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STATISTICAL PROPERTIES OF TYPE D DISPERSING BILLIARDS

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Abstract. We consider dispersing billiard tables whose boundary is piecewise smooth and the free flight function is unbounded. We also assume there are no cusps. Such billiard tables are called type D in the monograph of Chernov and Markarian [9]. For a class of non-degenerate type D dispersing billiards, we prove exponential decay of correlation and several other statistical properties.

1. **Introduction.** Consider a collection of disjoint open sets on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (called scatterers in the sequel) with piecewise C^3 boundary which are locally convex with bounded from below curvature at regular points. We assume that there are no cusps. To define the Sinai billiard flow [19], let a point particle fly freely with constant speed on the complement of the scatterers (called the billiard table) and be subject to elastic collision upon reaching their boundaries. Depending on the geometry of the scatterers, the free flight time may or may not be bounded. A partial classification of dispersing billiard tables is given by [9] as follows: assume first that the boundary of the billiard table is $\mathcal{C}^3$. If the free flight is bounded, then the table is of type A, otherwise of type B. Now assume that the boundary of the billiard table is only piecewise $\mathcal{C}^3$. Points where the boundary is $\mathcal{C}^3$ are called regular. The finitely many non-regular points are called corner points. If the free flight is bounded, then the table is of type C, otherwise of type D. (In case of cusps, type E and F.) Statistical properties were first proven for type A billiards (see the Central Limit Theorem in [3–5], exponential decay of correlations [21]). Next, type B tables were also extensively studied (see [2,7,20]). Although there are early works for types C and D [5], the more recent theory (such as the construction of Young towers [21]) was not studied in these classes until recently. For type C billiard tables, [14] proves the m-step expansion estimate, which together with other estimates (that can be proved as in type A) yield the statistical properties

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mentioned above. There are fewer results available in types D-F (in fact, these classes are labelled as "hard" in [9]).

It is standard that long free flights are only possible after a collision in a small vicinity of finitely many points, which we call boundary points of corridors. We now distinguish two classes of type D billiard tables: if all boundary points of corridors are regular, we say that the billiard table is of type D1, otherwise of type D2. The main result of the present work can be informally stated as follows (precise definitions, in particular those of (A1) and (A2) are given in Section 2).

Theorem 1.1. *Consider a billiard table of type D1 or type D2. In the latter case we also require that assumptions (A1) and (A2) hold. Then the correlations of bounded dynamically Hölder observables decay exponentially fast and the Central Limit Theorem holds for such observables.*

Before proceeding further, let us explain the main novelty of this paper. Theorem 1.1 has been long known *under some complexity hypotheses.* The strongest version of the complexity hypothesis states that the number of singular collisions that any orbit can ever experience, is uniformly bounded. A careful study of type D billiards appears in the early work [4]. There, the authors construct a Markov partition which can be used to prove the Central Limit Theorem under this complexity hypothesis (called hypothesis B there). Shortly after the the exponential decay of correlation was settled in [21] for type A billiards, Chernov [6] proved the same for type D billiard tables under a slightly more general form of the complexity hypothesis, which allows the complexity to grow exponentially as long as it grows slower than the expansion, see hypothesis (9.1) in [6]. We note that [10] also proves Theorem 1.1 under the complexity hypothesis, see hypothesis (6.3) in [10].

Up to our best knowledge, no version of the complexity hypotheses is known to hold for *any* billiard table of type C, let alone for type D. The main novelty of our Theorem 1.1 is that it does not assume any version of the complexity hypothesis. The story closely parallels that of type C, where the complexity hypothesis is unknown to hold but the conclusion of Theorem 1.1 holds by [14].

In the preset work, partially based on ideas from [14], we prove Theorem 1.1 using directly the m-step expansion (10) without assuming the complexity hypothesis. As a matter of fact, we also have two hypothesis (A1) and (A2) but those are very different from the complexity hyoptheses. On the one hand, it is easy to prove that (A1) and (A2) hold on an open and dense set of billiard tables (for completeness we provide a proof in Section 6) and on the other hand, they are easy to verify for any given billiard table as they only depend on the boundary points of the corridors. Furthermore, we believe that our approach could in theory be applied in cases when (A1) or (A2) fails at the expense of increasing the length of the proof but without essential new ideas.

The rest of this paper is organized as follows. In Section 2 we collect the necessary background information needed in this work. None of the results of Section 2 are new. In Section 3 we state our main technical theorems. Theorem 3.1 implies Theorem 1.1 in type D1. Theorem 3.2 implies Theorem 1.1 in type D2. Section 4 contains the proof of Theorem 3.1. Section 5 contains the proof of Theorem 3.2. This proof is substantially more complex then that of Theorem 3.1 as we need a careful study of the geometry of long free flights in case of type D2 configurations. Finally, in Section 6, we prove that the conditions (A1), (A2) hold on an open and dense set of billiard tables.

FIGURE 1. A scatterer with a convex corner point (left) and a concave corner point (right)

We mention that very recently there has been an increasing interest in the detailed description of possible orbits in infinite corridors in cases of hyperbolic billiards [1, 16] and in some similar hyperbolic systems with singularities [15].

2. Preliminaries.

Here we review the preliminaries needed for our work. All results in this section are known, see [6, 8, 9]. More specific references will be given for the most important statements.

2.1. Billiards of type D.

Let $\mathbb{T}^2$ be the 2-torus and $\mathcal{D} \subset \mathbb{T}^2$ be a dispersing billiard table. That is, the complement of $\mathcal{D}$ consists of finitely many (say d) connected components $\mathcal{B}_i$ (called scatterers). For convenience we also label the scatterers. Each scatterer $i = 1, ..., d$ is bounded by a finite union of curves $\Gamma_{i,j}$, $j = 1, ..., J_i$. It is assumed that $\Gamma_{i,j}$ is a $\mathcal{C}^3$ curve, that is there is a $\mathcal{C}^3$ function $f_{i,j} : [0, 1] \to \mathbb{T}^2$ which is a bijection between $[0, 1]$ and $\Gamma_{i,j}$. Furthermore, $f_{i,j}(1) = f_{i,j+1}(0)$ where $j + 1$ is interpreted modulo J_i (that is, $f_{i,J_i}(1) = f_{i,1}(0)$). The endpoints of $\Gamma_{i,j}$ are called corner points, all other points of $\Gamma_{i,j}$ are regular points. We require that one of the first three one-sided derivatives at $f_{i,J_i}(1)$ differ from the corresponding derivative at $f_{i,1}(0)$, that is no regular point is labelled as corner point. We also require that the curvature of $\Gamma_{i,j}$ is positive with uniform upper and lower bounds at all regular points. The orientation of $\Gamma_{i,j}$ is assumed to be so that when following $\Gamma_{i,1}$, $\Gamma_{i,2}$, ..., Γ_{i,J_i}, we follow clockwise orientation and $\mathcal{D}$ is to the left hand side. The region enclosed by $\Gamma_{i,1}, ..., \Gamma_{i,J_i}$ (a subset of $\mathbb{T}^2 \setminus \mathcal{D}$) is one scatterer. If the boundary of the scatterer i is $\mathcal{C}^3$ smooth, i.e. does not contain corner points, then the scatterer is necessarily strictly convex and $J_i = 1$. We also assume no cusps, that is the tangent lines of $\Gamma_{i,j}$ and $\Gamma_{i,j+1}$ have an angle of at least α_0 at their common endpoint $f_{i,j}(1)$. The interiors of $\Gamma_{i,j}$ are disjoint for all i, j. Furthermore, at each corner point exactly two curves meet. Any billiard table satisfying these assumptions is called admissible.

Given a corner point, let γ be the angle between the two half tangent lines at it, measured at the interior of $\mathcal{D}$. The admissible property implies that $\gamma \neq 0$ for all corner points (the case $\gamma = 0$ is called a cusp). We say that the corner point is *convex* if $0 < \gamma \leq \pi$. Note that $\gamma = \pi$ is possible, in this case we assume that either the second or the third derivatives on the two sides of the corner points differ. We say that the corner point is *concave* if $\pi < \gamma < 2\pi$ (noting that $\gamma = 2\pi$ is impossible due to local convexity of the scatterers at regular points). See Figure 1.

Given two admissible billiard tables $\mathcal{D}_1, \mathcal{D}_2$ with the same combinatorial data (that is the same number of scatterers d and the same number of smooth pieces J_i, $i = 1, .., d$), we define their distance as

$$d(\mathcal{D}_1, \mathcal{D}_2) = \inf_{\{f^1\},\{f^2\}} \max_{i,j} d_{\mathcal{C}^3}(f^1_{i,j}, f^2_{i,j}),$$

where the infimum is taken over admissible parametrizations, i.e. collections of $\mathcal{C}^3$ functions $f^k_{i,j}$ so that $f^k_{i,j}$ is a bijection between $[0,1]$ and $\Gamma^k_{i,j}$ where $k = 1, 2$ indicates the two billiard tables. This makes the set of labelled admissible billiard tables with given combinatorial data a metric space $\boldsymbol{D}_{d,J_1,\dots,J_d}$. Let $\boldsymbol{D}$ denote the space of all (labelled) admissible billiard tables, that is $\boldsymbol{D} = \cup_{d,J_1,\dots,J_d} \boldsymbol{D}_{d,J_1,\dots,J_d}$. The space $\boldsymbol{D}$ is also a metric space by defining the distance between two tables of different combinatorial data to be infinite (mind the labelling).

The billiard dynamics on a fixed admissible billiard table $\mathcal{D}$ prescribes the motion of a point particle that flies with constant speed 1 in a given direction v until it reaches the boundary $\partial\mathcal{D}$, where it undergoes an elastic collision (meaning the angle of reflection equals the angle of incidence). The phase space of the billiard flow is $\Omega = \mathcal{D} \times \mathcal{S}^1 / \sim$, where $\sim$ means identifying pre-collisional and post-collisional data (that is, if $q \in \partial\mathcal{D}$ is a regular point, then v and $-v$ are identified unless v is tangent to $\partial\mathcal{D}$ at q. We will discuss the case of corner points in more detail later). We use the notation $(q, v) \in \Omega$ with v being the velocity vector. We say that $\mathcal{D}$ is the configuration space and $q = \Pi_{\mathcal{D}}(q, v)$ is the configurational component of (q, v). The billiard flow is denoted by $\Phi^t : \Omega \mapsto \Omega$ for every $t \in \mathbb{R}$.

Note that the dynamics may not be well defined upon reaching a corner point. Such trajectories have Lebesgue measure zero, so the definition of Φ on this set is irrelevant for physical properties. It is convenient, though, to define the flow to be possibly multi-valued upon reaching a corner point, corresponding to possible limit points of nearby regular orbits. One way of defining the flow is as follows. First, we say that a collision is improper if the trajectory can be approximated by trajectories missing the collision. In the case of smooth scatterers, an improper collision is the same as a grazing collision. In the case of a concave corner point, we may have an improper collision which is not grazing (such as a horizontal flight touching the corner point on the right of Figure 1). A proper collision is a collision that is not improper. For example, a vertical flight hitting the corner on the right of Figure 1 is proper, and nearby regular trajectories have two possible continuations. All trajectories hitting a convex corner point are proper. Furthermore, we may have a sequence of short flights near the convex corner point (also known as corner series), but the number of short collisions (the length of the corner series) is bounded due to the assumption that there are no cusps (see [6, Section 9]). Now given a point (q, v), put $\tau(q, v) = \inf\{t > 0 : \Pi_{\mathcal{D}}\Phi^t(q, v) \in \partial\mathcal{D}\}$. Now assuming that $\lim_{t \nearrow \tau(q,v)} \Pi_{\mathcal{D}}\Phi^t(q, v)$ is a corner point $\tilde{q} \in \partial\mathcal{D}$, we define $\Phi^{\tau(q,v)}(q, v)$ as

$$\lim_{\varepsilon \searrow 0} \lim_{q' \to q, v' \to v} \Phi^{\tau(q,v)+\varepsilon}(q', v')$$

where q', v' are points that can only experience collisions at regular points up to time $\tau(q, v) + \varepsilon$ and the second limit is to be interpreted as the set of all possible limit points. With this definition, $\Phi^{\tau(q,v)}(q, v)$ can take one or two values. This is trivial in case of concave corner points; for convex points see [9, Section 2.8]. The flow Φ^t preserves the Lebesgue measure ν on Ω (we assume by normalization that ν is a probability measure).

We will also study the billiard map. Let $\mathcal{M}$ be a cross-section of post-collisional points. Then $\mathcal{M}$ can be identified with a union of cylinders and rectangles. For any curve $\Gamma_{i,j}$, we define $\mathcal{M}_{i,j} = [a_{i,j}, b_{i,j}] \times [-\pi/2, \pi/2]$ where $b_{i,j} - a_{i,j} = |\Gamma_{i,j}|$ and the intervals $[a_{i,j}, b_{i,j}]$ are disjoint. If the scatterer i is smooth (in this case necessarily $J_i = 1$), then we identify the endpoints of the interval $[a_{i,1}, b_{i,1}]$, so $\mathcal{M}_{i,1}$ becomes a cylinder. Finally, we put $\mathcal{M} = \cup_{i,j} \mathcal{M}_{i,j}$. Coordinates in $\mathcal{M}$ are denoted by (r, φ): r is arclength parameter along the boundary of the scatterer in clockwise direction; φ is the angle of the postcollisional velocity and the normal to $\mathcal{D}$ at q pointing into $\mathcal{D}$. The angle φ is also measured in the clockwise direction with $\varphi \in [-\pi/2, \pi/2]$ (see [9, Figure 2.14]). The billiard map, denoted by $F : \mathcal{M} \to \mathcal{M}$ takes a postcollisional phase point to the next postcollisional phase point. Note that F can be multivalued at points when the next collision is at a corner point. The special case of unbounded free flight near a corner point will be discussed in the next section (in particular, the definition will be given in (3)). F preserves the physical invariant measure μ defined by $d\mu = C_\mu \cos\varphi dr d\varphi$, where C_μ is a normalizing constant (μ is obtained as the projection of ν to the Poincaré section). The flow is now a suspension over the base map F with roof function τ, which is the free flight time.

For ease of notation, we will identify $\mathcal{M}$ with a subset of Ω in the natural way. For example, we will write $\Pi_{\mathcal{D}} x$ for $x \in \mathcal{M}$.

2.2. **Structure of corridors.** Next we study corridors. We say that an admissible billiard table has *infinite horizon* if the free flight is unbounded. In this case, there are finitely many "corridors". A corridor H by definition is a direction $v = v_H \in [0, \pi)$ and a connected subset Q_H of $\mathcal{D}$ with non-empty interior

$$Q_H = \{q \in \mathcal{D} : \forall t \in \mathbb{R} : q + tv \in \mathcal{D}\}. \tag{1}$$

There are only finitely many corridors (see [9, Exercise 4.51]). Let us say that an admissible billiard table is of type D1 if all corridors are bounded by grazing orbits at regular points (such orbits are necessarily periodic). In other words, a billiard table is of type D1 if no corner point is in the corridors. If the billiard table is of type D but not of type D1, then we call it type D2. We say that a corridor H is simple if $B_H := \partial Q_H \cap \partial \mathcal{D}$ consist of exactly 2 points, one on both sides of Q_H, that is $B_H = \{q_{H,l}, q_{H,r}\}$. Here, l and r stand for left and right points, when viewed from the direction v. We say that an admissible billiard table is simple if all corridors are simple. For simple billiard tables, we consider the four points in $\mathcal{M}$, whose trajectory up to the next collision projects onto ∂Q_H in the configuration space. The set of these four points is denoted by

$$A_H = \{(r_{H,l,1}, \varphi_{H,l,1}), (r_{H,l,2}, \varphi_{H,l,2}), (r_{H,r,1}, \varphi_{H,r,1}), (r_{H,r,2}, \varphi_{H,r,2})\}. \tag{2}$$

We will say that the elements of A_H are boundary points of the corridor H. Note that if $q_{H,s}$ is a regular point (for $s = l, r$), then $r_{H,s,1} = r_{H,s,2}$ and the set $\{r = r_{H,s,1}\} = \{r = r_{H,s,2}\}$ corresponds to a vertical line segment in the interior of $\mathcal{M}_{i,j}$. On the other hand, if $q_{H,l}$ is a corner point, then $r_{H,l,1}$ corresponds to the right side of $\mathcal{M}_{i,j}$ and $r_{H,l,2}$ corresponds to the left side of $\mathcal{M}_{i,j+1}$ (and vice versa for the right boundary: if $q_{H,r}$ is a corner point, then $r_{H,r,2}$ corresponds to the right side of $\mathcal{M}_{i',j'}$ and $r_{H,r,1}$ corresponds to the left side of $\mathcal{M}_{i',j'+1}$). In this case and with a slight abuse of notation, we will also say that the elements of A_H are corner points. Note that whenever $q_{H,s}$ is a corner point, it is necessarily concave. See Figure 2 for a typical arrangement in the case when both sides are bounded by a corner point (for simplicity, we depict $i = j = j' = 1$, $i' = 2$, v is horizontal, so by convention pointing to the right). The figure represents a part of the scatterer configuration lifted from $\mathbb{T}^2$ to $\mathbb{R}^2$. The point $q_{H,r} = \Gamma_{1,1} \cap \Gamma_{1,2}$ is the corner point on the bottom left as well as the bottom right of the figure. The two corresponding signed angles $\varphi_{H,r,k}$, $k = 1, 2$ are between the dashed lines (normals to the curves) and the lower dotted line. Similarly, observe the left boundary of the corridor on the top part

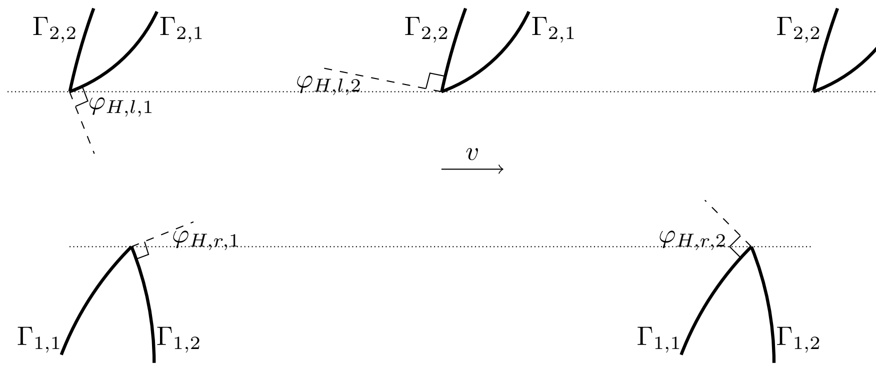

FIGURE 2. A simple corridor bounded by two corner points

of the figure. In this particular case, we have $\varphi_{H,r,1} > 0$, $\varphi_{H,r,2} < 0$, $\varphi_{H,l,1} < 0$, $\varphi_{H,l,2} < 0$ although these signs may be different for other corridors bounded by two corner points.

According to our previous definition,

$$\Phi^{\tau(r_{H,r,1},\varphi_{H,r,1})}(r_{H,r,1},\varphi_{H,r,1})$$

takes two values: $(r_{H,r,2}, -\varphi_{H,r,2})$ and $(r_{H,r,1}, \varphi_{H,r,1})$ (where we identified $\mathcal{M}$ with a subset of Ω, in particular, $r_{H,r,1}$ is identified with $r_{H,r,2}$ as points of $\mathcal{D}$). There are two possible types of free flights from regular points close to $(r_{H,r,1}, \varphi_{H,r,1})$. One possibility is a flight of bounded length, terminating on $\Gamma_{1,1}$. For such points (q, v), $F(q, v)$ is close to $(r_{H,r,2}, -\varphi_{H,r,2})$. The other possibility is a very long flight in the corridor which eventually terminates on $\Gamma_{2,2}$. For such points (q, v), $F(q, v)$ is close to $(r_{H,l,2}, -\varphi_{H,l,2})$. The local geometry of such orbits will be studied more carefully later. Correspondingly, we define the map F at the point $x = (r_{H,r,1}, \varphi_{H,r,1})$ by

$$\begin{aligned} &F_{short}(x) = (r_{H,r,2}, -\varphi_{H,r,2}), \quad F_{long}(x) = (r_{H,l,2}, -\varphi_{H,l,2}) \\ &F(x) = \{F_{short}(x), F_{long}(x)\}. \end{aligned} \tag{3}$$

Now we define $A = \cup_{H \text{ corridors}} A_H$. With these notations, we are ready to introduce our assumptions

(A1) $\mathcal{D}$ is simple.
(A2) For any $(r, \varphi) \in A$, if r corresponds to a corner point, then $|\varphi| \neq \pi/2$.

It seems likely that our results remain true if we remove (A1) and (A2) but the proof becomes more complicated so we assume them for convenience.

2.3. **Definitions.** We will denote by C any constant only depending on $\mathcal{D}$, whose explicit value is irrelevant. In particular, the value of C may change from line to line.

The billiard map F is hyperbolic and ergodic. In particular, there exists uniformly transversal families of stable and unstable cones. Specifically, there are cones $\mathcal{C}_x^{u/s}$ for every $x \in \mathcal{M}$ so that $D_xF(\mathcal{C}_x^u) \subset \mathcal{C}_{F(x)}^u$and $D_xF^{-1}(\mathcal{C}_x^s) \subset \mathcal{C}_{F^{-1}(x)}^s$. These cones were constructed in [6, Section 9]: the unstable conefield C^u is constructed by applying dF to the non-negative cones $drd\varphi \geq 0$ and the stable conefield C^s is constructed by applying dF^{-1} to the non-positive cones $drd\varphi \leq 0$. We recall the most important properties of the conefields here (see [6, Section 9]). First, let us fix any open neighborhood $\mathcal{N}$ of the convex corner points. Then there is a constant B_1 depending on the billiard table and $\mathcal{N}$ so that for all $x \in \mathcal{M} \setminus \mathcal{N}$, and for all $(dr_1, d\varphi_1) \in \mathcal{C}^u_x$, $(dr_2, d\varphi_2) \in \mathcal{C}^s_x$, we have

$$B_1^{-1} \le d\varphi_1/dr_1 \le B_1, \quad -B_1 \le d\varphi_2/dr_2 \le -B_1^{-1}. \tag{4}$$

Next, for all $x \in \mathcal{N}$ there are two possibilities. Either for all $(dr_1, d\varphi_1) \in \mathcal{C}^u_x$, $(dr_2, d\varphi_2) \in \mathcal{C}^s_x$,

$$B_1^{-1} \le d\varphi_1/dr_1, \quad -B_1 \le d\varphi_2/dr_2 \le -B_1^{-1}. \tag{5}$$

or for all $(dr_1, d\varphi_1) \in \mathcal{C}^u_x$, $(dr_2, d\varphi_2) \in \mathcal{C}^s_x$,

$$B_1^{-1} \le d\varphi_1/dr_1 \le B_1, \quad d\varphi_2/dr_2 \le -B_1^{-1}. \tag{6}$$

The intuitive meaning of the above properties is as follows. First, away from convex corner points, a picture familiar from type A billiards (4) holds. In a vicinity of a convex corner point there may be short free flights (recall the definition of corner series). If x is in such a series which starts with a nearly grazing collision (also called left-singular series), then we have (5). If x is in such a series which ends with a nearly grazing collision (also called right-singular series), then we have (6).

Let us summarize the key properties of the conefield that we are going to use: first, there exists some number $\gamma > 0$ so that at any point $x \in \mathcal{M}$, the angle between any stable and unstable vectors is bounded from below by γ. Second, no horizontal vector (that is $d\varphi = 0$) can be in the stable/unstable cones. Third, any point not in a small neighborhood of convex corner points (in particular, any point whose configurational component is in a vicinity of B_H, a boundary of a corridor B_H) satisfies (4). We will briefly refer to these properties as transversality.

Hyperbolicity needs to be understood in the sense that for almost every point there is an unstable and a stable manifold through this point, however they can be arbitrarily short. This is due to the singularities. The hyperbolicity is uniform in the sense that there are constants $C_\#$ and $\Lambda_* > 1$ so that for any $n \ge 1$, for every unstable vector u,

$$\|DF^n(u)\| \ge C_\# \Lambda_*^n \|u\| \tag{7}$$

and for any stable vector v,

$$\|DF^{-n}(v)\| \ge C_\# \Lambda_*^n \|v\|.$$

Let us write

$$S_0 = \cup_{i,j} \Gamma_{i,j} \times \{\pm\pi/2\}, \quad V_0 = \cup_i \cup_j \partial\Gamma_{i,j} \times [\pi/2, \pi/2],$$

that is S_0 is the set of grazing collisions and V_0 is the set of collisions at the corner points, and $R_0 = S_0 \cup V_0$. Furthermore, let $R_{m,n} = \cup_{l=m}^n F^l R_0$. Then for any $n \ge 1$ (including $n = \infty$), the singularity set of F^n is $R_{-n,0}$ and the singularity set of F^{-n} is $R_{0,n}$. Furthermore, as usual, we introduce secondary (artificial) singularities

$$\hat{S}_{\pm k} = \{(r, \phi) : \phi = \pm\pi/2 \mp k^{-2}\}$$

for some $k \ge k_0$ to control distortion. We denote

$$\mathbb{H}_0 = \{r, \varphi \in \mathcal{M} : -\pi/2 - k_0^{-2} \le \varphi \le \pi/2 - k_0^{-2}\}$$
$$\mathbb{H}_k = \{r, \varphi \in \mathcal{M} : \pi/2 - k^{-2} \le \varphi \le \pi/2 - (k+1)^{-2}\}$$
$$\mathbb{H}_{-k} = \{r, \varphi \in \mathcal{M} : -\pi/2 + (k+1)^{-2} \le \varphi \le -\pi/2 + k^{-2}\}.$$

The extended set of singularities is

$$R_0^{\mathbb{H}} = S_0^{\mathbb{H}} \cup V_0, \text{ where } S_0^{\mathbb{H}} = S_0 \cup (\cup_{k \ge k_0} \hat{S}_{\pm k})$$

and likewise $R^{\mathbb{H}}_{m,n} = \cup_{l=m}^{n} F^l R^{\mathbb{H}}_0$. We say that a C^2 curve in $W \subset \mathcal{M}$ is unstable if at every point $x \in W$, the tangent line $T_x W$ is in the unstable cone, and W has a uniformly bounded curvature and is disjoint to R_0 (except possibly for its endpoints). We say that an unstable curve is homogeneous if it lies entirely in $\mathbb{H}_k$ for some k ($k = 0$ or $|k| \geq k_0$). It is useful to think about unstable curves as smooth curves in the northeast-southwest direction on $\mathcal{M}$.

As in [8, section 4], we say that $\ell = (W, \rho)$ is a standard pair if W is a homogeneous unstable curve and ρ is a probability measure supported on W that satisfies

$$\left| \log \frac{d\rho}{dLeb}(x) - \log \frac{d\rho}{dLeb}(y) \right| \leq C_0 \frac{|W(x,y)|}{|W|^{2/3}}.$$

Note that there is some constant C so that for any standard pair $\ell = (W, \rho)$ and for any $x, x' \in W$

$$e^{-C|W|^{1/3}} \leq \frac{\rho(x)}{\rho(x')} \leq e^{C|W|^{1/3}}. \tag{8}$$

The image of a standard pair is a weighted average of standard pairs. More precisely, if $\ell = (W, \rho)$ is a standard pair and ν_ℓ is the measure on W with density ρ, then $F(W) = \cup_i W_i$, where W_i are homogeneous unstable curves and $F_*(\nu_\ell) = \sum_i c_i \nu_{\ell_i}$, where $\ell_i = (W_i, \rho_i)$ are standard pairs (see [8, Proposition 4.9]). We will also write $F_*(\ell) = \sum_i c_i \ell_i$.

Substandard families are weighted averages of standard pairs where the sum of the weights is ≤ 1. That is, $\mathcal{G} = ((W_\alpha, \rho_\alpha)_{\alpha \in \mathfrak{A}}, \lambda)$ is a substandard family if (W_α, ρ_α)'s are standard pairs and λ is a subprobability measure on $\mathfrak{A}$. Here, $\mathfrak{A}$ is an arbitrary index set. It can even be uncountable. We assume that the W_α's are disjoint. Given a substandard family $\mathcal{G}$, it induces a measure $\nu_\mathcal{G}$ on $\mathcal{M}$ by

$$\nu_\mathcal{G}(B) = \int_{\alpha \in \mathfrak{A}} \nu_\alpha(B \cap W_\alpha) d\lambda(\alpha) \text{ for } B \subset \mathcal{M} \text{ Borel sets,}$$

where ν_α is the measure on W_α with density ρ_α. In case $\lambda_\mathcal{G}$ is a probability measure, we call $\mathcal{G}$ a standard family. Now given a point $x \in W_\alpha$, denote by $r_\mathcal{G}(x)$ the distance between x and the closest endpoint of W_α (measured along W_α with respect to arclength). We introduce the notion of the $\mathcal{Z}_q$ function for $q \in (0, 1]$ by

$$\mathcal{Z}_q(\mathcal{G}) = \sup_{\varepsilon > 0} \frac{\nu_\mathcal{G}(r_\mathcal{G} < \varepsilon)}{\varepsilon^q}.$$

For example if $\mathcal{G}$ consists of only one standard pair (W, ρ), then

$$\mathcal{Z}_q(\mathcal{G}) \sim 2^q / |W|^q \tag{9}$$

as $|W| \to 0$. A fundamental fact about the class of standard families is that they are preserved under iterations of the map F.

Given a homogeneous unstable curve W, its image $F^m(W)$ will consist of a collection of homogeneous unstable curves W_i. For each i, let $\Lambda_i = \Lambda_{i,m}$ be the minimal expansion factor of F^m on $F^{-m} W_i$. We say that the m-step expansion holds if

$$\lim_{\delta \to 0} \sup_{W : |W| < \delta} \sum_i \frac{1}{\Lambda_{i,m}} < 1, \tag{10}$$

where the supremum is taken over homogeneous unstable curves. We note that the above limit is traditionally written as $\liminf$, however the sequence as $\delta \to 0$ is non-increasing and bounded below, so the limit always exists.

For given $n \geq 0$, let ξ^n be the partition of $\mathcal{M} \setminus R^{\mathbb{H}}_{0,n}$ into connected components and let ξ^{-n} be the partition of $\mathcal{M} \setminus R^{\mathbb{H}}_{-n,0}$ into connected components. Now the forward separation time of points $x, y \in \mathcal{M}$, denoted by $s_+(x,y)$, is defined as the smallest n so that x and y belong to different partition elements of ξ^n. Likewise, we define $s_-(x,y)$, the backward separation time, as the smallest non-negative integer n so that x and y belong to different partition elements of ξ^{-n}. We say that $f : \mathcal{M} \to \mathbb{R}$ is dynamically Hölder if there are constants $C = C(f)$ and $\vartheta = \vartheta(f) < 1$ so that for any x, y on the same unstable manifold W^u,

$$|f(x) - f(y)| \leq C\vartheta^{s_+(x,y)}$$

and for any x, y on the same stable manifold W^s,

$$|f(x) - f(y)| \leq C\vartheta^{s_-(x,y)}.$$

We will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$ and $a_n \approx b_n$ if there is a constant C so that $1/C \leq a_n/b_n \leq C$ for all n.

3. **Results.**

Now we can state our results.

Theorem 3.1. *For all type D1 billiard tables there is some m_0 so that the m_0-step expansion holds.*

Theorem 3.2. *For all type D2 billiard tables satisfying (A1) and (A2) and for all $q \in (0,1)$ we have the following. There are constants $M \in \mathbb{N}$, $\varkappa < 1$ and $\delta_0 > 0$ so that for any standard pair $\ell = (W, \rho)$ with $|W| < \delta_0$, $\mathcal{Z}_q(F^M_*(\ell)) < \varkappa \mathcal{Z}_q(\ell)$.*

Theorem 3.3. *The set of billiard tables satisfying (A1) and (A2) is open and dense in $\boldsymbol{D}$.*

Proof of Theorem 1.1 assuming Theorems 3.1, 3.2. In the case of type D1 billiard tables, the key estimate is the m_0-step expansion, provided by Theorem 3.1. Once it is established, the exponential decay of correlations and the Central Limit Theorem follow from the theory developed in [6] as it was also noted in case of type C billiards in [14].

In case of type D2 billiard tables when the assumptions (A1) and (A2) are satisfied, the proof follows from Theorem 3.2 and the results of [11]. In [11], the exponential decay of correlation is proven from some abstract assumptions denoted by (H1) - (H5). In our case, assumptions (H1) - (H4) are standard as is usual for billiards (see e.g. [6, section 9]). We do not know how to prove (H5) (see Remark 4 below) however we have Theorem 3.2 instead. In the proof of [11], (H5) is only used to derive the Growth Lemma (see [11, Lemma 3(a)]) which is standard once our Theorem 3.2 holds. That being said, we can replace (H5) by Theorem 3.2 and conclude the result of Theorem 1.1. □

Remark 1. Under the same assumptions as Theorm 1.1, several other results follow immediately from the abstract theory. Indeed, a ”magnet” is constructed in [11] which implies the existence of a Young tower [21] with exponential return times. Thus the Central Limit Theorem [21], Large Deviation Principles [18], almost sure invariance principle, law of iterated logarithm [7, 17], etc. follow.

Remark 2. It is important to note that the test functions in the setup of Theorem 1.1 and Remark 1 are assumed to be bounded and dynamically Hölder. Important functions of interest are the free flight time $\tau : \mathcal{M} \to \mathbb{R}$ and the displacement vector $\kappa : \mathcal{M} \to \mathbb{R}^2$ defined as $\Pi_{\mathcal{D}}(F(x)) - \Pi_{\mathcal{D}}(x)$, lifted to the universal cover $\mathbb{R}^2$. Both of these functions are dynamically Hölder but unbounded. In particular, we do not claim the Central Limit Theorem for the position of the billiard particle in the periodic extension of $\mathcal{D}$ from $\mathbb{T}^2$ to $\mathbb{R}^2$ (also known as Lorentz gas). In fact, we expect that this position will converge to the normal distribution under the scaling $\sqrt{t \log t}$ (where t is continuous time in case the flow is considered, and collision time in case of the map) as in [20], but we do not study this question here.

4. **Proof of Theorem 3.1.** The proof of Theorem 3.1 is based on similar proofs for type C as in [14] and type B as in [9]. As such, we only give detailed arguments at places where our proof differs from these references and otherwise cite the necessary lemmas from [9, 14].

First we review the structure of corridors and singularities, see [9, Section 4.10] for details. Let us fix a regular billiard table. There are finitely many points,

$$A = \{x_h = (r_h, \varphi_h), h = 1, ..., h_{\max}, \varphi_h = \pm\pi/2\}$$

that are periodic and whose trajectories bound the corridors. The singularity structure of F and F^{-1} near x_h is as follows. There are infinitely many singularity curves accumulating at x_h. Specifically, there are connected components $D^+_{h,n}$ for $n \geq 1$ in a neighborhood of x_h so that F is smooth on $D^+_{h,n}$ and the trajectories of the points in $D^+_{h,n}$ pass by n copies of the given scatterer before the next collision. Likewise, $D^-_{h,n}$ is a set on which F^{-1} is continuous, and $F(D^+_{h,n}) = D^-_{h',n}$ for some h'. The size of $D^+_{h,n}$ is $\approx n^{-2}$ in the unstable direction and $\approx n^{-1/2}$ in the stable direction. Likewise, the size of $D^-_{h,n}$ is $\approx n^{-1/2}$ in the unstable direction and $\approx n^{-2}$ in the stable direction. Consequently, $D^-_{h,n}$ intersects with $\mathbb{H}_k$ if $|k| \geq Cn^{1/4}$. The rate of expansion of F on $D^+_{h,n} \cap F^{-1}(\mathbb{H}_k)$ is $\approx nk^2$.

We start by the following simple lemma, which is proven in [12, page 816].

Lemma 4.1. *There is a constant C so that for any unstable curve W and for any homogeneous connected component $W' \subset F(W)$, we have*

$$|W'| \leq C|W|^{1/3}.$$

Remark 3. In the case of finite horizon if we do *not* require W' to be homogeneous, we have $|W'| \leq |W|^{1/2}$ by [9, Exercise 4.50]. Furthermore, in case of finite horizon and if W' is required to be homogeneous, then $|W'| \leq |W|^{3/5}$ holds (see [13, Formula (6.9)]). Finally, in the case of infinite horizon if W' is not required to be homogeneous, then the weaker bound $|W'| \leq |W|^{1/4}$ holds which can be proved similarly.

We will also need an estimate on the growth of the free flight function along orbits, which is provided by the next lemma.

Lemma 4.2. *There are constants C_1, t_0, t_1 only depending on $\mathcal{D}$ so that for any point $x \in \mathcal{M}$ with $\tau(x) > t_1$, there are two possibilities:*

- *either $\tau(F(x)) \in [C_1^{-1}\sqrt{\tau(x)}, C_1(\tau(x))^2]$*
- *or $\tau(F(x)) < t_0$ in which case $\tau(F^2(x)) \in [C_1^{-1}\sqrt{\tau(x)}, C_1(\tau(x))^2]$.*

We don't give a formal proof of Lemma 4.2, as the first case was proved in [20, Proposition 9] and the second case is similar. Instead, we give an explanation. A long flight needs to happen in a corridor, e.g. in the northeast direction in a horizontal corridor with an angle $\alpha \approx 1/\tau(x) \ll 1$. Let us assume that the corridor is simple (this is not necessary for the lemma to hold but simplifies the explanation). Now let $P, P' \in \partial\mathcal{D}$ be two consecutive points on the boundary of the corridor. Then the simplicity of the corridor means that P and P' project to the same point on the torus, i.e. $P \in \partial\mathcal{B}$, $P' \in \partial\mathcal{B}'$, where $\mathcal{B} = \mathcal{B}_i + m$ and $\mathcal{B}' = \mathcal{B}_i + m'$ for some $i = 1, ..., d$ and $m, m' \in \mathbb{Z}^2$. The configuration space is invariant under the shift by $P' - P$ and so without loss of generality, we can assume that the long free flight crosses the line segment PP' at a point R. Let us write $P \leq R < P'$ where $X < Y$ means that Y is to the right of X (see the dashed lines on either panel of Figure 3).

If $R = P$, then the the postcollisional flight has an angle α in the southeast direction and consequently $\tau(F(x)) \approx \tau(x)$. Let us start to move the point R to the right along the line segment P, P' and study $\tau(F(x))$. Initially, the postcollisional angle in the southeast direction is rotated towards the and crosses the corridor again before the next collision. In particular, $\tau(F(x)) \gtrapprox \tau(x)$. This is true until the trajectory becomes tangent to $\mathcal{B}'$. In this case, let us denote R by Q (see the bottom panel of Figure 3). A simple computation (see [20, Proposition 9]) shows that whenever $P \leq R < Q$, $\tau(F(x)) < C_1(\tau(x))^2$. As we move R beyond Q, the trajectory will collide on both $\mathcal{B}$ and $\mathcal{B}'$ and so $\tau(F(x)) < t_0$ where t_0 can be chosen as the maximum over both sides of all corridors of $2\text{dist}(P, P')$. Moving R to the right, eventually the trajectory will be tangent to $\mathcal{B}$. When this happens, let R be denoted by Q' (see the top panel of Figure 3). Again, an elementary argument shows that the second bullet point of the lemma holds whenever $Q \leq R \leq Q'$. Finally, the last case is $Q' < R < P$. Note that this is the typical case in since $\text{dist}(P, Q') \ll 1$. In this last typical case, $C_1^{-1}\sqrt{\tau(x)} \leq \tau(F(x)) \lessapprox \tau(x)$.

Figure 3 represents the singular trajectories through Q and Q' as in the proof of Lemma 4.2. Since the points $P < Q < Q'$ are close, their vicinity is magnified in on both panels.

Finally, in case the billiard table is not simple, the boundary of the corridor is decomposed into a periodically repeating finite finite set of intervals joining consecutive boundary points and the above argument can be repeated on each of these intervals.

Now fix some $\varepsilon_0 > 0$, n_0 and $\bar{\tau}$ so that the following is true: For any point $x = (r, \varphi) \in \mathcal{M}$ that has a free flight longer than $\bar{\tau}$, there is some $h = 1, ..., h_{\max}$ so that $d(x, x_h) < \varepsilon_0$ and $x \in D^+_{h,n}$ for some $n \geq n_0$. Furthermore, the trajectory of x under the billiard flow until the next collision avoids the ε_0 neighborhood of all corner points in $\mathbb{T}^2$.

Let us write $\mathcal{M}_{\mathfrak{n}} = \mathcal{M} \setminus \cup_{h=1}^{h_{\max}} \cup_{n \geq \mathfrak{n}} D^+_{h,n}$ and $\mathcal{M}_{m,\mathfrak{n}} = \cap_{l=0}^{m} F^{-l}\mathcal{M}_{\mathfrak{n}}$. Note that for m fixed and for $\mathfrak{n}$ large enough, we have

$$\mathcal{M}_{(\hat{c}\mathfrak{n})^{1/(2^m)}} \subset \mathcal{M}_{m,\mathfrak{n}} \subset \mathcal{M}_{\mathfrak{n}}. \tag{11}$$

Indeed, the first inequality follows from Lemma 4.2 and the second one is trivial.

Following [14], we introduce the following definitions. Let W be a homogenenous unstable curve and $W_{i,m}$ the homogeneous components (H-components) of $F^m(W)$. We say that $W_{i,m}$ is m-regular if $F^{-l}(W_{l,m}) \in \mathbb{H}_0$ for all $0 \leq l < m$. If $W_{i,m}$ is not m-regular, it is called m-nearly grazing.

Given some $\mathfrak{n}$, m and $z \in \mathcal{M}_{m,\mathfrak{n}}$, we define $\mathcal{K}^{reg}_{m,\mathfrak{n}}(z)$ as the number of connected components of $\mathcal{M}' \subset \mathcal{M}_{\mathfrak{n}} \setminus R^{\mathbb{H}}_{0,m}$ so that the closure of $\mathcal{M}'$ contains z and some

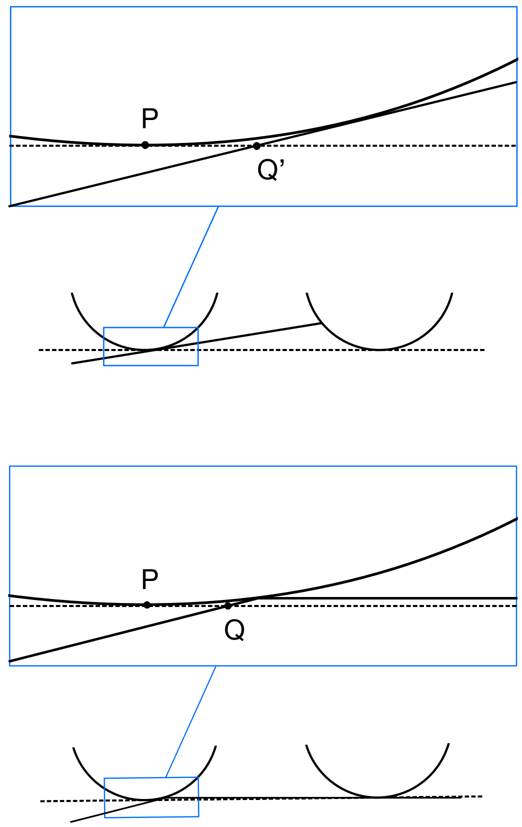

FIGURE 3. Singular trajectories after a long flight. The trajectory on the top panel is tangent to the scatterer on the left and the trajectory on the bottom panel is tangent to the scatterer on the right. A neighborhood of the first collision point is magnified for better visibility.

(and consequently all) points $x \in \mathcal{M}'$ satisfy

$$F^l(z) \in \hat{\mathbb{H}}_0 \text{ for all } l = 0, ..., m, \tag{12}$$

where

$$\hat{\mathbb{H}}_0 := \mathbb{H}_0 \cup \mathbb{H}_{k_0} \cup \mathbb{H}_{-k_0}.$$

Let

$$\mathcal{K}_m^{reg} = \sup_{\mathfrak{n} \geq 1} \sup_{z \in \mathcal{M}_{m,\mathfrak{n}}} \mathcal{K}_{m,\mathfrak{n}}^{reg}(z)$$

Now we have

Lemma 4.3. *There is some Ξ depending only on $\mathcal{D}$ (and in particular not depending on k_0) so that*

$$\mathcal{K}_m^{reg} = \sup_{\mathfrak{n} \geq 1} \sup_{z \in \mathcal{M}_{\mathfrak{n}}} \mathcal{K}_{m,\mathfrak{n}}^{reg}(z) \leq \Xi(m+1)$$

Proof. By (11), the first equality follows. The inequality follows from [14, Lemma 3.5]. Although that lemma is only proved in the case of finite horizon, the proof applies in our case as well. Let us fix $\mathfrak{n}_0 = \max\{n_0, Ck_0^4\}$. Then the proof of [14] implies $\sup_{z \in \mathcal{M}_{\mathfrak{n}_0}} \mathcal{K}^{reg}_{m,\mathfrak{n}_0}(z) \le \Xi(m+1)$. We just need to replace $\tau_{\max}$ by $\bar{\tau}$ in Lemma 3.6; in particular $\Xi = 4\bar{\tau}/\tau_* + 6$ works, where τ_* is the length of the minimal free flight between two improper collisions (a geometric constant only depending on $\mathcal{D}$). Indeed, whenever $\tau(x) > \bar{\tau}$ (this can happen if $\mathfrak{n} > n_0$), the trajectory up to the next collision avoids the ε_0 neighborhood of the corner points by the choice of n_0, so Lemma 3.6 remains valid. Now if $\mathfrak{n} > \mathfrak{n}_0$, then by the choice of $\mathfrak{n}_0$ and by Lemma 4.2, all points $z \in \mathcal{M}_{m,\mathfrak{n}} \setminus \mathcal{M}_{m,\mathfrak{n}_0}$ satisfy $F(z) \in \mathbb{H}_k$ for some k with $|k| > k_0$. Thus $\mathcal{K}^{reg}_{m,\mathfrak{n}}(z) = 0$.

□

Let k_0 be fixed and let $K^{reg}_m(W)$ denote the number of m-regular H-components of $F^m(W)$. Then we have

Lemma 4.4. *There exists some m_0 only depending on $\mathcal{D}$ so that for any k_0,*

$$\lim_{\delta \to 0} \sup_{W : |W| < \delta} K^{reg}_{m_0}(W) < \frac{1}{3} C_\# \Lambda_*^{m_0}$$

where $C_\#$ and Λ_* are defined by (7).

Proof. This lemma is proved as [14, Lemma 2.12]. We fix m_0 so that $\Xi(m_0+1) < \frac{1}{3} C_\# \Lambda_*^{m_0}$. Now since k_0 is fixed, we can choose $n_1 > \max\{n_0, k_0^4\}$. Then as in Lemma 4.3, if $z \notin \mathcal{M}_{m_0,n_1}$, then $F(z) \in \mathbb{H}_k$ for $|k| > k_0$, so the H-component $W' \subset F^{m_0}(W)$ containing $F^{m_0}(z)$ must be m_0-nearly grazing. Once n_1 is fixed, there are only finitely many points $\mathcal{Z} = \{z_1, ..., z_Z\}$ where $\mathcal{K}^{reg}_{m_0,n_1}(z) > 2$. By choosing $\delta = \delta(n_1)$ small, we can assume that our curve W is only close to one of these points and so by transversality and by the fact that we used $\hat{\mathbb{H}}_0$ in (12), we find $K^{reg}_{m_0}(W) \le \mathcal{K}^{reg}_{m_0}$, which by Lemma 4.3 completes the proof. □

Next, we bound the contribution of nearly grazing components for $m = 1$.

Lemma 4.5. *For any $\varepsilon > 0$ there exists k_0 so that*

$$\lim_{\delta \to 0} \sup_{W : |W| < \delta} \sum_i^* \frac{1}{\Lambda_{i,1}} < \varepsilon,$$

where $\sum^*$ means that the sum is restricted to nearly grazing H-components $W_{i,1}$.

Proof. This lemma is analogous to [14, Lemma 2.13] but the proof differs substantially as the free flight now is unbounded. However, we can use [9, Remark 5.59].

Let us write $W_1 = W \setminus \mathcal{M}_{n_0}$ and $W_2 = W \cap \mathcal{M}_{n_0}$. Let S^{*j} be the sum corresponding to the images of W_j for $j = 1, 2$. As before, for any k_0 there exists some $\delta > 0$ so that if $|W| < \delta$, then $F(W_1)$ have at most $L = \bar{\tau}/\tau_{\min} + 2$ connected components. Each of these components could be further cut by secondary singularities, so $S^{*1} \le L \sum_{k \ge k_0} C k^{-2} \le CLk_0^{-1}$ which is less then $\varepsilon/2$ assuming that $k_0 > 2CL/\varepsilon$. For any fixed k_0 and n_0, we can make $S^{*2} \le \varepsilon/2$ by further reducing δ if needed, exactly as in [9, Remark 5.59]. □

Now Theorem 3.1 follows from Lemmas 4.3, 4.4 and 4.5 as in [14]. Note that there is a typo at the middle of page 1231 in [14] as one only has

$$\mathcal{L}_{n+m}(W) \le \sum_i \frac{1}{\Lambda_{i,n}} \mathcal{L}_m(W_{i,n}), \tag{13}$$

where

$$\mathcal{L}_n(W) = \sum_i \frac{1}{\Lambda_{i,n}} \tag{14}$$

(and the equation in (13) may not hold), but this is enough since Theorem 3.1 only gives an upper bound.

5. **Proof of Theorem 3.2.** By (A1) and (A2), we can group the corridors into three categories: $H \in \mathcal{H}_1$ if H is bounded by two regular points, $H \in \mathcal{H}_2$ if H is bounded by a regular point and a corner point and $H \in \mathcal{H}_3$ if H is bounded by two corner points. We say that the corridor is of type $1, 2, 3$, respectively. We also say that a boundary point $x \in A_H$ with $H \in \mathcal{H}_j$ is of type j.

The reason we assume (A1) and (A2) is to guarantee that there are only 3 types here. Without these assumptions, there would be many more types (see [4] for a description of all types in general).

Let us fix an enumeration of the set $\cup_H A_H$ as $\{x_1, ..., x_{h_{\max}}\}$. Note that it is possible that $\Pi_{\mathcal{D}}(x_h) = \Pi_{\mathcal{D}}(x_{h'})$ for some $h \in H$, $h' \in H'$, $H \neq H'$ in case $\Pi_{\mathcal{D}}(x_h)$ is a corner point.

Recall that in case of type 1 corridors

$$A_H = \{(r_{H,r}, \pi/2), (r_{H,r}, -\pi/2), (r_{H,l}, \pi/2), (r_{H,l}, -\pi/2)\}.$$

Also recall the notation introduced in Section 4: for any point $x_h \in A_H$ for some $H \in \mathcal{H}_1$, we denote by $D^{\pm}_{h,n}$ the domains where $F^{\pm}$ is smooth in a neighborhood of x_h. We also note that $F(D^+_{h,n}) = D^-_{h',n}$ with $x_h = (r_{H,l/r}, \pm\pi/2)$ and $x_{h'} = (r_{H,r/l}, \mp\pi/2)$.

If $H \in \mathcal{H}_3$, then A_H is of the form (2). To emphasize the difference from the previous case, we will denote by $E^+(h,n)$ the set of points $x \in \mathcal{M}$ in a vicinity of x_h so that the free flight of x passes by n copies of the scatterer before colliding on the other side of the corridor whenever x_h is of type 3. A simple geometric argument shows that $E^+(h,n)$ is of size $\approx n^{-2}$ in the unstable direction and $\approx n^{-1}$ in the stable direction, see [4, Section 4], and [4, Figure 11]. Since this figure will be used a lot, we reproduced it in our Figure 4, left panel. $E^+(h,n)$ is a topological rectangle. Let us make the convention that it contains two of its for sides: the left and the bottom side on the left panel of Figure 4. Namely,

$$E^+(h,n) = int(E^+(h,n)) \cup [\partial E^+(h,n) \cap (\partial E^+(h,n-1) \cup \{r = \Pi_{\mathcal{D}} x_h\})]$$

This way, the sets $E^+(h,n)$ are disjoint. The image of $E^+(h,n)$ is in a small neighborhood of a point $(q,-v)$, where $(q,v) \in A_H$.

Type 2 corridors will require special consideration. Let $x_h = (r_h, \varphi_h) \in A_H$ with $H \in \mathcal{H}_2$. If r_h corresponds to a regular point, then we define $D^{\pm}_{h,n}$ with the same asymptotic size as in case $\mathcal{H}_1$ and if r_h corresponds to a corner point, then we define $E^+(h,n)$ with the same asymptotic size as in case $\mathcal{H}_3$.

Recall that

$$A = \cup_H A_H \text{ and } A' = \{x_h \in A : \Pi_{\mathcal{D}} x_h \text{ is a corner point}\}. \tag{15}$$

Let us write

$$\mathcal{E}(h,n) = \mathcal{E}(x_h,n) = \cup_{N \geq n} E^+(h,N), \quad \mathcal{E}_n = \cup_{h: x_h \in A'} \mathcal{E}(h,n). \tag{16}$$

Note that $\mathcal{E}_n$ is obtained as a countable disjoint union. On the left panel of Figure 4, $\mathcal{E}_n$ is the big wedge. It contains the bottom boundary of the wedge and the

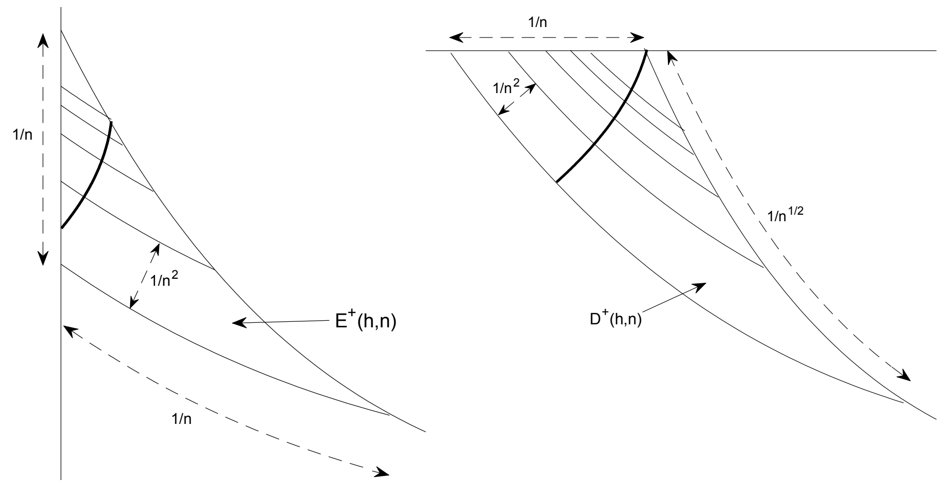

FIGURE 4. Singularity structure near type 3 and type 1 boundary points Similar figures can be found in [4, Figure 11]. An unstable curve is indicated with bold on both panels.

left boundary except for the top vertex x_h. It does not contain neither the right boundary, nor the point x_h. For type 2 corridors, we need some further definitions:

$$\begin{aligned}\mathcal{B}(h,n) &= \mathcal{B}(x_h,n) = \cup_{N\geq n} D^+_{h,N},\\ \mathcal{B}_n &= \cup_{h: x_h \text{regular boundary point of a type 2 corridor}} \mathcal{B}(h,n).\end{aligned} \tag{17}$$

and finally

$$\mathcal{E}_{n_1}\mathcal{B}_{n_2} = \mathcal{E}_{n_1} \cup \mathcal{B}_{n_2}$$

That is, $\mathcal{E}_{n_1}\mathcal{B}_{n_2}$ is the set of points that experience a long free flight in type 2 or type 3 corridors. This set is the disjoint union of $\mathcal{E}_{n_1}$ and $\mathcal{B}_{n_2}$, where $\mathcal{E}_{n_1}$ is contained in a neighborhood of A' and $\mathcal{B}_{n_2}$ is contained in a neighborhood of $A \setminus A'$. Note that when there are no type 2 corridors we have $\mathcal{B}_{n_2} = \emptyset$. In this case, the forthcoming proof could be simplified substantially.

Let us fix a large integer N_1 so that the sets $D^+_{h,n}$ and $E^+(h',n')$ are disjoint whenever $n, n' \geq N_1$, $h \neq h'$. We start with a geometric lemma.

Lemma 5.1. *There is a constant C_2 so that for every unstable curve W with $W \subset \mathcal{E}_{N_1}$, there is some $T = T(W)$ so that for all $x \in W$, $T \leq \tau(x) \leq C_2 T$ holds.*

Proof. For every h with $x_h = (r_h, \varphi_h) \in A'$, we will show that the desired C_2 exists for unstable curves in $\mathcal{E}(x_h, N_1)$. This is sufficient as there are finitely many corridors and we can take the biggest C_2. Let us assume that r_h is the left endpoint of the corresponding boundary curve Γ (the other case is similar).

Without loss of generality we can assume that the endpoint of W is a point (r_h, φ_0). Indeed, if the curve does not stretch all the way to the left boundary of $\mathcal{M}$, we can smoothly extend it to the "southwest". A key observation is that $\varphi_0 < \varphi_h$. Indeed, if $\varphi_0 \geq \varphi_h$, then since unstable curves are in the "northeast direction", W could not intersect $\mathcal{E}_{N_1}$. Now we choose $T = T(W) = \tau(r_h, \varphi_0)$. We can assume that the other endpoint of W is $(r', \varphi') \in \partial\mathcal{E}_{N_1}$ so that the next collision after leaving (r', φ') is at the same corner point (and so is improper). Strictly speaking, (r', φ') is not contained in W as $(r', \varphi') \notin \mathcal{E}_{N_1}$, that is, W is a curve that does not contain one of its endpoints. Indeed, if W does not fully cross $\mathcal{E}_{N_1}$, we can extend it smoothly to the northeast.

Next, we claim that all angles of the triangle with vertices (r_h, φ_h),(r_h, φ_0),(r', φ') are bounded from below by a positive constant only depending on $\mathcal{D}$. Note that the claim implies the lemma. Indeed, since $\varphi_h - \varphi_0 \approx 1/T$, the claim implies $\tau(r', \varphi') < C_2 T$ for a geometric constant C_2.

It remains to prove the claim. To this end, assume first, as we can, that N_1 is large so that $\mathcal{E}_{N_1}$ is disjoint to an open neighborhood of all convex corner point. Then (4) holds throughout $\mathcal{E}_{N_1}$. The angle at (r_h, φ_h) is bounded from below because the stable cone is transversal to the vertical direction. The angle at (r', φ') is bounded from below because stable and unstable cones are transversal. Finally, the angle at (r_h, φ_0) is bounded from below because the unstable cone is also transveral to the vertical direction. The lemma follows.

□

Figure 4 shows unstable curves (indicated with bold) with long flight in corridors of type 1 and 3. In case of type 1 corridors, the free flight function restricted to an unstable curve W may be unbounded (see the right side of figure), but any long flight is necessarily followed by a nearly grazing collision. This nearly grazing collision makes it easier to control the sum of expansion factors, as in the proof of Theorem 3.1. As seen on the left side of the figure and proven by Lemma 5.1, for any unstable curve W, the free flight function restricted to W is bounded in type 3 corridors (with a bound depending on W). We will leverage this fact in Lemma 5.2 to show that such a flight can only increase the $\mathcal{Z}_1$ function of a standard family by a bounded factor.

Most work is required in case of type 2 corridors. In this case, given an unstable curve near the regular boundary point (as on the right panel of Figure 4) the free flight is unbounded and after the collision, the expansion is *not* large. In particular, the sum in (14) is infinite. To overcome this difficulty, we prove in Lemma 5.3 that the $\mathcal{Z}_q$ function remains finite for any $q < 1$. Then, we show that multiple visits into corridors of type 2 or 3 in a short succession are not possible (Lemma 5.4).

Lemma 5.2. *There is some integer $N_2 \geq N_1$ and a constant C_3 so that for every standard pair $\ell = (W, \rho)$ with $W \subset \mathcal{E}_{N_2}$,*

$$\mathcal{Z}_1(F_*(\ell)) \leq C_3 \mathcal{Z}_1(\ell). \tag{18}$$

Proof. Let $x_h = (r_h, \varphi_h)$ and $\ell = (W, \rho)$ be such that $W \subset \mathcal{E}(x_h, N_2)$ for some N_2 to be specified later. Assume first that x_h is a type 3 boundary point. Let us write $V_n = W \cap E^+(x_h, n)$ and $W_n = F(V_n)$ for $n \geq N_2$. Next we claim that there is a constant $\beta > 0$ only depending on $\mathcal{D}$ such that for all $(r, \varphi) \in W$ and with the notation $F(r, \varphi) = (r', \varphi')$, we have $\cos\varphi' \geq \beta$. Indeed, let $\beta = \alpha/2$, where α is the minimal angle between the half tangents of the boundary points bounding the corridors and the directions of the corresponding corridor v_H ($\alpha > 0$ by assumption (A2)). Then by choosing N_2 sufficiently large, we can guarantee that the angle between the line segment emanating from (r, φ) and v_H is less than $\alpha/2$, which implies the claim. Without loss of generality we can assume that $k_0 > \beta^{-1}$ and so for all $n \geq N_2$, W_n is a homogeneous unstable curve and the expansion of F on V_n is $\approx n$. Furthermore, a simple geometric argument shows $|W_n| \approx n^{-1}$ whence $|V_n| \approx n^{-2}$. By Lemma 5.1, there is some n_W so that $F(W) = \cup_{n=n_W}^{C_2 n_W} W_n$. We conclude

$$\mathcal{L}_1(W) = \sum_{n=n_W}^{C_2 n_W} \frac{1}{\Lambda_{n,1}} \leq C \sum_{n=n_W}^{C_2 n_W} \frac{1}{n} \leq 2C \ln C_2. \tag{19}$$

We obtained the variant of (18), where ρ is constant. Generalizing it to all admissible densities ρ is standard only using (8) and (9) and so we omit the proof (see e.g. [11]).

In case x_h is a type 2 boundary corner point, we write $V_n = W \cap E^+(x_h, n)$ and $W_n = F(V_n)$ as before. Now W_n is not a homogeneous curve as it is further cut into infinitely many pieces by secondary singularities. However, on any of these pieces, the expansion factor is large and so $\mathcal{L}_1(W)$ is bounded as in [9, Remark 5.59]. The lemma follows. □

Lemma 5.3. *There is some integer $N_3 \geq N_2$ so that for every $q < 1$ there is a constant C_4 so that for every standard pair $\ell = (W, \rho)$ with $W \subset \mathcal{B}_{N_3}$,*

$$\mathcal{Z}_q(F_* \ell) \leq C_4 \mathcal{Z}_q(\ell).$$

Proof. Let $x_h = (r_h, \pm\pi/2)$ be a regular boundary point of a type 2 corridor and let $\ell = (W, \rho)$ be such that $W \subset \mathcal{B}(x_h, N_3)$ for some N_3 to be specified later. As in the case of type 3 corridors (cf. the proof of Lemma 5.2), we find that $V_n = W \cap D^+_{h,n}$, $W_n = F(V_n)$, the expansion of F on V_n is $\approx n$, $|W_n| \approx n^{-1}$, and $|V_n| \approx n^{-2}$. The main difference from the case of type 3 corridors is that now Lemma 5.1 fails to hold. Indeed, the curve W may be cut into *infinitely many* pieces (see the right panel of Figure 4) and so the sum in (19) diverges. We are going to prove that

$$\sum_{n \geq N_3} \left(\frac{|W|}{|W_n|} \right)^q \frac{|V_n|}{|W|} \leq C_4. \tag{20}$$

First note that (20) implies the lemma when ρ is constant. The general case can be proven using (8) and (9). Thus it remains to verify (20). To prove (20), we distinguish three cases.

First assume that W is cut into infinitely many pieces, that is, $V_n \neq \emptyset$ for infinitely many n's. Then x_h is necessarily an endpoint of W. Let M be the smallest integer n so that W fully crosses $D^+_{h,n}$. Then we have $|W| \approx M^{-1}$ and so

$$\sum_{n \geq N_3} \left(\frac{|W|}{|W_n|} \right)^q \frac{|V_n|}{|W|} \leq C|W|^{q-1} \sum_{n \geq M-1} n^{-2+q} \leq C|W|^{q-1} \sum_{n \geq \bar{C}|W|^{-1}} n^{-2+q} \leq C_4.$$

Next, assume that $W \subset D^+_{h,M}$ for some M. Then the left hand side of (20) is $(|W|/|W_M|)^q \approx M^{-q}$, which is also bounded.

Finally, assume that there are positive integers $M_1 \leq M_2$ so that $V_n \neq \emptyset$ if and only if $M_1 \leq n \leq M_2$. The contribution of $n = M_1$ and $n = M_2$ is bounded as in the second case. Thus it suffices to bound the contribution of $n = M_1 + 1, ..., M_2 - 1$, that is the set of n's so that W fully crosses $D^+_{h,n}$. To simplify the notation, we replace M_1+1 by M_1 and M_2-1 by M_2. We obtain

$$\sum_{n=M_1}^{M_2}\left(\frac{|W|}{|W_n|}\right)^q\frac{|V_n|}{|W|}\le C|W|^{q-1}\sum_{n=M_1}^{M_2}n^{-2+q}\le C(M_1^{-1}-M_2^{-1})^{q-1}(M_1^{q-1}-M_2^{q-1}).$$

Writing $a=M_2/M_1-1$, we find that the above display is bounded by

$$C(M_2^{-1}a)^{q-1}M_2^{q-1}\left[\left(\frac{M_1}{M_2}\right)^{q-1}-1\right]=Ca^{q-1}[(1+a)^{1-q}-1]=:Cf(a).$$

Now f is a continuous function on $\mathbb{R}_+$ with $\lim_{a\to\infty}f(a)=1$ and $\lim_{a\to 0}f(a)=0$ (in fact $f(a)\sim(1-q)a^q$ as $a\to 0$). Consequently, f is bounded and so (20) follows. □

Remark 4. The proof of Theorem 3.2 could be simplified if we knew that the constant C_4 given by Lemma 5.3 is less than 1. Although the 1-step expansion as required by (H5) of [11] would not follow, as it does not even hold in type C billiards, at least long flights in a short succession would not cause a trouble. Unfortunately we do not know whether $C_4<1$ and so we need Lemma 5.4, which says that two long free flights in short succession emanating near a corner point are not possible.

Lemma 5.4. *For every $K\in\mathbb{N}$ there exists $N_4=N_4(K)$ so that for all $k=1,...,K$,*

$$\mathcal{E}_{N_4}\cap F^{-k}(\mathcal{E}_{N_4})=\emptyset.$$

Proof. Let

$$B=\cup_{H\in\mathcal{H}_3}A_H,\quad B'=\{x_h\in A:\Pi_{\mathcal{D}}x_h\text{ is a type 2 corner point}\}.$$

Recall that $A'=B\cup B'$. The idea of the proof is the following. We consider the set $A'_L:=\cup_{k=1}^{L}F^k(A')$ for some large $L=L(K)$. Clearly, A'_L is a finite set. We will show that if N_4 is large, then for any $k=1,..,K$ and for any $y\in F^k(\mathcal{E}_{N_4})$, y is close to a point in $z\in A'_L$ in an unstable direction. Furthermore, observe that every point in $\mathcal{E}_{N_4}$ is close to some $z\in A'$ in a stable direction. Then $\mathcal{E}_{N_4}\cap F^{-k}(\mathcal{E}_{N_4})$ must be empty even if $A'\cap A'_L\neq\emptyset$. To highlight the main ideas of the proof, we give the details first in a special case, namely when $A'=B$ and $A'\cap A'_L=\emptyset$. Then we proceed to the general case.

Special case: no type 2 corridors and $F^k(B)\cap B=\emptyset$ for all $k\ge 1$. Recall the definition of proper collision from Section 2.1. First, we claim that there is a constant M only depending on the billiard table, so that for any point $x\in\mathcal{M}\setminus A$ (that is, x is not a boundary point of a corridor of any type), any branch of $F^M(x)$ must contain a proper collision.

To prove this claim, assume first that x is in a small neighborhood $\mathcal{A}$ of A (but x is not in A). Then simple geometry shows that x can only experience at most two improper collisions before a proper one. Next assume that $x\notin\mathcal{A}$. Improper collisions cannot take the orbit into $\mathcal{A}$, that is if the first m collisions are improper, then $F^i(x)\notin\mathcal{A}$ for all $i=0,...,m$. Next observe that any two improper collisions are necessarily separated by a universal constant time τ_* (this is obvious in case the collisions happen on different scatterers, and by [6], corner series can only contain one improper, i.e. grazing, collision). Since the free flight is bounded by some constant $\bar\tau$ on $\mathcal{M}\setminus\mathcal{A}$, we conclude that x can only experience at most $\lceil\bar\tau/\tau_*\rceil$ improper collisions before a proper one. The claim follows.

The above claim implies that for any $x \in F(B)$, any branch of the future orbit of x sufficiently long to contain at least $L := M(K+1)$ collisions, necessarily contains at least $K+1$ proper collisions. Indeed, we can just use the claim inductively as we assumed that $F^k(B) \cap B = \emptyset$ (and so $F^k(B) \cap A = \emptyset$ too since there are no type 2 corridors).

Recall Figure 2. Let us prove that

$$\mathcal{E}(x_0, N_4) \cap F^{-k}(\mathcal{E}_{N_4}) = \emptyset$$

for $x_0 = (r_{H,r,1}, \varphi_{H,r,1})$ (the proof for any other $x_h \in B$ is identical). Let $(p_0, \psi_0) \in \mathcal{E}(x_0, N)$ for some N large and $(p_1, \psi_1) \in F(p_0, \psi_0)$. Then most of the time (p_1, ψ_1) is uniquely defined (in which case we identify $F(p_0, \psi_0)$ with $\{F(p_0, \psi_0)\}$). The only case when (p_1, ψ_1) is not uniquely defined is when $(p_0, \psi_0) \in \partial E(x, n) \cap \partial E(x, n+1)$ for some n. In either case, we see that (p_1, ψ_1) is in a small neighborhood of $(r_{H,l,2}, -\varphi_{H,l,2}) = F_{long}(r_{H,r,1}, \varphi_{H,r,1})$. Now let $W \subset \mathcal{M}$ be the line segment between the points $(r_{H,l,2}, -\varphi_{H,l,2})$ and (p_1, ψ_1). Then the convexity of $\Gamma_{2,2}$ implies that the tangent of W satisfies $d\varphi/dr \geq 0$. Also note that $d\varphi/dr = \infty$ is possible in case $p_1 = r_{H,l,2}$. Although W may not be an unstable curve yet but for any $i \geq 1$, $F^i(W)$ will be a union of unstable curves.

We say that a multivalued map T is multicontinuous at x if for every ε there is some δ so that for any y with $dist(x, y) < \delta$ there is a mapping $g_{x,y}$ from $T(y)$ to $T(x)$ (recall that these are sets now!) so that for any $z \in T(y)$, $dist(z, g_{x,y}(z)) < \varepsilon$. Next, we claim that Φ^t is multicontinuous on $F(B)$ for any $t \geq 0$. Indeed, the values of the flow were defined as the possible limit points of nearby regular trajectories and so multicontinuity follows from [9, Exercise 2.27].

Now let

$$\mathcal{T} = \max_{x \in B} \max_{branches} \sum_{k=1}^{L} \tau(F^k(x_0)), \tag{21}$$

where "branches" means any branch of the orbit. Again, as the forward orbit of B is disjoint to A, $\mathcal{T}$ is finite. Note that there are finitely many branches and so the set $B_L := \cup_{k=1}^{L} F^k(B)$ is finite.

Now we fix a small ε so that $dist(B_L, B) > 2\varepsilon$. Such an ε exists because both B_L, B are finite and they are disjoint. Then using multicontinuity of $\Phi^{\mathcal{T}}$, we can choose N_4 so large that whenever $(p_0, \psi_0) \in \mathcal{E}_{N_4}$, then any element of $\Phi^{\mathcal{T}}(p_1, \psi_1)$ is in the ε neighborhood of B_L. Note that the ε neighborhood is taken in Ω even though $B_L \subset \mathcal{M}$ as nearby points do not collide at the exact same time. By the definition of L and further reducing ε and then increasing N_4 if necessary, we can assume that all points of $\mathcal{E}_{N_4}$ experience at least K proper collisions before $\mathcal{T}$. We conclude that for any $k = 0, ..., K-1$ and for any point $(p_k, \psi_k) \in F^k(p_1, \psi_1)$, (p_k, ψ_k) is in the ε neighborhood of B_L. By the choice of ε, (p_k, ψ_k) is at least ε distance away from B and so $(p_k, \psi_k) \notin \mathcal{E}_{N_4}$. This completes the proof.

General case. Now we prove the lemma in general, i.e. type 2 corridors are allowed and $F^k(A') \cap A'$ may not be empty. Since the proof is similar to the above special case, we only discuss the differences.

First, we define $L = (M+1)K$ as before. Note however that this time some branches of the orbit of B (or A') containing L collisions may not contain K proper collisions. To give a specific example, recall Figure 2 and let $\varphi_{H,r,1} = \varphi_{H,l,2} = 0$.

Then

$$F_{long}(r_{H,r,1},0) = (r_{H,l,2},0) \in B, \quad F_{long}(r_{H,l,2},0) = (r_{H,r,1},0) \in B. \tag{22}$$

However the following weaker statement is still valid (and is proved exactly as before): for any $x \in F(A')$ any branch of the future orbit of x that can only use F_{short} upon reaching A', necessarily contains at least K proper collisions.

We have already discussed the definition of (p_1, ψ_1) in case $(p_0, \psi_0) \in \mathcal{E}(x_h, N)$ for N large with x_h a type 3 boundary point. Assume now that $x_h = (r_h, \varphi_h) \in A' \setminus B$. Then the boundary points of the corresponding corridor H are

$$A_H = \{x_h = (r_h, \varphi_h), x_{h'} = (r_{h'}, \varphi_{h'}), (r_{h''}, -\pi/2), (r_{h''}, \pi/2)\}.$$

Here, r_h and $r_{h'}$ correspond to the corner point on one side of the corridor, and the points $(r_{h''}, \pm\pi/2)$ correspond to the regular boundary point of the corridor. Let $(p_0, \psi_0) \in \mathcal{E}(x_h, N)$ for some N large. As in Lemma 4.2, we find that either $F^2(p_0, \psi_0)$ or $F^3(p_0, \psi_0)$ is in a small neighborhood of $(r_{h'}, -\varphi_{h'})$. Indeed, the particle starting from (p_0, ψ_0) experiences a long free flight, after which it collides once or twice in a small neighborhood of the regular boundary point of the corridor, and then has another long free flight terminating in a small neighborhood of the boundary corner point. Let us now denote by (p_1, ψ_1) either $F^2(p_0, \psi_0)$ or $F^3(p_0, \psi_0)$, whichever is close to $(r_{h'}, -\varphi_{h'})$. Then, as before, the tangent of the line segment W connecting (p_1, ψ_1) to either $F^2(p_0, \psi_0)$ or $F^3(p_0, \psi_0)$ satisfies $d\varphi/dr \geq 0$.

To simplify notation, let us extend the definition of F_{short} from A' to $\mathcal{M}$ by setting $F_{short} = F$ on $\mathcal{M} \setminus A'$. Now, we update the definition of $\mathcal{T}$ in (21) by replacing F by F_{short}.

Next, we put $A'_L = \cup_{k=1}^{L} F_{short}^k(A')$. A'_L is a finite set just like B_L however $A'_L \cap A'$ may not be empty. Now we use the fact (p_0, ψ_0) is not just close to z for some $z \in A'$, but also is in an unstable direction from z. By induction, we see that for each k, and for each $(p_k, \psi_k) \in F(p_{k-1}, \psi_{k-1})$, (p_k, ψ_k) is close to a point in $F_{short}^k(A')$ and is in an unstable direction. Indeed, if (p_{k-1}, ψ_{k-1}) is close to $z \in F_{short}^{k-1}(A') \cap A'$, and is in the unstable direction, then (p_{k-1}, ψ_{k-1}) must not have a long flight as the entire set $\mathcal{E}(z, N)$ is in a *stable* direction form z. Then we can complete the proof just like in the special case.

□

Note that in the case of type 2 corridors, two long flights are possible. Specifically, we have

Lemma 5.5. *For every $K \in \mathbb{N}$ there exists $N_5 = N_5(K)$ so that for all $k = 1, 2, ..., K$*

$$\mathcal{B}_{N_5} \cap F^{-k}(\mathcal{E}_{N_4(K)} \mathcal{B}_{N_5}) = \emptyset$$

and for all $x \in \mathcal{E}(x_h, N_4(K))$, the set

$$\mathfrak{k} = \{k = 1, ..., K : F^k(x) \in \mathcal{B}_{N_5}\}$$

can only be non empty if x_h is of type 2. In this case, $\mathfrak{k} = \{1\}$ or $\mathfrak{k} = \{2\}$.

Proof. Let

$$N_5(K) = \frac{\max_{x \in A} \tau(x)}{\min_{x \in A} \tau(x)} C_1 N_4^2(K+2),$$

where C_1 is defined in Lemma 4.2. If $x \in \mathcal{B}(x_h, N_5)$, then as in Lemma 4.2, we have either $F^{-1}(x) \in \mathcal{E}_{N_4(K+2)}$ or $F^{-2}(x) \in \mathcal{E}_{N_4(K+2)}$. Now the result follows from Lemma 5.4.

□

For the remaining part of the proof let us fix some $q<1$. Recalling (9), there exists a constant $\bar{c}$ only depending on $\mathcal{D}$ so that for any standard pair $\ell=(W,\rho)$,

$$1/\bar{c} \le \mathcal{Z}_q(\ell)|W|^q \le \bar{c}, \quad 1/\bar{c} \le \mathcal{Z}_1(\ell)|W| \le \bar{c}. \tag{23}$$

For a given homogeneous unstable curve W, we write

$$\boldsymbol{T}_{N,N'}(W) = \boldsymbol{T}_{N,N',\mathcal{D}}(W) = \min\{m \ge 0 : F^{-m}(W) \subset \mathcal{E}_N \mathcal{B}_{N'}\}.$$

The next lemma states a weaker version of Theorem 3.2 as we only count those H-components in $F^m(\ell)$ that avoid type 2 and type 3 corridors.

Lemma 5.6. *There exists $m_0 \in \mathbb{N}$, and C_5 so that for every N, N' and for every K there is some $\delta_0 = \delta_0(N,N',K)$ such that the following holds for every standard pair $\ell=(W,\rho)$ with $|W|<\delta_0$ and for all $m=1,2,...,2Km_0+6$*

$$\mathcal{Z}_1(F_*^m(\ell)|_{W_i:\boldsymbol{T}_{N,N'}(W_i)>m}) < C_5 \mathcal{Z}_1(\ell), \tag{24}$$

$$\mathcal{Z}_1(F_*^{Km_0}(\ell)|_{W_i:\boldsymbol{T}_{N,N'}(W_i)>Km_0}) < 2^{-K} \mathcal{Z}_1(\ell). \tag{25}$$

Proof. First, assume that ρ is constant. Now we claim the following: there is some m_0 and C_5 so that for any N, N'

$$\lim_{\delta\to 0} \sup_{W:|W|<\delta} \sum_{W_i \in F^{m_0}(W):\boldsymbol{T}_{N,N'}(W_i)>m_0} \frac{1}{\Lambda_{i,m_0}} < \frac{1}{2}, \tag{26}$$

and for any m,

$$\lim_{\delta\to 0} \sup_{W:|W|<\delta} \sum_{W_i \in F^{m}:\boldsymbol{T}_{N,N'}(W_i)>m} \frac{1}{\Lambda_{i,m}} < C_5. \tag{27}$$

To prove this claim, let us replace a small neighborhood (of diameter $< cN^{-1}$) of the corner points bounding the corridors by a smooth curve so as the new billiard table $\mathcal{D}'=\mathcal{D}'(N)$ contains $\mathcal{D}$. By construction, for all W_i H-component of $F^m_{\mathcal{D}'}(W)$ with $\boldsymbol{T}_{N,N',\mathcal{D}'}(W_i) > m$ and for all $x \in W_i$, the orbits $F_{\mathcal{D}}^{-m}(x), ..., F_{\mathcal{D}}^{-1}(x), x$ and $F_{\mathcal{D}'}^{-m}(x), ..., F_{\mathcal{D}'}^{-1}(x), x$ coincide. Then Theorem 3.1 implies that the left hand side of (26) is bounded by some number $\beta<1$. Replacing m_0 by $m_0^{\ln(1/2)/\ln\beta}$ and using (13) and Lemma 4.1, we obtain (26). Although we only proved Lemma 4.1 under the conditions of Theorem 3.1, it is valid under the more general conditions of Theorem 3.2. Indeed, a long flight and an almost grazing collision expands an unstable curve more than just a long flight. Likewise, we obtain

$$\lim_{\delta\to 0} \sup_{W:|W|<\delta} \sum_{W_i \in F^{Km_0}:\boldsymbol{T}_{N,N'}(W_i)>Km_0} \frac{1}{\Lambda_{i,Km_0}} < 2^{-K}. \tag{28}$$

Also observe that (27) follows from the proof of Theorem 3.1 for $m \le m_0$. Then it also follows for $m \ge m_0$ by (26) and by (13).

Since our construction did not depend on N', it remains to prove that m_0 and C_5 are uniform in N. Although the curvature of $\partial\mathcal{D}'$ is not uniformly bounded in N, points visiting the part of the phase space with unbounded curvature are discarded. Then it remains to observe that the constants Ξ and $C_\#$ appearing in the proof of Theorem 3.1 are uniform in $\mathcal{D}'$ and so is m_0 and C_5.

Now (28) and (27) combined with (8) and (9) imply (24) and (25).

Finally, if ρ is not constant, we just need to apply (8) once more to complete the proof. □

In the setup of Lemma 5.6, we discard the points one step before reaching $\mathcal{E}_N\mathcal{B}_{N'}$. The next lemma says that we can iterate the map once more and only discard the points upon reaching $\mathcal{E}_N\mathcal{B}_{N'}$. Let

$$\boldsymbol{T}'_{N,N'}(W) = \min\{m \geq 1 : F^{-m}(W) \subset \mathcal{E}_N\mathcal{B}_{N'}\}.$$

Lemma 5.7. *Let m_0 be as in Lemma 5.6. There exists C_6 so that for every K and for every large N, N', there is some $\delta_0' = \delta_0'(N, N', K)$ such that the following holds for every standard pair $\ell = (W, \rho)$ with $|W| < \delta_0'$ and for all $m = 1, 2, ..., 2Km_0 + 6$*

$$\mathcal{Z}_1(F_*^m(\ell)|_{W_i:\boldsymbol{T}'_{N,N'}(W_i)>m}) < C_6\mathcal{Z}_1(\ell) \tag{29}$$

$$\mathcal{Z}_1(F_*^{Km_0+m}(\ell)|_{W_i:\boldsymbol{T}'_{N,N'}(W_i)>Km_0+m}) < C_6 2^{-K}\mathcal{Z}_1(\ell) \tag{30}$$

Proof. First we claim that

$$\mathcal{Z}_1(F_*^{Km_0+m}(\ell)|_{W_i:\boldsymbol{T}_{N,N'}(W_i)>Km_0+m}) < C_5 2^{-K}\mathcal{Z}_1(\ell) \tag{31}$$

for all standard pairs $\ell = (W, \rho)$ with $|W| < (\delta_0)^{3^{Km_0}}$. Indeed, by Lemma 4.1, any H-component $W_i \subset F^{Km_0}(W)$ satisfies $|W_i| < \delta_0$. By applying (24) to the H-components $W_i \subset F^{Km_0}(W)$, (24) and (25) imply (31).

To derive (29), we write

$$\mathcal{Z}_1(F_*^m(\ell)|_{W_i:\boldsymbol{T}'_{N,N'}(W_i)>m})$$

$$= \mathcal{Z}_1(F_*^m(\ell)|_{W_i:\boldsymbol{T}_{N,N'}(W_i)>m}) + \mathcal{Z}_1(F_*^m(\ell)|_{W_i:W_i\subset\mathcal{E}_N\mathcal{B}_{N'},\boldsymbol{T}'_{N,N'}(W_i)>m}) =: Z_{11} + Z_{12}$$

By (24), $Z_{11} \leq C_5\mathcal{Z}_1(\ell)$. Let us write $j \in \mathcal{J}$ if the H-component $W_{j,m-1} \subset F^{m-1}(W)$ contains a point $x \in W_{j,m-1}$ with $F(x) \in \mathcal{E}_N\mathcal{B}_{N'}$ and $\boldsymbol{T}_{N,N'}(W_j) > m-1$. Also write $\ell_j = (W_j, \rho_{j,m-1})$. Choosing $\delta_0' \leq (\tilde{\delta})^{3^{2(Km_0+1)}}$ for some $\tilde{\delta} \leq \delta_0$, we have $|W_j| \leq \tilde{\delta}$.

We claim that there is some $\tilde{\delta} < \delta_0$ and C_7 so that

$$\mathcal{Z}_1(F_*(\ell_j)) \leq C_7\mathcal{Z}_1(\ell_j) \text{ for all } j \in \mathcal{J}. \tag{32}$$

To prove (32), we first claim that there is a constant C only depending on $\mathcal{D}$ so that for any $N > N_4(1)$ and $N' > N_5(1)$ fixed, and for any $x \in \ell_j$, $\tau(x) < C$. Indeed, it is not possible for x to have a long flight in a type 2 or type 3 corridor since $x \notin \mathcal{E}_N\mathcal{B}_{N'}$. Let x be so that $F(x) \in \mathcal{E}_N\mathcal{B}_{N'}$. Then it is also not possible for x to have a long flight in a type 1 corridor, because in this case $F(x)$ would be close to a boundary point of a type 1 corridor and we could not have $F(x) \in \mathcal{E}_N\mathcal{B}_{N'}$. Thus $\tau(x) < C$. This estimate can be extended to all $x \in \ell_j$ by choosing $\tilde{\delta} < \delta_0$ (for example, smaller than half of the smallest distance between two distinct points in $A = \cup_{corridors}A_H$). Now since the free flight function on ℓ_j is uniformly bounded, (32) follows from [14].

We conclude

$$Z_{12} \leq \sum_{j\in\mathcal{J}} c_j\mathcal{Z}_1(F_*(\ell_j)) \leq C_7\sum_{j\in\mathcal{J}} c_j\mathcal{Z}_1(\ell_j)$$

$$\leq C_7\mathcal{Z}_1(F_*^{m-1}(\ell)|_{W_j:\boldsymbol{T}_{N,N'}(W_j)>m-1}) \leq C_7C_5\mathcal{Z}_1(\ell).$$

Thus (29) follows with $C_6 = C_5(1+C_7)$. The derivation of (30) from (31) is similar. ∎

Recall that two long flights are possible in a type 2 corridor, right after one another or separated by exactly one short flight. Our next lemma says that the $\mathcal{Z}_q$ function can be controlled throughout the course of these two long flights.

Lemma 5.8. *There is some C_8 so that for any standard pair $\tilde\ell = (\tilde W, \tilde\rho)$ with $\tilde W \subset \mathcal{E}_{N_4(4)}\mathcal{B}_{N_5(4)}$,*

$$\mathcal{Z}_q(F_*^3(\tilde\ell)) \le C_8 \mathcal{Z}_q(\tilde\ell).$$

We finish the proof of the theorem first and then will prove Lemma 5.8. First, we fix a large constant K so that

$$C_6 \bar c^4 2^{-qK}(1 + (2Km_0)C_8C_6\bar c^4) < \frac{1}{2} \tag{33}$$

holds. Next, we choose $M = 2Km_0 + 6$. Then, we choose $\tilde N = N_4(M)$, $\tilde N' = N_5(M)$. Note that by Lemma 5.5, for every $x \in \mathcal{M}$ there exists $m = 0, ..., M$ so that

$$\{n : 1 \le n \le M, F^n(x) \in \mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}\} \subset \{m, m+1, m+2\}. \tag{34}$$

Finally, we fix $\delta_0' = \delta_0'(\tilde N, \tilde N', K)$ as given by Lemma 5.7. Then we choose δ_1 so small that for any W with $|W| < \delta_1$, for any $m = 1, ..., M$, any H-component of $F^m(W)$ is shorter than δ_0' (e.g., $\delta_1 = (\delta_0')^{3^M}$ works by Lemma 4.1).

We are going to prove Theorem 3.2 with $\varkappa = 1/2$, $\delta = \delta_1$ and M as chosen above.

Note that all standard pairs in the proof are shorter than δ_0'. Thus, by further reducing δ_0' if necessary, we can assume that any standard pair intersecting $\mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$ is fully contained in $\mathcal{E}_{\tilde N-1}\mathcal{B}_{\tilde N'-1}$. To simplify notations, we will assume that once a standard pair intersects $\mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$, it is also fully contained in $\mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$.

Let us fix a standard pair $\ell = (W, \rho)$ with $|W| < \delta_1$, let $W_{i,m}$ denote an H-component of $F^m(W)$ and write $F_*^M(\ell) = \sum_i c_{i,M}\ell_i$, where $\ell_i = (W_{i,M}, \rho_{i,M})$. The idea of the proof is now the following. Let the time of the first visit to $\mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$ be m. Then no visit to $\mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$ is possible any time after $m+2$ by Lemma 5.5. If $m \le M/2$, then $M - m - 3 > Km_0$ and so we can use (30) after the last visit to show that the Z function does not grow. Likewise, if $m > M/2$, we will use (30) at time zero (before the visit).

To make this idea precise, for all standard pairs $\ell_{i,M}$ we associate a set $\mathcal{T}_i$ of integers so that for any $x \in F^{-M}(W_{i,M})$, we have $F^k(x) \in \mathcal{E}_{\tilde N}\mathcal{B}_{\tilde N'}$ for $k = 0, 1, ..., M-1$ if and only if $k \in \mathcal{T}_i$. By Lemma 5.5, all associated sets $\mathcal{T}$ can contain up to 2 integers. Furthermore, if $\mathcal{T}$ contains exactly 2 numbers, then $\mathcal{T} = \{m, m+1\}$ or $\mathcal{T} = \{m, m+2\}$ for some $m = 0, ..., M-1$. We have $\mathcal{Z}_q(F_*^M(\ell)) = \sum_i c_{i,M}\mathcal{Z}_q(\ell_{i,M})$. Now let

$$Z_m = \mathcal{Z}_q(F_*^M(\ell)|_{W_i:\min \mathcal{T}_i = m})$$

and

$$Z_{M+} = \mathcal{Z}_q(F_*^M(\ell)|_{W_i:\mathcal{T}_i=\emptyset}) = \mathcal{Z}_q(F_*^M(\ell)|_{W_i:\boldsymbol{T}'_{\tilde N,\tilde N'}(W_i)>M}).$$

Clearly, we have

$$\mathcal{Z}_q(F_*^M(\ell)) = \left[\sum_{m=0}^{M-1} Z_m\right] + Z_{M+}. \tag{35}$$

Given a substandard family $\mathcal{G} = (\ell_\alpha = (W_\alpha, \rho_\alpha)_{\alpha\in\mathbb{N}}, \lambda)$ (recall that substandard means $s = \sum_{\alpha=1}^\infty \lambda_\alpha \le 1$), we have

$$\mathcal{Z}_q(\mathcal{G}) = \sum_{\alpha=1}^\infty \lambda_\alpha \mathcal{Z}_q(\ell_\alpha) \le \bar c s \sum_{\alpha=1}^\infty \frac{\lambda_\alpha}{s}\frac{1}{|W_\alpha|^q}$$
$$\le \bar c s \left[\sum_{\alpha=1}^\infty \frac{\lambda_\alpha}{s}\frac{1}{|W_\alpha|}\right]^q \le \bar c^2 s^{1-q}[\mathcal{Z}_1(\mathcal{G})]^q$$

$$\leq \bar{c}^2[\mathcal{Z}_1(\mathcal{G})]^q, \tag{36}$$

where we used (23) in the first two lines and Jensen's inequality in the second line. Now combining (30) with (36), we find

$$Z_{M+} \leq \bar{c}^2[C_6 2^{-K}\mathcal{Z}_1(\ell)]^q \leq \bar{c}^2\left[C_6 2^{-K}\bar{c}\frac{1}{|W|}\right]^q \leq C_6\bar{c}^4 2^{-qK}\mathcal{Z}_q(\ell). \tag{37}$$

Next, fix some $m \in [0, M/2]$ and consider the substandard family

$$\mathcal{G}_m = (\ell_{i,m} = (W_{i,m}, \rho_{i,m})_{i\in\mathcal{I}_m}, \lambda_i = c_{i,m}),$$

where $i \in \mathcal{I}_m$ if

$$W_{i,m} \in \mathcal{E}_{\tilde{N}}\mathcal{B}_{\tilde{N}'}, \boldsymbol{T}'_{\tilde{N},\tilde{N}'}(W_{i,m}) > m.$$

Note that $\mathcal{G}_m$ corresponds to the image under F^m of the points in W whose first hitting time of the set $\mathcal{E}_{\tilde{N}}\mathcal{B}_{\tilde{N}'}$ is exactly m and so

$$Z_m = \mathcal{Z}_q(F_*^{M-m}(\mathcal{G}_m)).$$

By (29),

$$\mathcal{Z}_1(\mathcal{G}_{M-m}) \leq C_6\mathcal{Z}_1(\ell).$$

Now using (36) we compute as in (37) that

$$\mathcal{Z}_q(\mathcal{G}_m) \leq \bar{c}^2[\mathcal{Z}_1(\mathcal{G}_m)]^q \leq \bar{c}^2[C_6\mathcal{Z}_1(\ell)]^q \leq C_6\bar{c}^4\mathcal{Z}_q(\ell). \tag{38}$$

Now fix some $\ell_{i,m} = (W_{i,m}, \rho_{i,m}) \in \mathcal{G}_m$. By Lemma 5.8, we have

$$\mathcal{Z}_q(F_*^3(\ell_{i,m})) \leq C_8\mathcal{Z}_q(\ell_{i,m}). \tag{39}$$

Now fix any $W_{j,m+3} \in F^3(W_{i,m})$. By (36),

$$\mathcal{Z}_q(F_*^{M-m-3}(\ell_{j,m+3})) \leq \bar{c}^2[\mathcal{Z}_1(F_*^{M-m-3}(\ell_{j,m+3}))]^q.$$

By (34), no points of $W_{j,m+3}$ can visit $\mathcal{E}_{\tilde{N}}\mathcal{B}_{\tilde{N}'}$ for $M-m-3$ iterations. Combining this observation with the fact that $M-m-3 \geq Km_0$ and with (30), we find

$$\mathcal{Z}_q(F_*^{M-m-3}(\ell_{j,m+3})) \leq \bar{c}^2[C_6 2^{-K}\mathcal{Z}_1(\ell_{j,m+3}))]^q \leq C_6\bar{c}^4 2^{-qK}\mathcal{Z}_q(\ell_{j,m+3}). \tag{40}$$

Now combining (39) and (40) we find

$$Z_m = \mathcal{Z}_q(F_*^{M-m}(\mathcal{G}_m)) \leq C_8C_6\bar{c}^4 2^{-qK}\mathcal{Z}_q(\mathcal{G}_m),$$

and so by (38)

$$Z_m \leq C_8(C_6)^2\bar{c}^8 2^{-qK}\mathcal{Z}_q(\ell). \tag{41}$$

Finally, let us consider the case $m \in [M/2+1, M]$ and define $\mathcal{G}_m$ as before. Since $m > Km_0$, we have by (30) that

$$\mathcal{Z}_1(\mathcal{G}_m) \leq C_6 2^{-K}\mathcal{Z}_1(\ell),$$

and so by (36),

$$\mathcal{Z}_q(\mathcal{G}_m) \leq C_6 2^{-qK}\bar{c}^4\mathcal{Z}_q(\ell).$$

Now Lemma 5.8 implies

$$\mathcal{Z}_q(F_*^3(\mathcal{G}_m)) \leq C_8\mathcal{Z}_q(\mathcal{G}_{M-m}).$$

Finally, for any $\ell_j \in F_*^3(\mathcal{G}_m)$, we combine (36), (34) and (29) to conclude

$$\mathcal{Z}_q(F_*^{M-m-3}(\ell_j)) \leq \bar{c}^2[\mathcal{Z}_1(F_*^{M-m-3}(\ell_j))]^q \leq \bar{c}^2[C_6\mathcal{Z}_1(\ell_j)]^q \leq C_6\bar{c}^4\mathcal{Z}_q(\ell_j).$$

Combining the last three displayed inequalities, we obtain

$$Z_m \leq C_8(C_6)^2\bar{c}^8 2^{-qK}\mathcal{Z}_q(\ell). \tag{42}$$

Now we substitute the estimates (37), (41) and (42) into (35) to conclude

$$\mathcal{Z}_q(F_*^M(\ell)) \le C_6\bar{c}^4 2^{-qK}(1+(2Km_0)C_8C_6\bar{c}^4)\mathcal{Z}_q(\ell). \tag{43}$$

The right hand side of (43) is bounded by $\mathcal{Z}_q(\ell)/2$ by (33). This completes the proof of Theorem 3.2 assuming Lemma 5.8. In the remaining part of this section, we give a proof of this lemma.

Proof of Lemma 5.8. Let us write $N_5 = N_5(4)$ and $N_4 = N_4(4)$. We will distinguish three cases.

Case 1: $\tilde{W} \subset \mathcal{B}_{N_5}$. By Lemma 5.3, we have

$$\mathcal{Z}_q(F_*(\tilde{\ell})) < C_4\mathcal{Z}_q(\tilde{\ell})$$

and by Lemma 5.5, we have

$$(F(\tilde{W}) \cup F^2(\tilde{W})) \cap (\mathcal{E}_{N_4}\mathcal{B}_{N_5}) = \emptyset. \tag{44}$$

Consequently, as in (32), for any standard pair $\tilde{\ell}' = (\tilde{W}', \tilde{\rho}')$ in the standard family $F_*(\tilde{\ell})$, we have

$$\mathcal{Z}_1(F_*^2(\tilde{\ell}')) \le C_7^2\mathcal{Z}_1(\tilde{\ell}'). \tag{45}$$

Using (36), we conclude

$$\mathcal{Z}_q(F_*^3(\tilde{\ell})) \le \bar{c}^4 C_4 C_7^2 \mathcal{Z}_q(\tilde{\ell}).$$

Case 2: $\tilde{W} \subset \mathcal{E}(x_h, N_4)$ for some type 3 boundary point x_h. By Lemma 5.2, we have

$$\mathcal{Z}_1(F_*(\tilde{\ell})) \le C_3\mathcal{Z}_1(\tilde{\ell}). \tag{46}$$

As in case 1, (44) and (45) hold. Thus $\mathcal{Z}_1(F_*^3(\tilde{\ell})) \le C_7^2 C_3 \mathcal{Z}_1(\tilde{\ell})$ and so

$$\mathcal{Z}_q(F_*^3(\tilde{\ell})) \le \bar{c}^4 C_7^2 C_3 \mathcal{Z}_q(\tilde{\ell}).$$

Case 3: $\tilde{W} \subset \mathcal{E}(x_h, N_4)$ for some type 2 boundary point x_h. As in case 2, (46) holds. By Lemma 4.2 and by the choice of N_5, we can write

$$F(\tilde{W}) = \left(\cup_{i\in I_1}\tilde{W}_i\right) \cup \left(\cup_{i\in I_2}\tilde{W}_i\right) \cup \left(\cup_{i\in I_3}\tilde{W}_i\right),$$

where $\tilde{W}_i \subset \mathcal{B}_{N_5}$ for all $i \in I_1$, $F(\tilde{W}_i) \subset \mathcal{B}_{N_5}$ for all $i \in I_2$ and $(\tilde{W}_i \cup F(\tilde{W}_i)) \cap (\mathcal{E}_{N_4}\mathcal{B}_{N_5}) = \emptyset$ for all $i \in I_3$. As in case 1, we derive

$$\mathcal{Z}_q(F_*^2(\tilde{\ell}_i)) < \bar{c}^4 C_4 C_7^2 \mathcal{Z}_q(\tilde{\ell}_i)$$

for all $i \in I_1$. For $i \in I_2$, we have

$$\mathcal{Z}_1(F_*(\tilde{\ell}_i)) \le C_7\mathcal{Z}_1(\tilde{\ell}_i)$$

as in (32). Next, for any $\tilde{\ell}_{i,j} \in F_*(\tilde{\ell}_i)$ with $i \in I_2$,

$$\mathcal{Z}_q(F_*(\tilde{\ell}_{i,j})) < C_4\mathcal{Z}_q(\tilde{\ell}_{i,j})$$

by Lemma 5.3. Finally, for $i \in I_3$, we have

$$\mathcal{Z}_1(F_*^2(\tilde{\ell}_i)) \le C_7^2\mathcal{Z}_1(\tilde{\ell}_i)$$

as in (32).

Combining the above estimates, we obtain

$$\mathcal{Z}_q(F_*^3(\tilde{\ell})) \le \bar{c}^6 C_3[3C_4C_7^2]\mathcal{Z}_q(\tilde{\ell}).$$

The lemma follows with $C_8 = 3\bar{c}^6 C_3 C_4 C_7^2$.

□

6. **Proof of Theorem 3.3.** Theorem 3.3 is quite intuitive. Indeed, conditions (A1) and (A2) prescribe degeneracies in the geometry which can be easily destroyed by a small perturbation (e.g., the genericity of (A1) was stated in [20] without a proof). It is not difficult to turn this intuition into a rigorous proof, but we decided to include such a proof for completeness.

Let us fix some combinatorial data $(d, J_1, ..., J_d)$. Since $\boldsymbol{D}$ is a disjoint union of the open sets $\boldsymbol{D}_{d,J_1,...,J_d}$, it is enough to prove the theorem for $\boldsymbol{D}_{d,J_1,...,J_d}$. To simplify the notation, we will drop the subscript and only write $\boldsymbol{D}$ instead of $\boldsymbol{D}_{d,J_1,...,J_d}$ in the sequel.

We say that an *incipient corridor* H is a direction $v = v_H \in [0, \pi)$ and a connected subset Q_H of $\mathcal{D}$ with empty interior satisfying (1). The difference between corridors and incipient corridors is that in case of the latter one, Q_H has empty interior. That is, the configurational component of an infinite orbit that only experiences grazing collisions, but does so on both sides of the flight, constitutes an incipient corridor.

Now define the set $\boldsymbol{D}_0 \subset \boldsymbol{D}$ as the set of billiard tables $\mathcal{D}$ that satisfy (A1) and (A2) and do not have incipient corridors. We are going to prove that $\boldsymbol{D}_0$ is open and dense. This clearly implies the theorem.

Step 1: $\boldsymbol{D}_0$ is open

Given $\mathcal{D} \in \boldsymbol{D}_0$, we need to find $\varepsilon > 0$ so that U, the ε neighborhood of $\mathcal{D}$, is contained in $\boldsymbol{D}_0$. For $\mathcal{D} \in \boldsymbol{D}_0$, let κ_+ denote the maximal curvature at regular points. Then $\mathbb{T}^2 \setminus \mathcal{D}$ contains a disc of radius κ_+^{-1}. By choosing $\varepsilon < \kappa_+^{-1}/2$, we ensure that for all $\mathcal{D}' \subset U$ there is a disc of radius $\kappa_+^{-1}/2$ inside $\mathbb{T}^2 \setminus \mathcal{D}'$.

Next we claim that there is a finite set $\mathcal{V} \subset \mathcal{S}^1$ so that for any $\mathcal{D}' \subset U$ and for any corridor H on $\mathcal{D}'$, the direction of H satisfies $v_H \in \mathcal{V}$. To prove the claim, first observe that for any direction v_H, $\tan v_H$ is rational. Indeed, if it was not rational, then the set $\{q+tv_H\}_{t\in\mathbb{R}}$ would be dense in $\mathbb{T}^2$. Now assume that $v_H \in [0, \pi/4]$ (the other cases are similar). Let us write $\tan v_H = P/Q$ where $0 < P < Q$ are coprime integers. Then necessarily $Q < 3\kappa_+$ because otherwise the set $\mathbb{T}^2 \setminus \{q + tv_H\}_{t\in\mathbb{R}}$ would not contain a ball of radius $\kappa_+^{-1}/2$. The claim follows.

Let $\mathcal{V}_0 \subset \mathcal{V}$ be the set of directions in which there is a corridor on $\mathcal{D}$ and let $v \in \mathcal{V} \setminus \mathcal{V}_0$. Now we claim that there is some $\delta_v > 0$ so that for any $q \in \mathbb{T}^2$, the line $q + tv$, $t \in \mathbb{R}$ intersects with the complement of the δ_v neighborhood of $\mathcal{D}$. Indeed, this follows from the assumption that $\mathcal{D}$ does not have incipient corridors and from compactness. Likewise, for any $v \in \mathcal{V}_0$ there is some $\delta_v > 0$ so that for any $q \notin \cup_{H:v_H=v}\boldsymbol{B}_{\delta_v}(Q_H)$ the line $q + tv$, $t \in \mathbb{R}$ intersects with the complement of the δ_v neighborhood of $\mathcal{D}$. Here, $\boldsymbol{B}_\rho(Q)$ means the ρ neighborhood of $Q \subset \mathbb{T}^2$. Further reducing ε if necessary, we can assume that $\varepsilon < \delta_v$ for all $v \in \mathcal{V}$. Consequently, for all $\mathcal{D}' \in U$, there is an injection from the set of corridors of $\mathcal{D}'$ to the set of corridors of $\mathcal{D}$ preserving the angle of the corridors. Indeed, by the choise of ε, no new corridor can open up if we perturb $\mathcal{D}$ with an ε small $\mathcal{C}^3$ (in fact $\mathcal{C}^0$) perturbation. It may be possible at this point that some corridors close during the perturbation, which we rule out next.

Now since $\mathcal{D}$ satisfies (A1), the following is true. For any corridor H on $\mathcal{D}$, we can find some $\varepsilon_H > 0$ so that for any q in the ε_H neighborhood of Q_H,

$$\{q + tv_H : t \in \mathbb{R}\} \cap (\mathbb{T}^2 \setminus \mathcal{D}) \subset \boldsymbol{B}_{\varepsilon_H}(B_H).$$

Here, $B_H = \partial Q_H \cap \partial\mathcal{D}$ has two elements by (A1). Further reducing ε as necessary, we can assume $\varepsilon < \varepsilon_H/2$ for all corridors H on $\mathcal{D}$. Now by construction for any corridor H on $\mathcal{D}$ and for any $\mathcal{D}' \in U$, we can find a correspoding corridor H' on $\mathcal{D}'$ so that $v_H = v_{H'}$ and the symmetric difference of Q_H and $Q_{H'}$ is contained in the ε neighborhood of the boundary of Q_H. In particular, the injection constructed in the previous paragraph is now a bijection. Furthermore, $B_{H'} = \partial Q_{H'} \cap \partial \mathcal{D}'$ has two elements. We conclude that $\mathcal{D}'$ satisfies (A1) and has no incipient corridors.

Finally, since $\mathcal{D}$ satisfies (A2), there is some angle $\alpha > 0$ so that for any type 2 or 3 corridor H and for any boundary corner point $q_H \in B_H$, the angle between v_H and any one-sided tangent to $\partial \mathcal{D}$ at q_H is bigger than α. Further reducing ε if necessary, we can assume $\varepsilon < \alpha/2$. This guarantees that all $\mathcal{D}' \in U$ satisfy (A2). It follows that $\boldsymbol{D}_0$ is open.

Step 2: $\boldsymbol{D}_0$ is dense

We will need the following simple lemma.

Lemma 6.1. *(Local enlargement) Let $\mathcal{D}$ be an admissible billiard table, $q \in \partial \mathcal{D}$ and $\varepsilon > 0$. Then there exists another admissible billiard table $\tilde{\mathcal{D}}$ so that*

- $d(\mathcal{D}, \tilde{\mathcal{D}}) < \varepsilon$
- *$\mathcal{D}$ and $\tilde{\mathcal{D}}$ coincide on the complement of the ε neighborhood of q*
- *$\tilde{\mathcal{D}} \subset \mathcal{D}$ with q being in the interior of $\mathbb{T}^2 \setminus \tilde{\mathcal{D}}$*

Proof. Assume that $q \in \Gamma_{i,j}$ is a regular point. Then we can represent $\Gamma_{i,j}$ in a small neighborhood of q in local coordinates as a graph of a concave function $f : [-1, 1] \to \mathbb{R}^2$ with $f(0) = 0$. Fix a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ so that $\phi(0) = 1$ and ϕ is identically zero outside of $(-1/2, 1/2)$. Let the curvature of Γ at q be κ and $\varepsilon' = \min\{\varepsilon, \kappa\}/(10\|\phi\|_{\mathcal{C}^3})$. Now define $\tilde{\mathcal{D}}$ to be the same as $\mathcal{D}$ except that the image of f is replaced by the image of $\tilde{f} = (1 + \phi) f$. By construction, $\tilde{\mathcal{D}}$ is an admissible table satisfying the requirements.

The case of corner points is similar, we just need to perturb both curves meeting at the corner point. □

To prove that $\boldsymbol{D}_0$ is dense, fix an arbitrary $\mathcal{D} \in \boldsymbol{D}$ and $\varepsilon > 0$. We need to find some $\hat{\mathcal{D}} \in \boldsymbol{D}_0$ with $d(\mathcal{D}, \hat{\mathcal{D}}) < \varepsilon$. In the remaining part of the proof, the term corridor can stand for either non-incipient or incipient corridor.

Let us denote by U the ε neighborhood of $\mathcal{D}$. Reducing ε if necessary, we can assume as in Step 1 that there is a finite set $\mathcal{V}$ so that for any $\mathcal{D}' \in U$ and for any corridor H on $\mathcal{D}'$, $v_H \in \mathcal{V}$. Furthermore, for any given $v \in V$, there may only be a bounded number of corridors with direction v. Let us fix some ordering of the corridors. E.g. fix arbitrary ordering on $\mathcal{V}$ and define $H_1 < H_2$ if $v_{H_1} < v_{H_2}$. For corridors H_1, H_2 with $v_{H_1} = v_{H_2}$, project Q_{H_i} to the direction perpendicular to v_{H_1} (when $\mathbb{T}^2$ is identified with the unit square). If the projections are denoted by πQ_{H_1}, πQ_{H_2}, then define $H_1 < H_2$ if the origin is closer to πQ_{H_1} than to πQ_{H_2}.

We are going to consider billiard tables $\mathcal{D}' \in U$ with $\mathcal{D}' \subset \mathcal{D}$. This guarantees that no new corridors open up by the perturbation, that is there is an injection $\iota_{\mathcal{D}'}$ from the set of corridors on $\mathcal{D}'$ to the set of corridors on $\mathcal{D}$ that preserves the angle and the ordering. Note however that this time ι may not be a bijection as we want to eliminate incipient corridors. Let $H_1 < H_2 < ... < H_k$ be the ordered list of corridors of $\mathcal{D}$. Let us say that a corridor on a billiard table $\mathcal{D}'$ is good if it is non-incipient and does not violate (A1) and (A2).

We are going to define $\mathcal{D} = \mathcal{D}'_0, \mathcal{D}'_1, ..., \mathcal{D}'_k = \hat{\mathcal{D}}$ in a way that for every $i = 1, ..., k$,

- $d(\mathcal{D}'_i, \mathcal{D}'_{i+1}) < \varepsilon/2k$

- the corridors in $\iota^{-1}_{\mathcal{D}'_i}(\{H_1, ..., H_i\})$ are all good.

If these items can be guaranteed, then it follows that $\hat{\mathcal{D}} \in \boldsymbol{D}_0$ and $d(\mathcal{D}, \hat{\mathcal{D}}) < \varepsilon$, which completes the proof. We prove the above items by induction. Assume they hold for i. If $\iota^{-1}_{\mathcal{D}'_i}(\{H_{i+1}\}) = \emptyset$, then we define $\mathcal{D}'_{i+1} = \mathcal{D}'_i$. Next assume that there is a corridor H' on $\mathcal{D}'_i$ with $\iota_{\mathcal{D}'_i}(H') = H_{i+1}$. If H' is good, then we define $\mathcal{D}'_{i+1} = \mathcal{D}'_i$. Let us now assume that H' is either incipient or violates (A1) or (A2). In all cases, we can apply the local enlargement lemma with $\mathcal{D}, \varepsilon$ replaced by $\mathcal{D}_i, \delta_{i+1} < \varepsilon/2k$ at some point q_{i+1} to produce another billiard table $\mathcal{D}'_{i+1}$ with either $\iota^{-1}_{\mathcal{D}'_{i+1}}(\{H_{i+1}\}) = \emptyset$ (in case H' was incipient) or $\iota^{-1}_{\mathcal{D}'_{i+1}}(H_{i+1})$ is a good corridor. Indeed, if H' is incipient, then we apply the local enlargement lemma at a point $q_{i+1} \in H' \cap \partial \mathcal{D}'_i$. If H' violates (A1), then it has several boundary points on at least one of its sides. Now we apply the local enlargement lemma at one of these boundary points. Finally, if H' violates (A2), then we apply the local enlargement lemma at the given boundary corner point. Clearly, the perturbation can be made in a way that the direction of the half-tangents is modified and so $\iota^{-1}_{\mathcal{D}'_{i+1}}(H_{i+1})$ will not violate (A2).

Finally, we claim that by choosing δ_{i+1} small, we can guarantee that the corridors in $\iota^{-1}_{\mathcal{D}'_{i+1}}(\{H_1, ..., H_i\})$ are all good, too. Note that this is not entirely obvious as a corner point can be on the boundary of multiple corridors (with different directions) and so the perturbation at iteration $i+1$ may change $\iota^{-1}_{\mathcal{D}'_i}(H_j)$ with $j \leq i$. However, Step 1 ensures that there is some $\delta_{i+1} \in (0, \varepsilon/2k)$ so that δ_{i+1} small $\mathcal{C}^3$ perturbations preserve the goodness of corridors. This completes the poof of the induction. It follows that $\boldsymbol{D}_0$ is dense.

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