SATURATION BOUNDS FOR SMOOTH VARIETIES

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Introduction

The purpose of this paper is to prove some saturation bounds for the ideals of non-singular complex projective schemes and their powers.

We begin with some background. Consider the polynomial ring $S = \mathbf{C}[x_0, \ldots, x_r]$ in $r+1$ variables, and fix homogeneous polynomials

$$f_0\,,\, f_1\,,\, \ldots\,,\, f_p \in S \ \text{ with } \ \deg(f_i) = d_i.$$

We assume that $d_0 \geq d_1 \geq \ldots \geq d_p$, and we denote by

$$J \ = \ \big(\, f_0\,,\, f_1\,,\, \ldots\,,\, f_p \,\big) \ \subseteq \ S$$

the ideal that the polynomials span. Suppose now that J is primary for the irrelevant maximal ideal $\mathfrak{m} = (x_0, \ldots, x_r)$, or equivalently that $\dim_{\mathbf{C}} S/J < \infty$. In this case J contains all monomials of sufficiently large degree, and it is a classical theorem of Macaulay [5, Theorem 7.4.1] that

$$J_t \ = \ S_t \quad \text{for} \quad t \ \geq \ d_0 + \ldots + d_r - r. \tag{1}$$

Moreover this bound is (always) sharp when $p = r$. Although less well known, a similar statement holds for powers of J:

$$(J^a)_t \ = \ S_t \quad \text{for} \quad t \ \geq \ ad_0 + d_1 + \ldots + d_r - r. \tag{2}$$

This again is always sharp when $p = r$.

It is natural to ask whether there are analogous results for more general homogeneous ideals J, in particular when

$$X \ =_{\text{def}} \ \text{Zeroes}(J) \ \subseteq \ \mathbf{P}^r$$

is a smooth complex projective scheme. Of course if J has non-trivial zeroes, then it does not contain any power of the maximal ideal. However if one interprets (1) and (2) as saturation bounds, then the question makes sense more generally. Specifically, recall that the *saturation* of a homogeneous ideal J is defined by

$$J^{\text{sat}} \ = \ \big\{\, f \in S \mid \mathfrak{m}^k \cdot f \subseteq J \text{ for some } k \geq 0 \,\big\}.$$

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The quotient J^{sat}/J has finite length, and in particular

$$(J^{\text{sat}})_t \;=\; J_t \quad \text{for} \quad t \gg 0.$$

The least such integer t is called the *saturation degree* $\operatorname{sat.deg}(J)$ of J. Observing that $J^{\text{sat}} = S$ if and only if J is $\mathfrak{m}$-primary, statements (1) and (2) are equivalent to estimates for the saturation degrees of J and J^a. So the problem becomes to bound the saturation degree of an ideal in terms of the degrees of its generators.

It is instructive to consider some examples. Let $X \subseteq \mathbf{P}^r$ be a hyperplane defined by a linear form $\ell \in S$, and set

$$f_i \;=\; x_i^{d-1}\cdot \ell \quad , \quad J = (f_0, \ldots, f_r) \;\subseteq\; S. \tag{3}$$

Then $J^{\text{sat}} = (\ell)$, and it follows from Macaulay's theorem that

$$\operatorname{sat.deg}(J) \;=\; (r+1)(d-1) - r + 1 \;=\; (r+1)d - 2r,$$

which is very close to the bound (1). On the other hand, it is not the case that the saturation degree of an arbitrary ideal is bounded linearly in the degrees of its generators. For instance, the ideals

$$J \;=\; \big(x^d, y^d, xz^{d-1} - yw^{d-1}\big) \;\subseteq\; \mathbf{C}[x,y,z,w]$$

considered by Caviglia [6, Example 4.2.1] have $\operatorname{sat.deg}(J) \approx d^2$.

Our first main result asserts that for ideals defining smooth varieties, the Macaulay bounds remain true without modification.

Theorem A. *As above, suppose that*

$$J \;=\; \big(f_0\,,\, f_1\,,\, \ldots\,,\, f_p\big) \;\subseteq\; S$$

is generated by forms of degrees $d_0 \geq \ldots \geq d_p$, and assume that the projective scheme

$$X \;=_{def}\; \text{Zeroes}(J) \;\subseteq\; \mathbf{P}^r$$

cut out by the f_i is non-singular. Then $\operatorname{sat.deg}(J) \leq d_0 + \ldots + d_r - r$*, and more generally*

$$\operatorname{sat.deg}(J^a) \;\leq\; ad_0 + d_1 + \ldots + d_r - r. \tag{4}$$

(If $p < r$, one takes $d_{p+1} = \ldots = d_r = 0$.) We do not know whether the stated bound is best possible, but in any event it is asymptotically sharp. Indeed, if J is the ideal considered in (3), then the Theorem predicts that $\operatorname{sat.deg}(J^a) \leq (a+r)d - r$, whereas in fact $\operatorname{sat.deg}(J^a) = (a+r)d - 2r$.

Given a reduced algebraic set $X \subseteq \mathbf{P}^r$ denote by $I_X \subseteq S$ the saturated homogeneous ideal of X. Recall that the *symbolic powers* of I_X are

$$I_X^{(a)} \;=\; \big\{ f \in S \mid \operatorname{ord}_x(f) \geq a \text{ for general (or every) } x \in X \big\}.$$

Evidently $I_X^a \subseteq I_X^{(a)}$, and there has been a huge amount of interest in recent years in understanding the connections between actual and symbolic powers (cf [12], [17], [3], [10]). If X is non-singular, then $I_X^{(a)} = (I_X^a)^{\text{sat}}$. Therefore Theorem A implies

Corollary B. *Assume that $X \subseteq \mathbf{P}^r$ is smooth, and that I_X is generated in degrees $d_0 \geq d_1 \geq \ldots \geq d_p$. Then*

$$\big(I_X^{(a)}\big)_t \ = \ (I_X^a)_t \quad \textit{for } t \ \geq \ ad_0 + d_1 + \ldots + d_r - r.$$

For example, suppose that $X \subseteq \mathbf{P}^2$ consists of the three coordinate points, so that $I_X = (xy, yz, zx) \subseteq \mathbf{C}[x,y,z]$. The Corollary guarantees that I_X^a and $I_X^{(a)}$ agree in degrees $\geq 2a+2$, whereas in reality $\text{sat.}\deg(I_X^a) = 2a$. So here again the statement is asymptotically but not precisely sharp.

In the case of finite sets, results of Geramita-Gimigliano-Pitteloud [15], Chandler [7] and Sidman [23] provide an alternative bound that is often best-possible. Recall that a scheme $X \subseteq \mathbf{P}^r$ is said to be m-regular in the sense of Castelnuovo–Mumford if its ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^r}$ satisfies the vanishings:

$$H^i\big(\mathbf{P}^r, \mathcal{I}_X(m-i)\big) \ = \ 0 \quad \text{for} \ \ i > 0.$$

This is equivalent to asking that I_X be generated in degrees $\leq m$, that the first syzygies among minimal generators of I_X appear in degrees $\leq m+1$, the second syzygies in degrees $\leq m+2$, and so on.[1] The authors just cited show that if $X \subseteq \mathbf{P}^r$ is an m-regular finite set, then

$$\text{sat.}\deg(I_X^a) \ \leq \ am.$$

This is optimal for the example of the three coordinate points in $\mathbf{P}^2$.

Our second main result asserts that the same statement holds when $\dim X = 1$.

Theorem C. *Let $X \subseteq \mathbf{P}^r$ be a smooth m-regular curve. Then*

$$\big(I_X^a\big)_t \ = \ \big(I_X^{(a)}\big)_t \ \ \textit{for} \ \ t \ \geq \ am.$$

In fact, for the saturation bound it suffices that the curve X be reduced. The statement is optimal (for all a) for instance when $X \subseteq \mathbf{P}^4$ is a rational normal curve. We also show that if $X \subseteq \mathbf{P}^r$ is a reduced surface, then $\text{reg}(\mathcal{I}_X^a) \leq a \cdot \text{reg}(\mathcal{I}_X)$. We do not know any examples where the analogous statements fail for smooth varieties of higher dimension.

Returning to the setting of Theorem A, the first and third authors showed with Bertram some years ago [2] that if $X \subseteq \mathbf{P}^r$ is a smooth complex projective variety of codimension e cut out as a scheme by homogeneous polynomials of degrees $d_0 \geq \ldots \geq d_p$, then $\mathcal{I}_X^a$ is $(ad_0+d_1+\ldots+d_{e-1}-e)$-regular in the sense of Castelnuovo-Mumford. Note however that this does not address the questions of saturation required to control the arithmetic (Eisenbud–Goto) regularity of I_X^a.[2] In fact, one can view Theorem A as promoting the results of [2] to statements about arithmetic regularity:

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[1] For saturated ideals, Castelnuovo–Mumford regularity of I_X agrees with an algebraic notion of regularity introduced by Eisenbud and Goto [13] that we propose to call *arithmetic regularity.* An arbitrary ideal $J \subseteq S$ is arithmetically m-regular if and only if J^{sat} is m-regular and $\text{sat.}\deg(J) \leq m$. Given that we are interested in establishing bounds on saturation degree, unless otherwise stated we always refer to regularity in the geometric sense.

[2] In particular, the proof of Proposition 2.2 in [1] seems to be erroneous.

Corollary D. *Assume that $J \subseteq S$ satisfies the hypotheses of Theorem A. Then*
$$\text{arith. reg}(J^a) \ \le \ ad_0 + (d_1 + \ldots + d_r - r).$$

It is known ([20], [9]) that if $J \subseteq S$ is an arbitrary homogeneous ideal then
$$\text{arith. reg}(J^a) \ = \ ad + b \quad \text{when} \ \ a \gg 0,$$
where d is the maximal degree needed to generate a reduction of J – which coincides with the generating degree of J when it is equigenerated – and b is some constant. However computing the constant term b has proven elusive, and the Corollary gives a bound in the case at hand.

The proofs of these results revolve around using complexes of sheaves to study the image in $H^0_*\big(\mathbf{P}^r, \mathcal{I}_X^a\big) = (I_X^a)^{\text{sat}}$ of the powers of the ideal spanned by generators of I_X or J: this approach was inspired in part by geometrizing the arguments of Cooper and coauthors for codimenson two subvarieties in [8]. Specifically, suppose that
$$\varepsilon : U_0 \ =_{\text{def}} \ \oplus \mathcal{O}_{\mathbf{P}^r}(-d_i) \longrightarrow \mathcal{I}_X$$
is the surjective map of sheaves determined by generators of I_X or J. If X is m-regular, then this sits in an exact complex $U_\bullet$ of bundles:
$$0 \longrightarrow U_{r-1} \longrightarrow U_{r-2} \longrightarrow \ldots \longrightarrow U_1 \longrightarrow U_0 \xrightarrow{\ \varepsilon\ } \mathcal{I}_X \longrightarrow 0$$
where $\text{reg}(U_i) \le m+i$. Weyman [26] (see also [25]) constructs a new complex $L_\bullet = \text{Sym}^a(U_\bullet)$ that takes the form
$$\ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow S^a(U_0) \longrightarrow \mathcal{I}_X^a \longrightarrow 0$$
where $\text{reg}(L_i) \le am + i$. This complex is exact only off X, but as in [16] when $\dim X = 1$ one can still read off the surjectivity of
$$H^0\big(\mathbf{P}^r, S^a(U_0)(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{I}_X^a(t)\big)$$
for $t \ge am$. This gives Theorem C.

Turning to Theorem A, a natural idea is to start with the Koszul complex
$$\ldots \longrightarrow \Lambda^3 U_0 \longrightarrow \Lambda^2 U_0 \longrightarrow U_0 \longrightarrow \mathcal{I}_X \longrightarrow 0.$$
As established by Buchsbaum–Eisenbud [4], this determines a new complex
$$(*) \qquad \ldots \longrightarrow S^{a,1^2}(U_0) \longrightarrow S^{a,1}(U_0) \longrightarrow S^a(U_0) \longrightarrow \mathcal{I}_X^a \longrightarrow 0,$$
where $S^{a,1^k}(U_0)$ denotes the Schur power of U_0 corresponding to the Young diagram $(a, 1^k)$. We observe that
$$\text{reg}\big(S^{a,1^i}(U_0)\big) \ \le \ ad_0 + d_1 + \ldots + d_i,$$
so if (*) were exact then the statement of the Theorem would follow immediately. Unfortunately (*) is exact only if X is a complete intersection, but by blowing up X this construction yields an exact complex whose cohomology groups one can control with some effort. At the end of the day, the computation boils down to using Kodaira–Nakano vanishing on X to prove a vanishing statement for symmetric powers of the normal bundle to X in $\mathbf{P}^r$:

Proposition E. *Let $X \subseteq \mathbf{P}^r$ be a smooth complex projective variety, and denote by $N = N_{X/\mathbf{P}^r}$ the normal bundle to X in $\mathbf{P}^r$. Then*

$$H^i\Big(X\,,\, S^kN \otimes \det N \otimes \mathcal{O}_X(\ell)\Big) \ =\ 0 \quad \text{for } i > 0$$

and every $k \geq 0$, $\ell \geq -r$.

(Similar but slightly different vanishings were established by Schneider and Zintl in [22].) We hope that some of these ideas may find other applications in the future.[3]

The paper is organized as follows. The first section is devoted to Theorem C. We collect in §2 some preliminary results towards the Macaulay-type bounds. Specifically, we discuss the Buchsbaum–Eisenbud powers of Koszul complexes, the computation of some push-forwards from a blowing-up, and Proposition E. The proof of Theorem A occupies §3.

Concerning our assumptions: we work throughout over the complex numbers. As the referee points out, the main results stated above do not require X to be irreducible or even pure dimensional. However the essential ideas occur for irreducible varieties, and we generally leave it to the reader to think through this technical improvement.

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1. Saturation and regularity

The present section is devoted to the proof of Theorem C from the Introduction.

We start with some general remarks. Let $X \subseteq \mathbf{P}^r$ be a complex projective scheme, with ideal sheaf $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^r}$ and homogeneous ideal $I_X \subseteq S$. Denote by $U_\bullet$ the locally free resolution of $\mathcal{I}_X$ obtained by sheafifying a minimal graded free resolution of I_X:

$$0 \longrightarrow U_r \longrightarrow U_{r-1} \longrightarrow \ldots \longrightarrow U_1 \longrightarrow U_0 \xrightarrow{\ \varepsilon\ } \mathcal{I}_X \longrightarrow 0. \tag{1.1}$$

Thus each U_i is a direct sum of line bundles, and we recover the original resolution as the the complex $H^0_*\big(\mathbf{P}^r\,,\,U_\bullet\big)$ obtained from $U_\bullet$ by taking global sections of all twists.

Consider now the surjective homomorphism of sheaves

$$S^a(\varepsilon)\ :\ S^aU_0 \longrightarrow \mathcal{I}_X^a.$$

For any $t \geq 0$ one has

$$H^0\big(\mathbf{P}^r, \mathcal{I}_X^a(t)\big) \ =\ \big((I_X^a)^{\mathrm{sat}}\big)_t\,.$$

On the other hand, the fact that U_0 is constructed from minimal generators of I_X implies that

$$\mathrm{Im}\Big(H^0\big(\mathbf{P}^r, S^a(U_0)(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{I}_X^a(t)\big)\Big) \ =\ \big(I_X^a\big)_t.$$

Therefore

[3] We remark that some of the auxiliary results appearing here – for example the Proposition just stated – were known to the first and third authors some years ago in connection with their work on [2]. However they were put aside in favor of the simpler arguments with vanishing theorems that eventually appeared in that paper.

Lemma 1.1. *The degree t pieces of I_X^a and $(I_X^a)^{\text{sat}}$ coincide if and only if the homomorphism*

$$H^0\big(\mathbf{P}^r, S^a(U_0)(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{I}_X^a(t)\big)$$

determined by $S^a(\varepsilon)$ is surjective. □

The plan is to study $S^a(\varepsilon)$ by realizing it as the last map of a complex $S^a(U_\bullet)$.

Specifically, consider a smooth variety M, a subvariety $X \subseteq M$, and a locally free resolution $U_\bullet$ of $\mathcal{I}_X \subseteq \mathcal{O}_M$ as above:

$$0 \longrightarrow U_r \longrightarrow U_{r-1} \longrightarrow \ldots \longrightarrow U_1 \longrightarrow U_0 \xrightarrow{\varepsilon} \mathcal{I}_X \longrightarrow 0. \tag{1.2}$$

As explained by Weyman [26] and Tchernev [25], $U_\bullet$ determines for fixed $a \geq 1$ a new complex $L_\bullet = S^a(U_\bullet)$ having the shape

$$\ldots \longrightarrow L_4 \longrightarrow L_3 \longrightarrow \begin{matrix} S^{a-2}U_0 \otimes \Lambda^2 U_1 \\ \oplus \\ S^{a-1}U_0 \otimes U_2 \end{matrix} \longrightarrow S^{a-1}U_0 \otimes U_1 \longrightarrow S^a U_0 \longrightarrow \mathcal{I}_X^a \longrightarrow 0. \tag{1.3}$$

The last map on the right is $S^a(\varepsilon)$, and the homomorphism $S^{a-1}U_0 \otimes U_1 \longrightarrow S^a U_0$ is the natural one arising as the composition

$$S^{a-1}U_0 \otimes U_1 \longrightarrow S^{a-1}U_0 \otimes U_0 \longrightarrow S^a U_0.$$

The L_i are determined by setting

$$C^k(U_j) \;=\; \begin{cases} S^k U_j & \text{if } j \text{ is even} \\ \Lambda^k U_j & \text{if } j \text{ is odd} \end{cases}, \tag{1.4}$$

and then taking

$$L_i \;=\; \bigoplus_{\substack{k_0+\ldots+k_r=a \\ k_1+2k_2+\ldots+rk_r=i}} C^{k_0}(U_0) \otimes C^{k_1}(U_1) \otimes \ldots \otimes C^{k_r}(U_r). \tag{1.5}$$

It follows from [26, Theorem 1] or [25, Theorem 2.1] that:

(1.6) The complex (1.3) is exact away from X.

In general one does not expect exactness at points of X, but when X is smooth the right-most terms at least are well-behaved:

Lemma 1.2. *Assume that X is non-singular. Then the sequence*

$$S^{a-1}U_0 \otimes U_1 \longrightarrow S^a U_0 \longrightarrow \mathcal{I}_X^a \longrightarrow 0$$

is exact.

Proof. The question being local, we can work over the local ring $\mathcal{O} = \mathcal{O}_{M,x}$ of M at a point $x \in X$. Since X is smooth, $\mathcal{I} = \mathcal{I}_{X,x} \subseteq \mathcal{O}$ is generated by a regular sequence of length $e = \operatorname{codim} X$. Thus $\mathcal{I}$ has a minimal presentation

$$\Lambda^2 \mathcal{U} \longrightarrow \mathcal{U} \longrightarrow \mathcal{I} \longrightarrow 0$$

given by the beginning of a Koszul complex, where $\mathcal{U} = \mathcal{O}^e$ is a free module of rank e. Here one checks by hand the exactness of

$$S^{a-1}\mathcal{U} \otimes \Lambda^2\mathcal{U} \longrightarrow S^a\mathcal{U} \longrightarrow \mathcal{I}^a \longrightarrow 0.$$

(Compare Proposition 2.3 below.) An arbitrary free presentation of $\mathcal{I}$ then has the form

$$\Lambda^2\mathcal{U} \oplus \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{U} \oplus \mathcal{A} \longrightarrow \mathcal{I} \longrightarrow 0,$$

where $\mathcal{A}$ is a free module mapping to zero in $\mathcal{I}$, $\mathcal{B}$ is a free module mapping to zero in $\mathcal{U} \oplus \mathcal{A}$, and the left-hand map is the identity on $\mathcal{A}$. It suffices to verify the exactness of

$$S^{a-1}\big(\mathcal{U} \oplus \mathcal{A}\big) \otimes \big(\Lambda^2\mathcal{U} \oplus \mathcal{A}\big) \longrightarrow S^a\big(\mathcal{U} \oplus \mathcal{A}\big) \longrightarrow \mathcal{I}^a \longrightarrow 0,$$

and this is clear upon writing $S^a\big(\mathcal{U} \oplus \mathcal{A}\big) = S^a\mathcal{U} \,\oplus\, \mathcal{A} \otimes S^{a-1}\big(\mathcal{U} \oplus \mathcal{A}\big)$. □

With these preliminaries out of the way, we now prove (a slight strengthening of) Theorem C from the Introduction.

Theorem 1.3. *Let $X \subseteq \mathbf{P}^r$ be a reduced (but possibly singular) curve, and assume that X is m-regular in the sense of Castelnuovo–Mumford. Denote by $I_X \subseteq S$ the homogeneous ideal of X. Then*

$$\text{sat. deg}(I_X^a) \ \leq \ am.$$

Proof. The m-regularity of X means that we can take a resolution $U_\bullet$ of $\mathcal{I}_X$ as in (1.1) where U_i is a direct sum of line bundles of degrees $\geq -m-i$, ie $\text{reg}(U_i) \leq m+i$. Consider the resulting Weyman complex $L_\bullet = S^a(U_\bullet)$:

$$(*) \qquad \longrightarrow L_3 \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow \mathcal{I}_X^a \longrightarrow 0,$$

the last map being the surjection $S^a(\varepsilon) : L_0 = S^aU_0 \longrightarrow \mathcal{I}_X$. In view of Lemma 1.1, the issue is to establish the surjectivity of the homomorphism

$$(**) \qquad H^0\big(\mathbf{P}^r, L_0(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{I}_X^a(t)\big)$$

for $t \geq am$. To this end, observe first from (1.4) and (1.5) that

$$\text{reg}(L_i) \ \leq \ am + i.$$

Consider next the homology sheaves $\mathcal{H}_i = \mathcal{H}_i(L_\bullet \longrightarrow \mathcal{I}_X^a)$ of the augmented complex (*). (So for $i = 0$ we understand $\mathcal{H}_0 = \ker(L_0 \longrightarrow \mathcal{I}_X^a)/\text{Im}(L_1 \longrightarrow L_0)$.) Thanks to (1.6), these are all supported on the one-dimensional set X. Moreover it follows from Lemma 1.2 that $\mathcal{H}_0$ is supported on the finitely many singular points of X. Therefore the required surjectivity (**) is a consequence of the first statement of the following Lemma. □

Lemma 1.4. *Consider a complex $L_\bullet$ of coherent sheaves on $\mathbf{P}^r$ sitting in a diagram*

$$(1.7) \qquad \ldots \longrightarrow L_3 \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \stackrel{\varepsilon}{\longrightarrow} \mathcal{F} \longrightarrow 0,$$

and denote by $\mathcal{H}_i = \mathcal{H}_i(L_\bullet \longrightarrow \mathcal{F})$ the i^{th} homology sheaf of the augmented complex (1.7).[4] *Assume that ε is surjective, and let p be an integer with the property that L_i is $(p+i)$-regular for every i.*

[4]So as above, the group of zero-cycles used to compute $\mathcal{H}_0$ is $\ker(\varepsilon)$.

(*i*) *If each* $\mathcal{H}_i$ *is supported on a set of dimension* $\leq i$*, then the homomorphism*

$$H^0\big(\mathbf{P}^r, L_0(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{F}(t)\big)$$

is surjective for $t \geq p$*.*

(*ii*) *If each* $\mathcal{H}_i$ *is supported on a set of dimension* $\leq i+1$*, then* $\mathcal{F}$ *is* p*-regular.*

Proof. This is established by chopping $L_\bullet$ into short exact sequences in the usual way and chasing through the resulting diagram. (Compare [21, B.1.2, B.1.3], but note that the sheaf $\mathcal{H}_0$ there should refer to the augmented complex, as above.) □

Remark 1.5. The argument just completed shows that Theorem 1.3 remains true if X has several irreducible components, as well as possibly isolated points.

We conclude this section by observing that the same argument proves that Castelnuovo–Mumford regularity of surfaces behaves submultiplicatively in powers. For curves, this has been known for some time [7], [23].

Proposition 1.6. *Let* $X \subseteq \mathbf{P}^r$ *be a reduced* (*but possibly singular*) *surface, and denote by* $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}^r}$ *the ideal sheaf of* X*. If* $\mathcal{I}_X$ *is* m*-regular, then* $\mathcal{I}_X^a$ *is* am*-regular.*

Sketch of Proof. One argues just as in the proof of Theorem 1.3, reducing to statement (ii) of the previous Lemma. □

2. Macaulay-type bounds: preliminaries

This section is devoted to some preliminary results that will be used in the proof of Theorem A from the Introduction. In the first subsection, we discuss symmetric powers of a Koszul complex. The second is devoted to the computation of some direct images from a blow-up. Finally §2.3 gives the proof of Proposition E form the Introduction. We will focus mostly on the case when X is a variety.

2.1. Powers of Koszul complexes.

In this subsection we review the construction of symmetric powers of a Koszul complex. In the local setting this (and much more) appears in the paper [4] of Buchsbaum and Eisenbud, and it was revisited by Srinivasan in [24]. However for the convenience of the reader we give here a quick sketch of the particular facts we require. We continue to work over the complex numbers.

Let M be a smooth algebraic variety, and let V be a vector bundle of rank e on M. Fix integers $a, k \geq 1$. We denote by $S^{a,1^{k-1}}(V)$ the Schur power of V corresponding to the partition $(a, 1, \ldots, 1)$ ($k-1$ repetitions of 1). It follows from Pieri's rule that

$$\begin{aligned} S^{a,1^{k-1}}(V) \ &= \ \ker\Big(\Lambda^{k-1}V \otimes S^aV \longrightarrow \Lambda^{k-2}V \otimes S^{a+1}V\Big) \\ &= \ \operatorname{im}\Big(\Lambda^kV \otimes S^{a-1}V \longrightarrow \Lambda^{k-1}V \otimes S^aV\Big). \end{aligned} \tag{2.1}$$

Remark 2.1 (Properties of $S^{a,1^{k-1}}(V)$)**.** We collect some useful observations concerning this Schur power.

(i). If $k=1$ then $S^{a,1^{k-1}}(V)=S^aV$, while if $a=1$ then $S^{a,1^{k-1}}(V)=\Lambda^kV$. Moreover
$$S^{a,1^{k-1}}(V) \;=\; 0 \quad \text{when } k > \text{rank } V.$$

(ii). The bundle $S^{a,1^{k-1}}(V)$ is actually a summand of $S^{a-1}V\otimes\Lambda^kV$. In fact, Pieri shows that
$$S^{a-1}V\otimes\Lambda^kV \;=\; S^{a,1^{k-1}}(V)\,\oplus\,S^{a-1,1^k}(V).$$

(iii). If L is a line bundle on M, then it follows from (2.1) or (ii) that
$$S^{a,1^{k-1}}(V\otimes L) \;=\; S^{a,1^{k-1}}(V)\,\otimes\,L^{\otimes a+k-1}.$$

(iv). Suppose that $M=\mathbf{P}^r$ and
$$V \;=\; \mathcal{O}_{\mathbf{P}^r}(-d_0)\oplus\ldots\oplus\mathcal{O}_{\mathbf{P}^r}(-d_p)$$
with $d_0\geq\ldots\geq d_p$. Then it follows from (ii) that $S^{a,1^{k-1}}(V)$ is a direct sum of line bundles of degrees $\geq\; -(ad_0+d_1+\ldots+d_{k-1})$, and moreover a summand of this degree appears. In other words,
$$\text{reg}\big(\,S^{a,1^{k-1}}(V)\,\big) \;=\; ad_0+d_1+\ldots+d_{k-1}.$$

One can also realize $S^{a,1^{k-1}}(V)$ geometrically, à la Kempf [19].

Lemma 2.2. *Let $\pi:\mathbf{P}(V)\longrightarrow M$ be the projective bundle of one-dimensional quotients of V, and denote by F the kernel of the canonical quotient $\pi^*V\longrightarrow\mathcal{O}_{\mathbf{P}(V)}(1)$, so that F sits in the short exact sequence*

$$0\longrightarrow F\longrightarrow\pi^*V\longrightarrow\mathcal{O}_{\mathbf{P}(V)}(1)\longrightarrow 0 \tag{*}$$

of bundles on $\mathbf{P}(V)$. Then
$$S^{a,1^{k-1}}(V) \;=\; \pi_*\Big(\,\Lambda^{k-1}F\otimes\mathcal{O}_{\mathbf{P}(V)}(a)\,\Big).$$

Proof. In fact, (*) gives rise to a long exact sequence
$$0\longrightarrow\Lambda^{k-1}F\otimes\mathcal{O}_{\mathbf{P}(V)}(a)\longrightarrow\Lambda^{k-1}(\pi^*V)\otimes\mathcal{O}_{\mathbf{P}(V)}(a)\longrightarrow\Lambda^{k-2}(\pi^*V)\otimes\mathcal{O}_{\mathbf{P}(V)}(a+1)\longrightarrow\ldots\ .$$
The assertion follows from (2.1) upon taking direct images. □

Now suppose given a map of bundles

$$\varepsilon: V\longrightarrow\mathcal{O}_M \tag{2.2}$$

whose image is the ideal sheaf $\mathcal{I}\subseteq\mathcal{O}_M$ of a subscheme $Z\subseteq X$: equivalently, ε is dual to a section $\mathcal{O}_M\longrightarrow V^*$ whose zero-scheme is Z. We allow the possibility that ε is surjective, in which case $\mathcal{I}=\mathcal{O}_M$ and $Z=\varnothing$.

If Z has the expected codimension $e=\text{rank }(V)$, then $\mathcal{I}$ is resolved by the Koszul complex associated to ε. The following result of Buchsbaum and Eisenbud gives the resolution of powers of $\mathcal{I}$.

Proposition 2.3 ([4, Theorem 3.1], [24, Theorem 2.1])**.** *Fix $a \geq 1$. Then ε determines a complex*

$$\ldots \longrightarrow S^{a,1^2}(V) \longrightarrow S^{a,1}(V) \longrightarrow S^a V \xrightarrow{S^a(\varepsilon)} \mathcal{I}^a \longrightarrow 0 \tag{2.3}$$

of vector bundles on M. This complex is exact provided that either ε is surjective, or that Z has codimension = rank (V).

Observe from Remark 2.1 (i) that this complex has the same length as the Koszul complex of ε.

Proof. Returning to the setting of Lemma 2.2, denote by $\tilde{\varepsilon} : F \longrightarrow \mathcal{O}_{\mathbf{P}(V)}$ the composition of the inclusion $F \hookrightarrow \pi^*V$ with $\pi^*\varepsilon : \pi^*V \longrightarrow \pi^*\mathcal{O}_M$. The zero-locus of $\tilde{\varepsilon}$ defines the natural embedding of $\mathbf{P}(\mathcal{I})$ in $\mathbf{P}(V)$. Now consider the Koszul complex of $\tilde{\varepsilon}$. After twisting by $\mathcal{O}_{\mathbf{P}(V)}(a)$ this has the form:

$$\ldots \longrightarrow \Lambda^2 F \otimes \mathcal{O}_{\mathbf{P}(V)}(a) \longrightarrow F \otimes \mathcal{O}_{\mathbf{P}(V)}(a) \longrightarrow \mathcal{O}_{\mathbf{P}(V)}(a) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{I})}(a) \longrightarrow 0. \tag{*}$$

In view of Lemma 2.2, (2.3) arises by taking direct images. If ε is surjective, or defines a regular section of V^*, then the Koszul complex (*) is exact. Since the higher direct images of all the terms vanish, (*) pushes down to an exact complex. Furthermore, in this case $\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{I})}(a) = \mathcal{I}^a$ (cf [14, Theorem IV.2.2]), and the exactness of (2.3) follows. □

Example 2.4 (Macaulay's Theorem)**.** Suppose as in the Introduction that $f_0, \ldots, f_p \in \mathbf{C}[x_0, \ldots, x_r]$ are homogeneous polynomials of degrees $d_0 \geq \ldots \geq d_p$ that generate a finite colength ideal J. This gives rise to a surjective map

$$V \ = \ \oplus \mathcal{O}_{\mathbf{P}^r}(-d_i) \longrightarrow \mathcal{O}_{\mathbf{P}^r} \longrightarrow 0$$

of bundles on projective space. Keeping in mind Remark 2.1 (iv), Macaulay's statements (1) and (2) follow by looking at the cohomology of the resulting complex (2.3). When $p = r$ this complex has length $r + 1$, so one can also read off the non-surjectivity of

$$H^0\big(\mathbf{P}^r, S^a V(t)\big) \longrightarrow H^0\big(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(t)\big)$$

when $t < ad_0 + d_1 + \ldots + d_r - r$.

Example 2.5 (Complete intersection ideals)**.** Suppose that $Z \subseteq \mathbf{P}^r$ is a complete intersection of dimension ≥ 0. Applying Proposition 2.3 to the Koszul resolution of its homogeneous ideal I_Z, one sees that I_Z^a is saturated for every $a \geq 1$. This is a result of Zariski.

2.2. **Push-forwards from a blowing up.**

We compute here the direct images of multiples of the exceptional divisor under the blowing-up of a smooth subvariety.

Consider then a smooth variety M and a non–singular subvariety $X \subseteq M$ having codimension $e \geq 2$ and ideal sheaf $\mathcal{I} = \mathcal{I}_X \subseteq \mathcal{O}_M$. We consider the blowing-up

$$\mu : M' = \mathrm{Bl}_X(M) \longrightarrow M$$

of M along X. Write $\mathbf{E} \subseteq M'$ for the exceptional divisor of M', so that $\mathcal{I} \cdot \mathcal{O}_{M'} = \mathcal{O}_{M'}(-\mathbf{E})$. We recall that if $a > 0$ then

$$\mu_* \mathcal{O}_{M'}(-a\mathbf{E}) \ = \ \mathcal{I}^a \ \text{ and } \ R^j \mu_* \mathcal{O}_{M'}(-a\mathbf{E}) \ = \ 0 \text{ for } j > 0. \tag{2.4}$$

The following Proposition gives the analogous computation for positive multiples of $\mathbf{E}$.

Proposition 2.6. *Fix $a > 0$. Then*

$$R^j \mu_* \mathcal{O}_{M'}(a\mathbf{E}) \ = \ \mathcal{E}xt^j_{\mathcal{O}_M}\Big(\mathcal{I}^{a-e+1}\,,\,\mathcal{O}_M\Big).^5 \tag{2.5}$$

In particular, $\mu_* \mathcal{O}_{M'}(a\mathbf{E}) = \mathcal{O}_M$, $R^j \mu_* \mathcal{O}_{M'}(a\mathbf{E}) = 0$ if $j \neq 0, e-1$, and

$$R^{e-1} \mu_* \mathcal{O}_{M'}(a\mathbf{E}) \ = \ \mathcal{E}xt^{e-1}_{\mathcal{O}_M}\big(\mathcal{I}^{a-e+1}\,,\,\mathcal{O}_M\big).$$

Proof. This is a consequence of duality for μ, which asserts that

$$R\mu_* \, R\mathcal{H}om_{\mathcal{O}_{M'}}\big(\mathcal{F}\,,\,\omega_\mu\big) \ = \ R\mathcal{H}om_{\mathcal{O}_M}\big(\, R\mu_* \mathcal{F}\,,\mathcal{O}_M\,\big) \tag{*}$$

for any sheaf $\mathcal{F}$ on M', where ω_μ denotes the relative dualizing sheaf for μ ([18, (3.19) on page 86]). We apply this with

$$\mathcal{F} \ = \ \mathcal{O}_{M'}\big(\,(e-1-a)\mathbf{E}\,\big).$$

Then $R\mu_* \mathcal{F} = \mathcal{I}^{a-e+1}$ thanks to (2.4) (and a direct computation when $0 < a < e-1$), and $\omega_\mu = \mathcal{O}_{M'}\big((e-1)\mathbf{E}\big)$. Therefore the first assertion of the Proposition follows from (*). The vanishing of $\mathcal{E}xt^j_{\mathcal{O}_M}(\mathcal{I}^{a-e+1}, \mathcal{O}_M)$ for $j \neq 0, e-1$ follows from the perfection of powers of the ideal of a smooth variety (which in turn is a consequence eg of Proposition 2.3). ☐

Remark 2.7 (Generalization to multiplier ideal sheaves)**.** Let $\mathfrak{b} \subseteq \mathcal{O}_M$ be an arbitrary ideal sheaf, and let $\mu : M' \longrightarrow M$ be a log resolution of $\mathfrak{b}$, with $\mathfrak{b} \cdot \mathcal{O}_{M'} = \mathcal{O}_{M'}(-\mathbf{E})$. A completely parallel argument shows that for $a > 0$:

$$R^j \mu_* \mathcal{O}_{M'}(a\mathbf{E}) \ = \ \mathcal{E}xt^j_{\mathcal{O}_M}\big(\mathcal{J}\big(\mathfrak{b}^a\big)\,,\,\mathcal{O}_M\big),$$

where $\mathcal{J}\big(\mathfrak{b}^a\big)$ is the multiplier ideal of $\mathfrak{b}^a$. The formula (2.5) is a special case of this.

Corollary 2.8. *Continuing to work in characteristic zero, fix $a \geq 1$ and denote by $N = N_{X/M}$ the normal bundle to X in M. If $a \leq e-1$, then*

$$R^{e-1} \mu_* \, \mathcal{O}_{M'}(a\mathbf{E}) \ = \ 0.$$

If $a \geq e$, then $R^{e-1} \mu_* \, \mathcal{O}_{M'}(a\mathbf{E})$ has a filtration with successive quotients

$$S^k N \otimes \det N \quad \text{for} \ \ 0 \ \leq \ k \ \leq \ a-e.$$

Proof. The first statement follows directly from the previous Proposition. For the second, recall first that if E is any locally free $\mathcal{O}_X$-module, then – X being non-singular of codimension e in M –

$$\mathcal{E}xt^e_{\mathcal{O}_M}\big(\,E\,,\,\mathcal{O}_M\,\big) \ = \ E^* \otimes \det N,$$

[5] When $0 < a < e-1$ we take $\mathcal{I}^{a-e+1} = \mathcal{O}_M$.

while all the other $\mathcal{E}xt^j$ vanish. The claim then follows from Proposition 2.6 using the exact sequences

$$0 \longrightarrow \mathcal{I}^{k+1} \longrightarrow \mathcal{I}^k \longrightarrow S^k N^* \longrightarrow 0$$

together with the isomorphism $\big(S^k(N^*)\big)^* = S^k N$ valid in characteristic zero. ☐

Remark 2.9. Recalling that $\mathbf{E} = \mathbf{P}(N^*)$, one can inductively prove the Corollary directly, circumventing Proposition 2.6, by pushing forward the exact sequences

$$0 \longrightarrow \mathcal{O}_{M'}\big((k-1)\mathbf{E}\big) \longrightarrow \mathcal{O}_{M'}\big(k\mathbf{E}\big) \longrightarrow \mathcal{O}_{\mathbf{E}}(k\mathbf{E}) \longrightarrow 0.$$

However it seemed to us that the Proposition may be of independent interest.

2.3. **A vanishing theorem for normal bundles.** This final subsection is devoted to the proof of

Proposition 2.10. *Let $X \subseteq \mathbf{P}^r$ be a smooth complex projective variety of dimension n, and denote by $N = N_{X/\mathbf{P}^r}$ the normal bundle to X. Then*

$$H^i\Big(X\,,\, S^k N \otimes \det N \otimes \mathcal{O}_X(\ell)\Big) \ = \ 0$$

for all $i > 0$, $k \geq 0$ and $\ell \geq -r$.

Here $\mathcal{O}_X(k)$ denotes $\mathcal{O}_{\mathbf{P}^r}(k)|X$. We remark that similar statements were established by Schneider and Zintl in [22], but this particular vanishing does not seem to appear there. Other vanishings for normal bundles played a central role in [11].

Proof of Proposition 2.10. We use the abbreviation $\mathbf{P} = \mathbf{P}^r$. Starting from the exact sequence $0 \longrightarrow TX \longrightarrow T\mathbf{P}|X \longrightarrow N \longrightarrow 0$, we get a long exact sequence

$$(*) \qquad \ldots \longrightarrow S^{k-2}T\mathbf{P}|X \otimes \Lambda^2 TX \longrightarrow S^{k-1}T\mathbf{P}|X \otimes TX \longrightarrow S^k T\mathbf{P}|X \longrightarrow S^k N \longrightarrow 0.$$

By adjunction, $\det N \otimes \mathcal{O}_X(\ell) = \omega_X \otimes \mathcal{O}_X(\ell + r + 1)$. So after twisting through by $\det N \otimes \mathcal{O}_X(\ell)$ in (*), we see that the Proposition will follow if we prove:

$$(**) \qquad H^i\Big(X\,,\, S^{k-j}T\mathbf{P}|X \otimes \Lambda^j TX \otimes \omega_X \otimes \mathcal{O}_X(\ell + r + 1))\Big) \ = \ 0 \quad \text{for} \ \ i \ \geq \ j+1$$

when $0 \leq j \leq k$ and $\ell \geq -r$. It follows from the Euler sequence that $S^m T\mathbf{P}|X$ has a presentation of the form

$$0 \longrightarrow \oplus\, \mathcal{O}_X(m-1) \longrightarrow \oplus\, \mathcal{O}_X(m) \longrightarrow S^m T\mathbf{P}|X \longrightarrow 0,$$

so for (**) it suffices in turn to verify that

$$H^i\big(X, \Lambda^j TX \otimes \omega_X \otimes \mathcal{O}_X(\ell_1)\big) \ = \ 0$$

for $i \geq j+1$ and $\ell_1 > 0$. But $\Lambda^j TX \otimes \omega_X = \Omega_X^{n-j}$, so finally we're asking that

$$H^i\big(X, \Omega_X^{n-j} \otimes \mathcal{O}_X(\ell_1)\big) \ = \ 0 \ \ \text{for} \ \ i \geq j+1 \text{ and } \ell_1 > 0,$$

and this follows from Nakano vanishing. ☐

3. Proof of Theorem A

We now turn to the proof of Theorem A from the Introduction.

Consider then a non-singular scheme $X \subseteq \mathbf{P}^r$ that is cut out as a scheme by hypersurfaces of degrees $d_0 \geq \ldots \geq d_p$. Equivalently, we are given a surjective homomorphism of sheaves:

$$\varepsilon : U \longrightarrow \mathcal{I}_X \quad , \quad U \;=\; \oplus \mathcal{O}_{\mathbf{P}^r}(-d_i).$$

Let $\mu : \mathbf{P}' = \mathrm{Bl}_X(\mathbf{P}^r) \longrightarrow \mathbf{P}^r$ be the blowing up of X, with exceptional divisor $\mathbf{E} \subseteq \mathbf{P}'$, so that $\mathcal{I}_X \cdot \mathcal{O}_{\mathbf{P}'} = \mathcal{O}_{\mathbf{P}'}(-\mathbf{E})$. Write H for the pull-back to $\mathbf{P}'$ of the hyperplane class on $\mathbf{P}^r$, and set $U' = \mu^* U$. Thus on $\mathbf{P}'$ we have a surjective map of bundles:

$$\varepsilon' : U' \longrightarrow \mathcal{O}_{\mathbf{P}'}(-\mathbf{E}). \tag{3.1}$$

Noting that

$$H^0\Big(\mathbf{P}' \, , \, \mathcal{O}_{\mathbf{P}'}(tH - a\mathbf{E})\Big) \;=\; H^0\Big(\mathbf{P}^r \, , \, \mathcal{I}_X^a \otimes \mathcal{O}_{\mathbf{P}^r}(t)\Big),$$

one sees as in Lemma 1.1 that the question is to prove the surjectivity of

$$H^0\Big(\mathbf{P}' \, , \, S^a U' \otimes \mathcal{O}_{\mathbf{P}'}(tH)\Big) \longrightarrow H^0\Big(\mathbf{P}' \, , \, \mathcal{O}_{\mathbf{P}'}(tH - a\mathbf{E})\Big) \tag{3.2}$$

for $t \geq ad_0 + d_1 + \ldots + d_r - r$.

To this end, we pass to the Buchsbaum–Eisenbud complex (2.3) constructed from

$$U' \otimes \mathcal{O}_{\mathbf{P}'}(\mathbf{E}) \stackrel{\varepsilon'}{\longrightarrow} \mathcal{O}_{\mathbf{P}'} \longrightarrow 0.$$

Twisting through by $\mathcal{O}_{\mathbf{P}'}(tH - a\mathbf{E})$, we arrive at a long exact sequence of vector bundles on $\mathbf{P}'$ having the form:

(3.3)

$$\ldots \to \underbrace{S^{a,1^2}U' \otimes \mathcal{O}_{\mathbf{P}'}(tH + 2\mathbf{E})}_{C_2} \to \underbrace{S^{a,1}U' \otimes \mathcal{O}_{\mathbf{P}'}(tH + \mathbf{E})}_{C_1} \to \underbrace{S^a U' \otimes \mathcal{O}_{\mathbf{P}'}(tH)}_{C_0} \to \mathcal{O}_{\mathbf{P}'}(tH - a\mathbf{E}) \to 0.$$

With indexing as indicated, the i^{th} term of this sequence is given by

$$C_i \;=\; S^{a,1^i}(U') \otimes \mathcal{O}_{\mathbf{P}'}(tH + i\mathbf{E}).$$

In order to establish the surjectivity (3.2) it suffices upon chasing through (3.3) to prove that

$$H^i\big(\mathbf{P}', C_i\big) \;=\; 0 \quad \text{for} \quad 1 \leq i \leq r \tag{3.4}$$

provided that $t \geq ad_0 + d_1 + \ldots + d_r - r$. But now recall (Remark 2.1) that if $i \leq r$ then $S^{a,1^i}(U')$ is a sum of line bundles $\mathcal{O}_{\mathbf{P}'}(mH)$ with

$$m \;\geq\; -ad_0 - d_1 - \ldots - d_i \geq -ad_0 - d_1 - \ldots - d_r.$$

Hence when $t \geq ad_0 + d_1 + \ldots + d_r - r$, C_i is a sum terms of the form

$$\mathcal{O}_{\mathbf{P}'}(\ell H + i\mathbf{E}) \;\; \text{with } \ell \geq -r.$$

Therefore (3.4) – and with it Theorem A – is a consequence of

Proposition 3.1. *If* $\ell \geq -r$, *then*

$$H^i\big(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(\ell H + i\mathbf{E})\big) \ = \ 0 \ \ \textit{for} \ \ i > 0.$$

Proof. Thanks to the Leray spectral sequence, it suffices to show:

$$(*) \qquad H^j\big(\mathbf{P}^r, R^k\mu_*\mathcal{O}_{\mathbf{P}'}(\ell H + i\mathbf{E})\big) \ = \ 0 \ \text{ when } j + k = i > 0.$$

For $k = 0$, observe that $\mu_*\mathcal{O}_{\mathbf{P}'}(\ell H + i\mathbf{E}) = \mathcal{O}_{\mathbf{P}^r}(\ell)$, and these sheaves have no higher cohomology when $\ell \geq -r$. Suppose that $k > 0$. Suppose first that X is smooth and irreducible of codimension e. By Proposition 2.6, the only non-vanishing higher direct images are the $R^{e-1}\mu_*\mathcal{O}_{\mathbf{P}'}(\ell H + i\mathbf{E})$, which do not appear when $i \leq e - 1$. So (*) holds when $j = 0, k = e - 1$. It remains to consider the case $k = e - 1$ and $i \geq e$, so $j = i - (e - 1) > 0$. Here Corollary 2.8 implies that the R^{e-1} have a filtration with quotients

$$S^\alpha N \otimes \det N \otimes \mathcal{O}_X(\ell),$$

where as above $N = N_{X/\mathbf{P}^r}$ is the normal bundle to X in $\mathbf{P}^r$. But since we are assuming $\ell \geq -r$, Proposition 2.10 guarantees that these sheaves have vanishing higher cohomology. This completes the proof when X is irreducible. When X has several components of possibly different dimensions one argues similarly one component at a time: we leave the details to the interested reader. □

Remark 3.2. Observe that if X is defined by $p < r$ equations, then the argument just completed goes through taking $d_{p+1} = \ldots = d_r = 0$.

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