Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 336 (2022) 1–43

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic behavior of solutions to a chemotaxis-logistic model with transitional end-states

Yanni Zeng [a,*], Kun Zhao [b]

[a] Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294, USA

[b] Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

Received 3 June 2022; revised 1 July 2022; accepted 10 July 2022

Abstract

We study Cauchy problem of a Keller-Segel type chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate. Our Cauchy data connect two different end-states for the chemical signal while the cell density takes its typical carrying capacity at the far fields. We are interested in the time-asymptotic behavior of the solution. We show that in the borderline, the component representing the chemical signal converges to a permanent, diffusive background wave, which connects the two end-states monotonically. On the other hand, the cell component converges to the spatial derivative of a heat kernel. The asymptotic solution has explicit formulation and is common to all solutions sharing the same end-states. Optimal L^2 and L^∞ convergence rates are obtained. We first convert the model into a 2×2 hyperbolic-parabolic system via inverse Hopf-Cole transformation. Then we apply Chapman-Enskog expansion to identify the asymptotic solution. After extracting the asymptotic solution, we use a variety of analytic tools to study the remainder and obtain optimal rates. These include time-weighted energy method, spectral analysis, Green's function estimate and iterations. Our results apply to a general class of Cauchy data for the model and for its transformed system. In particular, our results apply to large data solutions.

MSC: 35B40; 35Q92; 35M31

Keywords: Asymptotic behavior; Hyperbolic-parabolic system; Large data solution; Generic Cauchy data; Keller-Segel model; Logistic growth

* Corresponding author.
E-mail addresses: ynzeng@uab.edu (Y. Zeng), kzhao@tulane.edu (K. Zhao).

https://doi.org/10.1016/j.jde.2022.07.013
0022-0396/

1. Introduction

We consider Cauchy problem of a Keller-Segel type chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate:

$$\begin{cases} s_t = -\mu us - \sigma s, \\ u_t = Du_{xx} - \chi[u(\ln s)_x]_x + au(1 - \frac{u}{K}), \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$(s, u)(x, 0) = (s_0, u_0)(x), \quad x \in \mathbb{R}. \tag{1.2}$$

In (1.1) the unknown functions are $s = s(x, t)$ and $u = u(x, t)$ for the concentration of a chemical signal and density of a cellular population, respectively. We have assumed that the chemical signal is non-diffusive. Meanwhile, the system parameters have the following meaning:

- $\mu \neq 0$: coefficient of density-dependent production/consumption rate of chemical signal;
- $\sigma \geq 0$: natural degradation rate of chemical signal;
- $D > 0$: diffusion coefficient of cellular population;
- $\chi \neq 0$: coefficient of chemotactic sensitivity;
- $a > 0$: natural growth rate of cellular population;
- $K > 0$: typical carrying capacity of cellular population.

In (1.2) we assume $s_0 > 0$ and $u_0 \geq 0$. We are interested in the situation that the chemical signal has transitional end-states while the cellular population takes its typical carrying capacity at the far fields,

$$\lim_{x\to\pm\infty} (s_0, u_0)(x) = (s_\pm, K), \qquad s_\pm > 0. \tag{1.3}$$

Our goal is to study the time-asymptotic behavior of solutions to (1.1)-(1.3) under a very general set of hypotheses on the Cauchy data. In particular, we impose neither decay rates as $x \to \pm\infty$ nor smallness assumption on the data. We identify a time-asymptotic solution, and obtain optimal convergence rates towards it for the solution to (1.1)-(1.3).

It turns out that the s-component of the asymptotic solution is a monotonic curve connecting s_- to s_+ for each $t \geq 0$ while diffusive in the t-direction. Meanwhile, the u-component is the spatial derivative of a heat kernel atop of the carrying capacity. The asymptotic solution is uniquely determined by the end-states. That is, all Cauchy data satisfying (1.3) with the same $s_\pm$ give rise to solutions of (1.1)-(1.3) with the same asymptotic solution. Optimal convergence rates towards the asymptotic solution are then obtained in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$.

System (1.1) is an appended version of the Othmer-Stevens model [21,8]:

$$\begin{cases} s_t = -\mu us - \sigma s, \\ u_t = Du_{xx} - \chi[u(\ln s)_x]_x, \end{cases} \tag{1.4}$$

which interprets the dynamical behavior of chemotactic movement of random walkers that deposit non-diffusive or slow-moving chemical signals for succeeding passages to modify the local environment. Mathematical properties of (1.4), along with its companion with chemical diffusion, have been studied abundantly in recent years. These include, but are not limited to, global well-posedness [3,4], asymptotic stability of equilibria [1,9,10,12,20,23,24,26,35], existence and stability of traveling wave solutions [2,7,11,13–16,22,25], and vanishing chemical diffusivity limit [5,6,23,26].

One of the important features of (1.4) is the logarithmic sensitivity function, which is incorporated to serve as potential of the gradient flow driving the biased movement of cellular population, based on the assumption that the detection of chemical signal by cellular population follows the Weber-Fechner law. Though the logarithmic sensitivity has achieved success in biological science, its singular nature brings difficulties to mathematical analysis of the model. The key to resolve the issue is the inverse Hopf-Cole transformation [8]:

$$v = (\ln s)_x = \frac{s_x}{s}, \tag{1.5}$$

which has been extensively employed in qualitative analysis of (1.4). Under the new variables v and u, the reaction-diffusion-advection system (1.1) becomes a system of hyperbolic-parabolic balance laws:

$$\begin{cases} v_t + \mu u_x = 0, \\ u_t + \chi(uv)_x = Du_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{1.6}$$

Doing so allows us to utilize sophisticated analytic tools originally developed for hyperbolic-parabolic systems [17–19] and dissipative hyperbolic systems [27,31] for our current treatment of technical difficulties associated with $s_- \neq s_+$.

In this paper we assume

$$\chi\mu > 0. \tag{1.7}$$

This includes two scenarios: $\chi > 0$ and $\mu > 0$, or $\chi < 0$ and $\mu < 0$. The former is interpreted as cells are attracted to and consume the chemical. On the other hand, the latter describes cells depositing the chemical to modify the local environment for succeeding passages [21]. Mathematically, the non-diffusive, non-reactive part of (1.6) is hyperbolic in biologically relevant regimes when $\chi\mu > 0$, while it may change type when $\chi\mu < 0$ [32].

Under (1.7), we introduce rescaled and dimensionless variables:

$$\bar{t} = \frac{\chi\mu K}{D} t, \qquad \bar{x} = \frac{\sqrt{\chi\mu K}}{D} x, \qquad \bar{v} = \text{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}\, v, \qquad \bar{u} = \frac{u}{K}. \tag{1.8}$$

This simplifies (1.6) to

$$\begin{cases} \bar{v}_{\bar{t}} + \bar{u}_{\bar{x}} = 0, \\ \bar{u}_{\bar{t}} + (\bar{u}\bar{v})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}} + r\bar{u}(1 - \bar{u}), \end{cases} \qquad x \in \mathbb{R},\ t > 0. \tag{1.9}$$

Here the new parameter r is

$$r = \frac{aD}{\chi\mu K} > 0. \tag{1.10}$$

From (1.5) and (1.8), the corresponding Cauchy data for (1.9) are

$$(\bar{v}_0, \bar{u}_0)(\bar{x}) \equiv (\bar{v}, \bar{u})(\bar{x}, 0), \quad \bar{v}_0(\bar{x}) = \mathrm{sign}(\chi)\sqrt{\frac{\chi}{\mu K}\frac{s_0'(x(\bar{x}))}{s_0(x(\bar{x}))}}, \quad \bar{u}_0(\bar{x}) = \frac{1}{K}u_0(x(\bar{x})). \quad (1.11)$$

In particular, (1.11) implies

$$m_0 \equiv \int_{\mathbb{R}} \bar{v}_0(\bar{x})\mathrm{d}\bar{x} = \frac{\chi}{D}\ln\frac{s_+}{s_-}. \quad (1.12)$$

Thus, in the more biologically relevant situation when the chemical concentration experiences transition between two different end-states $s_- \neq s_+$, the initial mass of $\bar{v}$ is nonzero. Since $(1.9)_1$ is a conservation law, the mass of $\bar{v}(\bar{x}, \bar{t})$ is nonzero for all $\bar{t} \geq 0$ in this case.

Dropping the bar accent, we write the converted Cauchy problem as

$$\begin{cases} v_t + u_x = 0, \\ u_t + (uv)_x = u_{xx} + ru(1-u), \end{cases} \quad x \in \mathbb{R}, \ t > 0, \quad (1.13)$$

$$\begin{aligned} &(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}, \\ &\lim_{x \to \pm\infty} (v_0, u_0)(x) = (0, 1). \end{aligned} \quad (1.14)$$

Our strategy is to use hyperbolic theory (including Chapman-Enskog expansion from fluid dynamics) to study the converted problem (1.13), (1.14), and then translate the results via the transformations (1.5) and (1.8) into those of the original problem (1.1)-(1.3). The converted system is considered under generic perturbations of the constant equilibrium state $(0, 1)$. We do not impose zero-mass assumption on the variables. We do not make smallness assumption on perturbations either.

Mathematical study of (1.1) is recently initiated to understand the influence of logistic damping on the global dynamics of solutions to (1.4) through studying the transformed system (1.13). Cauchy problem (1.13), (1.14) has been studied in [32], where global well-posedness and asymptotic stability of the equilibrium state $(0, 1)$ are obtained. In particular, explicit time decay rates of the solution to the equilibrium are identified against general initial perturbation without smallness assumption, by using weighted energy method. This is one of the major discoveries of the enhanced dissipation induced by logistic damping, comparing to the non-growth model (1.4), as the same decay rates were previously established for (1.4) but under certain smallness assumption on the initial perturbation [9]. We stress that to obtain time decay rates by weighted energy method, however, the zero-mass assumption on v_0 is imposed in [32]. By (1.12), it is equivalent to considering the case $s_- = s_+$ there.

The decay rates in [32] are up to the capacity of weighted energy method but can be improved to the optimal ones by more sophisticated analytic tools such as a combination of spectral analysis, Green's function estimate and Duhamel's principle under an additional L^1 assumption on the data. This has been done in [33] (also see [34]). The optimal rates provide us a clear picture on how the solution of the original model (1.1), (1.2) behaves even in the borderline case of $\mu K + \sigma = 0$. In that case, $s(x, t)$ neither exponentially grows nor exponentially decays but algebraically decays to the background state $(\bar{s}, K)$, where $\bar{s} \equiv s_- = s_+$. This is another demonstration of the enhanced dissipation of logistic damping, since the same result has never been built for (1.4) (see Remark 1.2 in [9]).

The present paper is to remove the zero-mass assumption on v_0 in [32–34], and hence allowing transitional end-states $s_- \neq s_+$. For this we note that a key step in [32–34] is to utilize the antiderivative of v:

$$\psi(x,t) = \int_{-\infty}^{x} v(y,t)\mathrm{d}y. \tag{1.15}$$

Under the zero-mass assumption on v_0,

$$\int_{\mathbb{R}} v_0(x)\mathrm{d}x = 0, \tag{1.16}$$

and observing that the mass of $v(x,t)$ is a conserved quantity by $(1.13)_1$, we have $\psi(\pm\infty,t) = 0$. Therefore, we expect $\psi \in L^2(\mathbb{R})$ under appropriate assumptions on the initial data, and are able to perform energy estimate for it. Once we remove the restriction (1.16), $\psi(\infty,t)$ is the mass of $v(\cdot,t)$ or v_0 and hence nonzero. This means $\psi \notin L^2(\mathbb{R})$, and the analysis immediately breaks down.

A strategy to circumvent the obstacle is to extract a conserved quantity of the same mass from v and take the antiderivative of the difference instead. The conserved quantity needs to be constructed carefully as it is an asymptotic solution of v. The construction has been developed successfully for hyperbolic-parabolic systems [17–19] and dissipative hyperbolic systems [27, 31] in the spirit of Chapman-Enskog expansion, commonly used in fluid dynamics. In fact, the construction of such an asymptotic solution has been done in [28] for (1.13).

On the other hand, the strategy of constructing asymptotic solutions for hyperbolic-parabolic systems, dissipative hyperbolic systems and (1.13) with nonzero mass Cauchy data has worked only for small data solutions so far. In the present paper we extend such a Chapman-Enskog expansion based strategy to large data solutions. The key of success is to obtain the time-weighted energy estimates in Theorem 2.1 below.

Recently, the Cauchy problem with transitional end-states (1.1)-(1.3) has been considered in a very different scenario, where one of $s_\pm$ or both are zero. It is considered near a weak diffusive contact wave of (1.13), connecting two different end-states $v_\pm$ for v with $|v_+ - v_-| \ll 1$ [29]. The background wave for s is then special, the Hopf-Cole transformation of the weak diffusive contact wave of (1.13). Due to the difficulty of logarithmic singularity, s_0 needs to be a small perturbation of the background wave and u_0 a small perturbation of K. The perturbation in s_0 needs to stay away from the singularity as not to upset the exponential decay of s_0 to the zero end-state [30].

In the present paper we have $s_\pm > 0$ and hence there is no logarithmic singularity. The payoff is that our results cover a very general class of Cauchy data. As to be seen in Theorem 2.4, we only require $s_0 - s_- \in L^2((-\infty,0)) \cap L^1((-\infty,0))$, $s_0 - s_+ \in L^2((0,\infty)) \cap L^1((0,\infty))$, $s_0' \in H^2(\mathbb{R})$ and $u_0 - K \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$, besides $s_0 > 0$ and $u_0 \geq 0$ for physical relevance. There is no convergence rates of s_0 to $s_\pm$ or smallness assumption attached to (1.1)-(1.3) or (1.13), (1.14). In particular, by (1.12) a smallness assumption on the mass of $\bar{v}_0$ is equivalent to the closeness of s_+ and s_-. Without such an assumption, our results apply to data with arbitrary positive $s_\pm$.

The plan of the paper is as follows. In Section 2 we give needed preliminaries and identify a time-asymptotic solution. Then we state and comment on the main results. In Section 3 we prove Theorem 2.1 by time-weighted energy method. In Section 4 we iterate the decay rates in Theorem 2.1 by a different set of analytic tools to obtain optimal rates. That proves Theorem 2.2. In Section 5 we prove Theorem 2.4 concerning the original variables in (1.1)-(1.3).

2. Main results

We start with the transformed problem (1.13), (1.14). Our first step is to identify an asymptotic solution. We introduce a new variable for the perturbation in the u-component,

$$\tilde{u} = u - 1. \tag{2.1}$$

Then (1.13) becomes

$$\begin{cases} v_t + \tilde{u}_x = 0, \\ \tilde{u}_t + v_x + (\tilde{u}v)_x = \tilde{u}_{xx} - r\tilde{u}(\tilde{u}+1), \end{cases} \qquad x \in \mathbb{R}, \ t > 0. \tag{2.2}$$

We take expansion according to time decay rates (in the spirit of Chapman-Enskog expansion) for $(2.2)_2$. The leading terms give us

$$v_x \approx -r\tilde{u}. \tag{2.3}$$

Substituting (2.3) into $(2.2)_1$ we have

$$v_t \approx \frac{1}{r} v_{xx}. \tag{2.4}$$

Therefore, we define the v-component of the asymptotic solution as the self-similar solution θ of

$$\theta_t = \frac{1}{r}\theta_{xx}, \tag{2.5}$$

carrying the same mass of v. That is, $\theta(x,t)$ is a heat kernel,

$$\theta(x,t) = \frac{m_0}{\sqrt{4\pi(t+1)/r}} \exp\left\{-\frac{rx^2}{4(t+1)}\right\}, \tag{2.6}$$

where

$$m_0 = \int_{\mathbb{R}} v_0(x)\mathrm{d}x = \int_{\mathbb{R}} v(x,t)\mathrm{d}x, \tag{2.7}$$

noting the mass of v is a conserved quantity. From (2.3), the asymptotic solution for $\tilde{u}$ is given by $-\frac{1}{r}\theta_x$. Finally, the asymptotic profile for (v,u) is defined as $(\theta, 1 - \frac{1}{r}\theta_x)$.

Next, we set up a decomposition formula for the solution to (1.13), (1.14). Since $(2.2)_1$ and (2.5) are conservation laws, we have

$$\int_{\mathbb{R}} [v(x,t) - \theta(x,t)]\mathrm{d}x = \int_{\mathbb{R}} [v_0(x) - \theta(x,0)]\mathrm{d}x = m_0 - m_0 = 0.$$

This allows us to define a new variable

$$\phi(x,t) \equiv \int_{-\infty}^{x} [v(y,t) - \theta(y,t)]\mathrm{d}y, \tag{2.8}$$

which gives us

$$\phi_x(x,t) = v(x,t) - \theta(x,t) \tag{2.9}$$

and $\phi(\pm\infty, t) = 0$ for $t \geq 0$. For convenience we set

$$\phi_0(x) \equiv \phi(x,0) = \int_{-\infty}^{x} [v_0(y) - \theta(y,0)]\mathrm{d}y. \tag{2.10}$$

Let

$$\begin{aligned} \mathrm{V}(x,t) &\equiv \phi_x(x,t) = v(x,t) - \theta(x,t), \\ \mathrm{U}(x,t) &\equiv \tilde{u}(x,t) + \frac{1}{r}\theta_x(x,t) = u(x,t) - 1 + \frac{1}{r}\theta_x(x,t). \end{aligned} \tag{2.11}$$

Then we have the following decomposition for the solution to (1.13), (1.14),

$$\begin{aligned} v(x,t) &= \theta(x,t) + \mathrm{V}(x,t), \\ u(x,t) &= 1 - \frac{1}{r}\theta_x(x,t) + \mathrm{U}(x,t). \end{aligned} \tag{2.12}$$

It is worth mentioning that the main results of this paper are concerned with the explicit decay rates of large data classical solutions to (1.1)-(1.3) and (1.13), (1.14). The proofs are based upon the global well-posedness and long time behavior of the solutions, which have been established in [32] for $u_0 > 0$. However, it can be readily checked that by adapting the arguments in [9], the results of [32] can be produced when u_0 is not strictly positive, i.e., $u_0 \geq 0$ and $u_0 \not\equiv 0$. The idea is essentially to add a positive constant to u_0 and its background state to avoid zero value in the solution, then take the limit as the constant tends to zero. Since the adaptation can be made in a straightforward fashion, we shall not go through the technical details in this paper, but rather focus on deriving the explicit decay rates.

Our first theorem is on time-decay rates for the remainder (V, U) under the L^2 framework.

Theorem 2.1. *Suppose that the initial data satisfy $\phi_0 \in L^2(\mathbb{R})$, $v_0 \in H^2(\mathbb{R})$, $u_0 \geq 0$, and $u_0 - 1 \in H^2(\mathbb{R})$. Then there exists a unique solution to (1.13), (1.14) for all $t > 0$. The solution satisfies $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$, with the following decay properties:*

- $\|\phi(t)\|_{L^2}^2 + \sum_{k=1}^{3}(1+t)^k\|\partial_x^{k-1}\mathrm{V}(t)\|_{L^2}^2 + \sum_{k=0}^{2}\int_0^t (1+\tau)^k\|\partial_x^k\mathrm{V}(\tau)\|_{L^2}^2 \mathrm{d}\tau \le C,$
- $\sum_{k=0}^{1}(1+t)^{k+2}\|\partial_x^k\mathrm{U}(t)\|_{L^2}^2 + (1+t)^3\|\mathrm{U}_{xx}(t)\|_{L^2}^2 \le C,$ (2.13)
- $\int_0^t \Big[\sum_{k=0}^{2}(1+\tau)^{k+1}\|\partial_x^k\mathrm{U}(\tau)\|_{L^2}^2 + (1+\tau)^3\|\mathrm{U}_{xxx}(\tau)\|_{L^2}^2\Big]\mathrm{d}\tau \le C,$

where $C > 0$ is a constant.

Remark 2.1. We compare time-decay rates on the right-hand side of (2.12). From (2.13), V decays at the rates $(1+t)^{-\frac{1}{2}}$ and $(1+t)^{-\frac{3}{4}}$ in L^2 and L^∞, respectively. Here we have applied Sobolev inequality to obtain the L^∞ norm. The corresponding rates for θ are $(1+t)^{-\frac{1}{4}}$ and $(1+t)^{-\frac{1}{2}}$, respectively. Similarly, the rates for U are $(1+t)^{-1}$ and $(1+t)^{-\frac{5}{4}}$, which are compared with the rates $(1+t)^{-\frac{3}{4}}$ and $(1+t)^{-1}$ for θ_x. Therefore, Theorem 2.1 justifies $(\theta, 1-\frac{1}{r}\theta_x)$ as an asymptotic profile for the solution (v,u) to (1.13), (1.14). While representing the leading terms in (v,u) for large time, the asymptotic solution has explicit formulation. Theorem 2.1 implies that the solution (v,u) to (1.13), (1.14) converges to the constant equilibrium state $(0,1)$ at the same rates as $(\theta, -\frac{1}{r}\theta_x)$. They are $(1+t)^{-\frac{1}{4}}$ and $(1+t)^{-\frac{3}{4}}$ in L^2, and $(1+t)^{-\frac{1}{2}}$ and $(1+t)^{-1}$ in L^∞.

With L^1-L^2 initial data, the decay rates in Theorem 2.1 can be iterated to optimal ones. This is our next theorem.

Theorem 2.2. *Under the hypotheses of Theorem 2.1 and with the additional assumption $\phi_0 \in L^1(\mathbb{R})$ and $u_0 - 1 \in L^1(\mathbb{R})$, the unique solution to (1.13), (1.14) satisfies the following decay property:*

$$\begin{aligned}(1+t)^{\frac{1}{4}}\|\phi(t)\|_{L^2} + (1+t)^{\frac{3}{4}}\|\mathrm{V}(t)\|_{L^2} + (1+t)^{\frac{5}{4}}(\|\mathrm{V}_x(t)\|_{L^2} + \|\mathrm{U}(t)\|_{L^2}) \\ + (1+t)^{\frac{7}{4}}\|\mathrm{U}_x(t)\|_{L^2} \le C,\end{aligned} \tag{2.14}$$

where $C > 0$ is a constant.

Remark 2.2. The time-decay rates in L^∞ is a direct consequence of Sobolev inequality:

$$\|\phi(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}}, \quad \|\mathrm{V}(t)\|_{L^\infty} \le C(1+t)^{-1}, \quad \|\mathrm{U}(t)\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}}, \tag{2.15}$$

where $C > 0$ is a constant.

The decay rates are optimal in the sense that they are the best possible rates for generic Cauchy data. They are determined by the rates of the Green's function and its derivatives. These rates cannot be improved even when initial data (after subtracting the far field states) decay faster as $x \to \pm\infty$. For instance, in the case that initial data have compact support we still have the same rates. As determined by the corresponding rates of the Green's function, $\mathrm{V}(x,t)$ decays with the same rates as the first derivative of a heat kernel and the rates for $\mathrm{U}(x,t)$ are the same as those of V_x, or the second derivative of a heat kernel. These can be roughly seen from (2.11).

The rest of the section concerns the original variables (s,u), the solution to (1.1)-(1.3). We are particularly interested in the s-component of the asymptotic solution and the convergence of s towards it. Noting that the Cauchy problem (1.1)-(1.3) is related to the transformed one, (1.13), (1.14), by the inverse Hopf-Cole transformation (1.5) and the rescaling (1.8), we recover the bar accent for all variables, dependent and independent ones, for the transformed problem. That is, we use (1.9), (1.11) to replace (1.13), (1.14).

From (1.5) and (1.8) we have

$$s(x,t)=s(-\infty,t)\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)}\bar{v}(\bar{y},\bar{t}(t))\mathrm{d}\bar{y}\right).$$

Solving $(1.1)_1$ for $x\to-\infty$ and noting (1.3) give us

$$s(-\infty,t)=s_-e^{-(\mu K+\sigma)t}.$$

Therefore,

$$\begin{aligned}&s(x,t)=e^{-(\mu K+\sigma)t}\tilde{s}(x,t),\\&\tilde{s}(x,t)=s_-\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)}\bar{v}(\bar{y},\bar{t}(t))\mathrm{d}\bar{y}\right).\end{aligned}\tag{2.16}$$

In the critical case of $\mu K+\sigma=0$, $s(x,t)=\tilde{s}(x,t)$, which possesses an interesting wave pattern to be identified here. From $(2.12)_1$,

$$\bar{v}(\bar{x},\bar{t})=\theta(\bar{x},\bar{t})+\bar{\mathrm{V}}(\bar{x},\bar{t}),\tag{2.17}$$

where θ is the leading term in terms of time-decay. Thus, substituting (2.17) into (2.16) gives us

$$\tilde{s}(x,t)=s_-\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)}[\theta(\bar{y},\bar{t}(t))+\bar{\mathrm{V}}(\bar{y},\bar{t}(t))]\mathrm{d}\bar{y}\right).\tag{2.18}$$

The leading term in (2.18) with respect to time-decay is

$$\Theta(x,t)=s_-\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)}\theta(\bar{y},\bar{t}(t))\mathrm{d}\bar{y}\right).\tag{2.19}$$

Next we study properties of Θ, which is the s-component in the time asymptotic solution. Substituting (2.6) into (2.19) gives us

$$\Theta(x,t) = s_- \exp\left(\frac{D}{\chi} \frac{m_0}{\sqrt{4\pi(\bar{t}+1)/r}} \int_{-\infty}^{\bar{x}(x)} e^{-\frac{r\bar{y}^2}{4(\bar{t}+1)}} \mathrm{d}\bar{y} \right). \tag{2.20}$$

Here m_0 is the total mass of $\bar{v}_0$, given by (1.12). Thus, we substitute (1.12) into (2.20) and simplify to arrive at

$$\begin{aligned} \Theta(x,t) &= s_- \exp\left(\ln(\frac{s_+}{s_-}) \frac{1}{\sqrt{4\pi(\bar{t}+1)/r}} \int_{-\infty}^{\bar{x}(x)} e^{-\frac{r\bar{y}^2}{4(\bar{t}+1)}} \mathrm{d}\bar{y} \right) \\ &= s_- \exp\left(\ln(\frac{s_+}{s_-}) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\bar{x}(x)}{\sqrt{4(\bar{t}+1)/r}}} e^{-z^2} \mathrm{d}z \right) \\ &= s_- \exp\left(\frac{1}{2} \ln(\frac{s_+}{s_-}) \Big[1 + \mathrm{erf}\big(\frac{\bar{x}(x)}{\sqrt{4(\bar{t}(t)+1)/r}} \big) \Big] \right) > 0. \end{aligned} \tag{2.21}$$

It is clear that

$$\lim_{x\to\pm\infty} \Theta(x,t) = s_\pm.$$

It is also clear that

$$\Theta_x(x,t) = \Theta(x,t) \ln(\frac{s_+}{s_-}) e^{-\frac{r\bar{x}^2}{4(\bar{t}+1)}} \frac{1}{\sqrt{4\pi(\bar{t}+1)/r}} \frac{\sqrt{\chi\mu K}}{D}. \tag{2.22}$$

Thus, $\Theta_x(x,t) > 0$ if $s_+ > s_-$, and $\Theta_x(x,t) < 0$ if $s_+ < s_-$. This gives us the following proposition.

Proposition 2.3. *For each $t \geq 0$, $\Theta(x,t)$ monotonically connects s_- to s_+ on $\mathbb{R}$.*

Remark 2.3. As the background wave of $s(x,t)$, $\Theta(x,t)$ is permanent but diffusive in time. It is a function of $\bar{x}/\sqrt{\bar{t}+1}$ while $\bar{x}$ and $\bar{t}$ are scalings of x and t, respectively. It is interesting to observe that for $x \in \mathbb{R}$ fixed, $\Theta(x,t)$ approaches the geometric mean of $s_\pm$ time asymptotically,

$$\lim_{t\to\infty} \Theta(x,t) = s_- \exp\left(\frac{1}{2} \ln \frac{s_+}{s_-} \right) = \sqrt{s_- s_+} = \Theta(0,t).$$

Besides, $\Theta(x,t)$ is independent of Cauchy data sharing the same end-states $s_\pm$. That is, all those Cauchy problems have the same asymptotic solution.

The u-component is simpler. From (1.8) and (2.12) we have

$$u(x,t) = K\bar{u}(\bar{x}(x), \bar{t}(t)) = K[1 - \frac{1}{r}\theta_{\bar{x}}(\bar{x}(x), \bar{t}(t)) + \bar{\mathrm{U}}(\bar{x}(x), \bar{t}(t))]. \tag{2.23}$$

Thus, the u-component of the asymptotic solution against the background equilibrium state K is

$$\theta^*(x,t) \equiv -\frac{K}{r}\theta_{\bar{x}}(\bar{x}(x), \bar{t}(t)). \tag{2.24}$$

Our next theorem concerns the convergence of $(\tilde{s}, u)$ to $(\Theta, K + \theta^*)$.

Theorem 2.4. *Suppose that the initial data satisfy $s_0, s_\pm > 0$, $u_0 \geq 0$, $s_0 - s_- \in L^1((-\infty, 0)) \cap L^2((-\infty, 0))$, $s_0 - s_+ \in L^1((0,\infty)) \cap L^2((0,\infty))$, $s_0' \in H^2(\mathbb{R})$, and $u_0 - K \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a unique solution to (1.1)-(1.3) for all $t > 0$. The solution satisfies $s(x,t) > 0$, $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. With the decomposition,*

$$\begin{aligned} s(x,t) &= e^{-(\mu K+\sigma)t}[\Theta(x,t) + \mathcal{S}(x,t)], \\ u(x,t) &= K + \theta^*(x,t) + \mathcal{U}(x,t), \end{aligned} \tag{2.25}$$

the solution has the following decay properties,

$$\begin{aligned} &\|\partial_x^k \mathcal{S}(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}},\ 0 \leq k \leq 2; \quad \|\partial_x^k \mathcal{S}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}},\ k = 0, 1; \\ &\|\partial_x^k \mathcal{U}(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}},\ k = 0, 1; \qquad \|\mathcal{U}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}; \end{aligned} \tag{2.26}$$

where $C > 0$ is a constant.

Remark 2.4. From (2.25) we see that $s(x,t)$ exponentially grows if $\mu K + \sigma < 0$, and exponentially decays if $\mu K + \sigma > 0$. In the critical case of $\mu K + \sigma = 0$, which may happen in chemotactic repulsion, $s(x,t) = \tilde{s}(x,t)$ exhibits an interesting wave pattern. In this case, $s(x,t)$ time-asymptotically converges to a permanent, diffusive background wave $\Theta(x,t)$, which is common to all Cauchy solutions sharing the same end-states $s_\pm$. The convergence rates are $(1+t)^{-\frac{1}{4}}$ in L^2 and $(1+t)^{-\frac{1}{2}}$ in L^∞. On the other hand, $u(x,t)$ converges to the asymptotic solution $K + \theta^*(x,t)$ at the rates $(1+t)^{-\frac{5}{4}}$ in L^2 and $(1+t)^{-\frac{3}{2}}$ in L^∞. Since $\theta^*(x,t)$ is the spatial derivative of a heat kernel, $u(x,t)$ converges to the background, constant equilibrium state K at the rates $(1+t)^{-\frac{3}{4}}$ and $(1+t)^{-1}$ in L^2 and L^∞, respectively. These are faster than the convergence rates of $\tilde{s}(x,t)$ to $\Theta(x,t)$.

3. Proof of Theorem 2.1

In this section for convenience and without loss of generality we set $r = 1$. We consider (2.2), an equivalent form of (1.13),

$$\begin{cases} v_t = -\tilde{u}_x, \\ \tilde{u}_t = \tilde{u}_{xx} - (\tilde{u}v)_x - v_x - \tilde{u}(\tilde{u}+1), \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{3.1}$$

subject to the initial conditions:

$$(v, \tilde{u})(x, 0) = (v_0, \tilde{u}_0)(x) \equiv (v_0, u_0 - 1)(x), \quad x \in \mathbb{R}, \tag{3.2}$$

which satisfy $\tilde{u}_0(x) + 1 \geq 0$ for $x \in \mathbb{R}$. From (2.11) and (3.1) we can show that

$$\begin{cases} \mathrm{V}_t = -\mathrm{U}_x, \\ \mathrm{U}_t = \mathrm{U}_{xx} - \mathrm{U} - \mathrm{V}_x - [(\mathrm{U} - \theta_x)(\mathrm{V} + \theta)]_x - (\mathrm{U} - \theta_x)^2, \end{cases} \quad x \in \mathbb{R},\ t > 0. \tag{3.3}$$

3.1. Preliminaries

Before implementing time-weighted energy estimates, we need to establish the uniform temporal integrability of $\|\mathrm{V}(t)\|_{L^2}^2$, which is the foundation for the subsequent asymptotic analysis. For this purpose, we first collect the uniform *a priori* estimates of $(\tilde{u}, v)$, which are recorded in [32].

Lemma 3.1. *Under the conditions of Theorem 2.1, there exists a unique solution $(v, \tilde{u})$ to Cauchy problem (3.1), (3.2), such that $\tilde{u}(x,t) + 1 \geq 0$ for $x \in \mathbb{R}$, $t > 0$, and*

$$\|\tilde{u}(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \int_0^t \left(\|\tilde{u}(\tau)\|_{H^3}^2 + \|v_x(\tau)\|_{H^1}^2 + \|\tilde{u}_t(\tau)\|_{L^2}^2 + \|\tilde{v}_t(\tau)\|_{L^2}^2 \right) \mathrm{d}\tau \leq C,$$

where $C > 0$ is a constant. Moreover,

$$\lim_{t\to\infty} \left(\|\tilde{u}(t)\|_{H^2(\mathbb{R})} + \|v_x(t)\|_{H^1(\mathbb{R})} + \|\tilde{u}(t)\|_{C^1(\mathbb{R})} + \|v(t)\|_{C^1(\mathbb{R})} \right) = 0.$$

As a consequence of Lemma 3.1 and the properties of θ, we have

Lemma 3.2. *Under the conditions of Theorem 2.1, it holds that*

$$\|\mathrm{U}(t)\|_{H^2}^2 + \|\mathrm{V}(t)\|_{H^2}^2 + \int_0^t \left(\|\mathrm{U}(\tau)\|_{H^3}^2 + \|\mathrm{V}_x(\tau)\|_{H^1}^2 + \|\mathrm{U}_t(\tau)\|_{L^2}^2 + \|\mathrm{V}_t(\tau)\|_{L^2}^2 \right) \mathrm{d}\tau \leq C,$$

where $\mathrm{U} = \tilde{u} + \theta_x$, $\mathrm{V} = v - \theta$, and the constant is independent of t. Moreover,

$$\lim_{t\to\infty} \left(\|\mathrm{U}(t)\|_{H^2(\mathbb{R})} + \|\mathrm{V}_x(t)\|_{H^1(\mathbb{R})} + \|\mathrm{U}(t)\|_{C^1(\mathbb{R})} + \|\mathrm{V}(t)\|_{C^1(\mathbb{R})} \right) = 0.$$

The next lemma establishes the uniform temporal integrability of the zeroth frequency of the perturbed function $\mathrm{V} = \phi_x = v - \theta$.

Lemma 3.3. *Under the conditions of Theorem 2.1, there is a constant $C > 0$ such that*

$$\|\phi(t)\|_{L^2}^2 + \int_0^t \|\mathrm{V}(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C.$$

Proof. Integrating $(3.3)_1$ over $(-\infty, x)$ and using $(3.1)_2$ give us

$$\begin{aligned}\phi_t = -\mathrm{U} = -\tilde{u} - \theta_x = \tilde{u}_t - \tilde{u}_{xx} + (\tilde{u}v)_x + v_x + \tilde{u}^2 - \theta_x \\ = \tilde{u}_t - \tilde{u}_{xx} + (\tilde{u}v)_x + \phi_{xx} + \tilde{u}^2.\end{aligned} \tag{3.4}$$

Taking L^2 inner product of (3.4) with ϕ and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2 + \|\phi_x\|_{L^2}^2 &= \int_{\mathbb{R}} (\tilde{u}_x - \tilde{u}v)\phi_x \mathrm{d}x + \int_{\mathbb{R}} \tilde{u}^2\phi \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \tilde{u}\phi \mathrm{d}x\right) - \int_{\mathbb{R}} \tilde{u}\phi_t \mathrm{d}x \\ &= \int_{\mathbb{R}} (\tilde{u}_x - \tilde{u}v)\phi_x \mathrm{d}x + \int_{\mathbb{R}} \tilde{u}^2\phi \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \tilde{u}\phi \mathrm{d}x\right) + \int_{\mathbb{R}} \tilde{u}(\tilde{u}+\theta_x) \mathrm{d}x.\end{aligned}$$

After rearranging terms, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\phi\|_{L^2}^2 - \int_{\mathbb{R}} \tilde{u}\phi \mathrm{d}x\right) + \|\phi_x\|_{L^2}^2 = \underbrace{\int_{\mathbb{R}} (\tilde{u}_x - \tilde{u}v)\phi_x \mathrm{d}x}_{\equiv R_{0a}} + \underbrace{\int_{\mathbb{R}} \tilde{u}^2\phi \mathrm{d}x}_{\equiv R_{0b}} + \underbrace{\int_{\mathbb{R}} \tilde{u}(\tilde{u}+\theta_x) \mathrm{d}x}_{\equiv R_{0c}}. \tag{3.5}$$

Using Lemma 3.1 alongside Sobolev and Young inequalities, we can show that

$$\begin{aligned}|R_{0a}| &\le \frac{1}{4}\|\phi_x\|_{L^2}^2 + 2\|\tilde{u}_x\|_{L^2}^2 + 2\|v\|_{L^\infty}^2\|\tilde{u}\|_{L^2}^2 \\ &\le \frac{1}{4}\|\phi_x\|_{L^2}^2 + 2\|\tilde{u}_x\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}|R_{0b}| \le \|\phi\|_{L^\infty}\|\tilde{u}\|_{L^2}^2 &\le C\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi_x\|_{L^2}^{\frac{1}{2}}\|\tilde{u}\|_{L^2}^{\frac{3}{2}} \\ &\le \frac{1}{4}\|\phi_x\|_{L^2}^2 + C\|\phi\|_{L^2}^{\frac{2}{3}}\|\tilde{u}\|_{L^2}^2 \\ &\le \frac{1}{4}\|\phi_x\|_{L^2}^2 + \|\phi\|_{L^2}\|\tilde{u}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2.\end{aligned} \tag{3.7}$$

Using the expression of θ, we can show that

$$|R_{0c}| \le \frac{3}{2}\|\tilde{u}\|_{L^2}^2 + \frac{1}{2}\|\theta_x\|_{L^2}^2 \le \frac{3}{2}\|\tilde{u}\|_{L^2}^2 + C(1+t)^{-\frac{3}{2}}. \tag{3.8}$$

Substituting (3.6)-(3.8) into (3.5) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\phi\|_{L^2}^2 - \int_{\mathbb{R}} \tilde{u}\phi \mathrm{d}x\right) + \frac{1}{2}\|\phi_x\|_{L^2}^2 \le \|\phi\|_{L^2}\|\tilde{u}\|_{L^2}^2 + C\left[\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\right]. \tag{3.9}$$

Fix an arbitrary $T > 0$. For any $0 \le t \le T$, integrating (3.9) over $[0, t]$ yields

$$\frac{1}{2}\|\phi(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\phi_x\|_{L^2}^2 \mathrm{d}\tau \le \int_{\mathbb{R}} \tilde{u}(x,t)\phi(x,t)\mathrm{d}x + \int_0^T \|\phi\|_{L^2}\|\tilde{u}\|_{L^2}^2 \mathrm{d}t + C, \tag{3.10}$$

where we used Lemma 3.1, and the constant is independent of time. Note that by Lemma 3.1,

$$\int_{\mathbb{R}} \tilde{u}(x,t)\phi(x,t)\mathrm{d}x \le \frac{1}{4}\|\phi(t)\|_{L^2}^2 + \|\tilde{u}(t)\|_{L^2}^2 \le \frac{1}{4}\|\phi(t)\|_{L^2}^2 + C, \tag{3.11}$$

and

$$\int_0^T \|\phi\|_{L^2}\|\tilde{u}\|_{L^2}^2 \mathrm{d}t \le \big(\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}\big) \int_0^T \|\tilde{u}\|_{L^2}^2 \mathrm{d}t \le C\big(\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}\big), \tag{3.12}$$

where the constants are independent of time. Substituting (3.11) and (3.12) into (3.10) yields

$$\frac{1}{4}\|\phi(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\phi_x\|_{L^2}^2 \mathrm{d}t \le C\big(\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}\big) + C, \quad \forall\, t \in [0, T]. \tag{3.13}$$

Taking the supremum of the left-hand side of (3.13) gives us

$$\begin{aligned}\frac{1}{4}\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}^2 + \frac{1}{2}\int_0^T \|\phi_x\|_{L^2}^2 \mathrm{d}t &\le C\big(\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}\big) + C \\ &\le \frac{1}{8}\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}^2 + C,\end{aligned}$$

which implies

$$\frac{1}{8}\sup_{t\in[0,T]} \|\phi(t)\|_{L^2}^2 + \frac{1}{2}\int_0^T \|\phi_x\|_{L^2}^2 \mathrm{d}t \le C,$$

where the constant is independent of time. Since $T > 0$ is arbitrary, we conclude that

$$\|\phi(t)\|_{L^2}^2 + \int_0^t \|\phi_x(\tau)\|_{L^2}^2 \mathrm{d}\tau \le C,$$

where the constant is independent of time. This completes the proof of the lemma. □

3.2. Decay rate of zeroth frequency

Lemma 3.4. *Under the conditions of Theorem 2.1, there is a constant $t_1 > 0$ such that*

$$(1+t)\left(\|\mathrm{U}(t)\|_{L^2}^2 + \|\mathrm{V}(t)\|_{L^2}^2\right) + \int_{t_1}^t (1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2 + \|\mathrm{U}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \le C, \quad \forall\, t > t_1,$$

where the constant $C > 0$ is independent of $t > t_1$.

Proof. Taking L^2 inner product of $(3.3)_2$ with U, $(3.3)_1$ with V, then adding the results, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}\|_{L^2}^2 + \|\mathrm{V}\|_{L^2}^2\right) + \|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^2}^2 \\
&= \underbrace{\int_{\mathbb{R}} (\mathrm{U}-\theta_x)(\mathrm{V}+\theta)\mathrm{U}_x \mathrm{d}x}_{\equiv R_{1a}} - \underbrace{\int_{\mathbb{R}} (\mathrm{U}-\theta_x)^2 \mathrm{U}\mathrm{d}x}_{\equiv R_{1b}}.
\end{aligned} \tag{3.14}$$

By Hölder's inequality, we can show that

$$|R_{1a}| \le (\|\mathrm{V}\|_{L^\infty} + \|\theta\|_{L^\infty})\,\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} + \left(\|\theta_x\|_{L^\infty}\|\mathrm{V}\|_{L^2} + \|\theta\|_{L^\infty}\|\theta_x\|_{L^2}\right)\|\mathrm{U}_x\|_{L^2}, \tag{3.15}$$

and

$$|R_{1b}| \le (\|\mathrm{U}\|_{L^\infty} + 2\|\theta_x\|_{L^\infty})\,\|\mathrm{U}\|_{L^2}^2 + \|\theta_x\|_{L^\infty}\|\theta_x\|_{L^2}\|\mathrm{U}\|_{L^2}. \tag{3.16}$$

Since $\|(\mathrm{U},\mathrm{V},\theta,\theta_x)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, there is a constant $t_1 > 0$, such that for $t \ge t_1$,

$$(\|\mathrm{V}\|_{L^\infty} + \|\theta\|_{L^\infty})\,\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} \le \frac{1}{3}\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} \le \frac{1}{6}\|\mathrm{U}\|_{L^2}^2 + \frac{1}{6}\|\mathrm{U}_x\|_{L^2}^2, \tag{3.17}$$

and

$$(\|\mathrm{U}\|_{L^\infty} + 2\|\theta_x\|_{L^\infty})\,\|\mathrm{U}\|_{L^2}^2 \le \frac{1}{6}\|\mathrm{U}\|_{L^2}^2. \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.15) and (3.16), respectively, then taking the sum of the results and applying Cauchy inequality, we can show that

$$
\begin{aligned}
&|R_{1a}|+|R_{1b}| \\
&\le \frac{1}{2}\|\mathrm{U}\|_{L^2}^2+\frac{1}{2}\|\mathrm{U}_x\|_{L^2}^2+\frac{3}{2}\left(\|\theta_x\|_{L^\infty}^2\|\mathrm{V}\|_{L^2}^2+\|\theta\|_{L^\infty}^2\|\theta_x\|_{L^2}^2+\|\theta_x\|_{L^\infty}^2\|\theta_x\|_{L^2}^2\right), \quad t\ge t_1.
\end{aligned} \tag{3.19}
$$

Substituting (3.19) into (3.14) gives us

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}\|_{L^2}^2+\|\mathrm{V}\|_{L^2}^2\right)+\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}\|_{L^2}^2 \\
&\le 3\left(\|\theta_x\|_{L^\infty}^2\|\mathrm{V}\|_{L^2}^2+\|\theta\|_{L^\infty}^2\|\theta_x\|_{L^2}^2+\|\theta_x\|_{L^\infty}^2\|\theta_x\|_{L^2}^2\right), \quad t\ge t_1.
\end{aligned} \tag{3.20}
$$

Multiplying (3.20) by $(1+t)$, then integrating the result with respect to time, we have

$$
\begin{aligned}
&(1+t)\left(\|\mathrm{U}(t)\|_{L^2}^2+\|\mathrm{V}(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2+\|\mathrm{U}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \\
&\le 3\int_{t_1}^{t}(1+\tau)\big(\underbrace{\|\theta_x(\tau)\|_{L^\infty}^2\|\mathrm{V}(\tau)\|_{L^2}^2}_{\equiv R_{2a}}+\underbrace{\|\theta(\tau)\|_{L^\infty}^2\|\theta_x(\tau)\|_{L^2}^2}_{\equiv R_{2b}}+\underbrace{\|\theta_x(\tau)\|_{L^\infty}^2\|\theta_x(\tau)\|_{L^2}^2}_{\equiv R_{2c}}\big)\mathrm{d}\tau \\
&\quad+\int_{t_1}^{t}\left(\|\mathrm{U}(\tau)\|_{L^2}^2+\|\mathrm{V}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau+(1+t_1)\left(\|\mathrm{U}(t_1)\|_{L^2}^2+\|\mathrm{V}(t_1)\|_{L^2}^2\right).
\end{aligned} \tag{3.21}
$$

Note that

$$\|\theta(\tau)\|_{L^\infty}^2\le C(1+\tau)^{-1}, \tag{3.22}$$

$$\|\theta_x(\tau)\|_{L^\infty}^2\le C(1+\tau)^{-2}, \tag{3.23}$$

$$\|\theta_x(\tau)\|_{L^2}^2\le C(1+\tau)^{-\frac{3}{2}}, \tag{3.24}$$

which imply

$$R_{2a}\le C(1+\tau)^{-2}\|\mathrm{V}(\tau)\|_{L^2}^2, \tag{3.25}$$

$$R_{2b}\le C(1+\tau)^{-\frac{5}{2}}, \tag{3.26}$$

$$R_{2c}\le C(1+\tau)^{-\frac{7}{2}}. \tag{3.27}$$

Substituting (3.25)-(3.27) into (3.21) gives us

$$
(1+t)\left(\|\mathrm{U}(t)\|_{L^2}^2+\|\mathrm{V}(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2+\|\mathrm{U}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau
$$

$$\leq C\int_{t_1}^{t}\left[(1+\tau)^{-\frac{3}{2}}+(1+\tau)^{-\frac{5}{2}}\right]\mathrm{d}\tau+C\int_{t_1}^{t}\left(\|\mathrm{U}(\tau)\|_{L^2}^2+\|\mathrm{V}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau$$
$$+(1+t_1)\left(\|\mathrm{U}(t_1)\|_{L^2}^2+\|\mathrm{V}(t_1)\|_{L^2}^2\right). \tag{3.28}$$

In view of Lemma 3.2 and Lemma 3.3 we see that

$$\int_{0}^{t}\left(\|\mathrm{U}(\tau)\|_{L^2}^2+\|\mathrm{V}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\leq C,\quad \forall\, t>0, \tag{3.29}$$

where the constant is independent of t. Applying (3.29) to (3.28) yields

$$(1+t)\left(\|\mathrm{U}(t)\|_{L^2}^2+\|\mathrm{V}(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2+\|\mathrm{U}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\leq C,\quad \forall\, t>t_1, \tag{3.30}$$

where the constant is independent of $t>t_1$. This completes the proof of the lemma. □

3.3. Decay rate of first frequency

Lemma 3.5. *Under the conditions of Theorem 2.1, for the same constant $t_1>0$ as in Lemma 3.4, we have*

$$\int_{t_1}^{t}(1+\tau)\|\mathrm{V}_x(\tau)\|_{L^2}^2\mathrm{d}\tau\leq C,\quad \forall\, t>t_1,$$

and

$$(1+t)^2\left(\|\mathrm{U}_x(t)\|_{L^2}^2+\|\mathrm{V}_x(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)^2\left(\|\mathrm{U}_{xx}(\tau)\|_{L^2}^2+\|\mathrm{U}_x(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\leq C,$$
$$\forall\, t>t_1,$$

where the constants are independent of $t>t_1$.

Proof. We split the proof into two steps.

Step 1. We first establish the weighted temporal integrability of $\|\mathrm{V}_x\|_{L^2}^2$. Substituting $(3.3)_1$ into $(3.3)_2$ gives us

$$\mathrm{V}_{xt}=-\mathrm{U}_t-\mathrm{U}-\mathrm{V}_x-[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x-(\mathrm{U}-\theta_x)^2. \tag{3.31}$$

Taking L^2 inner product of (3.31) with V_x, we can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{V}_x\|_{L^2}^2 + \|\mathrm{V}_x\|_{L^2}^2 = -\int_{\mathbb{R}} \left\{\mathrm{U}_t + \mathrm{U} + [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\} \mathrm{V}_x \mathrm{d}x. \tag{3.32}$$

Note that by $(3.3)_1$,

$$\begin{aligned} -\int_{\mathbb{R}} \mathrm{U}_t \mathrm{V}_x \mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) + \int_{\mathbb{R}} \mathrm{U}\mathrm{V}_{xt}\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) - \int_{\mathbb{R}} \mathrm{U}\mathrm{U}_{xx}\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) + \|\mathrm{U}_x\|_{L^2}^2. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.32), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathrm{V}_x\|_{L^2}^2 + \int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) + \|\mathrm{V}_x\|_{L^2}^2 \\ &= -\int_{\mathbb{R}} \left\{\mathrm{U} + [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\} \mathrm{V}_x \mathrm{d}x + \|\mathrm{U}_x\|_{L^2}^2. \end{aligned} \tag{3.34}$$

The integral on the right-hand side of (3.34) is estimated as

$$\begin{aligned} &\left|\int_{\mathbb{R}} \left\{\mathrm{U} + [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\} \mathrm{V}_x \mathrm{d}x\right| \\ &\le C\Big[\underbrace{\left(\|\mathrm{V}\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2\right)\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2\right)}_{\equiv R_{3a}} + \underbrace{\left(\|\mathrm{U}\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\right)\left(\|\mathrm{V}_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2\right)}_{\equiv R_{3b}} \\ &\quad + \underbrace{\|\mathrm{U}\|_{L^\infty}^2\|\mathrm{U}\|_{L^2}^2}_{\equiv R_{3c}} + \underbrace{\|\theta_x\|_{L^\infty}^2\|\theta_x\|_{L^2}^2}_{\equiv R_{3d}}\Big] + \|\mathrm{U}\|_{L^2}^2 + \frac{1}{2}\|\mathrm{V}_x\|_{L^2}^2. \end{aligned} \tag{3.35}$$

Note that

$$\|\theta_{xx}(\tau)\|_{L^2}^2 \le C(1+\tau)^{-\frac{5}{2}}. \tag{3.36}$$

Using Sobolev inequality, (3.22), (3.30) and (3.36), we can show that

$$R_{3a} \le C\left[(1+t)^{-\frac{1}{2}}\|\mathrm{V}_x\|_{L^2} + (1+t)^{-1}\right]\left[\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{5}{2}}\right]. \tag{3.37}$$

Similarly, using (3.23) and (3.24), we can show that

$$R_{3b} \le C\left[\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} + (1+t)^{-2}\right]\left[\|\mathrm{V}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\right]. \tag{3.38}$$

Moreover, we can show that

$$R_{3c} \le C\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}\|_{L^2}^2, \tag{3.39}$$

and

$$R_{3d} \le C(1+t)^{-\frac{7}{2}}. \tag{3.40}$$

Substituting (3.37)-(3.40) into (3.35) yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left\{\mathrm{U} + [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\}\mathrm{V}_x \mathrm{d}x\right| \\
&\le C\Big[(1+t)^{-\frac{1}{2}}\|\mathrm{V}_x\|_{L^2}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-3}\|\mathrm{V}_x\|_{L^2} + (1+t)^{-1}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{7}{2}} \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{V}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} + (1+t)^{-2}\|\mathrm{V}_x\|_{L^2}^2 \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}\|_{L^2}^2\Big] + \|\mathrm{U}\|_{L^2}^2 + \frac{1}{2}\|\mathrm{V}_x\|_{L^2}^2.
\end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.34) gives us

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathrm{V}_x\|_{L^2}^2 + \int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) + \frac{1}{2}\|\mathrm{V}_x\|_{L^2}^2 \\
&\le C\Big[(1+t)^{-\frac{1}{2}}\|\mathrm{V}_x\|_{L^2}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-3}\|\mathrm{V}_x\|_{L^2} + (1+t)^{-1}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{7}{2}} \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{V}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} + (1+t)^{-2}\|\mathrm{V}_x\|_{L^2}^2 \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}\|_{L^2}^2\Big] + \|\mathrm{U}\|_{L^2}^2 + \|\mathrm{U}_x\|_{L^2}^2.
\end{aligned} \tag{3.42}$$

Note that according to Lemma 3.2, $\|\mathrm{V}_x\|_{L^2}$, $\|\mathrm{U}\|_{L^2}$ and $\|\mathrm{U}_x\|_{L^2}$ are uniformly bounded in time. Using such information, we update (3.42) as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{V}_x\|_{L^2}^2 + 2\int_{\mathbb{R}} \mathrm{U}\mathrm{V}_x \mathrm{d}x\right) + \|\mathrm{V}_x\|_{L^2}^2 \\
&\le C\left[\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^2}^2 + (1+t)^{-3} + (1+t)^{-1}\|\mathrm{V}_x\|_{L^2}^2\right].
\end{aligned} \tag{3.43}$$

Multiplying (3.43) by $(1+t)$, then integrating the result with respect to time, we have

$$(1+t)\left(\|\mathrm{V}_x(t)\|_{L^2}^2 + 2\int_{\mathbb{R}} \mathrm{U}(x,t)\mathrm{V}_x(x,t)\mathrm{d}x\right) + \int_{t_1}^{t}(1+\tau)\|\mathrm{V}_x(\tau)\|_{L^2}^2 \mathrm{d}\tau$$

$$
\leq C\int_{t_1}^{t}\left[(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2+\|\mathrm{U}(\tau)\|_{L^2}^2\right)+(1+\tau)^{-2}+\|\mathrm{V}_x(\tau)\|_{L^2}^2\right]\mathrm{d}\tau
$$

$$
+\int_{t_1}^{t}\left(\|\mathrm{V}_x(\tau)\|_{L^2}^2+2\int_{\mathbb{R}}\mathrm{U}(x,\tau)\mathrm{V}_x(x,\tau)\mathrm{d}x\right)\mathrm{d}\tau
$$

$$
+(1+t_1)\left(\|\mathrm{V}_x(t_1)\|_{L^2}^2+2\int_{\mathbb{R}}\mathrm{U}(x,t_1)\mathrm{V}_x(x,t_1)\mathrm{d}x\right). \tag{3.44}
$$

Note that according to (3.30) and Lemma 3.2, the first integral on the right-hand side of (3.44) is uniformly bounded in time. The second and third terms are also uniformly bounded, thanks to Cauchy's inequality and Lemma 3.2. Therefore, we obtain

$$
(1+t)\left(\|\mathrm{V}_x(t)\|_{L^2}^2+2\int_{\mathbb{R}}\mathrm{U}(x,t)\mathrm{V}_x(x,t)\mathrm{d}x\right)+\int_{t_1}^{t}(1+\tau)\|\mathrm{V}_x(\tau)\|_{L^2}^2\mathrm{d}\tau\leq C,\quad \forall\, t>t_1,
$$

where the constant is independent of $t>t_1$. Moreover, it can be shown that

$$
\begin{aligned}
(1+t)\|\mathrm{V}_x(t)\|_{L^2}^2+\int_{t_1}^{t}(1+\tau)\|\mathrm{V}_x(\tau)\|_{L^2}^2\mathrm{d}\tau &\leq C-2(1+t)\int_{\mathbb{R}}\mathrm{U}(x,t)\mathrm{V}_x(x,t)\mathrm{d}x\\
&\leq C+2(1+t)\|\mathrm{U}(t)\|_{L^2}^2+\frac{1}{2}(1+t)\|\mathrm{V}_x(t)\|_{L^2}^2,
\end{aligned}
$$

which yields

$$
\frac{1}{2}(1+t)\|\mathrm{V}_x(t)\|_{L^2}^2+\int_{t_1}^{t}(1+\tau)\|\mathrm{V}_x(\tau)\|_{L^2}^2\mathrm{d}\tau\leq C,\quad \forall\, t>t_1, \tag{3.45}
$$

where we applied (3.30).

Step 2. We now apply (3.45) to prove the decay rate of the first order spatial derivative of the perturbed functions. Taking L^2 inner products of $(3.3)_2$ with $-\mathrm{U}_{xx}$ and $(3.3)_1$ with $-\mathrm{V}_{xx}$, respectively, we can show that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{V}_x\|_{L^2}^2\right)+\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{U}_x\|_{L^2}^2\\
&=\int_{\mathbb{R}}\left\{[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x+(\mathrm{U}-\theta_x)^2\right\}\mathrm{U}_{xx}\mathrm{d}x.
\end{aligned} \tag{3.46}
$$

The integral on the right-hand side of (3.46) can be estimated by using the arguments between (3.35) and (3.41) as

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \left\{[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\} \mathrm{U}_{xx}\mathrm{d}x\right| \\
&\le C\big[(1+t)^{-\frac{1}{2}}\|\mathrm{V}_x\|_{L^2}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-3}\|\mathrm{V}_x\|_{L^2} + (1+t)^{-1}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{7}{2}} \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{V}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2} + (1+t)^{-2}\|\mathrm{V}_x\|_{L^2}^2 \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}\|_{L^2}^2\big] + \frac{1}{2}\|\mathrm{U}_{xx}\|_{L^2}^2. \qquad (3.47)
\end{aligned}
$$

Using (3.30) and (3.45), we update (3.47) as

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \left\{[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x + (\mathrm{U}-\theta_x)^2\right\} \mathrm{U}_{xx}\mathrm{d}x\right| \\
&\le C\big[(1+t)^{-1}\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^2}^2\right) + (1+t)^{-\frac{7}{2}} + (1+t)^{-2}\|\mathrm{V}_x\|_{L^2}^2\big] + \frac{1}{2}\|\mathrm{U}_{xx}\|_{L^2}^2. \qquad (3.48)
\end{aligned}
$$

Substituting (3.48) into (3.46) gives us

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{V}_x\|_{L^2}^2\right) + \|\mathrm{U}_{xx}\|_{L^2}^2 + \|\mathrm{U}_x\|_{L^2}^2 \\
&\le C\big[(1+t)^{-1}\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^2}^2\right) + (1+t)^{-\frac{7}{2}} + (1+t)^{-2}\|\mathrm{V}_x\|_{L^2}^2\big]. \qquad (3.49)
\end{aligned}
$$

Multiplying (3.49) by $(1+t)^2$, then integrating the result with respect to time, we can show that

$$
\begin{aligned}
&(1+t)^2\left(\|\mathrm{U}_x(t)\|_{L^2}^2 + \|\mathrm{V}_x(t)\|_{L^2}^2\right) + \int_{t_1}^{t}(1+\tau)^2\left(\|\mathrm{U}_{xx}\|_{L^2}^2 + \|\mathrm{U}_x\|_{L^2}^2\right)\mathrm{d}\tau \\
&\le C\int_{t_1}^{t}\big[(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2 + \|\mathrm{U}(\tau)\|_{L^2}^2\right) + (1+\tau)^{-\frac{3}{2}} + \|\mathrm{V}_x(\tau)\|_{L^2}^2\big]\mathrm{d}\tau \\
&\quad + 2\int_{t_1}^{t}(1+\tau)\left(\|\mathrm{U}_x(\tau)\|_{L^2}^2 + \|\mathrm{V}_x(\tau)\|_{L^2}^2\right)\mathrm{d}\tau + (1+t_1)^2\left(\|\mathrm{U}_x(t_1)\|_{L^2}^2 + \|\mathrm{V}_x(t_1)\|_{L^2}^2\right). \qquad (3.50)
\end{aligned}
$$

Note that according to (3.30) and Lemma 3.2, the first integral on the right-hand side of (3.50) is uniformly bounded in time. By (3.30) and (3.45), the second integral is uniformly bounded in time. Hence,

$$(1+t)^2\left(\|\mathrm{U}_x(t)\|_{L^2}^2+\|\mathrm{V}_x(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)^2\left(\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{U}_x\|_{L^2}^2\right)\mathrm{d}\tau\le C,\quad\forall\, t>t_1,\tag{3.51}$$

where the constant is independent of $t > t_1$. This completes the proof of the lemma. □

3.4. Decay rate of second frequency

Lemma 3.6. *Under the conditions of Theorem 2.1, for the same constant $t_1 > 0$ as in Lemma 3.4, we have*

$$\int_{t_1}^{t}(1+\tau)^2\|\mathrm{V}_{xx}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>t_1,$$

and

$$(1+t)^3\left(\|\mathrm{U}_{xx}(t)\|_{L^2}^2+\|\mathrm{V}_{xx}(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)^3\left(\|\mathrm{U}_{xxx}(\tau)\|_{L^2}^2+\|\mathrm{U}_{xx}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\le C,$$
$$\forall\, t>t_1,$$

where the constants are independent of $t > t_1$.

Proof. We prove the lemma by using the same strategy as in Lemma 3.5.

Step 1. Taking ∂_x of (3.31) gives us

$$\mathrm{V}_{xxt}=-\mathrm{U}_{xt}-\mathrm{U}_x-\mathrm{V}_{xx}-[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}-2(\mathrm{U}-\theta_x)(\mathrm{U}_x-\theta_{xx}).\tag{3.52}$$

Taking L^2 inner product of (3.52) with V_{xx}, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathrm{V}_{xx}\|_{L^2}^2+\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x\right)+\|\mathrm{V}_{xx}\|_{L^2}^2\\
&=-\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x-\int_{\mathbb{R}}[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}\mathrm{V}_{xx}\mathrm{d}x-2\int_{\mathbb{R}}(\mathrm{U}-\theta_x)(\mathrm{U}_x-\theta_{xx})\mathrm{V}_{xx}\mathrm{d}x+\|\mathrm{U}_{xx}\|_{L^2}^2\\
&=-\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x-\underbrace{\int_{\mathbb{R}}(\mathrm{U}_{xx}\mathrm{V}+2\mathrm{U}_x\mathrm{V}_x+\mathrm{U}\mathrm{V}_{xx}+2\mathrm{U}\mathrm{U}_x)\mathrm{V}_{xx}\mathrm{d}x}_{\equiv R_{4a}}\\
&\quad\underbrace{-\int_{\mathbb{R}}(\mathrm{U}_{xx}\theta-\mathrm{U}\theta_{xx}-\mathrm{V}_{xx}\theta_x-2\mathrm{V}_x\theta_{xx}-\mathrm{V}\theta_{xxx})\mathrm{V}_{xx}\mathrm{d}x}_{\equiv R_{4b}}
\end{aligned}$$

$$+\underbrace{\int_{\mathbb{R}} (\theta_x\theta_{xx}+\theta\theta_{xxx})\mathrm{V}_{xx}\mathrm{d}x}_{\equiv R_{4c}} + \|\mathrm{U}_{xx}\|_{L^2}^2. \tag{3.53}$$

The first term on the right-hand side of (3.53) is simply estimated as

$$\left|\int_{\mathbb{R}} \mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x\right| \le 2\|\mathrm{U}_x\|_{L^2}^2 + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2. \tag{3.54}$$

By Cauchy-Schwarz and Sobolev inequalities, we can show that

$$\begin{aligned}
|R_{4a}| &\le C\left(\|\mathrm{U}_{xx}\|_{L^2}^2\|\mathrm{V}\|_{L^\infty}^2 + \|\mathrm{U}_x\|_{L^\infty}^2\|\mathrm{V}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^\infty}^2\|\mathrm{V}_{xx}\|_{L^2}^2 + \|\mathrm{U}\|_{L^\infty}^2\|\mathrm{U}_x\|_{L^2}^2\right) \\
&\quad + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2 \\
&\le C\big(\|\mathrm{U}_{xx}\|_{L^2}^2\|\mathrm{V}\|_{L^2}\|\mathrm{V}_x\|_{L^2} + \|\mathrm{U}_x\|_{L^2}\|\mathrm{U}_{xx}\|_{L^2}\|\mathrm{V}_x\|_{L^2}^2 + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{V}_{xx}\|_{L^2}^2 \\
&\quad + \|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}_x\|_{L^2}^2\big) + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2.
\end{aligned} \tag{3.55}$$

The four triple products on the right-hand side of (3.55) are estimated by using (3.30) and (3.51) as

$$\|\mathrm{U}_{xx}\|_{L^2}^2\|\mathrm{V}\|_{L^2}\|\mathrm{V}_x\|_{L^2} \le C(1+t)^{-\frac{3}{2}}\|\mathrm{U}_{xx}\|_{L^2}^2, \tag{3.56}$$

$$\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}_{xx}\|_{L^2}\|\mathrm{V}_x\|_{L^2}^2 \le C(1+t)^{-2}\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}_{xx}\|_{L^2}^2\right), \tag{3.57}$$

$$\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{V}_{xx}\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}}\|\mathrm{V}_{xx}\|_{L^2}^2, \tag{3.58}$$

$$\|\mathrm{U}\|_{L^2}\|\mathrm{U}_x\|_{L^2}\|\mathrm{U}_x\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}}\|\mathrm{U}_x\|_{L^2}^2. \tag{3.59}$$

Substituting (3.56)-(3.59) into (3.55) gives us

$$|R_{4a}| \le C(1+t)^{-\frac{3}{2}}\left(\|\mathrm{U}_x\|_{L^2}^2 + \|\mathrm{U}_{xx}\|_{L^2}^2 + \|\mathrm{V}_{xx}\|_{L^2}^2\right) + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2. \tag{3.60}$$

In a similar fashion, by the properties of θ, we can show that

$$\begin{aligned}
|R_{4b}| &\le C(1+t)^{-1}\|\mathrm{U}_{xx}\|_{L^2}^2 + C(1+t)^{-2}\|\mathrm{V}_{xx}\|_{L^2}^2 \\
&\quad + C(1+t)^{-3}\left(\|\mathrm{U}\|_{L^2}^2 + \|\mathrm{V}_x\|_{L^2}^2\right) + C(1+t)^{-4}\|\mathrm{V}\|_{L^2}^2 + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2,
\end{aligned} \tag{3.61}$$

and

$$|R_{4c}| \le C(1+t)^{-\frac{9}{2}} + \frac{1}{8}\|\mathrm{V}_{xx}\|_{L^2}^2. \tag{3.62}$$

Substituting (3.54), (3.60)-(3.62) into (3.53) gives us

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathrm{V}_{xx}\|_{L^2}^2+\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x\right)+\frac{1}{2}\|\mathrm{V}_{xx}\|_{L^2}^2\\
&\le C(1+t)^{-\frac{3}{2}}\left(\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-1}\|\mathrm{U}_{xx}\|_{L^2}^2+C(1+t)^{-2}\|\mathrm{V}_{xx}\|_{L^2}^2+C(1+t)^{-3}\left(\|\mathrm{U}\|_{L^2}^2+\|\mathrm{V}_x\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-4}\|\mathrm{V}\|_{L^2}^2+C(1+t)^{-\frac{9}{2}}+2\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2.
\end{aligned}\tag{3.63}$$

Note that we haven't established any time-weighted estimate for $\|\mathrm{V}_{xx}\|_{L^2}^2$. However, similar to the derivation of (3.45) and based on (3.63), we can show that

$$(1+t)\|\mathrm{V}_{xx}(t)\|_{L^2}^2+\int_{t_1}^{t}(1+\tau)\|\mathrm{V}_{xx}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>t_1.\tag{3.64}$$

Next, we shall upgrade (3.64). Multiplying (3.63) by $(1+t)^2$, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^2\left(\frac{1}{2}\|\mathrm{V}_{xx}\|_{L^2}^2+\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x\right)\right]+\frac{1}{2}(1+t)^2\|\mathrm{V}_{xx}\|_{L^2}^2\\
&\le C(1+t)^{\frac{1}{2}}\left(\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)\\
&\quad+C(1+t)\|\mathrm{U}_{xx}\|_{L^2}^2+C\|\mathrm{V}_{xx}\|_{L^2}^2+C(1+t)^{-1}\left(\|\mathrm{U}\|_{L^2}^2+\|\mathrm{V}_x\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-2}\|\mathrm{V}\|_{L^2}^2+C(1+t)^{-\frac{5}{2}}+2(1+t)^2\|\mathrm{U}_x\|_{L^2}^2+(1+t)^2\|\mathrm{U}_{xx}\|_{L^2}^2\\
&\quad+(1+t)\left(\|\mathrm{V}_{xx}\|_{L^2}^2+2\int_{\mathbb{R}}\mathrm{U}_x\mathrm{V}_{xx}\mathrm{d}x\right).
\end{aligned}\tag{3.65}$$

The terms on the right-hand side of (3.65) are uniformly integrable with respect to t for $t>t_1$, thanks to Lemma 3.2–Lemma 3.5, and (3.64). This implies

$$(1+t)^2\|\mathrm{V}_{xx}(t)\|_{L^2}^2+\int_{t_1}^{t}(1+\tau)^2\|\mathrm{V}_{xx}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>t_1.\tag{3.66}$$

Step 2. Taking ∂_x of (3.3) gives us

$$\mathrm{U}_{xt}=\mathrm{U}_{xxx}-\mathrm{U}_x-\mathrm{V}_{xx}-[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}-2(\mathrm{U}-\theta_x)(\mathrm{U}_x-\theta_{xx}),\tag{3.67a}$$

$$\mathrm{V}_{xt}=-\mathrm{U}_{xx}.\tag{3.67b}$$

Taking L^2 inner products of (3.67a) with $-\mathrm{U}_{xxx}$ and (3.67b) with $-\mathrm{V}_{xxx}$, respectively, then adding the results, we can show that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)+\|\mathrm{U}_{xxx}\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2\\
&=\int_{\mathbb{R}}[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}\mathrm{U}_{xxx}\mathrm{d}x+2\int_{\mathbb{R}}(\mathrm{U}-\theta_x)(\mathrm{U}_x-\theta_{xx})\mathrm{U}_{xxx}\mathrm{d}x. \qquad (3.68)
\end{aligned}$$

Note that the two integrals on the right-hand side of (3.68) are similar to the second and third integrals in the second line of (3.53), respectively. Hence, by adapting the estimates of R_{4a}, R_{4b} and R_{4c}, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)+\|\mathrm{U}_{xxx}\|_{L^2}^2+2\|\mathrm{U}_{xx}\|_{L^2}^2\\
&\le C(1+t)^{-\frac{3}{2}}\left(\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-1}\|\mathrm{U}_{xx}\|_{L^2}^2+C(1+t)^{-2}\|\mathrm{V}_{xx}\|_{L^2}^2+C(1+t)^{-3}\left(\|\mathrm{U}\|_{L^2}^2+\|\mathrm{V}_x\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-4}\|\mathrm{V}\|_{L^2}^2+C(1+t)^{-\frac{9}{2}}. \qquad (3.69)
\end{aligned}$$

Multiplying (3.69) by $(1+t)^3$, we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^3\left(\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)\right]+(1+t)^3\|\mathrm{U}_{xxx}\|_{L^2}^2+2(1+t)^3\|\mathrm{U}_{xx}\|_{L^2}^2\\
&\le C(1+t)^{\frac{3}{2}}\left(\|\mathrm{U}_x\|_{L^2}^2+\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right)\\
&\quad+C(1+t)^2\|\mathrm{U}_{xx}\|_{L^2}^2+C(1+t)\|\mathrm{V}_{xx}\|_{L^2}^2+C\left(\|\mathrm{U}\|_{L^2}^2+\|\mathrm{V}_x\|_{L^2}^2\right)\\
&\quad+C(1+t)^{-1}\|\mathrm{V}\|_{L^2}^2+C(1+t)^{-\frac{3}{2}}+3(1+t)^2\left(\|\mathrm{U}_{xx}\|_{L^2}^2+\|\mathrm{V}_{xx}\|_{L^2}^2\right). \qquad (3.70)
\end{aligned}$$

Note that the right-hand side of (3.70) is uniformly integrable with respect to time for $t>t_1$, by virtue of Lemma 3.2–Lemma 3.5 and (3.66). This implies for $\forall\, t>t_1$,

$$(1+t)^3\left(\|\mathrm{U}_{xx}(t)\|_{L^2}^2+\|\mathrm{V}_{xx}(t)\|_{L^2}^2\right)+\int_{t_1}^{t}(1+\tau)^3\left(\|\mathrm{U}_{xxx}(\tau)\|_{L^2}^2+\|\mathrm{U}_{xx}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\le C, \qquad (3.71)$$

where the constant is independent of $t>t_1$. This completes the proof of the lemma. □

3.5. Improved decay rates

In this section, we improve the decay rates of the U-component.

Lemma 3.7. *Under the conditions of Theorem 2.1, there is a constant $t_2 \geq t_1$ such that*

$$(1+t)^2\|\mathrm{U}(t)\|_{L^2}^2 + (1+t)^3\|\mathrm{U}_x(t)\|_{L^2}^2 \leq C, \quad \forall\, t > t_2,$$

where the constant $C > 0$ is independent of $t > t_2$.

Proof. We prove the lemma by deriving the decay rates of the temporal derivative of the perturbed functions. Taking ∂_t of (3.3) gives us

$$\mathrm{U}_{tt} = \mathrm{U}_{xxt} - \mathrm{U}_t - \mathrm{V}_{xt} - [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xt} - 2(\mathrm{U}-\theta_x)(\mathrm{U}_t - \theta_{xt}), \tag{3.72a}$$

$$\mathrm{V}_{tt} = -\mathrm{U}_{xt}. \tag{3.72b}$$

Taking L^2 inner product of (3.72a) with U_t and (3.72b) with V_t, then adding the results, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_t\|_{L^2}^2 + \|\mathrm{V}_t\|_{L^2}^2\right) + \|\mathrm{U}_{xt}\|_{L^2}^2 + \|\mathrm{U}_t\|_{L^2}^2 \\ &= \underbrace{-\int_{\mathbb{R}} [(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xt}\mathrm{U}_t \mathrm{d}x}_{\equiv R_{5a}} \underbrace{-2\int_{\mathbb{R}} (\mathrm{U}-\theta_x)(\mathrm{U}_t-\theta_{xt})\mathrm{U}_t \mathrm{d}x}_{\equiv R_{5b}}. \end{aligned} \tag{3.73}$$

By a direct calculation, we can show that

$$\begin{aligned} R_{5a} &= \int_{\mathbb{R}} \big(\mathrm{U}_{xt}\mathrm{V} + \mathrm{U}_x\mathrm{V}_t + \mathrm{U}_t\mathrm{V}_x + \mathrm{U}\mathrm{V}_{xt} + \mathrm{U}_{tx}\theta + \mathrm{U}_x\theta_t + \mathrm{U}_t\theta_x + \mathrm{U}\theta_{xt} \\ &\qquad - \mathrm{V}_{xt}\theta_x - \mathrm{V}_x\theta_{xt} - \mathrm{V}_t\theta_{xx} - \mathrm{V}\theta_{xxt} - 2\theta_{xt}\theta_x - \theta_t\theta_{xx} - \theta\theta_{xxt}\big)\mathrm{U}_t \mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(-\mathrm{U}_x\mathrm{U}_x + \frac{1}{2}\mathrm{U}_t\mathrm{V}_x - \mathrm{U}\mathrm{U}_{xx} + \mathrm{U}_x\theta_t + \frac{1}{2}\mathrm{U}_t\theta_x + \mathrm{U}\theta_{xt} \\ &\qquad + \mathrm{U}_{xx}\theta_x - \mathrm{V}_x\theta_{xt} + \mathrm{U}_x\theta_{xx} - \mathrm{V}\theta_{xxt} - 2\theta_{xt}\theta_x - \theta_t\theta_{xx} - \theta\theta_{xxt}\Big)\mathrm{U}_t \mathrm{d}x, \end{aligned}$$

where we integrated by parts and replaced V_t by $-\mathrm{U}_x$. Using Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} |R_{5a}| &\leq \frac{1}{2}(\|\mathrm{V}_x\|_{L^\infty} + \|\theta_x\|_{L^\infty})\|\mathrm{U}_t\|_{L^2}^2 + \big(\|\mathrm{U}_x\|_{L^\infty}\|\mathrm{U}_x\|_{L^2} + \|\mathrm{U}\|_{L^\infty}\|\mathrm{U}_{xx}\|_{L^2} \\ &\quad + \|\mathrm{U}_x\|_{L^\infty}\|\theta_t\|_{L^2} + \|\mathrm{U}\|_{L^\infty}\|\theta_{xt}\|_{L^2} + \|\theta_x\|_{L^\infty}\|\mathrm{U}_{xx}\|_{L^2} + \|\mathrm{V}_x\|_{L^\infty}\|\theta_{xt}\|_{L^2} \\ &\quad + \|\mathrm{U}_x\|_{L^\infty}\|\theta_{xx}\|_{L^2} + \|\mathrm{V}\|_{L^\infty}\|\theta_{xxt}\|_{L^2} + \|\theta_x\|_{L^\infty}\|\theta_{xt}\|_{L^2} + \|\theta_t\|_{L^\infty}\|\theta_{xx}\|_{L^2} \\ &\quad + \|\theta\|_{L^\infty}\|\theta_{xxt}\|_{L^2}\big)\|\mathrm{U}_t\|_{L^2}. \end{aligned} \tag{3.74}$$

Using Lemma 3.4–Lemma 3.6 and Sobolev inequality, we can show that for $t > t_1$,

$$\begin{aligned} &\|\mathrm{U}\|_{L^\infty}^2 \le C(1+t)^{-\frac{3}{2}}; \qquad \|\mathrm{U}_x\|_{L^\infty}^2 \le C(1+t)^{-\frac{5}{2}};\\ &\|\mathrm{V}\|_{L^\infty}^2 \le C(1+t)^{-\frac{3}{2}}; \qquad \|\mathrm{V}_x\|_{L^\infty}^2 \le C(1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.75}$$

Moreover, by direct calculations, we can show that

$$\begin{aligned} &\|\theta_t\|_{L^2}^2 \le C(1+t)^{-\frac{5}{2}}; \qquad \|\theta_{xt}\|_{L^2}^2 \le C(1+t)^{-\frac{7}{2}}; \qquad \|\theta_{xxt}\|_{L^2}^2 \le C(1+t)^{-\frac{9}{2}};\\ &\|\theta\|_{L^\infty}^2 \le C(1+t)^{-1}; \qquad \|\theta_x\|_{L^\infty}^2 \le C(1+t)^{-2}; \qquad \|\theta_{xx}\|_{L^\infty}^2 \le C(1+t)^{-3}. \end{aligned} \tag{3.76}$$

Using (3.75)-(3.76), we update (3.74) as

$$\begin{aligned} |R_{5a}| \le{}& \frac{1}{2}\left(\|\mathrm{V}_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}\right)\|\mathrm{U}_t\|_{L^2}^2 + \frac{1}{8}\|\mathrm{U}_t\|_{L^2}^2 + C\big[(1+t)^{-5} + (1+t)^{-\frac{11}{2}} + (1+t)^{-6}\\ &+ (1+t)^{-\frac{5}{2}}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|\mathrm{U}_{xx}\|_{L^2}^2 + (1+t)^{-2}\|\mathrm{U}_{xx}\|_{L^2}^2\big]. \end{aligned}$$

Similarly, we can show that

$$|R_{5b}| \le 2\left(\|\mathrm{U}\|_{L^\infty} + \|\theta_x\|_{L^\infty}\right)\|\mathrm{U}_t\|_{L^2}^2 + \frac{1}{8}\|\mathrm{U}_t\|_{L^2}^2 + C\big[(1+t)^{-5} + (1+t)^{-\frac{11}{2}}\big].$$

Since $\|(\mathrm{U}, \mathrm{V}_x, \theta_x)(t)\|_{L^\infty} \to 0$ as $t \to \infty$, there is a constant $t_2 \ge t_1$, such that

$$\begin{aligned} |R_{5a}| + |R_{5b}| \le{}& \frac{1}{2}\|\mathrm{U}_t\|_{L^2}^2 + C\big[(1+t)^{-5} + (1+t)^{-\frac{11}{2}} + (1+t)^{-6}\\ &+ (1+t)^{-\frac{5}{2}}\|\mathrm{U}_x\|_{L^2}^2 + (1+t)^{-\frac{3}{2}}\|\mathrm{U}_{xx}\|_{L^2}^2 + (1+t)^{-2}\|\mathrm{U}_{xx}\|_{L^2}^2\big]. \end{aligned} \tag{3.77}$$

Substituting (3.77) into (3.73) gives us

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{U}_t\|_{L^2}^2 + \|\mathrm{V}_t\|_{L^2}^2\right) + \|\mathrm{U}_{xt}\|_{L^2}^2 + \frac{1}{2}\|\mathrm{U}_t\|_{L^2}^2\\ &\le C\big[(1+t)^{-5} + (1+t)^{-\frac{11}{2}} + (1+t)^{-6} + (1+t)^{-\frac{5}{2}}\|\mathrm{U}_x\|_{L^2}^2\\ &\quad + (1+t)^{-\frac{3}{2}}\|\mathrm{U}_{xx}\|_{L^2}^2 + (1+t)^{-2}\|\mathrm{U}_{xx}\|_{L^2}^2\big]. \end{aligned} \tag{3.78}$$

Multiplying (3.78) by $(1+t)$ and applying Lemma 3.2, we can show that for $t > t_2$,

$$(1+t)\left(\|\mathrm{U}_t(t)\|_{L^2}^2 + \|\mathrm{V}_t(t)\|_{L^2}^2\right) + \int_{t_2}^{t}(1+\tau)\left(\|\mathrm{U}_{xt}(\tau)\|_{L^2}^2 + \|\mathrm{U}_t(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \le C, \tag{3.79}$$

where the constant is independent of $t > t_2$. Multiplying (3.78) by $(1+t)^2$ and applying (3.79) and Lemma 3.5, and noting $\mathrm{V}_t = -\mathrm{U}_x$, we can show that for $t > t_2$,

$$(1+t)^2\left(\|\mathrm{U}_t(t)\|_{L^2}^2 + \|\mathrm{V}_t(t)\|_{L^2}^2\right) + \int_{t_2}^{t}(1+\tau)^2\left(\|\mathrm{U}_{xt}(\tau)\|_{L^2}^2 + \|\mathrm{U}_t(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \le C,$$

where the constant is independent of $t > t_2$. Iterating one more time, we can show that

$$(1+t)^3 \left(\|U_t(t)\|_{L^2}^2 + \|V_t(t)\|_{L^2}^2 \right) + \int_{t_2}^{t} (1+\tau)^3 \left(\|U_{xt}(\tau)\|_{L^2}^2 + \|U_t(\tau)\|_{L^2}^2 \right) d\tau \le C. \quad (3.80)$$

Since $V_t = -U_x$, we get in particular from (3.80):

$$(1+t)^3 \|U_x(t)\|_{L^2}^2 \le C. \quad (3.81)$$

In view of $(3.3)_2$, (3.80), (3.71) and (3.51), we see that

$$\begin{aligned}\|U(t)\|_{L^2}^2 &\le C \left(\|U_t(t)\|_{L^2}^2 + \|U_{xx}(t)\|_{L^2}^2 + \|V_x(t)\|_{L^2}^2 + \|\text{N.T.}\|_{L^2}^2 \right) \\ &\le C \left[(1+t)^{-3} + (1+t)^{-2} + \|\text{N.T.}\|_{L^2}^2 \right],\end{aligned}$$

where N.T. stands for the nonlinear terms on the right-hand side of $(3.3)_2$, i.e.,

$$\text{N.T.} = [(U - \theta_x)(V + \theta)]_x + (U - \theta_x)^2.$$

According to (3.37)-(3.40), Lemma 3.4–Lemma 3.5 and (3.81), $\|\text{N.T.}\|_{L^2}^2$ decays faster than $(1+t)^{-2}$. Therefore, we conclude

$$\|U(t)\|_{L^2}^2 \le C(1+t)^{-2}, \quad \forall\, t > t_2,$$

where the constant is independent of $t > t_2$. This completes the proof of the lemma. □

Finally, noting that $t_2 \ge t_1$ are constants and applying Lemma 3.2, we replace the lower limits of temporal integrals in the conclusions of Lemmas 3.4-3.6 by 0. The combination of Lemmas 3.3-3.7 gives us (2.13) for $t > t_2$. The case of $0 \le t \le t_2$ is a direct consequence of Lemmas 3.2 and 3.3. With Lemma 3.1 we thus have proved Theorem 2.1.

4. Proof of Theorem 2.2

In this section we obtain the optimal decay rates in (2.14). Under the additional assumption that $\phi_0 \in L^1(\mathbb{R})$ and $\tilde{u}_0 = u_0 - 1 \in L^1(\mathbb{R})$, we iterate the rates in Theorem 2.1 by a different set of analytic tools. It is a combination of linearization, spectral analysis, Green's function estimate, Plancherel theorem and Duhamel's principle. The strategy is detailed in [33,34] for the case of zero-mass for the v-component, i.e., $\theta = 0$. Thus here we focus on the terms induced by the asymptotic solution $(\theta, 1 - \frac{1}{r}\theta_x)$ in the estimates, and outline or cite results from [33,34] as needed.

For our convenience and without loss of generality we continue to set $r = 1$, and use C as a generic positive constant.

We write (3.3) in matrix form,

$$W_t + AW_x = BW_{xx} + LW + F, \quad (4.1)$$

where $\mathrm{W} = (\mathrm{V}, \mathrm{U})^{\mathrm{T}}$, the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.2}$$

and the nonlinearity is given by

$$F = F_{1x} + F_2, \qquad F_1 = \begin{pmatrix} 0 \\ -(\mathrm{U} - \theta_x)(\mathrm{V} + \theta) \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 \\ -(\mathrm{U} - \theta_x)^2 \end{pmatrix}. \tag{4.3}$$

Consider Fourier transform with respect to x:

$$\begin{aligned} \widehat{\mathrm{W}}(\xi, t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{W}(x, t) e^{-ix\xi} \mathrm{d}x, \\ \mathrm{W}(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\mathrm{W}}(\xi, t) e^{ix\xi} \mathrm{d}\xi. \end{aligned} \tag{4.4}$$

Taking Fourier transform of (4.1) with respect to x, we have

$$\widehat{\mathrm{W}}_t = E(i\xi)\widehat{\mathrm{W}} + \widehat{F}, \tag{4.5}$$

where $E(i\xi) = -i\xi A - \xi^2 B + L$ with A, B, L given in (4.2). The solution of (4.5) is

$$\widehat{\mathrm{W}}(\xi, t) = e^{tE(i\xi)}\widehat{\mathrm{W}}_0(\xi) + \int_0^t e^{(t-\tau)E(i\xi)} \widehat{F}(\xi, \tau)\, \mathrm{d}\tau, \tag{4.6}$$

where $\widehat{\mathrm{W}}_0$ denotes Fourier transform of

$$\mathrm{W}_0 = \begin{pmatrix} \mathrm{V}_0 \\ \mathrm{U}_0 \end{pmatrix} \equiv \begin{pmatrix} v_0 - \theta_0 \\ u_0 - 1 - \theta_{0x} \end{pmatrix} \tag{4.7}$$

and $\theta_0(x) = \theta(x, 0)$.

The solution operator in (4.6) is studied in [33] with details. For small ξ it is studied via spectral analysis of the matrix

$$E(i\xi) = \begin{pmatrix} 0 & -i\xi \\ -i\xi & -\xi^2 - 1 \end{pmatrix}.$$

Otherwise, energy estimation for the linear system is used to study the global decay property of the solution operator in the Fourier space. Here we cite the following lemma for the needed properties. Interested readers are referred to Lemma 3.3 in [33].

Lemma 4.1. *Let $k \geq 0$ be an integer, $h \in L^1(\mathbb{R})$, $D_x^k h \in L^2(\mathbb{R})$, and*

$$H_1(x) = \begin{pmatrix} h(x) \\ 0 \end{pmatrix}, \qquad H_2(x) = \begin{pmatrix} 0 \\ h(x) \end{pmatrix}.$$

Let $(e^{tE(i\xi)})_{1,2}$ denote the first/second row of $e^{tE(i\xi)}$. Then for $t \geq 0$,

$$\|(e^{tE(i\xi)})_1 (i\xi)^k \widehat{H}_1(\xi)\|_{L^2} \leq C(t+1)^{-\frac{1}{4}-\frac{k}{2}} \|h\|_{L^1} + Ce^{-ct} \|D_x^k h\|_{L^2}, \tag{4.8}$$

$$\|(e^{tE(i\xi)})_1 (i\xi)^k \widehat{H}_2(\xi)\|_{L^2} \leq C(t+1)^{-\frac{3}{4}-\frac{k}{2}} \|h\|_{L^1} + Ce^{-ct} \|D_x^k h\|_{L^2}, \tag{4.9}$$

$$\|(e^{tE(i\xi)})_2 (i\xi)^k \widehat{H}_1(\xi)\|_{L^2} \leq C(t+1)^{-\frac{3}{4}-\frac{k}{2}} \|h\|_{L^1} + Ce^{-ct} \|D_x^k h\|_{L^2}, \tag{4.10}$$

$$\|(e^{tE(i\xi)})_2 (i\xi)^k \widehat{H}_2(\xi)\|_{L^2} \leq C(t+1)^{-\frac{5}{4}-\frac{k}{2}} \|h\|_{L^1} + Ce^{-ct} \|D_x^k h\|_{L^2}, \tag{4.11}$$

where C and c are positive constants.

For $k = 0, 1$, we apply Plancherel theorem and (4.6) to obtain

$$\begin{aligned}
\|\partial_x^k \mathrm{V}(t)\|_{L^2} &= \|(i\xi)^k \widehat{\mathrm{V}}(t)\|_{L^2} \\
&\leq \|(i\xi)^k (e^{tE(i\xi)})_1 \widehat{\mathrm{W}}_0(\xi)\|_{L^2} + \int_0^t \|(i\xi)^k (e^{(t-\tau)E(i\xi)})_1 \widehat{F}(\xi,\tau)\|_{L^2} \,\mathrm{d}\tau \qquad (4.12) \\
&\equiv I_{1a} + I_{1b}, \\
\|\partial_x^k \mathrm{U}(t)\|_{L^2} &= \|(i\xi)^k \widehat{\mathrm{U}}(t)\|_{L^2} \\
&\leq \|(i\xi)^k (e^{tE(i\xi)})_2 \widehat{\mathrm{W}}_0(\xi)\|_{L^2} + \int_0^t \|(i\xi)^k (e^{(t-\tau)E(i\xi)})_2 \widehat{F}(\xi,\tau)\|_{L^2} \,\mathrm{d}\tau, \qquad (4.13) \\
&\equiv I_{2a} + I_{2b}
\end{aligned}$$

where $(e^{tE(i\xi)})_j$, $j = 1, 2$, denotes the jth row of $e^{tE(i\xi)}$, and F is given in (4.3).

Lemma 4.2. *Under the conditions of Theorem 2.2, for $k = 0, 1$ and $t \geq 0$ we have*

$$I_{1a} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{4.14}$$

$$I_{2a} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}}. \tag{4.15}$$

Proof. From (4.7) and (2.10) we have

$$\begin{aligned}
I_{1a} &\leq \|(i\xi)^k (e^{tE(i\xi)})_1 (\widehat{\mathrm{V}}_0(\xi), 0)^{\mathrm{T}}\|_{L^2} + \|(i\xi)^k (e^{tE(i\xi)})_1 (0, \widehat{\mathrm{U}}_0(\xi))^{\mathrm{T}}\|_{L^2} \\
&= \|(i\xi)^{k+1} (e^{tE(i\xi)})_1 (\widehat{\phi}_0(\xi), 0)^{\mathrm{T}}\|_{L^2} + \|(i\xi)^k (e^{tE(i\xi)})_1 (0, \widehat{\mathrm{U}}_0(\xi))^{\mathrm{T}}\|_{L^2}.
\end{aligned}$$

Applying (4.8) and (4.9) to the first and second terms on the right-hand side of the above inequality, respectively, we further have

$$I_{1a} \le C(1+t)^{-\frac{1}{4}-\frac{k+1}{2}}\|\phi_0\|_{L^1} + Ce^{-ct}\|D_x^{k+1}\phi_0\|_{L^2} + C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|\mathrm{U}_0\|_{L^1} + Ce^{-ct}\|D_x^k\mathrm{U}_0\|_{L^2}. \tag{4.16}$$

From the hypotheses of Theorem 2.2, (4.7) and (2.6) we know

$$\|\phi_0\|_{L^1} < \infty \qquad \text{and} \qquad \|\mathrm{U}_0\|_{L^1} \le \|u_0 - 1\|_{L^1} + \|\theta_{0x}\|_{L^1} < \infty,$$

where $\theta_0(x) = \theta(x, 0)$. Similarly, the assumption that $(v_0, u_0 - 1) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ implies

$$\|D_x^{k+1}\phi_0\|_{L^2} = \|D_x^k(v_0 - \theta_0)\|_{L^2} \le \|D_x^k v_0\|_{L^2} + \|\partial_x^k\theta_0\|_{L^2} < \infty$$

and

$$\|D_x^k\mathrm{U}_0\|_{L^2} = \|D_x^k(u_0 - 1 - \theta_{0x})\|_{L^2} \le \|D_x^k(u_0 - 1)\|_{L^2} + \|\partial_x^{k+1}\theta_0\|_{L^2} < \infty$$

for $k = 0, 1$. These simplify (4.16) to

$$I_{1a} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}} + Ce^{-ct} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \qquad k = 0, 1.$$

We thus have proved (4.14). Using (4.10) and (4.11) one can prove (4.15) in a similar way. □

Lemma 4.3. *Under the conditions of Theorem 2.2, for $k = 0, 1$ and $t \ge 0$ we have*

$$I_{1b} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{4.17}$$

$$I_{2b} \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}. \tag{4.18}$$

Proof. From (4.3) and (4.12) we have

$$\begin{aligned} I_{1b} &= \int_0^t \|(i\xi)^k(e^{(t-\tau)E(i\xi)})_1(i\xi\widehat{F}_1(\xi,\tau) + \widehat{F}_2(\xi,\tau))\|_{L^2}\,\mathrm{d}\tau \\ &\le \int_0^t \|(i\xi)^k(e^{(t-\tau)E(i\xi)})_1 i\xi\widehat{F}_1(\xi,\tau)\|_{L^2}\,\mathrm{d}\tau + \int_0^t \|(i\xi)^k(e^{(t-\tau)E(i\xi)})_1\widehat{F}_2(\xi,\tau)\|_{L^2}\,\mathrm{d}\tau. \end{aligned}$$

Applying (4.9) to the right-hand side gives us

$$\begin{aligned} I_{1b} &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x(\tau)\|_{L^1}\,\mathrm{d}\tau \\ &\quad + C\int_0^t e^{-c(t-\tau)}\|\partial_x^{k+1}[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)](\tau)\|_{L^2}\,\mathrm{d}\tau \end{aligned}$$

$$+ C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|(\mathrm{U}-\theta_x)^2(\tau)\|_{L^1} \, d\tau$$
$$+ C \int_0^t e^{-c(t-\tau)} \|\partial_x^k (\mathrm{U}-\theta_x)^2(\tau)\|_{L^2} \, d\tau. \tag{4.19}$$

For the first integral in (4.19), using (2.13), (2.6), we can show that

$$\|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x(\tau)\|_{L^1} \le (\|\mathrm{U}_x\|_{L^2} + \|\theta_{xx}\|_{L^2})(\|\mathrm{V}\|_{L^2} + \|\theta\|_{L^2}) + (\|\mathrm{U}\|_{L^2} + \|\theta_x\|_{L^2})(\|\mathrm{V}_x\|_{L^2} + \|\theta_x\|_{L^2}) \le C(1+\tau)^{-\frac{3}{2}}. \tag{4.20}$$

For the second integral, using the same tools alongside Sobolev inequality, we can show that

$$\|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x(\tau)\|_{L^2} \le (\|\mathrm{U}_x\|_{L^2} + \|\theta_{xx}\|_{L^2})(\|\mathrm{V}\|_{L^\infty} + \|\theta\|_{L^\infty}) + (\|\mathrm{U}\|_{L^2} + \|\theta_x\|_{L^2})(\|\mathrm{V}_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}) \le C(1+\tau)^{-\frac{7}{4}} \tag{4.21}$$

and

$$\|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}(\tau)\|_{L^2} \le C(1+\tau)^{-2}. \tag{4.22}$$

Moreover, it can be shown that

$$\|(\mathrm{U}-\theta_x)^2(\tau)\|_{L^1} \le C(1+\tau)^{-\frac{3}{2}}, \tag{4.23}$$
$$\|(\mathrm{U}-\theta_x)^2(\tau)\|_{L^2} \le C(1+\tau)^{-\frac{7}{4}}, \qquad \|\partial_x(\mathrm{U}-\theta_x)^2(\tau)\|_{L^2} \le C(1+\tau)^{-\frac{9}{4}}. \tag{4.24}$$

Substituting (4.20)-(4.24) into (4.19), we arrive at

$$I_{1b} \le C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-\frac{3}{2}} \, d\tau + C \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{7}{4}-\frac{k}{4}} \, d\tau$$
$$\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \qquad k = 0, 1.$$

This is (4.17).

Similarly, from (4.3) and (4.13) we have

$$I_{2b} \le \int_0^{\frac{t}{2}} \|(i\xi)^k (e^{(t-\tau)E(i\xi)})_2 i\xi \widehat{F}_1(\xi,\tau)\|_{L^2} \, d\tau + \int_{\frac{t}{2}}^t \|(e^{(t-\tau)E(i\xi)})_2 (i\xi)^{k+1} \widehat{F}_1(\xi,\tau)\|_{L^2} \, d\tau$$

$$+\int_0^{\frac{t}{2}} \|(i\xi)^k (e^{(t-\tau)E(i\xi)})_2 \widehat{F}_2(\xi,\tau)\|_{L^2}\,\mathrm{d}\tau + \int_{\frac{t}{2}}^{t} \|(e^{(t-\tau)E(i\xi)})_2 (i\xi)^k \widehat{F}_2(\xi,\tau)\|_{L^2}\,\mathrm{d}\tau.$$

Applying (4.11) to the right-hand side gives us

$$\begin{aligned} I_{2b} \le\, & C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} \|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_x(\tau)\|_{L^1}\,\mathrm{d}\tau \\ & + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{5}{4}} \|\partial_x^{k+1}[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)](\tau)\|_{L^1}\,\mathrm{d}\tau \\ & + C\int_0^{t} e^{-c(t-\tau)} \|\partial_x^{k+1}[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)](\tau)\|_{L^2}\,\mathrm{d}\tau \\ & + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} \|(\mathrm{U}-\theta_x)^2(\tau)\|_{L^1}\,\mathrm{d}\tau \\ & + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{5}{4}} \|\partial_x^{k}(\mathrm{U}-\theta_x)^2(\tau)\|_{L^1}\,\mathrm{d}\tau \\ & + C\int_0^{t} e^{-c(t-\tau)} \|\partial_x^{k}(\mathrm{U}-\theta_x)^2(\tau)\|_{L^2}\,\mathrm{d}\tau. \end{aligned} \tag{4.25}$$

Besides (4.20)-(4.24), we have the following estimates based on (2.13), (2.6) and Sobolev inequality,

$$\|[(\mathrm{U}-\theta_x)(\mathrm{V}+\theta)]_{xx}(\tau)\|_{L^1} \le C(1+\tau)^{-\frac{7}{4}}, \tag{4.26}$$

and

$$\|\partial_x(\mathrm{U}-\theta_x)^2(\tau)\|_{L^1} \le C(1+\tau)^{-2}. \tag{4.27}$$

Substituting (4.20)-(4.24) and (4.26)-(4.27) into (4.25) gives us

$$I_{2b} \le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}}(1+\tau)^{-\frac{3}{2}}\,\mathrm{d}\tau + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{2}-\frac{k}{4}}\,\mathrm{d}\tau$$

$$+C\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{7}{4}-\frac{k}{4}}\,\mathrm{d}\tau \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \qquad k=0,1.$$

We have proved (4.18). □

Substituting (4.14), (4.15), (4.17) and (4.18) into (4.12) and (4.13), we arrive at

$$\begin{aligned} &\|\partial_x^k \mathrm{V}(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \\ &\|\partial_x^k \mathrm{U}(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \end{aligned} \qquad k=0,1. \tag{4.28}$$

To finish the proof of Theorem 2.2 we need the following lemma.

Lemma 4.4. *Under the conditions of Theorem 2.2, for $t \ge 0$ we have*

$$\|\phi(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}}. \tag{4.29}$$

Proof. We write (3.4) for ϕ as a heat equation with a nonlinear source term,

$$\phi_t = \phi_{xx} + \tilde{F}(x,t),$$

where

$$\tilde{F}(x,t) = \tilde{u}_t - \tilde{u}_{xx} + (\tilde{u}v)_x + \tilde{u}^2 \equiv \sum_{i=1}^4 \tilde{F}_i(x,t).$$

Denote the heat kernel as

$$H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \tag{4.30}$$

Then by Duhamel's principle,

$$\phi(x,t) = \int_{\mathbb{R}} H(x-y,t)\phi_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} H(x-y,t-\tau)\tilde{F}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \equiv \sum_{i=0}^4 I_i, \tag{4.31}$$

where

$$\begin{aligned} I_0 &= \int_{\mathbb{R}} H(x-y,t)\phi_0(y)\,\mathrm{d}y, \\ I_i &= \int_0^t \int_{\mathbb{R}} H(x-y,t-\tau)\tilde{F}_i(y,\tau)\,\mathrm{d}y\mathrm{d}\tau, \qquad 1 \le i \le 4. \end{aligned}$$

If $0 \le t \le 2$, (4.29) is a consequence of (2.13). Thus, we consider the case $t > 2$ below. By Young's inequality, (4.30) and the assumption on ϕ_0 in Theorem 2.2,

$$\|I_0\|_{L^2} \le \|H(t)\|_{L^2}\|\phi_0\|_{L^1} = C(1+t)^{-\frac{1}{4}}. \tag{4.32}$$

By integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^{t-1}\int_{\mathbb{R}} H(x-y,t-\tau)\tilde{u}_\tau(y,\tau)\,\mathrm{d}y\mathrm{d}\tau + \int_{t-1}^{t}\int_{\mathbb{R}} H(x-y,t-\tau)\tilde{u}_\tau(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &= \underbrace{\int_{\mathbb{R}} H(x-y,1)\tilde{u}(y,t-1)\,\mathrm{d}y}_{\equiv I_{1a}} - \underbrace{\int_{\mathbb{R}} H(x-y,t)\tilde{u}(y,0)\,\mathrm{d}y}_{\equiv I_{1b}} \\ &\quad + \underbrace{\int_0^{t-1}\int_{\mathbb{R}} H_t(x-y,t-\tau)\tilde{u}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau}_{\equiv I_{1c}} \\ &\quad + \underbrace{\int_{t-1}^{t}\int_{\mathbb{R}} H(x-y,t-\tau)\tilde{u}_\tau(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau}_{\equiv I_{1d}}. \end{aligned} \tag{4.33}$$

By Young's inequality, (2.6), (4.28), and noticing $t > 2$, we have

$$\begin{aligned} \|I_{1a}\|_{L^2} \le \|H(1)\|_{L^1}\|\tilde{u}(t-1)\|_{L^2} &\le C(\|\theta_x(t-1)\|_{L^2} + \|\mathrm{U}(t-1)\|_{L^2}) \\ &\le C(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.34}$$

and by the assumption on $\tilde{u}_0 = u_0 - 1$ in Theorem 2.2,

$$\|I_{1b}\|_{L^2} \le \|H(t)\|_{L^2}\|\tilde{u}_0\|_{L^1} \le C(1+t)^{-\frac{1}{4}}. \tag{4.35}$$

Similarly, we can show that

$$\begin{aligned} \|I_{1c}\|_{L^2} \le \|H_t(t-\tau)\|_{L^1}\|\tilde{u}(\tau)\|_{L^2} &\le C(1+t-\tau)^{-1}(\|\theta_x(\tau)\|_{L^2} + \|\mathrm{U}(\tau)\|_{L^2}) \\ &\le C(1+t-\tau)^{-1}(1+\tau)^{-\frac{3}{4}}. \end{aligned} \tag{4.36}$$

For I_{1d} we apply $(3.1)_2$ and by integration by parts to get

$$I_{1d} = \int_{\mathbb{R}} H(x-y,t-\tau)[\tilde{u}_{yy} - (\tilde{u}v)_y - v_y - \tilde{u}(\tilde{u}+1)](y,\tau)\mathrm{d}y$$

$$= \underbrace{\int_{\mathbb{R}} H_x(x-y,t-\tau)\tilde{u}_y(y,\tau)\mathrm{d}y}_{\equiv R_{1d1}} - \underbrace{\int_{\mathbb{R}} H(x-y,t-\tau)[(\tilde{u}v)_y + v_y + \tilde{u}(\tilde{u}+1)](y,\tau)\mathrm{d}y}_{\equiv R_{1d2}}. \tag{4.37}$$

Then again by Young's inequality, (2.6) and (4.28) we have

$$\begin{aligned} \|I_{1d1}\|_{L^2} &\le \|H_x(t-\tau)\|_{L^1}\|\tilde{u}_x(\tau)\|_{L^2} \le \|H_x(t-\tau)\|_{L^1}(\|\theta_{xx}(\tau)\|_{L^2} + \|\mathrm{U}_x(\tau)\|_{L^2}) \\ &\le C(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} \|I_{1d2}\|_{L^2} &\le \|H(t-\tau)\|_{L^1}\left(\|(\tilde{u}v)_x(\tau)\|_{L^2} + \|v_x(\tau)\|_{L^2} + \|(\tilde{u}(\tilde{u}+1))(\tau)\|_{L^2}\right) \\ &\le C\left(\|(\tilde{u}_x v)(\tau)\|_{L^2} + \|(\tilde{u}v_x)(\tau)\|_{L^2} + \|v_x(\tau)\|_{L^2} + \|(\tilde{u}(\tilde{u}+1))(\tau)\|_{L^2}\right). \end{aligned} \tag{4.39}$$

By Sobolev inequality, the first two terms on the right-hand side of (4.39) are estimated as

$$\|(\tilde{u}_x v)(\tau)\|_{L^2} \le (\|\theta_{xx}(\tau)\|_{L^2} + \|\mathrm{U}_x(\tau)\|_{L^2})(\|\theta(\tau)\|_{L^\infty} + \|\mathrm{V}(\tau)\|_{L^\infty}) \le C(1+\tau)^{-\frac{7}{4}}, \tag{4.40}$$

and

$$\|(\tilde{u}v_x)(\tau)\|_{L^2} \le (\|\theta_x(\tau)\|_{L^\infty} + \|\mathrm{U}(\tau)\|_{L^\infty})(\|\theta_x(\tau)\|_{L^2} + \|\mathrm{V}_x(\tau)\|_{L^2}) \le C(1+\tau)^{-\frac{7}{4}}. \tag{4.41}$$

The third term satisfies

$$\|v_x(\tau)\|_{L^2} \le \|\theta_x(\tau)\|_{L^2} + \|\mathrm{V}_x(\tau)\|_{L^2} \le C(1+\tau)^{-\frac{3}{4}}. \tag{4.42}$$

For the last term, we can show that

$$\begin{aligned} \|(\tilde{u}(\tilde{u}+1))(\tau)\|_{L^2} &\le (\|\theta_x(\tau)\|_{L^2} + \|\mathrm{U}(\tau)\|_{L^2})(\|\theta_x(\tau)\|_{L^\infty} + \|\mathrm{U}(\tau)\|_{L^\infty} + 1) \\ &\le C(1+\tau)^{-\frac{3}{4}}. \end{aligned} \tag{4.43}$$

Substituting (4.40)-(4.43) into (4.39) gives us

$$\|I_{1d2}\|_{L^2} \le C(1+\tau)^{-\frac{3}{4}}. \tag{4.44}$$

Substituting (4.38) and (4.44) into (4.37), we have

$$\|I_{1d}\|_{L^2} \le (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}} + C(1+\tau)^{-\frac{3}{4}}. \tag{4.45}$$

Feeding (4.34)-(4.36) and (4.45) into (4.33) and invoking Bochner inequality imply

$$
\begin{aligned}
\|I_1\|_{L^2} &\le \|I_{1a}\|_{L^2} + \|I_{1b}\|_{L^2} + \int_0^{t-1} \|I_{1c}\|_{L^2}\,\mathrm{d}\tau + \int_{t-1}^{t} \|I_{1d}\|_{L^2}\,\mathrm{d}\tau \\
&\le C(1+t)^{-\frac{1}{4}} + C\int_0^{t-1} (1+t-\tau)^{-1}(1+\tau)^{-\frac{3}{4}}\,\mathrm{d}\tau \\
&\quad + C\int_{t-1}^{t} (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}\,\mathrm{d}\tau + C\int_{t-1}^{t} (1+\tau)^{-\frac{3}{4}}\,\mathrm{d}\tau \le C(1+t)^{-\frac{1}{4}}. \quad (4.46)
\end{aligned}
$$

Similarly,

$$
I_2 = -\int_0^t \int_{\mathbb{R}} H(x-y,t-\tau)\tilde{u}_{yy}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau = -\int_0^t \int_{\mathbb{R}} H_x(x-y,t-\tau)\tilde{u}_y(y,\tau)\,\mathrm{d}y\mathrm{d}\tau,
$$

and hence

$$
\begin{aligned}
\|I_2\|_{L^2} \le \int_0^t \|H_x(t-\tau)\|_{L^1}\|\tilde{u}_x(\tau)\|_{L^2}\,\mathrm{d}\tau &\le \int_0^t (t-\tau)^{-\frac{1}{2}}(\|\theta_{xx}(\tau)\|_{L^2} + \|\mathrm{U}_x(\tau)\|_{L^2})\,\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}\,\mathrm{d}\tau \le C(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{4.47}
$$

We can also show that

$$
\begin{aligned}
\|I_3\|_{L^2} &\le \int_0^t \|H_x(t-\tau)\|_{L^2}\|(\tilde{u}v)(\tau)\|_{L^1}\,\mathrm{d}\tau \\
&\le \int_0^t \|H_x(t-\tau)\|_{L^2}(\|\theta_x(\tau)\|_{L^2} + \|\mathrm{U}(\tau)\|_{L^2})(\|\theta(\tau)\|_{L^2} + \|\mathrm{V}(\tau)\|_{L^2})\,\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}\,\mathrm{d}\tau \le C(1+t)^{-\frac{3}{4}}\ln(1+t), \quad (4.48)
\end{aligned}
$$

and

$$\|I_4\|_{L^2} \leq \int_0^t \|H(t-\tau)\|_{L^2}\|\tilde{u}^2(\tau)\|_{L^1}\,\mathrm{d}\tau \leq \int_0^t \|H(t-\tau)\|_{L^2}(\|\theta_x(\tau)\|_{L^2} + \|\mathrm{U}(\tau)\|_{L^2})^2\,\mathrm{d}\tau$$
$$\leq C\int_0^t (t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{3}{2}}\,\mathrm{d}\tau \leq C(1+t)^{-\frac{1}{4}}. \tag{4.49}$$

Substituting (4.32) and (4.46)-(4.49) into (4.31) gives us (4.29). □

5. Proof of Theorem 2.4

In this section we continue to use C as a generic positive constant. First we verify that under the assumptions of Theorem 2.4, the regularity requirements in Theorem 2.2 are satisfied, and hence the theorem applies.

Lemma 5.1. *Under the conditions of Theorem 2.4, the Cauchy data of the transformed system satisfy $\bar{\phi}_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $\bar{v}_0 \in H^2(\mathbb{R})$, $\bar{u}_0 \geq 0$ and $\bar{u}_0 - 1 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$.*

Proof. From (2.10), (1.11), (1.12), (2.6), (1.8),

$$\bar{\phi}_0(\bar{x}) = \int_{-\infty}^{\bar{x}} [\bar{v}_0(\bar{y}) - \theta(\bar{y},0)]\mathrm{d}\bar{y}$$
$$= \mathrm{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}\int_{-\infty}^{\bar{x}} (\ln s_0)'(x(\bar{y}))\mathrm{d}\bar{y} - \frac{\chi}{D}\ln(\frac{s_+}{s_-})\frac{1}{\sqrt{4\pi/r}}\int_{-\infty}^{\bar{x}} e^{-\frac{r\bar{y}^2}{4}}\mathrm{d}\bar{y}$$
$$= \frac{\chi}{D}\ln(\frac{s_0(x(\bar{x}))}{s_-}) - \frac{\chi}{D}\ln(\frac{s_+}{s_-})\frac{1}{\sqrt{4\pi/r}}\int_{-\infty}^{\bar{x}} e^{-\frac{r\bar{y}^2}{4}}\mathrm{d}\bar{y}$$
$$\equiv \bar{\phi}_{01}(\bar{x}) + \bar{\phi}_{02}(\bar{x}). \tag{5.1}$$

By Taylor expansion,

$$\bar{\phi}_{01}(\bar{x}) = \frac{\chi}{D}\ln\left(1 + \frac{s_0(x(\bar{x})) - s_-}{s_-}\right) = \frac{\chi}{D}\frac{s_0(x(\bar{x})) - s_-}{s_- + \eta(\bar{x})[s_0(x(\bar{x})) - s_-]}, \tag{5.2}$$

where $0 < \eta(\bar{x}) < 1$. Noting that

$$s_- + \eta(\bar{x})[s_0(x(\bar{x})) - s_-] = [1-\eta(\bar{x})]s_- + \eta(\bar{x}))s_0(x(\bar{x}))$$

is a convex combination of s_- and $s_0(x(\bar{x}))$, it takes a value between s_- and $s_0(x(\bar{x}))$. On the other hand, the assumptions on s_0 in Theorem 2.4 imply that

$$m \equiv \inf_{x\in\mathbb{R}} s_0(x) > 0. \tag{5.3}$$

Therefore, (5.2) gives us

$$|\bar{\phi}_{01}(\bar{x})| \le C|s_0(x(\bar{x})) - s_-|, \tag{5.4}$$

which implies

$$\bar{\phi}_{01} \in L^1((-\infty,0)) \cap L^2((-\infty,0)) \tag{5.5}$$

by the assumption on $s_0 - s_-$ in Theorem 2.4.

For the second term in (5.1), by change of variables, we have

$$\bar{\phi}_{02}(\bar{x}) = -\frac{\chi}{D}\ln(\frac{s_+}{s_-})\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{\bar{x}}{\sqrt{4/r}}} e^{-\bar{y}^2}\mathrm{d}\bar{y} = -\frac{\chi}{2D}\ln(\frac{s_+}{s_-})\mathrm{erfc}\left(-\frac{\bar{x}}{\sqrt{4/r}}\right).$$

By properties of complementary error function we have

$$\bar{\phi}_{02} \in L^1((-\infty,0)) \cap L^2((-\infty,0)). \tag{5.6}$$

Combining (5.1), (5.5) and (5.6) gives us $\bar{\phi}_0 \in L^1((-\infty,0)) \cap L^2((-\infty,0))$.

Noting $\bar{\phi}_0(+\infty) = 0$, we can write

$$\bar{\phi}_0(\bar{x}) = -\int_{\bar{x}}^{\infty} [\bar{v}_0(\bar{y}) - \theta(\bar{y},0)]\mathrm{d}\bar{y}.$$

Similar to the above argument we may prove $\bar{\phi}_0 \in L^1((0,\infty)) \cap L^2((0,\infty))$. Therefore, $\bar{\phi}_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

From (1.11),

$$\bar{v}_0(\bar{x}) = \mathrm{sign}(\chi)\sqrt{\frac{\chi}{\mu K}\frac{s_0'(x(\bar{x}))}{s_0(x(\bar{x}))}}.$$

With (5.3) we have

$$|\bar{v}_0(\bar{x})| \le C|s_0'(x(\bar{x}))|,$$

which implies $\bar{v}_0 \in L^2(\mathbb{R})$ since $s_0' \in H^2(\mathbb{R})$. Similarly, one can show that the first and second derivatives of $\bar{v}_0$ are in $L^2(\mathbb{R})$. Therefore, $\bar{v}_0 \in H^2(\mathbb{R})$.

By (1.11),

$$\bar{u}_0(\bar{x}) = \frac{1}{K}u_0(x(\bar{x})),$$

and the assumptions for u_0 in Theorem 2.4, all conclusions on $\bar{u}_0$ are valid. We thus have proved the lemma. □

The hypotheses of Theorem 2.4 imply those of Theorems 2.1 and 2.2. Thus, we apply Theorem 2.1 to conclude that there is a unique solution $(\bar{v}, \bar{u})$ for the transformed problem (1.9), (1.11). The transformation (2.16) and (2.23) gives us the unique solution (s, u) of (1.1), (1.2). The solution satisfies $s(x,t) > 0$, $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$.

Now we need to prove (2.26) to finish this section.

Lemma 5.2. *Under the conditions of Theorem 2.4, with the decomposition (2.25),*

$$
\begin{aligned}
s(x,t) &= e^{-(\mu K+\sigma)t}[\Theta(x,t) + \mathcal{S}(x,t)],\\
u(x,t) &= K + \theta^*(x,t) + \mathcal{U}(x,t),
\end{aligned}
$$

the solution of (1.1), (1.2) has the following decay properties,

$$
\begin{aligned}
&\|\partial_x^k \mathcal{S}(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}},\ 0 \leq k \leq 2; \quad \|\partial_x^k \mathcal{S}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}},\ k = 0, 1;\\
&\|\partial_x^k \mathcal{U}(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}},\ k = 0, 1; \qquad \|\mathcal{U}(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{5.7}
$$

Proof. Lemma 5.1 implies that the conclusion of Theorem 2.2 on the transformed system is valid,

$$
\begin{aligned}
(1+\bar{t})^{\frac{1}{4}}\|\bar{\phi}(\bar{t})\|_{L^2} + (1+\bar{t})^{\frac{3}{4}}\|\bar{\mathrm{V}}(\bar{t})\|_{L^2} + (1+\bar{t})^{\frac{5}{4}}(\|\bar{\mathrm{V}}_{\bar{x}}(\bar{t})\|_{L^2} + \|\bar{\mathrm{U}}(\bar{t})\|_{L^2})&\\
+ (1+\bar{t})^{\frac{7}{4}}\|\bar{\mathrm{U}}_{\bar{x}}(\bar{t})\|_{L^2} \leq C.&
\end{aligned}
\tag{5.8}
$$

From (2.16), (2.18), (2.19) and (2.11), we have

$$
\begin{aligned}
\mathcal{S}(x,t) = \tilde{s}(x,t) - \Theta(x,t) &= \Theta(x,t)\left[\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)} \bar{\mathrm{V}}(\bar{y}, \bar{t}(t))\mathrm{d}\bar{y}\right) - 1\right]\\
&= \Theta(x,t)\left[\exp\left(\frac{D}{\chi}\int_{-\infty}^{\bar{x}(x)} \bar{\phi}_{\bar{y}}(\bar{y}, \bar{t}(t))\mathrm{d}\bar{y}\right) - 1\right]\\
&= \Theta(x,t)\left[\exp\left(\frac{D}{\chi}\bar{\phi}(\bar{x}(x), \bar{t}(t))\right) - 1\right].
\end{aligned}
\tag{5.9}
$$

Recall Proposition 2.3. For each $t \geq 0$, $\Theta(x,t)$ monotonically connects s_- to s_+ on $\mathbb{R}$, and hence $0 < \Theta(x,t) < \max\{s_-, s_+\}$. Then applying Taylor expansion we further have

$$
\begin{aligned}
|\mathcal{S}(x,t)| &\leq C\left|\exp\left(\frac{D}{\chi}\bar{\phi}(\bar{x}(x), \bar{t}(t))\right) - 1\right|\\
&\leq C\exp\left(\frac{D}{|\chi|}\|\bar{\phi}(\bar{t}(t))\|_{L^\infty}\right)|\bar{\phi}(\bar{x}(x), \bar{t}(t))|.
\end{aligned}
\tag{5.10}
$$

From (5.8) and by Sobolev inequality,

$$\begin{aligned} &\|\bar{\phi}(\bar{t})\|_{L^2} \le C(1+\bar{t})^{-\frac{1}{4}}, \quad \|\bar{\phi}_{\bar{x}}(\bar{t})\|_{L^2} = \|\bar{\mathrm{V}}(\bar{t})\|_{L^2} \le C(1+\bar{t})^{-\frac{3}{4}}, \\ &\|\bar{\phi}(\bar{t})\|_{L^\infty} \le C(1+\bar{t})^{-\frac{1}{2}}. \end{aligned} \tag{5.11}$$

Substituting (5.11) into (5.10) gives us

$$|\mathcal{S}(x,t)| \le C|\bar{\phi}(\bar{x}(x), \bar{t}(t))|.$$

Therefore,

$$\begin{aligned} &\|\mathcal{S}(t)\|_{L^2} \le C\|\bar{\phi}(\bar{t}(t))\|_{L^2} \le C(1+\bar{t})^{-\frac{1}{4}} \le C(1+t)^{-\frac{1}{4}}, \\ &\|\mathcal{S}(t)\|_{L^\infty} \le C\|\bar{\phi}(\bar{t}(t))\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}}. \end{aligned}$$

We take the first spatial derivative to (5.9) to have

$$\begin{aligned} \mathcal{S}_x(x,t) = \Theta_x(x,t)&\left[\exp\left(\frac{D}{\chi}\bar{\phi}(\bar{x}(x), \bar{t}(t))\right) - 1\right] \\ &+ \Theta(x,t)\exp\left(\frac{D}{\chi}\bar{\phi}(\bar{x}(x), \bar{t}(t))\right)\frac{D}{\chi}\bar{\phi}_{\bar{x}}(\bar{x}(x), \bar{t}(t))\frac{d\bar{x}}{dx}, \end{aligned}$$

which implies

$$|\mathcal{S}_x(x,t)| \le C|\Theta_x(x,t)||\bar{\phi}(\bar{x}(x), \bar{t}(t))| + C|\bar{\phi}_{\bar{x}}(\bar{x}(x), \bar{t}(t))|.$$

Applying (2.22) and (5.11) gives us

$$\|\mathcal{S}_x(t)\|_{L^2} \le C\|\Theta_x(t)\|_{L^\infty}\|\bar{\phi}(\bar{t}(t))\|_{L^2} + C\|\bar{\phi}_{\bar{x}}(\bar{t}(t))\|_{L^2} \le C(1+t)^{-\frac{3}{4}}.$$

Similarly, we obtain the estimate for $\|S_{xx}(t)\|_{L^2}$. By Sobolev inequality we also have $\|S_x(t)\|_{L^\infty}$.

Finally, applying the transformation (2.23) it is clear that

$$\mathcal{U}(x,t) = K\bar{\mathrm{U}}(\bar{x}(x), \bar{t}(t)).$$

Thus, the estimates on $\mathcal{U}$ in (5.7) are straightforward from (5.8) and Sobolev inequality. □

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to thank the anonymous referee for constructive comments and suggestions which helped improve the quality of the paper. The research of Y. Zeng was partially supported by the National Science Foundation under grant DMS-1908195. Support for K. Zhao came in part from the Simons Foundation Collaboration Grant for Mathematicians No. 413028.

References

[1] J.A. Carrillo, J. Li, Z. Wang, Boundary spike-layer solutions of the singular Keller–Segel system: existence and stability, Proc. Lond. Math. Soc. 122 (2021) 42–68.
[2] K. Choi, M. Kang, Y. Kwon, A. Vasseur, Contraction for large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model, Math. Models Methods Appl. Sci. 30 (2020) 387–437.
[3] M.A. Fontelos, A. Friedman, B. Hu, Mathematical analysis of a model for the initiation of angiogenesis, SIAM J. Math. Anal. 33 (2002) 1330–1355.
[4] J. Guo, J. Xiao, H. Zhao, C. Zhu, Global solutions to a hyperbolic-parabolic coupled system with large initial data, Acta Math. Sci. Ser. B Engl. Ed. 29 (2009) 629–641.
[5] Q. Hou, C. Liu, Y. Wang, Z. Wang, Stability of boundary layers for a viscous hyperbolic system arising from chemotaxis: one dimensional case, SIAM J. Math. Anal. 50 (2018) 3058–3091.
[6] Q. Hou, Z. Wang, K. Zhao, Boundary layer problem on a hyperbolic system arising from chemotaxis, J. Differ. Equ. 261 (2016) 5035–5070.
[7] H. Jin, J. Li, Z. Wang, Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity, J. Differ. Equ. 255 (2013) 193–219.
[8] H.A. Levine, B.D. Sleeman, A system of reaction diffusion equations arising in the theory of reinforced random walks, SIAM J. Appl. Math. 57 (1997) 683–730.
[9] D. Li, R. Pan, K. Zhao, Quantitative decay of a one-dimensional hybrid chemotaxis model with large data, Nonlinearity 28 (2015) 2181–2210.
[10] H. Li, K. Zhao, Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis, J. Differ. Equ. 258 (2015) 302–338.
[11] J.Li.T. Li, Z. Wang, Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity, Math. Models Methods Appl. Sci. 24 (2014) 2819–2849.
[12] T. Li, R. Pan, K. Zhao, Global dynamics of a hyperbolic-parabolic model arising from chemotaxis, SIAM J. Appl. Math. 72 (2014) 417–443.
[13] T. Li, Z. Wang, Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis, SIAM J. Appl. Math. 7 (2009) 1522–1541.
[14] T. Li, Z. Wang, Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic-parabolic system arising in chemotaxis, Math. Models Methods Appl. Sci. 20 (2010) 1967–1998.
[15] T. Li, Z. Wang, Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis, J. Differ. Equ. 250 (2011) 1310–1333.
[16] T. Li, Z. Wang, Steadily propagating waves of a chemotaxis model, Math. Biosci. 240 (2012) 161–168.
[17] T.-P. Liu, Y. Zeng, Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws, Mem. Am. Math. Soc. 125 (599) (1997), viii+120 pp.
[18] T.-P. Liu, Y. Zeng, Time-asymptotic behavior of wave propagation around a viscous shock profile, Commun. Math. Phys. 290 (2009) 23–82.
[19] T.-P. Liu, Y. Zeng, Shock waves in conservation laws with physical viscosity, Mem. Am. Math. Soc. 234 (1105) (2015), vi+168 pp.
[20] V. Martinez, Z. Wang, K. Zhao, Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology, Indiana Univ. Math. J. 67 (2018) 1383–1424.
[21] H. Othmer, A. Stevens, Aggregation, blowup and collapse: the ABC's of taxis in reinforced random walks, SIAM J. Appl. Math. 57 (1997) 1044–1081.
[22] H. Peng, Z. Wang, Nonlinear stability of strong traveling waves for the singular Keller-Segel system with large perturbations, J. Differ. Equ. 265 (2018) 2577–2613.
[23] H. Peng, Z. Wang, K. Zhao, C. Zhu, Boundary layers and stabilization of the singular Keller-Segel model, Kinet. Relat. Models 11 (2018) 1085–1123.

[24] Y. Tao, L. Wang, Z. Wang, Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension, Discrete Contin. Dyn. Syst., Ser. B 18 (2013) 821–845.
[25] Z. Wang, Mathematics of traveling waves in chemotaxis, Discrete Contin. Dyn. Syst., Ser. B 18 (2013) 601–641.
[26] Z. Wang, K. Zhao, Global dynamics and diffusion limit of a parabolic system arising from repulsive chemotaxis, Commun. Pure Appl. Anal. 12 (2013) 3027–3046.
[27] Y. Zeng, Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation, Arch. Ration. Mech. Anal. 150 (1999) 225–279.
[28] Y. Zeng, Hyperbolic-parabolic balance laws: asymptotic behavior and a chemotaxis model, Commun. Appl. Anal. 23 (2019) 209–232.
[29] Y. Zeng, Nonlinear stability of diffusive contact wave for a chemotaxis model, J. Differ. Equ. 308 (2022) 286–326.
[30] Y. Zeng, Time asymptotic behavior of solutions to a chemotaxis model with logarithmic singularity, preprint.
[31] Y. Zeng, J. Chen, Pointwise time asymptotic behavior of solutions to a general class of hyperbolic balance laws, J. Differ. Equ. 260 (8) (2016) 6745–6786.
[32] Y. Zeng, K. Zhao, On the logarithmic Keller-Segel-Fisher/KPP system, Discrete Contin. Dyn. Syst. 39 (2019) 5365–5402.
[33] Y. Zeng, K. Zhao, Optimal decay rates for a chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate, J. Differ. Equ. 268 (2020) 1379–1411.
[34] Y. Zeng, K. Zhao, Corrigendum to "Optimal decay rates for a chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate", J. Differ. Equ. (2020) 1379–1411, J. Differ. Equ. 269 (2020) 6359–6363.
[35] M. Zhang, C. Zhu, Global existence of solutions to a hyperbolic-parabolic system, Proc. Am. Math. Soc. 135 (2006) 1017–1027.