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THRESHOLD DYNAMICS FOR COROTATIONAL WAVE MAPS

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We study the dynamics of corotational wave maps from $\mathbb{R}^{1+2} \to \mathbb{S}^2$ at threshold energy. It is known that topologically trivial wave maps with energy $< 8\pi$ are global and scatter to a constant map. We prove that a corotational wave map with energy equal to 8π is globally defined and scatters in one time direction, and in the other time direction, either the map is globally defined and scatters, or the map breaks down in finite time and converges to a superposition of two harmonic maps. The latter behavior stands in stark contrast to higher equivariant wave maps with threshold energy, which have been proven to be globally defined for all time. Using techniques developed in this paper, we also construct a corotational wave map with energy $= 8\pi$ which blows up in finite time. The blow-up solution we construct provides the first example of a minimal topologically trivial nondispersing solution to the full wave map evolution.

1. Introduction

1A. *Wave maps.* We study the dynamics of energy critical wave maps which are defined as follows. Let η be the Minkowski metric on $\mathbb{R}^{1+2}_{t,x}$, and let $\mathcal{N}$ be a Riemannian manifold with metric h. A map $u : \mathbb{R}^{1+2} \to \mathcal{N}$ is a *wave map* if it is a critical point of the action

$$\mathcal{A}(u) = \frac{1}{2} \int_{\mathbb{R}^{1+2}} \langle \partial^\mu u, \partial_\mu u \rangle_h \, dx \, dt,$$

where we raise and lower indices using the Minkowski metric η. The associated Euler–Lagrange equations are the *wave maps equations* given in local coordinates by

$$\partial^\mu \partial_\mu u^a + \Gamma^a_{bc}(u) \, \partial^\mu u^b \, \partial_\mu u^c = 0. \tag{1-1}$$

Here the Γ^a_{bc} are the Christoffel symbols associated to the metric h on $\mathcal{N}$. The time translational symmetry of Minkowski space and Noether's theorem provide a conserved energy for the evolution

$$\mathcal{E}(u(t), \partial_t u(t)) := \frac{1}{2} \int_{\mathbb{R}^2} |\partial_t u(t,x)|^2_h + |\nabla u(t,x)|^2_h \, dx = \text{const.} \tag{1-2}$$

We study wave maps as solutions to the Cauchy problem (1-1) with prescribed finite-energy initial data $\vec{u}(0) = (u_0, u_1)$, where

$$u_0(x) \in \mathcal{N}, \quad u_1(x) \in T_{u_0(x)}\mathcal{N}, \quad x \in \mathbb{R}^2.$$

Here and throughout the paper we use the notation $\vec{u}(t)$ to denote the pair of functions

$$\vec{u}(t) := (u(t, \cdot), \partial_t u(t, \cdot)).$$

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We also assume that there exists $u_\infty \in \mathcal{N}$ such that

$$u_0(x) \to u_\infty \quad \text{as } |x| \to \infty.$$

Due to the conformal symmetry of Minkowski space, we also have the following scaling symmetry: if $\vec{u}(t)$ is a wave map and $\lambda > 0$, then

$$\vec{u}_\lambda(t,x) = (u_\lambda(t,x), \partial_t u_\lambda(t,x)) := \left(u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda}\partial_t u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right)$$

is also a wave map. The energy is scale-invariant,

$$\mathcal{E}(\vec{u}_\lambda) = \mathcal{E}(\vec{u}),$$

and for this reason, the wave maps equations in (1+2)-dimensions are said to be *energy critical*. Wave maps have been extensively studied over the past several decades, and we refer the reader to [Shatah and Struwe 1998; Geba and Grillakis 2017] for reviews of the work that has been done.

In this work we specialize to the case $\mathcal{N} = \mathbb{S}^2$ (with the usual round metric) and wave maps which respect the rotational symmetry of the background and target. More precisely, we fix an origin in $\mathbb{R}^2$ and north pole $N \in \mathbb{S}^2$. We say a map $u : \mathbb{R}^{1+2} \to \mathbb{S}^2$ is *corotational* or 1-*equivariant* if $u \circ \rho = \rho \circ u$ for all $\rho \in \mathrm{SO}(2)$. Here ρ acts on $\mathbb{S}^2$ by rotation about the axis determined by N. Choosing $N = (0,0,1)$ without loss generality, we can write a corotational map as

$$u(t,r,\theta) = (\sin\psi(t,r)\cos\theta, \sin\psi(t,r)\sin\theta, \cos\psi(t,r)) \in \mathbb{S}^2 \subset \mathbb{R}^3, \tag{1-3}$$

where (t,r,θ) are polar coordinates on $\mathbb{R}^{1+2}$, and (ψ,θ) are spherical coordinates on $\mathbb{S}^2$. For corotational maps, the Cauchy problem (1-1) reduces to a single equation for the azimuth angle $\psi = \psi(t,r)$:

$$\partial_t^2\psi - \partial_r^2\psi - \frac{1}{r}\partial_r\psi + \frac{\sin 2\psi}{2r^2} = 0, \quad \vec{\psi}(0) = (\psi_0, \psi_1). \tag{1-4}$$

The conserved energy (1-2) is given by

$$\mathcal{E}(\vec{\psi}(t)) = \pi\int_0^\infty \left((\partial_t\psi(t,r))^2 + (\partial_r\psi(t,r))^2 + \frac{\sin^2\psi(t,r)}{r^2}\right) r\,dr.$$

The expression for the energy implies that there exist $m, n \in \mathbb{Z}$ such that $\lim_{r\to 0}\psi_0(r) = m\pi$ and $\lim_{r\to\infty}\psi_0(r) = n\pi$. By continuity of the flow $\vec{\psi}(t)$,

$$\lim_{r\to 0}\psi(t,r) = m\pi, \quad \lim_{r\to\infty}\psi(t,r) = n\pi \quad \text{for all } t.$$

Without loss of generality, we may assume that $m = 0$ and $n \in \mathbb{N}\cup\{0\}$. Thus, finite-energy solutions to (1-4) are split into disjoint classes given by

$$\mathcal{H}_n := \left\{(\psi_0,\psi_1) \,\middle|\, \mathcal{E}(\psi_0,\psi_1) < \infty \text{ and } \lim_{r\to 0}\psi_0(r) = 0,\ \lim_{r\to\infty}\psi_0(r) = n\pi\right\}.$$

We refer to the parameter $n \in \mathbb{N}\cup\{0\}$ as the *degree* of the map, and it can be thought of as parametrizing the minimal number of times the map $\psi(t)$ (more precisely, $u(t)$ given by (1-3)) wraps $\mathbb{R}^2$ around the sphere. We study those corotational initial data $(\psi_0, \psi_1) \in \mathcal{H}_0$, i.e., those which satisfy

$$\lim_{r\to 0} \psi_0(r) = \lim_{r\to\infty} \psi_0(r) = 0.$$

A corotational ansatz reduces the complexity of the wave maps equations greatly and is possible in the more general case when $\mathcal{N}$ is a surface of revolution. Choosing $\mathcal{N} = \mathbb{S}^2$ is motivated by what is known about stationary wave maps, or *harmonic maps*, in this setting. By an ODE argument, the unique (up to scaling) nontrivial corotational harmonic map is given explicitly by

$$Q(r) = 2\arctan r,$$

with energy

$$\mathcal{E}(\vec{Q}) = 4\pi.$$

We note that

$$\lim_{r\to 0} Q(r) = 0, \qquad \lim_{r\to\infty} Q(r) = \pi,$$

so that $\vec{Q} \in \mathcal{H}_1$. In fact, it can be shown that Q minimizes the energy in $\mathcal{H}_1$ (see Section 2). As we will soon discuss, these harmonic maps play a fundamental role in the long-time dynamics of wave maps with large initial data.

We conclude this subsection by discussing k-equivariant maps, a generalization of our corotational reduction. For $k \in \mathbb{N}$, we say a map $u : \mathbb{R}^{1+2} \to \mathbb{S}^2$ is *k-equivariant* if $u \circ \rho = \rho^k \circ u$ for all $\rho \in \mathrm{SO}(2)$, where SO(2) acts on the $\mathbb{R}^{1+2}$ and $\mathbb{S}^2$ as before. Then we may write

$$u(t, r, \theta) = (\sin\psi(t, r)\cos k\theta, \sin\psi(t, r)\sin k\theta, \cos\psi(t, r)),$$

and the wave maps equations reduce to the single equation

$$\partial_t^2\psi - \partial_r^2\psi - \frac{1}{r}\partial_r\psi + k^2\frac{\sin 2\psi}{2r^2} = 0, \qquad \vec{\psi}(0) = (\psi_0, \psi_1). \tag{1-5}$$

The conserved energy (1-2) is given by

$$\mathcal{E}^k(\vec{\psi}(t)) = \pi\int_0^\infty \left((\partial_t\psi(t, r))^2 + (\partial_r\psi(t, r))^2 + k^2\frac{\sin^2\psi(t, r)}{r^2}\right) r\,dr.$$

As in the corotational setting, the unique (up to scaling) nontrivial k-equivariant harmonic map is given by

$$Q^k(r) = 2\arctan(r^k).$$

The harmonic map $\vec{Q}^k$ is in $\mathcal{H}_1$, $\mathcal{E}^k(\vec{Q}^k) = 4\pi k$ and $\vec{Q}^k$ minimizes the energy $\mathcal{E}^k(\,\cdot\,)$ in the class $\mathcal{H}_1$. In particular, the corotational harmonic map $Q = Q^1$ has the least energy of all nontrivial equivariant harmonic maps.

We now turn to motivating our main results.

1B. ***History and motivation.*** Strichartz estimates suffice to prove global existence for equivariant wave maps evolving from small degree-0 data (see Section 2), so recent work has been dedicated to understanding the long-time dynamics of wave maps evolving from large initial data. It is here that the family of harmonic maps plays a fundamental role. Indeed, a classical result of [Struwe 2003] states that if a smooth k-equivariant wave map $\vec{\psi}(t)$ breaks down at time $t = 1$, say, then $\vec{\psi}(t)$ converges to the harmonic map $\vec{Q}^k$ in a local spacetime norm. Moreover, $\vec{\psi}(t, r)$ must concentrate energy in excess of $\mathcal{E}^k(\vec{Q}^k)$ at the tip of the inverted light cone centered at $(T_+, r) = (1, 0)$. Thus, a k-equivariant wave map $\vec{\psi}(t)$ with energy less than $\mathcal{E}^k(\vec{Q}_k)$ is globally defined and smooth. The works [Krieger, Schlag, and Tataru 2008; Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012] constructed examples of degree-1 wave maps that blow-up by bubbling off a harmonic map, i.e.,

$$\vec{\psi}(t) = \vec{Q}^k_{\lambda(t)} + \vec{\varphi}(t),$$

with $\lambda(t) \to 0$ as $t \to T_+ < \infty$ and $\varphi(t)$ regular up to $t = T_+$.

As we've discussed, harmonic maps play a key role in singularity formation for wave maps, but in fact they should be fundamental in describing the dynamics of *arbitrary* wave maps. Indeed, according to the *soliton resolution conjecture*, one expects the following beautiful simplification of the dynamics: smooth wave maps asymptotically break up into a sum of dynamically rescaled harmonic maps and a free radiation term (a solution to the linearized equations). The problem of describing the dynamics of corotational wave maps with energy $= 2\mathcal{E}(\vec{Q})$ we address in this paper is motivated by several recent advances made in establishing this conjecture for equivariant wave maps. We first state the following refined threshold theorem proved in [Côte, Kenig, Lawrie, and Schlag 2015a].

Theorem 1.1 [Côte, Kenig, Lawrie, and Schlag 2015a]. *For smooth initial data $(\psi_0, \psi_1) \in \mathcal{H}_0$, with*

$$\mathcal{E}^k(\psi_0, \psi_1) < 2\mathcal{E}^k(\vec{Q}^k),$$

there exists a unique global smooth k-equivariant wave map $\vec{\psi} \in C(\mathbb{R}; \mathcal{H}_0)$ with $\vec{\psi}(0) = (\psi_0, \psi_1)$. Moreover, $\vec{\psi}(t)$ scatters both forward and backward in time; i.e., there exist solutions $\vec{\varphi}_L^{\pm}$ to the linearized equation

$$\partial_t^2 \varphi - \partial_r^2 \varphi - \frac{1}{r} \partial_r \varphi + \frac{k^2}{r^2} \partial_r \varphi = 0 \tag{1-6}$$

such that

$$\vec{\psi}(t) = \vec{\varphi}_L^{\pm}(t) + o_{\mathcal{H}_0}(1) \quad \text{as } t \to \pm\infty.$$

The intuition for the threshold energy being $2\mathcal{E}^k(\vec{Q}^k)$ rather than $\mathcal{E}^k(\vec{Q}^k)$ is the following. If a k-equivariant map $\vec{\psi}(t) \in \mathcal{H}_0$ wraps the plane around the sphere once, then it must also unwrap the sphere once more in order to have degree 0. Since the minimum amount of energy needed for a k-equivariant map to wrap the plane around the sphere once is equal to $\mathcal{E}^k(\vec{Q}^k)$, it follows that if $\mathcal{E}^k(\vec{\psi}) < 2\mathcal{E}^k(\vec{Q}^k)$ then $\psi(t)$ is bounded away from the south pole (i.e., $\psi(t, r) < \pi - \epsilon$ for all t, r). Thus, $\vec{\psi}(t)$ cannot converge locally to a harmonic map $\vec{Q}^k$, which by Struwe's bubbling result implies $\vec{\psi}(t)$ is globally regular.

A result analogous to Theorem 1.1 for the full wave map system, with no symmetry assumptions, was established in [Lawrie and Oh 2016]. More precisely, we say initial data (u_0, u_1) (with target $\mathbb{S}^2$) is *topologically trivial* if

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} u_0^* \omega_{\mathbb{S}^2} = 0,$$

where $\omega_{\mathbb{S}^2}$ is the volume form on $\mathbb{S}^2$. It can be checked that the above condition is propagated by the wave map evolution, and an equivariant map $\vec{u}$ with associated azimuth angle $\vec{\psi} \in \mathcal{H}_0$ is topological trivial. The authors obtain the following result as a consequence of the analysis from [Sterbenz and Tataru 2010].

Theorem 1.2 [Lawrie and Oh 2016]. *Suppose that* (u_0, u_1) *is smooth topologically trivial finite-energy initial data with*

$$\mathcal{E}(u_0, u_1) < 8\pi = 2\mathcal{E}(\vec{Q}^1).$$

Then there exists a unique global solution $u : \mathbb{R}^{1+2} \to \mathbb{S}^2$ *to the wave maps equations* (1-1) *with* $\vec{u}(0) = (u_0, u_1)$. *Moreover,* $\vec{u}(t)$ *scatters to the constant map as* $t \to \pm\infty$.

The works [Côte, Kenig, Lawrie, and Schlag 2015a; 2015b] also established soliton resolution for corotational wave maps in $\mathcal{H}_1$ with energy below $3\mathcal{E}(\vec{Q})$. In this setting only one concentrating bubble is possible, and these works showed that for any such wave map there exists a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the free equation (1-6) (the radiation) and a continuous dynamical scale $\lambda(t) \in (0, \infty)$ such that

$$\vec{\psi}(t) = \vec{Q}_{\lambda(t)} + \vec{\varphi}_L(t) + o_{\mathcal{H}_0}(1) \quad \text{as } t \to T_+.$$

Proving soliton resolution above $3\mathcal{E}(\vec{Q})$ is very challenging since one can conceivably have multiple harmonic maps concentrating at different scales and interacting. However, there has been exciting recent progress in establishing a weaker form of the conjecture. The work [Côte 2015] (for 1-equivariant maps) and [Jia and Kenig 2017] (for all equivariant maps) established the following soliton resolution result along a well-chosen sequence of times.

Theorem 1.3 [Côte 2015; Jia and Kenig 2017]. *Let* $\vec{\psi}(t) \in \mathcal{H}_n$ *be a smooth k-equivariant wave map on* $[0, T_+)$. *Then there exists a sequence of times* $t_n \to T_+$, *an integer* $J \in \mathbb{N} \cup \{0\}$, *a solution* $\vec{\varphi}_L(t) \in \mathcal{H}_0$ *to* (1-6), *sequences of scales* $\lambda_{n,j}$ *which satisfy* $0 < \lambda_{n,1} \ll \lambda_{n,2} \ll \cdots \ll \lambda_{n,J}$ *and signs* $\iota_j \in \{-1, 1\}$ *for* $j \in \{1, \dots, J\}$, *so that*

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} \iota_j \vec{Q}^k_{\lambda_{n,j}} + \vec{\varphi}_L(t_n) + o_{\mathcal{H}_0}(1) \quad \textit{as } n \to \infty. \tag{1-7}$$

If $T_+ < \infty$ *then* $J \geq 1$, $0 < \lambda_{n,1} \ll \cdots \ll \lambda_{n,J} \ll T_+ - t_n$, *and if* $T_+ = \infty$ *then* $0 < \lambda_{n,1} \ll \cdots \ll \lambda_{n,J} \ll t_n$. *The signs* ι_j *are required to satisfy the topological constraint* $\vec{\psi}(t) \in \mathcal{H}_n$, *i.e.,*

$$\lim_{r \to \infty} \sum_{j=1}^{J} \iota_j Q^k_{\lambda_{n,j}}(r) = n\pi.$$

We remark that [Côte, Kenig, Lawrie, and Schlag 2015a; 2015b; Côte 2015; Jia and Kenig 2017; Jendrej and Lawrie 2018] use ideas and techniques inspired by the seminal papers on the focusing quintic nonlinear wave equation in three space dimensions by [Duyckaerts, Kenig, and Merle 2011; 2012a; 2012b; 2013] (see also [Kenig 2015] for an account of the important techniques and ideas in these papers).

Jendrej [2019] showed it is possible for more than one bubble to form in the decomposition (1-7).

Theorem 1.4 [Jendrej 2019]. *Fix an equivariance class $k > 2$. There exists a solution $\vec{\psi} : (-\infty, T_+) \to \mathcal{H}_0$ of* (1-5) *such that*

$$\lim_{t\to-\infty} \|\vec{\psi}(t) - (\vec{Q}_{c_k|t|^{-2/(k-2)}} - \vec{Q})\|_{\mathcal{H}_0} = 0,$$

where $c_k > 0$ is explicit. □

A similar construction is possible when $k = 2$ with an explicit exponentially decaying scale as $t \to -\infty$. By Theorem 1.1, these solutions are examples of nondispersing *threshold solutions* to (1-5) for $k \geq 2$.

Jendrej and Lawrie [2018] classified the dynamics of k-equivariant wave maps $\vec{\psi}(t)$ with *threshold energy* $\mathcal{E}^k(\vec{\psi}(t)) = 2\mathcal{E}^k(\vec{Q}^k)$ for $k \geq 2$. Their work provided the primary motivation and roadmap for establishing our main results. To state their results concisely, we first introduce some terminology. Let $\vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0$ be a k-equivariant wave map with $\mathcal{E}^k(\vec{\psi}) = 2\mathcal{E}^k(\vec{Q}^k)$. We say that $\vec{\psi}(t)$ is a *two-bubble in the forward time direction* if there exist $\iota \in \{1, -1\}$ and continuous functions $\lambda(t), \mu(t) > 0$ such that

$$\lim_{t\to T_+} \|(\psi(t) - \iota(Q^k_{\lambda(t)} - Q^k_{\mu(t)}), \psi_t(t))\|_{\mathcal{H}_0} = 0, \qquad \lambda(t) \ll \mu(t) \quad \text{as } t \to T_+.$$

The notion of a *two-bubble in the backward time direction* is defined similarly.

Theorem 1.5 [Jendrej and Lawrie 2018]. *Let $k \geq 2$, and let $\vec{\psi} : (T_-, T_+) \to \mathcal{H}_0$ be a k-equivariant wave map such that*

$$\mathcal{E}^k(\vec{\psi}) = 2\mathcal{E}^k(\vec{Q}^k) = 8\pi k.$$

Then $T_- = -\infty$, $T_+ = \infty$ and one the following alternatives holds:

- *$\vec{\psi}(t)$ scatters in both time directions.*
- *$\vec{\psi}(t)$ scatters in one time direction and is a two-bubble in the other time direction. Moreover if $\vec{\psi}(t)$ is a two-bubble in the forward time direction, then there exists $C = C(k) > 0$, $\mu_0 > 0$ such that $\mu(t) \to \mu_0$ and*

$$\begin{aligned} \mu_0 \exp(-Ct) \leq \lambda(t) \leq \mu_0 \exp(-t/C) \quad & \text{if } k = 2, \\ \frac{\mu_0}{C} t^{-2/(k-2)} \leq \lambda(t) \leq C\mu_0 t^{-2/(k-2)} \quad & \text{if } k \geq 3. \end{aligned}$$

An analogous estimate holds if $\vec{\psi}(t)$ is a two-bubble in the backward time direction.

1C. *Main results.* The two-bubble solutions given by Theorem 1.4 and the classification result Theorem 1.5 are for k-equivariant wave maps with $k \geq 2$. The first main result of this paper establishes the existence of a corotational two-bubble solution. In contrast to higher equivariant wave maps, our solution is in fact a threshold *blow-up solution*.

Theorem 1.6 (Main Theorem 1). *There exists a corotational wave map $\vec{\psi}_c : (0, T_+) \to \mathcal{H}_0$, a continuous scale $\lambda_c(t) > 0$, and constant $C > 0$ such that*

$$\frac{1}{C} t^2 \leq \lambda_c(t) |\log \lambda_c(t)| \leq Ct^2$$

and

$$\lim_{t\to 0^+} \|\vec{\psi}_c(t) - (\vec{Q}_{\lambda_c(t)} - \vec{Q})\|_{\mathcal{H}_0} = 0.$$

In particular, $\mathcal{E}(\vec{\psi}_c) = 8\pi$ and $T_- = 0$.

By Theorem 1.1, $\vec{\psi}_c$ is a minimal energy nondispersing solution to (1-4). Moreover, by Theorem 1.2 the map $u_c : (0, T_+) \times \mathbb{R}^2 \to \mathbb{S}^2$ given by

$$u_c(t, r, \theta) = (\sin \psi_c(t, r) \cos \theta, \sin \psi_c(t, r) \sin \theta, \cos \psi_c(t, r))$$

is a topologically trivial minimal energy nondispersing solution to the *full wave map equations*. The existence of such a solution has been an open question up until now. The proof of Theorem 1.6 is a byproduct of estimates we derive to prove our second main result and the general scheme for constructing multisoliton solutions introduced in [Martel 2005; Merle 1990]. We remark that our construction does not require as precise as an ansatz for $\vec{\psi}_c$ as in those works but is closer in spirit to the two-bubble construction for NLS in [Jendrej 2017].

Theorem 1.7 (Main Theorem 2). *Let $\vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0$ be a solution to* (1-4) *such that*

$$\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q}) = 8\pi.$$

Then either $T_- = -\infty$ or $T_+ = \infty$. Assume that $T_- = -\infty$. Then $\vec{\psi}(t)$ scatters in backward time, while in forward time one of the following holds:

- *$T_+ = \infty$ and $\vec{\psi}(t)$ scatters in forward time.*
- *$T_+ < \infty$, $\vec{\psi}(t)$ is a two-bubble in the forward time direction, and there exists an absolute constant $C > 0$ such that the scales of the bubbles $\lambda(t)$, $\mu(t)$ satisfy*

$$\lim_{t \to T_+} \mu(t) = \mu_0 \in (0, \infty), \qquad \frac{1}{C}(T_+ - t)^2 \le \frac{\lambda(t)}{\mu_0} \left| \log\left(\frac{\lambda(t)}{\mu_0} \right) \right| \le C(T_+ - t)^2.$$

If we assume initially that $\vec{\psi}(t)$ satisfies $T_+ = \infty$, then $\vec{\psi}(t)$ scatters in forward time, and one of analogous alternatives formulated in backward time must hold.

Overall, our main results state that the dynamics of corotational wave maps at threshold energy are very different from the those of higher equivariant wave maps at threshold energy.

We remark that by Theorem 1.7, the blow-up solution $\vec{\psi}_c$ from Theorem 1.6 is global in forward time, $T_+ = \infty$, and scatters. Thus, $\vec{\psi}_c$ is a trajectory connecting asymptotically free behavior to blow-up behavior. Theorem 1.7 also asserts that for (1-4) the collision of two bubbles produces only radiation and is therefore inelastic. This is consistent with what is known and expected for nonintegrable dispersive equations; see [Martel and Merle 2011a; 2011b; 2018; Jendrej and Lawrie 2018]. Our main results are in the spirit of the classification results at threshold energy by [Duyckaerts and Merle 2008; 2009; Jendrej and Lawrie 2018], but one may also draw parallels to the study of minimal blow-up solutions for dispersive equations; see for example [Merle 1993; Raphaël and Szeftel 2011]. Finally, we remark that apart from the seminal work [Duyckaerts, Kenig, and Merle 2013] which verified the soliton resolution conjecture for the 3-dimensional radial energy critical wave equation and Theorem 1.5 due to Jendrej and Lawrie, Theorem 1.7 is the only other result which proves soliton resolution continuously in time at an energy level that a priori allows two solitons in the asymptotic decomposition. In fact, Theorem 1.7 shows that solutions with two concentrating bubbles cannot occur, and any nonscattering solution must blow up precisely one bubble while radiating a second stationary harmonic map outside the inverted light cone.

1D. ***Outline.*** The general framework for proving Theorem 1.7 is inspired by [Jendrej and Lawrie 2018] on higher equivariant wave maps, but due to the slow convergence to π of the corotational harmonic map $Q(r) = 2\arctan r$, there are serious technical challenges not found in the higher equivariant setting that arise. The main source of these obstacles will be elaborated on below.

A rough outline of the proof of Theorem 1.7 is as follows. By Theorem 1.3, a corotational wave map $\vec{\psi}$ that does not scatter forward in time must approach the space of two-bubbles along a sequence of times. Towards a contradiction, we assume that $\vec{\psi}$ does not approach the space of two-bubbles continuously in time. We then split time into a sequence of intervals $[a_m, b_m]$ so that $\vec{\psi}(t)$ is close to the space of two-bubbles on $[a_m, b_m]$ (bad intervals), and $\vec{\psi}(t)$ stays away from the space of two-bubbles on $[b_m, a_{m+1}]$ (good intervals). By concentration compactness techniques, the trajectory $\vec{\psi}(t)$ has a certain *compactness property* on the union of good intervals (see Sections 2 and 4). Past experience suggests that $\vec{\psi}(t)$ converges to a degree-0 stationary solution to (1-4) along a sequence of times in the good intervals (see [Duyckaerts, Kenig, and Merle 2016] for example). Since the only degree-0 stationary solution to (1-4) is 0, we conclude $\vec{\psi} = 0$, a contradiction.

To prove that $\vec{\psi}(t)$ approaches a stationary solution to (1-4), we use a virial identity for wave maps (see Section 2) which bounds an integral of $\|\partial_t \psi(t)\|_{L^2}^2$ over certain good intervals by small error terms plus an integral of $\boldsymbol{d}(\vec{\psi}(t))^{1/2}$ over certain bad intervals. Here $\boldsymbol{d}(\cdot)$ is a measure of the distance to the space of two-bubbles (see Section 2). The errors can be made small because $\vec{\psi}$ is close to a two-bubble on the bad intervals and has the compactness property on the good intervals. The time integral of $\boldsymbol{d}(\vec{\psi}(t))^{1/2}$ can be absorbed into the left-hand side, which shows that $\|\partial_t \psi(t)\|_{L^2}^2$ converges to 0 in a certain averaged (over the good intervals) sense. The compactness property then allows us to conclude that $\vec{\psi}(t)$ must approach a stationary solution. The fact that the integral of $\boldsymbol{d}(\vec{\psi}(t))^{1/2}$ can be absorbed into the left-hand side is due to the following informal fact: leaving the space of two-bubbles on a bad interval causes an appreciable amount of kinetic energy, $\|\partial_t \psi(t)\|_{L^2}^2$, to be present on the neighboring good interval (see Proposition 3.12 and Section 4). We prove this fact by studying the interaction of corotational two-bubbles using the modulation method (see Section 3). This is one of the main novelties of this paper.

On a time interval where a corotational wave map $\vec{\psi}$ is close to a two-bubble, we decompose the solution as

$$\vec{\psi}(t) = (Q_{\lambda(t)} - Q_{\mu(t)} - g(t), \partial_t \psi(t)),$$

where the modulation parameters $\lambda(t)$ and $\mu(t)$ are chosen by imposing certain orthogonality conditions on g. The choice also ensures that $\boldsymbol{d}(\vec{\psi}(t))$ is comparable to $\lambda(t)/\mu(t)$. The goal of Section 3 is to show and control growth of the ratio $\lambda(t)/\mu(t)$ in the future of a time t_0, where $\frac{d}{dt}(\lambda(t)/\mu(t))|_{t=t_0} > 0$ (see Propositions 3.3 and 3.12). In contrast to [Jendrej and Lawrie 2018] on higher equivariant wave maps, the function $(r\partial_r Q)_{\lambda(t)}$, which is the tangent vector to the curve $t \mapsto Q_{\lambda(t)}$ is not in $L^2(\mathbb{R}^2)$. This function plays a key role in the scheme since $\lambda(t)^{-1}\langle (r\partial_r Q)_{\lambda(t)} \mid \partial_t \psi(t)\rangle_{L^2}$ should heuristically be proportional to $\lambda'(t)$, so we may then differentiate it and use (1-4) to get information about $\lambda''(t)$. The fact that $r\partial_r Q \notin L^2$ is the major obstacle in deriving the estimates, and the technique we introduce in Section 3 to overcome this challenge is a central contribution of this work.

We conclude our discussion of the proof of Theorem 1.7 with the following remarks. Our overall scheme of proving Theorem 1.7 may also be summarized as showing that a threshold wave map that leaves a small neighborhood of the space of two-bubbles can never return. This type of *ejection* result is similar in appearance to those obtained by Krieger, Nakanishi and Schlag [2013; 2015] in their study of the dynamics near the unstable ground state for the energy critical wave equation. However, the ejection of a near two-bubble wave map is due to a purely nonlinear mechanism (the interaction of the harmonic maps).

We now briefly outline the proof of Theorem 1.6. The construction of the blow-up solution $\vec{\psi}_c(t)$ is quite short due to the results proved in Section 3. We consider initial data at time t_n of the form

$$\vec{\psi}_n(t_n) = (Q_{\ell(t_n)} - Q, -\ell'(t_n)\ell(t_n)^{-1}(r\partial_r Q)_{\ell(t_n)}\chi_n),$$

where χ_n is a cutoff that ensures $\mathcal{E}(\vec{\psi}(t_n)) = 8\pi$. The function $\ell(t)$ is chosen to satisfy $\ell'(t_n) > 0$ and to essentially saturate the bounds on the modulation parameters in Proposition 3.3. Let $\vec{\psi}_n(t)$ denote the solution to (1-4) with data $\vec{\psi}_n(t_n)$ at time $t = t_n$. By our choice of the data, the control of the growth of the modulation parameters obtained in Proposition 3.3 and a bootstrap argument, we conclude that there exist absolute constants $\alpha, C, T > 0$ with T small such that $T_+(\vec{\psi}_n) > T$ and

$$\inf_{\substack{\mu\in[1/2,2] \\ \lambda|\log\lambda|\in[t^2/C, Ct^2]}} \|\vec{\psi}_n(t) - (Q_\lambda - Q_\mu)\|_{\mathcal{H}_0}^2 \le \alpha t^2 \quad \text{for all } n, \text{ for all } t \in [t_n, T].$$

Passing to a weak limit then finishes the proof. Full details are in Section 5.

2. Preliminaries

The purpose of this section is to recall preliminary facts about solutions to (1-4) that will be required in our analysis. Before recalling these facts, we establish some notation. For two quantities A and B, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$, and we write $A \sim B$ if $A \lesssim B \lesssim A$. For the paper, we denote by χ a smooth cutoff $\chi \in C^\infty_{\mathrm{rad}}(\mathbb{R}^2)$, so that, writing $\chi = \chi(r)$ we have

$$\chi(r) = 1 \quad \text{if } r \le 1 \qquad \text{and} \qquad \chi(r) = 0 \quad \text{if } r \ge 2 \qquad \text{and} \qquad |\chi'(r)| \le 2 \quad \text{for all } r \ge 0.$$

We define $\chi_R(r) := \chi(r/R)$. The L^2 pairing of two radial functions is denoted by

$$\langle f \mid g\rangle := \frac{1}{2\pi}\langle f \mid g\rangle_{L^2(\mathbb{R}^2)} = \int_0^\infty f(r)g(r)r\,dr.$$

The $\dot{H}^1$ and L^2 rescalings of a radial function f are denoted by

$$f_\lambda(r) = f(r/\lambda), \quad f_{\underline{\lambda}}(r) = \frac{1}{\lambda}f(r/\lambda),$$

and the corresponding infinitesimal generators are given by

$$\Lambda f := -\frac{\partial}{\partial\lambda}\Big|_{\lambda=1} f_\lambda = r\partial_r f \qquad (\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^2) \text{ scaling}),$$

$$\Lambda_0 f := -\frac{\partial}{\partial\lambda}\Big|_{\lambda=1} f_{\underline{\lambda}} = (1 + r\partial_r)f \quad (L^2_{\mathrm{rad}}(\mathbb{R}^2) \text{ scaling}).$$

Recall the definition of the space of degree-0 data with finite energy:

$$\mathcal{H}_0 := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty,\ \lim_{r \to 0} \psi_0(r) = \lim_{r \to \infty} \psi_0(r) = 0\}.$$

We define the norm H via

$$\|\psi_0\|_H^2 := \int_0^\infty \left((\partial_r \psi_0(r))^2 + \frac{(\psi_0(r))^2}{r^2} \right) r\,dr,$$

and for pairs $\vec{\psi} = (\psi_0, \psi_1) \in \mathcal{H}_0$ we write

$$\|\vec{\psi}\|_{\mathcal{H}_0} := \|(\psi_0, \psi_1)\|_{H \times L^2}.$$

Given $\psi_0 \in H$, if we define $\tilde{\psi}_0(x) := \psi(e^x),\ x \in \mathbb{R}$, we see that $\|\psi_0\|_H = \|\tilde{\psi}_0\|_{H^1(\mathbb{R})}$. Thus, by Sobolev embedding on $\mathbb{R}$ we conclude that

$$\|\psi_0\|_{L^\infty} \leq C \|\psi_0\|_H.$$

This fact will be used frequently in our analysis.

2A. ***Compactness of nonscattering threshold solutions.*** A central result we use in our work, obtained from the standard well-posedness theory for (1-4), concentration-compactness methods and Theorem 1.1, is the following compactness statement for nonscattering threshold solutions. The proof is the same as the higher equivariant analog found in [Jendrej and Lawrie 2018] and is omitted.

Lemma 2.1 [Jendrej and Lawrie 2018, Lemma 2.9]. *Let $\vec{\psi}(t) \in \mathcal{H}_0$ be a solution to* (1-4) *defined on* $[0, T_+(\vec{\psi}))$. *Suppose that $\mathcal{E}(\vec{\psi}) = 8\pi$ and $\vec{\psi}(t)$ does not scatter in forward time. Then if $t_n \to T_+$ is any sequence of times such that*

$$\sup_n \|\vec{\psi}(t_n)\|_{\mathcal{H}_0} \leq C < \infty,$$

there exist a subsequence, which we continue to denote by t_n, scales $\nu_n > 0$ and a nonzero $\vec{\varphi} \in \mathcal{H}_0$ such that

$$\vec{\psi}(t_n)_{1/\nu_n} \to \vec{\varphi} \in \mathcal{H}_0$$

strongly in $\mathcal{H}_0$. Moreover, $\mathcal{E}(\vec{\varphi}) = 8\pi$, and the solution $\vec{\varphi}(s)$ to (1-4) *with data $\vec{\varphi}(0) = \vec{\varphi}$ is nonscattering in forwards and backwards time.*

2B. ***Near two-bubble maps.*** We recall that the unique (up to scaling) nontrivial corotational harmonic map Q is given by

$$Q(r) = 2 \arctan r.$$

The harmonic map Q has a variational characterization as follows. As in the Introduction, let $\mathcal{H}_1$ be the set of all finite-energy corotational maps which map infinity to the south pole, i.e.,

$$\mathcal{H}_1 := \{(\phi_0, \phi_1) \mid \mathcal{E}(\vec{\phi}) < \infty,\ \phi_0(0) = 0,\ \lim_{r \to \infty} \varphi_0(r) = \pi\}.$$

Then for $(\varphi_0, \varphi_1) \in \mathcal{H}_1$, we have the following Bogomol'nyi factorization of the nonlinear energy:

$$\begin{aligned}\mathcal{E}(\varphi_0, \varphi_1) &= \pi \|\varphi_1\|_{L^2}^2 + \pi \int_0^\infty \left(\partial_r \varphi_0 - \frac{\sin(\varphi_0)}{r}\right)^2 r\,dr + 2\pi \int_0^\infty \sin(\varphi_0)\partial_r \varphi_0\,dr \\ &= \pi \|\varphi_1\|_{L^2}^2 + \pi \int_0^\infty \left(\partial_r \varphi_0 - \frac{\sin(\varphi_0)}{r}\right)^2 r\,dr + 2\pi \int_{\varphi_0(0)}^{\varphi_0(\infty)} \sin(\rho)\,d\rho \\ &= \pi \|\varphi_1\|_{L^2}^2 + \pi \int_0^\infty \left(\partial_r \varphi_0 - \frac{\sin(\varphi_0)}{r}\right)^2 r\,dr + 4\pi.\end{aligned}$$

By solving the differential equation in the parentheses, we see that $\mathcal{E}(\varphi_0, \varphi_1) \geq 4\pi$ with equality if and only if $(\varphi_0, \varphi_1) = (Q_\lambda, 0)$ for some $\lambda > 0$.

In our analysis, we will need several technical facts related to the distance of a map $\vec{\psi}$ to the set of two-bubbles. More precisely, given a map $\vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_0$ we define its distance $\boldsymbol{d}(\vec{\phi})$ to the set of two-bubbles by

$$\boldsymbol{d}(\vec{\phi}) := \inf_{\lambda,\mu>0,\iota\in\{+1,-1\}} (\|(\phi_0 - \iota(Q_\lambda - Q_\mu), \phi_1)\|_{\mathcal{H}_0}^2 + (\lambda/\mu)). \tag{2-1}$$

To distinguish between the two cases of a map being close to a pure two-bubble ($\iota = +1$ above) or an anti-two-bubble ($\iota = -1$ above), we define

$$\boldsymbol{d}_\pm(\vec{\phi}) := \inf_{\lambda,\mu>0} (\|(\phi_0 \mp (Q_\lambda - Q_\mu), \phi_1)\|_{\mathcal{H}_0}^2 + (\lambda/\mu)).$$

The next two lemmas follow from the same arguments given in [Jendrej and Lawrie 2018] for higher equivariant wave maps, and the proofs will be omitted. The first lemma shows that the size of a map $\vec{\psi}$ with threshold energy can be controlled by its distance to the surface of two-bubbles. The second lemma proves the intuitive fact that a map $\vec{\psi}$ cannot simultaneously be close to a pure two-bubble and anti-two-bubble.

Lemma 2.2 [Jendrej and Lawrie 2018, Lemma 2.13]. *Suppose that* $\vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_0$ *and*

$$\mathcal{E}(\vec{\phi}) = 2\mathcal{E}(\vec{Q}) = 8\pi.$$

Then for each $\beta > 0$ *there exists* $C(\beta) > 0$ *such that*

$$\boldsymbol{d}(\vec{\phi}) \geq \beta \quad \Longrightarrow \quad \|(\phi_0, \phi_1)\|_{\mathcal{H}_0} \leq C(\beta).$$

Conversely, for each $A > 0$ *there exists* $\alpha = \alpha(A)$ *such that*

$$\boldsymbol{d}(\vec{\phi}) \leq \alpha(A) \quad \Longrightarrow \quad \|(\phi_0, \phi_1)\|_{\mathcal{H}_0} \geq A.$$

Lemma 2.3 [Jendrej and Lawrie 2018, Lemma 2.14]. *There exists an absolute constant* $\alpha_0 > 0$ *such that for any* $\vec{\phi} \in \mathcal{H}_0$

$$\boldsymbol{d}_\pm(\vec{\phi}) \leq \alpha_0 \quad \Longrightarrow \quad \boldsymbol{d}_\mp(\vec{\phi}) \geq \alpha_0.$$

The final preliminary results we will need for our analysis are related to a virial identity for solutions to (1-4). The following virial identity follows easily from (1-4) and integration by parts.

Lemma 2.4. *Let $\vec{\psi}(t)$ be a solution to* (1-4) *on a time interval I. Then for any time $t \in I$ and $R > 0$ fixed we have*

$$\frac{d}{dt}\langle \psi_t \mid \chi_R\, r\partial_r\psi\rangle_{L^2}(t) = -\int_0^\infty \psi_t^2(t,r)r\,dr + \Omega_R(\vec{\psi}(t)), \tag{2-2}$$

where

$$\Omega_R(\vec{\psi}(t)) := \int_0^\infty \psi_t^2(t)(1-\chi_R)r\,dr - \frac{1}{2}\int_0^\infty \left(\psi_t^2(t)+\psi_r^2(t) - \frac{\sin^2\psi(t)}{r^2}\right)\frac{r}{R}\chi'(r/R)r\,dr \tag{2-3}$$

satisfies

$$\begin{aligned}|\Omega_R(\vec{\psi}(t))| &\lesssim \int_R^\infty \psi_t^2(t,r)r\,dr\,dt + \int_R^\infty \left|\psi_r^2 - \frac{\sin^2\psi}{r^2}\right| r\,dr\,dt \\ &\lesssim \int_R^\infty \left(\psi_t^2(t,r)+\psi_r^2(t,r)+\frac{\sin^2\psi(t,r)}{2r^2}\right)r\,dr.\end{aligned}$$

Finally, using Lemma 2.2, one can bound the virial and the error for threshold solutions by its distance to the set of two-bubbles. The proof of this fact is the same as in [Jendrej and Lawrie 2018] and is omitted.

Lemma 2.5 [Jendrej and Lawrie 2018, Lemma 2.16]. *There exists a number $C_0 > 0$ such that for all $\vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\phi}) = 2\mathcal{E}(Q)$ and all $R > 0$ we have*

$$\begin{aligned}|\langle \phi_1, \chi_R r\partial_r\phi_0\rangle| &\le C_0 R\sqrt{\boldsymbol{d}(\vec{\phi})}, \\ |\Omega_R(\vec{\phi})| &\le C_0\sqrt{\boldsymbol{d}(\vec{\phi})}.\end{aligned} \tag{2-4}$$

3. The modulation method for two-bubble solutions

In this section we analyze the modulation equations that govern the evolution of corotational near two-bubble solutions. As in the case of higher equivariant wave maps studied in [Jendrej and Lawrie 2018], the scale of the less concentrated bubble does not change, but it does affect the evolution of the more concentrated bubble. A central challenge which arises in the analysis of corotational maps which is not found in the higher equivariant setting is the fact that the zero mode of the operator obtained by linearizing about the harmonic map Q is a *resonance* rather than an eigenvalue. A rough outline of this section is as follows. For a solution $\vec{\psi}(t)$ with $\boldsymbol{d}(\vec{\psi}(t))$ small on a time interval J, we first use the implicit function theorem to find modulation parameters $\lambda(t)$, $\mu(t)$ defined on J such that $g(t) := \psi(t) - (Q_{\lambda(t)} - Q_{\mu(t)})$ satisfies appropriate orthogonality conditions and $\boldsymbol{d}(\vec{\psi}(t)) \simeq \lambda(t)/\mu(t)$. We would like to then prove that if the modulation parameters $\lambda(t)$, $\mu(t)$ are approaching each other in scale, i.e., if $\frac{d}{dt}(\lambda(t)/\mu(t))|_{t=t_0} \ge 0$, then $\lambda(t)/\mu(t)$continues to grow in a controlled way in forward time near t_0. In particular, this would imply that $\vec{\psi}(t)$ has to leave a small neighborhood of the set of two-bubbles. However, the slow decay of Q requires us to deal with additional technical obstacles not encountered in the case of higher equivariant wave maps. In particular, we must replace $\lambda(t)$ with a carefully chosen logarithmic correction.

3A. ***Modulation equations.*** In this section, we study solutions near two-bubble solutions $\vec{\psi}(t)$ to (1-4). More precisely, we consider maps such that $\boldsymbol{d}(\vec{\psi}(t))$ (defined by (2-1)) is small on a time interval J.

The operator corresponding to linearizing (1-4) about the harmonic map Q_λ is the Schrödinger operator

$$\mathcal{L}_\lambda := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\cos 2Q_\lambda}{r^2}.$$

For convenience we write $\mathcal{L} := \mathcal{L}_1$. Differentiating the equation

$$\partial_r^2 Q_\lambda + \frac{1}{r}\partial_r Q_\lambda - \frac{\sin 2Q_\lambda}{2r^2} = 0$$

with respect to λ and setting $\lambda = 1$ implies that ΛQ is a zero mode for $\mathcal{L}$, i.e.,

$$\mathcal{L}\Lambda Q = 0, \quad \Lambda Q \in L^\infty(\mathbb{R}^2).$$

Note that $\Lambda Q \sim 1/r$ as $r \to \infty$ so that ΛQ fails (logarithmically) to be in $L^2(\mathbb{R}^2)$. We say that ΛQ is a *resonance* of $\mathcal{L}$. In the k-equivariant setting with $k \geq 2$, we have $\Lambda Q \in L^2(\mathbb{R}^2)$. This weak decay of ΛQ requires more care when studying the modulation equations compared to the higher equivariant setting. We note that in general, we have

$$\mathcal{L}_\lambda Q_\lambda = 0.$$

Define

$$\mathcal{Z}(r) := \chi_L(r)\Lambda Q(r),$$

where, as before, χ is a smooth cutoff. The parameter $L > 0$ will be chosen later. We use $\mathcal{Z}$ to obtain a useful choice of modulation parameters (the scales) for the near two-bubble solution $\vec{\psi}(t)$. We first recall the following modulation lemma from [Jendrej and Lawrie 2018], which follows from standard arguments involving the implicit function theorem, an expansion of the nonlinear energy and coercivity properties of $\mathcal{L}_\lambda$.

Lemma 3.1 [Jendrej and Lawrie 2018, Lemma 3.1]. *There exist $\eta_0 = \eta_0(L) > 0$ and $C = C(L) > 0$ such that the following holds. Let $\psi(t)$ be a solution to* (1-4) *defined on a time interval $J \subset \mathbb{R}$, and assume that*

$$\boldsymbol{d}_+(\vec{\psi}(t)) \leq \eta_0 \quad \textit{for all } t \in J.$$

Then there exist unique $C^1(J)$ functions $\lambda(t)$, $\mu(t)$ so that the function

$$g(t) := \psi(t) - Q_{\lambda(t)} + Q_{\mu(t)} \in H$$

satisfies for all $t \in J$

$$\langle \mathcal{Z}_{\underline{\lambda(t)}} \mid g(t)\rangle = 0, \tag{3-1}$$

$$\langle \mathcal{Z}_{\underline{\mu(t)}} \mid g(t)\rangle = 0, \tag{3-2}$$

$$\boldsymbol{d}_+(\vec{\psi}(t)) \leq \|(g(t), \partial_t\psi(t))\|_{\mathcal{H}_0}^2 + \frac{\lambda(t)}{\mu(t)} \leq C\boldsymbol{d}_+(\vec{\psi}(t)).$$

Moreover,

$$\|(g(t), \partial_t\psi(t))\|_{\mathcal{H}_0} \leq C\left(\frac{\lambda(t)}{\mu(t)}\right)^{1/2}, \tag{3-3}$$

and hence

$$\boldsymbol{d}_+(\vec{\psi}(t)) \simeq \frac{\lambda(t)}{\mu(t)}. \tag{3-4}$$

Finally, we have the explicit bound for the kinetic energy

$$\|\partial_t \psi(t)\|_{L^2}^2 \le 16\frac{\lambda(t)}{\mu(t)} + o\left(\frac{\lambda(t)}{\mu(t)}\right). \tag{3-5}$$

Remark 3.2. The $o(\cdot)$ term in (3-5) depends on the parameter L, but it will be important that the leading-order term is *independent* of L.

Given the modulation parameters $\lambda(t)$, $\mu(t)$ we define

$$\begin{aligned} g(t) &:= \psi(t) - Q_{\lambda(t)} + Q_{\mu(t)}, \\ \dot{g}(t) &:= \partial_t \psi(t). \end{aligned}$$

Then the vector $\vec{g} := (g, \dot{g})$ satisfies the equations

$$\partial_t g = \dot{g} + \lambda' \Lambda Q_{\underline{\lambda}} - \mu' \Lambda Q_{\underline{\mu}}, \tag{3-6}$$

$$\partial_t \dot{g} = \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)). \tag{3-7}$$

As a first step towards understanding the behavior of the modulation parameters, we establish bounds on the first derivatives of $\lambda(t)$, $\mu(t)$. This information is not enough to study the interaction of the bubbles for the near two-bubble solution $\vec{\psi}(t)$ and achieve the goal outlined at the start of the section. This should also be intuitively clear since $\vec{\psi}(t)$ satisfies a second-order equation in time, and thus, the interaction of the bubbles should be governed by second derivatives of $\lambda(t)$, $\mu(t)$.

Proposition 3.3. *There exist a constant $C > 0$ and $\eta_0 = \eta_0(L) > 0$ with the following property. Let $J \subset \mathbb{R}$, and let $\vec{\psi}(t)$ be a solution to* (1-4) *on J such that*

$$\boldsymbol{d}(\vec{\psi}(t)) \le \eta_0 \quad \textit{for all } t \in J.$$

Let $\lambda(t)$, $\mu(t)$ be the modulation parameters given by Lemma 3.1. Then for all $t \in J$ we have

$$|\lambda'(t)| \le C(\log L)^{-1/2}\left(\frac{\lambda(t)}{\mu(t)}\right)^{1/2}, \tag{3-8}$$

$$|\mu'(t)| \le C(\log L)^{-1/2}\left(\frac{\lambda(t)}{\mu(t)}\right)^{1/2}. \tag{3-9}$$

Proof. Differentiating the orthogonality conditions (3-1) and (3-2) and using (3-6) we obtain the relations

$$\begin{aligned} -\langle \mathcal{Z}_{\underline{\lambda}} \mid \dot{g}\rangle &= \lambda'\left(\langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\lambda}}\rangle - \left\langle \frac{1}{\lambda}[\Lambda_0 \mathcal{Z}]_{\underline{\lambda}} \,\middle|\, g\right\rangle\right) - \mu'\langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}}\rangle, \\ -\langle \mathcal{Z}_{\underline{\mu}} \mid \dot{g}\rangle &= \lambda'\langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\lambda}}\rangle + \mu'\left(-\langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\mu}}\rangle - \left\langle \frac{1}{\mu}[\Lambda_0 \mathcal{Z}]_{\underline{\mu}} \,\middle|\, g\right\rangle\right). \end{aligned}$$

These two equations yield the following linear system for (λ', μ'):

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \begin{pmatrix} -\langle \mathcal{Z}_{\underline{\lambda}} \mid \dot{g}\rangle \\ -\langle \mathcal{Z}_{\underline{\mu}} \mid \dot{g}\rangle \end{pmatrix}, \tag{3-10}$$

where

$$A_{11} := \langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\lambda}} \rangle - \left\langle \frac{1}{\lambda}[\Lambda_0 \mathcal{Z}]_{\underline{\lambda}} \;\middle|\; g \right\rangle,$$
$$A_{12} := -\langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}} \rangle,$$
$$A_{21} := \langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\lambda}} \rangle,$$
$$A_{22} := -\langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\mu}} \rangle - \left\langle \frac{1}{\mu}[\Lambda_0 \mathcal{Z}]_{\underline{\mu}} \;\middle|\; g \right\rangle.$$

We now estimate the coefficients of the matrix $A = (A_{ij})$ so that we may invert (3-10) and obtain estimates for (λ', μ'). We define

$$\alpha_L := \langle \mathcal{Z} \mid \Lambda Q \rangle = \int_0^\infty \chi_L |\Lambda Q|^2 r\, dr.$$

Note that since $|\Lambda Q(r)| \lesssim 1/(1+r)$, we have for all $L > 0$ sufficiently large

$$\log L \lesssim \alpha_L \lesssim \log L,$$

where the implied constants are absolute.

Claim 3.4. *For λ/μ sufficiently small (depending on L), the diagonal terms satisfy*

$$A_{11} = \alpha_L[1 + O_L((\lambda/\mu)^{1/2})], \tag{3-11}$$
$$A_{22} = -\alpha_L[1 + O_L((\lambda/\mu)^{1/2})]. \tag{3-12}$$

To prove the claim we simply observe that

$$\left| \left\langle \frac{1}{\lambda}[\Lambda_0 \mathcal{Z}]_{\underline{\lambda}} \;\middle|\; g \right\rangle \right| \lesssim \|g\|_{L^\infty} \|\Lambda_0 \mathcal{Z}\|_{L^1(r\,dr)} \lesssim_L \|g\|_H \lesssim_L (\lambda/\mu)^{1/2}.$$

Thus,

$$A_{11} = \langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\lambda}} \rangle - \left\langle \frac{1}{\lambda}[\Lambda_0 \mathcal{Z}]_{\underline{\lambda}} \;\middle|\; g \right\rangle = \alpha_L + O_L((\lambda/\mu)^{1/2}),$$

which establishes (3-11). The estimate (3-12) is established analogously, and the claim is proved.

We now estimate the off-diagonal terms.

Claim 3.5. *For λ/μ sufficiently small (depending on L) we have*

$$|A_{12}| \lesssim_L (\lambda/\mu)^2, \quad |A_{21}| \lesssim \log L, \tag{3-13}$$

where the implied constant in the estimate for A_{21} is absolute.

Since $r\mathcal{Z}(r) \in C_0^\infty$, $\lambda/\mu \ll 1$ and $|\Lambda Q| \lesssim r$ for small r, we conclude that

$$|A_{12}| = |\langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}} \rangle| = \left| \int_0^{2L\lambda/\mu} \frac{\mu r}{\lambda} Z(r\mu/\lambda) \Lambda Q(r)\, dr \right| \lesssim_L \int_0^{2L\lambda/\mu} |\Lambda Q|\, dr \lesssim_L (\lambda/\mu)^2.$$

This proves the first estimate in (3-13). Let $\sigma = \lambda/\mu$. By a change of variables and the explicit expression for $\mathcal{Z}$ we have

$$|A_{21}| = |\langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\lambda}} \rangle| = |\langle \mathcal{Z} \mid \Lambda Q_{\underline{\lambda/\mu}} \rangle| \lesssim \frac{1}{\sigma} \int_0^{2L} \frac{r}{1+r^2} \frac{(r/\sigma)}{1+(r/\sigma)^2} r\, dr \lesssim \log L,$$

which proves the second estimate in (3-13) and the claim.

We now solve for (λ', μ') by inverting A:

$$\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} -A_{22} \langle \mathcal{Z}_{\underline{\lambda}} \mid \dot{g} \rangle + A_{12} \langle \mathcal{Z}_{\underline{\mu}} \mid \dot{g} \rangle \\ A_{21} \langle \mathcal{Z}_{\underline{\lambda}} \mid \dot{g} \rangle - A_{11} \langle \mathcal{Z}_{\underline{\mu}} \mid \dot{g} \rangle \end{pmatrix}.$$

The previous two claims imply that

$$\det A = A_{11}A_{22} - A_{12}A_{21} = -\alpha_L^2 [1 + O_L((\lambda/\mu)^{1/2})] \tag{3-14}$$

as long as λ/μ is sufficiently small. It is easy to see that the function $\mathcal{Z} = \chi_L \Lambda Q$ satisfies $\|\mathcal{Z}\|_{L^2} \lesssim (\log L)^{1/2}$. Then by Cauchy–Schwarz and (3-5) we have, for λ/μ sufficiently small,

$$|\langle Z_{\underline{\lambda}} \mid \dot{g} \rangle| + |\langle Z_{\underline{\mu}} \mid \dot{g} \rangle| \lesssim |\log L|^{1/2} (\lambda/\mu)^{1/2} \lesssim \alpha_L^{1/2} (\lambda/\mu)^{1/2}, \tag{3-15}$$

where the implied constant is absolute. Our two claims, (3-14) and (3-15), imply that as long as λ/μ is sufficiently small

$$\begin{aligned} |\lambda'| &\lesssim |\det A|^{-1} (|A_{22}| |\langle Z_{\underline{\lambda}} \mid \dot{g} \rangle| + |A_{12}| |\langle Z_{\underline{\mu}} \mid \dot{g} \rangle|) \\ &\lesssim \alpha_L^{-1/2} (\lambda/\mu)^{1/2} \lesssim (\log L)^{-1/2} (\lambda/\mu)^{1/2} \end{aligned}$$

as desired. A similar argument establishes

$$|\mu'| \lesssim (\log L)^{-1/2} (\lambda/\mu)^{1/2}$$

as well, which finishes the proof. □

3B. ***Refined control of the modulation parameters.*** As stated previously, information about the first derivatives of the modulation parameters is not enough to study the evolution of two-bubbles since (1-4) is second-order in time. Due to the slow decay of the ΛQ, we will in fact need to study second-order derivatives of $2\lambda |\log \lambda/\mu|$ and μ. Moreover, for technical reasons we will study a function $\zeta = \zeta(t)$ which approximates $2\lambda |\log \lambda/\mu|$ and a function $b = b(t)$ which approximates $\zeta'(t)$ (see Proposition 3.9).

We first define a truncated virial functional and state some relevant properties. This functional played a fundamental role in the work of Jendrej and Lawrie on threshold dynamics for higher equivariant wave maps [Jendrej and Lawrie 2018] and in the two-bubble construction in [Jendrej 2019]. It will play a very important role in our work as well. For the proofs of the following statements we refer the reader to [Jendrej 2019, Lemmas 4.6 and 5.5]. In what follows, we denote the nonlinearity by $f(\rho) := \frac{1}{2} \sin 2\rho$.

Lemma 3.6 [Jendrej 2019, Lemma 4.6]. *For each c, $R > 0$ there exists a function $q(r) = q_{c,R}(r) \in C^{3,1}((0, \infty))$ with the following properties*:

(P1) $q(r) = r^2/2$ *for* $r \le R$.

(P2) *There exists an absolute constant* $\kappa > 0$ *such that* $q(r) \equiv \text{const.}$ *for* $r \geq \widetilde{R} := \kappa e^{\kappa/c} R$.

(P3) $|q'(r)| \lesssim r$ *and* $|q''(r)| \lesssim 1$ *for all* $r > 0$, *with constants independent of* c, R.

(P4) $q''(r) \geq -c$ *and* $(1/r)q'(r) \geq -c$ *for all* $r > 0$.

(P5) $(d^2/dr^2 + (1/r)(d/dr)r)^2 q(r) \leq c \cdot r^{-2}$ *for all* $r > 0$.

(P6) $|r(q'(r)/r)'| \leq c$ *for all* $r > 0$.

For each $\lambda > 0$ we define the operators $\mathcal{A}(\lambda)$ and $\mathcal{A}_0(\lambda)$ as

$$[\mathcal{A}(\lambda)g](r) := q'\left(\frac{r}{\lambda}\right) \cdot \partial_r g(r), \tag{3-16}$$

$$[\mathcal{A}_0(\lambda)g](r) := \left(\frac{1}{2\lambda} q''\left(\frac{r}{\lambda}\right) + \frac{1}{2r} q'\left(\frac{r}{\lambda}\right)\right) g(r) + q'\left(\frac{r}{\lambda}\right) \cdot \partial_r g(r). \tag{3-17}$$

Since $q(r) = r^2/2$ for $r \leq R$, we have $\mathcal{A}(\lambda)g(r) = (1/\lambda)\Lambda g(r)$ and $\mathcal{A}_0(\lambda)g(r) = (1/\lambda)\Lambda_0 g(r)$ for $r \leq R$. One may intuitively think of $\mathcal{A}(\lambda)$ and $\mathcal{A}_0(\lambda)$ as extensions of $(1/\lambda)\Lambda$ and $(1/\lambda)\Lambda_0$ to $r \geq R$ which have good boundedness properties. The following lemma makes this precise. In what follows, we set

$$X := \{g \in H \mid g/r, \partial_r g \in H\}.$$

Lemma 3.7 [Jendrej 2019, Lemma 5.5]. *Let* $c_0 > 0$ *be arbitrary. There exists* $c > 0$ *small enough and* $R, \widetilde{R} > 0$ *large enough in Lemma 3.6 so that the operators* $\mathcal{A}(\lambda)$ *and* $\mathcal{A}_0(\lambda)$ *defined in* (3-16) *and* (3-17) *have the following properties*:

- *The families* $\{\mathcal{A}(\lambda) \mid \lambda > 0\}$, $\{\mathcal{A}_0(\lambda) \mid \lambda > 0\}$, $\{\lambda\partial_\lambda \mathcal{A}(\lambda) \mid \lambda > 0\}$ *and* $\{\lambda\partial_\lambda \mathcal{A}_0(\lambda) \mid \lambda > 0\}$ *are bounded in* $\mathscr{L}(H; L^2)$, *with the bound depending only on the choice of the function* $q(r)$.
- *For all* $g_1 \in X$
$$\langle g_1 \mid \mathcal{A}_0(\lambda) g_1 \rangle = 0. \tag{3-18}$$
- *For all* $\lambda > 0$ *and* $g_1, g_2 \in X$ *there holds*
$$\left| \left\langle \mathcal{A}(\lambda)g_1 \,\middle|\, \frac{1}{r^2}(f(g_1+g_2) - f(g_1) - f'(g_1)g_2) \right\rangle + \left\langle \mathcal{A}(\lambda)g_2 \,\middle|\, \frac{1}{r^2}(f(g_1+g_2) - f(g_1) - g_2) \right\rangle \right| \leq \frac{c_0}{\lambda}\|g_2\|_H^2. \tag{3-19}$$
- *For all* $g \in X$ *we have*
$$\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) g \right\rangle \leq \frac{c_0}{\lambda}\|g\|_H^2 - \frac{1}{\lambda}\int_0^{R\lambda} \left((\partial_r g)^2 + \frac{1}{r^2} g^2\right) dr. \tag{3-20}$$
- *Moreover*, *for* $\lambda, \mu > 0$ *with* $\lambda/\mu \ll 1$
$$\|\Lambda Q_{\underline{\lambda}} - \mathcal{A}(\lambda) Q_\lambda\|_{L^\infty} \leq \frac{c_0}{\lambda}, \tag{3-21}$$
$$\|\Lambda_0 \Lambda Q_{\underline{\lambda}} - \mathcal{A}_0(\lambda)\Lambda Q_\lambda\|_{L^2} \leq c_0, \tag{3-22}$$
$$\|\mathcal{A}(\lambda) Q_\mu\|_{L^\infty} + \|\mathcal{A}_0(\lambda) Q_\mu\|_{L^\infty} \lesssim \frac{1}{\mu}, \tag{3-23}$$

and, *for any* $g \in H$,

$$\left| \int_0^\infty \frac{1}{2}\left(q''\left(\frac{r}{\lambda}\right)+\frac{\lambda}{r}q'\left(\frac{r}{\lambda}\right)\right)\frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)g\,dr \right.$$
$$\left. -\int_0^\infty \frac{1}{r^2}(f'(Q_\lambda)-1)g^2\,dr \right| \le c_0(\|g\|_H^2+(\lambda/\mu)). \quad (3\text{-}24)$$

Remark 3.8. The argument for the estimate (3-22) from [Jendrej 2019] does not quite apply to our case due to the slow decay of Q. We provide a different argument here. We first note that $\Lambda_0\Lambda Q = 4r/(1+r^2)^2 \in L^2(\mathbb{R}^2)$ and the estimate (3-22) is scaling-invariant so we can take $\lambda = 1$. Since $\Lambda_0\Lambda Q = \mathcal{A}_0(1)\Lambda Q$ for $r \le R$ and $\mathcal{A}_0(1)\Lambda Q = 0$ for $r \ge \widetilde{R} = R\kappa e^{\kappa/c}$, we have

$$\|\Lambda_0\Lambda Q - \mathcal{A}_0(1)\Lambda Q\|_{L^2}^2 \le \int_R^\infty |\Lambda_0\Lambda Q|^2 r\,dr + \int_R^{\widetilde{R}} |\mathcal{A}_0(1)\Lambda Q|^2 r\,dr.$$

The first term on the right-hand side above can be made less than $c_0^2/2$ as long as $R > 0$ is sufficiently large since $\Lambda_0\Lambda Q \in L^2$. For the second term, we write

$$\mathcal{A}_0(1)\Lambda Q = \frac{r}{2}\left(\frac{q'(r)}{r}\right)'\Lambda Q + \frac{q'(r)}{r}\Lambda_0\Lambda Q.$$

Then by properties (P6) and (P3) in Lemma 3.6 we have

$$\int_R^{\widetilde{R}} |\mathcal{A}_0(1)\Lambda Q|^2 r\,dr \lesssim c^2 \int_R^{\widetilde{R}} |\Lambda Q|^2 r\,dr + \int_R^\infty |\Lambda_0\Lambda Q|^2 r\,dr$$
$$\lesssim c^2 \int_R^{R\kappa e^{\kappa/c}} \frac{1}{r}\,dr + \int_R^\infty \frac{1}{r^5}\,dr \lesssim c + R^{-4} \le c_0^2/2$$

as long as c is sufficiently small and R is sufficiently large. We conclude that for c, R chosen appropriately, we have

$$\|\Lambda_0\Lambda Q - \mathcal{A}_0(1)\Lambda Q\|_{L^2}^2 \le c_0^2,$$

as desired.

As before, we let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a smooth radial cutoff. We then define the function $b(t)$ by

$$b(t) := -\langle \chi_{M\sqrt{\lambda(t)\mu(t)}}\Lambda Q_{\underline{\lambda(t)}} \mid \dot g(t)\rangle - \langle \dot g(t) \mid \mathcal{A}_0(\lambda(t))g(t)\rangle. \quad (3\text{-}25)$$

Here $M > 0$ is a constant which we will later fix. Finally, we define

$$\zeta(t) := 2\lambda(t)|\log(\lambda(t)/\mu(t))| - \langle \chi_{M\sqrt{\lambda(t)\mu(t)}}\Lambda Q_{\underline{\lambda(t)}} \mid g(t)\rangle. \quad (3\text{-}26)$$

Note that $\zeta(t)$ is C^1 since $\partial_t g(t)$ is continuous in L^2 with respect to t. We will now show that we may roughly view $\zeta(t)$ as $2\lambda(t)\log\lambda(t)$ and $b(t)$ as a subtle correction to $\zeta'(t)$. The essential feature of this correction is that $b'(t)$ (which intuitively is connected to $\lambda''(t)$) is bounded from below. More precisely, we prove the following.

Proposition 3.9 (modulation control). *Assume the same hypothesis as in Proposition 3.3, and in addition, assume that there exists* $t_0 \in J$ *such that* $\frac{1}{2}\mu(t_0) \le \mu(t) \le 2\mu(t_0)$ *for all* $t \in J$. *Let* $0 < \delta < \frac{1}{2}$ *be arbitrarily small, and let* η_0 *be as in Lemma 3.1. There exist functions* $L_0 = L_0(\delta) > 0$, $M_0 = M_0(\delta, L) > 0$ *and* $\eta_1 = \eta_1(\delta, L, M) > 0$ *such that if* $L > L_0$, $M > M_0$ *and* $\boldsymbol{d}_+(\vec{\psi}(t)) \le \eta_1 < \eta_0$, *then for all* $t \in J$ *the functions* $\lambda(t)$, $\mu(t)$, $\zeta(t)$ *and* $b(t)$ *(which implicitly depend on* L *and* M*) satisfy*

$$\left|\frac{\zeta(t)}{2\lambda(t)|\log(\lambda(t)/\mu(t))|} - 1\right| \le \delta, \tag{3-27}$$

$$|\zeta'(t) - b(t)| \le \delta\left[\frac{\lambda(t)}{\mu(t)}\right]^{1/2}\left|\log\frac{\lambda(t)}{\mu(t)}\right|^{1/2} \le \delta\left[\frac{\zeta(t)}{\mu(t)}\right]^{1/2}, \tag{3-28}$$

$$|b(t)| \le 4\left[\frac{\lambda(t)}{\mu(t)}\right]^{1/2}\left|2\log\frac{\lambda(t)}{\mu(t)}\right|^{1/2} + \delta\left[\frac{\lambda(t)}{\mu(t)}\right]^{1/2}\left|\log\frac{\lambda(t)}{\mu(t)}\right|^{1/2} \le 5\left[\frac{\zeta(t)}{\mu(t)}\right]^{1/2}. \tag{3-29}$$

Moreover, $b(t)$ *is locally Lipschitz and there exists* $C_1 = C_1(L) > 0$ *such that*

$$|b'(t)| \le C_1/\mu(t), \tag{3-30}$$

$$b'(t) \ge (8-\delta)/\mu(t). \tag{3-31}$$

Proof. Since we will take $\eta_1 < \eta_0$, the modulation parameters are well-defined and C^1 on the interval J. We also note that by rescaling $\vec{\psi}(t_0)$ we can assume that $\frac{1}{2} \le \mu(t) \le 2$ on J. Throughout the argument, implied constants and big-oh terms will depend on the parameters L and M unless stated otherwise.

We first prove (3-27). By Proposition 3.3 we have $\|g\|_{L^\infty} \le \|g\|_H \lesssim \lambda^{1/2}$. Thus,

$$|\langle \chi_{M\sqrt{\mu\lambda}} \Lambda Q_{\underline{\lambda}} \mid g\rangle| \lesssim \lambda^{3/2}\int_0^{4M/\sqrt{\lambda}} |\Lambda Q| r\,dr \lesssim \lambda.$$

We conclude that

$$\frac{1}{2\lambda|\log(\lambda/\mu)|}|\langle \chi_{\sqrt{\mu\lambda}} \Lambda Q_{\underline{\lambda}} \mid g\rangle| \lesssim |\log\lambda|^{-1},$$

which can be made smaller than δ as long as λ/μ is sufficiently small compared to L and M. This proves (3-27).

Now we prove (3-28). From (3-6) we have

$$\begin{aligned}\frac{d}{dt}\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid g\rangle = \langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid \dot{g}\rangle + \lambda'\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid \Lambda Q_{\underline{\lambda}}\rangle - \mu'\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}}\rangle \\ - \frac{\lambda'}{\lambda}\langle \chi_{M\sqrt{\lambda\mu}} \Lambda_0 \Lambda Q_{\underline{\lambda}} \mid g\rangle - \left(\frac{\lambda'}{2\lambda} + \frac{\mu'}{2\mu}\right)\langle \Lambda \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid g\rangle.\end{aligned} \tag{3-32}$$

Since $|\Lambda Q| \lesssim r^{-1}$,

$$\int_{\sqrt{\lambda\mu}}^{2M\sqrt{\lambda\mu}} |\Lambda Q_{\underline{\lambda}}|^2 r\,dr \lesssim \int_{\sqrt{\mu/\lambda}}^{2M\sqrt{\mu/\lambda}} r^{-1}\,dr \lesssim 1.$$

Thus,

$$\lambda'\int_0^\infty \chi_{M\sqrt{\lambda\mu}}|\Lambda Q_{\underline{\lambda}}|^2 r\,dr = \lambda'\int_0^{\sqrt{\lambda\mu}} |\Lambda Q_{\underline{\lambda}}|^2 r\,dr + O(\lambda') = 2\lambda'|\log(\lambda/\mu)| + O(\lambda^{1/2}).$$

We now show that the remaining terms on the right-hand side of (3-32) are $\ll |\log\lambda|^{1/2}\lambda^{1/2}$ for all L and M large and λ/μ sufficiently small compared to L and M. Since we are assuming $\frac{1}{2} \le \mu \le 2$, we have by (3-9)

$$\begin{aligned}|\mu'||\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}}\rangle| &\lesssim \lambda^{1/2} \int_0^{4M\sqrt{\lambda}} \frac{1}{\lambda}\frac{r}{\lambda}|Q_r(r/\lambda)|\frac{1}{\mu}\frac{r}{\mu}|Q_r(r/\mu)| r\,dr \\ &\lesssim \lambda^{-1/2}\int_0^{4M\sqrt{\lambda}} \frac{(r/\lambda)}{1+(r/\lambda)^2}\frac{r^2}{1+r^2}\,dr \\ &\lesssim \lambda^{1/2}\int_0^{4M\sqrt{\lambda}} \frac{r^2}{\lambda^2+r^2}\frac{r}{1+r^2}\,dr \lesssim \lambda^{3/2}.\end{aligned}$$

Thus, the third term in (3-32) is $\ll \lambda^{1/2}|\log\lambda|^{1/2}$. For the fourth term, we have

$$\left|\frac{\lambda'}{\lambda}\langle \chi_{M\sqrt{\lambda\mu}}\Lambda_0\Lambda Q_{\underline{\lambda}} \mid g\rangle\right| \lesssim |\lambda'|\,\|g\|_{L^\infty}\|\chi_{M\sqrt{\mu/\lambda}}\Lambda_0\Lambda Q\|_{L^1} \lesssim \lambda\|\chi_{4M/\sqrt{\lambda}}\Lambda_0\Lambda Q\|_{L^1}.$$

Now $\Lambda_0\Lambda Q = 4r/(1+r^2)^2$ so

$$\|\chi_{4M/\sqrt{\lambda}}\Lambda_0\Lambda Q\|_{L^1} \lesssim 1.$$

Thus,

$$\left|\frac{\lambda'}{\lambda}\langle \chi_{M\sqrt{\lambda\mu}}\Lambda_0\Lambda Q_{\underline{\lambda}} \mid g\rangle\right| \lesssim \lambda \ll \lambda^{1/2}|\log\lambda|^{1/2}.$$

For the fifth term appearing in (3-32), we have

$$|\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid g\rangle| \lesssim \|g\|_{L^\infty}\int_{M\sqrt{\lambda\mu}}^{2M\sqrt{\lambda\mu}} |\Lambda Q_{\underline{\lambda}}| r\,dr \lesssim \lambda^{3/2}\int_{M/(2\sqrt{\lambda})}^{4M/\sqrt{\lambda}} |\Lambda Q| r\,dr \lesssim \lambda.$$

By (3-8) and (3-9) we conclude that for all λ sufficiently small depending on L and M

$$\left|\left(\frac{\lambda'}{2\lambda}+\frac{\mu'}{2\mu}\right)\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid g\rangle\right| \lesssim \lambda^{1/2} \ll \lambda^{1/2}|\log\lambda|^{1/2}.$$

From (3-32) and the previous bounds we conclude that

$$\left|2\lambda'\log(\lambda/\mu) - \frac{d}{dt}\langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid g\rangle + \langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle\right| \ll \lambda^{1/2}|\log\lambda|^{1/2}. \tag{3-33}$$

By (3-8) and (3-9)

$$\frac{d}{dt}2\lambda\log(\lambda/\mu) = 2\lambda'\log(\lambda/\mu) + 2(\lambda'\mu-\mu'\lambda)/\mu = 2\lambda'\log(\lambda/\mu) + O(\lambda^{1/2}).$$

From this estimate and (3-33) we obtain

$$|\zeta' + \langle \chi_{\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle| \ll \lambda^{1/2}|\log\lambda|^{1/2}. \tag{3-34}$$

Recall that

$$b(t) := -\langle \chi_{\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle - \langle \dot g \mid \mathcal{A}_0(\lambda) g\rangle.$$

By (3-3) and Lemma 3.7 we have

$$|\langle \dot g \mid \mathcal{A}_0(\lambda) g\rangle| \lesssim \|\dot g\|_{L^2}\|\mathcal{A}_0(\lambda)g\|_{L^2} \lesssim \|(g,\dot g)\|_{\mathcal{H}_0}^2 \lesssim \lambda \ll \lambda^{1/2}|\log\lambda|^{1/2}.$$

This estimate and (3-34) imply

$$|\zeta' - b| \ll \lambda^{1/2}|\log\lambda|^{1/2}$$

for L and M large and λ/μ sufficiently small depending on L and M. This completes the proof of (3-28).

To prove (3-29), we argue as above and obtain

$$|b(t)| \le \|\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}}\|_2\,\|\partial_t\psi\|_2 - O(\lambda) = [2\log(\lambda/\mu)+O(1)]^{1/2}\|\partial_t\psi\|_2 - O(\lambda).$$

By (3-5) we have

$$\|\partial_t\psi(t)\|_2^2 \le 16(\lambda/\mu)+o(\lambda).$$

The previous two estimates combined yield (3-29).

We now turn to proving (3-31) and (3-30). By approximating the initial data $\vec\psi(t_0)$ for some $t_0\in J$ by smooth functions and using the well-posedness theory, we may assume that $\vec\psi(t)$ is smooth on J. We differentiate $b(t)$ and use the formulae (3-6), (3-7) to obtain

$$\begin{aligned}
b'(t) &= \frac{\lambda'}{\lambda}\langle \chi_{M\sqrt{\lambda\mu}}[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot g\rangle - \langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \partial_t \dot g\rangle - \langle \partial_t\dot g \mid \mathcal{A}_0(\lambda) g\rangle \\
&\quad - \frac{\lambda'}{\lambda}\langle \dot g \mid \lambda\partial_\lambda\mathcal{A}_0(\lambda)g\rangle - \langle \dot g \mid \mathcal{A}_0(\lambda)\partial_t g\rangle + \left(\frac{\lambda'}{2\lambda}+\frac{\mu'}{2\mu}\right)\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle \\
&= \frac{\lambda'}{\lambda}\langle \chi_{M\sqrt{\lambda\mu}}[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot g\rangle - \left\langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \,\middle|\, \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu))\right\rangle \\
&\quad - \left\langle \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda)+f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda) g\right\rangle - \frac{\lambda'}{\lambda}\langle \dot g \mid \lambda\partial_\lambda \mathcal{A}_0(\lambda) g\rangle \\
&\quad - \langle \dot g \mid \mathcal{A}_0(\lambda)\dot g\rangle - \lambda'\langle \dot g \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}}\rangle + \mu'\langle \dot g \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\mu}}\rangle + \left(\frac{\lambda'}{2\lambda}+\frac{\mu'}{2\mu}\right)\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle.
\end{aligned}$$

We first discard those terms which are $\ll 1$ as long as $L>0$ is sufficiently large, $M>0$ is sufficiently large depending on L, and λ/μ is sufficiently small depending on L and M. Consider the last term appearing above. Here we will choose the size of L. For some absolute constant $C_2>0$ we have

$$\|\Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}}\|_{L^2} \le C_2. \tag{3-35}$$

If C is the constant in (3-8), then we choose $L>0$ large so that

$$80CC_2(\log L)^{-1/2} \le \frac{\delta}{100}. \tag{3-36}$$

Then by Cauchy–Schwarz, (3-35), (3-5) and (3-36), we conclude that

$$\left|\frac{\lambda'}{\lambda}\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot g\rangle\right| \le \frac{|\lambda'|}{\lambda}\|\Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}}\|_{L^2}\,\|\dot g\|_{L^2} \le \frac{2C(\log L)^{-1/2}\lambda^{1/2}}{\lambda}C_2 40\lambda^{1/2} \le \frac{\delta}{100}$$

as long as λ/μ is sufficiently small. Similarly, we have

$$\left|\frac{\mu'}{\mu}\langle \Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid \dot{g}\rangle\right| \le \frac{|\lambda'|}{\lambda}\|\Lambda\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}}\|_{L^2}\|\dot{g}\|_{L^2} \le 4C_1(\log L)^{-1/2}\lambda^{1/2}C_2 40\lambda^{1/2} \le \frac{\delta}{100}$$

as long as λ/μ is sufficiently small. Thus, the last term above can be made $\le \delta/100$. We now consider the first and sixth terms appearing above. By Cauchy–Schwarz and the fact that $\Lambda_0\Lambda Q \in L^2$, we have

$$\left|\frac{\lambda'}{\lambda}\langle(1-\chi_{M\sqrt{\lambda\mu}})[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot{g}\rangle\right| \lesssim \frac{|\lambda'|}{\lambda}\|\dot{g}\|_{L^2}\|\Lambda_0\Lambda Q\|_{L^2(r\ge M\sqrt{\mu/\lambda})} \lesssim \|\Lambda_0\Lambda Q\|_{L^2(r\ge M\sqrt{\mu/\lambda})} \ll 1.$$

Then the first term and the sixth term combined yield

$$\begin{aligned}\frac{\lambda'}{\lambda}\langle\chi_{M\sqrt{\lambda\mu}}[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot{g}\rangle - \lambda'\langle\dot{g} \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}}\rangle &= \frac{\lambda'}{\lambda}\langle[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot{g}\rangle - \lambda'\langle\dot{g} \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}}\rangle + o(1)\\ &= \frac{\lambda'}{\lambda}\langle[\Lambda_0\Lambda Q]_{\underline{\lambda}} - \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}} \mid \dot{g}\rangle + o(1),\end{aligned}$$

where the little-oh satisfies $|o(1)| \ll 1$ as long as $L > 0$ is sufficiently large, $M > 0$ is sufficiently large depending on L, and λ/μ is sufficiently small depending on L and M. By (3-22)

$$\frac{|\lambda'|}{\lambda}|\langle[\Lambda_0\Lambda Q]_{\underline{\lambda}} - \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}} \mid \dot{g}\rangle| \le C\lambda^{-1/2}\|\dot{g}\|_{L^2}\|[\Lambda_0\Lambda Q]_{\underline{\lambda}} - \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}}\|_{L^2} \lesssim c_0 \ll 1,$$

as long as c_0 is sufficiently small. We conclude that

$$\left|\frac{\lambda'}{\lambda}\langle\chi_{M\sqrt{\lambda\mu}}[\Lambda_0\Lambda Q]_{\underline{\lambda}} \mid \dot{g}\rangle - \lambda'\langle\dot{g} \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\lambda}}\rangle\right| \ll 1.$$

Since $(\lambda\partial_\lambda\mathcal{A}_0(\lambda)) : H \to L^2$ is bounded, we have that the fourth term satisfies

$$\left|\frac{\lambda'}{\lambda}\langle\dot{g} \mid (\lambda\partial_\lambda\mathcal{A}_0(\lambda))g\rangle\right| \lesssim \lambda^{-1/2}\|(g,\dot{g})\|_{\mathcal{H}_0}^2 \lesssim \lambda^{1/2} \ll 1.$$

Via (3-18) the fifth term appearing above vanishes:

$$\langle\dot{g} \mid \mathcal{A}_0(\lambda)\dot{g}\rangle = 0.$$

Finally, since $\frac{1}{2} \le \mu \le 2$ we have

$$|\mu'\langle\dot{g} \mid \mathcal{A}_0(\lambda)\Lambda Q_{\underline{\mu}}\rangle| = \frac{|\mu'|}{\mu}|\langle\dot{g} \mid \mathcal{A}_0(\lambda)\Lambda Q_{\mu}\rangle| \lesssim |\mu'|\|\dot{g}\|_{L^2} \lesssim \lambda \ll 1.$$

We now introduce some notation. Until the end of the proof, we write $A \simeq B$ if $A = B$ up to terms which can be made $< \delta$ as long as $L > 0$ is sufficiently large, $M > 0$ is sufficiently large depending on L, and λ/μ is sufficiently small depending on L and M. We have shown so far that

$$\begin{aligned}b'(t) \simeq -\Big\langle\chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}}\ \Big|\ \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu))\Big\rangle\\ -\Big\langle\partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu))\ \Big|\ \mathcal{A}_0(\lambda)g\Big\rangle. \quad (3\text{-}37)\end{aligned}$$

We now choose the size of $M > 0$ (depending on L). Recall that

$$\mathcal{L}_\lambda \Lambda Q_{\underline{\lambda}} := \left(-\partial_{rr} - \frac{1}{r}\partial_r + \frac{f'(Q_\lambda)}{r^2}\right)\Lambda Q_{\underline{\lambda}} = 0.$$

In fact, since we have the factorization $\mathcal{L}_\lambda = A_\lambda^* A_\lambda$ with $A_\lambda = -\partial_r + \cos Q_\lambda / r$, we must have

$$A_\lambda \Lambda Q_{\underline{\lambda}} = 0.$$

Thus,

$$\begin{aligned}\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{f'(Q_\lambda)}{r^2} g \right\rangle &= -\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid A_\lambda^* A_\lambda g\rangle \\ &= -\langle A_\lambda(\chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}}) \mid A_\lambda g\rangle \\ &= \frac{1}{M\sqrt{\lambda\mu}} \langle \chi'_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid A_\lambda g\rangle.\end{aligned}$$

Since $\chi'_{M\sqrt{\lambda\mu}}$ is bounded by 2 and is supported on the annulus $\{M\sqrt{\lambda\mu} \le r \le 2M\sqrt{\lambda\mu}\}$, Cauchy–Schwarz and Proposition 3.3 imply

$$\frac{1}{M\sqrt{\lambda\mu}} |\langle \chi'_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid A_\lambda g\rangle| \lesssim M^{-1}\lambda^{-1/2} \|A_\lambda g\|_{L^2} \lesssim_L M^{-1}\lambda^{-1/2}\|g\|_H \lesssim_L M^{-1}.$$

Thus, for $M > M_0(L)$, the above term is $\ll 1$. We conclude that

$$\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \partial_r^2 g + \frac{1}{r}\partial_r g \right\rangle \simeq \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{f'(Q_\lambda)}{r^2} g \right\rangle.$$

We now rewrite (3-37) as

$$\begin{aligned} b'(t) \simeq \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) - f'(Q_\lambda) g) \right\rangle \\ - \left\langle \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda) g \right\rangle. \qquad (3\text{-}38)\end{aligned}$$

We add, subtract and regroup to obtain

$$\begin{aligned} b'(t) \simeq &\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu)) \right\rangle \\ &+ \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f'(Q_\lambda - Q_\mu) - f'(Q_\lambda)) g \right\rangle \qquad (3\text{-}39) \\ &+ \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - f'(Q_\lambda - Q_\mu) g) \right\rangle \qquad (3\text{-}40) \\ &- \left\langle \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda) g \right\rangle. \end{aligned}$$

We now identify the first term above as the leading-order contribution.

Claim 3.10.

$$\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu)) \right\rangle \simeq \frac{8}{\mu}. \qquad (3\text{-}40)$$

By trigonometric identities

$$\begin{aligned} f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) &= \tfrac{1}{2}(\sin 2Q_\lambda(\cos 2Q_\mu - 1) + \sin 2Q_\mu(1 - \cos 2Q_\lambda)) \\ &= -\sin 2Q_\lambda \sin^2 Q_\mu + \sin 2Q_\mu \sin^2 Q_\lambda \\ &= -\sin 2Q_\lambda (\Lambda Q_\mu)^2 + \sin 2Q_\mu (\Lambda Q_\lambda)^2. \end{aligned} \tag{3-41}$$

We show that the first term in the above expansion gives a negligible contribution to the L^2 pairing on the left side of (3-40). Indeed, if we set $\sigma := \lambda/\mu$, then as long as $\sigma \ll 1$, depending on L and M,

$$\begin{aligned} \left|\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{\sin 2Q_\lambda}{r^2}(\Lambda Q_\mu)^2 \right\rangle\right| &\lesssim \frac{1}{\lambda}\int_0^{2M\sqrt{\lambda\mu}} |\Lambda Q_\lambda|^2 |\Lambda Q_\mu|^2 \frac{dr}{r} \lesssim \frac{1}{\sigma}\int_0^{2M\sqrt{\sigma}} |\Lambda Q_\sigma|^2 |\Lambda Q|^2 \frac{dr}{r} \\ &= \frac{1}{\sigma}\int_0^{2M\sqrt{\sigma}} \frac{(r/\sigma)^2}{(1+(r/\sigma)^2)^2} \frac{r^2}{(1+r^2)^2} \frac{dr}{r} \\ &\lesssim \sigma\left[\int_0^\sigma \sigma^{-4} r^3\, dr + \int_\sigma^{2M\sqrt{\sigma}} \frac{r^3}{(\sigma^2+r^2)^2}\, dr\right] \lesssim \sigma[|\log\sigma| + \log M] \ll 1. \end{aligned}$$

Thus,

$$\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \mid (f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu))\rangle \simeq \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(\Lambda Q_\lambda)^2 \sin 2Q_\mu \right\rangle. \tag{3-42}$$

We now compute

$$\begin{aligned} \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(\Lambda Q_\lambda)^2 \sin 2Q_\mu \right\rangle &= \frac{1}{\lambda}\int_0^\infty \chi_{M\sqrt{\sigma}} (\Lambda Q_\sigma)^3 \sin 2Q \frac{dr}{r} \\ &= \frac{1}{\lambda}\int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma)^3 \sin 2Q \frac{dr}{r} + \frac{1}{\lambda}\int_{\sqrt{\sigma}}^\infty \chi_{M\sqrt{\sigma}} (\Lambda Q_\sigma)^3 \sin 2Q \frac{dr}{r}. \end{aligned} \tag{3-43}$$

Since $|\Lambda Q| \lesssim r^{-1}$ for r large and $\sigma \sim \lambda$, we have

$$\frac{1}{\lambda}\int_{\sqrt{\sigma}}^\infty |\Lambda Q_\sigma|^3 \frac{dr}{r} \lesssim \frac{1}{\sigma}\int_{\sqrt{\sigma}}^\infty |\Lambda Q_\sigma|^3 \frac{dr}{r} \lesssim \frac{1}{\sigma}\int_{1/\sqrt{\sigma}}^\infty |\Lambda Q|^3 \frac{dr}{r} \lesssim \sigma^{1/2} \ll 1.$$

Thus, from (3-43) it follows that

$$\left\langle \chi_{\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(\Lambda Q_\lambda)^2 \sin 2Q_\mu \right\rangle \simeq \frac{1}{\lambda}\int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma)^3 \sin 2Q \frac{dr}{r}.$$

Since $\sigma = \lambda/\mu \ll 1$, on the interval $[0, \sqrt{\sigma}]$ we write

$$\sin 2Q = 4r\frac{1-r^2}{(1+r^2)^2} = 4r + O(r^3). \tag{3-44}$$

We compute

$$\begin{aligned} \frac{1}{\lambda}\int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma)^3 4r \frac{dr}{r} &= \frac{4\sigma}{\lambda}\int_0^{1/\sqrt{\sigma}} (\Lambda Q)^3\, dr \\ &= \frac{4\sigma}{\lambda}\int_0^\infty (\Lambda Q)^3\, dr - 4\frac{\sigma}{\lambda}\int_{1/\sqrt{\sigma}}^\infty (\Lambda Q)^3\, dr = \frac{8}{\mu} + O(\sigma), \end{aligned}$$

where the integral $\int_0^\infty (\Lambda Q)^3\,dr = 2$ is evaluated using substitution. By (3-44),

$$\left|\frac{1}{\lambda}\int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma)^3(\sin 2Q - 4r)\,\frac{dr}{r}\right| \lesssim \frac{1}{\lambda}\int_0^{\sqrt{\sigma}} |\Lambda Q_\sigma|^3 r^2\,dr = \frac{\sigma^3}{\lambda}\int_0^{1/\sqrt{\sigma}} |\Lambda Q|^3 r^2\,dr \lesssim \sigma^2|\log\sigma| \ll 1.$$

Thus,

$$\left\langle \chi_{\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(\Lambda Q_\lambda)^2 \sin 2Q_\mu \right\rangle \simeq \frac{1}{\lambda}\int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 \sin 2Q(r)\,\frac{dr}{r} \simeq \frac{8}{\mu}. \tag{3-45}$$

Combining (3-42) and (3-45) we conclude that

$$\langle \chi_{\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \mid (f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu))\rangle \simeq \frac{8}{\mu}$$

as desired. □

For what follows, we list the useful identities

$$\Lambda^2 Q = \tfrac{1}{2}\sin 2Q = 2r\frac{1-r^2}{(1+r^2)^2}, \tag{3-46}$$

$$\Lambda^3 Q = 2r\left(\frac{1+r^2-5r^4-r^6}{(1+r^2)^4}\right), \tag{3-47}$$

$$\Lambda_0\Lambda Q = (r\partial_r + 1)(r\partial_r Q) = 2\Lambda Q + r^2\partial_r^2 Q.$$

We now claim that the term (3-39) in the expansion of $b'(t)$ satisfies

$$\left|\left\langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f'(Q_\lambda - Q_\mu) - f'(Q_\lambda))g \right\rangle\right| \lesssim (\lambda/\mu)^{1/2}. \tag{3-48}$$

First note that we have

$$f'(Q_\lambda - Q_\mu) - f'(Q_\lambda) = \sin 2Q_\lambda \sin 2Q_\mu - 2\cos 2Q_\lambda \sin^2 Q_\mu = 4\Lambda^2 Q_\lambda \Lambda^2 Q_\mu - (\Lambda Q_\mu)^2 \cos 2Q_\lambda. \tag{3-49}$$

By (3-46) and (3-47) we have

$$|\Lambda Q| + |\Lambda^2 Q| \lesssim \frac{r}{1+r^2}.$$

We first estimate

$$\left|\left\langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{4}{r^2}\Lambda^2 Q_\lambda \Lambda^2 Q_\mu g \right\rangle\right| \lesssim \|g\|_{L^\infty}\frac{1}{\lambda}\int_0^{2M\sqrt{\lambda\mu}} |\Lambda Q_\lambda||\Lambda^2 Q_\lambda||\Lambda^2 Q_\mu|\,\frac{dr}{r}$$
$$\lesssim \|g\|_H \frac{1}{\sigma}\int_0^{2M\sqrt{\sigma}} |\Lambda Q_\sigma||\Lambda^2 Q_\sigma||\Lambda^2 Q|\,\frac{dr}{r}$$
$$\lesssim \|g\|_H \int_0^{2M/\sqrt{\sigma}} |\Lambda Q||\Lambda^2 Q|\,dr \lesssim \sigma^{1/2},$$

where $\sigma = \lambda/\mu$ as before. We then estimate

$$\left|\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}((\Lambda Q_\mu)^2 \cos 2Q_\lambda) g \right\rangle\right| \lesssim \|g\|_H \left(\int_0^\infty (\Lambda Q_\sigma)^2 (\Lambda Q)^4 \, \frac{dr}{r} \right)^{1/2} \lesssim \sigma^{1/2} \left(\int_0^\infty (\Lambda Q_\sigma)^2 \, \frac{dr}{r} \right)^{1/2} \lesssim \sigma^{1/2}.$$

The previous two bounds along with (3-49) imply (3-48).

In summary, we have shown thus far that

$$b'(t) - \frac{8}{\mu} \simeq \left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - f'(Q_\lambda - Q_\mu) g) \right\rangle \tag{3-50}$$

$$- \left\langle \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda) g \right\rangle. \tag{3-51}$$

We now rewrite (3-50) as

$$\begin{aligned}
&\left\langle \chi_{M\sqrt{\lambda\mu}} \Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g) \right\rangle \\
&\quad = -\left\langle \mathcal{A}(\lambda) g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - g) \right\rangle \\
&\qquad + \left\langle \mathcal{A}(\lambda) g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - g) \right\rangle \\
&\qquad + \left\langle \mathcal{A}(\lambda)(Q_\lambda - Q_\mu) \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g) \right\rangle \\
&\qquad + \left\langle \mathcal{A}(\lambda) Q_\mu \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g) \right\rangle \\
&\qquad + \left\langle \chi_{M\sqrt{\lambda\mu}} (\Lambda Q_{\underline{\lambda}} - \mathcal{A}(\lambda) Q_\lambda) \,\middle|\, \frac{1}{r^2}(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g) \right\rangle.
\end{aligned}$$

We remark that we used the fact that $\chi_{M\sqrt{\lambda\mu}} \mathcal{A}(\lambda) Q_\lambda = \mathcal{A}(\lambda) Q_\lambda$ (as long as λ/μ is small) to obtain the previous expression. The second and third terms above can be estimated using (3-19) with $g_1 = Q_\lambda - Q_\mu$ and $g_2 = g$:

$$\begin{aligned}
&\left| \left\langle \mathcal{A}(\lambda) g \,\middle|\, \frac{1}{r^2}\big(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - g\big) \right\rangle \right. \\
&\qquad \left. + \left\langle \mathcal{A}(\lambda)(Q_\lambda - Q_\mu) \,\middle|\, \frac{1}{r^2}\big(f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g\big) \right\rangle \right| \lesssim_L c_0,
\end{aligned}$$

which is $\ll 1$ as long as c_0 is taken sufficiently small. The pointwise bound

$$\begin{aligned}
&|f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - f'(Q_\lambda - Q_\mu) g| \\
&\qquad = \tfrac{1}{2} |\sin(2Q_\lambda - 2Q_\mu)[\cos 2g - 1] + \cos(2Q_\lambda - 2Q_\mu)[\sin 2g - 2g]| \lesssim |g|^2
\end{aligned}$$

and (3-23) imply that the second-to-last line of the above satisfies

$$\left|\left\langle \mathcal{A}(\lambda) Q_\mu \,\middle|\, \frac{1}{r^2}\big(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-f'(-Q_\mu+Q_\lambda)g\big)\right\rangle\right| \lesssim \frac{1}{\mu}\|g\|_H^2 \lesssim \lambda \ll 1.$$

Using (3-21) we estimate the last line of the expansion of (3-40) similarly:

$$\begin{aligned}\left|\left\langle \chi_{M\sqrt{\lambda\mu}}(\Lambda Q_{\underline{\lambda}}-\mathcal{A}(\lambda)Q_\lambda) \,\middle|\, \frac{1}{r^2}\big(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-f'(-Q_\mu+Q_\lambda)g\big)\right\rangle\right|& \\ \lesssim \|\Lambda Q_{\underline{\lambda}}-\mathcal{A}(\lambda)Q_\lambda\|_{L^\infty}\|g\|_H^2 \lesssim_L c_0 \ll 1.&\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}\left\langle \chi_{M\sqrt{\lambda\mu}}\Lambda Q_{\underline{\lambda}} \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-f'(-Q_\mu+Q_\lambda)g)\right\rangle& \\ \simeq -\left\langle \mathcal{A}(\lambda)g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)\right\rangle,&\end{aligned}$$

which by (3-50) implies

$$\begin{aligned}b'(t)-\frac{8}{\mu} \simeq -\left\langle \mathcal{A}(\lambda)g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)\right\rangle& \\ -\left\langle \partial_r^2 g+\frac{1}{r}\partial_r g-\frac{1}{r^2}(f(Q_\lambda-Q_\mu+g)-f(Q_\lambda)+f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda)g\right\rangle.& \quad (3\text{-}52)\end{aligned}$$

We now consider the line (3-52). By adding and subtracting terms and (3-20) we have

$$\begin{aligned}&-\left\langle \partial_r^2 g+\frac{1}{r}\partial_r g-\frac{1}{r^2}(f(Q_\lambda-Q_\mu+g)-f(Q_\lambda)+f(Q_\mu)) \mid \mathcal{A}_0(\lambda)g\right\rangle \\ &\quad = -\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \partial_r^2 g+\frac{1}{r}\partial_r g-\frac{1}{r^2}g\right\rangle+\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \frac{1}{r^2}(f(Q_\lambda-Q_\mu)-f(Q_\lambda)+f(Q_\mu))\right\rangle \\ &\qquad\qquad +\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \frac{1}{r^2}(f(Q_\lambda-Q_\mu+g)-f(Q_\lambda-Q_\mu)-g)\right\rangle \\ &\quad \geq -\frac{c_0}{\lambda}\|g\|_H^2+\frac{1}{\lambda}\int_0^{R\lambda}\Big((\partial_r g)^2+\frac{1}{r^2}g^2\Big)r\,dr+\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \frac{1}{r^2}(f(Q_\lambda-Q_\mu)-f(Q_\lambda)+f(Q_\mu))\right\rangle \\ &\qquad\qquad +\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)+f(-Q_\mu+Q_\lambda)-g)\right\rangle,\end{aligned}$$

where R is defined in the statement of Lemma 3.7. From (3-41) we have the pointwise estimate

$$|f(Q_\lambda-Q_\mu)-f(Q_\lambda)+f(Q_\mu)| \lesssim (\Lambda Q_\lambda)^2(\Lambda Q_\mu)+\Lambda Q_\lambda(\Lambda Q_\mu)^2.$$

By Lemma 3.7, $\|\mathcal{A}_0(\lambda)g\|_{L^2} \lesssim \|g\|_H$ and $\mathcal{A}_0(\lambda)g$ is supported on a ball of radius $CR\lambda$. Thus, the third term in the second-to-last line above satisfies

$$\begin{aligned}&\left|\left\langle \mathcal{A}_0(\lambda)g \,\middle|\, \frac{1}{r^2}(f(Q_\lambda-Q_\mu)-f(Q_\lambda)+f(Q_\mu))\right\rangle\right| \\ &\qquad \lesssim \|g\|_H\left[\left(\int_0^{CR\sigma} r^{-2}(\Lambda Q_\sigma)^4(\Lambda Q)^2\,\frac{dr}{r}\right)^{1/2}+\left(\int_0^{CR\sigma} r^{-2}(\Lambda Q)^4(\Lambda Q_\sigma)^2\,\frac{dr}{r}\right)^{1/2}\right] \\ &\qquad \lesssim \|g\|_H \lesssim \lambda^{1/2} \ll 1.\end{aligned}$$

Thus,

$$-\left\langle \mathcal{A}(\lambda)g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)\right\rangle$$
$$-\left\langle \partial_r^2 g+\frac{1}{r}\partial_r g-\frac{1}{r^2}(f(Q_\lambda-Q_\mu+g)-f(Q_\lambda)+f(Q_\mu)) \,\middle|\, \mathcal{A}_0(\lambda)g\right\rangle$$
$$\geq \frac{1}{\lambda}\int_0^{R\lambda}\left((\partial_r g)^2+\frac{1}{r^2}g^2\right)r\,dr$$
$$+\left\langle (\mathcal{A}_0(\lambda)-\mathcal{A}(\lambda))g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)\right\rangle+o(1). \quad (3\text{-}53)$$

The difference $\mathcal{A}_0(\lambda)-\mathcal{A}(\lambda)$ is given by the operator of multiplication by

$$\frac{1}{2\lambda}\left(q''\left(\frac{r}{\lambda}\right)+\frac{\lambda}{r}q'\left(\frac{r}{\lambda}\right)\right).$$

By (3-24) we have

$$\left\langle (\mathcal{A}_0(\lambda)-\mathcal{A}(\lambda))g \,\middle|\, \frac{1}{r^2}(f(-Q_\mu+Q_\lambda+g)-f(-Q_\mu+Q_\lambda)-g)\right\rangle$$
$$=\frac{1}{\lambda}\int_0^\infty \frac{1}{r^2}(f'(Q_\lambda)-1)g^2\,dr+O(c_0\lambda), \quad (3\text{-}54)$$

where $c_0>0$ is as in Lemma 3.7.

The estimates (3-52), (3-53) and (3-54) combine to yield

$$b'(t)-\frac{8}{\mu}\geq \frac{1}{\lambda}\int_0^{R\lambda}\left((\partial_r g)^2+\frac{1}{r^2}g^2\right)r\,dr+\frac{1}{\lambda}\int_0^\infty \frac{1}{r^2}(f'(Q_\lambda)-1)g^2\,dr+o(1).$$

The orthogonality condition $\langle \mathcal{Z}_{\underline{\lambda}} \mid g\rangle=0$ implies the localized coercivity estimate,

$$\frac{1}{\lambda}\int_0^{R\lambda}\left((\partial_r g)^2+\frac{1}{r^2}g^2\right)r\,dr+\frac{1}{\lambda}\int_0^\infty \frac{1}{r^2}(f'(Q_\lambda)-1)g^2\,dr\geq -\frac{c_1}{\lambda}\|g\|_H^2;$$

see [Jendrej 2019, Lemma 5.4, equation (5.28)] for the proof. The constant $c_1>0$ appearing above can be made small by choosing R sufficiently large. Since $\|g\|_H^2\lesssim\lambda$, we conclude that

$$b'(t)-\frac{8}{\mu}\geq -\frac{\delta}{2}\geq -\frac{\delta}{\mu}$$

as long as L is sufficiently large, M is sufficiently large depending on L and λ/μ is sufficiently small depending on L and M. □

From Propositions 3.3 and 3.9 we now show that, roughly, if the modulation parameters are approaching each other in scale, then the solution to (1-4) is ejected from a small neighborhood of the set of two-bubbles.

Remark 3.11. We now fix the parameters L and M used in the definition of $\zeta(t)$ for the remainder of the section. In particular, we fix $L=L_0$ and $M=M_0$ large enough so that the estimates in Proposition 3.9 hold for

$$\delta=\tfrac{1}{2020},$$

whenever $\boldsymbol{d}(\vec{\psi}(t))<\eta_1=\eta_1(L_0,M_0)<\eta_0$.

Proposition 3.12. *Let $C > 0$. Then for all $\epsilon_0 > 0$ sufficiently small and for any $\epsilon > 0$ sufficiently small relative to ϵ_0 the following holds. Let $\vec{\psi}(t) : [T_0, T_+) \to \mathcal{H}_0$ be a solution of* (1-4). *Assume that $t_0 \in [T_0, T_+)$ is so that $\boldsymbol{d}(\vec{\psi}(t_0)) \le \epsilon$ and $\frac{d}{dt}(\zeta(t)/\mu(t))|_{t=t_0} \ge 0$. Then there exist t_1 and t_2, $T_0 \le t_0 \le t_1 \le t_2 < T_+$, such that*

$$\boldsymbol{d}(\vec{\psi}(t)) \ge 2\epsilon \quad \text{for } t \in [t_1, t_2], \tag{3-55}$$

$$\boldsymbol{d}(\vec{\psi}(t)) \le \tfrac{1}{4}\epsilon_0 \quad \text{for } t \in [t_0, t_1], \tag{3-56}$$

$$\boldsymbol{d}(\vec{\psi}(t_2)) \ge 2\epsilon_0, \tag{3-57}$$

$$\int_{t_1}^{t_2} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \ge C \int_{t_0}^{t_1} \sqrt{\boldsymbol{d}(\vec{\psi}(t))} \, dt. \tag{3-58}$$

If we assume that $\frac{d}{dt}(\zeta(t)/\mu(t))|_{t=t_0} \le 0$, then analogous statements hold with times $t_2 \le t_1 \le t_0$.

Proof. The proof is along the lines of Proposition 3.10 from [Jendrej and Lawrie 2018]. From (3-4), (3-27) and Remark 3.11, it follows that if $\epsilon_1 > 0$ is sufficiently small and $\zeta(t)/\mu(t) \le 4\epsilon_1$, then the estimates in Proposition 3.9 hold with $\delta = \frac{1}{2020}$ in a neighborhood of t_0. In particular, we have

$$\frac{1}{4}\frac{\zeta(t)}{\mu(t)} \le \frac{\lambda(t)}{\mu(t)}\left|\log \frac{\lambda(t)}{\mu(t)}\right| \le \frac{\zeta(t)}{\mu(t)}. \tag{3-59}$$

Let t_2 be the first time $t_2 \ge t_0$ such that $\zeta(t_2)/\mu(t_2) = 4\epsilon_1$. If there is no such time, we set $t_2 = T_+$. Define

$$f(x) = x|\log x|,$$

which is smooth and increasing on $(0, 100\epsilon_1)$ for ϵ_1 sufficiently small and satisfies $\lim_{x\to 0^+} f(x) = 0$. Then (3-59) becomes

$$\frac{1}{4}\frac{\zeta(t)}{\mu(t)} \le f\left(\frac{\lambda(t)}{\mu(t)}\right) \le \frac{\zeta(t)}{\mu(t)}. \tag{3-60}$$

Then if $t_2 < T_+$ we have $f(\lambda(t_2)/\mu(t_2)) \ge \epsilon_1$, which by (3-4) implies (3-57) by taking ϵ_0 comparable to $f^{-1}(\epsilon_1)$. By the scaling symmetry of the equation, we can assume that $\mu(t_0) = 1$. Let $t_3 \le t_2$ be the last time such that $\mu(t) \in \left[\frac{1}{2}, 2\right]$ for all $t \in [t_0, t_3]$. If there is no such final time we set $t_3 = t_2$. We will see by a bootstrapping argument that we can always take $t_3 = t_2$ and that $t_2 < T_+$.

By Remark 3.11 and by taking ϵ_1 small enough, we have by (3-31)

$$b'(t) \ge 1. \tag{3-61}$$

We also obtain from (3-28)

$$\zeta'(t) \ge b(t) - \zeta(t)^{1/2}.$$

Consider $\xi(t) := b(t) + \zeta(t)^{1/2}$. Using the two inequalities above we obtain

$$\xi'(t) \ge 1 + \tfrac{1}{2}\zeta(t)^{-1/2}(b(t) - \zeta(t)^{1/2}) = \tfrac{1}{2}\zeta(t)^{-1/2}(b(t) + \zeta(t)^{1/2}) = \tfrac{1}{2}\zeta(t)^{-1/2}\xi(t).$$

By (3-29) and the fact that $\mu(t) \in \left(\frac{1}{2}, 2\right)$, we conclude that

$$\xi(t) \le 10\zeta(t)^{1/2}. \tag{3-62}$$

Let

$$\xi_1(t) := b(t) + \tfrac{1}{2}\zeta(t)^{1/2} = \tfrac{1}{2}b(t) + \tfrac{1}{2}\xi(t) = \xi(t) - \tfrac{1}{2}\zeta(t)^{1/2}.$$

Since $b'(t) \geq 0$, we have

$$\xi_1'(t) \geq \tfrac{1}{2}\xi'(t) \geq \tfrac{1}{4}\zeta(t)^{-1/2}\xi(t) \geq \tfrac{1}{4}\zeta(t)^{-1/2}\xi_1(t). \tag{3-63}$$

Since $\mu(t_0) = 1$, we have $0 \leq \frac{d}{dt}(\lambda(t)/\mu(t))|_{t=t_0} = \zeta'(t_0) - \zeta(t_0)\mu'(t_0)$, so (3-9) and (3-27) imply that $\zeta'(t_0) \geq -\frac{1}{8}\zeta(t_0)^{1/2}$ as long as ϵ is taken small enough. This fact and (3-28) gives $b(t_0) \geq -\frac{1}{4}\zeta(t_0)^{1/2}$, so $\xi_1(t_0) > 0$ and (3-63) yields $\xi_1(t) > 0$ for all $t \in [t_0, t_3]$. Thus

$$\xi(t) \geq \tfrac{1}{2}\zeta(t)^{1/2} \quad \text{for } t \in [t_0, t_3]. \tag{3-64}$$

This lower bound along with $\xi'(t) \geq \frac{1}{2}\zeta(t)^{-1/2}\xi(t)$ imply

$$\xi'(t) \geq \tfrac{1}{4}. \tag{3-65}$$

By (3-62) we see that $\zeta(t)$ and thus $\lambda(t)$ is far from 0 on $[t_0, t_3]$.

The bounds (3-62), (3-60) and (3-4) imply that there exists a constant α_0 such that $\xi(t) \geq 40[f(\alpha_0\epsilon)]^{1/2}$ forces $\boldsymbol{d}(\vec{\psi}(t)) \geq 2\epsilon$. Let $t_1 \in [t_0, t_3]$ be the last time such that $\xi(t_1) = 40[f(\alpha_0\epsilon)]^{1/2}$ (set $t_1 = t_3$ if no such time exists). Then by (3-64) and (3-60) we have

$$[f(\lambda(t)/\mu(t))]^{1/2} \leq \zeta(t)^{1/2} \leq 80[f(\alpha_0\epsilon)]^{1/2} \quad \text{for } t \in [t_0, t_1],$$

which yields (3-56) if ϵ is small enough.

We now claim that $\mu(t) \in \left(\frac{1}{2}, 2\right)$ for all $t \in [t_0, t_2]$ and that $t_2 < T_+$. Recall that on $[t_0, t_3]$ we have $\xi'(t) > 0$ as well as

$$\zeta(t)^{1/2} \leq 2\xi(t) \leq 20\zeta(t)^{1/2}, \quad \xi'(t) \geq \tfrac{1}{2}\zeta^{-1/2}(t)\xi(t), \quad \zeta(t) \leq 8\epsilon_1.$$

Thus, by (3-9)

$$\int_{t_0}^{t_3} |\mu'|\,dt \lesssim \int_{t_0}^{t_3} \zeta(t)^{1/2}\,dt \lesssim \int_{t_0}^{t_3} \xi(t)\,dt \lesssim \int_{t_0}^{t_3} \zeta(t)^{1/2}\xi'(t)\,dt \lesssim \sqrt{\epsilon_1}\int_{t_0}^{t_3} \xi'(t)\,dt \lesssim \sqrt{\epsilon_1}\xi(t_3) \lesssim \epsilon_1,$$

where the implied constant is absolute. Thus, we get $\mu(t_3) \in \left[\frac{2}{3}, \frac{3}{2}\right]$ if ϵ_1 is small enough, which implies that $t_3 = t_2$. Now suppose that there is no $t_2 \geq t_0$ such that $\zeta(t_2)/\mu(t_2) = \epsilon_1$. Then, since $\zeta(t)$ (and hence $\lambda(t)$) is far from 0, by [Struwe 2003] the solution is global and (3-65) implies that $\xi(t)$ is eventually $O(1)$. Thus $\zeta(t)$ is eventually $O(1)$, which contradicts our definition of t_2. This implies that there exists $t_2 < T_+$ such that $\zeta(t_2)/\mu(t_2) = \epsilon_1$, which implies (3-57) by choosing ϵ_0 comparable to $f^{-1}(\epsilon_1)$.

By (3-28) and (3-29) we have $|\zeta'(t)| \lesssim |\zeta(t)|$. Thus, there exists an absolute constant $\alpha_1 > 0$ such that $\zeta(t) \geq \frac{1}{4}\epsilon_1$ for $t \in [t_2 - \alpha_1, t_2]$. Since $\zeta(t) \lesssim f(\alpha_0\epsilon)$ on $[t_0, t_1]$, we must have $t_2 - t_1 \geq \alpha_1$ if $f(\alpha_0\epsilon) \ll \epsilon_1$. Then (3-61) yields

$$b(t) \geq b(t_1) + \alpha_1 \geq b(t_0) + \alpha_1 \quad \text{for } t \in [t_1, t_2].$$

Thus, if ϵ is small enough, we get

$$b(t) \geq \tfrac{1}{2}\alpha_1, \quad t \in [t_1, t_2]. \tag{3-66}$$

By Proposition 3.3, the Cauchy–Schwarz inequality and the definition of $b(t)$ we have

$$b(t) \lesssim |\log \lambda|^{1/2} \|\dot{g}\|_{L^2}.$$

Since $\lambda(t) \le \zeta(t) \le 8\epsilon_1$ on $[t_0, t_2]$, we conclude that there exists an absolute constant $\alpha_2 > 0$ such that on $[t_0, t_2]$

$$|b(t)| \le \alpha_2 |\log \epsilon_1|^{1/2} \|\dot{g}\|_{L^2}. \tag{3-67}$$

Integrating, from t_1 to t_2 the lower bound (3-66) and using (3-67) we obtain

$$\frac{\alpha_1^3}{4} \le \int_{t_1}^{t_2} |b(t)|^2 \, dt \le \alpha_2^2 |\log \epsilon_1| \int_{t_1}^{t_2} \|\dot{g}(t)\|_{L^2}^2 \, dt,$$

which implies

$$\frac{\alpha_1^3}{4\alpha_2^2 |\log \epsilon_1|} \le \int_{t_1}^{t_2} \|\partial_t \psi(t)\|_{L^2}^2 \, dt. \tag{3-68}$$

Recall that on $[t_0, t_1]$, we have $\xi'(t) \ge \frac{1}{4}$ and $|\xi(t)| + \zeta(t)^{1/2} \lesssim \sqrt{\epsilon \alpha_0 |\log \alpha_0 \epsilon|}$, where α_0 is an absolute constant. Thus,

$$\int_{t_0}^{t_1} \sqrt{\boldsymbol{d}(\vec{\psi}(t))} \, dt \lesssim \int_{t_0}^{t_1} \sqrt{\zeta(t)} \, dt \lesssim \int_{t_0}^{t_1} \sqrt{\zeta(t)} \xi'(t) \, dt \lesssim \sqrt{\epsilon |\log \epsilon|} \int_{t_0}^{t_1} \xi'(t) \, dt \lesssim \epsilon |\log \epsilon|,$$

where the implied constant is absolute. This estimate and (3-68) imply (3-58) after choosing ϵ sufficiently small. □

4. Dynamics of nonscattering threshold solutions

In this section we prove the main result, Theorem 1.7. We will obtain it as a consequence of the following proposition.

Proposition 4.1. *Let $\psi(t) : (T_-, T_+) \to \mathcal{H}_0$ be a corotational wave map with $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q})$ which does not scatter in forward time. Then*

$$\lim_{t \to T_+} \boldsymbol{d}(\vec{\psi}(t)) = 0.$$

As a first step, we state a direct consequence of Theorem 1.3.

Proposition 4.2. *Let $\vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0$ be a corotational wave map with $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q})$ which does not scatter in forward time. Then*

$$\liminf_{t \to T_+} \boldsymbol{d}(\vec{\psi}(t)) = 0.$$

4A. ***Proof of Proposition 4.1.*** Using the results from Sections 2 and 3, the proof of Proposition 4.1 is identical to that of the corresponding statement, Proposition 4.1, for higher equivariant threshold solutions from [Jendrej and Lawrie 2018]. Therefore, we will sketch the main ideas of the proof and refer the reader to Section 4 of [Jendrej and Lawrie 2018] for complete details. For the remainder of this section, we will always denote by $\vec{\psi}(t)$ a solution to (1-4), $\vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0$, such that $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q})$ and $\vec{\psi}(t)$ does not scatter in forward time.

We argue by contradiction. By our preliminary step Proposition 4.2, we know that $\boldsymbol{d}(\vec{\psi}(t))$ tends to 0 along a sequence of times. If Proposition 4.1 were false, then using Proposition 3.12 we split the maximal time interval of existence into a collection of *bad* intervals where $\vec{\psi}(t)$ is close to the set of two-bubbles, and *good* intervals where $\vec{\psi}(t)$ is far from the set of two-bubbles. A defining feature of these intervals is that the integral of $[\boldsymbol{d}(\vec{\psi}(t))]^{1/2}$ on a given bad interval is controlled by a small constant times the integral of $\|\partial_t \psi(t)\|_{L^2}^2$ on neighboring good intervals; see [Jendrej and Lawrie 2018, Lemma 4.6]. On the union of good intervals which we denote by I, we use Lemmas 2.2 and 2.1 to show that the $\vec{\psi}(t)$ has the following *compactness property*: there exists a continuous function $\nu(t) : I \to (0, \infty)$ such that the trajectory

$$\mathcal{K} = \{\vec{\psi}(t)_{1/\nu(t)} \mid t \in I\}$$

is precompact in $\mathcal{H}_0$; see [Jendrej and Lawrie 2018, Lemma 4.8]. Solutions with the compactness property do not radiate energy, and thus we expect that such solutions are given by rescalings of stationary solutions (harmonic maps). If this intuition is correct, we arrive at a contradiction since the only degree-0 harmonic map is the constant map, which has energy equal to $0 \neq 8\pi$.

To prove that a solution with the compactness property on the union of good intervals is stationary, we will use the virial identity. Integrating (2-2) from $t = \tau_1$ to $t = \tau_2$ yields

$$\int_{\tau_1}^{\tau_2} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \le |\langle \partial_t \psi \mid \chi_R r \partial_r \psi \rangle(\tau_1)| + |\langle \partial_t \psi \mid \chi_R r \partial_r \psi \rangle(\tau_2)| + \int_{\tau_1}^{\tau_2} |\Omega_R(\vec{\psi}(t))| \, dt,$$

where the error $\Omega_R(\vec{\psi}(t))$ is given by (2-3). By Lemma 2.5, we obtain

$$\int_{\tau_1}^{\tau_2} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \le C_0\big(R\sqrt{\boldsymbol{d}(\vec{\psi}(\tau_1))} + R\sqrt{\boldsymbol{d}(\vec{\psi}(\tau_2))}\big) + \int_{\tau_1}^{\tau_2} |\Omega_R(\vec{\psi}(t))| \, dt.$$

We then show that by the defining feature of the good intervals and by choosing the parameters R, τ_1, τ_2 appropriately, we can absorb the error term involving $\Omega_R(\vec{\psi}(t))$ from the right-hand side into the left-hand side; see [Jendrej and Lawrie 2018, Lemmas 4.9, 4.11, 4.12]. The resulting averaged smallness of $\|\partial_t \psi(t)\|_{L^2}^2$ and the compactness property allow us to conclude that $\vec{\psi}(t) = \vec{0}$, our desired contradiction. □

4B. ***Proof of Theorem 1.7.*** We first use Proposition 4.1 to prove $\vec{\psi}(t)$ converges to a pure two-bubble or anti-two-bubble as $t \to T_+$. Let $\epsilon > 0$ be sufficiently small. By Proposition 4.1 there exists a $T_0 \in (T_-, T_+)$ such that

$$\boldsymbol{d}(\vec{\psi}(t)) < \epsilon \quad \text{for all } t \ge T_0.$$

We further assume that $\epsilon < \alpha_0$, where α_0 is the constant from Lemma 2.3. Towards a contradiction, assume that $\vec{\psi}(t)$ alternates between being close to a pure two-bubble and anti-two-bubble, i.e., that there exist $t_1, t_2 \ge T_0,\ t_1 < t_2$, such that $\boldsymbol{d}_+(\vec{\psi}(t_1)) \le \epsilon$ and $\boldsymbol{d}_-(\vec{\psi}(t_2)) \le \epsilon$. By Lemma 2.3 we have $\boldsymbol{d}_+(\vec{\psi}(t_2)) \ge \alpha_0$ and $\boldsymbol{d}_-(\vec{\psi}(t_1)) \ge \alpha_0$. By continuity there exists $t_0 \in (t_1, t_2)$ such that $\boldsymbol{d}_+(\vec{\psi}(t_0)) = \boldsymbol{d}_-(\vec{\psi}(t_0))$. But then again by Lemma 2.3, we conclude that $\boldsymbol{d}_+(\vec{\psi}(t_0)) = \boldsymbol{d}_-(\vec{\psi}(t_0)) > \alpha_0 > \epsilon$. This contradicts our definition of T_0, which proves the desired convergence. Without loss of generality, we assume that $\boldsymbol{d}_+(\vec{\psi}(t)) \to 0$ as $t \to T_+$.

We now prove finite time blow-up and asymptotics of the scales. By taking T_0 larger if necessary, we may assume that

$$\boldsymbol{d}_+(\vec{\psi}(t)) < \epsilon \quad \text{for all } t \geq T_0.$$

We note that as long as $\epsilon > 0$ is sufficiently small, the modulation parameters $\lambda(t)$ and $\mu(t)$ are well-defined on $[T_0, T_+)$, and by Lemma 3.1

$$\vec{\psi}(t) = \vec{Q}_{\lambda(t)} + \vec{Q}_{\mu(t)} + o_{\mathcal{H}_0}(1) \quad \text{as } t \to T_+.$$

Let $\epsilon_0 > 0$ and choose ϵ smaller if necessary so that the conclusions of Proposition 3.12 hold. Let $\zeta(t)$ be as in (3-26) with L and M chosen as in Remark 3.11 so that $\zeta(t) \sim \lambda(t)|\log \lambda(t)/\mu(t)|$. By rescaling if necessary, we can assume that $\mu(T_0) = 1$.

Since $\boldsymbol{d}_+(\vec{\psi}(t)) \to 0$ as $t \to T_+$, there exists a sequence of times $\tau_n \to T_+$ such that

$$\frac{d}{dt}\Big|_{t=\tau_n}\left(\frac{\zeta(t)}{\mu(t)}\right) \leq 0.$$

Then there exist times $t_1 \leq t_0 =: \tau_n$ and $t_2 \leq t_1$ satisfying the conclusions of Proposition 3.12. By our choice of T_0 and (3-55) we have $t_1 \leq T_0$ for every $t_0 = \tau_n$. From the proof of Proposition 3.12 we recall that $\mu(t) \in \left[\frac{1}{2}, 2\right]$ on $[T_0, \tau_n]$, and the function

$$\xi(t) = -b(t) + \zeta(t)^{1/2}$$

satisfies for all $t \in [T_0, \tau_n]$

$$\zeta(t)^{1/2} \leq 2\xi(t) \leq 20\zeta(t)^{1/2}, \quad \xi'(t) \leq -\tfrac{1}{2}\zeta^{-1/2}(t)\xi(t). \tag{4-1}$$

Since $\tau_n \to T_+$, these same bounds hold on $[T_0, T_+)$. From (3-4), (3-29), (4-1) and the fact that $\boldsymbol{d}_+(\vec{\psi}(t)) \to 0$ as $t \to T_+$ we can conclude

$$\zeta(t) \to 0 \quad \text{and} \quad \xi(t) \to 0 \quad \text{as } t \to T_+.$$

From (4-1) we see that $\xi(t)$ is positive on $[T_0, T_+)$ and satisfies $\xi'(t) \leq -\frac{1}{4}$. Since $\xi(t) \to 0$ as $t \to T_+$, we conclude that $T_+ < \infty$, which proves finite time blow-up.

We now turn to the asymptotics of the scales. The estimates (4-1) and (3-9) imply that

$$\int_{T_0}^{T_+} |\mu'|\,dt \lesssim \int_{T_0}^{T_+} \zeta(t)^{1/2}\,dt \lesssim \int_{T_0}^{T_+} \xi(t)\,dt \lesssim \int_{T_0}^{T_+} \zeta(t)^{1/2}(-\xi'(t))\,dt \lesssim \int_{T_0}^{T_+} (-\xi'(t))\,dt \lesssim 1.$$

Thus, $\mu(t)$ converges to some $\mu_0 \in \left[\frac{1}{2}, 2\right]$. For the decay of $\lambda(t)$, we first recall that by (4-1) we have $\xi'(t) \lesssim -1$. By Proposition 3.3, we see that

$$|\xi'(t)| \lesssim |b'(t)| + \zeta^{-1/2}|\zeta'(t)| \lesssim 1.$$

Thus, there exists $C > 0$ such that

$$-C \leq \xi'(t) \leq -\frac{1}{C} \quad \text{for all } t \in [T_0, T_+),$$

which implies

$$\frac{1}{C}(T_+ - t) \leq \xi(t) \leq C(T_+ - t) \quad \text{for all } t \in [T_0, T_+).$$

Since $\xi(t) \sim \zeta(t)^{1/2} \sim [\lambda(t)|\log \lambda(t)|]^{1/2}$ on $[T_0, T_+)$, we conclude that

$$\lambda(t)|\log \lambda(t)| \sim (T_+ - t)^2 \quad \text{as } t \to T_+$$

as desired.

Finally, we show that $\vec{\psi}$ scatters backward in time. Suppose not. Then $-\infty < T_- < T_+ < \infty$, and $\int_{T_-}^{T_+} \sqrt{\boldsymbol{d}(\vec{\psi}(t))}\, dt < \infty$ by what we have shown up to this point. The virial identity (2-2), (2-4) and the fact that $\boldsymbol{d}(\vec{\psi}(t)) \to 0$ as $t \to T_\pm$ imply that

$$\int_{T_-}^{T_+} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \le \int_{T_-}^{T_+} |\Omega_R(\psi(t))| \, dt \quad \text{for all } R > 0.$$

For all $t \in (T_-, T_+)$, we have $|\Omega_R(\vec{\psi}(t))| \le C_0 \sqrt{\boldsymbol{d}(\vec{\psi}(t))} \in L^1(T_-, T_+)$ and $\lim_{R\to\infty} \Omega_R(\vec{\psi}(t)) = 0$. Thus, by the dominated convergence theorem

$$\int_{T_-}^{T_+} \|\partial_t \psi(t)\|_{L^2}^2 \, dt = 0.$$

We conclude that $\vec{\psi}$ is a degree-0 harmonic map, i.e., $\vec{\psi} = (0, 0)$. This contradicts $\mathcal{E}(\vec{\psi}) = 8\pi$ and finishes the proof. □

5. Construction of a minimal blow-up solution

5A. ***Proof of Theorem 1.6.*** Let $T > 0$ be small (to be determined later). We define a function $\ell(t) : [0, T) \to [0, \infty)$ implicitly by the relation

$$\ell(t)|\log \ell(t)| = 2t^2, \quad t \in (0, T),$$

with $\ell(0) = 0$. By elementary calculus it is easy to see that $\ell \in C^\infty(0, T)$, ℓ is increasing on $[0, T)$ and

$$\ell'(t)|\log \ell(t)| = 4t[1 + O(|\log \ell(t)|^{-1})].$$

In particular, this implies that

$$\frac{\ell(t)}{\ell'(t)} = \tfrac{1}{2} t[1 + O(|\log \ell(t)|^{-1})], \tag{5-1}$$

$$\frac{\ell(t)}{(\ell'(t))^2 |\log \ell(t)|} = \tfrac{1}{8} + O(|\log \ell(t)|^{-1}). \tag{5-2}$$

Let t_n be a sequence in $(0, T)$ which is monotonically decreasing to 0. We define a sequence of initial data at time $t = t_n$ via

$$\psi_{0,n} := Q_{\ell(t_n)} - Q,$$
$$\psi_{1,n} := -\ell'(t_n) \Lambda Q_{\underline{\ell(t_n)}} \chi_{\sqrt{R_n \ell(t_n)}},$$

where χ is now a sharp cutoff, $\chi(r) = 1$ for $0 \le r \le 1$ and $\chi(r) = 0$ for $r > 1$, and $R_n > 0$ is chosen so that

$$\mathcal{E}(\psi_{0,n}, \psi_{1,n}) = 2\mathcal{E}(\vec{Q}).$$

We first show that R_n exists and that $R_n + R_n^{-1}$ is bounded.

Lemma 5.1. *For $T > 0$ sufficiently small, for all n there exists $R_n > 0$ such that the pair of initial data $(\psi_{0,n}, \psi_{1,n})$ defined above satisfies $\mathcal{E}(\psi_{0,n}, \psi_{1,n}) = 2\mathcal{E}(Q)$. Moreover, there exists $R > 0$ such that*

$$\frac{1}{R} \le R_n \le R.$$

Proof. We expand the nonlinear energy and obtain (see Section 3 of [Jendrej and Lawrie 2018])

$$\begin{aligned} 2\mathcal{E}(Q) &= \mathcal{E}(\psi_{0,n}, \psi_{1,n}) \\ &= 2\mathcal{E}(Q) + \int_0^\infty \psi_{1,n}^2 r\,dr - 4\int_0^\infty \Lambda Q_{\ell(t_n)}(\Lambda Q)^3 \frac{dr}{r} + 2\int_0^\infty (\Lambda Q_{\ell(t_n)})^2(\Lambda Q)^2 r \frac{dr}{r}, \end{aligned}$$

so that

$$\int_0^\infty \psi_{1,n}^2 r\,dr = 4\int_0^\infty \Lambda Q_{\ell(t_n)}(\Lambda Q)^3 \frac{dr}{r} - 2\int_0^\infty (\Lambda Q_{\ell(t_n)})^2(\Lambda Q)^2 \frac{dr}{r}. \tag{5-3}$$

By a change of variables, the left side of (5-3) is readily computed to be

$$\int_0^\infty \psi_{1,n}^2 r\,dr = (\ell'(t_n))^2 \int_0^{\sqrt{R_n/\lambda_n}} |\Lambda Q|^2 r\,dr = 2(\ell'(t_n))^2\Big[\log\Big(1+\frac{R_n}{\ell(t_n)}\Big) + \frac{1}{1+R_n/\ell(t_n)} - 1\Big]. \tag{5-4}$$

For the right side of (5-3), we first consider the expression

$$\begin{aligned} 4\int_0^\infty \Lambda Q_\sigma(\Lambda Q)^3 \frac{dr}{r} &= 64\sigma \int_0^\infty \frac{r^3}{(\sigma^2+r^2)(1+r^2)^3}\,dr \\ &= 64\sigma \int_0^\sigma \frac{r^3}{(\sigma^2+r^2)(1+r^2)^3}\,dr + 64\sigma \int_\sigma^\infty \frac{r^3}{(\sigma^2+r^2)(1+r^2)^3}\,dr, \end{aligned}$$

where for brevity we have set $\sigma = \ell(t_n)$. Now

$$\int_0^\sigma \frac{r^3}{(\sigma^2+r^2)(1+r^2)^3}\,dr \lesssim \int_0^\sigma r\,dr \lesssim \sigma^2.$$

Since

$$\frac{1}{\sigma^2+r^2} = \frac{1}{r^2} + \frac{\sigma^2}{(\sigma^2+r^2)r^2},$$

we have

$$\begin{aligned} \int_\sigma^\infty \frac{r^3}{(\sigma^2+r^2)(1+r^2)^3}\,dr &= \int_\sigma^\infty \frac{r}{(1+r^2)^3}\,dr + \sigma^2\int_\sigma^\infty \frac{r}{(1+r^2)^3(\sigma^2+r^2)}\,dr \\ &= \tfrac{1}{4} + O(\sigma^2|\log\sigma|). \end{aligned}$$

We conclude that

$$4\int_0^\infty \Lambda Q_{\ell(t_n)}(\Lambda Q)^3 \frac{dr}{r} = 16\ell(t_n)[1 + O(\ell(t_n)^2|\log\ell(t_n)|)]. \tag{5-5}$$

By a similar argument we also obtain

$$\int_0^\infty (\Lambda Q_{\ell(t_n)})^2(\Lambda Q)^2 \frac{dr}{r} \lesssim \ell(t_n)^2|\log\ell(t_n)|. \tag{5-6}$$

Combining (5-3), (5-4), (5-5) and (5-6) we obtain

$$\log\Big(1+\frac{R_n}{\ell(t_n)}\Big) + \frac{1}{1+R_n/\ell(t_n)} - 1 = \frac{8\ell(t_n)}{(\ell'(t_n))^2}[1 + O(\ell(t_n)|\log\ell(t_n)|)].$$

Thus by (5-2)

$$\log\left(1+\frac{R_n}{\ell(t_n)}\right)+\frac{1}{1+R_n/\ell(t_n)}-1=|\log\ell(t_n)|[1+O(|\log\ell(t_n)|^{-1})]. \tag{5-7}$$

The function $f(x)=\log(1+x)+1/(1+x)-1$ is continuous, is equal to 0 when $x=0$ and tends to ∞ as $x\to\infty$. Thus, by the intermediate value theorem and as long as T is sufficiently small, there exist R_n satisfying (5-7) for all n. From (5-7) we see that $R_n/\ell(t_n)\to\infty$ as $n\to 0$. Rearranging the previous expression yields

$$\log R_n = 1-\log\left(1+\frac{\ell(t_n)}{R_n}\right)-\frac{1}{1+R_n/\ell(t_n)}+O(1).$$

Since $R_n/\ell(t_n)\to\infty$, the right side of the previous expression is bounded. □

Let $\vec{\psi}_n(t)$ denote the solution to (1-4) with initial data $\vec{\psi}_n(t_n)=(\psi_{0,n},\psi_{1,n})$. We remark that the previous computations yield

$$\|\psi_{1,n}\|_{L^2}^2=16\ell(t_n)[1+O(\ell(t_n)^2|\log\ell(t_n)|)]. \tag{5-8}$$

Therefore, as long as $T>0$ is small, for all t in a neighborhood of t_n the modulation parameters $\lambda_n(t)$ and $\mu_n(t)$ are well-defined for $\vec{\psi}_n(t)$ and

$$\lambda_n(t_n)=\ell(t_n),\quad \mu_n(t_n)=1.$$

If we set $g_n(t):=\psi_n(t)-(Q_{\lambda_n(t)}-Q_{\mu_n(t)})$ and $\dot g_n(t)=\partial_t\psi_n(t)$, then

$$g_n(t_n)=0,\quad \dot g_n(t_n)=-\ell'(t_n)\Lambda Q_{\underline{\ell(t_n)}}\chi_{\sqrt{R_n\ell(t_n)}}.$$

Let $\zeta_n(t)$ and $b_n(t)$ be defined as in (3-26), (3-25) for each $\vec{\psi}_n$; i.e.,

$$\begin{aligned}\zeta_n(t)&:=2\lambda_n(t)|\log(\lambda_n(t)/\mu_n(t))|-\langle\chi_{M\sqrt{\lambda_n(t)\mu_n(t)}}\Lambda Q_{\underline{\lambda_n(t)}}\mid g_n(t)\rangle,\\ b_n(t)&:=-\langle\chi_{M\sqrt{\lambda_n(t)\mu_n(t)}}\Lambda Q_{\underline{\lambda_n(t)}}\mid \dot g_n(t)\rangle-\langle\dot g_n(t)\mid\mathcal{A}_0(\lambda_n(t))g_n(t)\rangle.\end{aligned}$$

Corollary 5.2. *As long $M>0$ is sufficiently large we have*

$$b_n(t_n)=8t_n[1+O(|\log\ell(t_n)|^{-1})].$$

Proof. Let M^2 be larger than R given by Lemma 5.1. Then by (5-8) and (5-1) we have

$$\begin{aligned}b_n(t_n)&=-\langle\chi_{M\sqrt{\ell_n(t_n)}}\Lambda Q_{\underline{\ell(t_n)}}\mid\dot g_n(t_n)\rangle=\frac{1}{\ell'(t_n)}\|\psi_{1,n}(t_n)\|_{L^2}^2=\frac{16\ell(t_n)}{\ell'(t_n)}[1+O(\ell(t_n)^2|\log\ell(t_n)|)]\\ &=2\ell'(t_n)|\log\ell(t_n)|[1+O(|\log\ell(t_n)|^{-1})]=8t_n[1+O(|\log\ell(t_n)|^{-1})].\end{aligned}$$ □

Let $L=L_0>0$, $M=M_0>0$ and $\eta_1>0$ be chosen so that the conclusions of Proposition 3.9 hold with $\delta=\frac{1}{2018}$ and so that the conclusion of Corollary 5.2 holds. Let

$$T_n'=\sup\left\{t\in[t_n,T]\mid\vec{\psi}_n(s)\text{ exists, }\boldsymbol{d}_+(\vec{\psi}_n(s)<\eta_1,\text{ and }\mu_n(s)\in\left(\tfrac12,2\right)\text{ for all }s\in[t_n,t]\right\}.$$

We will show that $T_n'=T$ as long as T is sufficiently small.

Let $t \in [t_n, T_n']$. By (3-28), (3-29) and our assumption on $\mu_n(t)$

$$\zeta_n(t) = \zeta_n(t_n) + \int_{t_n}^{t} \zeta_n'(s)\, ds \le \zeta_n(t_n) + \int_{t_n}^{t} [|b_n(s)| + \zeta_n(s)^{1/2}]\, ds \le \zeta_n(t_n) + 6 \int_{t_n}^{t} \zeta_n(s)^{1/2}\, ds.$$

Thus,

$$\zeta_n(t) \le 2\zeta_n(t_n) + 36(t - t_n)^2.$$

Since $\zeta_n(t_n) = 2\ell(t_n)|\log \ell(t_n)| = 4t_n^2$, we conclude that

$$\zeta_n(t) \le 148t^2. \tag{5-9}$$

Then by (3-27)

$$\lambda_n(t)|\log \lambda_n(t)| \le 75t^2. \tag{5-10}$$

We now consider $\mu_n(t)$. By the fundamental theorem of calculus, (3-9), (3-27) and (5-9) there exists an absolute constant $\beta > 0$ such that

$$|\mu_n(t) - 1| \le \beta t^2.$$

By (3-3), (5-10) and our assumption on μ_n there exists a constant $\alpha > 0$ such that

$$\|\vec{\psi}_n(t) - (\vec{Q}_{\lambda_n(t)} - Q_{\mu_n(t)})\|_{\mathcal{H}_0}^2 \le \alpha t^2. \tag{5-11}$$

In summary, we have shown that

$$\lambda_n(t)|\log \lambda_n(t)| \le 75t^2, \quad |\mu_n(t) - 1| \le \beta t^2, \quad \boldsymbol{d}_+(\vec{\psi}_n(t)) \le (\alpha + 150)t^2.$$

By a continuity argument, it follows that $T_n' = T$ provided that $\vec{\psi}_n(t)$ is defined on $[t_n, T]$. We now prove this fact.

Let $t \in [t_n, T_n']$. By Corollary 5.2 and (3-31) we have

$$b_n(t) \ge \tfrac{1}{2}\big(8 - \tfrac{1}{2020}\big)(t - t_n) + 8t_n[1 + O(|\log \ell(t_n)|^{-1})] \ge 3(t - t_n) + 5t_n \ge 3t. \tag{5-12}$$

By (3-28), (5-12) and (5-9) we have

$$\zeta_n'(t) \ge b_n(t) - \tfrac{2}{2020}\zeta_n^{1/2}(t) \ge 3t - \tfrac{2\sqrt{148}}{2020}t \ge 2t.$$

By the fundamental theorem of calculus we conclude that

$$\zeta_n(t) \ge \zeta_n(t_n) + t^2 - t_n^2 = 4t_n^2 + t^2 - t_n^2 \ge t^2.$$

By (3-27), the previous inequality implies that

$$\lambda_n(t)|\log \lambda_n(t)| \ge \tfrac{1}{3}t^2. \tag{5-13}$$

The estimates (5-13), (5-10) and (5-11) imply

$$\inf_{\substack{\mu \in [1/2, 2] \\ \lambda|\log \lambda| \in [t^2/3, 75t^2]}} \|\vec{\psi}_n(t) - (Q_\lambda - Q_\mu)\|_{\mathcal{H}_0}^2 \le \alpha t^2 \tag{5-14}$$

on $[t_n, T_n']$. By Corollary A.4 of [Jendrej 2019] we conclude that the interval of existence of $\vec{\psi}_n$ strictly includes $[t_n, T_n']$ as long as T is small. Thus, we have proved that $T_n' = T$.

The bound (5-14) also implies that we may pass to a weak limit and obtain our desired blow-up solution. Indeed, for any $T_0 < T$

$$\inf_{\substack{\mu\in[1/2,2]\\ \lambda|\log\lambda|\in[T_0^2/3,75T^2]}} \|\vec{\psi}_n(t) - (Q_\lambda - Q_\mu)\|_{\mathcal{H}_0}^2 \le \alpha T^2 \quad \text{for all } t \in [T_0, T], \text{ for all } n.$$

By Corollary A.6 of [Jendrej 2019] we can conclude, after shrinking T and extracting subsequences if necessary, there exists a solution $\vec{\psi}_c(t)$ defined on $(0, T]$ such that $\vec{\psi}_n(t) \rightharpoonup_n \vec{\psi}_c(t)$ for all $t \in (0, T]$. By weak convergence and (5-14)

$$\inf_{\substack{\mu\in[1/2,2]\\ \lambda|\log\lambda|\in[t^2/3,75t^2]}} \|\vec{\psi}(t) - (Q_\lambda - Q_\mu)\|_{\mathcal{H}_0}^2 \le \alpha t^2.$$

Thus, $\vec{\psi}_c$ is the desired solution with blow-up time $T_- = 0$. □

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