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# A Pattern Avoidance Characterization for Smoothness of Positroid Varieties

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**Abstract.** Positroids are certain representable matroids originally studied by Postnikov in connection with the totally nonnegative Grassmannian and now used widely in algebraic combinatorics. The positroids give rise to determinantal equations defining positroid varieties as subvarieties of the Grassmannian variety. Rietsch, Knutson–Lam–Speyer and Pawlowski studied geometric and cohomological properties of these varieties. In this paper, we continue the study of the geometric properties of positroid varieties by establishing several equivalent conditions characterizing smooth positroid varieties using a variation of pattern avoidance defined on decorated permutations, which are in bijection with positroids. Furthermore, we give a combinatorial method for determining the dimension of the tangent space of a positroid variety at key points using an induced subgraph of the Johnson graph. We also give a Bruhat interval characterization of positroids.

**Keywords:** positroids, decorated permutations, pattern avoidance, Grassmannian

## 1 Introduction

*Positroids* are an important family of realizable matroids originally defined by Postnikov in [16] in the context of the totally nonnegative part of the Grassmannian variety. These matroids and the totally positive part of the Grassmannian variety have played a critical role in the theory of cluster algebras and soliton solutions to the KP equations and have connections to statistical physics, integrable systems, and scattering amplitudes [2, 17, 18]. Positroids are closed under restriction, contraction, duality, and cyclic shift of the ground set, and furthermore, they have particularly elegant matroid polytopes [1].

*Positroid varieties* were studied by Knutson, Lam, and Speyer in [9], building on the work of Postnikov [16] and Rietsch [17]. They are homogeneous subvarieties of the complex Grassmannian variety $Gr(k,n)$ that are defined by determinantal equations determined by the bases of a positroid. They can also be described as projections of Richardson varieties $X_u \cap X^v$ in the complete flag manifold to $Gr(k,n)$. These varieties

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have beautiful geometric, representation theory, and combinatorial connections [10, 15]. See the background section for notation and further background.

The positroids $\mathcal{M}$ of rank $k$ on a ground set of size $n$ are in bijection with many different combinatorial objects [14, 16], including

1. decorated permutations $w^\circ$ on $n$ elements with $k$ anti-exceedances,
2. Grassmann necklaces $(I_1, \ldots, I_n) \in \binom{[n]}{k}^n$, and
3. intervals $[u, v]$ in Bruhat order on $S_n$ such that $v$ is a $k$-Grassmannian permutation.

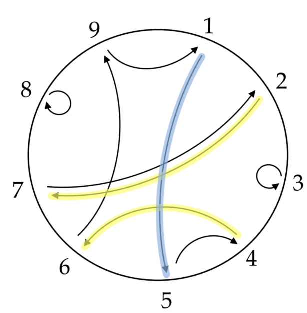
Here, a decorated permutation $w^\circ$ on $n$ elements is a permutation $w \in S_n$ together with an orientation clockwise or counterclockwise, denoted $\overrightarrow{i}$ or $\overleftarrow{i}$ respectively, on the fixed points of $w$. In addition to these, there are bijections to juggling sequences, Ⅎ-diagrams, equivalence classes of plabic graphs, and bounded affine permutations [1, 9, 16]. In Section 2, we will sketch the relevant bijections and terminology.

Many of the properties of positroid varieties can be "read off" from one or more of these bijectively equivalent definitions. Thus, we will index a positroid variety $\Pi_{\mathcal{M}} = \Pi_{w^\circ} = \Pi_{[u,v]}$, depending on the relevant context. For example, the codimension of $\Pi_{\mathcal{M}}$ in $Gr(k, n)$ is easy to read off from the decorated permutation as follows.

Let $S^\circ_{n,k}$ be the set of decorated permutations on $n$ elements with $k$ anti-exceedances. The *chord diagram* $D(w^\circ)$ of $w^\circ \in S^\circ_{n,k}$ is constructed by placing the numbers $1, 2, \ldots, n$ on $n$ vertices around a circle in clockwise order, and then, for each $i$, draw a directed arc from $i$ to $w(i)$ with a minimal number of crossings between distinct arcs while staying completely inside the circle. The arcs beginning at fixed points should be drawn clockwise or counterclockwise according to their orientation in $w^\circ$.

An *alignment* in $D(w^\circ)$ is a pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in the same direction. A pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in opposite directions is called a *misalignment*. A pair of directed edges which must cross if both are drawn inside the cycle is called a *crossing* [16, Sect. 5]. Let *Alignments*$(w^\circ)$ denote the set of alignments of $D(w^\circ)$.

*Example* 1.1. Let $w^\circ = 57\overleftarrow{3}6492\overrightarrow{8}1$ be the decorated permutation with a counterclockwise fixed point at 3 and a clockwise fixed point at 8. The chord diagram for $w^\circ$ is



Here, for example, $(2 \mapsto 7, 4 \mapsto 6)$ highlighted in yellow is an alignment, $(9 \mapsto 1, 4 \mapsto 6)$ is a misalignment, and both $(1 \mapsto 5, 2 \mapsto 7)$ and $(1 \mapsto 5, 5 \mapsto 4)$ are crossings. Note, $(4 \mapsto 6, 8 \mapsto 8)$ is an alignment, and $(4 \mapsto 6, 3 \mapsto 3)$ is a misalignment.

**Theorem 1.2** ([9, 16]). *For any decorated permutation $w^{\circ} \in S^{\circ}_{n,k}$ and associated Bruhat interval $[u, v]$, the codimension of $\Pi_{w^{\circ}}$ in $Gr(k, n)$ is*

$$\operatorname{codim}(\Pi_{w^{\circ}}) = \#Alignments(w^{\circ}) = k(n-k) - [\ell(v) - \ell(u)]. \tag{1.1}$$

Schubert varieties in the flag variety are indexed by permutations and are closely related to positroid varieties, as explained in Section 2. Smoothness of Schubert varieties is completely characterized by pattern avoidance of the corresponding permutations [12]. When studying the partially asymmetric exclusion process and its surprising connection to the Grassmannian, Sylvie Corteel posed the idea of considering patterns in decorated permutations [7]. This suggestion was the foundation for the present work.

We use the explicit equations defining a positroid variety in $Gr(k, n)$ to determine if the variety is smooth or singular. In general, a variety $X$ defined by polynomials $f_1, \ldots, f_s$ is *singular* if there exists a point $x \in X$ such that the Jacobian matrix, $Jac$, of partial derivatives of the $f_i$ satisfies $\operatorname{rank}(Jac|_x) < \operatorname{codim} X$. It is *smooth* if no such point exists. The value $\operatorname{rank}(Jac|_x)$ is the codimension of the tangent space to $X$ at the point $x$. Thus, $\operatorname{rank}(Jac|_x) < \operatorname{codim} X$ implies the dimension of the tangent space to the variety $X$ at $x$ is strictly larger than the dimension of the variety $X$, hence $x$ is a singularity like a cusp on a curve. In the case of a positroid variety $\Pi_{w^{\circ}}$, Theorem 1.2 implies that a point $x \in \Pi_{w^{\circ}}$ is a singularity of $\Pi_{w^{\circ}}$ if

$$\operatorname{rank}(Jac|_x) \; < \; \operatorname{codim} \Pi_{w^{\circ}} \; = \; \#Alignments(w^{\circ}). \tag{1.2}$$

Our first main theorem reduces the problem of finding singular points in a positroid variety to checking the rank of the Jacobian only at a finite number of $T$-fixed points. For any $J = \{j_1, \ldots, j_k\} \subseteq [n]$, let $A_J$ be the element in $Gr(k, n)$ spanned by the elementary row vectors $e_i$ with $i \in J$, or equivalently the subspace represented by a $k \times n$ matrix with a 1 in cell $(i, j_i)$ for each $j_i \in J$ and zeros everywhere else. These are the $T$-fixed points of $Gr(k, n)$, where $T \subset \mathrm{GL}(n)$ is the set of invertible diagonal matrices over $\mathbb{C}$. The reduction follows from the decomposition of $\Pi_{[u,v]}$ as a projected Richardson variety. Every point $A \in \Pi_{[u,v]}$ lies in the projection of some intersection of a Schubert cell with an opposite Schubert variety $C_y \cap X^v$ for $y \in [u, v]$. In particular, if $y = y_1 y_2 \cdots y_n \in [u, v]$ in one-line notation and we define $y[k] := \{y_1, y_2, \ldots, y_k\}$, then $A_{y[k]}$ is in the projection of $C_y \cap X^v$.

**Theorem 1.3.** *Assume $A \in \Pi_{[u,v]}$ is the image of a point in $C_y \cap X^v$ projected to $Gr(k, n)$ for some $y \in [u, v]$. Then the codimension of the tangent space to $\Pi_{[u,v]}$ at $A$ is bounded below by* $\operatorname{rank}(Jac|_{A_{y[k]}})$.

Theorem 1.3 indicates that the $T$-fixed points of the form $A_{y[k]}$ such that $y \in [u,v]$ are key for understanding the singularities of $\Pi_{[u,v]}$. In fact, the equations determining $\Pi_{[u,v]}$ and the bases of the positroid $\mathcal{M}$ associated with the interval $[u,v]$ can be determined from the permutations in the interval by the following theorem. Our proof of the following theorem depends on Theorem 2.5. It also follows from [11, Lemma 3.11].

**Theorem 1.4.** Let $w^\circ \in S^\circ_{n,k}$ have associated Bruhat interval $[u,v]$ and positroid $\mathcal{M}$. Then $\mathcal{M}$ is exactly the collection of initial sets of permutations in the Bruhat interval $[u,v]$,

$$\mathcal{M} = \{y[k] \,:\, y \in [u,v]\}.$$

Our next theorem provides a method to compute the rank of the Jacobian of $\Pi_{[u,v]}$ explicitly at the $T$-fixed points. Therefore, we can also compute the dimension of the tangent space of a positroid variety at those points.

**Theorem 1.5.** *Let $w^\circ \in S^\circ_{n,k}$ have associated Bruhat interval $[u,v]$ and positroid $\mathcal{M}$. For any $y \in [u,v]$, the codimension of the tangent space to $\Pi_{[u,v]} \subseteq Gr(k,n)$ at $A_{y[k]}$ is*

$$\operatorname{rank}(Jac|_{A_{y[k]}}) \;=\; \#\left\{I \in \binom{[n]}{k} \setminus \mathcal{M} \,:\, |I \cap y[k]| = k-1\right\}. \tag{1.3}$$

The formula in (1.3) is reminiscent of the *Johnson graph* $J(k,n)$ with vertices given by the $k$-subsets of $[n]$ such that two $k$-subsets $I,J$ are connected by an edge precisely if $|I \cap J| = k-1$. For a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $J(\mathcal{M})$ denote the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. We call $J(\mathcal{M})$ the *Johnson graph of* $\mathcal{M}$. Note, the Johnson graph of $\mathcal{M}$ is closely related to the Basis Exchange Property for matroids. Theorem 1.5 implies $J(\mathcal{M})$ encodes aspects of the geometry of the positroid varieties like the Bruhat graph in the theory of Schubert varieties [6].

To state our main theorem characterizing smoothness of positroid varieties, we need to define two types of patterns that may occur in a chord diagram. Given an alignment $(i \mapsto w(i), j \mapsto w(j))$ in $D(w^\circ)$, if there exists a third arc $(h \mapsto w(h))$ which forms a crossing with both $(i \mapsto w(i))$ and $(j \mapsto w(j))$, we say $(i \mapsto w(i), j \mapsto w(j))$ is a *crossed alignment* of $w^\circ$. In the example above, $(2 \mapsto 7, 4 \mapsto 6)$ is a crossed alignment; this alignment is crossed for instance by $(1 \mapsto 5)$, highlighted in blue. We call a permutation of the form $w(i) = i + t \pmod n$ for some fixed integer $1 \le t \le n$ a *spirograph permutation*, and we call its chord diagram a *spirograph*. We think of alignments, crossings, crossed alignments, and spirographs as subgraph patterns for decorated permutations.

**Theorem 1.6.** *For $w^\circ \in S^\circ_{n,k}$ with associated positroid $\mathcal{M}$, the following are equivalent.*

1. *The positroid variety $\Pi_{\mathcal{M}}$ is smooth.*

2. *For every $J \in \mathcal{M}$, $\#\{I \in \mathcal{M} \,:\, |I \cap J| = k-1\} \;=\; k(n-k) - \#Alignments(w^\circ)$.*

3. *The graph $J(\mathcal{M})$ is regular, and each vertex has degree $k(n-k) - \#Alignments(w^{\circ})$.*

4. *The decorated permutation $w^{\circ}$ has no crossed alignments.*

5. *The chord diagram of $w^{\circ}$ is a union of spirographs on a noncrossing partition of $[n]$.*

In this extended abstract, we outline the steps for the proof of Theorem 1.6 and leave the details of the remaining proofs as well as enumerative results to the forthcoming paper [4] and OEIS entries [13, A349413, A349456, A349457, A349458]. In Section 2, we provide background material and define our notation. In Section 3, we reduce the proof of Theorem 1.6 to derangements. We provide further reductions which state that $\Pi_{w^{\circ}}$ is smooth if and only if the positroid varieties defined by flipping, inverting, or rotating $D(w^{\circ})$ are also smooth. Finally, we use the fact that we can flip and rotate a chord diagram with a crossed alignment so that the crossing arc has its tail at 1 and crosses the alignment from starboard to port when the alignment is viewed as a boat moving forward, as shown in the highlighted crossed alignment in Example 1.1. In this configuration, we show that the $T$-fixed point associated with the anti-exceedance set of $w^{\circ}$ is singular in $\Pi_{w^{\circ}}$. If no crossed alignment exists, we show that $D(w^{\circ})$ is a disjoint union of spirographs, which leads to a regular Johnson graph of $\mathcal{M}$.

# 2 Background

We begin by giving notation and some background on several combinatorial objects and theorems from the literature. These objects will be used to index the varieties discussed throughout the paper. We will then introduce our notation for the flag variety, the Grassmannian, Schubert varieties, Richardson varieties, and positroid varieties. See Fulton's book on Young Tableaux for further background [8].

## 2.1 Combinatorial objects

For integers $i \leq j$, let $[i,j]$ denote the set $\{i, i+1, \ldots, j\}$, and write $[n] := [1,n]$. Let $\binom{[n]}{k}$ be the set of $k$-element subsets of $[n]$. Call $J \in \binom{[n]}{k}$ a $k$-subset of $[n]$. Let $A_J$ be the $k \times n$ matrix whose restriction to the columns indexed by $J$ is the identity matrix $I_k$, and whose other entries are zeros.

*Example* 2.1. For $J = \{2,4,8\} \in \binom{[9]}{3}$ , $A_J$ is the matrix

$$A_J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Define the *Gale partial order*, $\le$, on $k$-subsets of $[n]$ as follows. Let $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$. Then $I \le J$ if and only if $i_h \le j_h$ for all $h \in [k]$. This partial order is known by many other names; we are following [1] for consistency.

A *matroid* of rank $k$ on $[n]$, defined by its bases, is a nonempty subset $\mathcal{M} \subseteq \binom{[n]}{k}$ satisfying the following Basis Exchange Property: if $I, J \in \mathcal{M}$ such that $I \neq J$ and $a \in I \setminus J$, then there exists some $b \in J \setminus I$ such that $(I \setminus \{a\}) \cup \{b\} \in \mathcal{M}$. Compare the notion of matroid basis exchange to basis exchange in linear algebra.

For example, let $A$ be a full rank $k \times n$ matrix. The *matroid of $A$* is the set

$$\mathcal{M}_A := \left\{ J \in \binom{[n]}{k} \,:\, \Delta_J(A) \neq 0 \right\},$$

where $\Delta_J(A)$ is the determinant of the $k \times k$ submatrix of $A$ lying in column set $J$.

*Example* 2.2. The matroid of

$$A = \begin{bmatrix} 0 & 3 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

is

$$\mathcal{M}_A = \{\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,6\},\{5,6\}\} \subseteq \binom{[6]}{2}.$$

Let $S_n$ be the set of permutations of $[n]$, where we think of a permutation as a bijection from the set $[n]$ to itself. For $w \in S_n$, let $w_i = w(i)$, and write $w$ in *one-line notation* as $w = w_1 w_2 \cdots w_n$. A permutation with no fixed points $i = w(i)$ is a *derangement*. The permutation matrix $M_w$ of $w$ is the $n \times n$ matrix that has a 1 in cell $(i, w_i)$ for each $i \in [n]$ and zeros elsewhere. The *length of* $w \in S_n$ is the number of inversions in $w$,

$$\ell(w) := \#\{(i,j) \,:\, i < j \text{ and } w(i) > w(j)\}.$$

*Example* 2.3. For $w = 3124$, the length of $w$ is $\ell(w) = 2$, and $M_w$ is the matrix

$$M_{3124} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note, the permutation matrix of $w^{-1}$ is $M_w^T$. Permutation multiplication is given by function composition so that if $wv = u$, then $w(v(i)) = u(i)$. Hence, $M_w^T M_v^T = M_u^T$.

For $1 \le k \le n$, write $S_k \times S_{n-k}$ for the subgroup of $S_n$ consisting of permutations that send $[k]$ to $[k]$ and $[k+1, n]$ to $[k+1, n]$. For $0 \le k \le n$, a permutation $w \in S_n$ is a *$k$-Grassmannian permutation* if $w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_n$. This is equivalent to saying that $w$ is the minimal length element of its coset $w \cdot (S_k \times S_{n-k})$. The permutation $w = 35124$ is 2-Grassmannian.

**Definition 2.4** ([5, Ch. 2])**.** For $u, v \in S_n$, $u \le v$ in *Bruhat order* if $u[i] \le v[i]$ for all $i \in [n]$.

For each $u \le v$ in Bruhat order, the *interval* $[u, v]$ is defined to be $[u, v] := \{y \in S_n : u \le y \le v\}$. The intervals $[u, v]$ where $v$ is a $k$-Grassmannian permutation are key to this work. In this case, the following simpler criterion for Bruhat order follows from work of Bergeron–Sottile [3, Theorem A].

**Theorem 2.5.** *Let $u, v \in S_n$, and assume $v$ is $k$-Grassmannian. Then $u \le v$ if and only if*

*(i) for every $1 \le j \le k$, we have $u(j) \le v(j)$, and*

*(ii) for every $k < m \le n$, we have $u(m) \ge v(m)$.*

## 2.2 Grassmannian, Flag, and Richardson Varieties

For $0 \le k \le n$, the points in the *Grassmannian variety*, $Gr(k, n)$, are the $k$-dimensional subspaces of $\mathbb{C}^n$. Up to left multiplication by a matrix in $\mathrm{GL}_k$, we may represent $V \in Gr(k, n)$ by a full rank $k \times n$ matrix $A_V$ such that $V$ is the row span of $A_V$. Let $\mathrm{Mat}_{kn}$ be the set of full rank $k \times n$ matrices. One may think of $Gr(k, n)$ as the cosets $\mathrm{GL}_k \backslash \mathrm{Mat}_{kn}$. The Grassmannian varieties are smooth manifolds. This includes the case when $k = n = 0$, in which case $Gr(k, n)$ is the single point $(0)$.

Let $\mathcal{F}\ell(n)$ be the *complete flag variety* of nested subspaces of $\mathbb{C}^n$. A complete flag $V_\bullet = (V_1, \ldots, V_n)$ can be represented as an invertible $n \times n$ matrix where the row span of the first $j$ rows corresponds with the $j^{th}$ subspace in the flag. For a subset $J \subseteq [n]$, let $\mathrm{Proj}_J : \mathbb{C}^n \to \mathbb{C}^{|J|}$ be the projection map onto the indices specified by $J$. Then for every permutation, $w \in S_n$, there is a Schubert cell $C_w$ and an opposite Schubert cell $C^w$ in $\mathcal{F}\ell(n)$ defined by

$$C_w = \{V_\bullet \in \mathcal{F}\ell(n) \ : \ \dim(\mathrm{Proj}_{[j]}(V_i)) = |w[i] \cap [j]| \text{ for all } i, j\} \text{ and}$$
$$C^w = \{V_\bullet \in \mathcal{F}\ell(n) \ : \ \dim(\mathrm{Proj}_{[n-j+1,n]}(V_i)) = |w[i] \cap [n-j+1, n]| \text{ for all } i, j\}.$$

The *Schubert variety* $X_w$ is the closure of $C_w$ in the Zariski topology on $\mathcal{F}\ell(n)$, and similarly, the *opposite Schubert variety* $X^w$ is the closure of $C^w$. Bruhat order determines which Schubert cells are in a Schubert variety,

$$X_w = \biguplus_{y \ge w} C_y \quad \text{and} \quad X^w = \biguplus_{v \le w} C^v. \tag{2.1}$$

For permutations $u$ and $v$ in $S_n$, with $u \le v$, the *Richardson variety* is a nonempty variety in $\mathcal{F}\ell(n)$ and is defined as the intersection $X_u^v := X_u \cap X^v$. Then $\dim X_u^v = \ell(v) - \ell(u)$. The decompositions of $X_u$ and $X^v$ into Schubert cells and opposite Schubert cells yield

$$X_u^v = \biguplus_{u \le y \le v} (C_y \cap X^v) = \Big(\biguplus_{y \ge u} C_y\Big) \cap \Big(\biguplus_{t \le v} C^t\Big).$$

## 2.3 Positroids and Positroid Varieties

Postnikov and Rietsch considered an important cell decomposition of the totally nonnegative Grassmannian [16, 17]. The term *positroid* does not appear in either paper, but has become the name for the matroids that index the nonempty matroid strata in that cell decomposition. They also individually considered the closures of those cells, which determines an analog of Bruhat order. The cohomology classes for these cell closures was investigated by Knutson, Lam, and Speyer [9, 10] and Pawlowski [15].

**Definition 2.6.** Let $Gr(k,n)^{tnn}$ be the points in $Gr(k,n)$ that can each be represented by a real valued $k \times n$ matrix $A$ such that every minor $\Delta_I(A)$ satisfies $\Delta_I(A) \geq 0$. A *positroid* is a matroid of the form $\mathcal{M}_A$ for some matrix $A \in Gr(k,n)^{tnn}$.

Postnikov also made the following definitions in [16, Sect 16]. Given a decorated permutation $w^\circ$, as defined in the introduction, call $i \in [n]$ an *anti-exceedance* of $w^\circ$ if $i < w^{-1}(i)$ or if $w(i) = \overrightarrow{i}$ is a clockwise fixed point. Fix $r \in [n]$. Let $<_r$ be the shifted linear order on $[n]$ given by $r <_r (r+1) <_r \cdots <_r n <_r 1 <_r \cdots <_r (r-1)$. The *shifted anti-exceedance set* $I_r(w^\circ)$ of $w^\circ$ is the anti-exceedance set of $w^\circ$ with respect to the shifted linear order $<_r$ on $[n]$,

$$I_r(w^\circ) = \{i \in [n] \ : \ i <_r w^{-1}(i) \text{ or } w(i) = \overrightarrow{i}\}.$$

Thus, $I_1(w^\circ)$ is the set of anti-exceedances of $w^\circ$. Recall that $S^\circ_{n,k}$ is the set of decorated permutations with anti-exceedance set $I_1(w^\circ)$ of size $k$. The *Grassmann necklace* associated with $w^\circ$ is $(I_1(w^\circ), \ldots, I_n(w^\circ))$. By construction, $|I_1(w^\circ)| = \cdots = |I_n(w^\circ)| = k$.

Using the shifted linear order on $[n]$, we may define the *shifted Gale order* $\prec_r$ on $\binom{[n]}{k}$. Specifically, if $I, J \in \binom{[n]}{k}$, where $I = \{i_1 <_r \cdots <_r i_k\}$ and $J = \{j_1 <_r \cdots <_r j_k\}$, then $I \preceq_r J$ if $i_h \leq_r j_h$ for all $h \in [k]$. The positroid associated with a decorated permutation can be defined using the shifted anti-exceedance sets and the shifted Gale orders.

**Theorem 2.7** ([14, 16])**.** *For $w^\circ \in S^\circ_{n,k}$, the set*

$$\mathcal{M}(w^\circ) := \left\{ I \in \binom{[n]}{k} \ : \ I_r(w^\circ) \preceq_r I \text{ for all } r \in [n] \right\} \tag{2.2}$$

*is a positroid. Conversely, for every positroid $\mathcal{M}$ of rank $k$ on ground set $[n]$, there exists a unique decorated permutation $w^\circ \in S^\circ_{n,k}$ such that the sequence of minimal elements in the shifted Gale order on the subsets in $\mathcal{M}$ is the Grassmann necklace of $w^\circ$.*

To find the interval $[u,v]$ corresponding to $w^\circ$ following [9], compute the set $I_1 = I_1(w^\circ)$, say $|I_1| = k$. Let $w$ be the permutation associated with $w^\circ$ by forgetting the decorations. Let $v$ be the unique $k$-Grassmannian permutation such that $\{v_1, \ldots, v_k\} = w^{-1}(I_1)$. Then $wv = u$, so $I_1 = u[k] = \{u_1, \ldots, u_k\}$. To recover $w^\circ$ from $[u,v]$, compute

$w = uv^{-1}$ and orient all of the fixed points of $w$ which are in $u[k]$ to be clockwise, and orient all others to be counterclockwise.

For example, for the decorated permutation $w^\circ = 57\overleftarrow{3}6492\overrightarrow{8}1$ in Example 1.1, we have $k = 4$, the Grassmann necklace is

$$(I_1, \ldots, I_9) = (\{1248\}, \{2458\}, \{4578\}, \{4578\}, \{5678\}, \{4678\}, \{4789\}, \{2489\}, \{2489\}),$$

and the corresponding interval $[u, v]$ has $u = 428157369$ and $v = 578912346$. The corresponding positroid has 22 elements.

In Example 2.2, the matrix $A$ has all nonnegative $2 \times 2$ minors, so the associated matroid is a positroid. The minimal elements in shifted Gale order are $(\{24\}, \{24\}, \{34\}, \{46\}, \{56\}, \{26\})$, which is the Grassman necklace for the decorated permutation $\overleftarrow{1}36524$. The associated Bruhat interval is $[241365, 561234]$.

As mentioned in the introduction, there are many other objects in bijection with positroids and decorated permutations. We refer the reader to [1] for a nice survey of many other explicit bijections.

Let $\pi_k \colon \mathcal{F}\ell(n) \to Gr(k, n)$ be the projection map which sends a flag $V_\bullet = (0 \subset V_1 \subset \cdots \subset V_n)$ to the $k$-dimensional subspace $V_k$. Identifying a full rank $n \times n$ matrix $M$ with the point it represents in $\mathcal{F}\ell(n)$, then $\pi_k(M)$ denotes the span of the top $k$ rows of $M$.

**Theorem/Definition 2.8** ([9, Theorem 5.1])**.** *Given a decorated permutation $w^\circ \in S^\circ_{n,k}$ along with its associated Bruhat interval $[u, v]$ and positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, the following are equivalent definitions of the positroid variety $\Pi_{w^\circ} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$.*

1. *The positroid variety $\Pi_{\mathcal{M}}$ is the homogeneous subvariety of $Gr(k, n)$ whose vanishing ideal is generated by the Plücker coordinates $\{\Delta_I : I \notin \mathcal{M}\}$.*

2. *The positroid variety $\Pi_{[u,v]}$ is the projection of the Richardson variety $X_u^v$ to $Gr(k, n)$, so $\Pi_{[u,v]} = \pi_k(X_u^v)$.*

# 3 Outline of Proofs

In this section, we outline the steps for reducing the proof of Theorem 1.6 to equivalence classes of derangements under flip, inverses, and rotation. Theorem 1.4 leads to the first reduction, allowing us to effectively ignore fixed points. In particular, $w^\circ$ has a fixed point $j$ if and only if the corresponding $[u, v]$ has an index $i$ such that $u(i) = v(i) = j$. In this case, every $y \in [u, v]$ has $y(i) = j$, so the effect of removing the fixed point $j$ from $w^\circ$ is easily described in terms of the Bruhat interval, and hence the positroid. In turn, the equations defining the positroid variety for $w^\circ$ and the positroid variety indexed by the decorated permutation $v^\circ$ obtained by removing $j$ from $w^\circ$ are closely related.

**Lemma 3.1.** *Let $w^\circ \in S_n^\circ$ be a decorated permutation with fixed point $i$. Let $v^\circ \in S_{n-1}^\circ$ be obtained from $w^\circ$ by deleting the loop at $i$ from $D(w^\circ)$. Then $\Pi_{w^\circ}$ is smooth if and only if $\Pi_{v^\circ}$ is smooth.*

If $w^\circ$ fixes every point in $[n]$, then $\Pi_{w^\circ}$ is a single point, and hence is smooth. Otherwise, by Lemma 3.1, instead of considering $w^\circ \in S_n^\circ$, we may consider the derangement obtained by deleting all the fixed points in $w^\circ$. Thus, we may restrict our attention to derangements. For the remainder, we give all results for derangements in $S_n$, and we drop the decorated permutation notation. In [4], we further reduce to *stabilized interval-free permutations*, found in [13, A075834].

**Lemma 3.2.** *Let $w$ be a derangement in $S_n$, and let $w'$ be a derangement obtained from $w$ by (1) rotating the chord diagram of $w$, (2) reflecting $D(w)$ across the vertical axis, or (3) reversing the direction of all arcs. Then, in any case, $\Pi_w$ is smooth if and only if $\Pi_{w'}$ is smooth.*

The chord diagram of $w \in S_n$ is a disjoint union of $m$ graphs if the arcs can be partitioned into $m$ parts such that no arcs from distinct parts form a crossing. In this case, one may consider the restriction of the diagram to any of these parts and the corresponding decorated permutations.

**Lemma 3.3.** *Suppose $w \in S_n$ is a derangement whose chord diagram can be partitioned into a disjoint union of $m$ graphs. Let $w^{(1)}, \ldots, w^{(m)}$ be derangements corresponding to these parts. Then $\Pi_w$ is smooth if and only if all of the $\Pi_{w^{(i)}}$ are smooth.*

The proofs of the lemmas above rely on Theorem 1.5 and the Johnson graphs of the corresponding positroids. Explicit maps on edges in the Johnson graphs are given for each type of reduction.

Next, we give the outline of the proof of Theorem 1.6. The first three items are equivalent from the definition of smooth, Theorem 1.3, and Theorem 1.5. Next, we show parts $4 \iff 5$, $5 \Rightarrow 1$, and $2 \Rightarrow 4$.

A spirograph permutation $w$ of the form $w(i) = i + t \,(mod\, n)$ has no alignments, so it has no crossed alignments. For the reverse direction, one may reduce to a single connected component of $D(w)$ as in Lemma 3.3 and assume that $w$ does not have the form $w(i) = i + t\,(mod\, n)$. Then, one can directly find a crossed alignment. Thus, $4 \iff 5$.

The fact that $w$ satisfying $w(i) = i + t\,(mod\, n)$ has no alignments also implies that $\Pi_w = Gr(k,n)$ for the appropriate value of $k$. Thus, $\Pi_w$ is smooth. Lemma 3.3 then provides the implication $5 \Rightarrow 1$ in Theorem 1.6.

The final step in proving Theorem 1.6 is achieved by proving $2 \Rightarrow 4$ by the contrapositive. In particular, given a derangement $w$ with a crossed alignment, we identify a set $J \in \mathcal{M}$ such that the number of non-bases $I \in \mathcal{Q} = \binom{[n]}{k} \setminus \mathcal{M}$ for which $|I \cap J| = k-1$ is strictly less than the number of alignments of $w$. By applying Lemma 3.2 as necessary, we may assume that the crossing arc of the crossed alignment passes from starboard to port and has its tail at 1, *e.g.*, the highlighted crossed alignment in Example 1.1. In this regime, consider the anti-exceedance set $J = I_1(w)$.

**Lemma 3.4.** For a derangement $w \in S_n$ with corresponding positroid $\mathcal{M}$, let $J = I_1(w)$, and suppose that $a \in J$ and $b \notin J$. Then $I = (J \setminus \{a\}) \cup \{b\}$ is in $\mathcal{M}$ if and only if $a < b$ and for every $r \in [a+1, b]$, the following two conditions hold.

1. There exists some $x \in [a, r-1]$ such that $w^{-1}(x) \geq r$.
2. There exists $y \in [r, b]$ such that $w^{-1}(y) \leq r-1$.

Lemma 3.4 gives an exact condition on the sets $I \in \mathcal{Q}$ for which $|I \cap J| = k-1$. Any such set $I$ corresponds to the pair $(a, b)$ such that $I = (J \backslash \{a\}) \cup \{b\}$. We call such a pair $(a, b)$ an *anti-exchange pair* for $J$ and define a map $\Psi_w$ from the anti-exchange pairs for $J$ to $Alignments(w)$. In the case that $(a, b)$ is an anti-exchange pair for $J$ with $b < a$, then $\Psi_w(a, b) = (w^{-1}(b) \mapsto b, w^{-1}(a) \mapsto a)$. Otherwise, $a < b$ and at least one of condition 1 or 2 in Lemma 3.4 is not satisfied for some $r \in [a+1, b]$.

Conditions 1 and 2 of the lemma are symmetric. The map $\Psi_w$ inherits this symmetry, and thus we will only give the details of the map for anti-exchange pairs that do not satisfy condition (1). In this case, by Lemma 3.4, there exists some $r \in [a+1, b]$ such that for all $x \in [a, r-1]$, $w^{-1}(x) \leq r-1$. Choose r to be minimal such that condition 1 is not satisfied. By starting at $b$ and tracing in reverse the cycle containing $b$, there will be some first element $c \neq b$ such that $c \geq r$. Then $w(c) \in [c+1, n] \cup [a-1]$ and $w^{-1}(a) \in [a+1, r-1]$, since $a$ is an anti-exceedance. Therefore, we choose $\Psi_w(a, b) = (c \mapsto w(c), w^{-1}(a) \mapsto a)$.

**Lemma 3.5.** *Let $w \in S_n$ be a derangement. The map $\Psi_w$ from anti-exchange pairs of $I_1(w)$ to $Alignments(w)$ is injective. Furthermore, if w has a crossed alignment with crossing arc passing from starboard to port and whose tail is at 1, then $\Psi_w$ is not surjective.*

In particular, in the latter case, there is no anti-exchange pair for $I_1(w)$ that is mapped to the crossed alignment. Thus, if $w$ has a crossed alignment, then $\#Alignments(w)$ is strictly larger than the number of anti-exchange pairs for $I_1(w)$. Since $I_1(w) = u[k]$, where $[u, v]$ is the Bruhat interval corresponding to $w$, then $I_1(w) \in \mathcal{M} = \{y[k] : y \in [u, v]\}$ by Theorem 1.4. Therefore, $I_1(w)$ is an element of $\mathcal{M}$ for which part 2 of Theorem 1.6 is not satisfied. It then follows that $2 \Rightarrow 4$ in Theorem 1.6.

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# References


[1] F. Ardila, F. Rincón, and L. Williams. "Positroids and non-crossing partitions". *Trans. Amer. Math. Soc.* **368**.1 (2016), pp. 337–363.

[2] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka. *Grassmannian geometry of scattering amplitudes*. Cambridge University Press, 2016.

[3] N. Bergeron and F. Sottile. "Schubert polynomials, the Bruhat order, and the geometry of flag manifolds". *Duke Math. J.* **95(2)** (1998), pp. 373–423.

[4] S. Billey and J. Weaver. "Criteria for smoothness of positroid varieties via pattern avoidance, Johnson graphs, and spirographs". In preparation. 2022.

[5] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. New York: Springer, 2005.

[6] J. B. Carrell. "The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties". *Proc. Symposia Pure Math.* **56**.Part 1 (1994).

[7] S. Corteel. "Crossings and alignments of permutations". *Adv. in Appl. Math.* **38(2)** (2007), pp. 149–163.

[8] W. Fulton. *Young Tableaux. With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.

[9] A. Knutson, T. Lam, and D. Speyer. "Positroid Varieties: Juggling and Geometry". *Compos. Math.* **149(10)** (2013), pp. 1710–1752.

[10] A. Knutson, T. Lam, and D. E. Speyer. "Projections of Richardson varieties". *J. Reine Angew. Math.* **687** (2014), pp. 133–157.

[11] Y. Kodama and L. Williams. "The full Kostant–Toda hierarchy on the positive flag variety". *Comm. Math. Phys.* **335**.1 (2015), pp. 247–283.

[12] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$". *Proc. Indian Acad. Sci. Math. Sci.* **100(1)** (1990), pp. 45–52.

[13] OEIS. "The On-Line Encyclopedia of Integer Sequences". `http://oeis.org`.

[14] S. Oh. "Positroids and Schubert matroids". *J. Combin. Theory, Ser. A* **118(8)** (2011).

[15] B. Pawlowski. "Cohomology classes of interval positroid varieties and a conjecture of Liu". *Electron. J. Combin.* **25(4)** (2018).

[16] A. Postnikov. "Total positivity, Grassmannians, and networks". 2006. arXiv:0609764.

[17] K. Rietsch. "Closure relations for totally nonnegative cells in $G/P$". *Math. Res. Lett.* **13**.5-6 (2006), pp. 775–786.

[18] L. K. Williams. "The positive Grassmannian, the amplituhedron, and cluster algebras". To appear in the Proceedings of the 2022 ICM. arXiv:2110.10856.