ASYMPTOTIC EXPANSION OF CORRELATION FUNCTIONS FOR $\mathbb{Z}^d$ COVERS OF HYPERBOLIC FLOWS.

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Abstract. We establish expansion of an arbitrary order for the correlation function of sufficiently regular observables of $\mathbb{Z}^d$ extensions of some hyperbolic flows. Our examples include the $\mathbb{Z}^2$ periodic Lorentz gas and geodesic flows on abelian covers of compact manifolds with negative curvature.

Résumé. *Nous établissons des développements asymptotiques de tous ordres pour la fonction corrélation d'observables suffisamment régulières de $\mathbb{Z}^d$-extensions de flots hyperboliques. Nos résultats s'appliquent au gaz de Lorentz $\mathbb{Z}^2$-periodique et au flot géodesique sur des revêtements abéliens de variétés compactes de courbure négative.*

1. Introduction

1.1. **Setup.** Let (M, ν, T) be a probability preserving dynamical system. Consider $(\tilde{M}, \tilde{\nu}, \tilde{T})$– the $\mathbb{Z}^d$-extension of (M, ν, T) by $\kappa : M \to \mathbb{Z}^d$ for a positive integer d. Let $(\Phi_t)_{t\geq 0}$ be the suspension semiflow over (M, ν, T) with roof function $\tau : M \to (0, +\infty)$ and let $(\widetilde{\Phi}_t)_{t\geq 0}$ be the corresponding $\mathbb{Z}^d$ cover. That is, $(\widetilde{\Phi}_t)_{t\geq 0}$ is the semi-flow defined on

$$\widetilde{\Omega} := \{(x, \ell, s) \in M \times \mathbb{Z}^d \times [0, +\infty) \,:\, s \in [0, \tau(x))\}$$

such that $\widetilde{\Phi}_t(x, \ell, s)$ corresponds to $(x, \ell, s+t)$ by identifying (x, ℓ, s) with $(Tx, \ell+\kappa(x), s-\tau(x))$. We will consider throughout this article that κ and τ are bounded. The semi-flow $\widetilde{\Phi}$ preserves the restriction $\tilde{\mu}$ on $\widetilde{\Omega}$ of the product measure $\nu \otimes \mathfrak{m} \otimes \mathfrak{l}$, where $\mathfrak{m}$ is the counting measure on $\mathbb{Z}^d$ and $\mathfrak{l}$ is the Lebesgue measure on $[0, +\infty)$.

In the present paper we study the following correlation functions

$$C_t(f, g) := \int_{\widetilde{\Omega}} f.g \circ \widetilde{\Phi}_t \, d\tilde{\mu} \,,$$

as t goes to infinity, for suitable observables f, g. Our goal is to establish expansions of the form

$$C_t(f, g) = \sum_{k=0}^{K} C_k(f, g)\, t^{-\frac{d}{2}-k} + o(t^{-\frac{d}{2}-K})\,. \tag{1.1}$$

More precisely we assume that Φ_t is C^∞ away from singularities, which is a finite (possibly empty) union of positive codimension submanifolds. We say that $\tilde{\Phi}_t$ *admits a complete asymptotic expansion in inverse powers of* t if for f and g which are C^∞ and have compact support which is disjoint from the singularities of $\tilde{\Phi}$, the correlation function $C_t(f, g)$ admits the expansion (1.1) for each $K \in \mathbb{N}$.

The precise statement of our results will be given in Sections 2–4. Here we mention some important applications.

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Theorem 1.1. *Complete asymptotic expansion in inverse powers of t holds for finite horizon periodic Lorentz gases and geodesic flows on abelian covers of negatively curved manifolds.*

In fact, our results are more general than Theorem 1.1. Namely,

- we consider an abstract setup applicable to other hyperbolic flows;
- we allow the support of f and g to be unbounded (provided they decay sufficiently fast);
- we allow f and g to take non-zero values on the singularities of the flow. In addition, we allow them to be only Hölder continuous (note that continuity is required in the flow direction as well) with one of them being C^∞ in the flow direction.

1.2. **Related results.** The correlation function (1.1) has been studied by several authors. The leading term ($K = 0$) for hyperbolic maps (for functions of non-zero integral) is sometimes called mixing, Krickeberg mixing or local mixing. In case of $\mathbb{Z}^d$ extensions as above, it is a consequence of some versions of the local limit theorem. See related results in e.g. [1, 22–24, 44]. Less is known about higher order expansions for maps, but see the recent results in [42]. For flows, the leading term has been studied in e.g. [2, 15, 27, 45]. We also mention that there are other quantities besides the correlation functions whose asymptotic expansions are of interest. In particular, the asymptotic expansions have been obtained (using techniques similar to ones employed in the present paper) for the rate of convergence in the central limit theorem [19, 20] and for the number of periodic orbits in a given homology class [33, 41]. The relation of mixing with the other above mentioned problems is the following. First, the problem of counting periodic orbits can be reformulated as a special case of the mixing problem $\int A(x, \tilde{\Phi}_t x) d\mu(x)$ in case $A(\cdot, \cdot)$ is a distribution supported on the diagonal (see [30, 34]). Also given a function $\tau : M \to \mathbb{R}$ of zero mean, one can consider a skew product on $M \times \mathbb{R}$ defined by

$$F(x, s) = (Tx, s + \tau(x)).$$

Studying the higher order terms in the mixing local limit

$$\sqrt{n}\mu\left(f(x)g(T^n x)\phi\left(\tau_N - z\sqrt{N}\right)\right) = \Psi(z)\mu(f)\mu(g)\int_{\mathbb{R}}\phi(s)ds + \dots$$

where ϕ is a compactly supported test function, Ψ is a Gaussian density, gives, in the special case $z = 0$, the asymptotic expansions of correlation functions for F similar to the one considered in the present paper. While these relations are useful in computing the main terms in the asymptotic expansions, and consequently similar techniques may be applicable in all these cases, the study of a more complete expansion seems simpler when performed in each case, separately.

To comment on our smoothness assumption, we note that a comparison of the results of [19, 20] for smooth ϕ with the results of [7, 14] for the case where ϕ is an indicator shows that the assumption on the smoothness of ϕ is essential to derive expansions of the form (1.1). Therefore we expect that the assumption that some observable is smooth in the flow direction is essential. It is an interesting open problem to obtain corrections to our asymptotic expansions for non-smooth observables.

Other results are known for some hyperbolic systems preserving an infinite measure which may not be a $\mathbb{Z}^d$ cover and so the powers may be different from $-\frac{d}{2} - k$. See the leading term in e.g. [16, 38, 40] and expansions in e.g. [31, 36, 37]. We note that the expansions in the above papers are of the form $\phi(t)\tilde{\mu}(f)\tilde{\mu}(g)$ where $\phi(t)$ admits an expansion of the form $\phi(t) = \sum_{k=1}^{K} a_k t^{-\beta_k} + o\left(t^{-\beta_K}\right)$. Thus these expansions do not provide a leading term in the case $\tilde{\mu}(f)\tilde{\mu}(g) = 0$. In contrast, our expansion gives a leading term for a large class of zero mean observables. The main reason we are able to obtain the complete expansion in our problem is that the leading eigenvalue of the appropriate transfer operator is smooth near the origin in our setting but not in the setting considered in [31, 36, 37]. We note that recently [11] obtained asymptotic expansions of the leading eigenvalue in a non-smooth setting, so obtaining complete asymptotic expansions beyond the abelian covers is the natural next question.

We note that the results described above discuss mixing for the observables concentrated in a compact part of the phase space. Recently, a number of papers discuss mixing for extended observables. This topic is beyond the scope of our paper, we refer the readers to [17, 25, 32] and the references wherein for more information on this subject.

1.3. **Layout of the paper.** The rest of the paper is organized as follows. In Section 2, we present some abstract results on expansion of correlation functions for general suspension semi-flows and flows. Theorems 2.1 and 2.3 guarantee that under a list of technical assumptions, expansions of the kind (1.1) hold. The results are proved by a careful study of the twisted transfer operator. One major difference from the case of maps (cf. [42]) is an extra assumption called a *weak non-lattice property* along the lines of [12]. In Section 3 we study billiards and verify the abstract assumptions of Theorem 2.3 for the Lorentz gas obtaining a complete asymptotic expansion in inverse powers of t for that system. In Section 4, we verify the abstract assumptions for geodesic flows on $\mathbb{Z}^d$ covers of compact negatively curved Riemannian manifolds. Some technical computations are presented in the Appendix.

2. Abstract results.

2.1. **Notations.** We will work with symmetric multilinear forms. Let $\mathfrak{S}_m$ be the set of permutations of $\{1, ..., m\}$. We identify the set of symmetric m-linear forms on $\mathbb{C}^{d+1}$ with

$$\mathcal{S}_m := \left\{ A = (A_{i_1,...,i_m})_{(i_1,...,i_m)} \in \mathbb{C}^{\{1,...,d+1\}^m} \ : \ \forall i_1, ..., i_m, \quad \forall \mathfrak{s} \in \mathfrak{S}_m, \ A_{i_{\mathfrak{s}(1)},...,i_{\mathfrak{s}(m)}} = A_{i_1,...,i_m} \right\} .$$

For any $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_k$, we define $A \otimes B$ as the element C of $\mathcal{S}_{m+k}$ such that

$$\forall i_1, ..., i_{m+k} \in \{1, ..., d+1\}, \quad C_{i_1,...,i_{m+k}} = \frac{1}{(m+k)!} \sum_{\mathfrak{s} \in \mathfrak{S}_{m+k}} A_{i_{\mathfrak{s}(1)},...,i_{\mathfrak{s}(m)}} B_{i_{\mathfrak{s}(m+1)} \cdots, i_{\mathfrak{s}(m+k)}} .$$

Note that $\otimes$ is associative and commutative. For any $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_k$ with $k \le m$, we define $A * B$ as the element $C \in \mathcal{S}_{m-k}$ such that

$$\forall i_1, ..., i_{m-k} \in \{1, ..., d+1\}, \quad C_{i_1,,...,i_{m-k}} = \sum_{i_{m-k+1},...,i_m \in \{1,...,d+1\}} A_{i_1,...,i_m} B_{i_{m-k+1},...,i_m} .$$

Note that when $k = m = 1$, $A * B$ is simply the scalar product $A.B$. For any C^m-smooth function $F : \mathbb{C}^{d+1} \to \mathbb{C}$, we write $F^{(m)}$ for its differential of order m, which is identified with a m-linear form on $\mathbb{C}^{d+1}$. We write $A^{\otimes k}$ for the product $A \otimes ... \otimes A$. With these notations, Taylor expansions of F at 0 are simply written

$$\sum_{k=0}^{m} \frac{1}{k!} F^{(k)}(0) * x^{\otimes k} .$$

It is also worth noting that $A * (B \otimes C) = (A * B) * C$, for every $A \in \mathcal{S}_m$, $B \in \mathcal{S}_k$ and $C \in \mathcal{S}_\ell$ with $m \ge k + \ell$.

For any $\nu \otimes \mathfrak{l}$-integrable function $h_0 : M \times \mathbb{R} \to \mathbb{C}$, we set

$$\hat{h}_0(x, \xi) := \int_{\mathbb{R}} e^{i\xi s} h_0(x, s) \, ds ,$$

(this quantity is well defined for ν-a.e. x).

In both Sections 2.2 and 2.3, we will use notations $\lambda_0^{(k)}$, $a_0^{(k)}$, $\Pi_0^{(k)}$ for the k-th derivatives of λ, a and Π at 0. The function a is defined below in (2.4), whereas λ and Π will be introduced

in Assumptions (A1) (at the begining of Section 2.2) and (B3) (at Section 2.3).
We write P for the Perron-Frobenius operator of T with respect to ν, which is defined by:

$$\forall f, g \in L^2(\nu), \quad \int_M Pf \cdot g\, d\nu = \int_M f \cdot (g \circ T)\, d\nu. \tag{2.1}$$

We also consider the family $(P_{\theta,\xi})_{\theta\in[-\pi,\pi]^d,\xi\in\mathbb{R}}$ of operators given by

$$P_{\theta,\xi}(f) := P\left(e^{i\,\theta\cdot\kappa} e^{i\,\xi\tau} f\right). \tag{2.2}$$

Throughout this work, we assume that τ is bounded from both below and above by two positive numbers and $|\kappa|$ is bounded above. To simplify notations, we write $\nu(h) := \int_M h\, d\nu$.

Let Σ be a $(d+1)$-dimensional positive symmetric matrix. We will denote by $\Psi = \Psi_\Sigma$ the $(d+1)$-dimensional centered Gaussian density with covariance matrix Σ:

$$\Psi(s) = \Psi_\Sigma(s) := \frac{e^{-\frac{1}{2}\Sigma^{-1} * s^{\otimes 2}}}{(2\pi)^{\frac{d+1}{2}}\sqrt{\det \Sigma}}. \tag{2.3}$$

In particular, $\Psi^{(k)}$ is the differential of Ψ of order k. Let

$$a_s := e^{-\frac{1}{2}\Sigma * s^{\otimes 2}} \tag{2.4}$$

be the Fourier transform of Ψ. Given a non-negative integer α and a real number γ, we define

$$h_{\alpha,\gamma} : \mathbb{R}^2 \to \mathcal{S}_\alpha, \quad h_{\alpha,\gamma}(s,z) = z^\gamma \Psi^{(\alpha)}\left(\mathbf{0}, s/\sqrt{z/\nu(\tau)}\right) \tag{2.5}$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^d$. This function will appear in the expansion formulas (2.17) and (2.45).

We will use the notations

$$\kappa_n := \sum_{k=0}^{n-1} \kappa \circ T^k \quad \text{and} \quad \tau_n := \sum_{k=0}^{n-1} \tau \circ T^k.$$

Note that with this notation, we have

$$\widetilde{\Phi}_t(x,\ell,s) = (T^n x, \ell + \kappa_n(x), s + t - \tau_n(x)), \quad \text{with } n \text{ s.t.} \quad \tau_n(x) \le s + t < \tau_{n+1}(x).$$

It will be also useful to consider the suspension flow $(\Phi_t)_{t\ge 0}$ over (M,ν,T) with roof function τ which is defined on $\Omega := \{(x,s) \in M \times [0,+\infty) : s \in [0,\tau(x))\}$ and preserves the measure μ which is the restriction of the product measure $\nu \otimes \mathfrak{l}$ to Ω. Note that μ is a finite measure but not necessarily a probability measure.

2.2. **A general result under spectral assumptions.** We start by making some assumptions.

The first assumption is variant of the standard Perron-Frobenious type spectral condition.

(A1) Perron-Frobenius assumption. We say that (A1) holds with positive integers J and K if there is a Banach space of functions from M to $\mathbb{C}$ (denoted by $\mathcal{B}$) such that $\mathcal{B} \hookrightarrow L^1(M,\nu)$ and $\mathbf{1}_M \in \mathcal{B}$. Furthermore, $(P_{\theta,\xi})_{\theta\in[-\pi,\pi]^d,\xi\in\mathbb{R}}$ is a family of linear continuous operators on $\mathcal{B}$ such that there exist constants $b \in (0,\pi]$, $C > 0$, $\vartheta \in (0,1)$, $\beta > 0$ and three functions $\lambda_\cdot : [-b,b]^{d+1} \to \mathbb{C}$ (assumed to be C^{K+3}-smooth) and $\Pi_\cdot, R_\cdot : [-b,b]^{d+1} \to \mathcal{L}(\mathcal{B},\mathcal{B})$ (assumed to be C^{K+1}-smooth) such that $\Pi_0 = \mathbb{E}_\nu[\cdot]\mathbf{1}_M$, and $\tilde{\lambda}_{\theta,\xi} := \lambda_{\theta,\xi} e^{-i\xi\nu(\tau)}$ satisfies

$$\forall k < J, \quad \tilde{\lambda}_0^{(k)} = a_0^{(k)}, \tag{2.6}$$

with a positive definite $(d+1)\times(d+1)$ matrix Σ and, in $\mathcal{L}(\mathcal{B},\mathcal{B})$,

$$\forall s \in [-b,b]^{d+1}, \quad P_s = \lambda_s \Pi_s + R_s, \quad \Pi_s R_s = R_s \Pi_s = 0, \quad \Pi_s^2 = \Pi_s, \tag{2.7}$$

$$\sup_{s\in[-b,b]^{d+1}} \|R_s^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \sup_{\theta\in[-\pi,\pi]^d\setminus[-b,b]^d,\ |\xi|\le b} \|P_{\theta,\xi}^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \le C\vartheta^k. \tag{2.8}$$

Our next assumption helps control the Fourier transform for large values of ξ.

(A2) Weak non-lattice property. We say that $f, g : \widetilde{\Omega} \to \mathbb{C}$ and the Banach space $\mathbb{B}$ satisfy (A2) if there exist two families $(f_\ell)_{\ell\in\mathbb{Z}^d}$ and $(g_\ell)_{\ell\in\mathbb{Z}^d}$ of functions defined on $M \times \mathbb{R} \to \mathbb{C}$ and vanishing outside $\widetilde{\Omega}_0 := \Omega \cup \left(M \times \left[-\frac{\inf\tau}{10}, 0\right]\right)$ such that

$$\forall h \in \{f, g\} \quad \forall (x, \ell, s) \in \widetilde{\Omega}, \quad h(x, \ell, s) = h_\ell(x, s) + h_{\ell+\kappa(x)}(Tx, s - \tau(x)) \,. \tag{2.9}$$

and

$$\sup_{\theta\in[-\pi,\pi]^d} \|P^n_{\theta,\xi}\|_{\mathcal{L}(\mathbb{B},L^1)} \le C|\xi|^\alpha e^{-n\delta|\xi|^{-\alpha}} \tag{2.10}$$

for some suitable positive C, δ and α, and

$$\forall \gamma > 0, \quad \sum_{\ell,\ell'\in\mathbb{Z}^d} \left(\|\hat{f}_\ell(\cdot, -\xi)\|_{\mathbb{B}} \, \|\hat{g}_{\ell'}(\cdot, \xi)\|_\infty \right) = O(|\xi|^{-\gamma}) \,. \tag{2.11}$$

The last and most technical assumption is about regularity of observables: smoothness and quick decay at infinity. This assumption allows for a wide extension of compactly supported $\mathcal{C}^\infty$ test functions.

(A3) Regularity of the observables. We say that $f, g : \widetilde{\Omega} \to \mathbb{C}$ and the Banach spaces $\mathcal{B}$, $\mathbb{B}$ satisfy (A3) if (2.9) holds and one of the families $(f_\ell)_{\ell\in\mathbb{Z}^d}$ and $(g_\ell)_{\ell\in\mathbb{Z}^d}$ is made of functions continuous in the last variable. Furthermore, the following estimates hold true: [1]

$$\int_{\mathbb{R}} \sum_{\ell\in\mathbb{Z}^d} (1 + |\ell|^K)(\|f_\ell(\cdot, u)\|_{\mathcal{B}} + \|g_\ell(\cdot, u)\|_{\mathcal{B}'}) \, du < \infty \,, \tag{2.12}$$

$$\exists p_0, q_0 \in [1, +\infty] \; s.t. \; \frac{1}{p_0} + \frac{1}{q_0} = 1 \quad \text{and} \quad \sum_{\ell,\ell'\in\mathbb{Z}^d} \|f_\ell\|_{L^{p_0}(\nu\otimes\mathfrak{l})} \, \|g_{\ell'}\|_{L^{q_0}(\nu\otimes\mathfrak{l})} < \infty \,, \tag{2.13}$$

$$\sup_{\xi\in\mathbb{R}} \sum_{\ell,\ell'\in\mathbb{Z}^d} \|\hat{f}_\ell(\cdot, -\xi)\|_{\mathcal{B}} \|\hat{g}_{\ell'}(\cdot, \xi)\|_{\mathcal{B}'} < \infty \,. \tag{2.14}$$

Given two positive integers J and p, we write

$$\widetilde{\sum_p} = \sum_{\substack{m,r,q,k\ge 0, j\ge kJ \\ m+j+r+q-2k=2p}} \,, \tag{2.15}$$

i.e. $\widetilde{\sum_p}$ means the sum over $k = 0, ..., \lfloor 2p/(J - 2)\rfloor$, $j = kJ, ..., 2k + 2p$, and then $m, r, q \ge 0$ such that $m + r + q = 2p + 2k - j$.

Theorem 2.1. *Let K and J be two positive integers such that $3 \le J \le K + 3$. Assume (A1), (A2) and (A3) hold with Banach spaces $\mathcal{B}$, $\mathbb{B}$ and some functions $f, g : \widetilde{\Omega} \to \mathbb{C}$. Then*

$$C_t(f, g) = \sum_{p=0}^{\lfloor \frac{K}{2} \rfloor} \tilde{C}_p(f, g) \left(\frac{t}{\nu(\tau)}\right)^{-\frac{d}{2}-p} + o\left(t^{-\frac{K+d}{2}}\right) \,, \tag{2.16}$$

as $t \to +\infty$ where

$$\begin{aligned} \tilde{C}_p(f, g) \quad := \quad & \widetilde{\sum_p} \frac{1}{q!} \int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s\sqrt{\nu(\tau)}, 1)(-s)^q \, ds \\ & * \frac{i^{m+j}}{r!m!} \left(\sum_{\ell,\ell'} \int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot, v)\left(\Pi_0^{(m)}(f_\ell(\cdot, u))\right)\right) \otimes (\ell' - \ell, u - v)^{\otimes r} du dv \otimes A_{j,k} \right) . \end{aligned} \tag{2.17}$$

[1] The notation $\|G\|_{\mathcal{B}'}$ means here $\|G\|_{\mathcal{B}'} := \sup_{F\in\mathcal{B}, \ \|F\|_{\mathcal{B}}=1} |\mathbb{E}_\nu[G.F]|$.

Here, $\widetilde{\sum_p}$ *is defined by* (2.15), $\partial_2^q h_{\alpha,\gamma}$ *denotes the derivative of order* q *with respect to the second variable of* $h_{\alpha,\gamma}$ *(defined by* (2.5)*) and* $A_{j,k} \in \mathcal{S}_j$ *is given by* (A.2) *of Appendix* A *for* $k > 0$, $A_{0,0} = 1$ *and* $A_{j,0} = 0$ *for* $j > 0$.

Corollary 2.2. *Under the assumptions of Theorem* 2.1 *with* $K = 0$,

$$C_t(f,g) = \frac{\nu(\tau)^{\frac{d}{2}-1} t^{-\frac{d}{2}}}{\sqrt{(2\pi)^d \det \Sigma_d}} \int_{\widetilde{\Omega}} f \, d\widetilde{\mu} \int_{\widetilde{\Omega}} g \, d\widetilde{\mu} + o\left(t^{-\frac{d}{2}}\right),$$

where Σ_d *is the submatrix of* Σ *corresponding to its* d *first lines and rows. Equivalently*

$$\int_{\widetilde{\Omega}} f.g \circ \widetilde{\Phi}_t \, d\widetilde{\mu}_0 = \frac{t^{-\frac{d}{2}}}{\sqrt{(2\pi)^d \det \tilde{\Sigma}_d}} \int_{\widetilde{\Omega}} f \, d\widetilde{\mu}_0 \int_{\widetilde{\Omega}} g \, d\widetilde{\mu}_0 + o\left(t^{-\frac{d}{2}}\right),$$

where $\tilde{\mu}_0 := \frac{\tilde{\mu}}{\nu(\tau)}$ *is the measure proportional to* $\tilde{\mu}$ *such that* $\tilde{\mu}_0(M \times \{0\} \times [0, +\infty)) = 1$ *and* $\tilde{\Sigma}_d = \frac{\Sigma_d}{\sqrt{\nu(\tau)}}$ *is the variance matrix of the Gaussian distribution limit of* $\left(\frac{p_2 \circ \widetilde{\Phi}_t}{\sqrt{t}}\right)_t$ *(with respect to any probability measure* $\mathbb{P}$ *absolutely continuous with respect to* $\tilde{\mu}_0$*), with* $p_2 : \widetilde{\Omega} \to \mathbb{Z}^d$ *the canonical projection.*

Proof. We have to prove that

$$\widetilde{C}_0(f,g) = \frac{1}{\nu(\tau)\sqrt{(2\pi)^d \det \Sigma_d}} \int_{\widetilde{\Omega}} f \, d\widetilde{\mu} \int_{\widetilde{\Omega}} g \, d\widetilde{\mu} \, .$$

We assume $p = 0$. Then the sum $\widetilde{\sum_p}$ contains a single term corresponding to $m = j = r = q = k = 0$. Thus, using the fact that $A_{0,0} = 1$ and that $\Pi_0 = \nu(\cdot)\mathbf{1}_M$, the second part of the right hand side of (2.17) is then simply

$$\sum_{\ell,\ell'} \int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot, v)\nu[f_\ell(\cdot, u)]\right) du dv = \int_{\widetilde{\Omega}} f \, d\widetilde{\mu} \int_{\widetilde{\Omega}} g \, d\widetilde{\mu} \, ,$$

whereas the first part of the right hand side of (2.17) is

$$\int_{\mathbb{R}} h_{0,-\frac{d+1}{2}}(s\sqrt{\nu(\tau)}, 1) \, ds = \int_{\mathbb{R}} \Psi(0, s\nu(\tau)) \, ds = \frac{1}{\nu(\tau)} \int_{\mathbb{R}} \Psi(0, s') \, ds' = \frac{1}{\nu(\tau)\sqrt{(\Sigma^{-1})_{d+1,d+1}(2\pi)^d \det \Sigma}},$$

due to (2.5) and to (2.3). We conclude by noticing that $(\Sigma^{-1})_{d+1,d+1} \det \Sigma = \det \Sigma_d$.

The fact that our assumptions imply that both $(\kappa_n/\sqrt{n})_n$ and $((\tau_n - n\nu(\tau))/\sqrt{n})_n$ satisfy a central limit theorem with respective variances Σ_d and Σ_τ is well known (see for example [26]). We can thus deduce using e.g. [39, Theorem 1.1] (or, alternatively, using a functional central limit theorem) the convergence in distribution (with respect to $\nu \otimes \delta_0 \otimes \mathfrak{l}$) of $\left(\frac{p_2 \circ \widetilde{\Phi}_t}{\sqrt{t}}\right)_t$ to a centered Gaussian random variable with variance $\tilde{\Sigma}_d$. The fact that the convergence in distribution is valid for any probability measure absolutely continuous with respect to $\tilde{\mu}$ comes then from [47, Theorem 1]. □

Proof of Theorem 2.1. **Step 1: Fourier transform.**

Notice that

$$C_t(f,g) = \sum_{\ell,\ell'} \sum_{n \geq 0} \int_{M \times \mathbb{R}} f_\ell(x,s) \, g_{\ell'}\left(T^n x, s + t - \tau_n(x)\right) \mathbf{1}_{\{\kappa_n(x) = \ell' - \ell\}} \, d(\nu \otimes \mathfrak{l})(x,s) \, , \qquad (2.18)$$

due to the dominated convergence theorem, (2.13), and the fact that the sum over n is compactly supported, as explained below. Indeed $g_{\ell'}(T^n x, s+t-\tau_n(x)) \neq 0$ implies that

$$-\frac{\inf\tau}{10} \le s+t-\tau_n(x) < \tau(T^n x),\ \text{ i.e. } \tau_n(x)-\frac{\inf\tau}{10}-s \le t < \tau_{n+1}(x)-s \text{ with } -\frac{\inf\tau}{10} \le s < \tau(x)$$

and so the sum over n in (2.18) is in fact is supported in $\{t_-, t_-+1, ..., t_+\}$, where

$$t_- = \lceil t/\sup\tau\rceil - 2, \quad t_+ = \lfloor t/\inf\tau\rfloor + 2\,.$$

Note that

$$\mathbf{1}_{\{\kappa_n(x)=\ell'-\ell\}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i\,\theta\cdot(\ell'-\ell)} e^{i\,\theta\cdot\kappa_n}\,d\theta\,. \tag{2.19}$$

Moreover, for every $x\in M$ and every positive integer n,

$$h_{\ell,\ell',x,n}(\cdot) := \int_{\mathbb{R}} f_\ell(x,s)\, g_{\ell'}\left(T^n x, s+\cdot\right)\, ds$$

is the convolution of $f_\ell(x,-\cdot)$ with $g_{\ell'}(T^n x,\cdot)$. Due to (2.13), for ν-a.e. x and any choice of ℓ,ℓ',n, this $h_{\ell,\ell',x,n}(\cdot)$ well defined. Furthermore, it is continuous (since $f_\ell(x,\cdot)$ or $g_{\ell'}(T^n x,\cdot)$ is continuous) with compact support and its Fourier transform is

$$\hat f_\ell(x,-\cdot)\hat g_{\ell'}(T^n x,\cdot) \in L^\infty(\mathbb{R})\cap L^1(\mathbb{R}).$$

Consequently, $h_{\ell,\ell',x,n}$ is equal to its inverse Fourier transform, that is

$$h_{\ell,\ell',x,n}(t-\tau_n(x)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi(t-\tau_n(x))}\hat f_\ell(x,-\xi)\hat g_{\ell'}(T^n x,\xi)\,d\xi\,.$$

Combining this with (2.18) and with (2.19), we obtain

$$C_t(f,g) \tag{2.20}$$
$$= \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n\ge 0}\int_M\left(\int_{[-\pi,\pi]^d\times\mathbb{R}} e^{-i\xi t}\hat f_\ell(x,-\xi)\, e^{-i\theta\cdot(\ell'-\ell)} e^{i\theta\cdot\kappa_n(x)} e^{i\xi\tau_n(x)}\hat g_{\ell'}\left(T^n x,\xi\right)\,d\theta d\xi\right) d\nu(x)$$
$$= \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_M\left(\int_{[-\pi,\pi]^d\times\mathbb{R}} e^{-i\xi t} e^{-i\theta\cdot(\ell'-\ell)} P^n_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right)\hat g_{\ell'}\left(\cdot,\xi\right)\,d\theta d\xi\right) d\nu \tag{2.21}$$

where we used the fact that $P^n(e^{i\theta\cdot\kappa_n+i\xi\tau_n}F) = P^n_{\theta,\xi}F$. We split $(2\pi)^{d+1}C_t(f,g) = I_1+I_2$ where I_1 stands the contribution of $\xi\in[-b,b]$ and I_2 stands the contribution of $|\xi|>b$.

Step 2: Reduction to the integration over a compact domain.

Here we prove that $|I_2| = o\left(t^{-\frac{K+d}{2}}\right)$. Observe that

$$|I_2| \le \sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-\pi,\pi]^d\times([-\infty,-b]\cup[b,\infty])}\int_M \left|P^n_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right)\hat g_{\ell'}\left(\cdot,\xi\right)\right| d\nu d\theta d\xi$$
$$\le C' t\int_{[-\pi,\pi]^d\times([-\infty,-b]\cup[b,\infty])}\left(\sup_{n\in[t_-,t_+]}\sum_{\ell,\ell'}\left\|P^n_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right)\right\|_1 \|\hat g_{\ell'}\left(\cdot,\xi\right)\|_\infty\right) d\theta d\xi\,.$$

Now due to (2.10), we have

$$|I_2| \le C'' t\int_{[-\pi,\pi]^d}\int_{b<|\xi|}|\xi|^\alpha e^{-\delta t_-|\xi|^{-\alpha}}\sum_{\ell,\ell'}\left\|\hat f_\ell(\cdot,-\xi)\right\|_{\mathbb{B}}\|\hat g_{\ell'}(\cdot,\xi)\|_\infty\,d\xi d\theta\,.$$

We apply (2.11) to see that for any $\gamma > 0$ there is $\hat{C}''_\gamma, C''_\gamma > 0$ such that

$$|I_2| \le \hat{C}''_\gamma t \int_{b<|\xi|} e^{-\delta t - |\xi|^{-\alpha}} |\xi|^{\alpha-\gamma} d\xi \le C''_\gamma t^{2+\frac{1-\gamma}{\alpha}} \int_{\mathbb{R}} e^{-\delta |u|^{-\alpha}} |u|^{\alpha-\gamma} \, du.$$

Choosing γ large, we get $|I_2| = o\left(t^{-\frac{K+d}{2}}\right)$. In the remaining part of the proof, we compute I_1.

Step 3: Expansion of the leading eigenvalue and eigenprojector.

First, we use (2.7), (2.8) and (2.14) to write

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n \int_{[-b,b]^{d+1}} e^{-i\xi t} e^{-i\theta\cdot(\ell'-\ell)} \lambda^n_{\theta,\xi} \nu \left(\Pi_{\theta,\xi} \left(\hat{f}_\ell(\cdot,-\xi) \right) \hat{g}_{\ell'}(\cdot,\xi) \right) d(\theta,\xi) \, ,$$

where $\simeq$ means that the difference between the LHS and the RHS is $o\left(t^{-\frac{K+d}{2}}\right)$.

Now the change of variables $(\theta,\xi) \mapsto (\theta,\xi)/\sqrt{n}$ gives

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n n^{-\frac{d+1}{2}} \mathcal{I}(\ell,\ell',n)$$

where

$$\mathcal{I}(\ell,\ell',n) = \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\frac{\xi}{\sqrt{n}}t} e^{-i\theta\cdot\frac{\ell'-\ell}{\sqrt{n}}} \lambda^n_{(\theta,\xi)/\sqrt{n}} \nu \left(\Pi_{(\theta,\xi)/\sqrt{n}} \left(\hat{f}_\ell(\cdot, -\frac{\xi}{\sqrt{n}}) \right) \hat{g}_{\ell'} \left(\cdot, \frac{\xi}{\sqrt{n}} \right) \right) d\theta \, d\xi \, .$$

Next with an error $o\left(t^{-\frac{K+d}{2}}\right)$, we can replace $\mathcal{I}(\ell,\ell',n)$ in the last sum by

$$\int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\frac{\xi}{\sqrt{n}}t} e^{-i\theta\cdot\frac{\ell'-\ell}{\sqrt{n}}} \lambda^n_{(\theta,\xi)/\sqrt{n}} \sum_{m=0}^{K+1} \frac{1}{m!} \nu \left(\Pi_0^{(m)} \left(\hat{f}_\ell \left(\cdot, -\frac{\xi}{\sqrt{n}} \right) \right) \hat{g}_{\ell'} \left(\cdot, \frac{\xi}{\sqrt{n}} \right) \right) * \frac{(\theta,\xi)^{\otimes m}}{n^{\frac{m}{2}}} \, d\theta \, d\xi \, . \tag{2.22}$$

Indeed, for every $u \in \mathbb{R}^{d+1}$, there exist $\omega \in [0,1]$ and $x_u = \omega u$ such that

$$\Pi_u(\cdot) = \sum_{m=0}^{K} \frac{1}{m!} \Pi_0^{(m)}(\cdot) * u^{\otimes m} + \frac{1}{(K+1)!} \Pi^{(K+1)}_{x_u}(\cdot) * u^{\otimes(K+1)}.$$

Denote

$$E_n := \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} \left| \lambda^n_{s/\sqrt{n}} \right| \left\| \Pi^{(K+1)}_{x_{s/\sqrt{n}}} - \Pi^{(K+1)}_0 \right\| \, |s|^{K+1} \, ds \, .$$

Then $\lim_{n\to+\infty} E_n = 0$ by the Lebesgue dominated convergence theorem. Therefore

$$\lim_{t\to+\infty} t^{\frac{K+d}{2}} \sum_{n=t_-}^{t_+} n^{-\frac{d+1}{2}} \frac{E_n}{n^{\frac{K+1}{2}}} = 0 \, ,$$

justifying the replacement of Π by its jet.

Recalling elementary identities $a^n_{s/\sqrt{n}} = a_s$ and $a_s/a_{s/\sqrt{2}} = a_{s/\sqrt{2}}$, Lemma A.1 gives

$$\left| \tilde{\lambda}^n_{s/\sqrt{n}} - a_s \sum_{k=0}^{\lfloor (K+1)/(J-2) \rfloor} \sum_{j=kJ}^{K+1+2k} n^k A_{j,k} * (s/\sqrt{n})^{\otimes j} \right| \le a_{s/\sqrt{2}} n^{-\frac{K+1}{2}} (1+|s|^{K_0}) \eta(s/\sqrt{n}) \, ,$$

with $\lim_{t\to 0} \eta(t) = 0$ and $\sup_{[-b,b]^d} |\eta| < \infty$. Let

$$E'_n := \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} a_{s/\sqrt{2}} (1+|s|^{K_0}) \eta(s/\sqrt{n}) \, ds \, .$$

Since the Lebesgue dominated convergence theorem gives $\lim_{n\to\infty} E_n' = 0$, the same argument as above shows that the error term arising from replacing in (2.22) $\tilde\lambda^n_{s/\sqrt{n}}$ by the above sum is negligible. Since $\tilde\lambda_{\theta,\xi} = \lambda_{\theta,\xi} e^{-i\xi\nu(\tau)}$, we conclude

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n n^{-\frac{d+1}{2}} \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\xi\frac{t-n\nu(\tau)}{\sqrt{n}}} e^{-i\theta.\frac{\ell'-\ell}{\sqrt{n}}} a_{(\theta,\xi)}$$

$$\sum_{m=0}^{K+1} \frac{1}{m!} \nu\left(\hat g_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\Pi_0^{(m)}\left(\hat f_\ell\left(\cdot,-\frac{\xi}{\sqrt{n}}\right)\right)\right) * \frac{(\theta,\xi)^{\otimes m}}{n^{\frac{m}{2}}} \left(\sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor} \sum_{j=kJ}^{(K+1)+2k} n^k A_{j,k} * \frac{(\theta,\xi)^{\otimes j}}{n^{\frac{j}{2}}}\right) d\theta\, d\xi.$$

Step 4. Integrating by parts.

Note that $\forall A \in \mathcal{S}_j, \forall B \in \mathcal{S}_m$ and $s \in \mathbb{C}^{d+1}$, $(B * s^{\otimes m})(A * s^{\otimes j}) = (A \otimes B) * s^{\otimes(m+j)}$. We claim that

$$\begin{aligned}
&\frac{1}{(2\pi)^{d+1}} \int\limits_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\xi\frac{t-n\nu(\tau)}{\sqrt{n}} - i\theta.\frac{\ell'-\ell}{\sqrt{n}}} a_{(\theta,\xi)} \nu\left(\hat g_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\left(\Pi_0^{(m)}\left(\hat f_\ell\left(\cdot,-\frac{\xi}{\sqrt{n}}\right)\right)\otimes A_{j,k}\right)\right) \\
&\qquad\qquad * (\theta,\xi)^{\otimes(m+j)}\, d\theta d\xi \\
&= i^{m+j} \int_{\mathbb{R}^2} \Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right) * \nu\left(\Pi_0^{(m)}(f_\ell(\cdot,u))g_{\ell'}(\cdot,v)\otimes A_{j,k}\right) dudv \\
&\qquad + o\left(\rho^n \sup_{\xi\in\mathbb{R}} \left\|\hat f_\ell(\cdot,\xi)\right\|_{\mathcal{B}} \|\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{B}'}\right)
\end{aligned} \tag{2.23}$$

where Ψ is defined by (2.3) and $\rho < 1$. Note that the integration in the second line of (2.23) is over a compact set since f_ℓ and $g_{\ell'}$ vanish outside of a compact set.

To prove (2.23), we first note that, due to (2.14) by making an exponentially small error we can replace the integration in the first line to $\mathbb{R}^{d+1}$. Second, we observe that $\Pi_0^{(m)}\hat f_\ell = \widehat{f_{m,\ell}}$ where $f_{m,l} = \Pi_0^{(m)} f_\ell$ and that $\hat h(\xi/\sqrt{n}) = (\widehat{\sqrt{n}h(\sqrt{n}\cdot)})(\xi)$. Third, since a is the Fourier transform of Ψ, it follows that

$$(\theta,\xi)\mapsto (-i)^{\sum_{j=1}^{d+1}k_j}\,\theta_1^{k_1}\dots\theta_d^{k_d}\xi^{k_{d+1}}a_{(\theta,\xi)} \text{ is the Fourier transform of } s\mapsto \frac{\partial^{\sum_{j=1}^{d+1}k_j}}{(\partial s_1)^{k_1}\cdots(\partial s_{d+1})^{k_{d+1}}}\Psi.$$

Fourth, we use the inversion formula for the Fourier transform. To take the inverse Fourier transform with respect to ξ we note that we have a triple product, which is a Fourier transform of the triple convolution of the form

$$i^{m+j}\int_{\mathbb{R}^2}\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)}{\sqrt{n}} - t_1 - t_2\right) * n f_{m,\ell}(\cdot,-\sqrt{n}t_1)g_{\ell'}(\cdot,\sqrt{n}t_2)dt_1dt_2.$$

Making the change of variables $u = -\sqrt{n}t_1$, $v = \sqrt{n}t_2$ we obtain (2.23).

Formula (2.23) implies that

$$C_t(f,g) \simeq \sum_{m=0}^{K+1} \sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor} \sum_{j=kJ}^{K+1+2k} \frac{i^{m+j}}{m!} \sum_{\ell,\ell'}\sum_n n^{-\frac{m+j+d+1-2k}{2}} \tag{2.24}$$

$$\int_{[-\frac{\inf\tau}{10},\sup\tau)^2} \Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right) * \nu\left(\Pi_0^{(m)}(f_\ell(\cdot,u))g_{\ell'}(\cdot,v)\otimes A_{j,k}\right) dudv$$

Step 5: Simplifying the argument of Ψ.

Note that there exist $a_0, a_0', c_{m+j}, c'_{m+j} > 0$ such that, for every $\ell', \ell \in \mathbb{Z}^2$ and every $u, v \in (-\frac{\inf\tau}{10}, \sup\tau)$,

$$\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right) \le c_{m+j}e^{-\frac{a_0}{n}\left((\ell'-\ell)^2+(t-n\nu(\tau)+u-v)^2\right)} \le c'_{m+j}e^{-\frac{a_0'}{n}(t-n\nu(\tau))^2}. \tag{2.25}$$

Combining this estimate with

$$\sum_{n=t_-}^{t_+} e^{-\frac{a_0'}{n}(t-n\nu(\tau))^2} \le 2\sum_{n=0}^{t_+} e^{-\frac{a_0'(\nu(\tau))^2}{t_+}n^2} \le 2\int_0^\infty e^{-\frac{a_0'(\nu(\tau)s)^2}{t_+}}\,ds = O(\sqrt{t}). \tag{2.26}$$

Lemma A.3 (with $\alpha = 0$), we obtain that

$$\sup_{u,v\in(-\frac{\inf\tau}{10},\sup\tau)} \sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+1-2k}{2}} \left|\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right)\right| = O\left(t^{-\frac{m+j+d-2k}{2}}\right).$$

Therefore, the terms of (2.24) corresponding to (m,k,j) with $m+j-2k > K$ are in $o\left(t^{-\frac{K+d}{2}}\right)$ and so the third summation in (2.24) can be replaced by $\sum_{j=kJ}^{K-m+2k}$. The constraint $K-m+2k \ge kJ$ implies that we can replace the second summation in (2.24) by $\sum_{k=0}^{\lfloor K/(J-2)\rfloor}$.

Next let $p = K-m-j+2k$. We claim that we can replace $\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)-u-v}{\sqrt{n}}\right)$ in (2.24) by

$$\sum_{r=0}^{p} \frac{1}{r!\,n^{\frac{r}{2}}}\Psi^{(m+j+r)}\left(0, \frac{t-n\,\nu(\tau)}{\sqrt{n}}\right) * (\ell'-\ell, u-v)^{\otimes r}.$$

Indeed by Taylor's theorem, we just need to verify that for

$$\begin{aligned}&\lim_{t\to+\infty} t^{\frac{K+d}{2}} \sum_{\ell,\ell'}\int_{\mathbb{R}^2} \|f_\ell(\cdot,u)\|_{\mathcal{B}}\|g_{\ell'}(\cdot,v)\|_{\mathcal{B}'}|(\ell'-\ell,u-v)|^p \sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+1-2k+p}{2}} \\ &\sup_{x\in(0,1)}\left|\Psi^{(m+j+p)}\left(x\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+x(u-v)}{\sqrt{n}}\right) - \Psi^{(m+j+p)}\left(0, \frac{t-n\,\nu(\tau)}{\sqrt{n}}\right)\right| dudv \\ =\ & 0.\end{aligned} \tag{2.27}$$

By (2.25) and (2.26)

$$\begin{aligned}&\sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+1-2k-p}{2}} \sup_{x\in(0,1)}\left|\Psi^{(m+j+p)}\left(x\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+x(u-v)}{\sqrt{n}}\right)\right| \\ &\le c'_{m+j+p}\sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+1-2k+p}{2}} e^{-\frac{a_0'}{n}(t-n\,\nu(\tau))^2} = O\left(t^{-\frac{m+j+d-2k+p}{2}}\right)\end{aligned}$$

uniformly in $\ell, \ell' \in \mathbb{Z}^d$ and $u, v \in (-\frac{\inf\tau}{10}, \sup\tau)$. This combined with (2.12) shows that the LHS of (2.27) is dominated by an integrable function, so (2.27) follows by the dominated convergence theorem.

Therefore

$$\begin{aligned}C_t(f,g) \ \simeq\ & \sum_{\ell,\ell'}\sum_{m=0}^{K+1}\sum_{k=0}^{\lfloor K/(J-2)\rfloor}\sum_{j=kJ}^{K-m+2k}\sum_{r=0}^{K-m-j+2k}\frac{i^{m+j}}{r!m!}\sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+r+1-2k}{2}}\Psi^{(m+j+r)}\left(0,\frac{t-n\nu(\tau)}{\sqrt{n}}\right) * \\ &\int_{\mathbb{R}^2}\left(\nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\otimes(\ell'-\ell,+u-v)^{\otimes r}dudv\otimes A_{j,k}\right)\right).\end{aligned} \tag{2.28}$$

Step 6: Summing over n.

Performing the summation over n and using Lemma A.3 we obtain

$$C_t(f,g) \simeq \sum_{\ell,\ell'} \sum_{m=0}^{K+1} \sum_{k=0}^{\lfloor K/(J-2)\rfloor} \sum_{j=kJ}^{K-m+2k} \sum_{r=0}^{K-m-j+2k} \sum_{q=0}^{K+2k-m-j-r} \frac{i^{m+j}(t/\nu(\tau))^{-\frac{m+j+d+r+q-2k}{2}}}{r!m!\,q!\,(\nu(\tau))^{\frac{q+1}{2}}} \quad (2.29)$$

$$\int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s,1)(-s)^q\,ds$$

$$* \left(\int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right) \otimes (\ell'-\ell,u-v)^{\otimes r} dudv \otimes A_{j,k}\right).$$

Therefore $C_t(f,g) \simeq \sum_{p=0}^K \tilde{C}_{p/2}(f,g)\left(\frac{t}{\nu(\tau)}\right)^{-\frac{d+p}{2}}$ where

$$\tilde{C}_{p/2}(f,g) := \sum \frac{1}{q!}\int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s\sqrt{\nu(\tau)},1)(-s)^q\,ds \quad (2.30)$$
$$* \frac{i^{m+j}}{r!m!}\left(\sum_{\ell,\ell'}\int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right) \otimes (\ell'-\ell,u-v)^{\otimes r} dudv \otimes A_{j,k}\right),$$

and the first sum is taken over the nonnegative integers m, j, r, q, k satisfying $m+j+r+q-2k = p$. Applying Lemma A.4 with $b = m + j + r$, we see that $\tilde{C}_{p/2} = 0$ if p is an odd integer. This concludes the proof of Theorem 2.1. □

2.3. **A general result for hyperbolic systems.** Here we consider extensions of systems with good spectral properties. To define the setup, let (M,ν,T) be an extension, by $\mathfrak{p} : M \to \bar{\Delta}$, of a dynamical system $(\bar{\Delta},\bar{\nu},\bar{T})$ with Perron-Frobenius operator $\bar{P}$. In the applications considered in the present article, $\mathfrak{p}$ is essentially collapsing along stable manifolds.

Similarly to (A1)–(A3), we need to make assumptions to prove the complete asymptotic expansion in inverse powers of t. However this time we need more assumptions. On the one hand we need that the factor map $\bar{T}$ satisfies variants of (A1)–(A3) (see assumptions (B3), (B4), (B6)) and on the other hand we need assumptions relating the factor map to the extension (namely, assumptions (B1), (B2), (B5)). In particular, we will use Banach spaces $\mathcal{B}, \mathbb{B}$ of observables defined over $\bar{\Delta}$, the phase space of the factor map as before as well as another Banach space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ of functions $f : M \to \mathbb{C}$ with $\mathcal{V} \hookrightarrow L^\infty(\nu)$. Namely, we make the following assumptions.

(B1) κ **is independent of the past**. There exist a nonnegative integer m_0 and a $\bar{\nu}$-centered bounded function $\bar{\kappa} : \bar{\Delta} \to \mathbb{Z}^d$ such that $\bar{\kappa} \circ \mathfrak{p} = \kappa \circ T^{m_0}$.

(B2) τ **is quasi independent of the past**. There exist $\beta_0 \geq 0$, a function $\bar{\tau} : \bar{\Delta} \to \mathbb{R}$ and a function $\chi : M \to \mathbb{R}$ s.t. $\tau = \bar{\tau} \circ \mathfrak{p} + \chi - \chi \circ T$ and for every $\xi \in \mathbb{R}$, we have $e^{i\xi\chi} \in \mathcal{V}$ with $\|e^{i\xi\chi}\|_{\mathcal{V}} = O\left(|\xi|^{\beta_0}\right)$ and $(\bar{\tau}_{m_0})^q e^{-i\xi\bar{\tau}_{m_0}} \in \mathcal{B}$ for every $q \leq L = K+3$.

Our next assumption is a variant of (A1).

(B3) Perron-Frobenius assumption of the factor map. We say that (B3) holds with positive integers J and K and for the Banach space $\mathcal{B}$ of complex functions $f : \bar{\Delta} \to \mathbb{C}$ with $\mathcal{B} \hookrightarrow L^1(\bar{\Delta},\bar{\nu})$ and $\mathbf{1}_{\bar{\Delta}} \in \mathcal{B}$ if the following is true with the notation $L = K+3$. The family of linear continous operators on $\mathcal{B}$, defined by $(\bar{P}_{\theta,\xi} : \bar{f} \mapsto \bar{P}(e^{i\theta\cdot\bar{\kappa}}e^{i\xi\tau}\bar{f}))_{(\theta,\xi)\in[-\pi,\pi]^d\times\mathbb{R}}$ satisfies

$$\sup_{\theta,\xi,n} \|\bar{P}^n_{\theta,\xi}\| < \infty\,, \quad (2.31)$$

and there exist constants $b \in (0,\pi]$, $C > 0$, $\vartheta \in (0,1)$, $\beta > 0$ and three functions $\lambda_\cdot : [-b,b]^{d+1} \to \mathbb{C}$ and $\Pi_\cdot, R_\cdot : [-b,b]^{d+1} \to \mathcal{L}(\mathcal{B},\mathcal{B})$ (assumed to be C^L-smooth) such that $\tilde{\lambda}_{\theta,\xi} := \lambda_{\theta,\xi}e^{-i\xi\nu(\tau)}$

$$\forall k < J \quad \tilde{\lambda}_0^{(k)} = a_0^{(k)} \quad (2.32)$$

where a_s is given by (2.4) with a sutiable positive definite $(d+1)\times(d+1)$ matrix Σ, $\lambda_0=1$ and $\Pi_0=\mathbb{E}_{\bar\nu}[\cdot]\mathbf{1}_{\bar\Delta}$ and such that, in $\mathcal{L}(\mathcal{B},\mathcal{B})$,

$$\forall s\in[-b,b]^{d+1},\quad \bar P_s=\lambda_s\Pi_s+R_s,\quad \Pi_sR_s=R_s\Pi_s=0,\quad \Pi_s^2=\Pi_s\,, \tag{2.33}$$

$$\forall k\in\mathbb{N}\ \sup_{m=0,\dots,L}\ \sup_{s\in[-b,b]^{d+1}}\|(R_s^k)^{(m)}\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}+\sup_{\theta\in[-\pi,\pi]^d\setminus[-b,b]^d,\ |\xi|\le b}\|\bar P^k_{\theta,\xi}\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}\le C\vartheta^k. \tag{2.34}$$

The next assumption corresponds to (A2) for the factor map.

(B4) Weak non-lattice property of the factor map. We say that the Banach space $\mathbb{B}$ satisfies (B4) if

$$\exists C,\delta,\alpha>0,\qquad \sup_{\theta\in[-\pi,\pi]^d}\|\bar P^n_{\theta,\xi}\|_{\mathcal{L}(\mathbb{B},L^1)}\le C|\xi|^\alpha e^{-n\delta|\xi|^{-\alpha}}\,. \tag{2.35}$$

Our next assumption says that observables in $\mathcal{V}$ can be well approximated by regular functions only depending on the past.

(B5) Functions in $\mathcal{V}$ are well approximable by liftup of $\mathcal{B}\cap\mathbb{B}$. We say that L,m_0 and the Banach spaces $\mathcal{V},\mathcal{B}$ and $\mathbb{B}$ satsify (B5) if there exist $C_0>0$ and $\vartheta\in(0,1)$ and continuous linear maps $\mathbf{\Pi}_n:\mathcal{V}\to\mathcal{B}\cap\mathbb{B}$, such that, for every $f\in\mathcal{V}$ and every integer $n\ge m_0$ and for any $\theta\in[-\pi,\pi]^d,\xi\in\mathbb{R}$ and for any non-negative integer $j=0,...,L$,

$$\|f\circ T^n-\mathbf{\Pi}_n(f)\circ\mathfrak{p}\|_\infty\le C_0\|f\|_{\mathcal{V}}\,\vartheta^n\,, \tag{2.36}$$

$$\left\|\bar P^{2n}_{\theta,\xi}(e^{-i\theta.\bar\kappa_{n-m_0}-i\xi.\bar\tau_n}\mathbf{\Pi}_nf)\right\|_{\mathbb{B}}\le C_0(1+|\xi|)\|f\|_{\mathcal{V}}\,, \tag{2.37}$$

$$\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\bar P^{2n}_{\theta,\xi}(e^{-i\theta.\bar\kappa_{n-m_0}-i\xi.\bar\tau_n}\mathbf{\Pi}_nf))\right\|_{\mathcal{B}}\le C_0n^j(1+|\xi|)\|f\|_{\mathcal{V}}\,, \tag{2.38}$$

$$\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\mathbf{\Pi}_n(f)e^{i\theta\cdot\bar\kappa_{n-m_0}+i\xi.\bar\tau_n})\right\|_{\mathcal{B}'}\le C_0n^j\|f\|_{\mathcal{V}}\,, \tag{2.39}$$

with $\bar\kappa_n:=\sum_{k=0}^{n-1}\bar\kappa\circ\bar T^k$ and $\bar\tau_n:=\sum_{k=0}^{n-1}\bar\tau\circ\bar T^k$.

Finally we discuss the regularity of observables. As before, we allow a large class of observables, going well beyond compactly supported $\mathcal{C}^\infty$ functions.

(B6) Regularity of observables. We say that the observables $f,g:\widetilde\Omega\to\mathbb{C}$, Banach spaces $\mathcal{V},\mathcal{B},\mathbb{B}$ and the constants $K,L=K+3$ satsify (B6) if

$$\forall h\in\{f,g\}\quad\forall(x,\ell,s)\in\widetilde\Omega,\quad h(x,\ell,s)=h_\ell(x,s)+h_{\ell+\kappa(x)}(Tx,s-\tau(x))\,, \tag{2.40}$$

where $(f_\ell)_{\ell\in\mathbb{Z}^d}$ and $(g_\ell)_{\ell\in\mathbb{Z}^d}$ are two families of functions defined on $M\times\mathbb{R}\to\mathbb{C}$ and vanishing outside $\widetilde\Omega_0:=\widetilde\Omega\cup\left(M\times\left[-\frac{\inf\tau}{10},0\right]\right)$. Furthermore, one of these families is made of functions continuous in the last variable and that there exists β_0 such that $\xi\mapsto e^{i\xi\cdot\chi}\hat f_\ell(\cdot,\xi)$ and $\xi\mapsto e^{i\xi\cdot\chi}\hat g_\ell(\cdot,\xi)$ are C^L from $\mathbb{R}$ to $\mathcal{V}$ and for every $k=0,...,L$,

$$\sup_{|\xi|\le b}\sum_{\ell\in\mathbb{Z}^d}\left(\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi\cdot\chi}\hat f_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}+\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi\cdot\chi}\hat g_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}\right)<\infty\,, \tag{2.41}$$

$$\sum_\ell\int_{\mathbb{R}}(1+|\ell|)^K\left(\|f_\ell(\cdot,u)\|_{\mathcal{V}}+\|g_\ell(\cdot,u)\|_{\mathcal{V}}\right)\,du<\infty\,, \tag{2.42}$$

$$\forall\gamma>0,\quad\sum_{\ell,\ell'}\left(\|e^{i\xi\cdot\chi}\hat f_\ell(\cdot,-\xi)\|_{\mathcal{V}}\,\|e^{-i\xi\cdot\chi}\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{V}}\right)=O(|\xi|^{-\gamma})\,. \tag{2.43}$$

$$\sum_{\ell\in\mathbb{Z}^d}\|f_\ell\|_\infty<\infty\quad\text{or}\quad\sum_{\ell\in\mathbb{Z}^d}\|g_\ell\|_\infty<\infty\,, \tag{2.44}$$

Theorem 2.3. *Assume that (M,ν,T) is extension, by $\mathfrak{p}: M\to\bar\Delta$, of a dynamical system $(\bar\Delta,\bar\nu,\bar T)$ and K,J are two integers such that $3\le J\le L=K+3$. Assume furthermore that (B1)-(B6) hold with some Banach spaces $\mathcal{V},\mathcal{B},\mathbb{B}$, a constant m_0 and functions $f,g:\widetilde\Omega\to\mathbb{C}$. Then*

$$C_t(f,g)=\sum_{p=0}^{\lfloor\frac{K}{2}\rfloor}\tilde C_p(f,g)\left(\frac{t}{\nu(\tau)}\right)^{-\frac{d}{2}-p}+o\left(t^{-\frac{K+d}{2}}\right),$$

as $t\to+\infty$, where

$$\begin{aligned}\tilde C_p(f,g)\ &=\ \widetilde{\sum_p}\frac{1}{q!}\frac{1}{(\nu(\tau))^{\frac{q+1}{2}}}\int_{\mathbb{R}}\partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s,1)(-s)^q\,ds \\ &\quad *\frac{i^{m+j}}{r!m!}\left(\sum_{\ell,\ell'}\int_{\mathbb{R}^2}\mathcal{B}_m\left(f_\ell(\cdot,u),g_{\ell'}(\cdot,v)\right)\otimes(\ell'-\ell,u-v)^{\otimes r}\,dudv\otimes A_{j,k}\right).\end{aligned}\tag{2.45}$$

Here, $\widetilde{\sum_p}$ is defined by (2.15), h is defined in (2.5), $A_{j,k}$ for $k>0$ are the multilinear forms given by equation (A.2) from Appendix A, $A_{0,0}=1$ and $A_{j,0}=0$ for $j>0$ and $\mathcal{B}_m:\mathcal{V}\times\mathcal{V}\to\mathcal{S}_m$ are bilinear forms defined in (2.46) below.

To define $\mathcal{B}_m$ we need the following preliminary lemma, the proof of which is given at the end of this section, after the proof of Theorem 2.3.

Lemma 2.4. *Under the assumptions of Theorem 2.3, let $u,v:M\times([-\pi,\pi]^d]\times\mathbb{R})\to\mathbb{C}$ be two functions such that $(\theta,\xi)\mapsto e^{-i\xi\chi}u(\cdot,\theta,\xi)$ and $(\theta,\xi)\mapsto e^{-i\xi\chi}v(\cdot,\theta,\xi)$ are L times differentiable at 0 as functions from $[-\pi,\pi]^d\times\mathbb{R}$ to $\mathcal{V}$.*
Then, for every integer $N=0,...,L$, the quantity

$$\mathcal{A}_N(u,v):=\lim_{n\to+\infty}\left(\mathbb{E}_\nu\left[u(\cdot,-\theta,-\xi)e^{i\theta\cdot\kappa_n+i\xi\tau_n}v(T^n(\cdot),\theta,\xi)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}$$

is well defined and satisfies

$$|\mathcal{A}_N(u,v)|=O\left(\|u\|_{\mathcal{W}}\|v\|_{\mathcal{W}}\right).$$

Moreover for each $\bar L\in\mathbb{N}$ we have

$$\left|\mathcal{A}_N(u,v)-\left(\mathbb{E}_\nu\left[u(\cdot,-\theta,-\xi)e^{e^{i\theta\cdot\kappa_n+i\xi\tau_n}}v(\bar T^n(\cdot),\theta,\xi)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}\right|=O\left(\|u\|_{\mathcal{W}}\|v\|_{\mathcal{W}}n^{-\bar L}\right)$$

with

$$\|u\|_{\mathcal{W}}:=\sum_{m=0}^{L}\left\|\left(e^{-i\xi\chi}u(\cdot,\theta,\xi)\right)^{(m)}_{|(\theta,\xi)=0}\right\|_{\mathcal{V}}<\infty.$$

We let $\mathcal{B}_m$ to be the restriction of $\mathcal{A}_m$ on the space of functions depending on neither θ nor ξ. Thus

$$\mathcal{B}_m(F,G):=\lim_{n\to+\infty}\left(\mathbb{E}_\nu\left[F(\cdot)e^{i\theta\cdot\kappa_n(.)+i\xi(\tau_n(.)-n\nu(\tau))}G(T^n(\cdot))\right]\tilde\lambda_{\theta,\xi}^{-n}\right)^{(m)}_{|(\theta,\xi)=0}.\tag{2.46}$$

Observe that (2.45) has the same form as (2.17) with $\nu\left(G\Pi_0^{(m)}(F)\right)$ replaced by $\mathcal{B}_m(F,G)$. In fact these two quantities coincide under the assumptions of Theorem 2.1. More precisely, suppose that $(M,\nu,T)=(\bar\Delta,\bar\nu,\bar T)$. Then, for $(\theta,\xi)\in[-b,b]^{d+1}$,

$$\begin{aligned}\lim_{n\to+\infty}\left(\mathbb{E}_\nu\left[F(\cdot)e^{i\theta\cdot\kappa_n(.)+i\xi(\tau_n(.)-n\nu(\tau))}G(T^n(\cdot))\right]\tilde\lambda_{\theta,\xi}^{-n}\right)&=\lim_{n\to+\infty}\left(\mathbb{E}_\nu\left[\left(P_{\theta,\xi}^nF\right)G\right]\lambda_{\theta,\xi}^{-n}\right)\\ &=\lim_{n\to+\infty}\nu\left(G\left[\Pi_{\theta,\xi}F+\lambda_{\theta,\xi}^{-n}R_{\theta,\xi}^nF\right]\right)=\nu(G\Pi_{\theta,\xi}(F)).\end{aligned}$$

In particular, in this case $\mathcal{B}_0(F,G)=\nu(G\Pi_0(F))$. A similar argument shows that

$$\mathcal{B}_m(F,G)=\nu(G\Pi_0^{(m)}(F)),$$

see the proof of Lemma 2.4 for details.

We also note that due to mixing of T we have

$$\mathcal{B}_0(F,G)=\nu(F)\nu(G). \tag{2.47}$$

Let us mention that $\mathcal{B}_m(F,G)$ for $m\le 3$ as well as $\lambda_0^{(k)}$ for $k\le 4$ have been computed in [42] in the case of the Sinai billiard with finite horizon with κ_n instead of $(\kappa_n,\tau_n-n\nu(\tau))$. The formulas of [42, Propositions A.3] are also valid by replacing κ therein by $(\kappa,\tau-\nu(\tau))$ since (κ,τ) is dynamically Lipschitz. Moreover formulas for $\lambda_0^{(k)}$ can be obtained by adapting the proof of [42, Propositions A.4], up to replace the time-reversibility property of [42, Lemma 4.3] by the fact that $(((\kappa,\tau-\nu(\tau))\circ T^k)_k$ has the same distribution as $(((-\kappa,\tau-\nu(\tau))\circ T^{-k})_k$.

Proof of Theorem 2.3. The proof of Theorem 2.3 is in many places similar to the proof of Theorem 2.1 so below we mostly concentrate on the places requiring significant modifications. We note that we could have presented Theorem 2.3 without discussing Theorem 2.1 first, however, since the formulas are quite cumbersome in the present setting we prefer to discuss the argument in the simpler setup of Theorem 2.1 first. The strategy of the proof of Theorem 2.3 can be quickly summarized as follows. For $h\in\{f,g\}$, we will approximate the function $e^{-i\xi\chi(T^{k_t}(\cdot))}\hat h_\ell(T^{k_t}(\cdot),\xi)$ by $\mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat h_\ell(\cdot,\xi))$, with k_t large enough so that this approximation is good (see (2.36)), but with k_t not too large so that the controls in norm given by (2.38) and (2.39) are manageable. Then we will use the argument of the proof of Theorem 2.1 thanks to the nice properties of the transfer operator of $(\bar\Delta,\bar\nu,\bar T)$.

Decreasing the value of b if necessary, we can assume that

$$\forall s\in[-b,b]^{d+1},\ \vartheta^{\frac{1}{10L(d+1)}}\le|\lambda_s|\le a_{s/\sqrt2}\,, \tag{2.48}$$

where ϑ is given by (2.34). Let $k_t:=\lceil (L+\frac{L+1+d}{2})\log t/|\log\vartheta|\rceil$.

We consider $F_t,G_t:\bar\Delta\times\mathbb{Z}^d\times\mathbb{R}\to\mathbb{C}$ given by

$$\forall\ell\in\mathbb{Z}^d,\forall\xi\in\mathbb{R},\quad F_t(\cdot,\ell,\xi):=\mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat f_\ell(\cdot,\xi))\quad\text{and}\quad G_t(\cdot,\ell,\xi):=\mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat g_\ell(\cdot,\xi))\,.$$

As in (2.20), using (2.42) and (2.44), $C_t(f,g)$ is equal to

$$\frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_M\left(\int_{[-\pi,\pi]^d\times\mathbb{R}}e^{-i\xi t}\hat f_\ell(x,-\xi)\,e^{-i\theta\cdot(\ell'-\ell)}e^{i\theta\cdot\kappa_n(x)}e^{i\xi\tau_n(x)}\hat g_{\ell'}\left(T^nx,\xi\right)\,d\theta d\xi\right)d\nu(x)\,. \tag{2.49}$$

In order to apply the spectral method, as in the proof of Theorem 2.1, we want to reduce the integration over M in (2.49) to integration over $\bar\Delta$. Namely

$$\begin{aligned}
&\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\\
=\ &\mathbb{E}_\nu\left[e^{i\xi\chi\circ T^{k_t}}\hat f_\ell(T^{k_t}(\cdot),-\xi)\,e^{i\theta\cdot\bar\kappa_n\circ\bar T^{k_t-m_0}\circ\mathfrak{p}}e^{i\xi\bar\tau_n\circ\bar T^{k_t}\circ\mathfrak{p}}e^{-i\xi\chi\circ T^{k_t+n}}\hat g_{\ell'}\left(T^{k_t+n}\cdot,\xi\right)\right]\\
=\ &\mathbb{E}_\nu\Big[e^{i\xi\chi\circ T^{k_t}}\hat f_\ell(T^{k_t}(\cdot),-\xi)e^{-i\theta.\bar\kappa_{k_t-m_0}\circ\mathfrak{p}-i\xi.\bar\tau_{k_t}\circ\mathfrak{p}}\,e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p}}e^{i\xi\bar\tau_n\circ\mathfrak{p}}\\
&\quad e^{i\theta.\bar\kappa_{k_t-m_0}\circ\bar T^n\circ\mathfrak{p}+i\xi.\bar\tau_{k_t}\circ\bar T^n\circ\mathfrak{p}}e^{-i\xi\chi\circ T^{k_t+n}}\hat g_{\ell'}\left(T^{k_t+n}\cdot,\xi\right)\Big]\\
=\ &\mathbb{E}_{\bar\nu}\Big[F_t(\cdot,\ell,-\xi)e^{-i\theta.\bar\kappa_{k_t-m_0}-i\xi.\bar\tau_{k_t}}\,e^{i\theta\cdot\bar\kappa_n}e^{i\xi\bar\tau_n}\\
&\quad e^{i\theta.\bar\kappa_{k_t-m_0}\circ\bar T^n+i\xi.\bar\tau_{k_t}\circ\bar T^n}G_t\left(\bar T^n(\cdot),\ell',\xi\right)\Big]+O\left(\vartheta^{k_t}d_{\ell,\ell'}(\xi)\right),
\end{aligned} \tag{2.50}$$

with $d_{\ell,\ell'}(\xi) := \left(\|e^{i\xi\cdot\chi}\hat{f}_\ell(\cdot,-\xi)\|_{\mathcal{V}}\,\|e^{-i\xi\cdot\chi}\hat{g}_{\ell'}(\cdot,\xi)\|_{\mathcal{V}}\right)$ where we used

- the T-invariance of ν and the definitions of $\bar{\kappa}$ and $\bar{\tau}$ in the first equation,
- the identities $\bar{\kappa}_n\circ\bar{T}^{k_t-m_0}=\bar{\kappa}_n-\bar{\kappa}_{k_t-m_0}+\bar{\kappa}_{k_t-m_0}\circ\bar{T}^n$ and $\bar{\tau}_n\circ\bar{T}^{k_t}=\bar{\tau}_n-\bar{\tau}_{k_t}+\bar{\tau}_{k_t}\circ\bar{T}^n$ in the second one,
- (2.36) and $\mathcal{V}\hookrightarrow L^\infty(\nu)$ in the last one.

Now using the properties of Perron-Frobenius operator given by (2.1) and (2.2) we obtain

$$\begin{aligned}&\mathbb{E}_\nu\left[\hat{f}_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat{g}_{\ell'}\left(T^n\cdot,\xi\right)\right]\\ &=\mathbb{E}_{\bar{\nu}}\left[\bar{P}^n_{\theta,\xi}(\bar{F}_{t,-\theta}(\cdot,\ell,-\xi))\bar{G}_{t,\theta}(\cdot,\ell',\xi)\right]+O\left(\vartheta^{k_t}d_{\ell,\ell'}(\xi)\right),\end{aligned}\tag{2.51}$$

where

$$\bar{F}_{t,-\theta}(x,\ell,-\xi):=F_t(x,\ell,-\xi)e^{-i\theta\bar{\kappa}_{k_t-m_0}(x)}e^{-i\xi\bar{\tau}_{k_t}(x)}$$
$$\bar{G}_{t,\theta}(x,\ell',\xi):=G_t(x,\ell',\xi)e^{i\theta\bar{\kappa}_{k_t-m_0}(x)}e^{i\xi\bar{\tau}_{k_t}(x)}.$$

Due to (2.41) and (2.43), substituting (2.51) into (2.49) yields

$$\begin{aligned}C_t(f,g) &= \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-\pi,\pi]^d\times\mathbb{R}}\Big(e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)}\\ &\qquad\mathbb{E}_{\bar{\nu}}\left[\bar{P}^{n-2k_t}_{\theta,\xi}\left(\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)\right)\bar{G}_{t,\theta}(\cdot,\ell',\xi)\right]\Big)\,d\theta d\xi+O(\vartheta^{k_t}).\end{aligned}\tag{2.52}$$

Note that (2.52) is the analogue of (2.21) (with (M,ν), $P^n_{\theta,\xi}$, $\hat{f}_\ell(\cdot,-\xi)$ and $\hat{g}_{\ell'}(\cdot,\xi)$ being replaced by $(\bar{\Delta},\bar{\nu})$, $\bar{P}^{n-2k_t}_{\theta,\xi}$, $\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)$ and $\bar{G}_{t,\theta}(\cdot,\ell',\xi)$, respectively).

Due to (2.37) and (2.38)

$$\|\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathcal{B}}+\|\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathbb{B}}\le 2C_0(1+|\xi|)\|e^{i\xi\chi(\cdot)}\hat{f}_\ell(\cdot,-\xi))\|_{\mathcal{V}}.$$

Next, we estimate

$$\begin{aligned}\|\bar{G}_{t,\theta}(\cdot,\ell,\xi)\|_{\mathcal{B}'} &\le \|\bar{G}_{t,\theta}(\cdot,\ell',\xi)\|_\infty\\ &\le \|e^{-i\xi\chi(\cdot)}\hat{g}_{\ell'}(\cdot,\xi)\|_\infty+\|e^{-i\xi\chi\circ T^n}\hat{g}_{\ell'}(T^n(\cdot),\xi)-\mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat{g}_{\ell'}(\cdot,\xi))\circ\mathfrak{p}\|_\infty\\ &\le (1+C_0)\|e^{-i\xi\chi(\cdot)}\hat{g}_{\ell'}(\cdot,\xi)\|_{\mathcal{V}},\end{aligned}$$

where we used the fact that L^∞ is continuously embedded into $\mathcal{B}'$ in the first line, the definition of G_t and the triangle inequality in the second one and (2.36) and $\mathcal{V}\hookrightarrow L^\infty(\nu)$ in the third one. Therefore, due to (2.43),

$$\forall\gamma>0,\quad\sum_{\ell,\ell'\in\mathbb{Z}^d}\|\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathbb{B}}\|\bar{G}_{t,\theta}(\cdot,\ell',\xi)\|_\infty=O(|\xi|^{-\gamma})\,.$$

Hence, proceeding as in Step 2 of the proof of Theorem 2.1 we obtain that

$$\begin{aligned}C_t(f,g) &\simeq \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-b,b]^{d+1}}e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)}\\ &\qquad\mathbb{E}_{\bar{\nu}}\left[\bar{P}^{n-2k_t}_{\theta,\xi}\left(\bar{P}^{2k_t}_{\theta,\xi}\bar{F}_{t,-\theta}(\cdot,\ell,-\xi)\right)\bar{G}_{t,\theta}(\cdot,\ell',\xi)\right]\,d\theta d\xi\,.\end{aligned}\tag{2.53}$$

Using (2.51) again we obtain

$$\begin{aligned}C_t(f,g) &\simeq \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-b,b]^{d+1}}e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)}\\ &\qquad\mathbb{E}_\nu\left[\hat{f}_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat{g}_{\ell'}\left(T^n\cdot,\xi\right)\right]\,d\theta d\xi\,.\end{aligned}\tag{2.54}$$

Moreover, for every $(\theta,\xi)\in[-b,b]^{d+1}$ and every integer n satisfying $t_-\le n\le t_+$, using Taylor expansion, the following holds true

$$\begin{aligned}&\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\\ =&\sum_{N=0}^{L-1}\frac{1}{N!}\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}*(\theta,\xi)^{\otimes N}\\ &+O\left(\sup_{u\in[0,1],(\theta',\xi')=(u\theta,u\xi)}\left(\frac{\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi')\,e^{i\theta'\cdot\kappa_n}e^{i\xi'\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi'\right)\right]}{\lambda^n_{\theta,\xi}}\right)^{(L)}_{|(\theta',\xi')}|(\theta,\xi)|^L\right).\quad(2.55)\end{aligned}$$

Let us study the derivatives involved in this formula. First, since $\mathbf{\Pi}_{k_t}$ is linear and continuous, for every $m=0,...,L$, we have

$$\left(\mathbf{\Pi}_{k_t}\left(e^{-i\xi\chi}\hat h_\ell(\cdot,\theta,\xi)\right)\right)^{(m)}_{|(\theta,\xi)}=\mathbf{\Pi}_{k_t}\left(\left(e^{-i\xi\chi}\hat h_\ell(\cdot,\theta,\xi)\right)^{(m)}_{|(\theta,\xi)}\right).\quad(2.56)$$

Using (2.56) and (2.50) we obtain the following analogue of (2.51),

$$\left|\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda^{-n}_{\theta,\xi}\right)^{(L)}_{(\theta,\xi)}\right|=$$

$$\left(\mathbb{E}_{\bar\nu}\left[\bar P^{n-2k_t}_{\theta,\xi}\left(\bar P^{2k_t}_{\theta,\xi}\left(\bar F_{t,-\theta}(\cdot,\ell,-\xi)\right)\right)\bar G_{t,\theta}(\cdot,\ell',\xi)\right]\lambda^{-n}_{\theta,\xi}\right)^{(L)}_{(\theta,\xi)}+O\left(\vartheta^{k_t}n^L\widetilde d_{\ell,\ell'}(\xi)\left|\lambda^{-n}_{\theta,\xi}\right|\right)\quad(2.57)$$

with $\widetilde d_{\ell,\ell'}(\xi):=\sup_{m,m'=0,...,L}\left(\left\|\frac{\partial^m}{\partial\xi^m}\left(e^{i\xi\cdot\chi}\hat f_\ell(\cdot,-\xi)\right)\right\|_{\mathcal V}\left\|\frac{\partial^{m'}}{\partial\xi^{m'}}\left(e^{-i\xi\cdot\chi}\hat g_{\ell'}(\cdot,\xi)\right)\right\|_{\mathcal V}\right)$.

Using (2.33), (2.38), (2.39), we find that the first term of (2.57) is bounded from above by

$$C_0^2(1+|\xi|)\sup_{m=0,...,L}k_t^m\widetilde d_{\ell,\ell'}(\xi)\left\|\left((R^{n-2k_t}_{\theta,\xi}/\lambda^n_{\theta,\xi})+\lambda^{-2k_t}_{\theta,\xi})\Pi_{\theta,\xi}\right)^{(L-m)}_{(\theta,\xi)}\right\|_{\mathcal L(\mathcal B,\mathcal B)},$$

which is in $O\left(k_t^L\widetilde d_{\ell,\ell'}(\xi)\left(\frac{\vartheta^{n-2k}}{\vartheta^{\frac{n}{10}}}+\vartheta^{-\frac{k_t}{5L(d+1)}}\right)\right)$. This observation, combined with (2.55), (2.57) and our choice of k_t yields

$$\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda^{-n}_{\theta,\xi}\quad(2.58)$$

$$\begin{aligned}=&\sum_{N=0}^{L-1}\frac{1}{N!}\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda^{-n}_{\theta,\xi}\right)^{(N)}_{|(\theta,\xi)=0}*(\theta,\xi)^{\otimes N}+O\left(n^{\frac{2}{5}}\widetilde d_{\ell,\ell'}(\xi)|(\theta,\xi)|^L\right)\\ &+O\left(n^{-\frac{L+1+d}{2}}\widetilde d_{\ell,\ell'}(\xi)\left|\lambda^{-n}_{\theta,\xi}\right|\right),\quad(2.59)\end{aligned}$$

for $(\theta,\xi)\in[-b,b]^{d+1}$.

Now we apply Lemma 2.4 to conclude that (2.58) is equal to

$$\sum_{N=0}^{L-1}\frac{1}{N!}\mathcal A_N\left(\hat f_\ell,\hat g_{\ell'}\right)*(\xi,\theta)^{\otimes N}+O\left(\widetilde d_{\ell,\ell'}(\xi)\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|(\theta,\xi)|^L+n^{-\frac{L+1+d}{2}}\left|\lambda^{-n}_{\theta,\xi}\right|\right)\right).\quad(2.60)$$

Recalling the notation $a_s:=e^{-\frac12\Sigma*s^{\otimes2}}$ and Lemma A.1, we have

$$\begin{aligned}\lambda^n_s\ =&\ e^{in\xi\nu(\tau)}a_{s\sqrt n}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}n^kA_{j,k}*s^{\otimes j}\quad(2.61)\\ &+\ O\left(a_{s\sqrt n/\sqrt2}\,n^{-\frac{K+1}{2}}(1+|s\sqrt n|^{K_0})\eta(s)\right),\end{aligned}$$

where $\lim_{s\to 0}\eta(s)=0$. Note that the modulus of the dominating term of (2.60) is bounded by $O\left(\widetilde{d}_{\ell,\ell'}(\xi)\right)$ uniformly in $(\theta,\xi)\in[-b,b]^{d+1}$ and that the modulus of λ_s^n in (2.61) is bounded by $O(a_{s\sqrt{n}/\sqrt{2}})$ (the first one follows from Lemma 2.4, the second one follows from (2.48)). Thus multiplying (2.60) and (2.61) we conclude

$$\begin{aligned}
&\mathbb{E}_\nu\left[\hat{f}_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat{g}_{\ell'}\left(T^n\cdot,\xi\right)\right] \qquad (2.62)\\
=&\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\frac{e^{in\xi\nu(\tau)}a_{s\sqrt{n}}n^k}{N!}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*s^{\otimes(N+j)}\\
+&\;O\left(|\lambda_s^n|\widetilde{d}_{\ell,\ell'}(\xi)\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{L+1+d}{2}}\left|\lambda_s^{-n}\right|\right)\right)\\
+&\;O\left(\sum_{N=0}^{L-1}\frac{1}{N!}\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)*s^{\otimes N}a_{s\sqrt{n}/\sqrt{2}}\,n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right)
\end{aligned}$$

where $s=(\theta,\xi)$. This leads to the following error term

$$\begin{aligned}
&O\left(\widetilde{d}_{\ell,\ell'}(\xi)\left(a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L\right)+n^{-\frac{L+1+d}{2}}\right)\right)\\
+&\;O\left(\widetilde{d}_{\ell,\ell'}(\xi)a_{s\sqrt{n}/\sqrt{2}}\,n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right) \qquad (2.63)\\
=&\;O\left(\widetilde{d}_{\ell,\ell'}(\xi)\left(n^{-\frac{L+1+d}{2}}+a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{K+1}{2}}\left(1+|s\sqrt{n}|^{K_0}\right)\eta(s)\right)\right)\right),
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}}a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right)ds\\
=&\;n^{-\frac{d+1}{2}}\int_{\mathbb{R}^{d+1}}a_{s/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}-\frac{L}{2}}|s|^L+n^{-\frac{K+1}{2}}(1+|s|^{K_0})\eta(s/\sqrt{n})\right)ds\\
=&\;o\left(n^{-\frac{K+2+d}{2}}\right).
\end{aligned}$$

Therefore (2.41), (2.54) and (2.62), (2.63) imply

$$C_t(f,g)\simeq\frac{1}{(2\pi)^{d+1}}\sum_{N=0}^{L-1}\frac{1}{N!}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\mathcal{I}_{\ell,\ell',n}^{N,k,j}, \qquad (2.64)$$

where

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}=n^k\int_{[-b,b]^{d+1}}e^{-i\xi(t-n\nu(\tau))}\,e^{-i\theta\cdot(\ell'-\ell)}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*(\theta,\xi)^{\otimes(N+j)}a_{\sqrt{n}(\theta,\xi)}\,d\theta d\xi\,.$$

By changing variables, we see that

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}=n^{-\frac{d+1+N+j-2k}{2}}\int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*e^{-i\frac{\xi(t-n\nu(\tau))}{\sqrt{n}}}\,e^{-i\frac{\theta\cdot(\ell'-\ell)}{\sqrt{n}}}(\theta,\xi)^{\otimes(N+j)}a_{\theta,\xi}\,d\theta d\xi.$$

At first sight, this expression looks simpler than (2.23) since $\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)$ does not depend on ξ and so no convolution is involved when taking the inverse Fourier transform. Namely we obtain

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}\approx(2\pi)^{d+1}n^{-\frac{d+1+N+j-2k}{2}}i^{N+j}\Psi^{(N+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right), \quad (2.65)$$

where $\mathcal{I}\approx\mathcal{I}'$ means that (2.64) holds for $\mathcal{I}$ and $\mathcal{I}'$ at the same time (i.e. the difference obtained when substituting $\mathcal{I}$ and $\mathcal{I}'$ to (2.64) is in $o\left(t^{-\frac{K+d}{2}}\right)$). Now recall the definition $\mathcal{B}_N$ from (2.46).

Note that the difference between $\mathcal{A}_N$ and $\mathcal{B}_N$ is that the latter one is defined for function that do not depend on ξ. Thus

$$\mathcal{A}_N\left(\hat{f}_\ell, \hat{g}_{\ell'}\right) = \sum_{m_1+m_2+m_3=N} \frac{N!}{m_1!m_2!m_3!}(-1)^{m_1}\mathcal{B}_{m_2}\left((\hat{f}(.,\ell,\xi))^{(m_1)}_{|\xi=0}, (\hat{g}(.,\ell,\xi))^{(m_3)}_{|\xi=0}\right). \tag{2.66}$$

Note that

$$(\hat{f}(x,\ell,\xi))^{(m_1)}_{|\xi=0}(\hat{g}(y,\ell,\xi))^{(m_3)}_{|\xi=0} = \int_{\mathbb{R}^2}(iu)^{m_1}(iv)^{m_3}f(x,\ell,u)g(y,\ell,v)dudv.$$

Hence (2.66) is equal to

$$\sum_{m_1+m_2+m_3=N}\frac{N!}{m_1!m_2!m_3!}\int_{\mathbb{R}^2}(0,-iu)^{\otimes m_1}\otimes(0,iv)^{\otimes m_3}\otimes\mathcal{B}_{m_2}\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv.$$

Now using the binomial theorem, we find that (2.66) is equal to

$$\sum_{m=0}^{N}\frac{N!}{m!(N-m)!}\int_{\mathbb{R}^2}(0,i(v-u))^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv.$$

Substituting this into (2.65) and using (2.64) and the identity $(-1)^{N-m}i^{N+N-m}=i^m$, we find

$$C_t(f,g)\simeq\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{n=t_-}^{t_+}\frac{1}{m!(N-m)!}i^{m+j}n^{-\frac{d+1+N+j-2k}{2}}$$
$$\Psi^{(N+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

Now proceeding as in Step 5 of the proof of Theorem 2.1 we get

$$C_t(f,g)\simeq\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{r=0}^{K-N-j+2k}\sum_{n=t_-}^{t_+}\frac{i^{m+j}}{m!(N-m)!r!n^{\frac{d+1+N+j+r-2k}{2}}}$$
$$\Psi^{(N+j+r)}\left(0,\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*(\ell'-\ell)^{\otimes r}\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

Performing summation over n as in Step 6 of the proof of Theorem 2.1 (using again Lemma A.3), we derive

$$C_t(f,g)\simeq\sum_{N=0}^{K}\sum_{k=0}^{\lfloor K/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{r=0}^{K-N-j+2k}\sum_{q=0}^{K+2k-N-j-r}\frac{1}{m!(N-m)!r!q!}i^{m+j}$$
$$\frac{(t/\nu(\tau))^{-\frac{d+N+j+r+q-2k}{2}}}{(\nu(\tau))^{\frac{q+1}{2}}}\int_{\mathbb{R}}\partial_2^q h_{N+j+r,k-\frac{N+j+d+r+1}{2}}(s,1)(-s)^q\,ds$$
$$*(\ell'-\ell)^{\otimes r}\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

We will set $R=N-m+r$. The binomial theorem tells us that, m,j,k being fixed, for every $R=0,...,K-m-j+2k$, the following identity holds true

$$\sum_{(r,N)\,:\,N-m+r=R}\frac{R!}{(N-m)!r!}(\ell'-\ell)^{\otimes r}\otimes(0,u-v)^{\otimes N-m}=(\ell'-\ell,u-v)^{\otimes R}.$$

We conclude that

$$C_t(f,g)\simeq\sum_{\ell,\ell'}\sum_{m=0}^{K}\sum_{k=0}^{\lfloor K/(J-2)\rfloor}\sum_{j=kJ}^{K-m+2k}\sum_{R=0}^{K-m-j+2k}\sum_{q=0}^{K+2k-m-j-R}\frac{i^{m+j}(t/\nu(\tau))^{-\frac{m+j+d+R+q-2k}{2}}}{R!m!\,q!\,(\nu(\tau))^{\frac{q+1}{2}}}$$

$$\int_{\mathbb{R}} \partial_2^q h_{m+j+R,k-\frac{m+j+d+R+1}{2}}(s,1)(-s)^q\,ds$$
$$*\left(\int_{\mathbb{R}^2} \mathcal{B}_m\left(f_\ell(\cdot,u),g_{\ell'}(\cdot,v)\right)\otimes(\ell'-\ell,u-v)^{\otimes R}\,dudv\otimes A_{j,k}\right).$$

This implies the theorem. □

Proof of Lemma 2.4. Let $N\in\{0,...,L\}$ be fixed. Let us prove that, for every N,

$$\left(\mathcal{A}_{N,n}(u,v):=\left(\mathbb{E}_\nu\left[u(\cdot,-\theta,-\xi)e^{i\theta\cdot\kappa_n+i\xi\tau_n}v(T^n(\cdot),\theta,\xi)\right]\lambda_{(\theta,\xi)}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}\right)_n$$

is a Cauchy sequence. Observe that (2.50) is valid with k_t replaced by any integer k such that $m_0\le k\le n$. That is, for such k we have

$$\begin{aligned}\mathcal{A}_{N,n}(u,v) \;=\; &\Big(\mathbb{E}_\nu\Big[\left(e^{i\xi\chi\circ T^k}u(T^k(\cdot),-\theta,-\xi)e^{-i\theta\bar\kappa_{k-m_0}\circ\mathfrak{p}-i\xi\bar\tau_k\circ\mathfrak{p}}\right)e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p}+i\xi\bar\tau_n\circ\mathfrak{p}}\\ &e^{i\theta\bar\kappa_{k-m_0}\circ\bar T^n\circ\mathfrak{p}+i\xi\bar\tau_k\circ\bar T^n\circ\mathfrak{p}}e^{-i\xi\chi\circ T^{n+k}}v(T^{n+k}(\cdot),\theta,\xi)\Big]\lambda_{(\theta,\xi)}^{-n}\Big)^{(N)}_{|(\theta,\xi)=0}.\end{aligned}$$

Thus, we obtain

$$\mathcal{A}_{N,n}(u,v)=\widetilde{\mathcal{A}}_{N,n}\left(\tilde U_k,\tilde V_k\right), \tag{2.67}$$

where

$$\widetilde{\mathcal{A}}_{N,n}(U,V)=\left(\mathbb{E}_\nu\left[U(\cdot,-\theta,-\xi)e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p}+i\xi\bar\tau_n\circ\mathfrak{p}}V(T^n(\cdot),\theta,\xi)\right]\lambda_{(\theta,\xi)}^{-n}\right)^{(N)}_{|(\theta,\xi)=0},$$
$$\tilde U_k(\cdot,\theta,\xi):=(e^{-i\xi\chi}u(\cdot,\theta,\xi))\circ T^k.e^{i(\theta\cdot\bar\kappa_{k-m_0}+\xi\bar\tau_k)\circ\mathfrak{p}},$$

and

$$\tilde V_k(\cdot,\theta,\xi):=(e^{-i\xi\cdot\chi}v(\cdot,\theta,\xi))\circ T^k.e^{i(\theta\cdot\bar\kappa_{k-m_0}+\xi\cdot\bar\tau_k)\circ\mathfrak{p}}.$$

Recall (2.36) and denote

$$U_k(\cdot,\theta,\xi):=\mathbf{\Pi}_k(e^{-i\xi\chi}u(\cdot,\theta,\xi)).e^{i(\theta\cdot\bar\kappa_{k-m_0}+\xi\bar\tau_k)}\quad\text{and}\quad V_k(\cdot,\theta,\xi):=\mathbf{\Pi}_k(e^{-i\xi\chi}v(\cdot,\theta,\xi)).e^{i(\theta\cdot\bar\kappa_{k-m_0}+\xi\bar\tau_k)}.$$

Since $\mathbf{\Pi}_k$ is linear and continuous and since $(\theta,\xi)\mapsto e^{-i\xi\cdot\chi}u(\cdot,\theta,\xi)$ is L times differentiable at 0 as a $\mathcal{V}$-valued function, for every $m=0,...,L$, we have

$$\left(\mathbf{\Pi}_k\left(e^{-i\xi\chi}u(\cdot,\theta,\xi)\right)\right)^{(m)}_{|(\theta,\xi)=0}=\mathbf{\Pi}_k\left(\left(e^{-i\xi\chi}u(\cdot,\theta,\xi)\right)^{(m)}_{|(\theta,\xi)=0}\right). \tag{2.68}$$

Thus

$$\begin{aligned}&\left\|\left(e^{-i\xi\chi\circ T^k}u(T^k(\cdot),\theta,\xi)\right)^{(m)}_{|(\theta,\xi)=0}-\left(\mathbf{\Pi}_k(e^{-i\xi\chi}u(\cdot,\theta,\xi))\right)^{(m)}_{|(\theta,\xi)=0}\circ\mathfrak{p}\right\|_\infty\\ &\qquad\le C_0\vartheta^k\left\|\left(e^{-i\xi\chi}u(\cdot,\theta,\xi)\right)^{(m)}_{|(\theta,\xi)=0}\right\|_{\mathcal{V}}\le C_0\vartheta^k\|u\|_{\mathcal{W}},\end{aligned} \tag{2.69}$$

and idem by replacing u by v. Next, observe that

$$\|\bar\tau_n^m+|\bar\kappa_n|^m\|_\infty+\left|\left(\lambda^{-n}\right)^{(m)}_{|(\theta,\xi)=0}\right|=O(n^m). \tag{2.70}$$

Combining (2.68), (2.69), and (2.70) we obtain

$$\begin{aligned}&\mathcal{A}_{N,n}(u,v)-\widetilde{\mathcal{A}}_{N,n}(U_k\circ\mathfrak{p},V_k\circ\mathfrak{p})=\widetilde{\mathcal{A}}_{N,n}(\widetilde U_k,\widetilde V_k)-\widetilde{\mathcal{A}}_{N,n}(U_k\circ\mathfrak{p},V_k\circ\mathfrak{p}) \qquad (2.71)\\ =\;&\left(\mathbb{E}_\nu\left[e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\left(\tilde U_k(\cdot,-\theta,-\xi)\tilde V_k(T^n(\cdot),\theta,\xi)-U_k(\mathfrak{p}(\cdot),-\theta,-\xi)V_k(\mathfrak{p}(T^n(\cdot)),\theta,\xi)\right)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}\\ =\;&O\left(n^N\vartheta^k\|u\|_{\mathcal{W}}\|v\|_{\mathcal{W}}\right).\end{aligned}$$

Let $k_n := \lceil \log^2 n \rceil$. Take $n' \in [n, 2n]$. Using (2.71) we obtain

$$\begin{aligned}|\mathcal{A}_{N,n}(u,v) &- \mathcal{A}_{N,n'}(u,v)| \\ &\leq \left|\widetilde{\mathcal{A}}_{N,n}(U_{k_n}\circ\mathfrak{p}, V_{k_n}\circ\mathfrak{p}) - \widetilde{\mathcal{A}}_{N,n'}(U_{k_n}\circ\mathfrak{p}, V_{k_n}\circ\mathfrak{p})\right| + O\left(n^N\|u\|_{\mathcal{W}}\,\|v\|_{\mathcal{W}}\vartheta^{k_n}\right).\end{aligned}$$

The main term on the RHS equals to

$$\mathbb{E}_{\bar\nu}\left[\left((\lambda_t^{-n}\bar P_t^{n-2k_n} - \lambda_t^{-n'}\bar P_t^{n'-2k_n})\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)V_{k_n}(\cdot,t)\right)^{(N)}_{|t=0}\right]. \tag{2.72}$$

Since $\lambda_t^{-\tilde n}\bar P_t^{\tilde n-2k_n} = \lambda_t^{-2k_n}\Pi_t + \lambda_t^{-\tilde n}R_t^{\tilde n-2k_n}$ we can use the definition of $\mathcal{B}'$ to bound (2.72) by

$$\left\|\left((\lambda_t^{-n}R_t^{n-2k_n} - \lambda_t^{-n'}R_t^{n'-2k_n})\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)V_{k_n}(\cdot,t)\right)^{(N)}_{|t=0}\right\|_{L^1(\bar\nu)} \leq$$

$$\begin{aligned}\leq C_N &\max_{n'\in[n,2n],1\leq m_1\leq N}(\lambda_t^{-n'})^{(m_1)}_{|t=0}\left(\max_{1\leq m_2\leq N}\left\|(R_t^{n-2k_n})^{(m_2)}_{|t=0}\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \max_{1\leq m_2\leq N}\left\|(R_t^{n'-2k_n})^{(m_2)}_{|t=0}\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right)\\ &\times\left\|\max_{1\leq m_3\leq N}\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)^{(m_3)}_{|t=0}\right\|_{\mathcal{B}}\left\|\max_{1\leq m_4\leq N}V_{k_n}(\cdot,t)^{(m_4)}_{|t=0}\right\|_{\mathcal{B}'}.\end{aligned}$$

Now observe that the max over m_2 is bounded by $O(\vartheta^{n/2})$ by (2.34) and the other terms cannot grow faster than a polynomial in n. In particular, we use (2.38) to bound the max over m_3 and (2.39) to bound the max over m_4. We conclude that (2.72) is exponentially small.

Therefore, for each $\bar L \in \mathbb{N}$ we have

$$\begin{aligned}\sup_{\bar n\geq 0}|\mathcal{A}_{N,n}(u,v) - \mathcal{A}_{N,n+\bar n}(u,v)| &\leq \sum_{p\geq 0}\sup_{\bar n=0,\dots,2^pn}|\mathcal{A}_{N,2^pn}(u,v) - \mathcal{A}_{N,2^pn+\bar n}(u,v)|\\ \leq \left(\sum_{p\geq 0}(2^pn)^{-\bar L}\|u\|_{\mathcal{W}}\,\|v\|_{\mathcal{W}}\right) &= O\left(\|u\|_{\mathcal{W}}\,\|v\|_{\mathcal{W}}\,n^{-\bar L}\right).\end{aligned}$$

Hence $\mathcal{A}_N(u,v)$ is well defined and satisfies

$$|\mathcal{A}_{N,n}(u,v) - \mathcal{A}_N(u,v)| = O\left(\|u\|_{\mathcal{W}}\,\|v\|_{\mathcal{W}}\,n^{-\bar L}\right). \quad \square$$

3. Mixing expansion for the Sinai billiard flow

3.1. **Sinai billiards.** In the plane $\mathbb{R}^2$, we consider a $\mathbb{Z}^2$-periodic locally finite family of scatterers $\{O_i + \ell;\ i = 1,\dots,I,\ \ell\in\mathbb{Z}^2\}$. We assume that the sets $O_i+\ell$ are disjoint, open, strictly convex and their boundaries are C^3 smooth with strictly positive curvature.

The dynamics of the Lorentz gas can be described as follows. A point particle of unit speed is flying freely in the interior of $\tilde{\mathcal{Q}} = \mathbb{R}^2\setminus\cup_{\ell,i}(O_i+\ell)$ and undergoes elastic collisions on $\partial\tilde{\mathcal{Q}}$ (that is, the angle of reflection equals the angle of incidence). Throughout this paper we assume the so-called finite horizon condition, i.e. that the free flight is bounded. The same dynamics on the compact domain is called Sinai billiard. The position of the particle is a point $q\in\tilde{\mathcal{Q}}$ and its velocity is a vector $v\in\mathcal{S}^1$ (as the speed is identically 1). Since collisions happen instantaneously, the pre-collisional and post-collisional data are identified. By convention, we use the post-collisional data, i.e. whenever $q\in\partial\tilde{\mathcal{Q}}$, we assume that v satisfies $\vec n_q.v\geq 0$, where . stands for the scalar product and $\vec n_q$ is the unit vector normal to $\partial\tilde{\mathcal{Q}}$ directed inward $\tilde{\mathcal{Q}}$. The phase space, that is, the set of all possible positions and velocities, will be denoted by $\tilde{\boldsymbol{\Omega}} = \tilde{\mathcal{Q}}\times\mathcal{S}^1$.

The billiard flow is denoted by $\tilde{\boldsymbol{\Phi}}_t : \tilde{\boldsymbol{\Omega}}\to\tilde{\boldsymbol{\Omega}}$, where $t\in\mathbb{R}$. Let $\tilde{\boldsymbol{\mu}}_0$ be the Lebesgue measure on $\tilde{\boldsymbol{\Omega}}$ normalized so that $\tilde{\boldsymbol{\mu}}_0((\tilde{\mathcal{Q}}\cap[0,1]^2)\times S^1) = 1$.

The Sinai billiard is defined analogously on a compact domain. That is, we consider disjoint strictly convex open subsets $\bar{O}_i \subset \mathbb{T}^2$ (corresponding to the canonical projection of O_i), $i = 1, ..., I$, whose boundaries are $\mathcal{C}^3$ smooth with strictly positive curvature. Then we put $\boldsymbol{\mathcal{Q}} = \mathbb{T}^2 \setminus \cup_i O_i$. We define the billiard dynamics $(\boldsymbol{\Omega}, \boldsymbol{\Phi}_t, \boldsymbol{\mu}_0)$ exactly as $(\tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Phi}}_t, \tilde{\boldsymbol{\mu}}_0)$ except that we use the billiard table $\boldsymbol{\mathcal{Q}}$ instead of $\tilde{\boldsymbol{\mathcal{Q}}}$ and $\boldsymbol{\mu}_0$ is a probability measure.

Next, we represent the flow $\boldsymbol{\Phi}_t$ as a suspension over a map. This map is called the billiard ball map: the Poincaré section of $\boldsymbol{\Phi}_t$ corresponding to the collisions. That is, we define

$$\boldsymbol{M} = \{(q, v) \in \boldsymbol{\Omega} : q \in \partial \boldsymbol{\mathcal{Q}}\} = \{(q, v) \in \boldsymbol{\Omega} : q \in \partial \boldsymbol{\mathcal{Q}}, \vec{n}_q.v \geq 0\}.$$

$\boldsymbol{T} : \boldsymbol{M} \to \boldsymbol{M}$ is defined by $\boldsymbol{T}(x) = \boldsymbol{\Phi}_{\boldsymbol{\tau}}(x)$, where $\boldsymbol{\tau} = \boldsymbol{\tau}(x)$ is the smallest positive number such that $\boldsymbol{\Phi}_{\boldsymbol{\tau}}(x) \in \boldsymbol{M}$. The projection of $\boldsymbol{\mu}_0$ to the Poincaré section is denoted by $\boldsymbol{\nu}$. In fact, $\boldsymbol{\nu}$ has the density $c\vec{n}_q.vdqdv$, where $c = 2|\partial\boldsymbol{\mathcal{Q}}|$ is a normalizing constant such that $\boldsymbol{\nu}$ is a probability measure. Clearly, we can write

$$\boldsymbol{\Omega} = \{(x, t), x \in \boldsymbol{M},\ t \in [0, \boldsymbol{\tau}(x))\}.$$

With this notation, we have $\boldsymbol{\mu}_0 = \frac{1}{\boldsymbol{\nu}(\boldsymbol{\tau})}\boldsymbol{\nu} \otimes \mathfrak{l}$, where $\mathfrak{l}$ is the Lebesgue measure on $[0, +\infty)$. Note that the measure $\boldsymbol{\mu}_0$ is a probability measure unlike μ defined in Section 2.1.

Finally, we define the measure preserving dynamical system $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{\nu}})$ analogously to the Lorentz gas. For every $\ell \in \mathbb{Z}^2$, we define the ℓ-cell $\mathcal{C}_\ell$ as the set of the points with last reflection off $\widetilde{\boldsymbol{\mathcal{Q}}}$ took place in the set $\bigcup_{i=1}^I (O_i + \ell)$. Identifying $\mathbb{T}^2$ with the unit square $[0, 1)^2 \subset \mathbb{R}^2$, we see that $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{\nu}})$ is the $\mathbb{Z}^2$-extension of $(\boldsymbol{M}, \boldsymbol{T}, \boldsymbol{\nu})$ by $\boldsymbol{\kappa} : \boldsymbol{M} \to \mathbb{Z}^2$, where $\boldsymbol{\kappa}(x) = \ell$ if $\tilde{\boldsymbol{T}}(x) \in \mathcal{C}_\ell$.

The observable $(\boldsymbol{\kappa}, \boldsymbol{\tau}) : \boldsymbol{M} \to \mathbb{Z}^2 \times \mathbb{R}$ satisfies the central limit theorem (see e.g. [10]). That is, there exists a 3×3 positive definite matrix $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$ so that for any $A \subset \mathbb{R}^3$ whose boundary has zero Lebesgue measure

$$\boldsymbol{\nu}\left(x \in \boldsymbol{M} : \frac{(\boldsymbol{\kappa}_n, \boldsymbol{\tau}_n - n\nu(\tau))}{\sqrt{n}} \in A\right) = \int_A \Psi_{\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}$$

where Ψ is the Gaussian density defined by (2.3). Consequently, the central limit theorem holds for the observable $\boldsymbol{\kappa}$ with a covariance matrix $\Sigma_{\boldsymbol{\kappa}}$, which is obtained from $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$ by deleting the last row and the last column.

Denote

$$\|\mathfrak{h}\|_{\mathcal{H}^\eta_E} = \sup_{y \in E} |\mathfrak{h}(y)| + \sup_{y,z \in E,\ y \neq z} \frac{|\mathfrak{h}(y) - \mathfrak{h}(z)|}{d(y, z)^\eta}.$$

We will say that a function $\mathfrak{h} : \tilde{\boldsymbol{\Omega}} \to \mathbb{R}$ is *smooth in the flow direction* if

$$\forall N \geq 0 \quad \sum_\ell \left\| \frac{\partial^N}{\partial s^N} \left(\mathfrak{h} \circ \widetilde{\boldsymbol{\Phi}}_s\right)_{|s=0} \right\|_{\mathcal{H}^\eta_{\mathcal{C}_\ell}} < \infty\,. \tag{3.1}$$

Note that in order for (3.1) to hold, it is sufficient that $\mathfrak{h}$ is C^∞ in the position $q \in \widetilde{\boldsymbol{\mathcal{Q}}}$ and satisfies

$$\forall N \geq 0, \quad \sum_\ell \left\| \frac{\partial^N}{\partial q^N} \mathfrak{h} \right\|_{\mathcal{H}^\eta_{\mathcal{C}_\ell}} < \infty\,,$$

$$\forall (q, \vec{v}) \in \partial\widetilde{\boldsymbol{\mathcal{Q}}} \times S^1, \quad \frac{\partial^N}{\partial q^N}\mathfrak{h}(q, \vec{v}) = \frac{\partial^N}{\partial q^N}\mathfrak{h}\left(q, \vec{v} - 2(\vec{n}_q.\vec{v})\vec{n}_q\right)\,. \tag{3.2}$$

We say that $\mathfrak{h} : \tilde{\boldsymbol{\Omega}} \to \mathbb{R}$ is *η-Hölder continuous* if it is η-Hölder continuous on $\tilde{Q} \times S^1$ and satisfies (3.2) with $N = 0$.

Now we are ready to formulate the main result of this section.

Theorem 3.1. *Let $\mathfrak{f},\mathfrak{g}:\tilde{\boldsymbol{\Omega}}\to\mathbb{R}$ be two η-Hölder continuous functions with at least one of them smooth in the flow direction. Assume moreover that there exists an integer $K_0\ge 1$ such that*

$$\sum_{\ell}(1+|\ell|)^{2K_0}\left(\|\mathfrak{f}\|_{\mathcal{H}^{\eta}_{\mathcal{C}_\ell}}+\|\mathfrak{g}\|_{\mathcal{H}^{\eta}_{\mathcal{C}_\ell}}\right)<\infty\,. \tag{3.3}$$

Then there are real numbers $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g}),\mathfrak{C}_1(\mathfrak{f},\mathfrak{g}),...,\mathfrak{C}_{K_0}(\mathfrak{f},\mathfrak{g})$ so that we have

$$\int_{\tilde{\boldsymbol{\Omega}}}\mathfrak{f}\,\mathfrak{g}\circ\tilde{\Phi}_t d\tilde{\boldsymbol{\mu}}_0=\sum_{k=0}^{K_0}\mathfrak{C}_k(\mathfrak{f},\mathfrak{g})t^{-1-k}+o(t^{-1-K_0})\,, \tag{3.4}$$

as $t\to+\infty$. Furthermore, $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g})=\mathfrak{c}_0\int_{\tilde{\boldsymbol{\Omega}}}\mathfrak{f}d\tilde{\boldsymbol{\mu}}_0\int_{\tilde{\boldsymbol{\Omega}}}\mathfrak{g}d\tilde{\boldsymbol{\mu}}_0$ with

$$\mathfrak{c}_0=\frac{\boldsymbol{\nu}(\boldsymbol{\tau})}{2\pi\sqrt{\det\Sigma_{\boldsymbol{\kappa}}}} \tag{3.5}$$

and the coefficients $\mathfrak{C}_k$, as functionals over pairs of admissible functions, are bilinear.

We note that the bilinear forms $\mathfrak{C}_k$ are linearly independent. Namely in Appendix B we give examples of functions f_k,g_k such that $\mathfrak{C}_k(f_k,g_k)\neq 0$ while $\mathfrak{C}_j(f_k,g_k)=0$ for all $j<k$.

In the remaining part of Section 3, we derive Theorem 3.1 from Theorem 2.3. We will not be applying Theorem 2.3 directly to $(\boldsymbol{M},\boldsymbol{\nu},\boldsymbol{T})$, but instead we apply it to the Young tower extension of the Sinai billiard. Thus we first briefly review the Young tower construction in Section 3.2. Then we prove condition (B4) in Section 3.3 along the lines of [12]. We complete the proof of Theorem 3.1 in Section 3.4 and finally (3.5) is established in Section 3.5.

3.2. **Young towers.** Let $\mathcal{R}\subset\boldsymbol{M}$ be the hyperbolic product set constructed in [46, Section 8]. Furthermore, let (Δ,F) be the corresponding Young tower ("Markov extension"). There is a natural bijection ι between Δ_0, the base of the tower and $\mathcal{R}$. We will denote points of $\mathcal{R}$ by $x=(\gamma^u,\gamma^s)$, which is to be interpreted as $\gamma^u\cap\gamma^s$, where $\gamma^u=\gamma^u(x)$ and $\gamma^s=\gamma^s(x)$ are an unstable and a stable manifold containing x. Points of Δ_0 will be denoted by $\hat{x}=(\hat{\gamma}^u,\hat{\gamma}^s)$. Note that ι can be extended to π, a mapping from Δ to $\boldsymbol{M}$ (this map is in general not one-to-one).

We recall the most important ingredients of the construction of [46]. The base of the tower has the product structure $X=\Delta_0=\Gamma^u\times\Gamma^s$. The sets of the form $A\times\Gamma^s$, $A\subset\Gamma^u$ are called u-sets if $\iota(A\subset\Gamma^u)$ is compact. Similarly, sets of the form $\Gamma^u\times B$, $B\subset\Gamma^s$ are called s-sets if $\iota(B\subset\Gamma^u)$ is compact. Also, sets of the form $\Gamma^u\times\{\hat{\gamma}^s\}$ are called stable manifolds and sets of the form $\{\hat{\gamma}^u\}\times\Gamma^s$ are unstable manifolds as they are images of (un)stable manifolds (or rather, the intersections of (un)stable manifolds and $\mathcal{R}$) by the map ι^{-1}. Δ_0 has a partition $\Delta_0=\bigcup_{k\in\mathbb{Z}_+}\Delta_{0,k}$, where $\Delta_{0,k}=\Gamma^u\times\Gamma^s_k$ are s-sets. The return time to the base on the set $\Delta_{0,k}$ is identically r_k, that is $\Delta=\bigcup_{k\in\mathbb{Z}_+}\bigcup_{l=0}^{r_k-1}\Delta_{l,k}$, where $\Delta_{l,k}=\{(\hat{x},l):\hat{x}\in\Delta_{0,k}\}$. There is an F-invariant measure ν on Δ so that $\pi_*\nu=\mu$ and F is an isomorphism between $\Delta_{l,k}$ and $\Delta_{l+1,k}$ and $F(\hat{x},l)=(\hat{x},l+1)$ if $l<r_k-1$. Also F is an isomorphism between $\Delta_{r_k-1,k}$ and $F(\Delta_{r_k-1,k})$, the latter being a u-set of Δ_0. Furthermore, if $\hat{x}_1,\hat{x}_2\in\Delta_{0,k}$ belong to the same (un)stable manifold, so do $F^{r_k}(\hat{x}_1,0)$ and $F^{r_k}(\hat{x}_2,0)$. We write $\mathcal{F}=F^{r_k-l}$ on $\Delta_{l,k}$ and $r(\hat{\gamma}^u,\hat{\gamma}^s)=r(\hat{\gamma}^s)=r_k$ for $(\hat{\gamma}^u,\hat{\gamma}^s)\in\Delta_{0,k}$. Define Ξ on Δ by

$$\Xi((\hat{\gamma}^u,\hat{\gamma}^s),l)=((\hat{\mathfrak{I}}^u,\hat{\gamma}^s),l)\text{ with a fixed }\hat{\mathfrak{I}}^u\in\Gamma^u. \tag{3.6}$$

Let $\bar{\Delta}=\Xi(\Delta)$ and $\bar{\nu}=\Xi_*\nu$. There is a well defined $\bar{F}:\bar{\Delta}\to\bar{\Delta}$ such that $\Xi\circ F=\bar{F}\circ\Xi$. The dynamical system $(\bar{\Delta},\bar{\nu},\bar{F})$ is an expanding tower in the sense that it satisfies assumptions (E1)–(E5) below.

Let $(\bar{\Delta}, \bar{\nu}, \bar{F})$ be a probability preserving dynamical system with a partition $(\bar{\Delta}_{l,k})_{k\in I, l=0,\dots,r_k-1}$ into positive measure subsets, where I is either finite or countable and $r_k = r(\bar{\Delta}_{0,k})$ is a positive integer. We call it *an expanding tower* if

(E1) for every $i \in I$ and $0 \le j < r_i - 1$, $\bar{F}$ is a measure preserving isomorphism between $\bar{\Delta}_{j,i}$ and $\bar{\Delta}_{j+1,i}$.

(E2) for every $i \in I$, $\bar{F}$ is an isomorphism between $\bar{\Delta}_{r_i-1,i}$ and

$$\bar{X} := \bar{\Delta}_0 := \bigcup_{i\in I} \bar{\Delta}_{0,i}.$$

(Remark on convention: points of the space $\bar{X}$ are identified with and sometimes denoted as stable manifolds.)

(E3) Let $r(x) = r(\bar{\Delta}_{0,k})$ if $x \in \bar{\Delta}_{0,k}$ and $\bar{\mathcal{F}} : \bar{X} \to \bar{X}$ be the first return map to the base, i.e. $\bar{\mathcal{F}}(x) = \bar{F}^{r(x)}(x)$. Let $s(x, y)$, the separation time of $x, y \in \bar{X}$, be defined as the smallest integer n such that $\bar{\mathcal{F}}^n x \in \bar{\Delta}_{0,i}$, $\bar{\mathcal{F}}^n y \in \bar{\Delta}_{0,j}$ with $i \ne j$. As $\bar{\mathcal{F}} : \bar{\Delta}_{0,i} \to \bar{X}$ is an isomorphism, it has an inverse. Denote by α the logarithm of the Jacobian of this inverse (w.r.t. the measure $\bar{\nu}$). Then there are constants $\vartheta_\alpha < 1$ and $C > 0$ such that for every $x, y \in \bar{\Delta}_{0,i}$,

$$|\alpha(x) - \alpha(y)| \le C\vartheta_\alpha^{s(x,y)}. \tag{3.7}$$

(E4) Extend s to $\bar{\Delta}$ by setting $s(x, y) = 0$ if x, y do not belong to the same $\bar{\Delta}_{j,i}$ and $s(x, y) = s(\bar{F}^{-j}x, \bar{F}^{-j}y)+1$ if $x, y \in \bar{\Delta}_{j,i}$. $(\bar{\Delta}, \bar{\nu}, \bar{F})$ is exact (hence ergodic and mixing) with respect to the metric

$$d_\vartheta(x, y) := \vartheta^{s(x,y)}.$$

Furthermore, in case of Sinai billiards, we have

(E5) $\bar{\nu}(x : r(x) > n) \le C\rho^n$ with some $\rho < 1$.

3.3. **Weak non-lattice property for Sinai billiards.** In this section we verify condition (B4) for Sinai billiards. We note that the methods of this section are similar to those used in some earlier work [12,13,35]. Those methods are useful for proving that the mixing in some uniformly and non-uniformly hyperbolic flows are faster than any polynomial. In the context of Sinai billiard flows, more precise results are available. In particular, [9] proves stretched exponential bounds for the correlations functions. More recently, [4] showed that the correlations in fact decay exponentially. To prove this result, [4] uses Banach spaces which are more sophisticated than the spaces used here. We note that using the spaces from [4] would not improve our result since the decay of correlations for the infinite measure system is actually polynomial, not exponential.

Given a function $f : \boldsymbol{M} \to \mathbb{C}$, we define $\hat{f} : \Delta \to \mathbb{C}$ by $\hat{f} = f \circ \pi$. Now for a function $\hat{f} : \Delta \to \mathbb{C}$ (which may or may not be a lift-up of a function $f : \boldsymbol{M} \to \mathbb{C}$), we write $X = \Delta_0$ and define

$$\hat{f}_X : X \to \mathbb{C}, \quad \hat{f}_X(\hat{x}) = \sum_{j=0}^{r(\hat{x})-1} \hat{f}(F^j(\hat{x}, 0)),$$

$$\bar{f} : \bar{\Delta} \to \mathbb{C}, \quad \bar{f}(\hat{\gamma}^s, l) = \hat{f}(\hat{\mathfrak{I}}^u, \hat{\gamma}^s, l)\,,$$

$$\bar{f}_{\bar{X}} : \bar{X} \to \mathbb{C}, \quad \bar{f}_{\bar{X}}(\hat{\gamma}^s) = \sum_{j=0}^{r(\hat{\gamma}^s)-1} \hat{f}(F^j(\hat{\mathfrak{I}}^u, \hat{\gamma}^s, 0))\,.$$

Fix $\varkappa < 1$ and consider the space of dynamically Lipschitz functions on $\bar{X}$ (w.r.t. the metric $d_\varkappa$):

$$C_\varkappa(\bar{X}, \mathbb{C}) = \{f : \bar{X} \to \mathbb{C} \text{ bounded and } L(f) < \infty\},$$

where

$$L(f) = \inf\{C : \forall x, y \in \bar{X} : |f(x) - f(y)| \le C\varkappa^{s(x,y)}\}.$$

The larger $\varkappa$ is, the bigger the space $C_\varkappa(\bar{X}, \mathbb{C})$ is. It is best to take $\varkappa$ very close to 1 to include many functions in $C_\varkappa(\bar{X}, \mathbb{C})$. For example we assume that $\varkappa > \vartheta_\alpha$. More lower bounds on $\varkappa$ will be imposed in (3.49). The space $C_\varkappa(\bar{X}, \mathbb{C})$ is equipped with the norm

$$\|f\|_\varkappa = L(f) + \|f\|_\infty.$$

Let Q be the Perron-Frobenius-Ruelle operator associated with $\bar{\mathcal{F}}$, i.e.

$$(Qh)(x) = \sum_{y:\bar{\mathcal{F}}y=x} e^{\alpha(y)}h(y)$$

where e^α is the Jacobian defined in (E3). We have for h with $\|h\|_\varkappa < \infty$

$$Q^n h = \bar{\nu}(h)1 + R^n h, \tag{3.8}$$

where $\|R^n h\|_\varkappa \le C\rho^n \|h\|_\varkappa$ for some $\rho < 1$.

Now we introduce the (signed) temporal distance function D on $\mathcal{R}$ by defining

$$D(x,y) = \sum_{\ell=-\infty}^{\infty} [\boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(x), \gamma^s(x))) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(x), \gamma^s(y))) + \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(y), \gamma^s(y))) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(y), \gamma^s(x)))], \tag{3.9}$$

where $\boldsymbol{\tau}$ is defined in §3.1. Note that there is a lift-up $\hat{\boldsymbol{\tau}} : \Delta \to \mathbb{R}_+$ defined by $\hat{\boldsymbol{\tau}}(\hat{x}) = \boldsymbol{\tau}(\pi(\hat{x}))$ and corresponding functions $\hat{\boldsymbol{\tau}}_X, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\tau}}_{\bar{X}}$.

We also define the operators

$$Q_\xi h = Q(e^{i\xi\bar{\boldsymbol{\tau}}_{\bar{X}}} h)\,. \tag{3.10}$$

For real valued functions defined on $\bar{X}$, we will consider the norms

$$\|.\|_\infty, \quad \|.\|_\varkappa, \quad \|.\|_{(\xi)} := \max\left\{\|.\|_\infty, \frac{L(.)}{C_0\xi}\right\}, \tag{3.11}$$

where $\xi \gg 1$ and C_0 is a constant to be specified later.

Next we define several special points in the rectangle $\mathcal{R}$. Namely, let $x_0 \in \mathcal{R}$ be defined by the requirement

$$\iota^{-1}(\boldsymbol{T}^{r_1 k}(x_0)) \in \Delta_{0,1} \quad \text{ for all } k \in \mathbb{Z}$$

and for $m \in \mathbb{Z}_+$ we define $y_m \in \mathcal{R}$ by

$$\iota^{-1}(\boldsymbol{T}^{\sum_{j=0}^{l-1} r_{a_j}}(y_m)) \in \Delta_{0,a_l} \quad \text{ for all } l = 0, 1, 2, ...$$

$$\iota^{-1}(\boldsymbol{T}^{-\sum_{j=l}^{-1} r_{a_j}}(y_m)) \in \Delta_{0,a_l} \quad \text{ for all } l = -1, -2, ...$$

where the sequence $(a_j)_j$ depends on $m \in \mathbb{Z}_+$ and is defined by

$$a_j = \begin{cases} 2 \text{ if } j = -1 \text{ or } j = m-1 \\ 1 \text{ otherwise.} \end{cases}$$

The points $x, y_1, y_2, ...$ exist and are uniquely defined by Axioms (P1) and (P2) of [46] (compactness and Markov intersection). Note that upon each Markov return to Δ (in both positive and negative time), x_0 in fact returns to $\Delta_{0,1}$. Likewise y_m always returns to $\Delta_{0,1}$ except for times -1 and $m-1$, when it returns to $\Delta_{0,2}$. In fact, the forthcoming proof would work for any x_0 and y_m as long as they share their symbolic sequence for Markov return times $-m^{1+\varepsilon}, ..., m$ except for the times -1 and $m-1$ when they differ and their symbolic sequences are bounded. We chose the symbolic sequence to only contain 1 and 2 for simplicity and prescribed infinite orbit for the convenience of a unique definition.

To simplify notation, we write

$$[z_1, z_2] = (\gamma^u(z_1), \gamma^s(z_2)). \tag{3.12}$$

Let $\mathfrak{Q}_m$ be the solid rectangle with corners x_0, $[x_0, y_m]$, y_m, $[y_m, x_0]$, i.e. the unique topological rectangle inside the convex hull of $\mathcal{R}$ which is bounded by two stable and unstable manifolds, such that two of its corners are x_0 and y_m. We claim that there are two constants $0 < c_2 < c_1 < 1$ so that

$$c_2^m < \mu(\mathfrak{Q}_m) < c_1^m \tag{3.13}$$

for sufficiently large m.

To prove this claim, let $\mathfrak{Q}_{0,i}$ denote the smallest topological rectangle containing $\iota(\Delta_{0,i})$ for $i = 1, 2$. Note that $\boldsymbol{T}^{r_1}$ is a $\mathcal{C}^2$ map when restricted to the interior of $\mathfrak{Q}_{0,i}$. By construction, $\boldsymbol{T}^{jr_1}\mathfrak{Q}_m$ is a subset of $\mathfrak{Q}_{0,1}$ for $j = 0, 1, ..., m-2$. Now consider a foliation of $\mathfrak{Q}_m$ by unstable curves. Each such curve is expanded by a factor $\Lambda > 1$ by the $\mathcal{C}^2$-map $\boldsymbol{T}^{r_1}$ and so the upper bound follows. To prove the lower bound, first observe that $\gamma^u(y_m)$ and $\gamma^u(x_0)$ are independent of m (since the past itinerary of y_m does not depend on m) and consequently the stable size of $\mathfrak{Q}_m$ is bounded from below by a positive constant. Next, note that $\boldsymbol{T}^{(m-1)r_1}\mathfrak{Q}_m$ intersects both $\mathfrak{Q}_{0,1}$ and $\mathfrak{Q}_{0,2}$ and so, as we can assume that the distance between $\mathfrak{Q}_{0,1}$ and $\mathfrak{Q}_{0,2}$ is positive, the length of the image of each unstable curve in our foliation under the map $\boldsymbol{T}^{(m-1)r_1}$ is uniformly bounded from below. Furthermore, the expansion of $\boldsymbol{T}^{r_1}$ on $\mathfrak{Q}_{0,1}$ is bounded from above (since $\boldsymbol{T}^{r_1}$ is C^2 on $\mathfrak{Q}_{0,1}$) and so the lower bound follows as well. Next, Lemma 5.1 of [28] states that

$$\mu(\mathfrak{Q}_m) = |D(x_0, y_m)| \tag{3.14}$$

(see also [10, §6.11]). Note that $D(x_0, y_m)$ has another representation: it is the unique small number σ so that $\Phi^\sigma Y_1 = Y_5$, where Φ is the billiard flow, $Y_1, ..., Y_5$ are points whose last collisions were at x_0, $[x_0, y_m]$, y_m, $[y_m, x_0]$, x_0, respectively and the pairs $(Y_1, Y_2), (Y_3, Y_4)$ are on the same stable manifold of Φ while the pairs $(Y_2, Y_3), (Y_4, Y_5)$ are on the same unstable manifold of Φ (see Lemma 6.40 in [10]). The following property of the points x_0 and y_m is crucial.

Lemma 3.2. *There exist some $a_0 > 0$, and $c \in \mathbb{R}_+$ such that for any $\xi > 3$ the unique positive integer $m = m(\xi)$ defined by the property*

$$c_1^m < \xi^{-1} \le c_1^{m-1} \tag{3.15}$$

satisfies

$$|e^{i\xi D(x_0, y_m)} - 1| > c\xi^{-a_0}\,. \tag{3.16}$$

Proof. It is sufficient to prove the lemma for ξ large. Indeed, if we can prove the lemma for $\xi > \xi_0$, then we can extend it to any $\xi > 3$ by choosing c small enough unless there is some $\xi' \in [3, \xi_0]$ so that $\xi' D(x, y) = 0 \pmod{2\pi}$ for all x, y. Note that this cannot happen since this would imply $l\xi' D(x, y) = 0 \pmod{2\pi}$ where we can choose $l \in \mathbb{Z}_+$ so that $l\xi' > \xi_0$.

Now given ξ large, (3.13) and (3.14) imply that m satisfies $c_2^m < |D(x_0, y_m)| < c_1^m$. Thus

$$c_1 c_2 \xi^{1-\frac{\ln c_2}{\ln c_1}} \le c_1^{-m+1} c_2^m \le \xi |D(x_0, y_m)| \le 1$$

proving (3.16). □

Recall the definition of Q_ξ from (3.10). We have

Lemma 3.3. *Suppose that C_0 in (3.11) is large enough. Then there are constants a_1, C_1, d_1 so that for every $\xi > 3$,*

$$\left\| Q_\xi^{C_1 \ln \xi} \right\|_{(\xi)} < 1 - \frac{d_1}{\xi^{a_1}}\,. \tag{3.17}$$

Proof. The proof consists of several steps. We will need several large constants $C_2, C_3, \dots$, $b_1, b_2, \dots$ and small constants $d_2, d_3, \dots$ before being able to define the large constant C_1, a_1 and the small constant d_1.

Let h satisfy $\|h\|_{(\xi)} = 1$. By [8], there exists a constant C_2 such that for every n,

$$L(Q_\xi^n h) \le C_2[\xi\|h\|_\infty + \varkappa^n L(h)]\,, \tag{3.18}$$

(see also Proposition 3.7 in [35]). Thus fixing any $C_0 > 4C_2$ in the definition of the norm $\|.\|_{(\xi)}$ we have that for any $C_3 > -\frac{\ln 4}{(\ln 3)(\ln \varkappa)}$

$$L\left(Q_\xi^{C_3 \ln \xi} h\right) \le C_2[\xi + \varkappa^{C_3 \ln \xi} L(h)] \le C_2[\xi + C_0 \xi^{1 + C_3 \ln \varkappa}] \le C_0 \frac{\xi}{2}. \tag{3.19}$$

In order to prove the lemma, it remains to verify (3.17) for $\|\cdot\|_\infty$ norm.

This proof is divided into three parts:

Step 1. We show that given constants d_2 and b_1 there exist d_3 and b_2 so that for any $\ell = 0, 1, 2, \dots$, $\|Q_\xi^\ell h\|_{L^1} < 1 - \frac{d_3}{\xi^{b_2}}$ assuming the following hypothesis:

(H): for any $\xi > 3$ there is some $u \in \bar{X}_{\le 2} := \{\bar{x} \in \bar{X} : \bar{\mathcal{F}}^n(\bar{x}) \in \bar{\Delta}_{0,1} \cup \bar{\Delta}_{0,2} \text{ for all } n \in \mathbb{N}\}$ so that

$$|h(u)| < 1 - \frac{d_2}{\xi^{b_1}}. \tag{3.20}$$

Let U be the $d_2 \xi^{-b_1 - 1}/(2C_0)$ neighborhood of u (w.r.t the metric $d_\varkappa$) in $\bar{X}$. Since $L(h) \le C_0 \xi$, we have $|h(u')| < 1 - \frac{d_2}{2\xi^{b_1}}$ for any $u' \in U$. By the bounded distortion property and the fact that $u \in \bar{X}_{\le 2}$, there are constants d_4 and b_3 depending only on the billiard table so that

$$\frac{d_4}{\xi^{b_3}} \le \bar{\nu}(U). \tag{3.21}$$

Observing that

$$|Q_\xi^n h| \le Q^n |h| \tag{3.22}$$

holds pointwise (by definition of those operators), and using $\|h\|_\infty \le 1$, we derive that for any ℓ

$$\begin{aligned} \int |Q_\xi^\ell h| d\bar{\nu} &\le \int Q^\ell |h| d\bar{\nu} = \int |h| d\bar{\nu} = \int_U |h| d\bar{\nu} + \int_{\bar{X} \setminus U} |h| d\bar{\nu} \\ &\le \left(1 - \frac{d_2}{2\xi^{b_1}}\right) \bar{\nu}(U) + 1 - \bar{\nu}(U) \le 1 - \frac{d_2}{2\xi^{b_1}} \bar{\nu}(U) \le 1 - \frac{d_2 d_4/2}{\xi^{b_1 + b_3}}\,. \end{aligned}$$

This proves the statement of Step 1 with

$$d_3 = d_2 d_4/2 \quad \text{and} \quad b_2 = b_1 + b_3 \tag{3.23}$$

where d_4 and b_3 come from (3.21).

Step 2. Under hypothesis **(H)**, we show that if C_4 is sufficiently large then there is a constant d_5 so that

$$\left\| Q_\xi^{C_4 \ln \xi} h \right\|_\infty < 1 - \frac{d_5}{\xi^{b_2}}.$$

For any $u \in \bar{X}$, we have

$$\left| Q_\xi^{C_4 \ln \xi} h \right| (u) \le^{(3.22)} \left(Q^{C_4 \ln \xi} |h| \right)(u) \le \bar{\nu}(|h|) + C\xi(C_0 + 1)\rho^{C_4 \ln \xi}\,, \tag{3.24}$$

where the last inequality follows from (3.8) and the following computation

$$\|R^{C_4 \ln \xi} |h|\|_\infty \le \|R^{C_4 \ln \xi} |h|\|_\varkappa \le C\rho^{C_4 \ln \xi} \|h\|_\varkappa \le C\rho^{C_4 \ln \xi}(1 + \xi C_0)\|h\|_{(\xi)}.$$

Combining (3.24) with the result of Step 1 (with $\ell = 0$), we conclude

$$\left\| Q_\xi^{C_4 \ln \xi} h \right\|_\infty \le 1 - \frac{d_3}{\xi^{b_2}} + C\xi(C_0 + 1)\rho^{C_4 \ln \xi} \le 1 - \frac{d_3/2}{\xi^{b_2}},$$

where the last inequality holds if C_4 is so big that for all $\xi > 3$,

$$\xi^{1+C_4 \ln \rho + b_1} \leq \frac{d_3}{2C(C_0+1)}. \tag{3.25}$$

This completes Step 2.

Step 3. In Step 3, we will show the following:

($\star$): there exist constants C_4, d_2 and b_1 so that for any $\xi > 3$, there is some $v \in \bar{X}_{\leq 2}$ that either satisfies (3.20) or satisfies the following:

$$|Q_\xi^n h(v)| < 1 - \frac{d_2}{\xi^{b_1}} \text{ with } n = C_4 \ln \xi. \tag{3.26}$$

Before proving ($\star$), we first prove the lemma assuming ($\star$). Namley, we show that if C_4 is large enough, then for all $\xi > 3$,

$$\left\| Q_\xi^{C_5 \ln \xi} h \right\|_\infty < 1 - \frac{d_5}{\xi^{b_2}} \tag{3.27}$$

with $C_5 = 2C_4$. Indeed, if there is a v satisfying (3.20), then the proof in Step 2 applies with $b_2 = b_1 + b_3$ (see (3.23)). On the other hand, if there is a v satisfying (3.26), then since $\|Q_\xi\|_{(\xi)} \leq 1$, we have $\|Q_\xi^n h\|_{(\xi)} \leq 1$ and so we can apply the results of Step 2 for the function h replaced by $Q_\xi^n h$. Hence (3.27) holds. The estimate (3.27) implies the lemma because we can assume $C_5 > -\frac{\ln 4}{(\ln 3)(\ln \varkappa)}$, define $C_1 = C_5$, $a_1 = b_2$, $d_1 = d_5$ and combine (3.19) with (3.27).

In the remaining part of the proof, we verify ($\star$).

For a function $f : \bar{X} \to \mathbb{R}$, we write $f_n(x) = \sum_{j=0}^{n-1} f(\bar{\mathcal{F}}^j x)$.

Recall that for any fixed ξ, Lemma 3.2 tells us that x_0 and y_m with

$$m \sim (\ln(1/c_1))^{-1} \ln \xi \tag{3.28}$$

(see (3.15)) satisfy (3.16). Recalling the definition of n from (3.26), we note that

$$n/m \sim C_4 \ln(1/c_1). \tag{3.29}$$

Next, we write $(\hat{\gamma}^u(x), \hat{\gamma}^s(x)) = \iota^{-1}(x)$, $(\hat{\gamma}^u(y), \hat{\gamma}^s(y)) = \iota^{-1}(y)$, $v = \bar{\mathcal{F}}^{n/2}(\hat{\gamma}^s(x))$, $w = \bar{\mathcal{F}}^{n/2}(\hat{\gamma}^s(y))$. We will show that in case no point in $\bar{X}_{\leq 2}$ satisfies (3.20), then either v or w satisfies (3.26). This will complete the proof of Step 3. To this end, assume by contradiction that neither satisfies (3.26).

Writing $h(\bar{x}) = r(\bar{x}) e^{i\phi(\bar{x})}$, we have

$$\begin{aligned} (Q_\xi^n h)(v) &= \sum_{u \in \bar{X}: \bar{\mathcal{F}}^n u = v} e^{\alpha_n(u) + i\xi(\bar{\tau}_{\bar{X}})_n(u)} r(u) e^{i\phi(u)} \\ &= e^{\alpha_n(v'_{-n}) + i\xi(\bar{\tau}_{\bar{X}})_n(v'_{-n})} r(v'_{-n}) e^{i\phi(v'_{-n})} + e^{\alpha_n(v''_{-n}) + i\xi(\bar{\tau}_{\bar{X}})_n(v''_{-n})} r(v''_{-n}) e^{i\phi(v''_{-n})} + \ldots \end{aligned}$$

where

$$v'_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1 n/2}(\gamma^u(x), \gamma^s(x)))), v''_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1(n/2-1) - r_2}(\gamma^u(y), \gamma^s(x))))$$

and ... corresponds to all other preimages.

Thus $(Q_\xi^n h)(v)$ is expressed as a weighted sum of the unit vectors $z_u := e^{i[\xi(\bar{\tau}_{\bar{X}})_n(u) + \phi(u)]} \in \mathbb{C}$, with non-negative weights $\beta_u := e^{\alpha_n(u)} r(u)$. Next, we claim that if v violates (3.26), then any two unit vectors $z_u, z_{u'}$, with weights $\beta_{u'}, \beta_{u''} \geq \varepsilon$ necessarily satisfy $\arg(z_{u'}) - \arg(z_{u''}) < \varepsilon$, where

$$\varepsilon := (8d_2)^{1/3} \xi^{-b_1/3}. \tag{3.30}$$

To prove this claim, first note that $\sum_{u\in\bar X:\bar{\mathcal F}^n u=v} e^{\alpha_n(u)}=1$ and $|r|\le 1$. Now consider the special case $z_{u'}=e^{i\varepsilon/2}$, $z_{u''}=e^{i\varepsilon/2}$, $\beta_{u'}=\beta_{u''}=\varepsilon$ and $z_u=1$ for all $u\in\mathcal U:=\{u\in\bar X\setminus\{u',u''\}:\bar{\mathcal F}^n u=v\}$ and $\sum_{u\in\mathcal U}\beta_u=1-2\varepsilon$. In this case, $z_{u'}+z_{u''}=2\cos(\varepsilon/2)$ and so

$$|(Q_\xi^n h)(v)|=(1-2\varepsilon)+2\varepsilon\cos(\varepsilon/2)\le 1-\varepsilon^3/8$$

and so v satisfies (3.26). In any other case, whenever there is $z_u, z_{u'}$ with $\beta_{u'},\beta_{u''}\ge\varepsilon$ and $\arg(z_{u'})-\arg(z_{u''})\ge\varepsilon$, we have

$$\left|e^{i\arg(-z_{u'}-z_{u''})}\left[(\beta_{u'}-\varepsilon)z_{u'}+(\beta_{u''}-\varepsilon)z_{u''}+\sum_{u\in\mathcal U}\beta_u z_u\right]\right|\le 1-2\varepsilon$$

and

$$e^{i\arg(-z_{u'}-z_{u''})}(z_{u'}+z_{u''})\le 2\cos(\varepsilon/2).$$

Thus $|(Q_\xi^n h)(v)|$ cannot be bigger than in the above special case. This proves the claim.

If $r(v'_{-n})<1/2$ or $r(v''_{-n})<1/2$, then one of these points satisfies (3.20) and so the proof of Step 3 is complete.

Next consider the case when $r(v'_{-n})\ge 1/2$ and $r(v''_{-n})\ge 1/2$. Recall that $v'_{-n},v''_{-n}\in\bar X_{\le 2}$. Since α is a Hölder function (recall (3.7)), it is bounded from below on the compact set $\bar X_{\le 2}$ and so e^α is bounded from below by some $\eta>0$ on the set $\bar X_{\le 2}$. Consequently,

$$\min\left(e^{\alpha_n(v'_{-n})},e^{\alpha_n(v''_{-n})}\right)\ge\eta^n=\xi^{C_4\ln\eta}$$

where the last step relies on (3.26). Hence

$$\beta(v'_{-n}),\beta(v''_{-n})>\xi^{C_4\ln\eta}/2. \tag{3.31}$$

Next, we need to guarantee that

$$\frac12\xi^{C_4\ln\eta}\ge(8d_2)^{1/3}\xi^{-b_1/3}. \tag{3.32}$$

We now select

$$d_2=\frac{1}{64} \tag{3.33}$$

so that $(8d_2)^{1/3}=\frac12$. We will also assume that $b_1=b_1(C_4)$ satisfies

$$b_1=-3C_4\ln\eta \tag{3.34}$$

(the precise conditions C_4 will become clear at the end of Step 3). (3.33) and (3.34) ensure that (3.32) holds.

Now (3.31) and (3.32) imply that $\beta(v'_{-n}),\beta(v''_{-n})>\varepsilon$. Recall that we assumed by contradiction that v does not satisfy (3.26). Then our earlier claim implies that

$$|[\xi(\bar{\boldsymbol\tau}_{\bar X})_n(v'_{-n})-\xi(\bar{\boldsymbol\tau}_{\bar X})_n(v''_{-n})]-[\phi(v'_{-n})-\phi(v''_{-n})]|\le\varepsilon.$$

Repeating the above argument for w, and writing

$$w'_{-n}=\Xi(\iota^{-1}(\boldsymbol T^{-r_1(n/2-1)-r_2}(\gamma^u(y),\gamma^s(y)))),w''_{-n}=\Xi(\iota^{-1}(\boldsymbol T^{-r_1n/2}(\gamma^u(x),\gamma^s(y)))),$$

we find

$$|[\xi(\bar{\boldsymbol\tau}_{\bar X})_n(w'_{-n})-\xi(\bar{\boldsymbol\tau}_{\bar X})_n(w''_{-n})]-[\phi(w'_{-n})-\phi(w''_{-n})]|\le\varepsilon.$$

By construction, $s(v'_{-n},w''_{-n})\ge n/2$ and so further increasing C_4 if necessary, we can guarantee that $|\phi(v'_{-n})-\phi(w''_{-n})|\le\varepsilon$.

Similarly, we can assume $|\phi(v''_{-n})-\phi(w'_{-n})|\le\varepsilon$. Hence, denoting $d_6=4(8d_2)^{1/3}$,

$$b_4=b_1/3+1 \tag{3.35}$$

and recalling (3.30), we obtain

$$|A| \leq 4\varepsilon = d_6/\xi^{b_4}, \tag{3.36}$$

where

$$A = (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n}) - (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v''_{-n}) + (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w'_{-n}) - (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w''_{-n}). \tag{3.37}$$

The inequality (3.36) is a major step in deriving a contradiction with (3.16). As we will see, A is a good approximation of the finite chunk of the sum defining the temporal distance of x_0 and y_m (cf. (3.9)) corresponding to times ℓ with $|\ell| \lessapprox n$. We need to verify that the remaining part of the sum satisfies an inequlaity similar to (3.36).

Recall (3.6) and (3.9). Using the notations $z = (\gamma^u(z), \gamma^s(z)) \in \mathcal{R}$, $\hat{z} = \iota^{-1}(z) = (\hat{\gamma}^u(z), \hat{\gamma}^s(z))$ and

$$H(z) = \sum_{\ell=0}^{\infty} [\boldsymbol{\tau}(\boldsymbol{T}^\ell(z)) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\iota(\hat{\mathbf{J}}^u, \gamma^s(z)))], \tag{3.38}$$

observe that

$$\hat{\boldsymbol{\tau}}_X(\hat{\gamma}^u(z), \hat{\gamma}^s(z)) - \bar{\boldsymbol{\tau}}_{\bar{X}}(\hat{\gamma}^s(z)) = H(\gamma^u(z), \gamma^s(z)) - H(\boldsymbol{T}^{r(\hat{\gamma}^s(z))}(\gamma^u(z), \gamma^s(z))). \tag{3.39}$$

Recalling the notation (3.12), let us write

$$d_{\ell,f}(z_1, z_2) = f(\boldsymbol{T}^\ell([z_1, z_1]) - f(\boldsymbol{T}^\ell([z_1, z_2])) - f(\boldsymbol{T}^\ell([z_2, z_1])) + f(\boldsymbol{T}^\ell([z_2, z_2])). \tag{3.40}$$

Recall the dynamical Hölder continuity of $\boldsymbol{\tau}$: there are some constant $C_{\boldsymbol{\tau}}$ and $\vartheta_{\boldsymbol{\tau}} < 1$ so that if $z_1, z_2 \in \boldsymbol{M}$ are such that $\boldsymbol{T}^\ell(z_1)$ and $\boldsymbol{T}^\ell(z_1)$ stay on the same local unstable manifold for all $\ell \leq L$, then

$$|\boldsymbol{\tau}(z_1) - \boldsymbol{\tau}(z_2)| < C_{\boldsymbol{\tau}} \vartheta_{\boldsymbol{\tau}}^L. \tag{3.41}$$

Likewise, if $\boldsymbol{T}^\ell(z_1)$ and $\boldsymbol{T}^\ell(z_1)$ stay on the same local stable manifold for all $\ell \geq -L$, then $|\boldsymbol{\tau}(z_1) - \boldsymbol{\tau}(z_2)| < C_{\boldsymbol{\tau}} \vartheta_{\boldsymbol{\tau}}^L$.

With the above notation, we have

$$D(x_0, y_m) = \sum_{\ell=-\infty}^{\infty} d_{\ell,\boldsymbol{\tau}}(x_0, y_m).$$

The sum is absolutely convergent as both $|f(\boldsymbol{T}^\ell(z_1) - f(\boldsymbol{T}^\ell([z_1, z_2]))|$ and $|-f(\boldsymbol{T}^\ell([z_2, z_1])) + f(\boldsymbol{T}^\ell(, z_2))|$ are exponentially small in $|\ell|$ for $\ell < 0$ and both $|f(\boldsymbol{T}^\ell(z_1) - f(\boldsymbol{T}^\ell([z_2, z_1]))|$ and $|-f(\boldsymbol{T}^\ell([z_1, z_2])) + f(\boldsymbol{T}^\ell(z_2))|$ are exponentially small in ℓ for $\ell > 0$. We will decompose the above series as

$$\sum_{\ell=-\infty}^{\infty} d_{\ell,\boldsymbol{\tau}}(x_0, y_m) = S_1 + S_2 + S_3, \tag{3.42}$$

where $S_3 = \displaystyle\sum_{\ell=r_1(n/2-1)+1}^{\infty} d_{\ell,\boldsymbol{\tau}}(x_0, y_m)$ and

$$S_1 = \sum_{\ell=-\infty}^{-r_1 n/2-1} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x_0)) - \boldsymbol{\tau}(\boldsymbol{T}^\ell([x_0, y_m]))$$

$$+ \sum_{\ell=-\infty}^{-r_1(n/2-1)-r_2-1} -\boldsymbol{\tau}(\boldsymbol{T}^\ell([y_m, x_0])) + \boldsymbol{\tau}(\boldsymbol{T}^\ell(y_m)),$$

$$S_2 = \sum_{\ell=-r_1 n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x_0)) - \sum_{\ell=-r_1 n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell([x_0, y_m]))$$

$$-\sum_{\ell=-r_1(n/2-1)-r_2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell([y_m,x_0])) + \sum_{\ell=-r_1(n/2-1)-r_2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(y_m)).$$

Directly checking all four choices of $z,w\in\{x_0,y_m\}$, we see that the sum S_1+S_2 contains the term $(-1)^{1_{z\neq w}}\boldsymbol{\tau}(\boldsymbol{T}^\ell([z,w]))$ for every $\ell\le r_1(n/2-1)$ exactly once. Since S_2 is a finite sum, and since by our previous observation both series in S_1 converge absolutely, (3.42) holds.

Let us study S_2. First, we have by the definition of x_0 that $\iota^{-1}(\boldsymbol{T}^{r_1k}(x_0))\in\Delta_{0,1}$ for all $k\in\mathbb{Z}$. Consequently,

$$\sum_{\ell=-r_1n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x_0)) = \sum_{k=-n/2}^{n/2-1} \hat{\boldsymbol{\tau}}_X(\iota^{-1}(\boldsymbol{T}^{r_1k}(x_0))).$$

Next, recalling that $\Xi(\iota^{-1}(\boldsymbol{T}^{-r_1n/2}(x_0)))=v'_{-n}$, we have $\Xi(\iota^{-1}(\boldsymbol{T}^{r_1k}(x_0)))=\bar{\mathcal{F}}^{k+n/2}v'_{-n}$, for $k=-n/2,...,n/2-1$. Now applying (3.39), we conclude

$$\begin{aligned}\sum_{\ell=-r_1n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x_0)) &= (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n}) + \sum_{k=-n/2}^{n/2-1}\left[H(\boldsymbol{T}^{r_1k}(x_0)) - H(\boldsymbol{T}^{r_1(k+1)}(x_0))\right]\\ &= (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n}) + H(\boldsymbol{T}^{-r_1n/2}(x_0)) - H(\boldsymbol{T}^{r_1n/2}(x_0))\end{aligned}$$

Arguing similarly with the other three sums in S_2, we find

$$S_2 = A + d_{0,H}(\boldsymbol{T}^{-r_1n/2}(x_0),\boldsymbol{T}^{-r_1(n/2-1)-r_2}(y_m)) - d_{0,H}(\boldsymbol{T}^{r_1n/2}(x_0),\boldsymbol{T}^{r_1(n/2-1)+r_2}(y_m)), \quad (3.43)$$

where A is defined by (3.37), $d_{0,H}$ is defined by (3.40) with $\ell=0$ and H as in (3.38).

Next we claim that if $n>4m$ (which can be achieved by increasing C_4) then

$$|S_1|+|S_2-A|+|S_3|\le d_7\xi^{-b_5}, \quad (3.44)$$

where

$$d_7 = 4C_{\boldsymbol{\tau}}\frac{1}{1-\vartheta_{\boldsymbol{\tau}}}, \quad b_5 = -\frac{1}{4}r_1C_4\ln\vartheta_{\boldsymbol{\tau}}. \quad (3.45)$$

To prove (3.44), we first use the dynamical Hölder property of $\boldsymbol{\tau}$ to conculde that both series whose sum defines S_1 are absolutely convergent and in absolute value bounded by $C_{\boldsymbol{\tau}}\frac{1}{1-\vartheta_{\boldsymbol{\tau}}}\vartheta_{\boldsymbol{\tau}}^{r_1n/2}$. Thus by the definition of n (see (3.26)) and b_5 (see (3.45)), we have $|S_1|\le d_7\xi^{-b_5}/2$.

Estimating S_3 is simpler: since $n/2>m$ it follows that all of the points

$$\boldsymbol{T}^\ell(x_0),\boldsymbol{T}^\ell([x_0,y_m]),\boldsymbol{T}^\ell(y_m),\boldsymbol{T}^\ell([y_m,x_0])$$

lie on the same local stable manifold for $\ell>n/2$. Since $n/4>m$, the dynamical Hölder continuity of $\boldsymbol{\tau}$ implies $|S_3|\le C_{\boldsymbol{\tau}}\frac{1}{1-\vartheta_{\boldsymbol{\tau}}}\vartheta_{\boldsymbol{\tau}}^{n/4}$ and so by the definition of n and b_5, we have $|S_3|\le d_7\xi^{-b_5}/2$.

It remains to study S_2-A. Recall (3.43). Writing $z_1=\boldsymbol{T}^{-r_1n/2}(x_0)$, $z_2=\boldsymbol{T}^{-r_1(n/2-1)-r_2}(y_m)$, we note that by the definition of x_0 and y_m, z_1 and z_2 are on the same local unstable manifold. Thus $[z_1,z_2]=z_2$ and $[z_2,z_1]=z_1$ and so $d_{0,H}(z_1,z_2)=0$. Likewise,

$$d_{0,H}(\boldsymbol{T}^{r_1n/2}(x_0),\boldsymbol{T}^{r_1(n/2-1)+r_2}(y_m))=0$$

and so $S_2-A=0$. We have verified (3.44).

Finally, we combine (3.36) and (3.44) to conclude that

$$D(x_0,y_m)\le d_6\xi^{-b_4}+d_7\xi^{-b_5}. \quad (3.46)$$

By (3.35), (3.34) and (3.45), both b_4 and b_5 are a constant multiple of C_4 where this constant only depends on the geometry of the billiard. Thus we can increase C_4 if necessary to ensure that both b_4 and b_5 are bigger than a_0 given in Lemma 3.2. Then (3.46) is a contradiction with

Lemma 3.2. Thus ($\star$) is valid. This completes the proof of Step 3 and finishes the proof of Lemma 3.3. □

Now we revisit the tower $(\bar{\Delta}, \bar{F})$. Recall that a separation time s was defined in (E4). Let

$$\|f\|_{\mathbb{B}} = \|f\|_\infty + \sup\{C : \forall x, y \in \bar{\Delta} : |f(x) - f(y)| \le C\varkappa^{s(x,y)}\} . \tag{3.47}$$

Let us denote by $\bar{\boldsymbol{P}}$ the Perron-Frobenius operator associated with $\bar{F}$ and let $\bar{\boldsymbol{P}}_{\theta,\xi}$ be defined by $\bar{\boldsymbol{P}}_{\theta,\xi}(f) := \bar{\boldsymbol{P}}\left(e^{i\theta\cdot\bar{\boldsymbol{\kappa}}+i\xi\bar{\boldsymbol{\tau}}}f\right)$. We conclude this section by

Lemma 3.4. *There are constants C_3, α_2 and δ so that*

$$\sup_{\theta\in[-\pi,\pi]^d} \|\bar{\boldsymbol{P}}^n_{\theta,\xi}\|_{\mathcal{L}(\mathbb{B},L^1)} \le C_3|\xi|^{\alpha_2}e^{-n\delta|\xi|^{-\alpha_2}} . \tag{3.48}$$

Proof. The proof follows the lines of [35]. Consider the operator

$$Q_{\theta,s,z}h = Q(e^{i\theta\cdot\bar{\boldsymbol{\kappa}}_{\bar{X}}+s\bar{\boldsymbol{\tau}}_{\bar{X}}+zr}h)$$

where $\boldsymbol{\kappa} : \boldsymbol{M} \to \mathbb{Z}^2$ is defined in Section 3.1 and $r : \bar{X} \to \mathbb{Z}_+$ is defined by (E3). Here, s and z are complex numbers. In particular $Q_{0,i\xi,0} = Q_\xi$.

Assume first that $\theta = 0$. By Lemma 3.3, $\|(I - Q_{0,i\xi,0}^{C_1\ln\xi})^{-1}\|_{(\xi)} < d_1^{-1}\xi^{a_1}$ and so by the identity $(I - A)^{-1} = (I + A + ... + A^{m-1})(I - A^m)^{-1}$, we have

$$\|(I - Q_{0,i\xi,0})^{-1}\|_{(\xi)} < C\xi^{a_1'}$$

for any $a_1' > a_1$. This inequality can be extended from $Q_{0,i\xi,0}$ to $Q_{0,s,z}$ with $s = a+i\xi$, $z = \sigma+i\omega$ for $|a|, |\sigma| < \epsilon|\xi|^{-a_1'}$, $\omega \in [0, 2\pi)$ for some small ϵ as in Lemma 3.14 of [35].

Now we can repeat the proof by operator renewal theory as in Section 4 in [35]. Specifically, their Proposition 4.1 is applicable by (E5) and so their Lemma 4.4 gives (3.48) with $\theta = 0$.

Finally, since $\boldsymbol{\kappa}$ is constant on local stable manifolds, the proof can be extended to arbitrary $\theta \in [-\pi, \pi]^d$ as explained in the proof of Lemma 3.14 in [35]. □

3.4. **Proof of Theorem 3.1.** Let $\mathcal{S}_0 = \partial\boldsymbol{M} = \{(q, v) \in \boldsymbol{M} : \vec{n}_q.v = 0\}$ be the singularity set, i.e. the collection of points in the phase space corresponding to grazing collisions.

The transformation $\boldsymbol{T}$ defines a C^1 diffeomorphism from $\boldsymbol{M} \setminus (\mathcal{S}_0 \cup \boldsymbol{T}^{-1}\mathcal{S}_0)$ to $\boldsymbol{T} \setminus (\mathcal{S}_0 \cup \boldsymbol{T}\mathcal{S}_0)$.

Moreover there exist $C_0 > 0$ and $\theta_0 \in (0, 1)$ such that the diameter of every connected component of $\boldsymbol{M} \setminus \bigcup_{j=-n}^{n} \boldsymbol{T}^{-j}\mathcal{S}_0$ is less than $C_0\theta_0^n$. We consider now a suitable separation time $\hat{s}$ on Δ. The main difference between s and $\hat{s}$ is that counts the steps straight up in the tower, i.e. $\hat{s}((x, l), (y, l)) = \hat{s}((x, 0), (y, 0)) - l$. The exact definition of $\hat{s}$ is not important for us and can be found in [46].

Recall that, by construction of [46], for every $x, y \in \Delta$ in the same unstable manifold, $\pi(x)$ and $\pi(y)$ lie in the same connected component of $\boldsymbol{M} \setminus \bigcup_{j=-\infty}^{\hat{s}(x,y)} \boldsymbol{T}^{-j}\mathcal{S}_0$, with $\hat{s}(x, y) := \hat{s}(\Xi(x), \Xi(y))$.

We will prove that the assumptions of Theorem 2.3, namely (B1)–(B6), are satisfied with:

- $\Sigma = \Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$
- $K = 2K_0$
- $d = 2$,
- $(M, \nu, T) = (\Delta, \nu, F)$, $\tau := \hat{\boldsymbol{\tau}} = \boldsymbol{\tau} \circ \pi$, $\kappa := \hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa} \circ \pi$,
- $(\bar{\Delta}, \bar{\nu}, \bar{T}) = (\bar{\Delta}, \bar{\nu}, \bar{F})$, $\mathfrak{p} = \Xi$ and $\bar{P} = \bar{\boldsymbol{P}}$
- $\mathcal{V}$ the space of functions $f : \Delta \to \mathbb{C}$ such that the following quantity is finite

$$\|f\|_{\mathcal{V}} = \|f\|_\infty + \sup_{\gamma^u;\ x,y\in\gamma^u} \frac{|f(x) - f(y)|}{\varkappa^{\hat{s}(x,y)}} + \sup_{n\ge0,\ \gamma^s;\ x,y\in\gamma^s} \frac{|f(F^n(x)) - f(F^n(y))|}{\varkappa^n} ,$$

where $\varkappa$ is a fixed real number satisfying

$$\max\left(\theta_0^{1/4}, \theta_0^{\eta}, \vartheta_\alpha, \vartheta_{\boldsymbol{\tau}}\right) < \varkappa < 1, \tag{3.49}$$

where η, ϑ_α and $\vartheta_{\boldsymbol{\tau}}$ are defined by (3.3), (3.7) and (3.41), respectively.

- The space $\mathcal{B}$ is the Young space of complex-valued functions $f : \bar\Delta \to \mathbb{C}$ such that $\|f\|_{\mathcal{B}} < \infty$ with $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = \sup_l \|f|_{\bar\Delta_l}\|_\infty e^{-l\varepsilon_0} + \sup_l \operatorname{ess}\sup_{x,y\in\bar\Delta_l} \frac{|f(x)-f(y)|}{\varkappa^{\hat s(x,y)}} e^{-l\varepsilon_0}\,. \tag{3.50}$$

  with $\varkappa$ as in (3.49) and a suitable ε_0.
- The space $\mathbb{B}$ is the space of complex-valued bounded Lipschitz functions $f : \bar\Delta \to \mathbb{C}$ such that $\|f\|_{\mathbb{B}} < \infty$ with $\|\cdot\|_{\mathbb{B}}$ defined in (3.47) for the same choice of $\varkappa$.

In view of (E5),

$$\mathcal{B} \hookrightarrow L^{q_0}(\bar\nu) \text{ for some } q_0 \in (1, +\infty) \tag{3.51}$$

provided that ε_0 is small enough.

Observe that, with these notations $(\tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Phi}}_t, \tilde{\boldsymbol{\mu}}_0)$ can be represented by the suspension semiflow $(\tilde\Phi_t)_{t\ge0}$ (with roof function τ) over the $\mathbb{Z}^2$-extension of (M, ν, T) by τ.

We define

$$\|f\|_{\mathcal{B}_0} = \|f\|_\infty + \inf\{C : \forall x, y \in \bar\Delta : |f(x) - f(y)| \le C\varkappa^{\hat s(x,y)}\}.$$

Observe that $\mathcal{B}_0 \subset \mathcal{B} \cap \mathbb{B}$ and that the multiplication by an element of $\mathcal{B}_0$ defines a continuous linear operator on $\mathcal{B}$ and on $\mathbb{B}$.

Now we proceed to verifying assumptions (B1)–(B6). Since κ is constant on stable manifolds, there exists a $\bar\nu$-centered $\mathbb{Z}^2$-valued bounded function $\bar\kappa \in \mathbb{B}$ such that $\bar\kappa \circ \mathfrak{p} = \kappa$. Therefore, (B1) holds with $m_0 = 0$.

Next, since $\boldsymbol{\tau}$ is 1/2-Hölder on every connected component of $\boldsymbol{M} \setminus (\mathcal{S}_0 \cup T_0^{-1}(\mathcal{S}_0))$ and since $\sqrt{\theta_0} \le \varkappa$, we have $\tau \in \mathcal{V}$.

Now, on Δ, we define $\chi := \sum_{k\ge0}\left(\tau\circ F^k - \tau\circ F^k \circ \Xi\right)$. By construction,

$$\tau = \bar\tau \circ \mathfrak{p} + \chi - \chi\circ F, \text{ where } \bar\tau \circ \Xi(\hat x^u, l) = \bar{\boldsymbol{\tau}}(\hat x^u, l) = \hat{\boldsymbol{\tau}}(\hat x^u, \hat{\boldsymbol{x}}^s, l)\,. \tag{3.52}$$

Next, we claim that $\chi \in \mathcal{V}$ and $\bar\tau \in \mathcal{B}_0$.

Indeed, first,

$$\|\chi\|_\infty \le \sum_{k\ge0} \|\tau\circ F^k - \tau\circ F^k\circ\Xi\|_\infty \le \sum_{k\ge0}\|\tau\|_{\mathcal{V}}\varkappa^k < \infty\,.$$

Second, if $x, y \in \Delta$ are on the same stable manifold, then $\Xi(F^n(x)) = \Xi(F^n(y))$ and so, since τ is 1/2-Hölder, for every nonnegative integer n,

$$|\chi(F^n(x)) - \chi(F^n(y))| \le \sum_{k\ge0}\left|\tau(F^{k+n}(x)) - \tau(F^{k+n}(y))\right| \le C_\tau \sum_{k\ge0}\left(C_0\theta_0^{k+n}\right)^{\frac12} = O(\varkappa^n).$$

Third, if $x, y \in \Delta$ are on the same unstable manifold, then

$$|\tau(F^j(x)) - \tau(F^j(y))| + |\tau(F^j(\Xi(x))) - \tau(F^j(\Xi(y)))| \le 2C_\tau(C_0\theta_0^{\hat s(x,y)-j})^{\frac12}$$

and

$$|\tau(F^j(x)) - \tau(F^j(\Xi(x)))| + |\tau(F^j(y)) - \tau(F^j(\Xi(y)))| \le 2C_\tau(C_0\theta_0^j)^{\frac12}.$$

So, since $\theta_0^{\frac14} \le \varkappa$

$$|\chi(x) - \chi(y)| \le O\left(\sum_{0\le k\le \hat s(x,y)/2} \varkappa^{2(\hat s(x,y)-k)} + \sum_{k>\hat s(x,y)/2}\varkappa^{2k}\right) = O\left(\varkappa^{\hat s(x,y)}\right)\,.$$

This shows that $\chi \in \mathcal{V}$. Then clearly $\chi \circ F \in \mathcal{V}$ as well. Since $\tau \in \mathcal{V}$, (3.52) implies $\bar\tau \circ \mathfrak{p} \in \mathcal{V}$ which in turn gives $\bar\tau \in \mathcal{B}_0$.

Observe that $\left\|e^{i\xi.\chi}\right\|_{\mathcal{V}} = O(1+|\xi|)$ and that $(\bar\tau_{m_0})^k e^{-i\xi\bar\tau_{m_0}} \in \mathcal{B}$ for every k and $m_0 = 1$. Thus we have verified (B2).

The fact that $(\bar P_{\theta,\xi} : \bar f \mapsto \bar P(e^{i\theta\cdot\bar\kappa} e^{i\xi.\bar\tau}\bar f))_{(\theta,\xi)\in[-\pi,\pi]^d\times\mathbb{R}}$ satisfies (2.31), (2.32), (2.33), (2.34), with $J = 3$ follows from [44, 46] (see also [42]). This implies (B3). Condition (B4) is proved in Lemma 3.4.

Next, we check (B5). For any $f \in \mathcal{V}$ and any nonnegative integer n, we define $\mathbf{\Pi}_n f : \bar\Delta \to \mathbb{C}$ by

$$\forall x \in \Delta, \quad (\mathbf{\Pi}_n f)\circ \Xi(x) := \mathbb{E}_\nu[f\circ F^n | \hat s(\cdot, x) \geq 2n]\,.$$

Note that $\mathbf{\Pi}_n$ is linear and continuous from $\mathcal{V}$ to $\mathcal{B}_0$ with norm in $O\left(2\varkappa^{-2n}\right)$. By definition of $\mathcal{V}$, if $s(x,y)\geq 2n$, then by considering z in the stable manifold containing x and in the unstable manifold containing y, $F^n(z)$ is in the same unstable manifold as $F^n(y)$ with $\hat s(F^n(y), F^n(z)) \geq n$ and so

$$|f(F^n(x)) - f(F^n(y))| \leq |f(F^n(x)) - f(F^n(z))| + |f(F^n(z)) - f(F^n(y))| \leq \|f\|_{\mathcal{V}}\varkappa^n\,.$$

Therefore we have proved that

$$\forall f\in\mathcal{V}, \quad \|f\circ F^n - \mathbf{\Pi}_n(f)\circ\Xi\|_\infty \leq C_0\|f\|_{\mathcal{V}}\,\varkappa^n\,,$$

and so (2.36) holds for any $\vartheta \geq \varkappa$.

Recall that

$$\bar P^{2n}_{\theta,\xi} h(x) = \sum_{z\in\bar F^{-2n}(\{x\})} e^{\alpha_{2n}(z)+i\theta.\bar\kappa_{2n}(z)+i\xi.\bar\tau_{2n}(z)}h(z)\,,$$

with

$$\alpha_l := \sum_{k=0}^{l-1}\alpha\circ\bar F^k, \quad \bar\kappa_l := \sum_{k=0}^{l-1}\bar\kappa\circ\bar F^k, \quad \text{and} \quad \bar\tau_l := \sum_{k=0}^{l-1}\bar\tau\circ\bar F^k.$$

By construction of $(\bar\Delta,\bar\nu,\bar F)$, for every $x, y \in \bar\Delta$ with $\hat s(x,y) \geq 1$, there exists a bijection $W_{2n} : \bar F^{-2n}(\{x\}) \to \bar F^{-2n}(\{y\})$ such that $\hat s(z, W_{2n}(z)) \geq 2n$ and so $\mathbf{\Pi}_n f(z) = \mathbf{\Pi}_n f(W_{2n}(z))$. Moreover, since $\alpha,\bar\kappa,\bar\tau\in\mathcal{B}_0$, for $g\in\{\alpha,\bar\kappa,\bar\tau\}$ and for any x,y,z as above, we have

$$|g(\bar F^k(z)) - g(\bar F^k(W_n(z)))| \leq \|g\|_{\mathcal{B}_0}\varkappa^{\hat s(x,y)+2n-k}\,.$$

Hence

$$|g_n(\bar F^k(z)) - g_n(\bar F^k(W_n(z)))| \leq \|g\|_{\mathcal{B}_0}(1-\varkappa)^{-1}\varkappa^{\hat s(x,y)+n-k}\,.$$

We conclude that there exists $C_0 > 0$ such that, for every $\theta\in[-\pi,\pi]^d$, $\xi\in\mathbb{R}$ and for every non-negative integer j,

$$\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\bar P^{2n}_{\theta,\xi}(e^{-i\theta.\bar\kappa_n - i\xi.\bar\tau_n}\mathbf{\Pi}_n f))\right\|_{\mathcal{B}_0} \leq \left\|\frac{\partial^j}{\partial(\theta,\xi)^j}\bar P^{2n}(e^{i(\theta.\bar\kappa_n+\xi.\bar\tau_n)\circ\bar F^n}\mathbf{\Pi}_n f)\right\|_\infty +$$

$$\sup_{\substack{x,y\in\bar\Delta,\\ \hat s(x,y)\geq 1}}\varkappa^{-\hat s(x,y)}\left|\frac{\partial^j}{\partial(\theta,\xi)^j}\sum_{z\in\bar F^{-2n}(x)}\left(e^{\alpha_{2n}(z)+(i\theta\bar\kappa_n+i\xi\tau_n)\circ\bar F^n(z)} - e^{\alpha_{2n}(W_n(z))+(i\theta\bar\kappa_n+i\xi\tau_n)\circ\bar F^n(W_n(z))}\right)\mathbf{\Pi}_n f(z)\right|$$

$$\leq C_0 n^j(1+|\xi|)\|f\|_\infty$$

and

$$\begin{aligned}\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\mathbf{\Pi}_n(f)e^{i\theta\cdot\bar{\kappa}_{n-m_0}+i\xi.\bar{\tau}_n})\right\|_{\mathcal{B}'} &\leq \left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\mathbf{\Pi}_n(f)e^{i\theta\cdot\bar{\kappa}_{n-m_0}+i\xi.\bar{\tau}_n})\right\|_{L^p(\bar{\mu})}\\ &\leq \left\|\left(\frac{\partial^j}{\partial(\theta,\xi)^j}(\mathbf{\Pi}_n(f)e^{i\theta\cdot\bar{\kappa}_{n-m_0}+i\xi.\bar{\tau}_n})\right)\right\|_{\infty}\\ &\leq C_0 n^j\|f\|_\infty\,,\end{aligned}$$

where we used that $\bar{\kappa}$ and $\bar{\tau}$ are uniformly bounded and p is such that $\frac{1}{q_0}+\frac{1}{p}=1$ with q_0 defined in (3.51). Therefore we have proved (2.37), (2.38) and (2.39) and so verified (B5).

Next, we define f and g as follows: $f(x,\ell,s)=\mathfrak{f}(q+\ell+s\vec{v},\vec{v})$ and similarly $g(x,\ell,s)=\mathfrak{g}(q+\ell+s\vec{v},\vec{v})$ if $\pi(x)=(q,\vec{v})$. Note that $(q+\ell+s\vec{v},\vec{v})=\widetilde{\mathbf{\Phi}}_s(q+\ell,\vec{v})$ for $s\in[0,\tau(q,\vec{v}))$. Let $(\mathfrak{h},h)=(\mathfrak{f},f)$ or $(\mathfrak{g},g)$. We define

$$h_\ell(x,s):=\chi_0(s)\mathfrak{h}\left(\widetilde{\mathbf{\Phi}}_s(q+\ell,\vec{v})\right)(1-\chi_0(s-\tau(x)))\,,$$

with $\chi_0:\mathbb{R}\to[0,1]$ a fixed increasing C^∞ function such that $\chi_0(u)=0$ if $u\leq-\frac{\min\tau}{10}$ and $\chi_0(u)=1$ if $u\geq 0$.
Note that $h_\ell(x,\cdot)$ have support in $\left[-\frac{\min\tau}{10},\tau(x)\right]$, coincide with $h(x,\ell,\cdot)$ in $[0,\tau(x)-\frac{\min\tau}{10}]$, and satisfy (2.40). Let $u\in\mathbb{R}$ be fixed. Then $\|h_\ell(\cdot,u)\|_\infty\leq\sup_{|\ell'-\ell|\leq\max\tau}\left\|\mathfrak{h}\mathbf{1}_{\mathcal{C}_{\ell'}}\right\|_\infty$. Furthermore, since $\tau\in\mathcal{V}$, $\theta_0^\eta<\varkappa$, and $\mathfrak{h}\circ\widetilde{\mathbf{\Phi}}_s$ is uniformly η-Hölder continuous for $s\in[-\frac{\min\tau}{10},\max\tau]$, we obtain that there exists a uniform constant $\widetilde{C}>0$ such that

$$\|h_\ell(\cdot,u)\|_{\mathcal{V}}\leq\widetilde{C}\sup_{|\ell'-\ell|\leq\max\tau}\|\mathfrak{h}\|_{\mathcal{H}_{\mathcal{C}^\eta_{\ell'}}}\,. \tag{3.53}$$

Thus, (2.44) and (2.42) follow directly from (3.3). Recall that

$$\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi.\chi}\hat{h}_\ell(x,\xi)\right)=\sum_{m=0}^k\frac{k!}{m!\,(k-m)!}(-i\chi)^m e^{-i\xi\chi}\int_{(-\frac{\min\tau}{10},\tau(x))}(is)^{k-m}e^{i\xi s}h_\ell(x,s)\,ds\,. \tag{3.54}$$

Next, to prove (2.41) it suffices to show that

$$\sum_{\ell\in\mathbb{Z}^d}\left(\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi.\chi}\hat{f}_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}+\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi.\chi}\hat{g}_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}\right)<C(1+|\xi|)\,. \tag{3.55}$$

Observe that $\|e^{-i\xi\chi}\|_{\mathcal{V}}=O(1+|\xi|)$ and the integral in (3.54) is uniformly bounded by $2\max\tau\|h_\ell\|_\infty$. Furthermore, for $x,y\in\gamma^u$ such that $\hat{s}(x,y)\geq n$ (resp. for $x,y\in F^n(\gamma^s)$) and such that $\tau(x)\leq\tau(y)$, we have

$$\begin{aligned}&\left|\int_{(-\frac{\min\tau}{10},\tau(x))}\ldots h_\ell(x,s)\,ds-\int_{(-\frac{\min\tau}{10},\tau(y))}\ldots h_\ell(y,s)\,ds\right|\\ &\leq\int_{(-\frac{\min\tau}{10},\tau(x))}|\ldots|\;|h_\ell(x,s)-h_\ell(y,s)|\;ds+\int_{\tau(x)}^{\tau(y)}|\ldots|\;|h_\ell(y,s)|\;ds\\ &\leq\int_{(-\frac{\min\tau}{10},\tau(x))}C\;\|h_\ell(\cdot,s)\|_{\mathcal{V}}\,\varkappa^n\,ds+\|\tau\|_{\mathcal{V}}\,\varkappa^n C\;\|h_\ell(\cdot,s)\|_\infty\;ds\,.\end{aligned}$$

Now (3.55) follows from (3.53) and (3.3).

Assume next that $\mathfrak{h}$ satisfies (3.1), then the functions $h_\ell(x,\cdot)$ are C^∞ and there exists a uniform constant $\widetilde{C}_0>0$ such that

$$\forall N\in\mathbb{N},\quad \left\|\frac{\partial^N}{\partial s^N}h_\ell(\cdot,s)\right\|_{\mathcal{V}}\le \widetilde{C}_0\sup_{m=0,\dots,N}\ \sup_{|\ell'-\ell|\le\max\tau}\left\|\frac{\partial^m}{\partial s^m}\left(\mathfrak{h}\circ\widetilde{\mathbf{\Phi}}_s\right)_{|s=0}\right\|_{\mathcal{H}^\eta_{\mathcal{C}_{\ell'}}}.$$

Moreover, since h_ℓ is C^∞ with compact support, by classical integration by parts, we have

$$\forall N\in\mathbb{N},\quad \hat h_\ell(x,\xi)=(-i)^N\xi^{-N}\int_{\mathbb{R}}e^{i\xi\,s}\frac{\partial^N}{\partial s^N}h_\ell(\cdot,s)\,ds$$

Therefore, since $\chi\in\mathcal{V}$, we have proved that, if $\mathfrak{h}$ satisfies (3.1), we have

$$\forall\gamma>0,\quad \sum_\ell\|e^{-i\xi\,\chi}\hat h_\ell(\cdot,-\xi)\|_{\mathcal{V}}=O(|\xi|^{-\gamma})\,, \tag{3.56}$$

which, combined with (3.55) implies (2.43). We have finished the proof of (B6). Now Theorem 2.3 implies Theorem 3.1.

3.5. **Identifying $\mathfrak{C}_0$.** We can identify the constant $\mathfrak{C}_0$ by a computation similar to the proof of Corollary 2.2. Recall the notations $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}},\Sigma_{\boldsymbol{\kappa}}$ from Section 3.1 and that here $d=2$.

Set $\sigma:=\sqrt{\det\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}/\det\Sigma_{\boldsymbol{\kappa}}}$. Observe that $\Psi_{\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}(0,0,u)=\frac{e^{-\frac{u^2}{2\sigma^2}}}{(2\pi)^{\frac32}\sqrt{\det\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}}$.

Now the leading term of $C_t(f,g)$ can be obtained by taking $m=j=k=r=q=0$ in (2.45):

$$\begin{aligned}\lim_{t\to\infty}tC_t(f,g) &= \boldsymbol{\nu}(\boldsymbol{\tau})\tilde C_0(f,g) \qquad (3.57)\\ &= (\boldsymbol{\nu}(\boldsymbol{\tau}))^{\frac12}\int_{\mathbb{R}}\psi\left(0,0,s\sqrt{\boldsymbol{\nu}(\boldsymbol{\tau})}\right)ds\sum_{\ell,\ell'\in\mathbb{Z}^2}\int_{\mathbb{R}^2}\mathcal{B}_0(f_\ell(\cdot,u),g_{\ell'}(\cdot,v))\,dudv\\ &= \frac{\sigma}{2\pi\sqrt{\det\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}}\tilde\mu(f)\tilde\mu(g)=\frac{1}{2\pi\sqrt{\det\Sigma_{\boldsymbol{\kappa}}}}\tilde\mu(f)\tilde\mu(g)\end{aligned}$$

where we used $\mathcal{B}_0(u,v)=\nu(u)\nu(v)$ (see (2.47)).

Recalling that the left hand side of (3.4) is an integral with respect to $\tilde{\boldsymbol{\mu}}_0$ as opposed to $C_t(f,g)$ which is an integral with respect to $\tilde\mu$ and using $\tilde\mu=\boldsymbol{\nu}(\boldsymbol{\tau})\tilde{\boldsymbol{\mu}}_0$, we obtain (3.5).

4. Geodesic flows

Let Q be a compact Riemannian manifold with strictly negative curvature and $\tilde Q$ be a cover of Q with automorphism group $\mathbb{Z}^d$. Then $\tilde Q$ can be identified with $Q\times\mathbb{Z}^d$.

The unit tangent bundle of $\tilde Q$ is denoted by $\tilde\Omega$ and unit tangent bundle of Q is denoted by Ω.

The phase space of the geodesic flow $\tilde\Phi$ on $\tilde Q$ is $\tilde\Omega$ and likewise, the phase space of the geodesic flow Φ on Q is Ω. Thus $\tilde\Omega$ is a $\mathbb{Z}^d$ cover of Ω and we denote by $\mathfrak{p}$ the covering map. We endow $\tilde\Omega$ with the normalized Liouville measure $\tilde\mu_0$ so that $\tilde\mu_0(Q'\times S^1)=1$, where $Q'\subset\tilde Q$ corresponds to $Q\times\{0\}$. Geodesic flows are Anosov and can be represented as a suspension flows over a Poincaré section M such that $T:M\to M$, the first return map to M is Markov (see [5] and [6]). Thus M is a union of rectangles $M=\bigcup_{k=1}^K\Delta_k$ where Δ_k have product structure $\Delta_k=[\Delta_k^u\times\Delta_k^s]$ where Δ_k^u are u-sets and Δ_k^s are s-sets and $[\cdot,\cdot]$ is defined by (3.12).

Let τ be the first return to M. Choose a copy $\tilde M\subset\tilde\Omega$ such that $\mathfrak{p}(\tilde M)=M$ and $\mathfrak{p}:\tilde M\to M$ is one-to-one. As for billiards, we define $\mathcal{C}_\ell$ as the set of points in that $\tilde\Omega$ such that the last visit to the Poincaré section was in $\tilde M\times\{\ell\}$ for $\ell\in\mathbb{Z}^d$. We denote by $\tilde\mu$ the Liouville measure.

Now we have the following analogue of Theorem 3.1

Theorem 4.1. *Let $\mathfrak{f}, \mathfrak{g} : \tilde{\Omega} \to \mathbb{R}$ be two η-Hölder continuous functions with at least one of them being smooth in the flow direction. Assume moreover that there exists an integer $K_0 \geq 1$ such that (3.3) holds. Then there are real numbers $\mathfrak{C}_0(\mathfrak{f}, \mathfrak{g}), \mathfrak{C}_1(\mathfrak{f}, \mathfrak{g}), ..., \mathfrak{C}_{K_0}(\mathfrak{f}, \mathfrak{g})$ so that we have*

$$\int_{\tilde{\Omega}} \mathfrak{f}\, \mathfrak{g} \circ \tilde{\Phi}_t d\tilde{\mu}_0 = \sum_{k=0}^{K_0} \mathfrak{C}_k(\mathfrak{f}, \mathfrak{g}) t^{-\frac{d}{2}-k} + o\left(t^{-\frac{d}{2}-K_0}\right), \tag{4.1}$$

as $t \to +\infty$. Furthermore, $\mathfrak{C}_0(\mathfrak{f}, \mathfrak{g}) = \mathfrak{c}_0 \int_{\tilde{\Omega}} \mathfrak{f} d\tilde{\mu}_0 \int_{\tilde{\Omega}} \mathfrak{g} d\tilde{\mu}_0$ and the coefficients $\mathfrak{C}_k$, as functionals over pairs of admissible functions, are bilinear.

Proof. The proof of Theorem 4.1 is a simplified version of the proof of Theorem 3.1. Namely, we apply the abstract Theorem 2.3 to an appropriate symbolic system. This system is now a subshift of finite type that is constructed using a Markov partition $\{\Delta_k\}$. By mixing and by the Perron-Frobenius theorem, there exists r_0 so that for any $r \geq r_0$ and any $i, j = 1, ..., K$, $T^r(\Delta_i)$ and Δ_j have a non empty intersection. We define the spaces $\mathcal{V}, \mathcal{B}$, and $\mathbb{B}$ the same way as in Section 3 with

$$\Delta_0 = M \quad \text{and} \quad \bar{\Delta}_0 = \bigcup_{k=1}^{K} \Delta_k^u.$$

and with constant height r. Consequently, the norms $\|.\|_{\mathbb{B}}$ and $\|.\|_{\mathcal{B}}$ are equivalent. The assumptions of Theorem 2.3 are verified similarly to Section 3 with additional simplifications coming from the boundedness of the return time and the equivalence of $\mathcal{B}$ and $\mathbb{B}$.

The only point in the proof of Theorem 3.1 where we used the special properties of billiards is in the proof of Lemma 3.2, where we referred to Lemma 6.40 in [10] (which is specific to billiards). It remains to revisit this part of the argument (again, in a simplified version as the alphabet is finite and so the symbolic sequence of any specially chosen point is bounded).

Geodesic flows preserve the natural contact form α on the unit tangent bundle (corresponding to the symplectic structure on the tangent bundle). According to the results of [29] (Lemma B.6), there is some $\varepsilon > 0$ so that for any $z \in Q$ and for any sufficiently small unstable vector $v \in E^u(z)$ and stable vector $w \in E^s(z)$ with the notation $x = \exp_z(v)$, $y = \exp_z(w)$, the temporal distance function $D(x, y)$ (defined as in (3.9)) satisfies

$$D(x, y) = d\alpha(v, w) + O(\|v\|^\varepsilon \|w\|^2 + \|v\|^2 \|w\|^\varepsilon).$$

Since the contact form is non-degenerate, there is a constant R_0 such that for any z and any $v \in E^u(z)$, we can find some $w \in T_z Q$ such that $\frac{\|v\| \|w\|}{R_0} \leq d\alpha(v, w) \leq R_0 \|v\| \|w\|$. Let us decompose w into center unstable and stable components $w = w^{cu} + w^s$. By Lemma B.2 in [29], $d\alpha(v, w^{cu}) = 0$ and so we can assume $w = w^s \in E^s(z)$. We conclude that for fixed z, there are constants δ_0, R_0, so that for any $\delta < \delta_0$ there exist vectors $v \in E^u(z), w \in E^s(z)$ such that $\|v\| = \|w\| = \delta$ and

$$D(x, y) \in \left[\frac{\delta^2}{2R_0}, 2R_0 \delta^2\right].$$

Now we can complete the proof of the analogue of Lemma 3.2 as before by choosing δ in a way that for given ξ, $\delta^2 \approx \xi^{-1}$. □

Appendix A. Some facts about Taylor expansions.

Lemma A.1. *Let a be given by (2.4) and $\tilde{\lambda} : [-b, b]^{d+1} \to \mathbb{C}$ (for some $b > 0$) be a C^{K+3}-smooth function satisfying (2.6) for some $J \leq K+3$. Denote $\zeta_s = \frac{\tilde{\lambda}_s}{a_s}$, $M = \lfloor (K+1)/(J-2) \rfloor$. Then there are $A_{j,k} \in \mathcal{S}_j$ (where $j = 0, ... \lfloor J(K+1)/(J-2) \rfloor$, $k = 1, \dots, M$), $K_0 \in \mathbb{N}$ (depending on K and J) and a function $\eta : \mathbb{R}^{d+1} \to [0, +\infty)$ continuous at $\mathbf{0}$, satisfying $\eta(\mathbf{0}) = 0$ such that* *after, possibly, decreasing the value of b, for every n large enough, every $s \in [-b\sqrt{n}, b\sqrt{n}]^{d+1}$ and every $j = J, ..., K+3$, we have*

$$\sum_{k=1}^{M} \binom{n}{k} \sum_{j_1,...,j_k \geq J \,:\, j_1+...+j_k=j} \frac{1}{j_1!...j_k!} \left(\zeta_0^{(j_1)} \otimes ... \otimes \zeta_0^{(j_k)}\right) = \sum_{k=1}^{M} n^k A_{j,k} \tag{A.1}$$

and

$$\left| \zeta^n_{s/\sqrt{n}} - 1 - \sum_{k=1}^{M} \sum_{j=kJ}^{K+1+2k} n^k A_{j,k} * \left(\frac{s}{\sqrt{n}}\right)^{\otimes j} \right| \leq \frac{1}{a_{s/\sqrt{2}}} n^{-\frac{K+1}{2}} (1+|s|^{K_0}) \eta(s/\sqrt{n})\,. \tag{A.2}$$

Recalling that the first $J-1$ derivatives of ζ vanish at zero, we see that in case $\tilde{\lambda}$ is C^j (namely, if $j \leq K+3$), the LHS of (A.1) is simply equal to $\frac{1}{j!}(\zeta^n)_0^{(j)}$.

Proof. Decreasing if necessary the value of b, we may assume that $|\tilde{\lambda}_u| \leq a_{u/\sqrt{2.5}} \leq a_{u/\sqrt{2}}$ and $|\tilde{\lambda}_u - a_u| \leq C|u|^J$ for every $u \in \mathbb{R}^{d+1}$ with $|u| < b$ (the existence of b with these properties follows from our assumptions on J and $\tilde{\lambda}$). Applying Taylor's theorem to the function $x \mapsto x^n$ near 1 we conclude that for every $s \in \mathbb{R}^{d+1}$ with $|s| < b\sqrt{n}$,

$$\begin{aligned} &\left| \zeta^n_{s/\sqrt{n}} - \sum_{k=0}^{M} \binom{n}{k} \left(\zeta\left(\frac{s}{\sqrt{n}}\right) - 1\right)^k \right| \\ &\leq \binom{n}{M+1} \left| \zeta\left(\frac{s}{\sqrt{n}}\right) - 1 \right|^{M+1} (\max(1, |\zeta\left(\frac{s}{\sqrt{n}}\right)|))^{n-M-1}\,. \end{aligned} \tag{A.3}$$

Recall that $|\tilde{\lambda}_{s/\sqrt{n}}| \leq a_{s/\sqrt{1.5\,n}}$. This together with the fact that $a_{s/\sqrt{1.5\,n}}/a_{s/\sqrt{n}} = (a_{s/\sqrt{3n}})^{-1}$ implies that the RHS of (A.3) is bounded by

$$n^{M+1} \left|\zeta(s/\sqrt{n}) - 1\right|^{M+1} (a_{s/\sqrt{3n}})^{-(n-M-1)} = n^{M+1} \left|\tilde{\lambda}(s/\sqrt{n}) - a(s/\sqrt{n})\right|^{M+1} (a_{s/\sqrt{3n}})^{-n-M-1}.$$

Next, we use the identity $(a_{s/\sqrt{3n}})^n = a_{s/\sqrt{3}}$ and the inequality $|\tilde{\lambda}_u - a_u| \leq C|u|^J$ to conclude that the last displayed expression is bounded by

$$C_M n^{M+1} (a_{s/\sqrt{2}})^{-1} \left((s/\sqrt{n})^{J(M+1)}\right)\,,$$

for every s, for every n large enough since $(a_{s/\sqrt{3n}})^{-n-M-1} = \left(a_{s\sqrt{(1+\frac{M+1}{n})/3}}\right)^{-1} \leq (a_{s/\sqrt{2}})^{-1}$ for every n large enough. Now observe that by definition $(2-J)(M+1) < -K-1$ and so $(2-J)(M+1) \leq -K-2$. Thus the last display, and hence (A.3) is bounded by

$$C_M (a_{s/\sqrt{2}})^{-1} n^{-\frac{K+2}{2}} s^{J(M+1)}. \tag{A.4}$$

Clearly, (A.4) can be included in the RHS of (A.2). Thus it remains to compute the sum in the LHS of (A.3).

To do so, we fix some $k = 1, ..., M$. Let $L = K + 1 + 2k - J(k-1)$. Using the elementary estimate $|a^k - b^k| \le k \max(|a|, |b|)^{k-1}|a-b|$, we find

$$\binom{n}{k} \left| \left(\zeta(s/\sqrt{n}) - 1\right)^k - \left(\sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j} \right)^k \right| \tag{A.5}$$

$$\le \quad n^k k \max \left(|\zeta(s/\sqrt{n}) - 1|, \left| \sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j} \right| \right)^{k-1} \tag{A.6}$$

$$\times \quad \left| \zeta(s/\sqrt{n}) - 1 - \sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j} \right|. \tag{A.7}$$

Next by our choice of L

$$L = K + 1 + (2-J)k + J \le K + 1 + (2-J) + J = K + 3.$$

Recalling that $\tilde{\lambda}/a$ is C^{K+3} smooth and its first $J-1$ derivatives at zero vanish, Taylor's theorem implies that (A.7) is bounded by $(s/\sqrt{n})^L \eta_0(s/\sqrt{n})$, where $\eta_0(0) = 0$ and η is continuous at 0. On the other hand, (A.6) is bounded by $n^k k \, (s/\sqrt{n})^{J(k-1)}$. We conclude that (A.5) is bounded by

$$n^{-\frac{K+1}{2}} s^{K+1+2k} \eta_1(s/\sqrt{n}), \tag{A.8}$$

where $\eta_1 = k\eta_0$. Since $a_{s/\sqrt{2}}$ is bounded from above, (A.8) can be included in the RHS of (A.2). So we have approximated $\zeta^n_{s/\sqrt{n}}$ by

$$\begin{aligned}
& 1 + \sum_{k=1}^{M} \binom{n}{k} \left(\sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j} \right)^k \\
= \quad & 1 + \sum_{k=1}^{M} \binom{n}{k} \sum_{j_1,...,j_k=J}^{L} \frac{1}{j_1!...j_k!} \left(\zeta_0^{(j_1)} \otimes ... \otimes \zeta_0^{(j_k)} \right) * (s/\sqrt{n})^{\otimes(j_1+...+j_k)} \\
= \quad & 1 + \sum_{k=1}^{M} \binom{n}{k} \sum_{j=kJ}^{K+1+2k} \sum_{j_1,...,j_k \ge J \,:\, j_1+...+j_k=j} \frac{1}{j_1!...j_k!} \left(\zeta_0^{(j_1)} \otimes ... \otimes \zeta_0^{(j_k)} \right) \\
& *(s/\sqrt{n})^{\otimes j} + O\left(n^{-\frac{K+2}{2}} s^{K+1+2k+1} \right)
\end{aligned}$$

uniformly on $s \in [-b\sqrt{n}, b\sqrt{n}]^{d+1}$. Note that the last step above uses the observation that if $j_1, ..., j_k \ge J$ and $j_1 + \cdots + j_k \le K + 1 + 2k$, then necessarily $j_l \le L$ for all l. Again, the last error term can be included in the right hand side of (A.2) as $a_{s/\sqrt{2}}$ is bounded from above.

Finally, observe that

$$\binom{n}{k} \sum_{j_1,...,j_k \ge J \,:\, j_1+...+j_k=j} \frac{1}{j_1!...j_k!} \left(\zeta_0^{(j_1)} \otimes ... \otimes \zeta_0^{(j_k)} \right)$$

is a polynomial of degree k in n with values in $\mathcal{S}_j$. This ensures the existence of $A_{j,k}$. □

Lemma A.2. *If $H : \mathbb{R} \to \mathbb{R}$ is in the Schwartz space (i.e. $x^a H^{(b)}(x)$ is bounded for any positive integers a and b), then for any $L \in \mathbb{N}$ there is some constant $c_{H,L}$ such that*

$$\forall t \in \mathbb{R}, \forall \eta > 0, \qquad \left| \sum_{k \in \mathbb{Z}} \eta H(t + k\eta) - \int_{-\infty}^{\infty} H(x)dx \right| < c_{H,L}\eta^L. \tag{A.9}$$

Proof. We can assume without loss of generality that $t \in [0, 1)$. Given L, t and η, we choose A_L and B_L so that the above sum for $k \notin [A_L/\eta, B_L/\eta]$ and the above integral as well as the first L derivatives of H for $x \notin (A_L, B_L)$ are less than η^L. Such A_L and B_L exist since H is in the Schwartz space. Now Euler's summation formula (e.g. Theorem 4 in [3] with the notation $f(x) = \eta H(t + x\eta - A_L)$, $m = L$) implies that

$$\sum_{k=-A_L/\eta}^{B_L/\eta} \eta H(t + k\eta) - \int_{A_L}^{B_L} H(x)dx = \frac{1}{(2L+1)!} \int_{A_L}^{B_L} \mathcal{P}_{2L+1}(x/\eta) H^{(2L+1)}(x) dx \eta^{2L+1}$$
$$+ \sum_{r=1}^{L} \frac{\mathcal{B}_{2r}}{(2r)!} \left[H^{(2r-1)}(B_L) - H^{(2r-1)}(A_L) \right] \eta^{2r}$$
$$+ \frac{1}{2}\eta[H(B_L) - H(A_L)],$$

where $\mathcal{P}_k(x)$ are the periodic Bernoulli polynomials and $\mathcal{B}_k$ are Bernoulli numbers. Now (A.9) follows from the choice of A_L, B_L. □

Observe that (A.9) and the fact that H is in the Schwartz space imply

$$\forall K > 0, \quad \forall \varepsilon > 0, \quad \sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}} H\left(\frac{t - n\nu(\tau)}{\sqrt{t}}\right) = \frac{\sqrt{t}}{\nu(\tau)} \int_{\mathbb{R}} H(x)dx + O(t^{-K}) \tag{A.10}$$

(clearly, the constant in "O" depends on K and ε).

Lemma A.3. *For every* $\gamma \in \mathbb{R}$ *and* $Q \in \mathbb{Z}_+$,

$$\sum_{n=t_-}^{t_+} n^\gamma \Psi^{(\alpha)}\left(0, \frac{t - n\nu(\tau)}{\sqrt{n}}\right)$$

$$= \left(\frac{t}{\nu(\tau)}\right)^\gamma \sum_{q=0}^{Q} \frac{1}{q!} \frac{t^{-\frac{q-1}{2}}}{\nu(\tau)} \int_{\mathbb{R}} \partial_2^q h_{\alpha,\gamma}(s, 1) (-s)^q \, ds + O\left(t^{\gamma - \frac{Q}{2}}\right) \tag{A.11}$$

where $h_{\alpha,\gamma}$ *is defined by* (2.5) ∂_2^q *denotes the derivative of order* q *with respect to the second variable.*

Proof. For ease of notation, we prove the lemma coordinate-wise, i.e. we replace $\Psi^{(\alpha)}(s)$ by $\frac{\partial^\alpha}{\partial s_{j_1} \ldots \partial s_{j_\alpha}} \Psi(s)$.

Observe that due to the rapid decay of $\Psi^{(m+j+r)}(0, \cdot)$, we can replace $\sum_{n=t_-}^{t_+}$ by $\sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}}$, for any $\varepsilon > 0$ (for example, we can choose $\varepsilon = 1/4$).

Next, observe that by the definition (2.5),

$$\left(\frac{n}{t/\nu(\tau)}\right)^\gamma \Psi^{(\alpha)}\left(0, \frac{t - n\nu(\tau)}{\sqrt{n}}\right) = h_{\alpha,\gamma}\left(\frac{t - n\nu(\tau)}{\sqrt{t}}, \frac{n\nu(\tau)}{t}\right).$$

Thus it remains to estimate the sum

$$\sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}} h_{\alpha,\gamma}\left(\frac{t - n\nu(\tau)}{\sqrt{t}}, \frac{n\nu(\tau)}{t}\right). \tag{A.12}$$

Using Taylor expansion, we can rewrite (A.12) as

$$\left[\sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}} \sum_{q=0}^{Q} \frac{1}{q!}\partial_2^q h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},1\right)\left(-\frac{t-n\nu(\tau)}{t}\right)^q\right] + O\left(t^{-\frac{Q}{2}}\right). \tag{A.13}$$

Indeed, we control the error term using the estimate

$$\sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}} \sup_{|y-1|<1/2}\left|\partial_2^{Q+1}h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},y\right)\right|\left|\frac{t-n\nu(\tau)}{t}\right|^{Q+1} = O\left(t^{-\frac{Q}{2}}\right),$$

which can be derived similarly to (A.10). Performing summation over n in (A.13), using (A.10), we obtain that (A.12) (and thus the left hand side of (A.11)) equals to

$$\sum_{q=0}^{Q}\frac{1}{q!}\frac{t^{-\frac{q-1}{2}}}{\nu(\tau)}\int_{\mathbb{R}}\partial_2^q h_{\alpha,\gamma}(s,1)(-s)^q\,ds + O\left(t^{-\frac{Q}{2}}\right).$$

This completes the proof of the lemma. □

Lemma A.4. *Let b,q be non-negative integers. The function $s\mapsto \partial_2^q h_{b,\gamma}(s,1)(-s)^q$ is even if $b+q$ is even (and is odd if $b+q$ is odd).*

Proof. The lemma follows since if $P(x)$ is a polynomial with odd (even, resp.) leading term, then $\frac{d}{dx}(P(x)e^{cx^2}) = Q(x)e^{cx^2}$ where $Q(x)$ is a polynomial with even (odd, resp.) leading term. □

Appendix B. Correlation functions of coboundaries

Lemma B.1. *Let $\mathbf{G}^t:\mathbf{M}\to\mathbf{M}$ be a flow preserving a measure $\boldsymbol{\mu}$ (finite or infinite). Let $f,f',g:\mathbf{M}\to\mathbf{M}$ be bounded integrable observables such that $f'(x)=\frac{d}{dt}|_{t=0}f(\mathbf{G}^tx)$. Denote*

$$C_t = \int_{\mathbf{M}} f\left(g\circ\mathbf{G}^t\right)d\mu,\quad C'_t = \int_{\mathbf{M}} f'\left(g\circ\mathbf{G}^t\right)d\mu.$$

Assume that there exist real numbers $\alpha>0$, $c_0,\dots,c_{K-1},c'_0,\dots,c'_K$ satisfying:

$$C_t = t^{-\alpha}\left(\sum_{k=0}^{K-1}c_kt^{-k}+o\left(t^{-(K-1)}\right)\right)\quad\text{and}\quad C'_t = t^{-\alpha}\left(\sum_{k=0}^{K}c'_kt^{-k}+o\left(t^{-K}\right)\right). \tag{B.1}$$

Then $c'_0=0$ and $c'_k=-c_{k-1}(\alpha+k-1)$ for every $k=1,\dots,K-1$.

In particular if $K=1$ and $c_0\neq 0$, then $c'_0=0$ and

$$C_t(f',g)\sim -c_0\alpha t^{-\alpha-1} \tag{B.2}$$

We note that the fact that the rate of mixing for coboundaries is faster than for general observables is used, for example, in [18, 21].

Proof. By integration by parts

$$\begin{aligned} C'_t &= \int_{\mathbf{M}} f'\left(g\circ\mathbf{G}^t\right)d\boldsymbol{\mu} = -\int_{\mathbf{M}} f\left(g'\circ\mathbf{G}^t\right)d\boldsymbol{\mu}\\ &= -\int_{\mathbf{M}} f.\frac{\partial}{\partial t}\left(g\circ\mathbf{G}^t\right)d\boldsymbol{\mu} = -\frac{\partial}{\partial t}\int_{\mathbf{M}} f\left(g\circ\mathbf{G}^t\right)d\boldsymbol{\mu} = -\frac{\partial}{\partial t}C_t.\end{aligned}$$

Since $\lim_{t\to+\infty}C_t=0$

$$C_t = \int_t^{+\infty}C'_s\,ds = \int_t^{+\infty}\sum_{k=0}^{K}c'_ks^{-\alpha-k}+o(s^{-\alpha-K})\,ds\,.$$

It follows that $c'_k = 0$ if $\alpha + k \le 1$ and

$$C_t = \sum_{k=0}^{K} \frac{c'_k}{-\alpha - k + 1} t^{-\alpha+1-k} + o\left(t^{-\alpha-K+1}\right) .$$

The lemma follows by comparing the above expansion with the first equation in (B.1). □

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