mathcal{H}_1} dt + \|\Sigma(z_\lambda(t))\|_{\mathcal{L}_2(H_Q, \mathcal{H}_1)}^2 dt \\
 &\quad + \langle DK(z_\lambda(t)), \Sigma(z_\lambda(t)) dw^Q(t) \rangle_{\mathcal{H}_1} \\
 (3.19) \quad &= 2[\langle A^\lambda(z_\lambda(t))_1, (-\Delta)z_\lambda(t)_1 \rangle_H + \langle A^\lambda(z_\lambda(t))_2, z_\lambda(t)_2 \rangle_H] dt \\
 &\quad + \|\sigma(z_\lambda(t)_1)\|_{\mathcal{L}_2(H_Q, H)}^2 dt + 2\langle z_\lambda(t)_2, \sigma(z_\lambda(t)_1) dw^Q(t) \rangle_H \\
 &=: 2\Phi_\lambda(t) dt + \|\sigma(z_\lambda(t)_1)\|_{\mathcal{L}_2(H_Q, H)}^2 dt + 2\langle z_\lambda(t)_2, \sigma(z_\lambda(t)_1) dw^Q(t) \rangle_H.
 \end{aligned}$$

Recall that  $A^\lambda(z) = A(J_\lambda(z)) = \frac{1}{\lambda}[J_\lambda(z) - z]$ , which implies that, for every  $z \in \mathcal{H}$ ,

$$\begin{cases} J_\lambda(z)_1 + \lambda g(J_\lambda(z)_1) - \lambda J_\lambda(z)_2 = z_1, \\ J_\lambda(z)_2 - \lambda \Delta(J_\lambda(z)_1) - \lambda f(J_\lambda(z)_1) = z_2. \end{cases}$$

Therefore, we have

$$\begin{aligned}
 \Phi_\lambda &= \langle -g(J_\lambda(z_\lambda)_1) + J_\lambda(z_\lambda)_2, (-\Delta)[J_\lambda(z_\lambda)_1 + \lambda g(J_\lambda(z_\lambda)_1) - \lambda J_\lambda(z_\lambda)_2] \rangle_H \\
 &\quad + \langle \Delta(J_\lambda(z_\lambda)_1) + f(J_\lambda(z_\lambda)_1), J_\lambda(z_\lambda)_2 - \lambda \Delta(J_\lambda(z_\lambda)_1) - \lambda f(J_\lambda(z_\lambda)_1) \rangle_H \\
 &= -\langle \gamma(J_\lambda(z_\lambda)_1) \nabla(J_\lambda(z_\lambda)_1), \nabla(J_\lambda(z_\lambda)_1) \rangle_H - \lambda \|g(J_\lambda(z_\lambda)_1)\|_{H^1}^2 - \lambda \|J_\lambda(z_\lambda)_2\|_{H^1}^2 \\
 &\quad - \lambda \|J_\lambda(z_\lambda)_1\|_{H^2}^2 - \lambda \|f(J_\lambda(z_\lambda)_1)\|_H^2 - 2\lambda \langle \Delta(J_\lambda(z_\lambda)_1), f(J_\lambda(z_\lambda)_1) \rangle_H \\
 &\quad + \langle f(J_\lambda(z_\lambda)_1), J_\lambda(z_\lambda)_2 \rangle_H \\
 &< -\gamma_0 \|J_\lambda(z_\lambda)_1\|_{H^1}^2 - 2\lambda \langle \Delta(J_\lambda(z_\lambda)_1), f(J_\lambda(z_\lambda)_1) \rangle_H + \langle f(J_\lambda(z_\lambda)_1), J_\lambda(z_\lambda)_2 \rangle_H,
 \end{aligned}$$

where the last inequality uses Assumption 2. Since  $f: \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous, we have  $f \circ u \in H^1$ , for every  $u \in H^1$ , and

$$\|f \circ u\|_{H^1} \leq \|f\|_{\text{Lip}} \|u\|_{H^1}$$

which implies

$$|\langle \Delta(J_\lambda(z_\lambda)_1), f(J_\lambda(z_\lambda)_1) \rangle_H| \leq \|J_\lambda(z_\lambda)_1\|_{H^1} \|f(J_\lambda(z_\lambda)_1)\|_{H^1} \leq \|f\|_{\text{Lip}} \|J_\lambda(z_\lambda)_1\|_{H^1}^2,$$

and

$$\begin{aligned}
 |\langle f(J_\lambda(z_\lambda)_1), J_\lambda(z_\lambda)_2 \rangle_H| &\leq \|f(J_\lambda(z_\lambda)_1)\|_{H^1} \|J_\lambda(z_\lambda)_2\|_{H^{-1}} \\
 &\leq \|f\|_{\text{Lip}} \|J_\lambda(z_\lambda)_1\|_{H^1} \|J_\lambda(z_\lambda)_2\|_{H^{-1}}.
 \end{aligned}$$

Then, thanks to Young's inequality, we get

$$\begin{aligned}
 \Phi_\lambda &< -\gamma_0 \|J_\lambda(z_\lambda)_1\|_{H^1}^2 + 2\lambda \|f\|_{\text{Lip}} \|J_\lambda(z_\lambda)_1\|_{H^1}^2 \\
 (3.20) \quad &\quad + \|f\|_{\text{Lip}} \|J_\lambda(z_\lambda)_1\|_{H^1} \|J_\lambda(z_\lambda)_2\|_{H^{-1}} \\
 &\leq -\frac{\gamma_0}{2} \|J_\lambda(z_\lambda)_1\|_{H^1}^2 + c \|J_\lambda(z_\lambda)_2\|_{H^{-1}}^2
 \end{aligned}$$

for  $\lambda \in (0, \gamma_0/(8\|f\|_{\text{Lip}}))$ . Therefore, if we integrate (3.19) in time, apply (3.20) and use both the boundedness of  $\|\sigma(u)\|_{\mathcal{L}_2(H_Q, H)}^2$  and the Lipschitz continuity of  $J_\lambda$  on  $\mathcal{H}$ , we obtain

$$\begin{aligned}
 \|z_\lambda(t)\|_{\mathcal{H}_1}^2 &+ \gamma_0 \int_0^t \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2 ds \\
 (3.21) \quad &\leq \|z_0\|_{\mathcal{H}_1}^2 + c \int_0^t \|z_\lambda(s)\|_{\mathcal{H}}^2 ds + \sigma_\infty^2 t + 2 \int_0^t \langle z_\lambda(s)_2, \sigma(z_\lambda(s)_1) dw^Q(s) \rangle_H
 \end{aligned}$$

from which we can derive (3.18) after taking expectation.

*Step 2.* There exists  $c_T > 0$  such that, for every  $\lambda \in (0, \hat{\lambda}/2)$ ,

$$(3.22) \quad \mathbb{E} \sup_{t \in [0, T]} \|z_\lambda(t)\|_{\mathcal{H}_1}^2 \leq \|z_0\|_{\mathcal{H}_1}^2 + c_T.$$

*Proof of Step 2.* By taking the supremum in time for (3.21), we have

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \|z_\lambda(t)\|_{\mathcal{H}_1}^2 \\ & \leq \|z_0\|_{\mathcal{H}_1}^2 + c \int_0^T \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}}^2 ds + cT + 2 \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \langle z_\lambda(s)_2, \sigma(z_\lambda(s)_1) dw^Q(s) \rangle_H \right| \\ & \leq \|z_0\|_{\mathcal{H}_1}^2 + c \int_0^T \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}}^2 ds + cT + c \left( \mathbb{E} \int_0^T \|\sigma(z_\lambda(s)_1)\|_{\mathcal{L}_2(H_Q, H)}^2 \|z_\lambda(s)_2\|_H^2 ds \right)^{\frac{1}{2}} \\ & \leq \|z_0\|_{\mathcal{H}_1}^2 + c \int_0^T \mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}}^2 ds + cT + \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \|z_\lambda(t)\|_{\mathcal{H}_1}^2, \end{aligned}$$

where the last inequality follows from Assumption 1 and Young's inequality. Due to (3.17), this implies (3.22).

*Step 3.* There exists  $z \in L^\infty(0, T; L^2(\Omega, \mathcal{H}))$  such that

$$(3.23) \quad \lim_{\lambda \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E} \|z_\lambda(t) - z(t)\|_{\mathcal{H}}^2 = 0.$$

*Proof of Step 3.* For every  $\lambda, v \in (0, \hat{\lambda}/2)$ , we set

$$\varrho_{\lambda, v}(t) := z_\lambda(t) - z_v(t), \quad t \in [0, T].$$

Then, we have

$$d\varrho_{\lambda, v}(t) = [A^\lambda(z_\lambda(t)) - A^v(z_v(t))] dt + [\Sigma(z_\lambda(t)) - \Sigma(z_v(t))] dw^Q(t), \quad \varrho_{\lambda, v}(0) = 0.$$

Together with (3.10), that is,  $z = J_\lambda(z) - \lambda A(J_\lambda(z))$ , we have

$$\begin{aligned} d\|\varrho_{\lambda, v}(t)\|_{\mathcal{H}}^2 &= 2[A^\lambda(z_\lambda(t)) - A^v(z_v(t)), \varrho_{\lambda, v}(t)]_{\mathcal{H}} + \|\Sigma(z_\lambda(t)) - \Sigma(z_v(t))\|_{\mathcal{L}_2(H_Q, \mathcal{H})}^2 dt \\ &\quad + 2\langle \varrho_{\lambda, v}(t), [\Sigma(z_\lambda(t)) - \Sigma(z_v(t))] dw^Q(t) \rangle_{\mathcal{H}} \\ &= [A(J_\lambda(z_\lambda(t))) - A(J_v(z_v(t))), J_\lambda(z_\lambda(t)) - J_v(z_v(t))]_{\mathcal{H}} \\ &\quad - \langle A^\lambda(z_\lambda(t)) - A^v(z_v(t)), \lambda A^\lambda(z_\lambda(t)) - v A^v(z_v(t)) \rangle_{\mathcal{H}} \\ &\quad + \|\Sigma(z_\lambda(t)) - \Sigma(z_v(t))\|_{\mathcal{L}_2(H_Q, \mathcal{H})}^2 dt \\ &\quad + 2\langle \varrho_{\lambda, v}(t), [\Sigma(z_\lambda(t)) - \Sigma(z_v(t))] dw^Q(t) \rangle_{\mathcal{H}} \\ &\leq [\kappa \|J_\lambda(z_\lambda(t)) - J_v(z_v(t))\|_{\mathcal{H}}^2 + c(\lambda + v)(\|A^\lambda(z_\lambda(t))\|_{\mathcal{H}}^2 + \|A^v(z_v(t))\|_{\mathcal{H}}^2) \\ &\quad + c\|\varrho_{\lambda, v}(t)\|_{\mathcal{H}}^2] dt + 2\langle \varrho_{\lambda, v}(t), [\Sigma(z_\lambda(t)) - \Sigma(z_v(t))] dw^Q(t) \rangle_{\mathcal{H}} \\ &\leq c[\|\varrho_{\lambda, v}(t)\|_{\mathcal{H}}^2 + c(\lambda + v)(\|z_\lambda(t)\|_{\mathcal{H}_1}^2 + \|z_v(t)\|_{\mathcal{H}_1}^2 + 1)] dt \\ &\quad + 2\langle \varrho_{\lambda, v}(t), [\Sigma(z_\lambda(t)) - \Sigma(z_v(t))] dw^Q(t) \rangle_{\mathcal{H}}, \end{aligned}$$

where the first inequality follows from Lemma 3.1 and the Lipschitz continuity of  $\Sigma$ , and the second inequality follows from (3.6), (3.10) and (3.12). Therefore,

$$\mathbb{E} \|\varrho_{\lambda, v}(t)\|_{\mathcal{H}}^2 \leq c \int_0^t \mathbb{E} \|\varrho_{\lambda, v}(s)\|_{\mathcal{H}}^2 ds + c \int_0^t (\lambda + v)(\mathbb{E} \|z_\lambda(s)\|_{\mathcal{H}_1}^2 + \mathbb{E} \|z_v(s)\|_{\mathcal{H}_1}^2 + 1) ds.$$

Thanks to Grönwall's inequality, this yields

$$\sup_{t \in [0, T]} \mathbb{E} \| \varrho_{\lambda, \nu}(t) \|_{\mathcal{H}}^2 \leq c_T (\lambda + \nu) \int_0^T (\mathbb{E} \| z_{\lambda}(s) \|_{\mathcal{H}_1}^2 + \mathbb{E} \| z_{\nu}(s) \|_{\mathcal{H}_1}^2 + 1) ds.$$

In view of (3.22), we conclude that, for every sequence  $(\lambda_n)_{n \in \mathbb{N}}$  converging to zero, the sequence  $(z_{\lambda_n})_{n \in \mathbb{N}}$  is Cauchy in  $L^\infty(0, T; L^2(\Omega; \mathcal{H}))$ . In particular, there exists  $z \in L^\infty(0, T; L^2(\Omega; \mathcal{H}))$  such that (3.23) holds.

*Step 4.* For every  $t \in [0, T]$ , we have

$$(3.24) \quad z(t) = (u_0, g(u_0) + v_0) + \int_0^t A(z(s)) ds + \int_0^t \Sigma(z(s)) dw^{\mathcal{Q}}(s).$$

Moreover,  $z \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ .

*Proof of Step 4.* For every  $t \in [0, T]$ , we have

$$(3.25) \quad z_{\lambda}(t) = (u_0, g(u_0) + v_0) + \int_0^t A^{\lambda}(z_{\lambda}(s)) ds + \int_0^t \Sigma(z_{\lambda}(s)) dw^{\mathcal{Q}}(s).$$

If we define  $\mathcal{H}_{-1} = H^{-1} \times H^{-2}$ , we have

$$\|A(z_1) - A(z_2)\|_{\mathcal{H}_{-1}} \leq c \|z_1 - z_2\|_{\mathcal{H}}, \quad z_1, z_2 \in \mathcal{H}.$$

Since  $z(t) \in L^2(\Omega; \mathcal{H})$ , by (3.6) and (3.12) this implies

$$\begin{aligned} \|A^{\lambda}(z_{\lambda}(s)) - A(z(s))\|_{\mathcal{H}_{-1}} &= \|A(J_{\lambda}(z_{\lambda}(s))) - A(z(s))\|_{\mathcal{H}_{-1}} \\ &\leq c \|J_{\lambda}(z_{\lambda}(s)) - z_{\lambda}(s)\|_{\mathcal{H}} + c \|z_{\lambda}(s) - z(s)\|_{\mathcal{H}} \\ &\leq c\lambda(\|z_{\lambda}(s)\|_{\mathcal{H}_1} + 1) + c \|z_{\lambda}(s) - z(s)\|_{\mathcal{H}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t A^{\lambda}(z_{\lambda}(s)) ds - \int_0^t A(z(s)) ds \right\|_{\mathcal{H}_{-1}}^2 \\ \leq c_T \int_0^T [\lambda(\mathbb{E} \| z_{\lambda}(s) \|_{\mathcal{H}_1}^2 + 1) + \mathbb{E} \| z_{\lambda}(s) - z(s) \|_{\mathcal{H}}^2] ds. \end{aligned}$$

Thanks to (3.22) and (3.23), this implies

$$(3.26) \quad \lim_{\lambda \rightarrow 0} \int_0^t A^{\lambda}(z_{\lambda}(s)) ds = \int_0^t A(z(s)) ds, \quad \text{in } L^2(\Omega; C([0, T]; \mathcal{H}_{-1})).$$

Moreover,

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t [\Sigma(z_{\lambda}(s)) - \Sigma(z(s))] dw^{\mathcal{Q}}(s) \right\|_{\mathcal{H}}^2 &\leq c \int_0^T \|\Sigma(z_{\lambda}(s)) - \Sigma(z(s))\|_{\mathcal{L}_2(H_{\mathcal{Q}}, \mathcal{H})}^2 ds \\ &\leq c \int_0^T \mathbb{E} \| z_{\lambda}(s) - z(s) \|_{\mathcal{H}}^2 ds \end{aligned}$$

so that

$$\lim_{\lambda \rightarrow 0} \int_0^t \Sigma(z_{\lambda}(s)) dw^{\mathcal{Q}}(s) = \int_0^t \Sigma(z(s)) dw^{\mathcal{Q}}(s), \quad \text{in } L^2(\Omega; C([0, T]; \mathcal{H})).$$

This, together with (3.26), allows us to conclude that, for every  $t \in [0, T]$ , we can take the  $L^2(\Omega, C([0, T]; \mathcal{H}_{-1}))$ -limit on both sides of (3.25), as  $\lambda$  goes to zero, and we get

$$z(t) = (u_0, g(u_0) + v_0) + \int_0^t A(z(s)) ds + \int_0^t \Sigma(z(s)) dw^{\mathcal{Q}}(s).$$

Moreover, by proceeding as for  $z_{\lambda}$ , we have that  $z \in L^2(\Omega; L^\infty(0, T; \mathcal{H}_1))$ .

Now, in order to prove the continuity of trajectories in  $\mathcal{H}_1$ , we denote by  $\theta(t)$  the solution of the problem

$$d\theta(t) = L\theta(t) dt + \Sigma(z(t)) dw^Q(t), \quad \theta(0) = (u_0, g(u_0) + v_0),$$

where  $L(\theta_1, \theta_2) = (\theta_2, \Delta\theta_1)$ . Due to Assumption 1 and the fact that  $\theta(0) \in \mathcal{H}_1$ , we have that  $\theta$  belongs to  $L^2(\Omega; C([0, T]; \mathcal{H}_1))$  (for a proof, see [12], Theorem 5.11). Now, if we define  $\hat{z}(t) := z(t) - \theta(t)$ , for  $t \in [0, T]$ , and  $M(z_1, z_2) = (-g(z_1), f(z_1))$ , for  $(z_1, z_2) \in \mathcal{H}_1$ , we have

$$\frac{d}{dt} \hat{z}(t) = L\hat{z}(t) + M(z(t)), \quad \sigma(0) = 0$$

so that, by applying the variation of constants formula, we obtain

$$\hat{z}(t) = \int_0^t S(t-s)M(z(s)) ds,$$

where  $S(t)$  is the group generated by the operator  $L$ , endowed with Dirichlet boundary conditions, in  $\mathcal{H}_1$ . Since

$$\|M(z)\|_{\mathcal{H}_1} \leq c (\|z\|_{\mathcal{H}_1} + 1)$$

and  $z \in L^2(\Omega; L^\infty(0, T; \mathcal{H}_1))$ , we have that  $\hat{z} \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ . Then, as  $z = \hat{z} + \theta$ , we conclude that  $z \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ .

*Step 5. Uniqueness holds.*

*Proof of Step 5.* Let  $z_1$  and  $z_2$  be two solutions of equation (3.5). If we define  $\varrho(t) := z_1(t) - z_2(t)$ , we have

$$\begin{aligned} \|\varrho(t)\|_{\mathcal{H}}^2 &= \int_0^t [A(z_1(s)) - A(z_2(s)), \varrho(t)]_{\mathcal{H}} + \|\sigma(z_1(s)) - \sigma(z_2(s))\|_{\mathcal{L}_2(H_Q, \mathcal{H})}^2] ds \\ &\quad + 2 \int_0^t \langle \varrho(s), [\Sigma(z_1(s)) - \Sigma(z_2(s))] dw^Q \rangle_{\mathcal{H}}. \end{aligned}$$

Hence, by Lemma 3.1 and Assumption 1 we have

$$\mathbb{E} \|\varrho(t)\|_{\mathcal{H}}^2 \leq c \int_0^t \mathbb{E} \|\varrho(s)\|_{\mathcal{H}}^2 ds,$$

and this implies that  $z_1 = z_2$ .  $\square$

**4. Energy estimates.** In the previous section we have proved that, for any  $\mu > 0$  and any  $T > 0$ , there is a unique solution  $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0, T], \mathcal{H}_1))$  to system (3.1). In this section we prove some bounds for  $(u_\mu, \partial_t u_\mu)$  which are uniform with respect to  $\mu$ .

As we have already done in the proof of Theorem 3.2, if we apply the Itô formula to equation (3.1) and the function

$$K_\mu(u, v) = \|u\|_{H^1}^2 + \mu \|v\|_H^2,$$

we have

$$\begin{aligned} \frac{1}{2} dK_\mu(u_\mu, \partial_t u_\mu) &= \left[ \langle (-\Delta)u_\mu(t), \partial_t u_\mu(t) \rangle_H + \langle \Delta u_\mu(t), \partial_t u_\mu(t) \rangle_H \right. \\ &\quad - \langle \gamma(u_\mu(t)) \partial_t u_\mu(t), \partial_t u_\mu(t) \rangle_H \\ &\quad + \langle f(u_\mu(t), \partial_t u_\mu(t)) \rangle_H + \frac{1}{2\mu} \|\sigma(u_\mu(t))\|_{\mathcal{L}_2(H_Q, H)}^2 \Big] dt \\ &\quad + \langle \partial_t u_\mu(t), \sigma(u_\mu(s)) dw^Q(t) \rangle_H. \end{aligned}$$

This implies

$$\begin{aligned}
 & \frac{1}{2} d[\|u_\mu(t)\|_{H^1}^2 + \mu \|\partial_t u_\mu(t)\|_H^2] \\
 &= (\langle f(u_\mu(t)), \partial_t u_\mu(t) \rangle_H - \langle \gamma(u_\mu(t)) \partial_t u_\mu(t), \partial_t u_\mu(t) \rangle_H \\
 (4.1) \quad &+ \frac{1}{2\mu} \|\sigma(u_\mu(t))\|_{\mathcal{L}_2(H_Q, H)}^2) dt + \langle \partial_t u_\mu(t), \sigma(u_\mu(t)) dw^Q(t) \rangle_H \\
 &\leq \left( c(\|u_\mu(t)\|_H^2 + 1) - \frac{\gamma_0}{2} \|\partial_t u_\mu(t)\|_H^2 + \frac{\sigma_\infty^2}{2\mu} \right) dt \\
 &+ \langle \partial_t u_\mu(t), \sigma(u_\mu(t)) dw^Q(t) \rangle_H.
 \end{aligned}$$

In particular,

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} [\mathbb{E} \|u_\mu(t)\|_{H^1}^2 + \mu \mathbb{E} \|\partial_t u_\mu(t)\|_H^2] \\
 (4.2) \quad & \leq -\frac{\gamma_0}{2\mu} [\mathbb{E} \|u_\mu(t)\|_{H^1}^2 + \mu \mathbb{E} \|\partial_t u_\mu(t)\|_H^2 - \bar{c}] + c \left( \frac{1}{\mu} \mathbb{E} \|u_\mu(t)\|_{H^1}^2 + 1 \right),
 \end{aligned}$$

where  $\bar{c} = \sigma_\infty^2 / \gamma_0$ . Moreover, we have

$$(4.3) \quad \frac{d}{dt} \mu \|u_\mu(t)\|_H^2 = 2\mu \langle u_\mu(t), \partial_t u_\mu(t) \rangle_H$$

and

$$\begin{aligned}
 (4.4) \quad \mu d\langle u_\mu(t), \partial_t u_\mu(t) \rangle_H &= [\mu \|\partial_t u_\mu(t)\|_H^2 - \|u_\mu(t)\|_{H^1}^2 - \langle u_\mu(t), \gamma(u_\mu(t)) \partial_t u_\mu(t) \rangle_H \\
 &+ \langle f(u_\mu(t)), u_\mu(t) \rangle_H] dt + \langle u_\mu(t), \sigma(u_\mu(t)) dw^Q(t) \rangle_H.
 \end{aligned}$$

LEMMA 4.1. *Under Assumptions 1, 2 and 3, for every  $T > 0$  and  $(u_0, v_0) \in \mathcal{H}_1$ , there exists some constant  $c_T = c_T(\|u_0\|_{H^1}, \|v_0\|_H)$ , independent of  $\mu$ , such that*

$$\begin{aligned}
 (4.5) \quad & \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|u_\mu(s)\|_{H^1}^2 ds \\
 & \leq c_T \left( 1 + \mu \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds + \mu^2 \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 \right)
 \end{aligned}$$

for every  $\mu \in (0, 1)$  and  $t \in [0, T]$ .

PROOF. We define

$$\Gamma(r) = \int_0^r x \gamma(x) dx, \quad r \in \mathbb{R},$$

and

$$\Lambda(u) = \int_O \Gamma(u(x)) dx, \quad u \in H.$$

It is easy to see that (2.7) implies

$$(4.6) \quad 0 \leq \frac{\gamma_0}{2} r^2 \leq \Gamma(r) \leq \frac{\gamma_1}{2} r^2, \quad r \in \mathbb{R}$$

so that, for every  $u \in H$ ,

$$(4.7) \quad 0 \leq \frac{\gamma_0}{2} \|u\|_H^2 \leq \Lambda(u) \leq \frac{\gamma_1}{2} \|u\|_H^2.$$

Moreover, if  $v(t) = \partial_t u(t)$ , we have

$$(4.8) \quad \frac{d}{dt} \Lambda(u(t)) = \int_{\mathcal{O}} \gamma(u(t, x)) u(t, x) v(t, x) dx = \langle u(t), \gamma(u(t)) \partial_t u(t) \rangle_H.$$

Therefore, thanks to (4.4) and (4.7), this gives

$$\begin{aligned} \frac{\gamma_0}{2} \|u_\mu(t)\|_H^2 &\leq \Lambda(u_\mu(t)) \\ &= \Lambda(u_0) - \mu \langle u_\mu(t), \partial_t u_\mu(t) \rangle_H + \mu \langle u_0, v_0 \rangle_H + \mu \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds \\ &\quad - \int_0^t \|u_\mu(s)\|_{H^1}^2 ds + \int_0^t \langle f(u_\mu(s)), u_\mu(s) \rangle_H ds \\ &\quad + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s) \rangle_H. \end{aligned}$$

In particular, for every  $\mu \in (0, 1)$ , we have

$$\begin{aligned} \frac{\gamma_0}{4} \|u_\mu(t)\|_H^2 &\leq c + c\mu^2 \|\partial_t u_\mu(t)\|_H^2 + \mu \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds \\ &\quad - \int_0^t \|u_\mu(s)\|_{H^1}^2 ds + \int_0^t \langle f(u_\mu(s)), u_\mu(s) \rangle_H ds \\ &\quad + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s) \rangle_H. \end{aligned}$$

Now, by (2.3) and (2.9) we have

$$\begin{aligned} \langle f(u_\mu(s)), u_\mu(s) \rangle_H &\leq \lambda \|u_\mu(s)\|_H^2 + c(1 + \|u_\mu(s)\|_{L^{1+\delta}(\mathcal{O})}^{1+\delta}) \\ &\leq [\lambda/\alpha_1 + (1 - \lambda/\alpha_1)/2] \|u_\mu(s)\|_{H^1}^2 + c \end{aligned}$$

so that

$$\begin{aligned} \frac{\gamma_0}{4} \|u_\mu(t)\|_H^2 &+ \frac{1}{2} (1 - \lambda/\alpha_1) \int_0^t \|u_\mu(s)\|_{H^1}^2 ds \\ &\leq c_T + c\mu^2 \|\partial_t u_\mu(t)\|_H^2 + \mu \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s) \rangle_H. \end{aligned}$$

This implies

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_H^2 + \int_0^t \mathbb{E} \|u_\mu(s)\|_{H^1}^2 ds \\ &\leq c_T + c\mu^2 \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 + c\mu \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \\ &\quad + c \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \langle u_\mu(s), \sigma(u_\mu(s)) dw^{\mathcal{Q}}(s) \rangle_H \right| \\ &\leq c_T + c\mu^2 \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 + c\mu \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds + \frac{1}{2} \int_0^t \mathbb{E} \|u_\mu(s)\|_{H^1}^2 ds, \end{aligned}$$

and (4.5) follows.  $\square$



PROPOSITION 4.2. *Under Assumptions 1, 2 and 3, for every  $T > 0$  and  $(u_0, v_0) \in \mathcal{H}_1$ , there exist some constants  $c_T$  and  $\mu_T > 0$ , depending on  $\|u_0\|_{H^1}$ ,  $\|v_0\|_H$ , such that*

$$(4.9) \quad \mathbb{E} \sup_{r \in [0, T]} \|u_\mu(r)\|_{H^1}^2 + \mu \mathbb{E} \sup_{r \in [0, T]} \|\partial_t u_\mu(r)\|_H^2 + \int_0^T \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \leq \frac{c_T}{\mu}$$

for every  $\mu \in (0, \mu_T)$ .

PROOF. Due to (4.1), for every  $\mu \in (0, 1)$ , we have

$$\begin{aligned} & \|u_\mu(t)\|_{H^1}^2 + \mu \|\partial_t u_\mu(t)\|_H^2 + \frac{\gamma_0}{2} \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds \\ & \leq \frac{c_T}{\mu} + c \int_0^t \|u_\mu(s)\|_H^2 ds + \int_0^t \langle \partial_t u_\mu(s), \sigma(u_\mu(s)) dw^Q(s) \rangle_H. \end{aligned}$$

This implies

$$\begin{aligned} & \mathbb{E} \sup_{r \in [0, t]} \|u_\mu(r)\|_{H^1}^2 + \mu \mathbb{E} \sup_{r \in [0, t]} \|\partial_t u_\mu(r)\|_H^2 + \frac{\gamma_0}{2} \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds \\ & \leq \frac{c_T}{\mu} + c \int_0^t \mathbb{E} \|u_\mu(s)\|_H^2 ds + \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \langle \partial_t u_\mu(s), \sigma(u_\mu(s)) dw^Q(s) \rangle_H \right| \\ & \leq \frac{c_T}{\mu} + c \int_0^t \mathbb{E} \|u_\mu(s)\|_H^2 ds + \frac{\gamma_0}{4} \int_0^t \mathbb{E} \|\partial_t u_\mu(s)\|_H^2 ds. \end{aligned}$$

Therefore, we can conclude the proof by using (4.5).  $\square$

REMARK 4.3. Combining (4.5) and (4.9), we obtain that, for every  $T > 0$ , there exist  $c_T, \mu_T > 0$  such that

$$(4.10) \quad \mathbb{E} \sup_{r \in [0, T]} \|u_\mu(r)\|_H^2 + \int_0^T \mathbb{E} \|u_\mu(s)\|_{H^1}^2 ds \leq c_T$$

for all  $\mu \in (0, \mu_T)$ .

In fact, we can prove a better bound for the  $L^2(\Omega; L^\infty(0, T; \mathcal{H}_1))$ -norm of  $(u_\mu, \sqrt{\mu} \partial_t u_\mu)$  than the one in (4.9).

PROPOSITION 4.4. *Under Assumptions 1, 2 and 3, given any  $T > 0$ , there exist  $c_T, \mu_T > 0$  such that, for all  $\mu \in (0, \mu_T)$ ,*

$$(4.11) \quad \sqrt{\mu} \mathbb{E} \sup_{t \in [0, T]} (\|u_\mu(t)\|_{H^1}^2 + \mu \|\partial_t u_\mu(t)\|_H^2) \leq c_T.$$

PROOF. Assume (4.11) is not true. Then, we can find a sequence  $(\mu_k)_{k \in \mathbb{N}} \subset (0, 1)$  converging to 0, as  $k \rightarrow \infty$ , such that

$$(4.12) \quad \lim_{k \rightarrow \infty} \sqrt{\mu_k} \mathbb{E} \sup_{t \in [0, T]} (\|u_{\mu_k}(t)\|_{H^1}^2 + \mu_k \|\partial_t u_{\mu_k}(t)\|_H^2) = \infty.$$

In what follows, to simplify our notation we define

$$L_k(t) := \|u_{\mu_k}(t)\|_{H^1}^2 + \mu_k \|\partial_t u_{\mu_k}(t)\|_H^2, \quad t \in [0, T].$$

By Theorem 3.2, all  $L_k(t)$  are continuous in  $t$ ,  $\mathbb{P}$ -a.s.. Therefore, for every  $k \in \mathbb{N}$ , there exist a random time  $t_k \in [0, T]$  such that

$$L_k(t_k) = \sup_{t \in [0, T]} L_k(t).$$

For any random time  $s$  such that  $\mathbb{P}(s \leq t_k) = 1$ , from (4.1) we have

$$(4.13) \quad L_k(t_k) - L_k(s) \leq \int_s^{t_k} \left( c(\|u_{\mu_k}(\tau)\|_H^2 + 1) + \frac{\sigma_\infty^2}{\mu_k} \right) d\tau + M_k(t_k) - M_k(s),$$

where

$$M_k(t) = \int_0^t \langle \partial_t u_{\mu_k}(s), \sigma(u_{\mu_k}(s)) dw^\mathcal{Q}(s) \rangle_H.$$

If we define the random variables

$$M_k^* := \sup_{t \in [0, T]} |M_k(t)|, \quad U_k^* := \sup_{t \in [0, T]} \|u_{\mu_k}(t)\|_H^2,$$

then by Proposition 4.2 and (4.10) we have

$$(4.14) \quad \mathbb{E}(M_k^*) \leq c \left( \int_0^T \mathbb{E} \|\partial_t u_{\mu_k}(t)\|_H^2 dt \right)^{\frac{1}{2}} \leq \frac{cT}{\sqrt{\mu_k}}, \quad \mathbb{E}(U_k^*) \leq cT.$$

Due to the definition of  $M_k^*$  and  $U_k^*$ , there exists a constant  $\lambda_T$ , independent of  $k$ , such that  $\|u_0\|_{H^1}^2 + \|v_0\|_H^2 \leq \lambda_T$  and

$$L_k(t_k) - L_k(s) \leq \lambda_T(U_k^* + 1) + \frac{\sigma_\infty^2(t_k - s)}{\mu_k} + 2M_k^*.$$

If we take  $s = 0$ , then

$$t_k \geq \frac{\mu_k}{\sigma_\infty^2} (L_k(t_k) - \lambda_T - \lambda_T(U_k^* + 1) - 2M_k^*) =: \frac{\mu_k}{\sigma_\infty^2} \theta_k.$$

On the set  $E_k := \{\theta_k > 0\}$ , we consider  $s \in [t_k - \frac{\mu_k}{2\sigma_\infty^2} \theta_k, t_k]$  for which we have

$$L_k(s) \geq L_k(t_k) - \lambda_T(U_k^* + 1) - \frac{\theta_k}{2} - 2M_k^* = \frac{1}{2} [(L_k(t_k) - \lambda_T(U_k^* + 1) - 2M_k^*) + \lambda_T].$$

Finally, if we define

$$I_k := \int_0^T L_k(s) ds = \int_0^T (\|u_{\mu_k}(s)\|_{H^1}^2 + \mu_k \|\partial_t u_{\mu_k}(s)\|_H^2) ds,$$

we have

$$I_k \geq \int_{t_k - \frac{\mu_k}{2\sigma_\infty^2} \theta_k}^{t_k} L_k(s) ds \geq \frac{\mu_k}{4\sigma_\infty^2} [(L_k(t_k) - \lambda_T(U_k^* + 1) - 2M_k^*)^2 - \lambda_T^2]$$

on  $E_k$ . Thus, by taking expectation on both sides, we get

$$(4.15) \quad \mathbb{E}(I_k) \geq \mathbb{E}(I_k; E_k) \geq \mathbb{E} \left( \frac{\mu_k}{4\sigma_\infty^2} (L_k(t_k) - \lambda_T(U_k^* + 1) - 2M_k^*)^2; E_k \right) - \frac{\mu_k \lambda_T^2}{4\sigma_\infty^2}.$$

By (4.12) and (4.14) we know

$$\lim_{k \rightarrow \infty} \sqrt{\mu_k} \mathbb{E} \theta_k = \infty.$$

Moreover,

$$(4.16) \quad \mathbb{E}(\sqrt{\mu_k} \theta_k) \leq \mathbb{E}(\sqrt{\mu_k} \theta_k; E_k) \leq \mathbb{E}(\sqrt{\mu_k} (\theta_k + \lambda_T); E_k) \leq [\mathbb{E}(\mu_k (\theta_k + \lambda_T)^2; E_k)]^{\frac{1}{2}}.$$

Combine (4.15) and (4.16); we have

$$\mathbb{E}(I_k) \geq \frac{1}{4\sigma_\infty^2} \mathbb{E}(\mu_k (\theta_k + \lambda_T)^2; E_k) - \frac{\mu_k \lambda_T^2}{4\sigma_\infty^2} \geq \frac{1}{4\sigma_\infty^2} (\sqrt{\mu_k} \mathbb{E} \theta_k)^2 - \frac{\mu_k \lambda_T^2}{4\sigma_\infty^2}.$$

This implies that  $\lim_{k \rightarrow \infty} \mathbb{E}(I_k) = \infty$  which contradicts (4.9) and (4.10). Therefore, (4.11) must be true, and the proof is complete.  $\square$

**5. Tightness.** For every  $\mu > 0$  and  $T > 0$ , we shall define

$$\rho_\mu(t) = g(u_\mu(t)), \quad t \in [0, T].$$

In this section we study the tightness of the family of measures  $(\mathcal{L}(\rho_{\mu_k}))_{k \in \mathbb{N}}$ , for any sequence  $(\mu_k)_{k \in \mathbb{N}}$  converging to zero. According to Assumption 2 and the definition of  $g$ , we know that

$$|g(r)| \leq \gamma_1 |r|, \quad |g'(r)| \leq \gamma_1, \quad r \in \mathbb{R}.$$

Therefore, for every  $\mu > 0$  and  $t \in [0, T]$ ,

$$\|\rho_\mu(t)\|_H \leq \gamma_1 \|u_\mu(t)\|_H, \quad \|\rho_\mu(t)\|_{H^1} \leq \gamma_1 \|u_\mu(t)\|_{H^1}.$$

As a consequence of (4.10) and (4.11), this implies that there exist  $c_T, \mu_T > 0$  such that

$$(5.1) \quad \mathbb{E} \sup_{t \in [0, T]} (\|\rho_\mu(t)\|_H^2 + \sqrt{\mu} \|\rho_\mu(t)\|_{H^1}^2) + \mathbb{E} \int_0^T \|\rho_\mu(s)\|_{H^1}^2 ds \leq c_T, \quad \mu \in (0, \mu_T).$$

Since  $g(r)$  is a strictly increasing function, it is invertible, and for every  $\mu > 0$ ,

$$u_\mu(t) = g^{-1}(\rho_\mu(t)), \quad t \in [0, T].$$

This implies that

$$\Delta u_\mu(t) = \operatorname{div}[\nabla g^{-1}(\rho_\mu(t))] = \operatorname{div}\left[\frac{1}{\gamma(g^{-1}(\rho_\mu(t)))} \nabla \rho_\mu(t)\right].$$

Moreover, by the definition of  $\rho_\mu$ , we have that

$$(5.2) \quad \nabla \rho_\mu(t) = \gamma(u_\mu(t)) \nabla u_\mu(t), \quad \partial_t \rho_\mu(t) = \gamma(u_\mu(t)) \partial_t u_\mu(t).$$

Therefore, if we define

$$(5.3) \quad b(r) := \frac{1}{\gamma(g^{-1}(r))}, \quad F(r) := f(g^{-1}(r)), \quad r \in \mathbb{R}$$

and

$$(5.4) \quad \sigma_g(h) := \sigma(g^{-1} \circ h), \quad h \in H,$$

we can rewrite equation (3.1) into the following form:

$$(5.5) \quad \begin{aligned} \rho_\mu(t) + \mu \partial_t u_\mu(t) &= g(u_0) + \mu v_0 + \int_0^t \operatorname{div}[b(\rho_\mu(s)) \nabla \rho_\mu(s)] ds \\ &\quad + \int_0^t F(\rho_\mu(s)) ds + \int_0^t \sigma_g(\rho_\mu(s)) dw^Q(s) \end{aligned}$$

in space  $H^{-1}$ . From Assumption 2 we have

$$(5.6) \quad 0 < \frac{1}{\gamma_1} \leq b(r) \leq \frac{1}{\gamma_0}, \quad r \in \mathbb{R}.$$

In what follows, we shall define

$$(5.7) \quad X_1 := C\left([0, T]; \bigcap_{\delta > 0} H^{-\delta}\right), \quad X_2 := \bigcap_{p < \infty} L^p(0, T; H).$$

Both spaces turn out to be complete and separable metric spaces, endowed with the distances

$$(5.8) \quad d_{X_1}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} (|x - y|_{C([0, T]; H^{-\frac{1}{n}})} \wedge 1)$$

and

$$(5.9) \quad d_{X_2}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} (|x - y|_{L^n(0, T; H)} \wedge 1).$$

Notice that both  $X_1$  and  $X_2$  contain  $L^\infty(0, T, H)$  with proper inclusion.

**THEOREM 5.1.** *Assume that Assumptions 1, 2 and 3 are satisfied, and fix any initial datum  $(u_0, v_0) \in \mathcal{H}_1$  and any  $T > 0$ . Then, for any sequence  $(\mu_k)_{k \in \mathbb{N}}$  converging to zero, the family of probability measures*

$$(\mathcal{L}(g(u_{\mu_k})))_{k \in \mathbb{N}} \subset \mathcal{P}(X_1 \cap X_2)$$

*is tight.*

**PROOF.** For every  $\theta \in (0, 1)$ , let  $C^\theta([0, T]; H^{-1})$  denote the space of  $\theta$ -Hölder continuous functions, defined on  $[0, T]$ , with values in  $H^{-1}$ . As a first step, we prove that there exists some  $\theta \in (0, 1)$  such that the family

$$(\rho_\mu + \mu \partial_t u_\mu)_{\mu \in (0, \mu_T)} \subset L^1(\Omega; C^\theta([0, T]; H^{-1}))$$

is bounded. For the first integral term in (5.5), given any  $0 \leq t_1 < t_2 \leq T$ , by (5.1) and (5.6) we have

$$\begin{aligned} \mathbb{E} \int_{t_1}^{t_2} \|\operatorname{div}[b(\rho_\mu(s)) \nabla \rho_\mu(s)]\|_{H^{-1}} ds &\leq c \mathbb{E} \int_{t_1}^{t_2} \|b(\rho_\mu(s)) \nabla \rho_\mu(s)\|_H ds \\ (5.10) \qquad \qquad \qquad &\leq \frac{c}{\gamma_0} (t_2 - t_1)^{\frac{1}{2}} \left( \int_0^T \mathbb{E} \|\rho_\mu(s)\|_{H^1}^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\gamma_0} (t_2 - t_1)^{\frac{1}{2}}. \end{aligned}$$

For the second integral term in (5.5), thanks to (4.10), we have

$$\begin{aligned} \mathbb{E} \int_{t_1}^{t_2} \|F(\rho_\mu(s))\|_{H^{-1}} ds &= \mathbb{E} \int_{t_1}^{t_2} \|f(u_\mu(s))\|_{H^{-1}} ds \\ (5.11) \qquad \qquad \qquad &\leq c \int_{t_1}^{t_2} (\mathbb{E} \|u_\mu(s)\|_H + 1) ds \leq c(t_2 - t_1). \end{aligned}$$

Finally, due to the boundedness of  $\sigma_g$ , by proceeding as in [12], Theorem 5.11 and Theorem 5.15, and by using a factorization argument, we have that

$$\sup_{\mu > 0} \mathbb{E} \left\| \int_0^\cdot \sigma(u_\mu(s)) dw^Q(s) \right\|_{C^\theta(0, T; H^{-1})} < \infty$$

for any  $\theta \in (0, 1/2)$ . Therefore, by putting this together with (5.10) and (5.11), from (5.5) we can conclude that, for any  $\theta \in (0, 1/2)$ ,

$$(5.12) \qquad \sup_{\mu \in (0, \mu_T)} \mathbb{E} \|\rho_\mu + \mu \partial_t u_\mu\|_{C^\theta([0, T]; H^{-1})} < \infty.$$

Moreover, thanks to estimates (4.11) and (5.1), we have

$$(5.13) \qquad \sup_{\mu \in (0, \mu_T)} \mathbb{E} \|\rho_\mu + \mu \partial_t u_\mu\|_{C([0, T]; H)} < \infty.$$

Now, due to (5.12) and (5.13), for any  $\epsilon > 0$  there exist two constants  $L_1^\epsilon, L_2^\epsilon > 0$  such that, if we define

$$K_1^\epsilon = \{f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{C^\theta([0, T]; H^{-1})} \leq L_1^\epsilon\}$$

and

$$K_2^\epsilon = \{f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{C([0, T]; H)} \leq L_2^\epsilon\},$$

then

$$(5.14) \quad \inf_{\mu \in (0, \mu_T)} \mathbb{P}(\rho_\mu + \mu \partial_t u_\mu \in K_1^\epsilon \cap K_2^\epsilon) > 1 - \frac{\epsilon}{3}.$$

By the compact embedding of  $H$  into  $H^{-\delta}$ , we know that  $K_1^\epsilon \cap K_2^\epsilon$  is relatively compact in  $C([0, T]; H^{-\delta})$  for every  $\delta > 0$  (for a proof, see [30], Theorem 5). Therefore,  $K_1^\epsilon \cap K_2^\epsilon$  is relatively compact in  $X_1$ .

In Proposition 4.4 we have shown that

$$\lim_{\mu \rightarrow 0} \mathbb{E} \|\mu \partial_t u_\mu\|_{C([0, T]; H)}^2 = 0.$$

Hence, for every sequence  $(\mu_k)_{k \in \mathbb{N}} \subset (0, \mu_T)$  converging to zero, there is a compact set  $K_3^\epsilon$  in  $C([0, T]; H)$  such that

$$(5.15) \quad \mathbb{P}(-\mu_k \partial_t u_{\mu_k} \in K_3^\epsilon) > 1 - \frac{\epsilon}{6}, \quad k \in \mathbb{N}.$$

Since  $C([0, T]; H) \subset X_1$ ,  $K_3^\epsilon$  is also compact in  $X_1$ . Then,  $(K_1^\epsilon \cap K_2^\epsilon) + K_3^\epsilon$  is relatively compact in  $X_1$ , and, thanks to (5.14) and (5.15), for every  $k \in \mathbb{N}$ ,

$$(5.16) \quad \mathbb{P}(\rho_{\mu_k} \in (K_1^\epsilon \cap K_2^\epsilon) + K_3^\epsilon) \geq \mathbb{P}(\rho_{\mu_k} + \mu_k \partial_t u_{\mu_k} \in K_1^\epsilon \cap K_2^\epsilon, -\mu_k \partial_t u_{\mu_k} \in K_3^\epsilon) > 1 - \frac{\epsilon}{2}.$$

By the arbitrariness of  $\epsilon > 0$ , this means that the family of probability measures  $(\mathcal{L}(\rho_{\mu_k}))_{k \in \mathbb{N}}$  is tight in  $X_1$ .

Now, due to the characterization given in [30], Theorem 1, for compact sets in  $C([0, T]; H^{-\delta})$ , if for every  $h \in (0, T)$  we define

$$\tau_h f(t) = f(t + h), \quad t \in [-h, T - h],$$

we have

$$(5.17) \quad \lim_{h \rightarrow 0} \sup_{f \in (K_1^\epsilon \cap K_2^\epsilon) + K_3^\epsilon} \|\tau_h f - f\|_{C([0, T-h]; H^{-\delta})} = 0, \quad \delta > 0.$$

Next, due to (5.1), there exists  $L_4^\epsilon > 0$  such that if we define

$$K_4^\epsilon = \{f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{L^2(0, T; H^1)} \leq L_4^\epsilon\},$$

then

$$(5.18) \quad \inf_{\mu \in (0, \mu_T)} \mathbb{P}(\rho_\mu \in K_4^\epsilon) > 1 - \frac{\epsilon}{2}.$$

Thus, if we take

$$K^\epsilon := [(K_1^\epsilon \cap K_2^\epsilon) + K_3^\epsilon] \cap K_4^\epsilon,$$

from (5.16) and (5.18) we obtain

$$(5.19) \quad \inf_{k \in \mathbb{N}} \mathbb{P}(\rho_{\mu_k} \in K^\epsilon) > 1 - \epsilon.$$

Now, let us fix  $p \in (2, \infty)$ , and let us define

$$\delta_p = \frac{2}{p-2}, \quad \alpha_p = \frac{p-2}{p}.$$

It is immediate to check that

$$\|x\|_H \leq c_p \|x\|_{H^{-\delta_p}}^{\alpha_p} \|x\|_{H^1}^{1-\alpha_p}.$$

Due to (5.17), we have

$$\lim_{h \rightarrow 0} \sup_{f \in K^\epsilon} \|\tau_h f - f\|_{C([0, T-h]; H^{-\delta p})} = 0.$$

Moreover,  $K^\epsilon$  is bounded in  $L^2(0, T; H^1)$ . Then, since

$$\frac{\alpha_p}{\infty} + \frac{1 - \alpha_p}{2} = \frac{1}{p},$$

according to [30], Theorem 7, we have that  $K^\epsilon$  is relatively compact in  $L^p(0, T; H)$ . Due to the arbitrariness of  $p < \infty$ , we have that  $K^\epsilon$  is relatively compact in  $X_2$ . By the arbitrariness of  $\epsilon > 0$  and (5.19), this allows us to conclude that the family of probability measures  $(\mathcal{L}(\rho_{\mu_k}))_{k \in \mathbb{N}}$  is tight in  $X_2$ .  $\square$

**6. Uniqueness for the quasilinear parabolic equations.** In this section, we prove the uniqueness of solutions for the following quasilinear stochastic parabolic equation

$$(6.1) \quad \begin{cases} \partial_t \rho = \operatorname{div}[b(\rho) \nabla \rho] + F(\rho) + \sigma_g(\rho) dw^Q(t), & t > 0, x \in \mathcal{O}; \\ \rho(0, x) = g(u_0), & \rho(t, x) = 0, & x \in \partial \mathcal{O}, \end{cases}$$

where, we recall

$$b(r) = \frac{1}{\gamma(g^{-1}(r))}, \quad F(r) = (f \circ g^{-1})(r), \quad r \in \mathbb{R}$$

and

$$\sigma_g(h) = \sigma(g^{-1} \circ h), \quad h \in H.$$

Notice that, because of our assumptions on  $\gamma$  and  $f$ , the functions  $b$  and  $F$  are both globally Lipschitz continuous on  $\mathbb{R}$  and the mapping  $\sigma_g : H \rightarrow \mathcal{L}_2(H_Q, H)$  is bounded and Lipschitz continuous.

**DEFINITION 6.1.** An  $(\mathcal{F}_t)_{t \geq 0}$  adapted process  $\rho \in L^2(\Omega; C([0, T]; H^{-1})) \cap L^2(\Omega; L^2(0, T; H^1))$  is said to be a solution of equation (6.1) if, for every test function  $\psi \in C_0^\infty(\mathcal{O})$ ,

$$(6.2) \quad \begin{aligned} \langle \rho(t), \psi \rangle_H &= \langle g(u_0), \psi \rangle_H - \int_0^t \langle b(\rho(s)) \nabla \rho(s), \nabla \psi \rangle_H ds \\ &+ \int_0^t \langle F(\rho(s)), \psi \rangle_H ds + \int_0^t \langle \sigma_g(\rho(s)) dw^Q(s), \psi \rangle_H. \end{aligned}$$

**THEOREM 6.2.** Suppose Assumptions 1, 2 and 3 are satisfied. Then, there is, at most, one solution  $\rho \in L^2(\Omega; C([0, T]; H^{-1})) \cap L^2(\Omega; L^2(0, T; H^1))$  to equation (6.1).

**PROOF.** The proof is a slight modification of [20], Proof of Theorem 3.1, where Hofmanová and Zhang use a generalized Itô formula for the  $L^1$ -norm of solutions of the same class of stochastic quasilinear parabolic equations. In [20] the periodic boundary condition on the torus  $\mathbb{T}^n$  is considered, and this means that the authors can take the identity function on the torus as a test function. Since we are considering here Dirichlet boundary conditions, we have to use a different class of test functions.

Let  $(\varphi_n)_{n \in \mathbb{N}}$  be the sequence of functions constructed in [20], Proof of Theorem 3.1, which have bounded first and second order derivatives,

$$(6.3) \quad \varphi'_n(0) = 0, \quad |\varphi'_n(r)| \leq 1, \quad 0 \leq \varphi''_n(r) \leq \frac{2}{n|r|}, \quad r \in \mathbb{R},$$

and

$$(6.4) \quad \lim_{n \rightarrow \infty} \sup_{r \in \mathbb{R}} |\varphi_n(r) - |r|| = 0.$$

Now, suppose  $\rho_1, \rho_2 \in L^2(\Omega; C([0, T]; H^{-1})) \cap L^2(\Omega; L^2(0, T; H^1))$  are both solutions to (6.1). By the generalized Itô formula in Proposition A.1, for any test function  $\psi \in C_0^\infty(\mathcal{O})$ , we have

$$(6.5) \quad \langle \varphi_n(\rho_1(t) - \rho_2(t)), \psi \rangle_H =: \sum_{k=1}^5 I_{k,n}(t),$$

where

$$\begin{aligned} I_{1,n}(t) &:= \int_0^t \langle \varphi'_n(\rho_1(s) - \rho_2(s))(F(\rho_1(s)) - F(\rho_2(s))), \psi \rangle_H ds, \\ I_{2,n}(t) &:= - \int_0^t \langle \varphi''_n(\rho_1(s) - \rho_2(s))(\nabla \rho_1(s) - \nabla \rho_2(s)) \\ &\quad \cdot (b(\rho_1(s))\nabla \rho_1(s) - b(\rho_2(s))\nabla \rho_2(s)), \psi \rangle_H ds, \\ I_{3,n}(t) &:= - \int_0^t \langle \varphi'_n(\rho_1(s) - \rho_2(s))(b(\rho_1(s))\nabla \rho_1(s) - b(\rho_2(s))\nabla \rho_2(s)), \nabla \psi \rangle_H ds, \\ I_{4,n}(t) &:= \frac{1}{2} \int_0^t \left\langle \varphi''_n(\rho_1(s) - \rho_2(s)) \sum_{i=1}^\infty |[\sigma_g(\rho_1(s)) - \sigma_g(\rho_2(s))] \mathcal{Q}e_i|^2, \psi \right\rangle_H ds \end{aligned}$$

and

$$I_{5,n}(t) := \int_0^t \langle \varphi'_n(\rho_1(s) - \rho_2(s))[\sigma_g(\rho_1(s)) - \sigma_g(\rho_2(s))] dw^{\mathcal{Q}}(s), \psi \rangle_H.$$

By the boundedness of  $\varphi'_n$  and  $\varphi''_n$ , (6.5) is also valid for any  $\psi \in H^1 \cap C(\mathcal{O})$  with  $\psi = 0$  on  $\partial\mathcal{O}$  by approximation; that is, there exist  $\psi_n \in C_0^\infty(\mathcal{O})$  converging to  $\psi$  in both  $L^\infty$  and  $H^1$  norms. In particular, here we take the test function  $\psi$  to be positive superharmonic with nonpositive  $\Delta\psi \in L^2$ . Thanks to (6.3) and the Lipschitz continuity of  $F$ ,

$$I_{1,n}(t) \leq c \int_0^t \langle |\rho_1(s) - \rho_2(s)|, \psi \rangle_H ds.$$

For the second term, thanks to (5.6), (6.3) and the Lipschitz continuity of  $b$ ,

$$\begin{aligned} I_{2,n}(t) &= - \int_0^t \langle \varphi''_n(\rho_1(s) - \rho_2(s))b(\rho_1(s))(\nabla \rho_1(s) - \nabla \rho_2(s)) \cdot (\nabla \rho_1(s) - \nabla \rho_2(s)), \psi \rangle_H ds \\ &\quad - \int_0^t \langle \varphi''_n(\rho_1(s) - \rho_2(s))(b(\rho_1(s)) - b(\rho_2(s)))(\nabla \rho_1(s) - \nabla \rho_2(s)) \cdot \nabla \rho_2(s), \psi \rangle_H ds \\ &\leq \frac{c}{n} \int_0^t \langle |\nabla \rho_1(s) - \nabla \rho_2(s)| |\nabla \rho_2(s)|, \psi \rangle_H ds \\ &\leq \frac{c \|\psi\|_{L^\infty(\mathcal{O})}}{n} \int_0^t (\|\rho_1(s)\|_{H^1}^2 + \|\rho_2(s)\|_{H^1}^2) ds. \end{aligned}$$

For the third term and by the definition of  $b$ , we have  $b(\rho)\nabla\rho = \nabla g^{-1}(\rho)$ ; from which we have

$$\begin{aligned} I_{3,n}(t) &= - \int_0^t \langle \varphi'_n(\rho_1(s) - \rho_2(s))(\nabla g^{-1}(\rho_1(s)) - \nabla g^{-1}(\rho_2(s))), \nabla \psi \rangle_H ds \\ &= \int_0^t \langle \varphi''_n(\rho_1(s) - \rho_2(s))(\nabla \rho_1(s) - \nabla \rho_2(s))(g^{-1}(\rho_1(s)) - g^{-1}(\rho_2(s))), \nabla \psi \rangle_H ds \end{aligned}$$

$$+ \int_0^t \langle \varphi'_n(\rho_1(s) - \rho_2(s))(g^{-1}(\rho_1(s)) - g^{-1}(\rho_2(s))), \Delta\psi \rangle_H ds.$$

Thanks to (6.3) and the Lipschitz continuity of  $g^{-1}$ ,

$$|\varphi''_n(\rho_1(s) - \rho_2(s))(g^{-1}(\rho_1(s)) - g^{-1}(\rho_2(s)))| \leq c|\varphi''_n(\rho_1(s) - \rho_2(s))||\rho_1(s) - \rho_2(s)| \leq \frac{c}{n}.$$

Since  $\varphi'_n$  is increasing and  $\varphi'_n(0) = 0$ , we have  $\text{sign}\varphi'_n(r) = \text{sign}r$ . Then, as  $g^{-1}$  is also increasing, we have

$$\varphi'_n(r_1 - r_2)(g^{-1}(r_1) - g^{-1}(r_2)) \geq 0, \quad r_1, r_2 \in \mathbb{R}.$$

Together with  $\Delta\psi \leq 0$  on  $\mathcal{O}$ , for every  $t \in [0, T]$ , we have

$$\begin{aligned} I_{3,n}(t) &\leq \frac{c}{n} \int_0^t \langle |\nabla \rho_1(s) - \nabla \rho_2(s)|, |\nabla \psi| \rangle_H ds \\ &\leq \frac{cT \|\psi\|_{H^1}}{n} \int_0^t (\|\rho_1(s)\|_{H^1}^2 + \|\rho_2(s)\|_{H^1}^2) ds. \end{aligned}$$

For the fourth term and by Assumption 1, we have

$$\begin{aligned} I_{4,n}(t) &= \frac{1}{2} \int_0^t \left\langle \varphi''_n(\rho_1(s) - \rho_2(s)) \sum_{i=1}^{\infty} |\sigma_i(\cdot, g^{-1}(\rho_1(s))) - \sigma_i(\cdot, g^{-1}(\rho_2(s)))|^2, \psi \right\rangle_H ds \\ &\leq \frac{c}{2} \int_0^t \langle \varphi''_n(\rho_1(s) - \rho_2(s)) |\rho_1(s) - \rho_2(s)|^2, \psi \rangle_H ds \\ &\leq \frac{c \|\psi\|_H}{n} \int_0^t (\|\rho_1(s)\|_H + \|\rho_2(s)\|_H) ds \\ &\leq \frac{cT \|\psi\|_{H^1}}{n} \int_0^t (\|\rho_1(s)\|_{H^1}^2 + \|\rho_2(s)\|_{H^1}^2) ds. \end{aligned}$$

Therefore, we take the expectation of (6.5) and combine the estimates for  $I_{1,n}(t)$ ,  $I_{2,n}(t)$ ,  $I_{3,n}(t)$  and  $I_{4,n}(t)$  to obtain

$$\begin{aligned} \mathbb{E} \langle \varphi_n(\rho_1(t) - \rho_2(t)), \psi \rangle_H &\leq \frac{cT}{n} (\|\psi\|_{H^1} + \|\psi\|_{L^\infty(\mathcal{O})}) \int_0^t (\mathbb{E} \|\rho_1(s)\|_{H^1}^2 + \mathbb{E} \|\rho_2(s)\|_{H^1}^2) ds \\ &\quad + c \int_0^t \mathbb{E} \langle |\rho_1(s) - \rho_2(s)|, \psi \rangle_H ds. \end{aligned}$$

Now, we take the limit above, as  $n \rightarrow \infty$ , and we get

$$\mathbb{E} \langle |\rho_1(t) - \rho_2(t)|, \psi \rangle_H \leq c \int_0^t \mathbb{E} \langle |\rho_1(s) - \rho_2(s)|, \psi \rangle_H ds$$

which implies that

$$\langle |\rho_1(t) - \rho_2(t)|, \psi \rangle_H = 0, \quad \text{a.s. on } \Omega \times [0, T].$$

Since this is true for all positive superharmonic  $\psi \in C(\mathcal{O}) \cap H^1$  with zero boundary value and nonpositive  $\Delta\psi \in L^2$ , we have  $\rho_1 = \rho_2$ , and the uniqueness follows.  $\square$

**7. The convergence result.** Now, we are ready to prove the convergence of the solutions to (1.1) and to identify the limit as the unique solution of the quasilinear parabolic equation (1.6).



**THEOREM 7.1.** *Suppose Assumptions 1, 2 and 3 are satisfied and, for each  $\mu > 0$ , let  $(u_\mu, \partial_t u_\mu)$  denote the unique solution to equation (1.1) with the same initial condition  $(u_0, v_0) \in \mathcal{H}_1$ . Then, for every  $\delta > 0$  and  $p < \infty$ , and for every  $\eta > 0$ ,*

$$\lim_{\mu \rightarrow 0} \mathbb{P}(\|u_\mu - u\|_{C([0, T]; H^{-\delta})} + \|u_\mu - u\|_{L^p(0, T; H)} > \eta) = 0,$$

where  $u \in L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1))$  is the unique solution of equation (1.6) with initial datum  $u_0$ .

**REMARK 7.2.** Here, we only consider deterministic initial data  $(u_0, v_0) \in H^1(\mathcal{O}) \times L^2(\mathcal{O})$ , independent of  $\mu$ . Actually, it is easy to generalize our result to the cases of random initial data  $(u_\mu(0), \partial_t u_\mu(0)) \in H^1(\mathcal{O}) \times L^2(\mathcal{O})$ , depending on  $\mu$ , such that, for some  $u_0 \in H^1(\mathcal{O})$ ,

$$\lim_{\mu \rightarrow 0} (\mathbb{E}\|u_\mu(0) - u_0\|_{H^1(\mathcal{O})}^2 + \mu^2 \mathbb{E}\|\partial_t u_\mu(0)\|_{L^2(\mathcal{O})}^2) = 0.$$

**PROOF.** We recall that, in the previous section, we have introduced the two Polish spaces  $X_1$  and  $X_2$ , endowed with the distances  $d_1$  and  $d_2$ , defined in (5.8) and (5.9), respectively. Here, for every  $T > 0$ , we denote

$$\mathcal{K}_T := [X_1 \cap X_2]^2 \times [C([0, T]; H)]^2 \times C([0, T]; U),$$

where  $U$  is the Hilbert space containing  $H_Q$ , with Hilbert–Schmidt embedding (see (2.4)).

In Theorem 5.1 we have proved that, for any sequence  $(\mu_k)_{k \in \mathbb{N}}$  converging to zero, the sequence  $(\mathcal{L}(\rho_{\mu_k}, \mu_k \partial_t u_{\mu_k}))_{k \in \mathbb{N}}$  is tight in  $[X_1 \cap X_2] \times C([0, T]; H)$ . Hence, the Skorokhod theorem assures that, for any two sequences  $(\mu_k^1)_{k \in \mathbb{N}}$  and  $(\mu_k^2)_{k \in \mathbb{N}}$  converging to zero, there exist two subsequences, still denoted by  $(\mu_k^1)_{k \in \mathbb{N}}$  and  $(\mu_k^2)_{k \in \mathbb{N}}$ , a sequence of random variables

$$Y_k := ((\rho_k^1, \vartheta_k^1), (\rho_k^2, \vartheta_k^2), \hat{w}_k^Q), \quad k \in \mathbb{N}$$

in  $\mathcal{K}_T$ , and a random variable

$$Y := (\rho^1, \rho^2, \hat{w}^Q)$$

in  $[X_1 \cap X_2]^2 \times C([0, T]; U)$ ; all defined on some probability space  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ , such that

$$(7.1) \quad \mathcal{L}(Y_k) = \mathcal{L}((\rho_{\mu_k^1}^1, \mu_k^1 \partial_t u_{\mu_k^1}), (\rho_{\mu_k^2}^2, \mu_k^2 \partial_t u_{\mu_k^2}), w^Q), \quad k \in \mathbb{N},$$

and for  $i = 1, 2$

$$(7.2) \quad \lim_{k \rightarrow \infty} (\|\rho_k^i - \rho^i\|_{X_1} + \|\rho_k^i - \rho^i\|_{X_2} + \|\vartheta_k^i\|_{C([0, T]; H)} + \|\hat{w}_k^Q - \hat{w}^Q\|_{C([0, T]; U)}) = 0, \quad \hat{\mathbb{P}}\text{-a.s.}$$

Notice that, due to (4.10) and (7.2), we have

$$(7.3) \quad \rho^i \in L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1)), \quad i = 1, 2.$$

Next, a filtration  $(\hat{\mathcal{F}}_t)_{t \geq 0}$  is introduced in  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$  by taking the augmentation of the canonical filtration of  $(\rho^1, \rho^2, \hat{w}^Q)$ , generated by the restrictions of  $(\rho^1, \rho^2, \hat{w}^Q)$  to every interval  $[0, t]$ . Due to this construction,  $\hat{w}^Q$  is a  $(\hat{\mathcal{F}}_t)_{t \geq 0}$  Wiener process with covariance  $Q^*Q$  (for a proof, see [13], Lemma 4.8).

Now, if we show that  $\rho^1 = \rho^2$ , we have that  $\rho_\mu$  converges in probability to some  $\rho \in L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1))$ . Actually, as observed by Gyöngy and Krylov in [18], if  $E$  is

any Polish space equipped with the Borel  $\sigma$ -algebra, a sequence  $(\xi_n)_{n \in \mathbb{N}}$  of  $E$ -valued random variables converges in probability if and only if for every pair of subsequences  $(\xi_m)_{m \in \mathbb{N}}$  and  $(\xi_l)_{l \in \mathbb{N}}$  there exists an  $E^2$ -valued subsequence  $\eta_k := (\xi_{m(k)}, \xi_{l(k)})$  converging weakly to a random variable  $\eta$  supported on the diagonal  $\{(h, k) \in E^2 : h = k\}$ .

In order to show that  $\rho_1 = \rho_2$ , we prove that they are both a solution of equation (6.1), which has pathwise uniqueness due to Theorem 6.2. To this purpose, we use the general method introduced in [13].

Due to (7.1), both  $(\rho_k^1, \vartheta_k^1)$  and  $(\rho_k^2, \vartheta_k^2)$  satisfy equation (5.5) with  $w^Q$  replaced by  $\hat{w}_k^Q$ . Then, by first taking the scalar product in  $H$  of each term in (5.5) with an arbitrary but fixed  $\psi \in C_0^\infty(\mathcal{O})$  and then integrating by parts, we get

$$(7.4) \quad \begin{aligned} \langle \rho_k^i(t) + \vartheta_k^i(t), \psi \rangle_H &= \langle g(u_0) + \mu_k v_0, \psi \rangle_H - \int_0^t \langle b(\rho_k^i(s)) \nabla \rho_k^i(s), \nabla \psi \rangle_H ds \\ &\quad + \int_0^t \langle F(\rho_k^i(s)), \psi \rangle_H ds \\ &\quad + \int_0^t \langle \sigma_g(\rho_k^i(s)) d\hat{w}_k^Q(s), \psi \rangle_H, \quad i = 1, 2. \end{aligned}$$

Now, since  $g$  is invertible, we can define

$$u_k^i(t, x) = g^{-1}(\rho_k^i(t, x)), \quad u^i(t, x) = g^{-1}(\rho^i(t, x)), \quad (t, x) \in [0, T] \times \mathcal{O}.$$

Due to the Lipschitz continuity of  $g^{-1}$ , we have that  $u_k^i$  and  $u^i$  belong to  $L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1))$  and, in view of (7.2),

$$(7.5) \quad \lim_{k \rightarrow \infty} (\|u_k^i - u^i\|_{X_1} + \|u_k^i - u^i\|_{X_2}) = 0, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Moreover,

$$\nabla u_k^i(s) = b(\rho_k^i(s)) \nabla \rho_k^i(s), \quad \nabla u^i(s) = b(\rho^i(s)) \nabla \rho^i(s)$$

so that

$$\begin{aligned} &\int_0^t \langle b(\rho_k^i(s)) \nabla \rho_k^i(s), \nabla \psi \rangle_H ds - \int_0^t \langle b(\rho^i(s)) \nabla \rho^i(s), \nabla \psi \rangle_H ds \\ &= \int_0^t \langle \nabla u_k^i(s), \nabla \psi \rangle_H ds - \int_0^t \langle \nabla u^i(s), \nabla \psi \rangle_H ds \\ &= - \int_0^t \langle (u_k^i(s) - u^i(s)), \Delta \psi \rangle_H ds. \end{aligned}$$

In particular, due to (7.5), we have that

$$(7.6) \quad \lim_{k \rightarrow \infty} \int_0^t \langle b(\rho_k^i(s)) \nabla \rho_k^i(s), \nabla \psi \rangle_H ds = \int_0^t \langle b(\rho^i(s)) \nabla \rho^i(s), \nabla \psi \rangle_H ds, \quad \hat{\mathbb{P}}\text{-a.s.}$$

Now, for  $i = 1, 2$  and  $t \in [0, T]$ , we define

$$\begin{aligned} \hat{M}^i(t) &= \langle \rho^i(t), \psi \rangle_H - \langle g(u_0), \psi \rangle_H + \int_0^t \langle b(\rho^i(s)) \nabla \rho_k^i(s), \nabla \psi \rangle_H ds \\ &\quad - \int_0^t \langle F(\rho^i(s)), \psi \rangle_H ds. \end{aligned}$$

By proceeding as in the proof of [13], Lemma 4.9, thanks to (7.2), (7.6) and the Lipschitz continuity of  $F$ , we have that, for every  $t \in [0, T]$ ,

$$\left\langle \hat{M}^i - \int_0^\cdot \langle \sigma_g(\rho^i(s)) d\hat{w}^Q(s), \psi \rangle_H \right\rangle_t = 0, \quad \mathbb{P}\text{-a.s.},$$

where  $\langle \cdot \rangle_t$  is the quadratic variation process. This implies that both  $\rho^1$  and  $\rho^2$  satisfy equation (6.1). Namely, for every  $\psi \in C_0^\infty(\mathcal{O})$  and  $i = 1, 2$ ,

$$\begin{aligned} \langle \rho^i(t), \psi \rangle_H &= \langle g(u_0), \psi \rangle_H - \int_0^t \langle b(\rho^i(s)) \nabla \rho^i(s), \nabla \psi \rangle_H ds \\ &\quad + \int_0^t \langle F(\rho^i(s)), \psi \rangle_H ds + \int_0^t \langle \sigma_g(\rho^i(s)) d\hat{w}^Q(s), \psi \rangle_H. \end{aligned}$$

As we have recalled above, thanks to the remark by Gyöngy–Krylov in [18], this implies that  $\rho_\mu$  converges in probability to some random variable  $\rho$  taking values in  $X_1 \cap X_2$ , as  $\mu$  goes to zero. Due to (7.3), we also have that  $\rho$  belongs to  $L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1))$  and satisfies equation (6.1).

Now, we set

$$u = g^{-1}(\rho).$$

Due to the Lipschitz continuity of  $g^{-1}$ , we have that  $u \in L^2(\Omega; X_1 \cap X_2 \cap L^2(0, T; H^1))$  and  $u_\mu$  converges in probability to  $u$  in  $X_1 \cap X_2$ , as  $\mu$  goes to zero. In order to conclude, we have to identify  $u$  with the solution of equation (1.6). We apply the generalized Itô formula, stated in Proposition A.1, to  $u := g^{-1}(\rho)$  with

$$\mathfrak{U} = H_Q, \quad J_i(t) = \sigma_g(\rho(t)) Q e_i, \quad i \in \mathbb{N}$$

and

$$F(t) = F(\rho(t)), \quad G(t) = b(\rho(t)) \nabla \rho(t).$$

Actually, since

$$(g^{-1})'(r) = \frac{1}{\gamma(g^{-1}(r))}, \quad (g^{-1})''(r) = -\frac{\gamma'(g^{-1}(r))}{\gamma(g^{-1}(r))^3}, \quad r \in \mathbb{R},$$

for any  $\psi \in C_0^\infty(\mathcal{O})$  we can conclude that

$$\begin{aligned} \langle u(t), \psi \rangle_H &= \langle u_0, \psi \rangle_H - \int_0^t \left\langle \frac{\nabla u(s)}{\gamma(u(s))}, \nabla \psi \right\rangle_H ds - \int_0^t \left\langle \nabla \left( \frac{1}{\gamma(u(s))} \right) \cdot \nabla u(s), \psi \right\rangle_H ds \\ &\quad + \int_0^t \left\langle \frac{f(u(s))}{\gamma(u(s))}, \psi \right\rangle_H ds - \int_0^t \left\langle \frac{\gamma'(u(s))}{2\gamma(u(s))^3} \sum_{i=1}^\infty (\sigma(u(s)) Q e_i)^2, \psi \right\rangle_H ds \\ &\quad + \int_0^t \left\langle \frac{\sigma(u(s))}{\gamma(u(s))} d\hat{w}^Q(s), \psi \right\rangle_H \end{aligned}$$

which means that  $u$  is a solution to (1.6).

In order to prove the uniqueness of the solution of equation (1.6), if  $u_1$  and  $u_2$  are two solutions, we apply Proposition A.1 to  $\rho_j = g(u_j)$ ,  $j = 1, 2$  with

$$\mathfrak{U} = H_Q, \quad J_i^j(t) = \frac{\sigma(u_j)}{\gamma(u_j)} Q e_i, \quad G^j(t) = \frac{\nabla u_j(t)}{\gamma(u_j(t))}, \quad i \in \mathbb{N}$$

and

$$\begin{aligned} F^j(t) &= \frac{f(u_j(t))}{\gamma(u_j(t))} - \gamma(u_j(t)) \nabla \left( \frac{1}{\gamma(u_j(t))} \right) \cdot \nabla u_j(t) \\ &\quad - \frac{\gamma'(u_j(t))}{2\gamma(u_j(t))^3} \sum_{i=1}^\infty (\sigma(u_j(t)) Q e_i)^2. \end{aligned}$$

Then, it turns out that both  $g(u_1)$  and  $g(u_2)$  are solutions to (6.1). Thus, by the uniqueness result in Theorem 6.2, we can conclude that  $g(u_1) = g(u_2)$ , and this implies that  $u_1 = u_2$ .  $\square$

## APPENDIX: A GENERALIZED ITÔ FORMULA

In [13] it proved a generalized Itô formula for the weak solutions of the following general class of equations:

$$(A.1) \quad du(t) = F(t) dt + \operatorname{div} G(t) dt + J(t) dw(t), \quad u_0 \in H,$$

where  $H = L^2(\mathbb{T}^d)$ ,  $d \geq 1$ . In the present paper we are dealing with Dirichlet boundary conditions in general bounded open sets  $\mathcal{O}$ . In what follows, we adapt the formulation of [13], Proposition A.1, to our situation, and we briefly describe the modification we have to do in the proof.

**PROPOSITION A.1.** *Let  $\psi \in C_0^\infty(\mathcal{O})$  and  $\varphi \in C^2(\mathbb{R})$  with bounded second-order derivative. Suppose  $W$  is a space-time white noise, that is,*

$$w(t) = \sum_{i=1}^{\infty} e_i \beta_i(t),$$

where  $(\beta_i)_{i \in \mathbb{N}}$  are mutually independent standard Wiener processes on the stochastic basis quadruple  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  and  $(e_i)_{i \in \mathbb{N}}$  is a complete orthonormal system in a separable Hilbert space  $\mathfrak{U}$ . Assume that  $F$  and  $G_j$  are adapted processes in  $L^2(\Omega; L^2(0, T; H))$ ,  $j = 1, \dots, d$  and  $J$  is an adapted process in  $L^2(\Omega; L^\infty(0, T; L_2(\mathfrak{U}; H)))$ . For every  $i \in \mathbb{N}$ , let  $J_i(t) := J(t)e_i$ . If the process

$$u \in L^2(\Omega; C([0, T]; H^{-1})) \cap L^2(\Omega; L^2(0, T; H^1))$$

solves (A.1) in  $H^{-1}$ , then almost surely, for all  $t \in [0, T]$ ,

$$(A.2) \quad \begin{aligned} \langle \varphi(u(t)), \psi \rangle_H &= \langle \varphi(u_0), \psi \rangle_H + \int_0^t \langle \varphi'(u(s))F(s), \psi \rangle_H ds \\ &\quad - \int_0^t \langle \varphi''(u(s))\nabla u(s) \cdot G(s), \psi \rangle_H ds \\ &\quad - \int_0^t \langle \varphi'(u(s))G(s), \nabla \psi \rangle_H ds + \frac{1}{2} \int_0^t \left\langle \varphi''(u(s)) \sum_{i=1}^{\infty} J_i^2(s), \psi \right\rangle_H ds \\ &\quad + \int_0^t \langle \varphi'(u(s))J(s)dw(s), \psi \rangle_H ds. \end{aligned}$$

Moreover, if we further assume that  $\varphi$  has bounded first-order derivative, the assumption on  $F$  could be relaxed to  $L^1(\Omega; L^1(0, T; L^1(\mathcal{O})))$ , and we still have (A.2) to be true.

**PROOF.** It is enough to prove the result for any smooth  $\psi$  with compact support in  $\mathcal{O}$ . Given a fixed  $\psi \in C_0^\infty(\mathcal{O})$ , suppose it is supported on the compact set  $K \subset \mathcal{O}$ , and let  $\delta_0 := d(K, \mathcal{O}^c) > 0$ . We fix a positive smooth function  $\xi$  supported on the unit ball with integral equals to 1 and define  $\xi_\delta(x) = \frac{1}{\delta^d} \xi(\frac{x}{\delta})$ . Then, if for any  $f \in H$ , we define  $f^\delta = f * \xi_\delta$ ; for  $\delta < \delta_0$ , we have

$$\|f^\delta\|_{L^2(K)} \leq \|f\|_H, \quad \|f^\delta - f\|_{L^2(K)} \rightarrow 0.$$

Now, we apply the mollifiers  $\xi_\delta$  to  $u(t)$ , and we have

$$u^\delta(t, x) = u_0^\delta(x) + \int_0^t F^\delta(s, x) ds + \int_0^t \operatorname{div} G^\delta(s, x) ds + \sum_{i=1}^{\infty} \int_0^t J_i^\delta(s, x) d\beta_i(s),$$

for all  $x \in K$ . Thus, we can apply the Itô formula to  $\varphi(u^\delta(t, x))\psi(x)$ , and, after we integrate in  $x$ , we get

$$\begin{aligned}
 & \langle \varphi(u^\delta(t)), \psi \rangle_H \\
 &= \langle \varphi(u_0^\delta), \psi \rangle_H + \int_0^t \langle \varphi'(u^\delta(s)) F^\delta(s), \psi \rangle_H ds + \int_0^t \langle \varphi'(u^\delta(s)) \operatorname{div} G^\delta(s), \psi \rangle_H ds \\
 & \quad + \frac{1}{2} \sum_{i=1}^\infty \int_0^t \langle \varphi''(u^\delta(s)) (J_i^\delta(s))^2, \psi \rangle_H ds + \sum_{i=1}^\infty \int_0^t \langle \varphi'(u^\delta(s)) J_i^\delta(s), \psi \rangle_H d\beta_i(s) \\
 (A.3) \quad &= \langle \varphi(u_0^\delta), \psi \rangle_H + \int_0^t \langle \varphi'(u^\delta(s)) F^\delta(s), \psi \rangle_H ds + \int_0^t \langle \operatorname{div}(\varphi'(u^\delta(s)) G^\delta(s)), \psi \rangle_H ds \\
 & \quad - \int_0^t \langle \varphi''(u^\delta(s)) \nabla u^\delta(s) \cdot G^\delta(s), \psi \rangle_H ds + \frac{1}{2} \sum_{i=1}^\infty \int_0^t \langle \varphi''(u^\delta(s)) (J_i^\delta(s))^2, \psi \rangle_H ds \\
 & \quad + \sum_{i=1}^\infty \int_0^t \langle \varphi'(u^\delta(s)) J_i^\delta(s), \psi \rangle_H d\beta_i(s).
 \end{aligned}$$

At this point, using the same argument as in [13], we can take the limit above, as  $\delta$  goes to zero, and obtain (A.2).  $\square$

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