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Local equilibrium of particle density in planar Lorentz processes

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Abstract

Particles are injected into a large planar domain through the boundary and perform a random or sufficiently chaotic deterministic motion inside the domain. Our main example is the Sinai billiard, which periodically extended to our large planar domain, is referred to as the Lorentz process. Assuming that the particles move independently from one another and the boundary is also absorbing, we prove the emergence of local equilibrium of the particle density in the diffusive scaling limit in two scenarios. One scenario is an arbitrary domain with piecewise smooth boundary and a carefully chosen injection rule; the other scenario is a rectangular domain and a much more general injection mechanism. We study the latter scenario in an abstract framework that includes Lorentz processes and random walks and hopefully allows for more applications in the future.

Keywords: Sinai billiard, local equilibrium, heat equation

Mathematics Subject Classification numbers: 37C83, 82C05.

(Some figures may appear in colour only in the online journal)

1. Introduction

A major, and widely open problem in mathematical statistical mechanics is to rigorously derive macroscopic laws of physics from underlying *deterministic* microscopic principles [3]. One such law is Fourier's law of heat conduction. In this context, an important phenomenon in the emergence of local equilibrium in systems that are forced out of equilibrium. To fix ideas, assume that the boundary of a piece of metal is subject to a heat bath, i.e. all points on the boundary are kept at a temperature that is constant in time but not constant in space. Then one

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recommended by Dr Mark F Demers

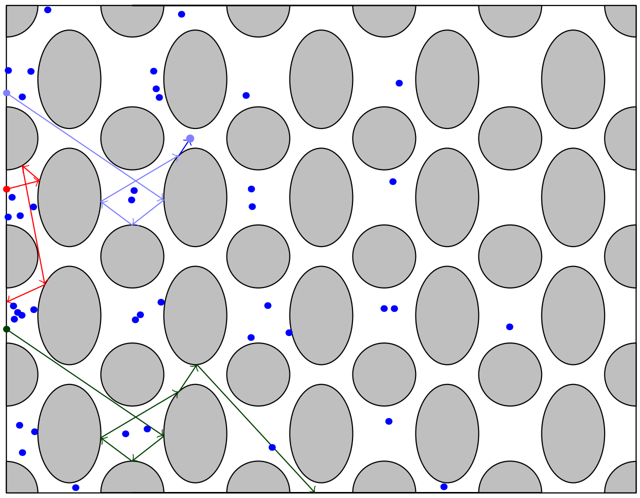

Figure 1. Particle configuration in a large rectangle. Point particles are enlarged for better visibility. A sample trajectory of one of the particles is indicated with light blue. Two more sample trajectories of absorbed particles are also indicated in red and green.

would like to first see that the temperature is locally well defined in the interior of the metal too (i.e. the system is at local equilibrium) and then study the temperature profile. A realistic microscopic model for this phenomenon should consist of a macroscopic domain inside which the microscopic particles are subject to some translation invariant local dynamics and interact with a heat bath on the boundary.

Proving that the bulk dynamics obey the heat equation, even without boundary effects, is notoriously difficult for realistic deterministic systems and consequently very few rigorous results are known. However, a notable realistic Hamiltonian system for which rigorous results are available is the Sinai billiard [25]. In Sinai billiards, point particles fly freely on the two-torus among fixed convex bodies and elastically collide on their boundaries (the planar infinite periodic extension of Sinai billiards is called the periodic Lorentz process). Thus, the point particles do not interact with one another and so there is no exchange of energies. By the results of [4–6] the trajectory of each particle satisfies the central limit theorem. Consequently, the scaling limit of the bulk dynamics in the infinite plane is given by the heat equation when ‘temperature’ is replaced by ‘particle density’.

The lack of interactions among particles in Sinai billiards is of course a serious limitation in modeling true heat conduction, let alone local equilibrium (as noted e.g. in [3, 24]) as the particles’ energies are fixed. Still, if we accept the idea of working with particle density instead of particle energy, then local equilibrium is feasible [17] (the above list of references to the physics literature is not exhaustive, they only serve as a sample).

1.1. Informal description of results

In this paper, we prove in a mathematically rigorous way, the local equilibrium property of particle density profiles in large domains of Sinai billiards by interpreting the ‘heat bath’ as a ‘varying chemical potential’. Furthermore, we develop this theory for an abstract class of non-interacting particle systems (which are composed of many copies of a process $\mathcal{Z}$) with the hope that this class will later include other interesting realistic systems. As of now, we show that this class is rich enough to contain two basic examples: (1) when $\mathcal{Z}$ is an iid random walk, and (2) when $\mathcal{Z}$ is given by the periodic Lorentz process. Our work is the first result of this kind for spatially extended deterministic systems.

To formulate our setup, let $D \subset \mathbb{R}^2$ be a bounded domain with piece-wise smooth boundary and let particles be injected into the large domain LD for $L \gg 1$ through its boundary. The particles will then perform some independent motion $\mathcal{Z}$ on a lattice inside LD. The boundary is also absorbing so most particles are killed (i.e. absorbed) shortly after injection. However, some will survive for a long time and find their way deep into the interior of LD. The problem now is to show that the limiting density profile of particles is governed by the heat equation when time is rescaled by L^2 and by the Laplace equation when time is infinite, where, in both cases, the boundary conditions are given by the injection rate. We will refer to the first case as the hydrodynamic limit and the second one as the non-equilibrium steady state. Specifically, we look at the problem of proving local equilibrium of the particle density profile in systems forced out of equilibrium when the particle injection rate varies along the boundary of the domain.

Our results in section 2 prove that in case D is a rectangle and the process $\mathcal{Z}$ satisfies some abstract hypotheses (H1)–(H3) (see section 2.1), then the local equilibrium holds in both the hydrodynamic limit and the non-equilibrium steady state. The main hypothesis is (H2), which is a conditional local invariance principle conditioned on the survival of the particle. See figure 1 for the case of the periodic Lorentz gas: particles, indicated by blue dots, are injected from the left ('West') side of a large rectangle while the entire boundary of the rectangle is absorbing.

Our results in section 4 exemplify that in some special cases, the above results can be generalized from a rectangular domain D to any domain with piece-wise smooth boundary. Specifically, if the process $\mathcal{Z}$ is such that its time-reversed process $\mathcal{Z}'$ converges to the Brownian motion, then the problem of local equilibrium can be reformulated in terms of the hitting probabilities of ∂D by $\mathcal{Z}'$ thus obtaining a simpler proof. We will refer to $\mathcal{Z}'$ as the dual process. The approach by duality thus gives similar results with two major differences: it is more general in the sense that D can be any domain with piece-wise smooth boundary, but it is more restrictive in the sense that it requires both the existence of a nice dual process and a very specific injection mechanism on the boundary. The utility of this special injection mechanism is limited since no reasonable heat bath is likely to preserve the invariant measure of the bulk dynamics (see e.g. [2]). Thus we decided to present the results of section 4 as a list of examples as opposed to providing an axiomatic framework.

To allow for more general injection mechanisms, it is essential to develop other tools which do not require such a rigid structure. Such tools are exemplified by our main results in sections 2 and 3. Indeed, our injection procedure (2.3) is quite general: besides the dependence on the macroscopic position we allow the injection rate to depend on the microscopic geometry through some function $\mathbf{A}$ and on time through another function $\mathbf{B}$. In the case of deterministic systems, the only source of randomness is the choice of the initial condition according to an initial probability measure. Once the initial condition is fixed, $\mathcal{Z}$ is deterministic. Here, we allow a lot of initial measures. Namely, we allow any 'standard pair' [10] in the case of Sinai billiards. In our context, a useful way of thinking about standard pairs is that they are conditional measures corresponding to a given past symbolic trajectory of the particle.

1.2. Related works

Let us compare our work with known results. Note that example (1) is random and Markovian. The study of Markovian microscopic dynamics is much easier than the deterministic ones and consequently much more results are known (including ones that go much beyond the derivation of the heat equations, such as second-order fluctuations or derivation of other PDEs even for interacting particle systems). Instead of reviewing any of the results on Markovian microscopic dynamics here, we refer the reader to the classical surveys [19, 27]. In the context of Markov processes, duality has been used to prove local equilibrium in systems that are more complicated than just iid particles (an important early reference for local equilibrium is [20], general classical references for duality in Markov chains are [23, 26]). Correspondingly, proposition 4.1, i.e. the approach by duality to random walks will not surprise experts, but we include it because on the one hand we could not locate a reference for this exact statement and on the other hand it simplifies some computations later in the more general framework in section 5. We believe that our results in case of more general injections prescribed by the functions **A** and **B** as discussed in section 2 are new even for random walks.

In the case of the periodic Lorentz gas, our results in section 2 provide a natural extension of [13] from one-dimensional domains (i.e. line segments) to two-dimensional rectangles. The proof by duality was not found in [13] and so our proposition 4.3 (with trivial changes to include a 1 dimensional macroscopic domain) gives a simple new proof of the main results of [13] in case of a very special injection mechanism, which is essentially given by the Lebesque measure.

1.3. Organization

The rest of this paper is organized as follows. In section 2, we provide the basic definitions and the main result theorem 2.1 in our abstract framework. In section 3, we present our two basic examples—namely, the random walk and the Lorentz gas. In section 4, we discuss the approach by duality. In section 5, we prove theorem 2.1. The most technical part of this work is the verification of the conditional local invariance principle (H2) for the Lorentz gas, which is presented in section 7. This section is heavily built upon the standard pair technique of Chernov and Dolgopyat [10] and tools from [13], such as the mixing local limit theorem (MLLT). The necessary background is summarized in section 6.

2. Abstract setup

2.1. Non-interacting particle systems

Let $\mathcal{L} \subset \mathbb{R}^2$ be a lattice (i.e. a discrete subgroup) of dimension 2. We consider the graph $\mathbf{G}$ with vertices $\mathcal{L}$ and edges joining l with $l + w_j$ for all $l \in \mathcal{L}$ and $j = 1, \ldots, J$ for a fixed set $\{w_1, \ldots, w_J\} \subset \mathcal{L}$. For $z \in \mathbb{R}^2$, let $\langle z \rangle$ be the closest $\mathfrak{l} \in \mathcal{L}$ to z with the property that $\mathfrak{l}_1 \geqslant z_1$ (if there are more than one such lattice points, then choose the smallest in lexicographic order).

Let $(\mathbf{S}, \mathbb{P})$ be a probability space and $\mathcal{Z}_t$ $(t \geqslant 0)$ be an $\mathcal{L}$ valued stochastic process. That is, $\mathcal{Z}_t : \mathbf{S} \to \mathcal{L}$ for every $t \geqslant 0$. We assume that $\mathcal{Z}$ is continuous from the right and has left limits. In other words, $\mathcal{Z}$ is a càdlàg function (i.e. for almost every $\mathbf{s} \in \mathbf{S}$ fixed, $\mathcal{Z}$ jumps at random times $\mathbf{t}$ from a lattice point $\mathcal{Z}_{\mathbf{t}-}$ to another lattice point $\mathcal{Z}_{\mathbf{t}}$). We do not assume that $\mathcal{Z}$ is Markovian.

Now let $D = [0, A] \times [0, 1]$ for a fixed positive real number A. Fix a non-negative continuous functions $f : [0, 1] \to \mathbb{R}$ and write $F : \partial D \to \mathbb{R}$,

$$F(z) = \begin{cases} f(y) & \text{if } z = (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We will consider the following Dirichlet problems

$$\Delta u = 0, \quad u|_{\partial D} = \varsigma F, \tag{2.1}$$

$$v_t = \frac{1}{2}\left[v_{xx} + v_{yy}\right], \quad v(t,x,y)|_{(x,y)\in\partial D} = \varsigma F, \quad v(0,x,y) = 0. \tag{2.2}$$

We are actually interested in the Dirichlet problem where F is permitted to be nonzero for all boundary points (as in (2.10)), but this case follows from linearity. We restrict to this case for now in order to simplify notations. By classical theory, there is a unique solution to both the Laplace equation (2.1) and the heat equation (2.2) and furthermore $\lim_{t\to\infty} v(t,x,y) = u(x,y)$. Of course, this is true for much more general domains D, e.g. when ∂D is piecewise-smooth with no cusps.

For $L \gg 1$, let $D_L = (LD) \cap \mathcal{L}$,

$$\partial D_L = \{\mathfrak{l} \in D_L : \mathfrak{l} \text{ is connected to a point outside of } D_L\},$$

and

$$\partial_{\mathrm{W}} D_L = \{\mathfrak{l} \in D_L : \mathfrak{l} \text{ is connected to a point } \mathfrak{l}' \text{ with } \mathfrak{l}'_1 < 0\}.$$

Here ∂_{W} stands for the West boundary as points in $\partial_{\mathrm{W}} D_L$ are close to the 'West' side of the rectangle D_L. Given $\mathfrak{l} \in \partial D_L$, let

$$\mathcal{J}(\mathfrak{l}) = \{j = 1,\ldots,J : \mathfrak{l} + w_j \notin D_L\}$$

We consider the following process for $L \gg 1$. First, for some $t \in \mathbb{R}_+ \cup \{\infty\}$, let Θ_t be a Poisson point process (PPP) on $(-t, 0] \times \partial_{\mathrm{W}} D_L$ with intensity measure

$$\mathbf{A}(\mathcal{J}(\mathfrak{l}))\mathbf{B}(s)f(\mathfrak{l}_2/L)\mathrm{dLeb}(s)\mathrm{dcounting}(\mathfrak{l}), \tag{2.3}$$

where $\mathfrak{l} \in \partial D_L$ and $\mathbf{A} : 2^{\{1,\ldots,J\}} \to \mathbb{R}_+$ and $\mathbf{B} : \mathbb{R} \to \mathbb{R}_+$ are fixed functions. We assume that $\mathbf{B}$ is continuous, periodic with period 1, and $\int_0^1 \mathbf{B} = 1$. One example is $\mathbf{A}(\mathcal{J}) = |\mathcal{J}|$ and $\mathbf{B} = 1$. However, we want to allow more general functions to accommodate for more general behavior of the heat bath.

For each point $(T, \mathfrak{l}) \in \Theta$, we start an iid copy of $\mathcal{Z}$ at time T from position $\mathfrak{l}$ and we kill it at

$$\tau^* = \inf\{t > T : \mathcal{Z}_t \notin D_L\}, \tag{2.4}$$

the first exit from D_L. In the case $\mathcal{Z}$ is not Markovian, the initial condition $\mathcal{Z}_T = \mathfrak{l}$ may not define the distribution of $\mathcal{Z}_{T+t}$ for $t > 0$ uniquely. In this case, we allow multiple choices of this distribution but we require that $\mathcal{Z}_{T+t} - \mathfrak{l}$ only depends on $\mathfrak{l}$ through $\mathcal{J}(\mathfrak{l})$. That is, if $\mathfrak{l}, \mathfrak{l}' \in \partial D_L$, with $\mathcal{J}(\mathfrak{l}) = \mathcal{J}(\mathfrak{l}')$ and $(T, \mathfrak{l}), (T', \mathfrak{l}') \in \Theta$, then we require that for all $t \geqslant 0$, and for all $\tilde{\mathfrak{l}} \in \mathcal{L}$,

$$\mathbb{P}(\mathcal{Z}_{T+t} = \mathfrak{l} + \tilde{\mathfrak{l}} | \mathcal{Z}_T = \mathfrak{l}) = \mathbb{P}(\mathcal{Z}_{T'+t} = \mathfrak{l}' + \tilde{\mathfrak{l}} | \mathcal{Z}_{T'} = \mathfrak{l}').$$

This procedure is to be interpreted as injecting a particle to the domain D_L at time T through an edge $(\mathfrak{l}_-, \mathfrak{l})$ of the graph $\mathbf{G}$, where $\mathfrak{l}_- \notin D_L$, $\mathfrak{l} \in D_L$ and letting particles evolve independently from one another until coming back to the absorbing boundary. The specific mechanism of injection through $(\mathfrak{l}_-, \mathfrak{l})$ only depends on $j = 1,\ldots,J$, where $\mathfrak{l} - \mathfrak{l}_- = w_j$. Let $\Lambda_t(\mathfrak{l})$ be the number of particles at site $\mathfrak{l}$ at time $T = 0$. We start with the following abstract result.

Theorem 2.1. *Assume that (H1)–(H3) (defined below) are satisfied. Then for any z in the interior of D,*

$$\lim_{L\to\infty} \mathbb{E}(\Lambda_\infty(\langle zL \rangle)) = u(z) \tag{2.5}$$

and

$$\lim_{L\to\infty} \mathbb{E}(\Lambda_{tL^2}(\langle zL\rangle)) = v(t,z), \tag{2.6}$$

where u and v are defined by (2.1) *and* (2.2) *with some* ς.

The proof of theorem 2.1 will be provided in section 5.

To define our hypotheses (H1)–(H3), we need some definitions.

Let W_t be a standard Brownian motion. Let

$$\phi(\eta,\gamma,\xi) = \lim_{\mathrm{d}t\to 0}\frac{1}{\mathrm{d}t}\mathbb{P}(W_1 \in [\gamma,\gamma+\mathrm{d}t],\ \min_{t\in[0,1]} W_t > 0,\ \max_{t\in[0,1]} W_t < \xi | W_0 = \eta).$$

It is known (see e.g. [16]) that for any $0 < \gamma, \eta < \xi$, the following formula holds

$$\phi(\eta,\gamma,\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\left(\exp\left(-\frac{(\gamma-\eta-2n\xi)^2}{2}\right) - \exp\left(-\frac{(\gamma+\eta+2n\xi)^2}{2}\right)\right). \tag{2.7}$$

Recall that the Brownian meander is a stochastic process on [0, 1] obtained by conditioning a standard Brownian motion to stay positive on [0, 1] (which has zero probability, but the definition still makes sense by conditioning on staying above $-\varepsilon$, letting $\varepsilon \to 0$ and taking the weak limit, see e.g. [15]). Let $\mathfrak{X}(t)$ be a Brownian meander and $\mathfrak{M}(t) = \max_{0\leqslant s\leqslant t}\mathfrak{X}(s)$ its maximum. Then it is proven in [12, theorem 5] that the function

$$\psi(\alpha,\beta) = \lim_{\mathrm{d}t\to 0}\frac{1}{\mathrm{d}t}\mathbb{P}(\mathfrak{X}(1)\in[\alpha,\alpha+\mathrm{d}t], \mathfrak{M}(1)<\beta)$$

for any $0 < \alpha < \beta$ satisfies

$$\psi(\alpha,\beta) = \sum_{k=-\infty}^{\infty} (2k\beta+\alpha)\exp\left(-\frac{(2k\beta+\alpha)^2}{2}\right). \tag{2.8}$$

Note that the formulas (2.7) and (2.8) are closely related as the Brownian meander is closely related to the Brownian motion. Indeed, by the definition of Brownian meander, $\psi(\alpha,\beta) = \lim_{\eta\to 0}\phi(\eta,\alpha,\beta)/\int_0^\beta \phi(\eta,\alpha',\beta)\mathrm{d}\alpha'$. We refer to [12] for more details.

Let us write $\mathcal{Z}_t = (\mathcal{X}_t, \mathcal{Y}_t)$. Denote

$$\tau_x^{\mathcal{X}} = \begin{cases} \min\{t>0 : \mathcal{X}_t > x\} & \text{if } x>0 \\ \min\{t>0 : \mathcal{X}_t < x\} & \text{if } x \leqslant 0. \end{cases}$$

We define $\tau_y^{\mathcal{Y}}$ analogously.

Now we make the following assumptions:

(H1) **Vertical rational dependence**. There is some $\mathfrak{l} \in \mathcal{L}$, $\mathfrak{l} \neq 0$ so that $\mathfrak{l}_1 = 0$.

Let $(0,0) = \mathfrak{l}^{(0)}, \mathfrak{l}^{(1)}, \mathfrak{l}^{(2)} \ldots$ be the enumeration of points $\mathfrak{l} \in \partial D_L$ which are connected to lattice points with negative first coordinate in increasing order of second coordinate (that is, $\mathfrak{l}_2^{(j)} \leqslant \mathfrak{l}_2^{(j+1)}$). If there are points $\mathfrak{l}^{(j)}, \mathfrak{l}^{(j+1)}$ with the same second coordinate, then we order them in increasing order of the first coordinate. Let K be the smallest positive integer so that

$$\mathfrak{l}_1^{(K)} = 0. \tag{2.9}$$

By condition (H1), K exists. Now we say that the lattice point $\mathfrak{l} \in \partial D_L$ is of type k with $k = 1,\ldots,K$ if there exists an integer m so that $\mathfrak{l} = \mathfrak{l}^{(mK+k)}$.

(H2) **Conditional local invariance principle**

There are constants $c_1, \ldots, c_K$ so that for any $0 < \alpha < \beta$ and for any $0 < \eta, \gamma < \xi$ the following holds. If $\mathfrak{l} \in \partial D_L$ is of type k, and $\mathfrak{l}_2 = \eta\sqrt{T}$, then

$$\lim_{T\to\infty} T^{3/2}\mathbb{P}\left(\mathcal{Z}_T = \langle(\alpha,\gamma)\sqrt{T}\rangle, \min\{\tau_0^{\mathcal{Y}}, \tau_{\xi\sqrt{T}}^{\mathcal{Y}}, \tau_0^{\mathcal{X}}, \tau_{\beta\sqrt{T}}^{\mathcal{X}}\} > T | \mathcal{Z}_0 = \mathfrak{l}\right)$$
$$= c_k\psi(\alpha,\beta)\phi(\eta,\gamma,\xi).$$

Furthermore, for any $\varepsilon > 0$, the convergence is uniform for $\varepsilon < \alpha < \alpha + \varepsilon < \beta < 1/\varepsilon$ and $\varepsilon < \eta < \eta + \varepsilon < \xi < 1/\varepsilon$, $\varepsilon < \gamma < \gamma + \varepsilon < \xi$.

(H3) **Moderate deviation bounds**. For any $x \in (0,1)$ and $y \in (-1,1)$, and for any $\mathfrak{l} = \mathfrak{l}^{(0)}, \ldots, \mathfrak{l}^{(K-1)}$

$$\lim_{\delta\to 0}\lim_{L\to\infty}\int_{[0,\delta L^2]\cup[L^2/\delta,\infty)} L\mathbb{P}\left(\mathcal{Z}_t = \langle(xL, yL)\rangle, \min\{\tau_0^{\mathcal{X}}, \tau_L^{\mathcal{X}}\} > t | \mathcal{Z}_0 = \mathfrak{l}\right) \mathrm{d}t = 0$$

2.2. *Local equilibrium*

Consider now the Dirichlet problems

$$\Delta\tilde{u} = 0, \tilde{u}|_{\partial D} = \tilde{F}, \tag{2.10}$$

$$\tilde{v}_t = \frac{1}{2}\left[\tilde{v}_{xx} + \tilde{v}_{yy}\right], \quad \tilde{v}(t,x,y)|_{(x,y)\in\partial D} = \tilde{F}, \tilde{v}(0,x,y) = 0. \tag{2.11}$$

Here $\tilde{F}$ is defined by $\tilde{F} : \partial D \to \mathbb{R}$,

$$\tilde{F}(z) = \begin{cases} \varsigma_{\mathrm{W}} f_{\mathrm{W}}(y) & \text{if } z = (0,y) \\ \varsigma_{\mathrm{S}} f_{\mathrm{S}}(x) & \text{if } z = (x,0) \\ \varsigma_{\mathrm{E}} f_{\mathrm{E}}(y) & \text{if } z = (A,y) \\ \varsigma_{\mathrm{N}} f_{\mathrm{N}}(y) & \text{if } z = (x,1), \end{cases}$$

where $f_{\mathrm{E}}, f_{\mathrm{W}} : [0,1] \to \mathbb{R}$, $f_{\mathrm{N}}, f_{\mathrm{S}} : [0,A] \to \mathbb{R}$ are given non-negative continuous functions and $\varsigma_{\mathrm{W/S/E/N}}$ are non-negative real numbers (W, S, E, N stand for West, South, North and East). We perform the same procedure of injecting particles and absorbing them on the boundary as before, but now we inject from all 4 sides of the rectangle. Let $\tilde{\Lambda}_t$ denote the resulting measure defined as Λ_t.

We say that $\mathcal{Z}$ satisfies that **local equilibrium** (LE) if for any $t \in \mathbb{R}_+ \cup \{\infty\}$, for any $k \in \mathbb{Z}_+$, for any $z_1, \ldots, z_k$ distinct points in the interior D, and for any distinct lattice points $\mathfrak{l}_1, \ldots, \mathfrak{l}_k \in \mathcal{L}$, the joint distribution of

$$\mathfrak{W}_{t,i,j,L} := \tilde{\Lambda}_{tL^2}(\langle z_i L\rangle + \mathfrak{l}_j), \quad i,j = 1, \ldots, k$$

converge weakly as $L \to \infty$ to independent Poisson random variables $\mathfrak{W}_{t,i,j,\infty}$ with expectation $\tilde{v}(t, z_i)$ (or $\tilde{u}(z_i)$ in case $t = \infty$), where $\tilde{v}$ is defined by (2.11) (and $\tilde{u}$ is defined by (2.10)) with some constants $\varsigma_{\mathrm{W/S/E/N}}$. The points $\langle z_i L\rangle + l_j$, $j = 1, \ldots, k$ can be thought of as lying in a microscopic region near $\langle z_i L\rangle$. In particular, each point $\langle z_i L\rangle + l_j$ is a finite distance from $\langle z_i L\rangle$ so that it is in a 'local' region of z_i as L becomes large. Indeed, the term *local equilibrium* refers to the fact that the limiting distribution does not depend on j. We call the case $t \in \mathbb{R}_+$ *local equilibrium in the hydrodynamic limit* and the case $t = \infty$ *local equilibrium in the non-equilibrium steady state*. Since in our case both hold at the same time, we simply refer to these properties as local equilibrium.

Finally, we say that a lattice $\mathcal{L}$ is **rational** if there are non-zero lattice points $\mathfrak{l}^{(K_1),1}$, $\mathfrak{l}^{(K_2),2}$ in $\mathcal{L}$ so that $\mathfrak{l}_1^{(K_1),1} = \mathfrak{l}_2^{(K_2),2} = 0$. Without loss of generality, we assume that $\mathfrak{l}_2^{(K_1),1} > 0$ and $\mathfrak{l}_2^{(K_1),1}$ is the smallest among such vectors with respect to the ordering introduced right after (H1) (and likewise for $\mathfrak{l}^{(K_2),2}$, except that in the ordering, the role of the first and second coordinates are swapped). Clearly, if $\mathcal{L}$ is rational, then (H1) holds with $K = K_1$ (and likewise, a variant of (H1), where the two coordinates are swapped, holds with $K = K_2$).

Before proceeding to the examples of the next section, where we can verify conditions (H1)–(H3) and also prove (LE), let us make some remarks. First, we believe that condition (H1) is not necessary for the main results ((2.5), (2.6) and (LE)) to hold but our proof does not apply in the general case when (H1) fails. The difficulty is the lack of periodicity in the local geometry on the boundary.

Let us now comment on the case of dimension $d \geqslant 3$. We have little doubt that theorem 2.1 could be extended to any dimension $d \geqslant 3$. However, the proof would be substantially longer and, more importantly, we do not know how to verify assumptions (H2) and (H3) in our main example, namely the finite horizon Lorentz gas in any configuration of dimension $\geqslant 3$ (even the classical CLT is only conditionally known, cf [1], and refinements along the lines of (H2) are widely open). Finally, the one dimensional case is much simpler and is essentially covered by [13] (although not in the axiomatic framework). This is why we decided to keep the abstract setup in planar domains.

3. Basic examples

3.1. Random walks

Let $\tilde{\mathcal{L}} \subset \mathbb{R}^2$ be a 2 dimensional lattice. Let $\tilde{\mathcal{P}}$ be a finitely supported probability measure on $\tilde{\mathcal{L}}$ with zero expectation. We assume that there are finitely many lattice points $\tilde{w}_1, \ldots, \tilde{w}_J$ so that $\tilde{\mathcal{P}}(\tilde{w}_j) > 0$ and $\sum \tilde{\mathcal{P}}(\tilde{w}_j) = 1$. To avoid degeneracy, we assume that the group generated by $\tilde{w}_j$'s is $\tilde{\mathcal{L}}$.

Let $\tilde{\mathcal{Z}}$ be a homogeneous Markov process: at exponential distributed times, $\tilde{\mathcal{Z}}$ jumps with a jump distribution given by $\tilde{\mathcal{P}}$. That is, the generator $\tilde{G}$ of $\tilde{\mathcal{Z}}$ is defined by

$$(\tilde{G}f)(\mathfrak{l}) = \sum_{j=1}^{J} \tilde{\mathcal{P}}(\tilde{w}_j)[f(\tilde{w}_j + \mathfrak{l}) - f(\mathfrak{l})] \tag{3.1}$$

for test functions $f : \tilde{\mathcal{L}} \to \mathbb{R}$. By the central limit theorem, $\tilde{\mathcal{Z}}_t/\sqrt{t}$ converges weakly to a Gaussian distribution with mean zero and some covariance matrix Σ. Furthermore, the non-degeneracy assumption ensures that Σ is positive definite. Now we define $\mathcal{L} = \Sigma^{-1/2}\tilde{\mathcal{L}}$, $w_j = \Sigma^{-1/2}\tilde{w}_j$, $\mathcal{P}(w_j) = \tilde{\mathcal{P}}(\tilde{w}_j)$, $\mathcal{Z} = \Sigma^{-1/2}\tilde{\mathcal{Z}}$.

Proposition 3.1. *Assume that in the above model of random walks, $\mathcal{L}$ is a rational lattice. Then* (*H1*)–(*H3*) *hold*.

We do not give a proof of proposition 3.1 as it follows from a simplified version of our proof of theorem 3.2. In fact, the one-dimensional version of (H2) and (H3) is known for random walks, see [7, 8]. We find it likely that the two-dimensional version is also known, but we could not find a reference.

3.2. Lorentz gas

3.2.1. Definitions. We start with the definition of Sinai billiards [25]. Consider a finite collection of strictly convex disjoint subsets $B_1, \ldots, B_{\mathfrak{k}}$ of the two-torus with C^3 boundary. The complement of these sets is denoted by $\mathcal{D}_0 = \mathbb{T}^2 \backslash \cup_{i=1}^{\mathfrak{k}} B_i$ and is called the *configuration space*. A point particle flies with constant speed inside $\mathcal{D}_0$ and undergoes specular reflection upon reaching $\partial\mathcal{D}_0$ (i.e. the angle of incidence equals the angle of reflection). Since the speed is conserved, we obtain a continuous time dynamical system Φ_0^t, $t \in \mathbb{R}$ on the *phase space* $\Omega_0 = \mathcal{D}_0 \times \mathcal{S}^1$. The Sinai billiard flow Φ_0 preserves the Lebesgue measure on Ω_0 (denoted by μ_0). We assume the finite horizon condition, i.e. that the sets B_i are chosen in such a way that the free flight time is bounded. Similarly, we define the periodic Lorentz gas when the phase space is lifted to the universal cover. That is, the configuration space is $\mathcal{D} = \mathbb{R}^2 \backslash \cup_{(m,n)\in\mathbb{Z}^2} \cup_{i=1}^{\mathfrak{k}} (B_i + (m, n))$, where we identify $\mathcal{D}_0$ with $\mathcal{D} \cap \left[-1/2, 1/2\right)^2$. We choose this identification in such a way that

$$(-1/2, -1/2) \notin \mathcal{D}. \tag{3.2}$$

The phase space is $\Omega = \mathcal{D} \times \mathcal{S}^1$ and the billiard flow is denoted by Φ^t. It preserves the σ-finite measure μ, which is μ_0 times the counting measure on $\mathbb{Z}^2$.

Now we construct the stochastic process, which is the projection of the billiard flow, Φ^t, onto $\mathbb{Z}^2$. Given $(q, v) \in \Omega$, let $\Pi_{\mathbb{Z}^2}(q, v) = (k, l) \in \mathbb{Z}^2$ if $q \in (k, l) + \left[-1/2, 1/2\right)^2$, let $\Pi_{\mathcal{D}_0}(q, v) = q_0$, and let $\Pi_{\Omega_0}(q, v) = (q_0, v)$ if $q = q_0 + \Pi_{\mathbb{Z}^2}(q, v)$. We also put $\tilde{\mathcal{Z}}_t(q, v) = \Pi_{\mathbb{Z}^2}(\Phi^t(q, v))$. Thus any probability measure on $\mathcal{D}_0$ induces a stochastic process $\tilde{\mathcal{Z}}_t$. It is important to note that here the randomness only appears in the initial condition. Once (q, v) is fixed, then $\tilde{\mathcal{Z}}_t$ is uniquely defined for every t.

We will also need the billiard map $\mathcal{F}_0$, which is defined as the Poincaré section corresponding to the collisions, that is $\mathcal{F}_0 : \mathcal{M}_0 \to \mathcal{M}_0$, where

$$\mathcal{M}_0 = \{(q, v) \in \partial\mathcal{D}_0 \times \mathcal{S}^1 : \langle v, n \rangle \geqslant 0\},$$

where n is normal to $\partial\mathcal{D}_0$ at q pointing inside $\mathcal{D}_0$. The phase space of the billiard map, $\mathcal{M}_0$, thus corresponds to collisions where by convention we use the post-collisional velocity v. $\mathcal{F}_0$ preserves the probability measure ν_0 defined by $\mathrm{d}\nu_0 = c \cos\phi \mathrm{d}r\mathrm{d}\phi$, where (r, ϕ) are coordinates on $\mathcal{M}_0$: r is arclength parameter and $\phi \in [-\pi/2, \pi/2]$ is the angle between v and n. The definitions of $\mathcal{M}, \mathcal{F}, \nu$ are analogous.

Fix a measure given by an arbitrary proper standard family (the exact definition standard family will be given in section 6; one example is the invariant measure ν). This measure induces a stochastic process $\tilde{\mathcal{Z}}_t$. Furthermore, $\tilde{\mathcal{Z}}_t$ satisfies the central limit theorem with a covariance matrix which is independent of the standard family. That is, there exists a positive definite 2×2 matrix Σ so that $\tilde{\mathcal{Z}}_T/\sqrt{T}$ converges weakly as $T \to \infty$ to the Gaussian distribution with mean zero and covariance matrix Σ (see e.g. [6]). Now let $\mathcal{L} = \Sigma^{-1/2}\mathbb{Z}^2$, $\mathcal{Z}_t = \Sigma^{-1/2}\tilde{\mathcal{Z}}_t$. The *invariance principle* holds as well. That is,

$$\left(\frac{\mathcal{Z}_{tT}}{\sqrt{T}}\right)_{t\in[0,1]} \text{converges weakly to a standard Brownian motion as}$$
$$T \to \infty \tag{3.3}$$

(see e.g. [9]).

Without loss of generality, we can assume that the length of the longest free flight is bounded by one. Indeed, pick any infinite periodic billiard table with finite horizon, that is the flight time being bounded above by some integer K. If $K > 1$, we just rescale the space by K, i.e. shrink the configuration in the square $\left[-K/2, K/2\right)^2$ to the square $\left[-1/2, 1/2\right)^2$ and call this configuration $\mathcal{D}_0$ and its infinite extension $\mathcal{D}$ (in other words, we choose the fundamental domain large enough). After this rescaling, condition (3.2) still holds. Since the billiard flow is continuous, we see that at each time t so that $\tilde{\mathcal{Z}}_{t-} \neq \tilde{\mathcal{Z}}_t$, we necessarily have $\tilde{\mathcal{Z}}_t = \tilde{\mathcal{Z}}_{t-} + e$ with some $e \in \{(-1,0),(1,0),(0,-1),(0,1)\}$. Indeed, by condition (3.2), the flow $\tilde{\mathcal{Z}}_t$ cannot pass through the corners and so e.g., $e = (1,1)$ is not possible. We made assumption (3.2) in order to guarantee that the resulting graph is the simplest possible, i.e. four-regular. Specifically, let $\tilde{\mathbf{G}}$ be the nearest neighbor graph on $\mathbb{Z}^2$, i.e. $\mathfrak{l}, \mathfrak{l}' \in \mathbb{Z}^2$ are connected in $\tilde{\mathbf{G}}$ if and only if $|\mathfrak{l} - \mathfrak{l}'| = 1$. This graph is thus 'induced' by the process $\tilde{\mathcal{Z}}_t$. Hence the graph $\mathbf{G} = \Sigma^{1/2}\tilde{\mathbf{G}}$ is also a transitive four-regular graph on the lattice $\mathcal{L}$. Even though we always obtain a transitive four-regular graph $\mathcal{L}$ when starting from a Lorentz gas where the fundamental domain is a torus, we still allow for graphs of higher degree in our abstract framework, motivated by e.g. the forthcoming example 3.5 where the fundamental domain is a hexagon.

Theorem 3.2. *Assume that in the above model of the Lorentz gas, $\mathcal{L}$ is a rational lattice. Then* (*H1*)–(*H3*) *hold*.

Theorem 3.2 does not claim that $\varsigma > 0$. In fact, there are standard families for which $\varsigma = 0$. This is not surprising since $\mathcal{Z}_t$ can be deterministic for a bounded time. In particular, we can choose a standard family so that $\mathcal{Z}_t < 0$ almost surely for a fixed t and so all particles will be absorbed within a bounded amount of time. However, there are standard families for which $\varsigma > 0$ (e.g. the invariant measure ν can be represented by such a standard family). In case of general standard families, we cannot compute ς even if it is positive.

Note that we assumed that $\mathcal{L}$ is a rational lattice, which immediately gives (H1) and the variant of (H1) when the vertical and the horizontal coordinates are swapped. This is a highly non-trivial assumption and we expect this not to hold for a typical billiard table. However, we have some examples when it does hold due to some extra symmetry. We discuss these examples in section 3.2.2. The proof of (H2) and (H3) will be given in section 7.

In case of deterministic systems like the Lorentz gas, a natural extension of (LE) is a finer counting problem: that is, to only count particles in a given nice subset of Ω_0 (for example, those that are close to a given scatterer). Let us fix and open set

$$A \subset \Omega_0 \quad \text{with} \quad \mu_0(\partial A) = 0 \tag{3.4}$$

and update the definition of $\tilde{\Lambda}_t$ so as we only count particles at phase (q, v) that satisfies $\Pi_{\Omega_0}(q, v) \in A$. Let the resulting measure be $\tilde{\Lambda}_t^A$ and let us say that detailed local equilibrium (DLE) holds if there is some ς so that for every A as in (3.4), the definition of (LE) with $\tilde{\Lambda}$ replaced by $\tilde{\Lambda}_A$ holds with the constant $\varsigma\mu_0(A)$.

Theorem 3.3. *Under the assumptions of theorem* 3.2*,* (*LE*) *and* (*DLE*) *hold*.

Proof of Theorem 3.3 *assuming theorem* 3.2.

As observed in [13], the derivation of (LE) from (2.5) and (2.6) is straightforward. Let $\mathbb{M} = (-tL^2, 0) \times \partial D_L \times \Omega_0$. Let $\mathbb{G} : \mathbb{M} \to D_L \times \Omega_0 \cup \{\infty\}$, where $\mathbb{G}(s, \mathfrak{l}, (q, v)) = \Phi^s(q + \Sigma^{1/2}\mathfrak{l}, v)$ if the particle has not been absorbed by time s and $\mathbb{G}(s, \mathfrak{l}, (q, v)) = \infty$ otherwise. Since the initial conditions of particles is given by a PPP on $\mathbb{M}$, the mapping and restriction theorems for PPP (see e.g. sections 2.2 and 2.3 in [18]) give that $\{\mathbb{G}(s_i, \mathfrak{l}_i, x_i)\}_{\mathbb{G}(s_i,\mathfrak{l}_i,x_i)\neq\infty}$ forms a PPP on $D_L \times \Omega_0$. Letting $L \to \infty$, the intensity measure of this PPP converges by theorems 2.1 and

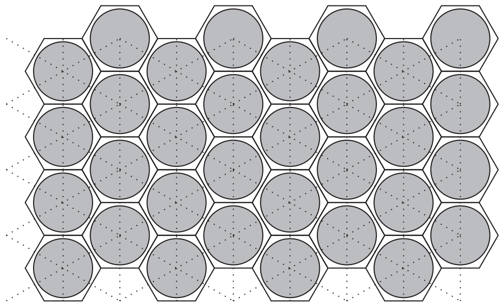

Figure 2. Billiard configuration on a hexagonal tiling.

3.2 (for particles injected on the West or on the East this is immediate. For particles injected on the North or the South, this follows from the variant of theorem 2.1 when the role of x and y are swapped. Since $\mathcal{L}$ is assumed to be rational, (H1) holds even in this case). Thus in the limit $L \to \infty$, we obtain a PPP with intensity measure as on the right-hand side of (2.5) and (2.6). This implies (LE). The proof of (DLE) is analogous, except that when we verify (H2), we only need to take into account particles at phase (q, v) that satisfy $\Pi_{\Omega_0}(q, v) \in A$. This requires a very minor change in the proof (see the remark after theorem 6.4). □

3.2.2. Symmetry conditions.

Example 3.4. Assume that $\mathcal{D}_0$ is invariant under a 90 degree rotation or a vertical or horizontal reflection of the unit square. Then $\mathcal{L}$ is rational.

Proof. Let us assume that $\mathcal{D}_0$ is invariant under a rotation by 90 degrees. Then the probability density function (pdf) of the limiting distribution of $\tilde{\mathcal{Z}}_t/\sqrt{t}$ also needs to be invariant under the rotation by 90 degrees. Since this is a normal distribution, the isocontours of the pdf are ellipses. The only ellipses invariant under the rotation by 90 degrees are circles. This means that there is a positive real number σ so that $\Sigma = \sigma^2 I_2$. Similarly, if $\mathcal{D}_0$ is invariant under reflection of the vertical or horizontal axis, then the isocontours of limiting normal distribution are ellipses with semi-axes parallel to the coordinate axes and so Σ is diagonal. □

In the above examples, $\mathcal{L}$ is generated by $\sigma_1^{-1}[1, 0]^{\mathrm{T}}$ and $\sigma_2^{-1}[0, 1]^{\mathrm{T}}$. Consequently, $K_1 = K_2 = 1$. In this sense, these examples are the simplest possible ones (figure 1 shows a configuration, which is symmetric with respect to the vertical axis, and is repeated over a 5×4 rectangle). Our next example is less trivial as $K_2 = 2$.

Example 3.5. Consider a scatterer configuration on the regular hexagon that is invariant under the rotation by 120 degrees and satisfies all other assumptions (that is, the scatterers are smooth, disjoint, strictly convex and the configuration has finite horizon). One such example is only one scatterer which is a disc, centered at the center of the hexagon and with a radius large enough to ensure that $\mathcal{D}$ is of finite horizon. By tiling the plane with regular hexagons, we obtain the Lorentz gas as before. As in the previous example, the isocontours of the limiting normal distribution are invariant under the rotation by 120 degrees; hence they are circles and $\Sigma = \sigma^2 I_2$. In this case, $\tilde{\mathcal{Z}}_t$ for any t takes values in the set of tiles of the hexagonal tiling. Let $\mathcal{L}$ be the lattice generated by the vectors $\sigma^{-1}[0, 1]^{\mathrm{T}}$ and $\sigma^{-1}[\sqrt{3}/2, 1/2]^{\mathrm{T}}$ and $\mathbf{G}$ be the graph with vertices $\mathcal{L}$ and edges between points at distance σ^{-1}. That is, $\mathbf{G}$ forms the triangular grid, dual to the hexagonal tiling (see figure 2, the edges of **G** are denoted by dotted lines). In this example, $K_1 = 1$ and $K_2 = 2$. Indeed, on the horizontal boundary, we see an alternating sequence of two kinds of hexagons (ignoring the very first and the very last one): one of them has 5 neighbors in D_L and the other one only has 3. A particle injected to a uniform random location on the first type of hexagon has a higher chance of staying in D_L than in the case of the second type of hexagon. Thus we expect that $c_1 \neq c_2$ in the variant of (H2) when the vertical and horizontal coordinates are swapped.

4. Duality

4.1. Random walks

The definitions given in section 2.1 easily extend to more general domains D with piece-wise smooth boundary. One minor difference is that in (2.3) instead of $f(\mathfrak{l}_2/L)$ we need to choose a slightly different argument of f as $\mathfrak{l}_2/L$ may not be on ∂D and f may not be defined (e.g. one can choose the closest point on ∂D to $\mathfrak{l}/L$). Since f is continuous, the exact choice is irrelevant as long as it is a bounded distance from $\mathfrak{l}/L$. To keep notations simple, we will write $f(\mathfrak{l}/L)$, where f is a continuous function defined on ∂D (there is no need to introduce F).

Proposition 4.1. *Consider a random walk as in section 3.1 and let*

$$\mathbf{A}(\mathcal{J}) = \sum_{j\in\mathcal{J}} \mathcal{P}(w_j) \tag{4.1}$$

and $\boldsymbol{B} = 1$. *Then the conclusion of theorem 2.1 and (LE) hold with* $\varsigma = 1$ *without assuming the rationality of* $\mathcal{L}$ *and for general bounded domains D with piece-wise smooth boundary and no cusps.*

Proof. We are only going to prove (2.6). A proof of (2.5) can be obtained by replacing t by ∞ in the proof below and (LE) can be proved as in theorem 3.3.

The key idea of the proof is duality. Let $\check{\mathcal{Z}}$ be the discretized version of $\mathcal{Z}$. That is, $\check{\mathcal{Z}}_0 = \mathcal{Z}_0$ and $\check{\mathcal{Z}}_n = \mathcal{Z}_{t_n}$ where t_n is the time of the nth jump of $\mathcal{Z}$. The reversed random walk $\mathcal{Z}'$ is defined by the generator

$$(G'f)(\mathfrak{l}) = \sum_{j=1}^{J} \mathcal{P}(w_j)[f(-w_j + \mathfrak{l}) - f(\mathfrak{l})]$$

and $\check{\mathcal{Z}}'$ is the discretized version of $\mathcal{Z}'$ (defined analogously to $\check{\mathcal{Z}}$).

Note that for any N, $\check{\mathcal{Z}}$ induces a measure $\mathbb{P}_{\check{\mathcal{Z}}}$ on $\mathcal{L}^N$ by

$$\mathbb{P}_{\check{\mathcal{Z}}}(\mathfrak{l}_0, \ldots, \mathfrak{l}_{N-1}) = \mathbb{P}(\check{\mathcal{Z}}_1 = \mathfrak{l}_1, \ldots, \check{\mathcal{Z}}_{N-1} = \mathfrak{l}_{N-1} | \check{\mathcal{Z}}_0 = \mathfrak{l}_0).$$

Let us define $\mathbb{P}_{\check{\mathcal{Z}}'}$ analogously. Then by definition of $\check{\mathcal{Z}}'$, for any sequence $\mathfrak{l}_0, \ldots, \mathfrak{l}_M \in \mathcal{L}$,

$$\mathbb{P}_{\check{\mathcal{Z}}}(\mathfrak{l}_0, \ldots, \mathfrak{l}_M) = \mathbb{P}_{\check{\mathcal{Z}}'}(\mathfrak{l}_M, \ldots, \mathfrak{l}_0). \tag{4.2}$$

For fixed L, $z \in D$, $t \in \mathbb{R}_+$, $\mathfrak{l} \in \partial D_L$ and M, let $\mathcal{A} = \mathcal{A}_{L,z,t,\mathfrak{l},M}$ be the set of length M trajectories from $\mathfrak{l}$ to $\langle zL\rangle$ staying inside D_L, i.e.

$$\mathcal{A} = \{(\mathfrak{l}_0, \dots, \mathfrak{l}_M) : \mathfrak{l}_0 = \mathfrak{l},\ \forall\ i = 0, \dots, M-1 : \exists j = 1, \dots, J : \mathfrak{l}_{i+1} - \mathfrak{l}_i = w_j, \mathfrak{l}_i \in D_L, \mathfrak{l}_M = \langle zL\rangle\}\ .$$

For a subset $\mathcal{B} \subset \mathcal{L}^{M+1}$ and a lattice point $\hat{\mathfrak{l}}$, let

$$\mathcal{B}' = \{(\mathfrak{l}_M, \dots, \mathfrak{l}_0) : (\mathfrak{l}_0, \dots, \mathfrak{l}_M) \in \mathcal{B}\},$$

and

$$\hat{\mathfrak{l}}\mathcal{B} = \{(\hat{\mathfrak{l}}, \mathfrak{l}_0, \dots, \mathfrak{l}_M) : (\mathfrak{l}_0, \dots, \mathfrak{l}_M) \in \mathcal{B}\}, \qquad \mathcal{B}\hat{\mathfrak{l}} = \{(\mathfrak{l}_0, \dots, \mathfrak{l}_M, \hat{\mathfrak{l}}) : (\mathfrak{l}_0, \dots, \mathfrak{l}_M) \in \mathcal{B}\}.$$

Then by (4.2), we have

$$\mathbb{P}_{\check{\mathcal{Z}}}(\mathcal{A}_{L,z,t,\mathfrak{l},M}) = \mathbb{P}_{\check{\mathcal{Z}}'}(\mathcal{A}'_{L,z,t,\mathfrak{l},M}).$$

Furthermore, for any $\mathfrak{l}_{-1} \notin D_L$, which is connected to $\mathfrak{l}$ in $\mathbf{G}$,

$$\mathbb{P}_{\check{\mathcal{Z}}}(\mathfrak{l}_{-1}\mathcal{A}_{L,z,t,\mathfrak{l},M}) = \mathbb{P}_{\check{\mathcal{Z}}'}(\mathcal{A}'_{L,z,t,\mathfrak{l},M}\mathfrak{l}_{-1}). \tag{4.3}$$

Let $\mathbf{T}$ be the first hitting time of $\mathcal{L}\backslash D_L$ by $\check{\mathcal{Z}}'$. Then (4.3) is equal to

$$\mathbb{P}(\mathbf{T} = M+1, \check{\mathcal{Z}}'_{M+1} = \mathfrak{l}_{-1}, \check{\mathcal{Z}}'_M = \mathfrak{l}|\check{\mathcal{Z}}'_0 = \langle zL\rangle).$$

To turn to continuous time, let τ'^* be the first time $\mathcal{Z}'$ leaves D_L. Then we have

$$\mathbb{P}(\tau'^* < tL^2, \mathcal{Z}'_{\tau'^*} = \mathfrak{l}_{-1}, \mathcal{Z}'_{\tau'^*-} = \mathfrak{l}|\mathcal{Z}'_0 = \langle zL\rangle) = \sum_{M=0}^{\infty} F_{M+1}(tL^2)\mathbb{P}_{\check{\mathcal{Z}}}(\mathfrak{l}_{-1}\mathcal{A}_{L,z,t,\mathfrak{l},M}), \tag{4.4}$$

where $F_N(.)$ is the cumulative distribution function of the Gamma distribution with shape parameter N and scale parameter 1 (that is, it is the sum of N iid exponential random variables, each with expectation 1). Indeed, (4.4) holds since the time of the jumps of the Markov process $\mathcal{Z}'$ are independent of the location of the jump. On the other hand, we have

$$\begin{aligned}\sum_{M=0}^{\infty} F_{M+1}(tL^2)\mathbb{P}_{\check{\mathcal{Z}}}(\mathfrak{l}_{-1}\mathcal{A}_{L,z,t,\mathfrak{l},M}) &= \mathcal{P}(\mathfrak{l}-\mathfrak{l}_{-1})\sum_{M=0}^{\infty} F_{M+1}(tL^2)\mathbb{P}_{\check{\mathcal{Z}}}(\mathcal{A}_{L,z,t,\mathfrak{l},M}) \\ &= \mathcal{P}(\mathfrak{l}-\mathfrak{l}_{-1})\int_0^{tL^2} \mathbb{P}\,(\mathcal{Z}_s \\ &= \langle zL\rangle,\ \forall\ s' \in [0,s], \mathcal{Z}_{s'} \in D_L|\mathcal{Z}_0 = \mathfrak{l}\big)\,\mathrm{d}s.\end{aligned} \tag{4.5}$$

Since $\mathbf{B} = 1$, we have

$$\Lambda_{tL^2}(\langle zL\rangle) = \sum_{\mathfrak{l}\in\partial D_L} \mathbf{A}(\mathcal{J}(\mathfrak{l}))f(\mathfrak{l}/L)\int_0^{tL^2} \mathbb{P}\left(\mathcal{Z}_s = \langle zL\rangle,\ \forall\ s' \in [0,s],\right. \\ \left. \times\ \mathcal{Z}_{s'} \in D_L|\mathcal{Z}_0 = \mathfrak{l}\right)\mathrm{d}s. \tag{4.6}$$

Thus by (4.1) and (4.5), we have

$$\Lambda_{tL^2}(\langle zL\rangle) = \sum_{\mathfrak{l}:\in\partial D_L} \sum_{\mathfrak{l}_{-1}\in\mathcal{L}\setminus D_L:(\mathfrak{l}_{-1},\mathfrak{l})\in\mathbf{G}} f(\mathfrak{l}/L)\sum_{M=0}^{\infty} F_{M+1}(tL^2)\mathbb{P}_{\tilde{\mathcal{Z}}}(\mathfrak{l}_{-1}\mathcal{A}_{L,z,t,\mathfrak{l},M})$$

and so by (4.4),

$$\Lambda_{tL^2}(\langle zL\rangle) = \mathbb{E}\left(f\left(\frac{\mathcal{Z}'_{\tau'^*-}}{L}\right)1_{\tau'^*<tL^2}|\mathcal{Z}'_0 = \langle zL\rangle\right). \tag{4.7}$$

Now the right-hand side of (4.7) converges, as $L\to\infty$ to

$$\mathbb{E}\left(f\left(W_{\mathcal{T}^*}\right)1_{\mathcal{T}^*<t}|W_0 = z\right), \tag{4.8}$$

where W_t is a standard planar Brownian motion and $\mathcal{T}^*$ is the hitting time of $\mathbb{R}^2\setminus D$ by W. (This follows from Donsker's theorem and the continuous mapping theorem. A more detailed proof of (4.8) for the case $t=\infty$ can be found in e.g. [22, proposition 3].) Let $\mathcal{W}$ be a diffusion process whose first coordinate is deterministic with constant 1 drift and whose second and third coordinates are independent standard Brownian motions. Applying Dynkin's formula for $\mathcal{W}$ with $\mathcal{W}_0 = (-t, z)$, the stopping time $\mathbf{t}$ as the first hitting time of $\mathbb{R}^3\setminus([-t,0]\times D)$, and with the test function $v(-s,\tilde{z})$, where v is defined by (2.2), we conclude that (4.8) satisfies (2.6) with $\varsigma = 1$. □

We record a remark for later reference:

Remark 4.2. Note that the proof of proposition 4.1 does not use theorem 2.1. Thus we already have an example (random walks), where both the assumptions and the conclusion of theorem 2.1 are verified (by propositions 3.1 and 4.1, respectively).

4.2. Lorentz gas

Let D be a bounded domain with piece-wise smooth boundary and no cusps. In the setup of section 3.2, given D, L, and $\mathfrak{l}\in\partial D_L$, we consider the following initial measure. For any $\mathfrak{l}_-\in\mathcal{L}\setminus D_L$ connected to $\mathfrak{l}$ in $\mathbf{G}$, $\tilde{\mathfrak{l}}' := \Sigma^{1/2}\mathfrak{l}'$ is a nearest neighbor of $\tilde{\mathfrak{l}} := \Sigma^{1/2}\mathfrak{l}$ in $\mathbb{Z}^2$ (that is, $\tilde{\mathfrak{l}}-\tilde{\mathfrak{l}}'\in\{w_1=(0,-1),\ w_2=(0,1),\ w_3=(-1,0),\ w_4=(1,0)\}$) by our assumption in section 3.2). Let $E = E_{\mathfrak{l},\mathfrak{l}'}\subset\mathbb{R}^2$ be the line segment on the boundary of $\tilde{\mathfrak{l}}+\left[-1/2,1/2\right)^2$ and $\tilde{\mathfrak{l}}'+\left[-1/2,1/2\right)^2$. Define

$$\mathcal{N} = \mathcal{N}_{\mathfrak{l},\mathfrak{l}'} = \{(q,v)\in\Omega : q\in E, \langle v,\tilde{\mathfrak{l}}-\tilde{\mathfrak{l}}'\rangle > 0\}.$$

Let type$(\mathfrak{l},\mathfrak{l}') = j$ if $\tilde{\mathfrak{l}}-\tilde{\mathfrak{l}}' = w_j$ and $\zeta_j : \mathcal{N}_{0,\Sigma^{-1/2}w_j}\to\mathbb{R}_+$ be the first return to $\mathcal{N}_{0,\Sigma^{-1/2}w_j}$ in the compact Sinai billiard. That is

$$\zeta_j = \min\{s : \Phi_0^s(q,v)\in\mathcal{N}_{0,\Sigma^{-1/2}w_j}\}.$$

Let us also write

$$\bar{\zeta}_j = \int_{\mathcal{N}_{0,\Sigma^{-1/2}w_j}} \zeta_j \mathrm{d}\varrho_{0,\Sigma^{-1/2}w_j}.$$

Next, we define the finite measure $\varrho = \varrho_{\mathfrak{l},\mathfrak{l}'}$ on $\mathcal{N}$ by

$$\mathrm{d}\varrho = \frac{1}{2\bar{\zeta}_j}\cos(\langle v, \tilde{\mathfrak{l}} - \tilde{\mathfrak{l}}'\rangle)\mathrm{d}q\mathrm{d}v,$$

where type$(\mathfrak{l}, \mathfrak{l}') = j$. Note that $\varrho(\mathcal{N}) = |E_{\mathfrak{l},\mathfrak{l}-w_j}|/\bar{\zeta}_j$ and so it may not be a probability measure. Now the initial condition $\mathcal{G}$ is given by the normalized sum of these measures for all neighbors $\mathfrak{l}'$. That is,

$$\nu_{\mathcal{G}} = \frac{1}{\sum_{j\in\mathcal{J}(\mathfrak{l})}\frac{|E_{\mathfrak{l},\mathfrak{l}-w_j}|}{\zeta_j}}\sum_{j\in\mathcal{J}(\mathfrak{l})}\varrho_{\mathfrak{l},\mathfrak{l}-w_j}.$$

By definition, $\nu_{\mathcal{G}}$ is a probability measure. Next, we choose

$$\mathbf{A}(\mathcal{J}(\mathfrak{l})) = \sum_{j\in\mathcal{J}(\mathfrak{l})}\frac{|E_{\mathfrak{l},\mathfrak{l}-w_j}|}{\zeta_j}$$

(which clearly depends on $\mathfrak{l}$ only through $\mathcal{J}(\mathfrak{l})$) and $\mathbf{B} = 1$. This choice guarantees that particles are being continuously injected through the entire boundary of D_L with a measure which is simply the projection of the invariant measure μ to the Poincaré section on the boundary of D_L. Because of this very special choice of $\nu_{\mathcal{G}}, \mathbf{A}, \mathbf{B}$, we have

Proposition 4.3. *With the above choice, the conclusion of theorem 2.1, (LE) and (DLE) hold with $\varsigma = 1$ without assuming the rationality of $\mathcal{L}$ and for general bounded domains D with piece-wise smooth boundary and no cusps.*

Proof. The proof is similar to that of proposition 4.1. We use duality and it is sufficient to verify (2.6).

We claim that there is some $s^* > 0$ so that for any $(q, v) \in \mathcal{N}_{\mathfrak{l},\mathfrak{l}'}$ and any $s \in [0, s^*]$, $\tilde{\mathcal{Z}}_s(q, v) \in \{\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}'\}$. Furthermore, if there is some $s \in [0, s^*]$ with $\tilde{\mathcal{Z}}_s(q, v) = \mathfrak{l}'$, then $\tilde{\mathcal{Z}}_{s^*}(q, v) = \mathfrak{l}'$. Indeed, the first statement follows from (3.2) and the second follows from the fact that visiting $\mathfrak{l}$, then $\mathfrak{l}'$ and then $\mathfrak{l}$ again requires at least 2 collisions and so we choose s^* shorter than the minimal free flight.

Next, for any $(\mathfrak{l}, \mathfrak{l}')$ as above, by the definition of ϱ and by the fact that $s^* < \min\zeta$, we have for measurable sets $B \subset \cup_{s\in[0,s^*]}\Phi^s(\mathcal{N}_{\mathfrak{l},\mathfrak{l}'})$

$$\int B\mathrm{d}\mu = \int_0^{s^*}\left(\int B\mathrm{d}\Phi^s_*(\varrho_{\mathfrak{l},\mathfrak{l}'})\right)\mathrm{d}s. \tag{4.9}$$

By the definition of $\nu_{\mathcal{G}}, \mathbf{A}$ and $\mathbf{B}$, we have

$$\Lambda_{tL^2}(\langle zL\rangle) = \sum_{\mathfrak{l}:\in\partial D_L}\sum_{\mathfrak{l}'\in\mathcal{L}\setminus D_L:(\mathfrak{l}',\mathfrak{l})\in\mathbf{G}} f(\mathfrak{l}/L)\int_0^{tL^2}\int_{\mathcal{N}_{\mathfrak{l},\mathfrak{l}'}}\{(q, v) : \forall\ s' \in [0, s], \mathcal{Z}_{s'}(q, v) \in D_L, \mathcal{Z}_s(q, v)$$
$$= \langle zL\rangle\}\ \mathrm{d}\varrho_{\mathfrak{l},\mathfrak{l}'}(q, v)\mathrm{d}s$$

For fixed t, L, let $K \in \mathbb{Z}_+$ so that $Ks^* \leqslant tL^2 < (K + 1)s^*$. To simplify formulas, let us assume that $Ks^* = tL^2$ holds (it is easy to check that the contribution of $s \in [Ks^*, tL^2]$ is negligible). Now for $k = 1, \ldots, K$ we apply (4.9) with

$$B_{\mathfrak{l},\mathfrak{l}',k} = \{(q, v) \in \cup_{s\in[0,s^*]}\Phi^s(\mathcal{N}_{\mathfrak{l},\mathfrak{l}'}) : \forall\ s' \in [0, (k-1)s^*], \mathcal{Z}_{s'}(q, v) \in D_L,$$
$$\times\, \mathcal{Z}_{(k-1)s^*}(q, v) = \langle zL\rangle\}$$

and the definition of s^* to conclude

$$\int_{(k-1)s^*}^{ks^*}\int_{\mathcal{N}_{\mathfrak{l},\mathfrak{l}'}} \{(q, v) : \forall\ s' \in [0, s], \mathcal{Z}_{s'}(q, v) \in D_L, \mathcal{Z}_s(q, v){=}\ \langle zL\rangle\}\ \mathrm{d}\varrho_{\mathfrak{l},\mathfrak{l}'}(q, v)\mathrm{d}s = \int B_{\mathfrak{l},\mathfrak{l}',k}\mathrm{d}\mu$$

and so

$$\Lambda_{tL^2}(\langle zL\rangle) = \sum_{\mathfrak{l}:\in\partial D_L}\ \sum_{\mathfrak{l}'\in\mathcal{L}\setminus D_L:(\mathfrak{l}',\mathfrak{l})\in\mathbf{G}} f(\mathfrak{l}/L)\sum_{k=1}^{K}\int(B_k)\mathrm{d}\mu. \tag{4.10}$$

Now we recall the involution (also known as time reversibility) property of the billiards. For $(q, v) \in \Omega$, let $\mathcal{I}(q, v) = (q, -v)$. Then $\mathcal{I}$ preserves μ and anticommutes with the flow. That is,

$$\Phi^{-s} \circ \mathcal{I} = \mathcal{I} \circ \Phi^s.$$

(see e.g. [11, section 2.14]). Thus

$$\int B_{\mathfrak{l},\mathfrak{l}',k}\mathrm{d}\mu = \int B'_{\mathfrak{l},\mathfrak{l}',k}\mathrm{d}\mu, \tag{4.11}$$

where

$$B'_{\mathfrak{l},\mathfrak{l}',k} = \{(q, v) \in \Omega : \mathcal{Z}_0(q, v) = \langle zL\rangle$$
$$\exists s \in [(k-1)s^*, ks^*] : \forall\ s' \in [0, s] : \mathcal{Z}_{s'} \in D_L, \Pi_{\mathcal{D}}\Phi^s(q, v) \in E_{\mathfrak{l},\mathfrak{l}'}\}\,. \tag{4.12}$$

Using the notation (2.4) and combining (4.10)–(4.12), we conclude

$$\Lambda_{tL^2}(\langle zL\rangle) = \int_{(q,v):\mathcal{Z}_0(q,v)=\langle zL\rangle} f\left(\frac{\mathcal{Z}_{\tau^*-}}{L}\right) 1_{\tau^*<tL^2}\mathrm{d}\mu \tag{4.13}$$

By the invariance principle (3.3), the right-hand side of (4.13) converges as $L \to \infty$ to (4.8). As in proposition 4.1, (2.6) follows. □

5. Proof of theorem 2.1

The simplify notations, we assume that $a = 1$ (the proof extends to any $a > 0$ with no new ideas). We will prove (2.5) first. Let $z = (x, y)$ be a point in the interior of D. By definition, we have

$$\mathbb{E}(\Lambda(\langle zL\rangle)) = \int_0^\infty \sum_{\mathfrak{l}\in\partial_{\mathrm{W}}D_L} \mathbf{A}(\mathcal{J}(\mathfrak{l}))\mathbf{B}(t)f\left(\frac{\mathfrak{l}_2}{L}\right)$$
$$\times\, \mathbb{P}\left(\mathcal{Z}_t = \langle(x, y)L\rangle,\ \min\{\tau_0^{\mathcal{Y}}, \tau_L^{\mathcal{Y}}, \tau_0^{\mathcal{X}}, \tau_L^{\mathcal{X}}\} > t|\mathcal{Z}_0 = \mathfrak{l}\right) \mathrm{d}t$$
$$= \int_{\delta L^2}^{L^2/\delta} \ldots \mathrm{d}t + \int_0^{\delta L^2} \ldots \mathrm{d}t + \int_{L^2/\delta}^{\infty} \ldots \mathrm{d}t =: I_1 + I_2 + I_3 \tag{5.1}$$

with $I_j = I_j(L, x, y, \delta)$ for $j = 1, 2, 3$. Noting that

$$\lim_{\delta\to 0}\lim_{L\to\infty} I_2 + I_3 = 0. \tag{5.2}$$

by (H3), it remains to prove

$$\lim_{\delta\to 0}\lim_{L\to\infty} I_1 = u(z). \tag{5.3}$$

Let $\Psi_{\delta'} : [0, 1] \to [0, 1]$ be defined by

$$\Psi_{\delta'}(y) = \begin{cases} 0 & \text{if } y < \delta' \\ \frac{1}{\delta'}y - 1 & \text{if } \delta' \leqslant y < 2\delta' \\ 1 & \text{if } 2\delta' \leqslant y < 1 - 2\delta' \\ -\frac{1}{\delta'}y - 1 + \frac{1}{\delta'} & \text{if } 1 - 2\delta' \leqslant y < 1 - \delta' \\ 0 & \text{if } y > 1 - \delta' \end{cases}$$

and write $f_{\delta'}(y) = f(y)\Psi_{\delta'}(y)$.

To prove (5.3), we first write $I_1 = I_{11} + I_{12}$ with $I_{1,k} = I_{1,k}(L, x, y, \delta, \delta')$ for $k = 1, 2$, where I_{11} and I_{12} are obtained from I_1 by replacing f by $f_{\delta'}$ and $f - f_{\delta'}$, respectively. To verify (5.3), it is sufficient to prove

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{L\to\infty} I_{11} = u(z) \tag{5.4}$$

and

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{L\to\infty} I_{12} = 0 \tag{5.5}$$

To simplify notations, we will write $I_{11}^{\infty} = \lim_{L\to\infty} I_{11}$ and $I_{11}^{\infty,0} := \lim_{\delta\to 0} I_{11}^{\infty}$.

Let us consider the following truncated version of (2.1)

$$\Delta\widehat{u} = 0, \quad \widehat{u}|_{\partial D} = \varsigma F_{\delta'}, \tag{5.6}$$

where $F_{\delta'}$ is defined as F except that f is replaced by $f_{\delta'}$.

Proposition 5.1. *For any $\delta' \in (0, 1/4)$, $I_{11}^{\infty,0}$ is the solution of* (5.6).

Proof. The proof consists of two steps. First, we prove that $I_{11}^{\infty,0}$ exists; then we show that it satisfies (5.6).

Step 1: $I_{11}^{\infty,0}$ **exists**

Let us define $B = \mathfrak{l}_2^{(K)}$, where K is defined by (2.9). To simplify formulas, let us write $\bar{\tau} = \min\{\tau_0^{\mathcal{Y}}, \tau_L^{\mathcal{Y}}, \tau_0^{\mathcal{X}}, \tau_L^{\mathcal{X}}\}$. Also observe that by transitivity of $\mathbf{G}$, there are constants $\mathbf{A}_1, \ldots, \mathbf{A}_K$ so that for any $m \in \mathbb{N}$ and for any $k = 1, \ldots, K$, $\mathbf{A}(\mathcal{J}(\mathfrak{l}^{(mK+k)})) = \mathbf{A}_k$. Now, we compute

$$I_{11} = \sum_{\mathfrak{l}\in\partial_{\mathrm{W}}D_L,\mathfrak{l}_2/L\in(\delta^{`},1-\delta^{`})} \mathbf{A}(\mathcal{J}(\mathfrak{l}))\int_{\delta L^2}^{L^2/\delta}\mathbf{B}(t)f_{\delta^{`}}\left(\frac{\mathfrak{l}_2}{L}\right)\mathbb{P}\left(\mathcal{Z}_t=\langle(x,y)L\rangle,\bar{\tau}>t|\mathcal{Z}_0=\mathfrak{l}\mathrm{d}t\right.$$

$$= \sum_{m=\delta^{`}L/B}^{(1-\delta^{`})L/B}\sum_{k=1}^{K}\mathbf{A}_k\int_{\delta L^2}^{L^2/\delta}\mathbf{B}(t)f_{\delta^{`}}\left(\frac{\mathfrak{l}_2^{(mK+k)}}{L}\right)\mathbb{P}\left(\mathcal{Z}_t=\langle(x,y)L\rangle,\bar{\tau}>t|\mathcal{Z}_0=\mathfrak{l}^{(mK+k)}\mathrm{d}t\right.$$

$$= \sum_{m=\delta^{`}L/B}^{(1-\delta^{`})L/B}\sum_{k=1}^{K}\mathbf{A}_k$$

$$\times\int_{\delta}^{1/\delta}\mathbf{B}(sL^2)f_{\delta^{`}}\left(\frac{\mathfrak{l}_2^{(mK+k)}}{L}\right)\mathbb{P}\,\mathcal{Z}_{sL^2}=\langle(x,y)L\rangle,\bar{\tau}>sL^2|\mathcal{Z}_0$$

$$=\mathfrak{l}^{(mK+k)}L^2\mathrm{d}s$$

Now using (H2) with $T=sL^2$, $\alpha=x/\sqrt{s}$, $\beta=1/\sqrt{s}$, $\eta=\mathfrak{l}_2^{(mK+k)}/(L\sqrt{s})$, $\gamma=y/\sqrt{s}$, $\xi=1/\sqrt{s}$, we obtain

$$I_{11}\sim\sum_{m=\delta' L/B}^{(1-\delta')L/B}\sum_{k=1}^{K}\mathbf{A}_k c_k\int_{\delta}^{1/\delta}\mathbf{B}(sL^2)f_{\delta'}\left(\frac{\mathfrak{l}_2^{(mK+k)}}{L}\right)s^{-3/2}L^{-1}\psi\left(\frac{x}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)$$
$$\times\,\phi\left(\frac{\mathfrak{l}_2^{(mK+k)}}{L\sqrt{s}},\frac{y}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\mathrm{d}s$$

by uniform convergence, where $a_L\sim b_L$ means that $\lim_{L\to\infty}a_L/b_L=1$. Let us write

$$\bar{c}=\frac{1}{K}\sum_{k=1}^{K}\mathbf{A}_k c_k.$$

Then

$$I_{11}\sim\frac{\bar{c}K}{B}\int_{\delta}^{1/\delta}\mathbf{B}(sL^2)s^{-3/2}\psi\left(\frac{x}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\left[\sum_{m=\delta' L/B}^{(1-\delta')L/B}\frac{B}{L}f_{\delta'}\left(\frac{\mathfrak{l}_2^{(mK)}}{L}\right)\phi\left(\frac{\mathfrak{l}_2^{(mK)}}{L\sqrt{s}},\frac{y}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\right]\mathrm{d}s.$$

Replacing the Riemann sum with the corresponding Riemann integral, we obtain

$$I_{11}\sim\frac{\bar{c}K}{B}\int_{\delta}^{1/\delta}\mathbf{B}(sL^2)s^{-3/2}\psi\left(\frac{x}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)$$
$$\times\left[\int_{\delta'}^{1-\delta'}f_{\delta'}(\sigma)\phi\left(\frac{\sigma}{\sqrt{s}},\frac{y}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\mathrm{d}\sigma\right]\mathrm{d}s$$

(We are permitted to do this because of uniform convergence of the bracketed expression in s). Since the integrand in the last formula is uniformly continuous in s and since $\mathbf{B}$ is periodic with period 1 and $\int_0^1 \mathbf{B} = 1$, we can take the limit $L \to \infty$ to conclude that I_{11}^{∞} exists and is equal to

$$\frac{\bar{c}K}{B}\int_{\delta}^{1/\delta} s^{-3/2}\psi\left(\frac{x}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\left[\int_{\delta'}^{1-\delta'} f_{\delta'}(\sigma)\phi\left(\frac{\sigma}{\sqrt{s}},\frac{y}{\sqrt{s}},\frac{1}{\sqrt{s}}\right)\mathrm{d}\sigma\right]\mathrm{d}s.$$

Now we substitute (2.7) and (2.8) to the above to conclude

$$I_{11}^{\infty} = \frac{\bar{c}K}{B}\int_{\delta'}^{1-\delta'}\int_{\delta}^{\frac{1}{\delta}}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{s^2}(2k+x)\exp\left(-\frac{(2k+x)^2}{2s}\right)\frac{1}{\sqrt{2\pi}}\right.$$
$$\left.\times f_{\delta'}(\sigma)\left[\exp\left(-\frac{(y-\sigma-2n)^2}{2s}\right)-\exp\left(-\frac{(y+\sigma+2n)^2}{2s}\right)\right]\right)\mathrm{d}s\,\mathrm{d}\sigma.$$

Clearly, the sum is absolutely and uniformly convergent and so we can write the sums in front of the integrals. Thus

$$I_{11}^{\infty} = \frac{\bar{c}K}{B}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_{\delta'}^{1-\delta'} R(k,n,\delta,\sigma,s,x,y)\mathrm{d}\sigma,$$

where

$$R(k,n,\delta,\sigma,s,x,y) = \frac{x+2k}{\sqrt{2\pi}} f_{\delta'}(\sigma)$$
$$\times\int_{\delta}^{\frac{1}{\delta}}\frac{1}{s^2}\left[\exp\left(-\frac{(2k+x)^2+(y-\sigma-2n)^2}{2s}\right)\right.$$
$$\left.-\exp\left(-\frac{(2k+x)^2+(y+\sigma+2n)^2}{2s}\right)\right]\mathrm{d}s.$$

Making the substitution $\omega = (2s)^{-\frac{1}{2}}$ (and so $4\omega\mathrm{d}\omega = -\mathrm{d}s/s^2$) and letting $P_1 = (2k+x)^2 + (y-\sigma-2n)^2$ and $P_2 = (2k+x)^2 + (y+\sigma+2n)^2$, we get:

$$R(k,n,\delta,\sigma,x,y) = \frac{4(x+2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma)\int_{\sqrt{\delta/2}}^{\frac{1}{\sqrt{2\delta}}}\omega\left[\exp(-P_1\omega^2)\right.$$
$$\left.-\exp(-P_2\omega^2)\right]\mathrm{d}\omega$$
$$= -\frac{2(x+2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma)\left[\frac{1}{P_1}\exp\left(-\frac{P_1}{2\delta}\right)-\frac{1}{P_2}\exp\left(-\frac{P_2}{2\delta}\right)\right]$$
$$+\frac{2(x+2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma)\left[\frac{1}{P_1}\exp\left(-\frac{P_1\delta}{2}\right)\right.$$
$$\left.-\frac{1}{P_2}\exp\left(-\frac{P_2\delta}{2}\right)\right]$$
$$=: R_1 + R_2.$$

Clearly, we have

$$\lim_{\delta\to 0}\sum_n\sum_k R_1 = 0$$

and as lemma 5.2 shows,

$$\lim_{\delta\to 0}\sum_n\sum_k R_2 = \sum_n\sum_k \lim_{\delta\to 0} R_2.$$

So we get

$$\lim_{\delta\to 0} R(k,n,\delta,\sigma,x,y) = R(k,n,\sigma,x,y) = \frac{2(x+2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma)\left[\frac{1}{P_1}-\frac{1}{P_2}\right].$$

and hence

$$I_{11}^{\infty,0} = \frac{\bar{c}K}{B}\int_{\delta'}^{1-\delta'}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R(k,n,\sigma,x,y)\mathrm{d}\sigma. \tag{5.7}$$

To complete step 1, it remains to verify

Lemma 5.2. *Let* $\mathfrak{u}(z)=\exp(-z)/z$. *And let* P_1 *and* P_2 *be as defined above. Then for* $\delta\in\mathbb{R}$, $x\in[0,1]$, $\sigma\in[0,1]$, *and* k,n *not both* 0, *the following sum converges uniformly in* δ, x, *and* σ *as* $M\to\infty$.

$$\sum_{k=-M}^{M}\sum_{n=-M}^{M}(2k+x)\delta[\mathfrak{u}(P_1\delta)-\mathfrak{u}(P_2\delta)].$$

Proof. Let us write

$$P_3=(2k+x)^2+(y-\sigma+2n)^2, \qquad P_4=(2k+x)^2+(y+\sigma-2n)^2.$$

We will show

$$\lim_{M\to\infty}\left\{\sum_{k:|k|>M}\sum_{n=1}^{\infty}+\sum_{k=-M}^{M}\sum_{n=M}^{\infty}\right\}|\mathcal{S}|=0, \tag{5.8}$$

where

$$\mathcal{S}=\mathcal{S}(k,n,\delta,\sigma,x,y)=(2k+x)\delta[\mathfrak{u}(P_1\delta)-\mathfrak{u}(P_2\delta)+\mathfrak{u}(P_3\delta)-\mathfrak{u}(P_4\delta)],$$

and the convergence is uniform in δ,x,σ. First, observe that

$$P_1-P_2=-4(\sigma+2n)y, \qquad P_3-P_4=4(2n-\sigma)y$$

By the mean value theorem, for some $P_1'\in(P_1,P_2)$ and $P_3'\in(P_4,P_3)$. Using the mean value theorem again, we conclude

$$\mathfrak{u}(P_1\delta)-\mathfrak{u}(P_2\delta)+\mathfrak{u}(P_3\delta)-\mathfrak{u}(P_4\delta) = -4\sigma y\delta[\mathfrak{u}'(P_1'\delta)+\mathfrak{u}'(P_3'\delta)] - 8ny\delta^2(P_1'-P_3')\mathfrak{u}''(P_1''\delta)$$

for some $P_1'' \in (P_4, P_2)$. In the sequel, C denotes a universal constant (independent of $k, n, x, y, \delta, \sigma, L$ or any other parameters), whose value is unimportant and can even change from line to line. Now using the estimates $|\mathrm{u}'(z)| < C/z^2$, $|\mathrm{u}''(z)| < C/z^3$ for any real number z, we have

$$|\mathcal{S}| \leqslant C\left(\frac{|k|}{(k^2+n^2)^2} + \frac{|k|n^2}{(k^2+n^2)^3}\right)$$

Thus we conclude

$$\sum_{k=M}^{\infty}\sum_{n=1}^{k}|\mathcal{S}| \leqslant C\sum_{k=M}^{\infty}\sum_{n=1}^{k}\frac{1}{k^3} \leqslant C/M$$

and likewise

$$\sum_{n=M}^{\infty}\sum_{k=0}^{n-1}|\mathcal{S}| \leqslant C\sum_{n=M}^{\infty}\sum_{k=0}^{n-1}\frac{1}{n^3} \leqslant C/M$$

We have verified (5.8). The lemma follows. □

Step 2: $I_{11}^{\infty,0}$ **satisfies** (5.6)

We give two independent proofs for step 2. The first proof is shorter and easily generalizes to the case of finite t. The second proof shows that the formulas derived above are tractable (at least in the case $t = \infty$).

Proof 1: step 1 shows that for any stochastic process $\mathcal{Z}_t$ satisfying (H1)–(H3), the limit (5.7) is the same. Recalling remark 4.2, we already have examples where (H1)–(H3) as well as the conclusion of the theorem hold. Thus $I_{11}^{\infty,0}$ has to satisfy (5.6). To finish the first proof, we identify the constant ς.

Let us consider the simplest possible random walk, called the simple symmetric random walk. That is, $w_1 = (0, -\sqrt{2})^{\mathrm{T}}$, $w_2 = (0, \sqrt{2})^{\mathrm{T}}$, $w_3 = (-\sqrt{2}, 0)^{\mathrm{T}}$, $w_4 = (\sqrt{2}, 0)^{\mathrm{T}}$ and

$$\mathcal{P}(w_i) = \frac{1}{4} \quad \text{for } i = 1, \ldots, 4.$$

In this case, $\mathcal{L} = (\sqrt{2}\mathbb{Z})^2$ and by the central limit theorem, $\mathcal{Z}_t/\sqrt{t}$ converges to a 2 dimensional standard normal random variable (we chose the normalization $\sqrt{2}$ so that the limiting covariance matrix is identity and so $\mathcal{Z}$ fits into the framework of proposition 4.1). In this case, we clearly have $K = 1$, $B = \sqrt{2}$, $\mathbf{A}_1 = 1/4$ (and $\mathbf{B} = 1$). Thus $\bar{c} = c_1/4$. Next we claim that now $c_1 = 4/\sqrt{\pi}$. To prove the claim, first note that

$$\lim_{T\to\infty}\sqrt{T}\mathbb{P}(\tau_0^{\mathcal{X}} > T | \mathcal{X}_0 = 0) = \frac{2}{\sqrt{\pi}} \tag{5.9}$$

(this follows from e.g., [21, proposition 5.1.2]). The proof of (H2) is based on the fact that, under the assumption that $\tau_0^{\mathcal{X}} > T$, $\mathcal{Z}_{\lfloor tT\rfloor}/\sqrt{T}$ with $0 \leqslant t \leqslant 1$ converges to a stochastic process whose first coordinate is a Brownian meander and the second coordinate is a Brownian motion. Furthermore, the local limit theorem also holds under the assumption $\tau_0^{\mathcal{X}} > T$ which gives (H2) (see the details in section 7). This local limit theorem combined with (5.9) gives $c_1 = \frac{2}{\sqrt{\pi}}\mathrm{covol}(\mathcal{L}) = 4/\sqrt{\pi}$ which proves the claim.

Thus in case $K = 1, B = \sqrt{2}, \bar{c} = 1/\sqrt{\pi}$, (5.7) satisfies (5.6) with $\varsigma = 1$. Since (5.7) is linear in $\bar{c}K/B$, we conclude that in case of general K, B and $\bar{c}$, (5.7) satisfies (5.6) with

$$\varsigma = \frac{\sqrt{2\pi}\bar{c}K}{B}. \tag{5.10}$$

Proof 2:

Step 2′a: $I_{11}^{\infty,0}$ is harmonic

An elementary computation shows that $R(k, n, \delta, \sigma, x, y)$, as a function of $x, y \in (0, 1)^2$ is harmonic for any k and n. Since the derivatives of $R(k, n, \sigma, x, y)$ with respect to x and y converge uniformly in a neighborhood of x, y, the Laplacian can be taken inside the sum in (5.7). It follows that $I_{11}^{\infty,0}$ is harmonic.

Step 2′b: $I_{11}^{\infty,0}$ satisfies the boundary conditions of (5.6)

Recall (5.7) from step 1. Let us first consider the case when $|n| + |k| > 0$. In this case, there is uniform convergence in x, y and σ so we can write the limit inside the sum and the integral:

$$\frac{\bar{c}K}{B} \sum_{k,n\in\mathbb{Z};|n|+|k|>0} \int_{\delta'}^{1-\delta'} \lim_{(x,y)\to(0,y_0)} R(k, n, \sigma, x, y)\mathrm{d}\sigma.$$

We can directly compute this limit as

$$\begin{aligned}\int_{\delta'}^{1-\delta'} \lim_{(x,y)\to(0,y_0)} R(k, n, \sigma, x, y)\mathrm{d}\sigma &= \int_{\delta'}^{1-\delta'} R(k, n, \sigma, 0, y_0)\mathrm{d}\sigma \\ &= \int_{\delta'}^{1-\delta'} f_{\delta'}(\sigma) \\ &\quad\times \left[\frac{16ky_0(\sigma + 2n)}{[(2k)^2 + (y_0 - \sigma - 2n)^2][(2k)^2 + (y_0 + \sigma + 2n)^2]}\right] \mathrm{d}\sigma.\end{aligned}$$

We see that for each n, these terms are antisymmetric in k, so that summing over k and n, with $|n| + |k| > 0$, all of the terms cancel. Now we consider the case $n = k = 0$. This term gives:

$$\lim_{(x,y)\to(0,y_0)} I_{11}^{\infty,0} = \frac{\bar{c}K}{B}\frac{8}{\sqrt{2\pi}} \lim_{(x,y)\to(0,y_0)} \int_{\delta'}^{1-\delta'} f_{\delta'}(\sigma) \left[\frac{\sigma xy}{[x^2 + (y - \sigma)^2][x^2 + (y + \sigma)^2]}\right] \mathrm{d}\sigma.$$

To compute the above integral assume first that $\delta' < y_0 < 1 - \delta'$, and decompose it as

$$\begin{aligned}\int_{\delta'}^{1-\delta'} \ldots \mathrm{d}\sigma &= \int_{y_0-Ax}^{y_0-Ax} \ldots \mathrm{d}\sigma + \int_{y\in[\delta',1-\delta']\setminus[y_0-Ax,y_0+Ax]} \ldots \mathrm{d}\sigma \\ &=: I_{111} + I_{112}\end{aligned}$$

for some large constant A.

First, we compute I_{111}. For y_0 and A fixed, and for x and $|y - y_0|$ small, $yf_{\delta'}(\sigma)/[x^2 + (y + \sigma)^2]$ is close to $f_{\delta'}(y_0)/(4y_0)$ uniformly in σ as in I_{111}. Indeed, this follows from the continuity of $f_{\delta'}$. Thus we can write this term in front of the integral. Now it remains to compute

$$\int_{y_0-Ax}^{y_0+Ax} x\sigma/[x^2 + (y_0 - \sigma)^2]\mathrm{d}\sigma.$$

Let us apply the substitution $\rho = (\sigma - y_0)/x$. Then the previous integral becomes

$$\int_{-A}^{A} x\rho/(1+\rho^2)\mathrm{d}\rho + \int_{-A}^{A} y_0/(1+\rho^2)\mathrm{d}\rho.$$

The first integral here is zero as the integrand is an odd function. The second integral is $\pi y_0(1 + o_A(1))$. We conclude

$$\lim_{(x,y)\to(0,y_0)} I_{111} = \frac{\pi}{4} f_{\delta'}(y_0)(1 + o_A(1)). \tag{5.11}$$

Next, we claim

$$\lim_{(x,y)\to(0,y_0)} I_{112} = o_A(1). \tag{5.12}$$

To prove (5.12), we compute

$$\begin{aligned}
&\int_{y_0+Ax}^{1-\delta'} f_{\delta'}(\sigma)\frac{\sigma x y_0}{[x^2+(y_0-\sigma)^2][x^2+(y_0+\sigma)^2]}\mathrm{d}\sigma \\
&\quad\leqslant \|f\|_\infty \sum_{i=1}^{\infty}\int_{y_0+Axi}^{y_0+Ax(i+1)} \frac{\sigma x y_0}{[x^2+(y_0-\sigma)^2][x^2+(y_0+\sigma)^2]}\mathrm{d}\sigma \\
&\quad\leqslant \|f\|_\infty \sum_{i=1}^{\infty}\int_{y_0+Axi}^{y_0+Ax(i+1)} \frac{\sigma x y_0}{[x^2+(Axi)^2][2y_0\sigma]}\mathrm{d}\sigma \\
&\quad\leqslant \frac{\|f\|_\infty}{2} \sum_{i=1}^{\infty}\int_{y_0+Axi}^{y_0+Ax(i+1)} \frac{x}{x^2[1+A^2i^2]}\mathrm{d}\sigma \\
&\quad= \frac{\|f\|_\infty}{2} \sum_{i=1}^{\infty}\frac{A}{1+A^2i^2} \leqslant \frac{\pi^2\|f\|_\infty}{12}\frac{1}{A}.
\end{aligned}$$

This estimate, combined with a similar computation for the domain $[\delta', y_0 - A\delta']$, verifies (5.12). Next, if $y_0 < \delta'$ or $y_0 > 1-\delta'$, then clearly $I_{111} = 0$ and $I_{112} = o_A(1)$. Now combining (5.11) and (5.12), we obtain the boundary conditions of (5.6) on the 'West side' (that is when $x = 0$) with the constant

$$\varsigma = \frac{\sqrt{2\pi}\bar{c}K}{B}.$$

which coincides with (5.10).

Checking the boundary conditions on the other three sides is easier. First, recall that

$$\begin{aligned}
R(n,k,\sigma,x,y) &= \frac{2(x+2k)}{\sqrt{2\pi}} \\
&\quad\times \frac{(y+\sigma+2n)^2-(y-\sigma-2n)^2}{[(2k+x)^2+(y-\sigma-2n)^2][(2k+x)^2+(y+\sigma+2n)^2]}.
\end{aligned}$$

Thus for every $k = 0, 1, 2, \ldots,$ we have $R(n,k,\sigma,1,y) = -R(n,-k-1,\sigma,1,y)$ and so $\sum_{k\in\mathbb{Z}} R(n,k,\sigma,1,y) = 0$ for every n. It follows that $\lim_{x\to1} I_{11}^{\infty,0} = 0$. Clearly, $R(n,k,\sigma,x,0) =$

0 for every n and k and so $\lim_{y\to 0} I_{11}^{\infty,0} = 0$. Finally, to prove $\lim_{y\to 1} I_{11}^{\infty,0} = 0$, let us write

$$\lim_{y\to 1} I_{11}^{\infty,0} = \sum_k \frac{2(x+2k)}{\sqrt{2\pi}} \sum_n \frac{1}{P_1(n)} - \frac{1}{P_2(n)},$$

where $P_1(n) = (2k+x)^2 + (1-\sigma-2n)^2$ and $P_2(n) = (2k+x)^2 + (1+\sigma+2n)^2$. Now observe that $P_2(n) = P_1(n+1)$. Thus the sum over n is telescopic and so by absolute convergence, $\lim_{y\to 1} I_{11}^{\infty,0} = 0$. We have finished the proof of step $2'$b. $\square$

Now we finish the proof of (2.5). First note that proposition 5.1 implies (5.4). Thus it remains to verify (5.5). Consider the following Dirichlet problem:

$$\begin{cases} \Delta U = 0 \quad \text{in}\,(0,1)\times(-1,2), \\ U(0,y) = \varsigma(f(y) - f_{\delta'}(y)), U(1,y) = U(x,-1) = U(x,2) = 0, \end{cases} \tag{5.13}$$

where f and $f_{\delta'}$ are identically zero on $[-1,0]\cup[1,2]$. Now the proof of proposition 5.1 applied on the domain $(0,1)\times(-1,2)$ with boundary condition given by $f - f_{\delta'}$ implies that for any δ', x, y fixed,

$$\lim_{\delta\to 0} \lim_{L\to\infty} I_{12} \leqslant U(x,y).$$

Indeed, on the one hand, if the particles are only killed upon leaving $(0,L)\times(-L,2L)$, then we obtain an upper bound on the number of surviving particles in the case when particles are killed upon leaving $(0,L)\times(0,L)$. On the other hand, the proof of proposition 5.1 is applicable on the larger domain since the boundary condition is identically zero in a neighborhood of the corners.

Now since the function $f - f_{\delta'}$ is supported on the union of two intervals with total length $4\delta'$ and is bounded uniformly in δ', we have $\lim_{\delta'\to 0} U(x,y) = 0$ for all x, y fixed. Thus (5.5) follows and the proof of (2.5) is complete.

The proof of (2.6) is similar, so we only explain the differences. First, the decomposition (5.1) now reads

$$\int_{\delta L^2}^{tL^2} \ldots \mathrm{d}t + \int_0^{\delta L^2} \ldots \mathrm{d}t =: I_1 + I_2.$$

In particular, I_3 is missing and I_2 is negligible as before. We decompose $I_1 = I_{11} + I_{12}$ as before. Proceeding as in step 1 of the proof of proposition 5.1, we obtain

$$\lim_{\delta\to 0} \lim_{L\to\infty} I_{11} = \frac{\bar{c}K}{B} \int_{\delta'}^{1-\delta'} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(t,k,n,\sigma,x,y)\mathrm{d}\sigma,$$

where

$$R(t,k,n,\sigma,x,y) = \frac{2(x+2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma) \left[\frac{1}{P_1} \exp\left(-\frac{P_1}{2t}\right) - \frac{1}{P_2} \exp\left(-\frac{P_2}{2t}\right)\right].$$

The first proof of step 2 in proposition 5.1 is the same as before. We prefer not to give a second proof of step 2 as in the time dependent case, the formulas in step $2'$a become substantially longer. Finally, the proof of (5.5) is again analogous to the previous case with U as in (5.13) replaced by the unique solution $V(t,x,y):\mathbb{R}_{\geqslant 0}\times(0,1)\times(-1,2)\to\mathbb{R}$ of

$$\begin{cases} V_t=\dfrac{1}{2}[V_{xx}+V_{yy}],\\ V(t,0,y)=\varsigma(f(y)-f_{\delta'}(y)), V(t,1,y)=V(t,x,-1)\\ \qquad\qquad =V(t,x,2)=0, V(0,x,y)=0.\end{cases}$$

We have finished the proof of theorem 2.1.

6. Background on Lorentz gas

6.1. Preliminaries

Here, we review some results for the Lorentz gas that are necessary to the proof of theorem 3.2. We refer the reader to [11] for an in depth discussion. Let us use the notation of section 3.2.

The map $\mathcal{F}_0$ is hyperbolic in the sense that there are stable and unstable cone fields $\mathcal{C}_x^{u/s}\subset T_x\mathcal{M}_0$ so that $D_x\mathcal{F}_0(\mathcal{C}_x^u)\subset\mathcal{C}^u_{\mathcal{F}_0(x)}$ and $D_x\mathcal{F}_0^{-1}(\mathcal{C}_s^u)\subset\mathcal{C}^2_{\mathcal{F}_0^{-1}(x)}$ and for all $v\in\mathcal{C}_x^u$, $\|D_x\mathcal{F}_0(v)\|\geqslant\Lambda\|v\|$ (and likewise for all $v\in\mathcal{C}_x^s$, $\|D_x\mathcal{F}_0^{-1}(v)\|\geqslant\Lambda\|v\|$). Furthermore, stable and unstable manifolds exist through almost every point, but not through every point because of singularities due to grazing collisions. In fact, the presence of these singularities makes the study of billiards particularly peculiar.

Let us use the coordinates (r,φ) on $\mathcal{M}_0$ where r is the arc length parameter of $\partial\mathcal{D}_0$ and $\varphi\in[-\pi/2,\pi/2]$ is the angle between the postcollisional velocity and the normal vector to $\mathcal{D}$. A curve $W\subset\mathcal{M}_0$ is called unstable if for every $x\in W$, T_xW is in the unstable cone. Furthermore, an unstable curve W is called weakly homogeneous if it does not intersect any singularity and there exists $k=0,k_0,k_0+1,\ldots$ so that for all $x=(r,\varphi)\varphi\in[(k+1)^{-2},k^{-2}]$ if $|k|>k_0$ or $|\varphi|<k_0^{-2}$. In other words, weakly homogeneous unstable curves are required to be disjoint from the real singularities of $\mathcal{F}_0$ as well as secondary singularities $\varphi=\pm k^{-2}$ for $|k|\geqslant k_0$. A weakly homogeneous unstable curve is called homogeneous if it satisfies certain extra regularity properties whose exact form are not needed for us (see the distortion and curvature bounds in [10, section 4.3]).

A pair $\ell=(W,\rho)$ is called a *standard pair* if W is a homogeneous unstable curve and ρ is a probability measure on W so that

$$\left|\log\frac{\mathrm{d}\rho}{\mathrm{dLeb}}(x)-\log\frac{\mathrm{d}\rho}{\mathrm{dLeb}}(y)\right|\leqslant C_0\frac{|W(x,y)|}{|W|^{2/3}},\qquad(6.1)$$

where C_0 is universal constant and $|.|$ stands for arc length. Here and in the sequel log stands for logarithm with base e. We will also use the notation $\log_2$ for the logarithm with base 2. Given ℓ, we denote by ν_ℓ the probability measure generated by ρ and length(ℓ) = length(W). Due to the singularities, an image of a homogeneous unstable curve will be a collection of unstable curves. Furthermore, the regularity of ρ in (6.1) is defined in a way that is preserved by $\mathcal{F}_0$. The exact exponent $2/3$ is related to the way the homogeneity strips are defined. The fact that the regularity (6.1) is preserved relies on distortion estimates and follows from e.g. [10, proposition 4.9]. Thus the image of a standard pair under $\mathcal{F}_0$ is the weighted average of standard pairs. It is convenient to introduce the notion of a *standard family*: a weighted average of standard pairs. Specifically, let us say that $\mathcal{G}=\{\{\ell_a=(W_a,\rho_a)\}_{a\in\mathfrak{A}},\lambda\}$ is a standard family if ℓ_a are standard pairs, W_a's are disjoint and λ is a probability measure on the index set $\mathfrak{A}$. The standard family $\mathcal{G}$ induces a measure $\nu_{\mathcal{G}}$ on $\mathcal{M}_0$ by

$$\nu_{\mathcal{G}}(B) = \int_{\mathfrak{A}} \nu_\ell(B \cap W_a) d\lambda(a)$$

for Borel sets $B \subset \mathcal{M}_0$. For a given homogeneous unstable curve W, and for $x \in W$, we denote by $r(x)$ the distance from x to the closest endpoint of W, measured along W. We denote by $r_n(x)$ the distance from $\mathcal{F}_0^n(x)$ to the closest endpoint of W', where W' is the maximal homogeneous curve in the image $\mathcal{F}^n(W)$ containing $\mathcal{F}_0^n(x)$. We define the Z function of a standard family by

$$Z_{\mathcal{G}} = \sup_{\varepsilon > 0} \frac{\nu_{\mathcal{G}}(r < \varepsilon)}{\varepsilon}.$$

Note that we assumed that the curves in a standard family are disjoint and so the function r is well defined. Now we are ready to state the last missing technical piece of theorem 3.2: $\mathcal{G}$ is any standard family with a finite Z function. Examples include any standard pair or the invariant measure ν_0.

A fundamental property of Sinai billiards is that the expansion wins over fragmentation. That is, most of the weight carried by the image of a standard pair is concentrated on long curves. The precise statement, called growth lemma is the following (see [10, propositions 4.9 and 4.10]):

Lemma 6.1. *For any standard pair $\ell = (W, \rho)$ and any $n \in \mathbb{Z}_+$,*

$$\nu_\ell(A \circ \mathcal{F}_0^n) = \sum_i c_{n,i} \nu_{\ell_{n,i}}(A), \tag{6.2}$$

where $c_{n,i} > 0$, $\sum_i c_{n,i} = 1$ and $\ell_{n,i} = (W_{n,i}, \rho_{n,i})$ are standard pairs so that $\cup_i W_{n,i} = \mathcal{F}_0^n(W)$ and $\rho_{n,i}$ is a constant times the push-forward of ρ by $\mathcal{F}_0^n$. Furthermore, there are universal constants $\varkappa$ and C so that for any $n > \varkappa \log \mathrm{length}(\ell)$ and for any $\varepsilon > 0$

$$\sum_{i:\mathrm{length}(\ell_{n,i}) < \varepsilon} c_{n,i} < C\varepsilon.$$

We will refer to (6.2) as Markov decomposition. A simple consequence of the growth lemma is the following lemma, which is proven in e.g. [11, proposition 7.17].

Lemma 6.2. *There are constants c_1, c_2 and $\theta < 1$ depending only on $\mathcal{D}_0$ so that for any standard family $\mathcal{G}$ with finite Z function and for any n,*

$$Z_{\mathcal{F}_0^n(\mathcal{G})} \leqslant c_1 \theta^n Z_{\mathcal{G}} + c_2.$$

Let $\kappa : \mathcal{M}_0 \to \mathbb{R}^2$ be the free flight vector and $\breve{\kappa} : \mathcal{M}_0 \to \mathbb{R}^2$ be the discrete free flight vector. That is, $\breve{\kappa}(q, v) = \Pi_{\mathbb{Z}^2}(\mathcal{F}_0(q, v)) - \Pi_{\mathbb{Z}^2}(q, v)$. Let us also write $\bar{\kappa} = \int |\kappa| \mathrm{d}\nu_0 \in \mathbb{R}_+$.

Let

$$\breve{\mathcal{Z}}_n(q, v) = \sum_{j=0}^{n-1} \breve{\kappa}(\mathcal{F}_0^j(q, v)). \tag{6.3}$$

Similarly to the flow, we write $\check{\mathcal{Z}}_n = (\check{\mathcal{X}}_n, \check{\mathcal{Y}}_n)$. Put

$$\tau_0^{\check{\mathcal{X}}} = \min\{n > 0 : \check{\mathcal{X}}_n < 0\}$$

and for $x \neq 0$, put

$$\tau_x^{\check{\mathcal{X}}} = \min\{n > 0 : \check{\mathcal{X}}_n = x\}$$

(and likewise with $\check{\mathcal{X}}$ replaced by $\check{\mathcal{Y}}$).

The next result is the extension of the invariance principle (3.3) to standard pairs (see e.g. [9]).

Theorem 6.3 (Invariance principle). *Fix a standard pair ℓ and consider the stochastic processes $\tilde{\mathcal{Z}}_t$, $\check{\mathcal{Z}}_n$ induced by the initial condition ℓ. Then*

(*a*) $\tilde{\mathcal{Z}}_{tT}/\sqrt{T}$, $t \in [0, 1]$ *converges weakly as $T \to \infty$ to a planar Brownian motion with zero mean and covariance matrix Σ (introduced in section 3.2) uniformly for ℓ satisfying* $|\log \text{length}(\ell)| > T^{1/4}$.

(*b*) *With the notation $\check{\Sigma} = \bar{\kappa}\Sigma$ we have $\check{\mathcal{Z}}_{\lfloor tN \rfloor}/\sqrt{N}$, $t \in [0, 1]$ converges weakly as $N \to \infty$ to a planar Brownian motion with zero mean and covariance matrix $\check{\Sigma}$ uniformly for ℓ satisfying* $|\log \text{length}(\ell)| > N^{1/4}$.

Another extension of the central limit theorem is the so-called MLLT, which we discuss next.

6.2. Mixing local limit theorem

Recall (6.3). Let us also define

$$F_n(q, v) = \sum_{j=0}^{n-1} |\kappa(\mathcal{F}_0^j(q, v))|.$$

Given $\mathbf{x} \in \mathbb{R}^2$, $y \in \mathbb{R}$ and a standard pair ℓ let us denote by ϑ_n the push-forward of ν_ℓ by the map

$$(q, v) \mapsto \left(\check{\mathcal{Z}}_n(q, v) - \langle \mathbf{x}\sqrt{n} \rangle, \qquad F_n(q, v) - n\bar{\kappa} - y\sqrt{n}, \mathcal{F}_0^n(q, v)\right).$$

That is, $\vartheta_n = \vartheta_n(\ell, \mathbf{x}, y)$ is a measure on $\mathbb{Z}^2 \times \mathbb{R} \times \mathcal{M}_0$. Fix an open set $A \subset \Omega_0$ as in (3.4) and define $\mathcal{A} \subset \mathbb{Z}^2 \times \mathbb{R} \times \mathcal{M}_0$ so that $((k, l), -t, (q, v)) \in \mathcal{A}$ if and only if $\Pi_{\mathbb{Z}^2}(q, v) = (k, l)$, $\Pi_{\mathbb{Z}^2}(\Phi^t(q, v)) = 0$, $\Phi^t(q, v) \in A$ and $|\kappa(q, v)| > t$. That is, $\mathcal{A}$ contains phase points $(q + (k, l), v)$ and corresponding flight times t so that a flight of length t from $(q + (k, l), v)$ is free and arrives in the set A. By the finite horizon assumption, $\mathcal{A}$ is bounded. Without loss of generality, we will later choose the fundamental domain large enough so that $|\check{\kappa}_i| \leqslant 1$ for $i = 1, 2$ and so the absolute value of both integer coordinates of $\mathcal{A}$ are bounded by 1.

Let $\mathfrak{g}_\Sigma$ denote the Gaussian density with zero mean and covariance matrix Σ. The version of the MLLT that we consider here is the following

Theorem 6.4. *There is a positive definite 3×3 matrix $\tilde{\Sigma}$ whose top left 2×2 minor is $\check{\Sigma}$ and constants C, C_1, C_2 so that for any standard pair ℓ with $|\log \text{length}(\ell)| < n^{1/4}$ the following hold* (*a*) *For any* $(\mathbf{x}, y) \in \mathbb{R}^3$ *and for any* A *as in* (3.4),

$$\lim_{n\to\infty} n^{3/2}\vartheta_n(\mathcal{A}) = \mu_0(A)\bar{\kappa}\mathfrak{g}_{\tilde{\Sigma}}(\mathbf{x}, y)$$

uniformly for $\boldsymbol{x}, y$ *in compact subsets of* $\mathbb{R}^3$.

(*b*) *For any* $(\mathbf{x}, y) \in \mathbb{R}^3$ *and for any* A *as in* (3.4) *and for any positive integer* n,

$$n^{3/2}\vartheta_n(\mathcal{A}) < C_1\mathfrak{g}_{C\tilde{\Sigma}}(\mathbf{x}, y) + C_2^{-1/2}.$$

A variant of theorem 6.4 was proved in [13, lemma 2.8]. Specifically, [13, lemma 2.8] covers the case when $\check{\mathcal{Z}}$ is replaced by $\check{\mathcal{X}}$ and $A = \Omega_0$ in the definition of ϑ_n (we included the more general case of A to accommodate for (DLE) as in theorem 3.3). Since the proof directly applies here as well (except for one minor adjustment), we only discuss this minor adjustment and do not repeat the entire proof.

Proof. First, we need some definitions. For a bounded Hölder function $f : \mathcal{M}_0 \to \mathbb{R}^d$, we define $S(f)$ as the smallest closed additive subgroup of $\mathbb{R}^d$ that supports the values of $f - r$ for some $r \in \mathbb{R}^d$. Let us write $f \sim g$ if f and g are cohomologous. That is, $f(x) = g(x) + h(x) - h(\mathcal{F}_0(x))$ for a measurable h and for all $x \in \mathcal{M}_0$. We say that f is minimal if $M(f) = S(f)$, where

$$M(f) = \cap_{g\sim f} S(g).$$

The only minor adjustment that is needed in the proof of [13, lemma 2.8] is that we need to show that

$$f := (\check{\kappa}, |\kappa| - \bar{\kappa}) : \mathcal{M}_0 \to \mathbb{R}^3$$

is minimal. That is, $M(f) = \mathbb{Z}^2 \times \mathbb{R}$. (Heuristically, there is a clear obstruction to the MLLT in its present form if $M(f)$ is a proper subgroup of $\mathbb{Z}^2 \times \mathbb{R}$. It turns out that, similarly to the case of IID random variables, this is the only possible obstruction.) This generalizes [13, lemma A.3], which shows that

$$\tilde{f} := (\check{\kappa}_1, |\kappa| - \bar{\kappa})$$

is minimal. That is,

$$M(\tilde{f}) = \mathbb{Z} \times \mathbb{R}. \tag{6.4}$$

To establish the minimality of f, it is enough to prove the following. If $M(f)$ is a proper subgroup of $\mathbb{Z}^2 \times \mathbb{R}$, then there are real numbers α, r and two measurable functions $h : \mathcal{M}_0 \to \mathbb{R}$, $g : \mathcal{M}_0 \to \mathbb{Z}$ so that

$$|\kappa(q, v)| = h(q, v) - h(\mathcal{F}_0(q, v)) + r + \alpha g(q, v). \tag{6.5}$$

Indeed, a contradiction follows from (6.5) as in [13]. To prove (6.5), we first recall that by [28, theorem 5.1], $\check{\kappa}$ is minimal. Thus the projection of $M(f)$ to the first two coordinates needs to be $\mathbb{Z}^2$. In particular, there exist $e_1 = (0, 0, \alpha)^{\mathrm{T}}$, $e_2 = (1, 0, \beta)^{\mathrm{T}}$, and $e_3 = (0, 1, \gamma)^{\mathrm{T}}$ in $M(f)$. If $M(f)$ is a proper subgroup of $\mathbb{Z}^2 \times \mathbb{R}$, then there exists a minimal $\alpha > 0$ with the property that $e_1 \in M(f)$. Now we claim that e_1, e_2, e_3 generate $M(f)$. Indeed, by the choice of α, e_1 generates $M(f) \cap \{(0, 0, z), z \in \mathbb{R}\}$ and so e_1, e_2, e_3 generate

$$M(f) \cap \{(x, y, z) : (x, y) \in \{(1, 0), (0, 1)\}\}.$$

Since the projection of $M(f)$ to the first two coordinates is in $\mathbb{Z}^2$, the claim follows.

Thus there are constants r_1, r_2, r_3 so that for every $(q,v) \in \mathcal{M}_0$ there are integers m, n, k (depending on (q,v)) so that

$$\begin{aligned}&(\breve{\kappa}_1(q,v), \breve{\kappa}_2(q,v), |\kappa(q,v)|)^{\mathrm{T}} - (r_1, r_2, r_3)^{\mathrm{T}} \\ &\quad = me_1 + ne_2 + ke_3 + (h_1,h_2,h_3)^{\mathrm{T}}(q,v) - (h_1,h_2,h_3)^{\mathrm{T}}(\mathcal{F}_0(q,v))\end{aligned} \tag{6.6}$$

From the first coordinate of (6.6) we have

$$n = \breve{\kappa}_1 - r_1 - h_1 + h_1 \circ \mathcal{F}_0$$

and likewise from the second coordinate we have

$$k = \breve{\kappa}_2 - r_2 - h_2 + h_2 \circ \mathcal{F}_0$$

Substituting these into to the third coordinate of the equation (6.6), we find

$$|\kappa(q,v)| - \tilde{r} - \beta\breve{\kappa}_1(q,v) - \gamma\breve{\kappa}_2(q,v) = m\alpha + \tilde{h}(q,v) - \tilde{h}(\mathcal{F}_0(q,v)), \tag{6.7}$$

where $\tilde{r} = r_3 - r_1\beta - r_2\gamma$ and $\tilde{h} = h_3 - \beta h_1 - \gamma h_2$. Fix now (q,v) and write $\mathcal{F}_0(q,v) = (q_1, v_1)$. Note that by reverting the free flight, we have $\mathcal{F}_0(q_1, -v_1) = (q, -v)$. Applying (6.7) to $(q_1, -v_1)$, we obtain

$$|\kappa(q,v)| - \tilde{r} + \beta\breve{\kappa}_1(q,v) + \gamma\breve{\kappa}_2(q,v) = m'\alpha + \tilde{h}(q_1, -v_1) - \tilde{h}(\mathcal{F}_0(q_1, -v_1)). \tag{6.8}$$

Finally, adding (6.7) to (6.8), we obtain (6.5) with $r = \tilde{r}$, $h(q,v) = \frac{1}{2}[\tilde{h}(q,v) + \tilde{h}(q_1 - v_1)]$ and $g(q,v) = \frac{m+m'}{2}$. This completes the proof of (6.5). □

7. Proof of theorem 3.2

7.1. Change of coordinates

Since $\mathcal{L}$ is rational, we have $\mathfrak{M} := \Sigma^{1/2}\mathfrak{l}^{(K_1)} \in \mathbb{Z}^2$ and $\mathfrak{N} := \Sigma^{1/2}\mathfrak{l}^{(K_2)} \in \mathbb{Z}^2$. Furthermore, $\mathfrak{M}$ and $\mathfrak{N}$ are primitive lattice vectors (i.e. their coordinates are coprime due to the definition of $(K_1), (K_2)$). Now we introduce an enlarged fundamental domain for the Lorentz gas. Let Z' be the subset of $\mathbb{Z}^2$ containing the origin and those points of $\mathbb{Z}^2$ that are in the interior of the parallelogram with vertices $0, \mathfrak{M}, \mathfrak{N}, \mathfrak{N} + \mathfrak{M}$. Let $T' = \cup_{z\in Z'}[z - 1/2, z + 1/2]^2/\sim$, where $P \sim Q$ if $P - Q$ is in the lattice generated by $\mathfrak{M}, \mathfrak{N}$. That is, T' is a union of unit squares and $\sim$ is a pairing of all parallel sides on the boundary of T'. In particular, T' is a flat torus. Now we put $\mathcal{D}'_0 = T' \backslash \cup_{z\in Z'} \cup_{i=1}^{\mathfrak{k}} (B_i + z)$. See figure 3 for the special case $\mathfrak{M} = (1,3)$ and $\mathfrak{N} = (2,1)$. T' is the polygon with bold boundary (modulo the identification).

We are going to study the Sinai billiard in $\mathcal{D}'_0$ and so we define Φ'^t_0, Ω'_0, μ'_0, $\mathcal{M}'_0$, $\mathcal{F}'_0$, ν'_0 exactly as before using the larger configuration space $\mathcal{D}'_0$. Note that Φ^t_0 is a factor of Φ'^t_0 by the map $\iota : \Omega'_0 \to \Omega_0$, $\iota : (q,v) \mapsto (\bar{q}, v)$, where $q \in \mathcal{D}'_0$, $\bar{q} \in \mathcal{D}_0$ and $\bar{q} = q (\mathrm{mod}\ \mathbb{Z}^2)$. Also, note that Φ^t is an extension of both Φ^t_0 and Φ'^t_0.

Given $(q,v) \in \Omega$, we write $\Pi'_{\mathbb{Z}^2}(q,v) = (m,n)$ if $q \in (m\mathfrak{M}, n\mathfrak{N}) + T'$ and $\Pi'_{\mathcal{D}'_0}(q,v) = q_0$ if $q = q_0 + \Pi'_{\mathbb{Z}^2}(q,v) * (\mathfrak{M}, \mathfrak{N})$, where $*$ means multiplication coordinate-wise. Let us write $\mathcal{Z}'_t(q,v) = \Pi'_{\mathbb{Z}^2}(\Phi^t(q,v))$.

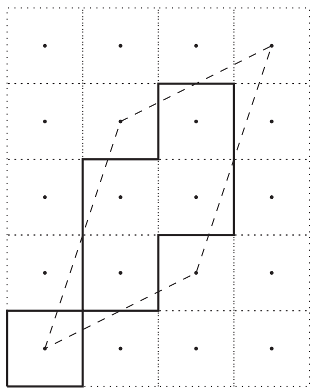

Figure 3. Enlarged fundamental domain.

Note that for any $(k, l) \in \mathbb{Z}^2$ we can find a unique $(k_0, l_0) \in Z'$ with $(k, l) \sim (k_0, l_0)$ and a unique (m, n) so that $(k, l) = (m\mathfrak{M}, n\mathfrak{N}) + (k_0, l_0)$. Let us write $[(k, l)] = (k_0, l_0)$ and $[[(k, l)]] = (m, n)$. Note that

$$[[\tilde{\mathcal{Z}}_t(q, v)]] = \Sigma^{-1/2}(\tilde{\mathcal{Z}}_t(q, v) - [\tilde{\mathcal{Z}}_t(q, v)]) = \mathcal{Z}'_t(q, v). \tag{7.1}$$

Given $(q, v) \in \Omega$, we write $\Pi_{Z'}(q, v) = [\Pi_{\mathbb{Z}^2}(q, v)]$. Let $\mathcal{E}_t(q, v) = \Pi_{T'}(\Phi^t(q, v)) = [\tilde{\mathcal{Z}}_t(q, v)]$ ($\mathcal{E}$ stands for extension). We will also write $[[\ell']] = [[\Pi_{\mathbb{Z}^2}(q, v)]]$ for any (q, v) in the support of $\nu_{\ell'}$ (we assume that the standard pairs are supported in one cell) and likewise $[[\mathcal{G}']]$ for standard families. All definitions and results in section 6 extend to Φ'_0. We will use those notations and results with a prime in the superscript.

7.2. Proof of (H2)

We claim that (H2) follows from

(H2′) For any proper standard family $\mathcal{G}'$ there is some $C_{\mathcal{G}'}$ so that for any $0 < \alpha < \beta$ and for any $0 < \eta, \gamma < \xi$ and for any $z' \in Z'$, if $[[\mathcal{G}']] = (0, \lfloor \eta\sqrt{T} \rfloor)$, then

$$\lim_{T\to\infty} T^{3/2}\nu_{\mathcal{G}'}\left(\mathcal{Z}'_T = \langle(\alpha, \gamma)\sqrt{T}\rangle, \mathcal{E}_T = z', \min\{\tau_0^{\mathcal{Y}'}, \tau_{\xi\sqrt{T}}^{\mathcal{Y}'}, \tau_0^{\mathcal{X}'}, \tau_{\beta\sqrt{T}}^{\mathcal{X}'}\} > T\right)$$

$$= C_{\mathcal{G}'}\psi(\alpha, \beta)\phi(\eta, \gamma, \xi)$$

Furthermore, for any $\varepsilon > 0$, the convergence is uniform for $\varepsilon < \alpha < \alpha + \varepsilon < \beta < 1/\varepsilon, \varepsilon < \eta < \eta + \varepsilon < \xi < 1/\varepsilon, \varepsilon < \gamma < \gamma + \varepsilon < \xi$.

To prove the claim, first recall that by (7.1), $\mathcal{Z}_t = \mathcal{Z}'_t + \Sigma^{-1/2}\mathcal{E}_t$. To compare the initial conditions in (H2) and (H2′), note that given any standard family $\mathcal{G}$ on $\mathcal{M}_0$, there are exactly $\mathbf{Z} := |Z'|$ corresponding standard families $\mathcal{G}'_1, \ldots, \mathcal{G}'_{\mathbf{Z}}$ on $\mathcal{M}'_0$ that project to $\mathcal{G}$ along ι. Indeed, for any point $(q, v) \in \Omega_0$, $\iota^{-1}((q, v)) = \{(q + z', v), z' \in Z'\}$. Recall that the free flight is bounded by 1 and so the initial condition in (H2), i.e. $\mathcal{Z}_0 = \mathfrak{l}$ and $\mathbb{P}$ being induced by a standard family $\mathcal{G}$, corresponds to an initial condition given by $\mathcal{G}'_{z'}$ for some $z' = 1, \ldots, \mathbf{Z}$ in (H2′). Indeed, the type of $\mathfrak{l}$ uniquely defines z'. Thus $\mathcal{G}$ and the type of $\mathfrak{l}$ in (H2′) is replaced by $\mathcal{G}'$ in (H2). Since $\mathcal{E}_t$ is bounded, the claim follows.

Note that for a given standard family $\mathcal{G}$ and two lift ups $\mathcal{G}'_{z'_1}, \mathcal{G}'_{z'_2}, z'_1 \neq z'_2 \in Z'$, the constants $C_{\mathcal{G}'_{z'_1}}, C_{\mathcal{G}'_{z'_2}}$ can be different. As we will see later,

$$C_{\mathcal{G}'_{z'}} = \lim_{T\to\infty} \nu_{\mathcal{G}'_{z'}}(\tau_0^{\mathcal{X}'} > T)/\sqrt{T}. \tag{7.2}$$

Thus e.g. in figure 3, $C_{\mathcal{G}'_{(1,1)}} \geqslant C_{\mathcal{G}'_{(1,2)}}$ for all standard families $\mathcal{G}$. This inequality is strict in case of some standard pairs. To prove this, note that in case of figure 3, $\tau_0^{\mathcal{X}'}(q, v) > T$ is equivalent to $(\tilde{\mathcal{Z}}_t)_2 \leqslant 3(\tilde{\mathcal{Z}}_t)_1$ for all $t \leqslant T$. Now observe that $\tau_0^{\mathcal{X}'}(q_0 + (1, 2), v) > T$ implies $\tau_0^{\mathcal{X}'}(q_0 + (1, 1), v) > T$, but the converse implication does not hold.

We will prove (H2′). The proof is built upon the results of [13, 14]. In particular, [13, proposition 3.8] gives that under the assumptions of (H2′),

$$\lim_{T\to\infty} T\nu_{\mathcal{G}'}\left(\mathcal{X}'_T = \lfloor \alpha\sqrt{T}\rfloor, \mathcal{E}_T = z', \min\{\tau_0^{\mathcal{X}'}, \tau_{\beta\sqrt{T}}^{\mathcal{X}'}\} > T\right) = C_{\mathcal{G}'}\psi(\alpha, \beta) \tag{7.3}$$

with $C_{\mathcal{G}'}$ defined by (7.2). Furthermore, [13, proposition 3.9] gives that under the assumptions of (H2′),

$$\lim_{T\to\infty} \sqrt{T}\nu_{\mathcal{G}'}\left(\mathcal{Y}'_T = \lfloor \gamma\sqrt{T}\rfloor, \mathcal{E}_T = z', \min\{\tau_0^{\mathcal{Y}'}, \tau_{\xi\sqrt{T}}^{\mathcal{Y}'}\} > T\right) = \phi(\eta, \gamma, \xi). \tag{7.4}$$

We interpret (7.3) as the one dimensional version of (H2′). If the events on the left hand sides of (7.3) and (7.4) were independent, then (H2′) would follow immediately. By the invariance principle, $\mathcal{X}'_T$ and $\mathcal{Y}'_T$ are asymptotically independent (since by the change of coordinates, the covariance matrix is identity) but this yet is not enough to conclude (H2′) as the events considered here have small probabilities. Thus we cannot derive (H2′) directly from (7.3) and (7.4); we instead have to revisit their proofs. Since we only need to make minor changes to their proofs, we give details only at places where changes are needed and otherwise refer to [13] (and sometimes give a sketch).

First we need some lemmas. Recall the notations introduced for the billiard ball map in section 6. To simplify notations, we will write

$$\tau_a^{|\check{\mathcal{X}}'|} = \min\{\tau_a^{\check{\mathcal{X}}'}, \tau_{-a}^{\check{\mathcal{X}}'}\}.$$

and likewise for $\check{\mathcal{X}}'$ replaced by $\check{\mathcal{Y}}'$.

Lemma 7.1. *There are constant C_3, C_4 depending only on $\mathcal{D}$ so that for every standard pair ℓ' with $[[\ell']] = (0, 0)$ and for every $m > C_3 \log \text{length}(\ell)$ and for every L,*

$$\nu_{\ell'}\left(\tau_{Lm}^{|\check{\mathcal{Y}}'|} < \tau_m^{|\check{\mathcal{X}}'|}\right) < 0.51^L + \frac{C_4 L}{m^{500}}. \tag{7.5}$$

Proof. Let us fix a positive constant η so that the probability that a standard planar Brownian motion W_t leaves the box $[-1, 1]^2$ through the North or South side (and not through the East or West side) is at most 0.505 whenever the y-coordinate of W_0, denoted by $(W_0)_2$, satisfies $|(W_0)_2| < \eta$. We are going to use the invariance principle and the above estimate inductively L times to derive the lemma. Each time the North or South side is reached, we apply a Markov decomposition and discard the curves that are too short (hence the second term on the right-hand side of (7.5)). The key to this argument is the fact that the limiting Brownian motion has a diagonal covariance matrix, which is guaranteed by the change of coordinates from section 7.1. Now we give the details of the proof.

Choosing C_3 large and using lemma 6.2, we can guarantee that the standard family $\mathcal{G} := \mathcal{F}_0^{\eta m}(\ell')$ has a bounded Z function (e.g. $Z_{\mathcal{G}} < 2c_2$, where c_2 is defined in lemma 6.2. Such standard families are sometimes called proper). Recall that we assumed that the free flight is bounded by 1. Thus for any standard pair $\ell'' = (W'', \rho'')$ in $\mathcal{G}$, $\|[[\ell'']]\| \leqslant \eta m$. If length$(\ell'') < m^{-500}$, then we estimate $\nu_{\ell''}(\mathcal{C}) \leqslant 1$, where $\mathcal{C} = \{\tau_{Lm}^{|\check{\mathcal{Y}}'|} < \tau_m^{|\check{\mathcal{X}}'|}\}$. By the growth lemma, the measure carried by such standard pairs in $\mathcal{G}$ is bounded by $C_4 m^{-500}$. Let us now assume that length$(\ell'') > m^{-500}$. Then by the choice of η and by invariance principle (assuming as we can that m is large enough),

$$\nu_{\ell''}(\tau_m^{|\check{\mathcal{Y}}'|} < \tau_m^{|\check{\mathcal{X}}'|}) \leqslant 0.51.$$

Now let $\ell''' = (W''', \rho''')$ be a standard pair in the standard family $\mathcal{G}_1 := \mathcal{F}_0^{\tau_m^{|\check{\mathcal{Y}}'|}}(\ell'')$.

Note that there exists a constant $T_{\ell'''}$ so that for any $x \in W''$ with $\mathcal{F}_0^{\tau_m^{|\check{\mathcal{Y}}'|}} \in W'''$, $\tau_m^{|\check{\mathcal{Y}}'|} = T_{\ell'''}$. Indeed, this follows from the definition of homogeneous unstable curves. Now we distinguish two cases. Let us say that ℓ''' is of type 1 if $T_{\ell'''} > m$ or length$(\ell''') < m^{-750}$. For type 1 standard pairs ℓ''', we use the trivial bound $\nu_{\ell'''}(\mathcal{C}) \leqslant 1$. By [13, lemma 5.1], the measure carried by standard pairs ℓ''' with $T_{\ell'''} > m^3$ is bounded by Cm^{-999}. Thus by the growth lemma, the measure carried by standard pairs ℓ''' with $T_{\ell'''} \leqslant m^3$ and length$(\ell''') < m^{-750}$ is bounded by Cm^{-747}. Thus the total contribution of type 1 standard pairs is bounded by $C_4 m^{-500}$. Let us say that ℓ''' is of type 2 if it is not of type 1. By the invariance principle and by the definition of η, for every type 2 standard pair ℓ''', we have

$$\nu_{\ell'''}(\tau_{2m}^{|\check{\mathcal{Y}}'|} < \tau_m^{|\check{\mathcal{X}}'|}) \leqslant 0.51.$$

Thus we have derived

$$\nu_{\ell'}(\tau_{2m}^{|\check{\mathcal{Y}}'|} < \tau_m^{|\check{\mathcal{X}}'|}) \leqslant 0.51^2 + \frac{2C_4}{m^{500}}.$$

Following the above procedure inductively, we obtain the lemma. □

Lemma 7.2. *For every $\delta > 0$ and for every $\xi > 0$ there exists M_0 and $\bar{L}$ so that for every standard pair ℓ' with $[[\ell']] = (0, 0)$ and length$(\ell') > \delta$, and for every $M > M_0$,*

$$\nu_{\ell'}\left(\tau_{\bar{L}M}^{|\check{\mathcal{Y}}'|} < \tau_M^{\check{\mathcal{X}}'} \,|\tau_M^{\check{\mathcal{X}}'} < \tau_0^{\check{\mathcal{X}}'}\right) < \xi$$

Proof. [14, lemma 11.1(a)] says that

$$\bar{c} = \bar{c}(\ell') = \lim_{M\to\infty} M\nu_{\ell'}(\tau_M^{\check{\mathcal{X}}'} < \tau_0^{\check{\mathcal{X}}'}) \tag{7.6}$$

is finite. We will use the proof of that lemma to prove our lemma. Let us recall the main steps of the proof.

Let $\mathbf{t}_k = \tau_{2^k}^{\check{\mathcal{X}}'}$ and

$$s_k = \min\{n > \mathbf{t}_k : \check{\mathcal{X}}'_n < 0 \,\text{or}\, \check{\mathcal{X}}'_n = 2^{k+1}\}.$$

Let now ℓ'' be a standard pair with

$$[[\ell'']]_1 = 2^k \quad \text{and} \quad \text{length}(\ell'') > 2^{-100k} \tag{7.7}$$

(we will consider ℓ'' in the image of ℓ' under the map $(\mathcal{F}')^{\mathbf{t}_k}$). The proof of [14, lemma 11.1(a)] is based on the following identity (see [14, lemma 11.2]):

$$\nu_{\ell''}\left(\mathbf{t}_{k+1} < \tau_0^{\check{\mathcal{X}}'} \text{ and } r'_{\mathbf{t}_{k+1}} \geqslant 2^{-100(k+1)}\right) = \frac{1}{2} + O(2^{-k\zeta}) \tag{7.8}$$

with a universal positive constant ζ. Fixing an arbitrary $\varepsilon > 0$, one can choose k_0 large enough so that an induction on $k = k_0, \ldots, \log_2 M$ using (7.8) gives that

$$|M\nu_{\ell'}(s_k = \mathbf{t}_{k+1}, r'_{s_k} \geqslant 2^{-100(k+1)} \quad \text{for } k = k_0, \ldots, \log_2 M) - \bar{c}| < \varepsilon, \tag{7.9}$$

which implies (7.6) (by the growth lemma, the measure of the points where $r_{s_k} < 2^{-100(k+1)}$ for some $k < \log_2 M$ can be neglected). We refer the reader to [14] for more details.

Now we turn to the proof of our lemma. Let us put $m_k = 2^k$, $\tilde{k} = (\log_2 M) - k$ and

$$L_k = \begin{cases} 2^k & \text{if } k_0 \leqslant k < \frac{1}{2}\log_2 M \\ K1.5^{\tilde{k}} & \text{if } \frac{1}{2}\log_2 M \leqslant k < \log_2 M \end{cases}$$

with some $K = K(\xi)$ to be specified later. Assuming that k_0 is bigger than a universal constant (as we can), we have $m_k > 100C_1 \log(1/m_k)$. Thus lemma 7.1 implies that for all standard pairs satisfying (7.7):

$$\nu_{\ell''}\left(\min\{\tau^{\check{\mathcal{Y}}'}_{[[\ell'']]_2 - L_k m_k}, \tau^{\check{\mathcal{Y}}'}_{[[\ell'']]_2 + L_k m_k}\} < \min\{\tau_0^{\check{\mathcal{X}}'}, \tau_{2m_k}^{\check{\mathcal{X}}'}\}\right) < 0.51^{L_k} + \frac{C_4 L_k}{m_k^{100}},$$

which combined with (7.8) gives

$$\nu_{\ell''}\left(\mathbf{t}_{k+1} < \min\{\tau_0^{\check{\mathcal{X}}'}, \tau^{\check{\mathcal{Y}}'}_{[[\ell'']]_2 - L_k 2^k}, \tau^{\check{\mathcal{Y}}'}_{[[\ell'']]_2 + L_k 2^k}\} \text{ and } r'_{\mathbf{t}_{k+1}} \geqslant 2^{-100(k+1)}\right) = \frac{1}{2} + E_{k,\ell''}, \tag{7.10}$$

where

$$-C' 2^{-k\zeta} - 0.51^{L_k} - \frac{C_4 L_k}{m_k^{1000}} < E_{k,\ell''} \leqslant C' 2^{-k\zeta}, \tag{7.11}$$

with a universal constant C'. Now we revisit the inductive proof of (7.9). Let us write

$$\mathcal{P} = \nu_{\ell'}\left(s_k = \mathbf{t}_{k+1}, r'_{s_k} \geqslant 2^{-100(k+1)}, \tau^{|\check{\mathcal{Y}}'|}_{M + \sum_{j=k_0}^{k} L_j 2^j} > s_k \text{ for } k = k_0, \ldots, \log_2 M - 1\right). \tag{7.12}$$

Using (7.10) inductively, we find

$$\mathcal{P} = \nu_{\ell'}(\tau^{\check{\mathcal{X}}'}_{2^{k_0}} < \min\{\tau_0^{\check{\mathcal{X}}'}, \tau_M^{|\check{\mathcal{Y}}'|}\}) \prod_{k=k_0}^{\log_2 M - 1} \frac{1}{2}(1 + E_k),$$

where E_k satisfies the same inequalities (7.11) as $E_{k,\ell''}$. As before, choosing k_0 and M large, we can guarantee

$$\mathcal{P} > \frac{\bar{c} - \xi'/10}{M} \prod_{k=k_0}^{\log_2 M - 1} (1 + E_k), \tag{7.13}$$

where $\xi' = \xi\bar{c}/2$. Let us write

$$\prod_{k=k_0}^{\log_2 M-1} (1+|E_k|) = \exp\left(\sum_{k=k_0}^{\log_2 M-1} \log(1+|E_k|)\right) \leqslant \exp\left(\sum_{k=k_0}^{\log_2 M-1} |E_k|\right). \tag{7.14}$$

Later we will show that

$$\sum_{k=k_0}^{\log_2 M} \left(C' 2^{-k\zeta} + 0.51^{L_k} + \frac{C_4 L_k}{m_k^{500}}\right) < \frac{\xi'}{10\bar{c}} = \frac{\xi}{20}. \tag{7.15}$$

Before proving (7.15), let us show how it implies the lemma. Combining (7.13)–(7.15), we find

$$\mathcal{P} > \frac{\bar{c} - \xi'}{M}. \tag{7.16}$$

Next observe that the event in (7.12) implies that

$$\tau_{\tilde{L}M}^{|\check{\mathcal{Y}}'|} > \tau_M^{\check{\mathcal{X}}'},$$

where $\tilde{L} = 1 + \frac{1}{M}\sum_{k=k_0}^{\log_2 M} L_k 2^k$. The next computation shows that $\tilde{L}$ is bounded by a constant $\bar{L} = \bar{L}(\xi)$ uniformly in M:

$$\begin{aligned} 1 + \frac{1}{M}\sum_{k=k_0}^{\log_2 M} L_k 2^k &= 1 + \frac{1}{M}\sum_{k=k_0}^{\frac{1}{2}\log_2 M} 4^k + \frac{K}{M}\sum_{k=\frac{1}{2}\log_2 M}^{\log_2 M} 1.5^{\tilde{k}} 2^k \\ &\leqslant 5 + \frac{K}{M}\sum_{\tilde{k}=0}^{\frac{1}{2}\log_2 M} 1.5^{\tilde{k}} 2^{\log_2 M - \tilde{k}} \\ &\leqslant 5 + K\sum_{\tilde{k}=0}^{\infty}\left(\frac{3}{4}\right)^{\tilde{k}} = 5 + 4K =: \bar{L}. \end{aligned}$$

Thus we find

$$\nu_{\ell'}\left(\tau_{\tilde{L}M}^{|\check{\mathcal{Y}}'|} < \tau_M^{\check{\mathcal{X}}'} \,|\tau_M^{\check{\mathcal{X}}'} < \tau_0^{\check{\mathcal{X}}'}\right) < 1 - \frac{\mathcal{P}}{\nu_{\ell'}(\tau_M^{\check{\mathcal{X}}'} < \tau_0^{\check{\mathcal{X}}'})} \leqslant 1 - \frac{\bar{c} - \xi'}{\bar{c} + \xi'} \leqslant \xi,$$

where the penultimate inequality uses (7.16) and the last one uses the definition of ξ'. This proves the lemma. It remains to verify (7.15).

To prove (7.15), first choose $K = K(\xi)$ large, so that

$$\sum_{k=\frac{1}{2}\log_2 M}^{\log_2 M} 0.51^{L_k} < \sum_{\tilde{k}=0}^{\infty} 0.51^{K 1.5^{\tilde{k}}} < \frac{\xi}{100}.$$

Then we compute

$$\sum_{k=k_0}^{\log_2 M} C' 2^{-k\zeta} < \frac{\xi}{100},$$

$$\sum_{k=k_0}^{\frac{1}{2}\log_2 M} 0.51^{L_k} < \sum_{k=k_0}^{\infty} 0.51^{2^k} < \frac{\xi}{100}$$

and

$$\sum_{k=k_0}^{\frac{1}{2}\log_2 M} \frac{L_k}{m_k^{500}} < \sum_{k=k_0}^{\infty} 2^{-499k} < \frac{\xi}{100}.$$

(Note that we can ensure the last inequality in all of the three displayed formulas above by increasing $k_0 = k_0(\xi)$ if necessary.) Finally, we have

$$\sum_{k=\frac{1}{2}\log_2 M}^{\log_2 M} \frac{L_k}{m_k^{500}} < \log_2 M \frac{1.5^{\log_2 M}}{2^{250\log_2 M}} = o(M^{-249}) < \frac{\xi}{100},$$

which completes the proof of (7.15). □

Lemma 7.3. *For every $\eta_1, \eta_2 > 0$, there exists ε_0 so that for every $\varepsilon < \varepsilon_0$ and for every $\delta > 0$, there is some N_0 so that for all $N > N_0$ and for all standard pairs ℓ', with $[[\ell']] = (0,0)$, $\mathrm{length}(\ell) > \delta$, we have*

$$\nu_{\ell'}\left(\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} < \min\{\tau^{\check{\mathcal{Y}}'}_{\eta_1\sqrt{N}}, \tau^{\check{\mathcal{Y}}'}_{-\eta_1\sqrt{N}}, \varepsilon N\} \mid \tau^{\check{\mathcal{X}}'}_0 > N\right) > 1 - \eta_2.$$

Proof. [13, lemma 5.2] implies that

$$\nu_{\ell'}\left(\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} < \varepsilon N, \middle| \tau^{\check{\mathcal{X}}'}_0 > N\right) > 1 - \frac{\eta_2}{2}.$$

and [14, theorem 8] implies that

$$\lim_{T\to\infty} \nu_{\ell'}(\tau^{\check{\mathcal{X}}'}_0 > N)/\sqrt{N} =: \check{C}_{\ell'} \tag{7.17}$$

is finite for all standard pairs and non-zero for some. Thus it suffices to prove

$$\nu_{\ell'}(\mathcal{ABC}) < \frac{\eta_2 \check{C}_{\ell'}}{4\sqrt{N}}, \tag{7.18}$$

where

$$\mathcal{A} = \{\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} > \min\{\tau^{\check{\mathcal{Y}}'}_{\eta_1\sqrt{N}}, \tau^{\check{\mathcal{Y}}'}_{-\eta_1\sqrt{N}}\}\}, \qquad \mathcal{B} = \{\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} < \varepsilon N\}, \qquad \mathcal{C} = \{\tau^{\check{\mathcal{X}}'}_0 > N\}.$$

To prove (7.18), let us write

$$\mathcal{D} = \{\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} < \tau^{\check{\mathcal{X}}'}_0\}$$

and

$$\nu_{\ell'}(\mathcal{ABC}) = \nu_{\ell'}(\mathcal{ABCD}) \leqslant \nu_{\ell'}(\mathcal{AD})\nu_{\ell'}(\mathcal{C}|\mathcal{ABD}) =: \mathrm{I} * \mathrm{II}.$$

To estimate II, we use the Markov decomposition at time $\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}}$. By the invariance principle, II is asymptotic (as $N \to \infty$) to the probability that the maximum of the standard Brownian motion before time 1 is less than ϵ which is bounded from above by $\widehat{c}\varepsilon$. Let $\bar{c} = \bar{c}(\ell')$ as in (7.6) and let $\xi = \frac{\eta_2 \check{C}_{\ell'}}{4\bar{c}\widehat{c}}$. Lemma 7.2 gives $\bar{L} = \bar{L}(\xi)$. Then we choose $\varepsilon_0 < \eta_1/\bar{L}$. Now lemma 7.2 implies that

$$I = \nu_{\ell'}(\mathcal{A}|\mathcal{D})\nu_{\ell'}(\mathcal{D}) \leqslant \xi \frac{\bar{c}}{\varepsilon\sqrt{N}}$$

and so (7.18) follows. □

Next, we have the following extension of [13, theorem 3.5] to two dimensions.

Proposition 7.4. *The process $\check{\mathcal{Z}}'_{tN}/(\sqrt{\bar{\kappa}N})$, $0 < t < 1$ induced by the measure $\nu_{\mathcal{G}'}(.|\tau_0^{\check{\mathcal{X}}'} > N)$ converges weakly as $N \to \infty$ to the planar stochastic process with independent coordinates, whose first coordinate is a Brownian meander and the second coordinate is a standard Brownian motion.*

The proof of proposition 7.4 is the same as that of [13, theorem 3.5] except that [13, lemma 5.2] is replaced by our lemma 7.3. The sketch of the proof is as follows. Under the assumption $\tau_0^{\check{\mathcal{X}}'} > N$, with high probability, we have $\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}} < \min\{\tau^{\check{\mathcal{Y}}'}_{\eta_1\sqrt{N}}, \tau^{\check{\mathcal{Y}}'}_{-\eta_1\sqrt{N}}, \varepsilon N\}$. Then we use the invariance principle starting at time $\tau^{\check{\mathcal{X}}'}_{\varepsilon\sqrt{N}}$. The invariance principle is applicable since $\nu_{\ell''}(\tau_0^{\check{\mathcal{X}}'} > N)$ is bounded from below for ℓ'' with $\ell'' > \delta_0$ and $[[\ell'']]_1 = \varepsilon\sqrt{N}$ for fixed ε. Thus we obtain a planar Brownian motion with identity covariance matrix, whose first coordinate starts from ε and does not reach 0 before time 1 and whose second coordinate starts from a position with absolute value less than η_1. Choosing η_1 small (and consequently ε small), the distribution of this process is close to the one described in the lemma.

(H2′) is a local version of proposition 7.4 in continuous time. The proof of (H2′) is again analogous to the one dimensional case given in [13, proposition 3.8]. Although the proof is quite lengthy, let us a give a short sketch. Let $N = T/\bar{\kappa}$ and $N_1 = (1 - \delta_t)N$, with a small δ_t, and partition the rectangle $R_T := [0, \beta\sqrt{T}] \times [0, \xi\sqrt{T}]$ into boxes B_k with side length $\delta_s\sqrt{T}$ with some fixed δ_s small. Proposition 7.4 gives the asymptotic probability (for T large and the other parameters fixed) of arriving in a box B_k after discrete time N_1. Then for any given box B_k and any given standard pair ℓ' in this box as an initial condition (with length(ℓ') $> \delta_0$ for some fixed δ_0), we need to find the probability that in the remaining continuous time before T, but after the first N_1 collisions, the particle arrives in the cell $\langle \alpha\sqrt{T}, \gamma\sqrt{T}\rangle$. To give an upper bound, we use the MLLT by simply ignoring the requirement that, in the remaining $\approx \delta_t T$ time, the particle has to stay inside R_T. Switching from discrete to continuous time is a nontrivial step. For a 'typical' number of collisions, theorem 6.4(a) is used. On the other hand, the contribution of non-typical number of collisions is negligible by theorem 6.4(b). This gives the upper bound in (H2′). To prove the lower bound, one needs to verify that the error made by ignoring the requirement that the particle has to stay inside R_T for the last $\approx \delta_t T$ time is negligible. If a particle leaves R_T and returns to $\langle \alpha\sqrt{T}, \gamma\sqrt{T}\rangle$, then in particular it has to travel a distance $\min\{\alpha, 1 - \alpha, \gamma, 1 - \gamma\}\sqrt{T}$ during the time $\delta_t T$. This has a small probability which gives the lower bound in (H2′) (in [13] d_t is chosen small given $\alpha \in (0, 1)$ and now we need to choose it small given $\alpha, \gamma \in (0, 1)$). No other substantial change is required.

7.3. Proof of (H3)

As in the case of (H2), we use the change of coordinates to reformulate (H3) as

(H3′) For any $x \in (0,1)$ and $y \in (-1,1)$, and for any proper standard family $\mathcal{G}'$ with $[[\mathcal{G}']] = (0,0)$

$$\lim_{\delta\to 0}\lim_{L\to\infty}\int_{[0,\delta L^2]\cup[L^2/\delta,\infty)} L\nu_{\mathcal{G}'}(\mathcal{Z}'_t = \langle (xL, yL)\rangle,\ \min\{\tau_0^{\mathcal{X}'}, \tau_L^{\mathcal{X}'}\} > t)\mathrm{d}t = 0.$$

The fact that (H3′) implies (H3) follows the same way as we proved that (H2′) implies (H2). In fact, this case is easier that the case of (H2). We only need an upper bound here and so we can ignore the requirement that $\mathcal{E}_t = z'$ at the cost of losing a constant multiplier.

As in the upper bound of (H2′), we can derive that for any given $(x, y) \in (0,1)^2$ and any $\varepsilon > 0$, there exists δ so that for large enough L and for any $t < \delta L^2$,

$$\nu_{\mathcal{G}'}(\mathcal{Z}'_t = \langle (xL, yL)\rangle, |\tau_{xL/2}^{\mathcal{X}'} < \tau_0^{\mathcal{X}'}) < \frac{\varepsilon}{L^2}.$$

Using this estimate, the proof of (H3′) follows as in [13, lemma 7.2].

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