Odd-frequency pairing and time-reversal symmetry breaking for repulsive interactions

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We study the pairing of fermions by an interaction consisting of a Hubbard repulsion, mimicking a screened Coulomb potential, and a dynamical phonon-mediated attraction. For such interaction, the gap equation allows even- and odd-frequency solutions Δ_e and Δ_o. We show that odd-frequency pairing does not develop within the Eliashberg approximation due to over-critical pair breaking from the self-energy. When vertex corrections are included, the pairing interaction gets stronger, and Δ_o can develop. We argue that even in this case keeping the self-energy is still a must as it cancels out the thermal piece in the gap equation. We further argue that Δ_o is not affected by Hubbard repulsion and for strong repulsion is comparable to a reduced Δ_e. The resulting superconducting state is a superposition $\Delta_e \pm i\Delta_o$, which spontaneously breaks the time-reversal symmetry, despite that the pairing symmetry is an ordinary s wave.

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I. INTRODUCTION

When two electrons in a superconductor form a Cooper pair with gap function $\Delta(\boldsymbol{r}, t)$, they must obey the Pauli principle. This iron fact enables a systematic symmetry classification of superconducting order parameters. Interestingly, the Pauli principle can be obeyed even when $\Delta(\boldsymbol{r}, t)$ is odd under time exchange $\Delta_o(\boldsymbol{r}, t) = -\Delta_o(\boldsymbol{r}, -t)$ [1–5]. The Fourier transform of such a gap function is an odd function of frequency along the Matsubara axis, where $\Delta_o(\boldsymbol{k}, \omega_m) = -\Delta_o(\boldsymbol{k}, -\omega_m)$ can be made real by a proper choice of the phase [3]. This implies that at $T = 0$, $\Delta_o(\boldsymbol{k}, 0) = 0$. Odd-frequency (OF) superconductivity has a number of profound physical consequences, e.g., it leads to an s-wave superconductivity with no gap in the density of states at $T = 0$ without magnetic impurities. OF pairing has been argued to develop in a two-channel Kondo model [6] and is particularly favorable in disordered electron liquids [7,8] and heterostructures [9,10], where it was argued to be observed in experiments [11,12]. It also has a rich interplay with topological effects [13]. OF pairing can also be induced by an external magnetic field [14–18].

However, in a regular bulk superconductor at zero field, OF superconductivity remains mostly elusive despite being intensively searched for over the last three decades. From a theoretical perspective, there are three obstacles to OF superconductivity. First, an OF solution $\Delta_o(\omega_m)$ does not emerge at weak coupling because the vanishing of $\Delta_o(0)$ implies that there is no enhancement of the pairing susceptibility by the Cooper logarithm. Second, OF pairing is eliminated by the development of even-frequency (EF) superconductivity, which reduces the kernel in the OF pairing channel [3]. Third, even if EF superconductivity does not develop for some reason, OF pairing is destroyed by pair-breaking effects from fermionic self-energy [19].

While the strong-coupling requirement cannot be avoided, we argue there are ways to overcome the two other obstacles. In this work we revisit them in the context of phonon-mediated superconductivity by studying a model of spin-1/2 fermions with an effective dynamical interaction

$$V(\Omega_m) \propto \lambda\left(f - \frac{\Omega_1^2}{\Omega_1^2 + \Omega_m^2}\right). \tag{1}$$

This model captures the competition between Hubbard repulsion, parametrized by f, and phonon-mediated attraction [20–28].

Phonon-mediated OF superconductivity has been analyzed before, most notably by Balatsky and co-workers (see [3] and references therein). They, however, focused on spin-singlet $\Delta_o(\boldsymbol{k}, \omega)$, which is odd in both $\boldsymbol{k}$ and ω. We follow the original proposal by Berezinskii [1] and consider OF superconductivity in the spin-triplet, spatially even channel $\Delta_o(\boldsymbol{k}, \omega) = \Delta_o(-\boldsymbol{k}, \omega)$, $\Delta_o(\boldsymbol{k}, \omega) = -\Delta_o(\boldsymbol{k}, -\omega)$. For such superconductivity the momentum dependence of the interaction is not relevant, and one can approximate the electron-phonon interaction by the interaction with an Einstein phonon, as in Eq. (1).

The model of Eq. (1) has been analyzed in Ref. [26] for particular λ and Λ and without including fermionic self-energy into consideration. We extend the analysis of [26] to arbitrary parameters and also analyze how the results change when the self-energy is included. A numerical analysis of the effects of fermionic self-energy and vertex corrections for phonon-mediated OF superconductivity has been recently performed in Ref. [29]. Where applicable, our results are in agreement with this work.

We first analyze OF superconductivity in the model of Eq. (1) on its own, assuming the EF superconductivity is not present. We prove a compact and fairly general theorem that

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OF superconductivity cannot develop within the canonical Eliashberg approximation, in which the interaction that gives rise to the pairing is exactly the same one that determines the fermionic self-energy. More specifically we show that thermal contributions from the static interaction to the pairing vertex $\Phi_o(\omega_m)$ and the self-energy $\Sigma(\omega_m)$ are exactly the same and cancel out in the gap equation for $\Delta_o(\omega_m) = \Phi_o(\omega_m)/[1 + \Sigma(\omega_m)/\omega_m]$, but the nonthermal piece is stronger for the self-energy, and this does not allow OF superconductivity to develop. We then go beyond this approximation, include vertex corrections, and show that the dressed interaction in the particle-particle channel becomes different from the one in the particle-hole channel. We show that in our model the pairing interaction gets relatively stronger and, as a result, OF superconductivity does develop at strong enough coupling. This is in agreement with Ref. [29], where OF superconductivity has been obtained numerically in "vertex-corrected" Eliashberg theory [30].

We compute the onset temperature for OF pairing T_c, and show that a nonzero OF condensate develops below T_c. We argue, however, that the self-energy cannot be neglected entirely as thermal contributions to the pairing vertex and the self-energy still cancel out even when vertex corrections are present.

We next analyze the interplay between OF and EF superconductivity. It has been shown previously [26,31] that static Hubbard repulsion suppresses EF superconductivity, but cancels out in the gap equation for OF pairing. Taken at face value, this would imply that OF superconductivity wins at sufficiently strong Hubbard repulsion. We show that the situation is more complex: While Hubbard repulsion is always detrimental to EF pairing, it does not eliminate it completely at strong coupling, which we need for OF pairing [by strong coupling we mean large overall coupling constant λ in Eq. (1)]. The argument is that the EF gap function $\Delta_e(\omega_m)$ changes sign between small and large frequencies, and for large enough λ the system chooses the position of the gap change to almost completely eliminate the effect of the Hubbard f. The outcome is that EF superconductivity survives even when f tends to infinity, and the corresponding T_c for EF pairing does not depend on f. We argue that in this situation the onset temperature for EF pairing is still higher, but the one for OF pairing becomes comparable when the Debye frequency Ω_1 in (1) becomes comparable to the Fermi energy, which acts as the UV cutoff for the model. This condition can be realized in low-density materials such as $SrTiO_3$ [23,25,32], Bi [33], and half-Heusler compounds [34]. When the onset temperatures for EF and OF pairing are comparable, both Δ_e and Δ_o become nonzero below a certain T. We show that the superconducting condensation energy is the largest when the phases of the two gap functions differ by $\pm\pi/2$, i.e., the order parameter is $\Delta_e \pm i\Delta_o$. Such an order spontaneously breaks time-reversal symmetry, even though the paring symmetry is still an ordinary s wave.

This paper is structured as follows: In Sec. II we introduce the model and the gap equation for the EF and OF components. In Sec. III A we momentarily neglect the self-energy and analyze the appearance of the OF solution for the gap once the coupling exceeds a certain threshold. Furthermore, we discuss the special role of the thermal term in the gap equation. In Sec. III B we include the self-energy, analyze the set of coupled Eliashberg equations for the pairing vertex and the self-energy and show that the thermal term gets canceled in the gap equation, and that there is no solution for a nonzero OF gap function. We then go beyond the Eliashberg approximation, evaluate vertex corrections at $T = 0$ and show that they increase the interaction in the particle-particle channel compared to that in the particle-hole channel and make OF superconductivity possible at strong enough coupling. In Sec. III C we analyze vertex corrections at a finite $T > 0$ and show that the thermal term in the gap equation still cancels out. In Sec. IV we discuss the interplay between EF and OF superconductivity: First, in Sec. IV A we analyze the suppression of the EF solution by a static repulsion and show that once the coupling λ exceeds a critical value, EF superconductivity exists for arbitrary strong Hubbard repulsion f. Then, in Sec. IV B we compare the couplings and critical temperatures for EF and OF solutions. Finally, in Sec. V we discuss the coexistence of EF and OF gap functions and argue that in such a state time-reversal invariance is spontaneously broken. Conclusion and outlook are presented in Sec. VI. Technical details are relegated to the Appendixes.

II. MODEL AND GAP EQUATION

We consider a system of spin-1/2 particles that interact via a dynamical interaction [20–25,27,28]:

$$V(\Omega_m) = \frac{2}{\rho}\chi(\Omega_m), \quad \chi(\Omega_m) = \lambda\left(f - \frac{\Omega_1^2}{\Omega_1^2 + \Omega_m^2}\right), \quad (2)$$

where Ω_m is a bosonic Matsubara frequency, ρ is the density of states, λ is a dimensionless coupling constant, f is a dimensionless measure of the Hubbard repulsion, and Ω_1 is a typical phonon energy scale, e.g., Debye energy. To distinguish between $T = 0$ and a finite T in the calculations on the Matsubara axis, we will label fermionic and bosonic frequencies as ω, Ω in the $T \to 0$ limit, and as ω_m, Ω_m at a finite T. We will measure all frequencies in units of Ω_1 and set $\Omega_1 \equiv 1$.

The dynamical interaction (2) gives rise to perturbations in both the particle-particle and particle-hole channels. Without vertex corrections [the terms that dress $V(\Omega_m)$], it yields a set of two coupled Eliashberg equations for the dynamical pairing vertex $\Phi(\omega)$ and the self-energy $\Sigma(\omega)$. These two equations can be combined into a closed-form equation for the dynamical gap function $\Delta(\omega) = \Phi(\omega)/[1 + \Sigma(\omega)/\omega]$ (see, e.g., Ref. [35] and Appendix B). We assume that the EF gap function Δ_e is spin singlet, and OF Δ_o is spin triplet and do not write spin indices explicitly. One can easily verify that the gap equation has the same form for both Δ_e and Δ_o. At $T = 0$,

$$\Delta(\omega) = -\int_{-\Lambda}^{\Lambda} d\omega' \chi(\omega - \omega') \frac{\Delta(\omega') - \Delta(\omega)\frac{\omega'}{\omega}}{\sqrt{(\omega')^2 + |\Delta(\omega')|^2}}, \quad (3)$$

where the second term in the numerator on the right-hand side (r.h.s.) is the contribution coming from the self-energy. The dimensionless UV cutoff Λ is generally of order of the Fermi energy in units of Ω_1. For most metals $\Lambda \gg 1$, but for low-density systems $\Lambda \simeq 1$. We will study both $\Lambda \gg 1$ and $\Lambda \simeq 1$. Since we consider a momentum-independent

interaction for simplicity, the resulting gap function $\Delta(\omega)$ has s-wave symmetry; more general interaction potentials can also lead to d-wave states, etc.

The gap equation can be reexpressed by introducing even and odd components: $\Delta_e(\omega) = \Delta_e(-\omega)$, $\Delta_o(\omega) = -\Delta_o(-\omega)$ [3,14,25,26]:

$$\Delta_{e/o}(\omega) = -\frac{1}{2}\int_{-\Lambda}^{\Lambda} d\omega' \frac{1}{\sqrt{(\omega')^2 + |\Delta_e(\omega') + \Delta_o(\omega')|^2}} \left(\chi_{e/o}(\omega,\omega')\Delta_{e/o}(\omega') - \chi_o(\omega,\omega')\Delta_{e/o}(\omega)\frac{\omega'}{\omega}\right), \tag{4}$$

$$\chi_e(\omega,\omega') = \chi(\omega-\omega') + \chi(\omega+\omega') = 2\lambda f - \frac{\lambda}{1+(\omega-\omega')^2} - \frac{\lambda}{1+(\omega+\omega')^2}, \tag{5}$$

$$\chi_o(\omega,\omega') = \chi(\omega-\omega') - \chi(\omega+\omega') = -\frac{4\lambda \times \omega\omega'}{[1+(\omega-\omega')^2][1+(\omega+\omega')^2]}. \tag{6}$$

Viewed separately, $\Delta_e(\omega)$ and $\Delta_o(\omega)$ can be made real. Observe that the Hubbard repulsion f is present in χ_e, but drops out of χ_o and that the self-energy contribution [the last term on the r.h.s. of Eq. (4)] contains χ_o for both $\Delta_{e/o}$, i.e., it does not contain f. This last result is a consequence of putting a symmetric cutoff on the fermionic ω' rather than on a bosonic $\omega' - \omega$, like in canonical Eliashberg theory. If we used the canonical expression, we would find that the Eliashberg self-energy does contain a term which depends on f. In the normal state this term is

$$\Sigma(\omega) = -\lambda f \int_{\Lambda-\omega}^{\Lambda+\omega} d\omega' \tag{7}$$

in the sign convention such that $G^{-1}(\boldsymbol{k},\omega) = i[\omega + \Sigma(\omega)] - \xi(\boldsymbol{k})$, with $\xi(\boldsymbol{k})$ the electron dispersion. For simplicity we work with a parabolic electron dispersion $\xi_{\boldsymbol{k}} = |\boldsymbol{k}|^2/2m - \mu$, which can be linearized near the Fermi surface. Equation (7) would yield $\Sigma(\omega) = -2\lambda f\omega$ and lead to the unphysical result that the quasiparticle residue $Z = 1/(1-2\lambda f) > 1$. We argue that this is an artifact. The issue can be traced back to the applicability of the canonical Eliashberg-type treatment of the self-energy for a model with frequency-independent Hubbard repulsion. We recall that the Eliashberg self-energy is obtained by integrating over $\xi(\boldsymbol{k})$ in infinite limits, *before* integrating over frequency. This procedure is well justified when the interaction drops starting from frequencies below the cutoff, which is the case for the electron-phonon term, but it is not justified for the frequency-independent Hubbard term. Indeed, if we compute the self-energy to first order in f by integrating over frequency first, we find that it is purely static and just renormalizes the chemical potential. The implication is that the dynamical $-2\lambda f\omega$ term in the self-energy is a parasitic one. It can be eliminated by either keeping the cutoff in the bosonic propagator, but adding a ghost counter-term to the Eliashberg self-energy, or by imposing a symmetric cutoff on the integral over fermionic ω' rather than bosonic $\omega' - \omega$. This is what we did in Eqs. (3) and (4). Either way, the Hubbard f term does not contribute to the dynamical self-energy, and the quasiparticle Z remains below 1. We verified that for the equation for the pairing vertex and for vertex corrections, which we consider later, the frequency integrals are UV convergent, and there is no difference between placing a symmetric cutoff on a fermionic or a bosonic frequency.

III. PROPERTIES OF THE GAP EQUATION FOR ODD-FREQUENCY PAIRING

In this section we solve the gap equation for Δ_o, assuming that Δ_e is absent.

A. Without fermionic self-energy

It is instructive to first solve the gap equation for $\Delta_o(\omega)$ without fermionic self-energy. We remind that the self-energy accounts for the second term in the numerator on the r.h.s. of Eq. (4). Neglecting this term, we obtain the truncated gap equation

$$\Delta_o(\omega) = -4\lambda\omega \int_0^{\Lambda} d\omega' \frac{\omega'\Delta_o(\omega')}{\sqrt{(\omega')^2 + \Delta_o^2(\omega')}} \times \frac{1}{[1+(\omega-\omega')^2][1+(\omega+\omega')^2]}. \tag{8}$$

This equation can be solved numerically by iteration. At small λ, there is no nonzero solution for $\Delta_o(\omega)$ because the pairing kernel is not logarithmically singular. However at large enough λ, exceeding the critical one λ_c^o, which depends on Λ, the solution does exist. We show λ_c^o as a function of Λ in Fig. 1. The critical coupling decreases with increasing Λ and saturates at $\lambda_c^o \simeq 0.88$ at $\Lambda \to \infty$.

In Fig. 2 we show Δ_o for $\Lambda = 10$ and representative $\lambda = 1.1 > \lambda_c^o$. We see that $\Delta_o(\omega)$ scales as ω at small frequencies, passes through a maximum at a higher ω, and at even higher ω decreases as $1/\omega^3$. This last behavior can be obtained

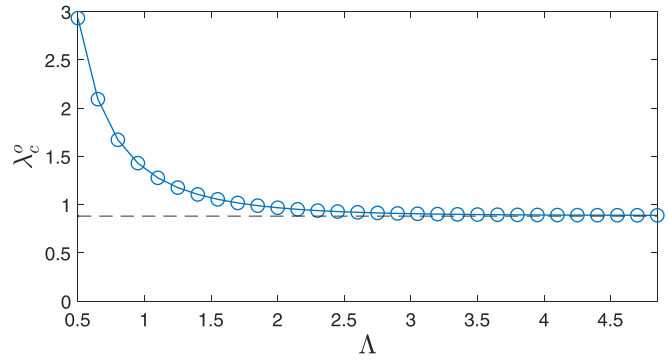

FIG. 1. Critical value of the coupling λ_c^o as a function of the cutoff Λ. Odd-frequency superconductivity at $T = 0$ develops when $\lambda > \lambda_c^o$.

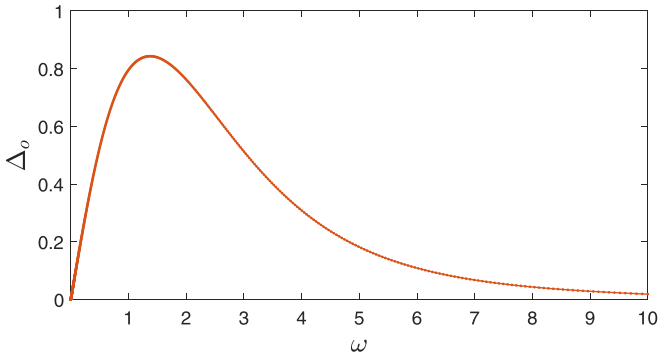

FIG. 2. Odd-frequency gap function at $T = 0$ for $\lambda = 1.1$, $\Lambda = 10$, obtained by solving the gap equation without fermionic self-energy.

analytically by extracting ω from the kernel on the r.h.s. of (8) and verifying *a posteriori* that the remaining integral over ω' converges.

We next move to finite T. To obtain the critical temperature T_c^o for OF pairing, it is convenient to treat the linearized gap equation as a matrix problem. A straightforward discretization of Eq. (8) leads to a matrix equation

$$\Delta_o(\omega_m) = \sum_{\omega'_m>0} K(\omega_m, \omega'_m)\Delta_o(\omega'_m), \tag{9}$$

where $\omega_m = (2m+1)\pi T$ are (positive) Matsubara frequencies and K is the matrix kernel

$$K(\omega_m, \omega'_m) = T\frac{8\pi\lambda\omega_m}{[1+(\omega'_m-\omega_m)^2][1+(\omega'_m+\omega_m)^2]}. \tag{10}$$

At the critical temperature T_c^o, the largest eigenvalue $\kappa(T)$ of the matrix K is equal to 1. It would be natural to expect that $\kappa(T)$ is a decreasing function of T, such that $\kappa(T) < 1$ at $T > T_c^o$ and $\kappa(T) > 1$ at $T < T_c^o$. However, the numerical analysis yields a different result: $\kappa(T)$ increases with T (the green line in Fig. 3). This leads to a quite exotic behavior: For

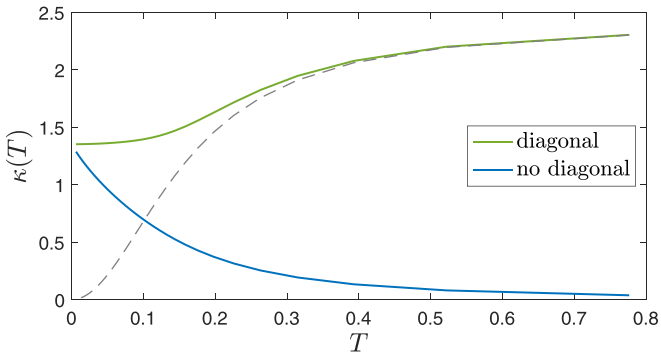

FIG. 3. Scaling of the largest eigenvalue $\kappa(T)$ of the gap equation kernel K, in the OF case for parameters $\Lambda = 100$, $\lambda = 1.2$. The green line shows $\kappa(T)$ when the diagonal elements of K are included, and the dashed gray line corresponds to the eigenvalue estimate of Eq. (11). The blue line shows $\kappa(T)$ when the diagonal elements are excluded. As expected, the green and blue lines approach the same limit as $T \to 0$, because the weight of the diagonal entries vanishes in this limit.

$\lambda > \lambda_c^o$, OF superconductivity exists at all T, and for $\lambda < \lambda_c^o$, it emerges at some finite T_c^o and exists at larger temperatures. For the model of Eq. (2) this behavior was first obtained in Refs. [26,36]. A similar behavior for spin-singlet superconductivity with gap function odd in both $\boldsymbol{k}$ and ω was obtained in the pioneering work of Ref. [2]. It was argued in [26] that $\kappa(T)$ is nonmonotonic and eventually drops at high enough T. This gives rise to a finite T_c^o for $\lambda > \lambda_c^o$, but to nonmonotonic temperature variation of Δ_o below this temperature, and to reentrant OF superconductivity at $\lambda < \lambda_c^o$, which exists in the window $T_c^o < T < T_{c,2}^o$ [for both end points, $\kappa(T) = 1$]. This behavior is reproduced in our model if we impose a UV cutoff on momentum integration, like it was done in [26].

The exotic behavior of $\kappa(T)$ is easy to understand from Eq. (10): because typical ω_m are of order T, off-diagonal elements of K scale as $1/T^2$, while the diagonal elements $K(\omega_m, \omega_m)$, which are thermal contributions from the static interaction $V(0)$, saturate at a finite value at large T. As a result, the largest eigenvalue of K at large enough T is determined by the largest diagonal element [26], the one at $\omega_m = \omega'_m = \pi T$:

$$T \gg 1: \quad \kappa(T) \to K(\pi T, \pi T) = \frac{8\lambda(\pi T)^2}{1+4(\pi T)^2}. \tag{11}$$

We present a numerical check of this behavior in Fig. 3. The gray line in this figure is $\kappa(T)$ obtained by keeping only the diagonal terms in $K(\omega_m, \omega'_m)$ [this is $\kappa(T)$ from Eq. (11)], the green line is the full $\kappa(T)$. We see that the two expressions coincide at large T. We argue below that this exotic behavior is an artifact of neglecting the self-energy. Indeed, one can see that in the full Eliashberg gap equation (3), which includes the self-energy, the thermal contribution with $\omega_m = \omega'_m$ cancels out. We show in Sec. III C that this cancellation holds beyond the Eliashberg approximation. This cancellation has a drastic effect on where OF superconductivity develops in the (λ, T) phase diagram, which we discuss in detail in the next section. As a preview, in Fig. 3 by a blue line we show the scaling of $\kappa(T)$ for $K(\omega_m, \omega'_m)$ still given by (10), but with diagonal terms set to zero. We see that $\kappa(T)$ has a conventional behavior: it decreases with increasing T. For such $\kappa(T)$, there is no OF superconductivity if $\lambda < \lambda_c^o$, and if $\lambda > \lambda_c^o$, the gap $\Delta_o(\omega_m)$ is nonzero for $T < T_c^o$.

B. Role of self-energy at $T = 0$

We now discuss the self-energy in more detail. Our first point is that when this term is kept, there is no nonzero solution of the full Eliashberg gap equation (4) for $\Delta_o(\omega_m)$ at any T, including $T = 0$. For interaction with acoustic phonons, this was first observed in Ref. [19]. In our case of purely dynamical interaction with an Einstein boson, we can prove this explicitly. Namely, we argue that if $\Delta_o(\omega_m)$ emerges at some T_c^o, the corresponding linearized gap equation at T immediately below T_c^o has to satisfy the inequality

$$\max_n \left|\frac{\Delta_o(\omega_n)}{\omega_n}\right| \leqslant \frac{\sum_m K^T_{n,m}}{1+\sum_m K^T_{n,m}}\left(\max_l \left|\frac{\Delta_o(\omega_l)}{\omega_l}\right|\right), \tag{12}$$

where $K^T_{n,m}$ is the transpose of the OF matrix kernel in Eq. (10) (see Appendix B for details). Because all components of K are positive, $\sum_m K^T_{n,m}/(1+\sum_m K^T_{n,m}) < 1$, hence a nonzero

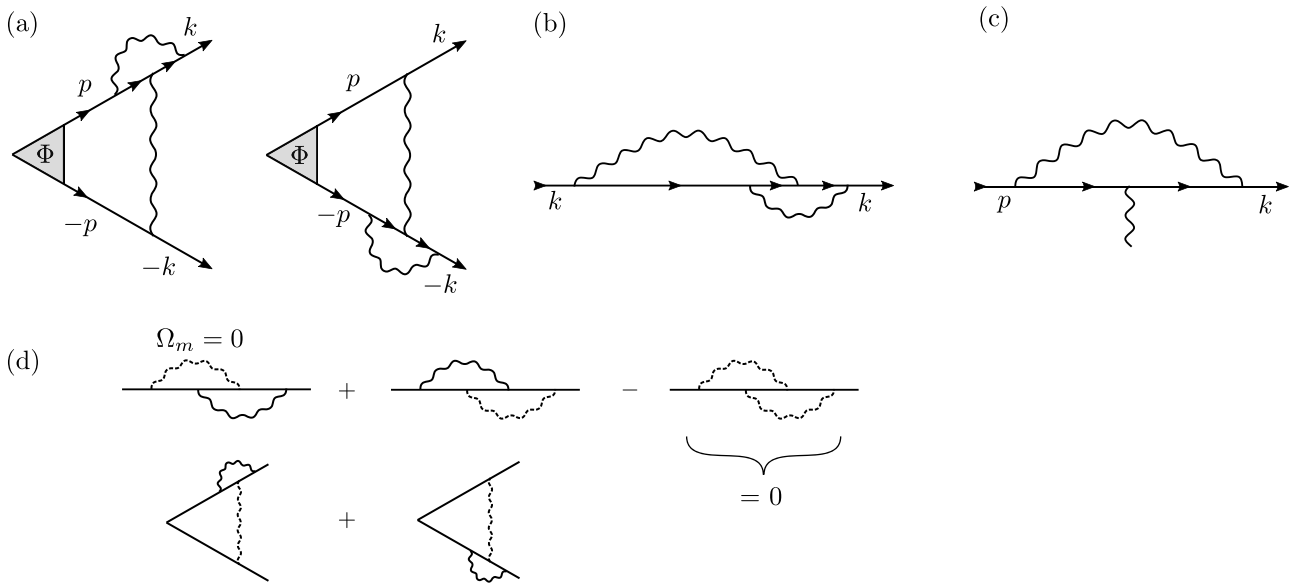

FIG. 4. (a) Relevant corrections for the pairing vertex Φ. Straight lines represent full Green's functions (including the self-energy), wiggly lines the interaction V. Four-momentum notation is used: $k = (\omega, \boldsymbol{k})$, $p = (\omega', \boldsymbol{p})$. (b) Relevant correction for the self-energy. (c) The vertex correction piece. (d) Relevant contributions with zero frequency transfer $\Omega_m = 0$, represented by dashed interaction lines. Full interaction lines imply a summation over all frequencies Ω_m.

$\Delta_o(\omega_m)$ has to satisfy the strict inequality

$$\max_n \left|\frac{\Delta_o(\omega_n)}{\omega_n}\right| < \max_n \left|\frac{\Delta_o(\omega_n)}{\omega_n}\right| \quad \text{if} \quad \Delta_o \neq 0, \tag{13}$$

which is impossible.

When vertex corrections are included, the interplay between the attraction in the odd-frequency pairing channel and pair breaking by the self-energy becomes more nuanced as the interaction in the particle-particle channel χ_{pp}, and the one in the particle-hole channel χ_{ph}, generally become different. Keeping the two interactions as separate variables in the gap equation at $T = 0$, we obtain the gap equation in the form

$$\Delta_o(\omega) = -\int d\omega' \chi_{pp}(\omega - \omega') \frac{\Delta_o(\omega') - \Delta_o(\omega)\frac{\omega'}{\omega}\alpha(\omega - \omega')}{\sqrt{\Delta_o^2(\omega') + (\omega')^2}}, \tag{14}$$

where

$$\alpha(\omega - \omega') = \frac{\chi_{ph}(\omega - \omega')}{\chi_{pp}(\omega - \omega')}\,. \tag{15}$$

For our purposes, this equation has to be projected to an odd-frequency channel.

That vertex corrections make χ_{pp} and χ_{ph} nonequivalent can be seen already in perturbation theory, by collecting vertex corrections for these two interactions to leading order in λ. We present the diagrams in Fig. 4, and discuss computational details in Appendix B. We emphasize that the result for the vertex correction diagram does not depend on whether we impose a symmetric cutoff on an internal fermionic frequency or on a bosonic frequency in the evaluation.

The key point is that there are two vertex correction diagrams for the pairing vertex but only one for the self-energy. In both cases, the integration over two fermionic and one bosonic propagator in the vertex correction piece in Fig. 4(c) yields $2\lambda f$. Then under vertex renormalization

$$\chi_{pp} \to \chi_{pp}(1 + 4\lambda f), \quad \chi_{ph} \to \chi_{ph}(1 + 2\lambda f). \tag{16}$$

Hence

$$\alpha = \frac{1 + 2\lambda f}{1 + 4\lambda f} < 1\,. \tag{17}$$

To simplify the analysis, below we treat $\alpha < 1$ as a phenomenological parameter. In Fig. 5 we show the behavior of the critical OF coupling λ_c^o as a function of α, in the limit of large Λ. At $\alpha = 1$ (the original model with no vertex corrections), $\lambda_c^o = \infty$, which implies that there is no OF pairing for any value of λ, as we already discussed. However, once α becomes smaller than 1, λ_c^o becomes finite, i.e., for strong enough λ, OF pairing does develop. At $\alpha \to 0$, λ_c^o approaches 0.88, as expected. The fact that OF pairing develops when vertex corrections are included has also been observed in a recent numerical work [29].

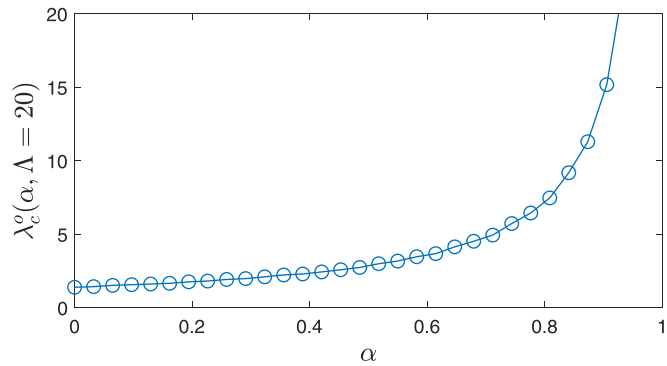

FIG. 5. Critical value of the coupling λ_c^o for large $\Lambda = 20$ as a function of the self-energy parameter α. When $\alpha = 1$, λ_c^o diverges and OF superconductivity cannot be realized.

We note in passing that for quantum-critical OF pairing by a gapless boson (the limit $\Omega_1 \to 0$, $\lambda \to \infty$, $\lambda\Omega_1^2$ tends to a constant), the system is at the boundary towards OF pairing already without vertex corrections. In this case OF superconductivity emerges already at infinitesimally small $1-\alpha$ [37].

Also, the authors of Ref. [19] argued that the effect of vertex corrections can be modeled by adding a spin-dependent component of the interaction that acts differently in the particle-particle and particle-hole channels. Accordingly, our results can be also modeled by introducing an extra spin-spin component of the interaction.

C. Role of self-energy at $T > 0$: Cancellation of thermal terms in the gap equation

So far we discussed vertex corrections at $T = 0$. The situation at a finite T is a bit more tricky. Namely, at a finite T we have to distinguish between vertex renormalization of the interaction at a finite frequency transfer Ω_m and at zero frequency $\Omega_m = 0$. The contributions from the latter to the pairing vertex and the self-energy are associated with thermal fluctuations. Vertex renormalizations to $\chi_{pp}(\Omega_m)$ and $\chi_{ph}(\Omega_m)$ at a finite Ω_m are essentially the same as at $T = 0$, and the interplay between vertex corrections to $\chi_{pp}(\Omega_m)$ and $\chi_{ph}(\Omega_m)$ is governed by $\alpha < 1$. For interactions with $\Omega_m = 0$, computations to the leading (first) order in λ yield a different result: there is no factor of 2 difference between vertex corrections to $\chi_{pp}(0)$ and $\chi_{ph}(0)$. To see this, in Fig. 4(d) we pictorially single out the interactions with $\Omega_m = 0$ by dashed interaction lines. For the pairing vertex at $\Omega_m = 0$, there are two different vertex correction diagrams, as before, hence there is an overall factor of 2. For the self-energy, there is only one diagram, but there are two choices to select which of the two interaction lines carries $\Omega_m = 0$ and hence is associated with $\chi_{ph}(0)$. This gives an extra factor of 2. Then the vertex corrections to $\chi_{pp}(0)$ and $\chi_{ph}(0)$ are the same. As a result, the thermal piece in the gap equation cancels out even in the presence of vertex corrections. We conjecture that this holds beyond first order in λ. We recall that this cancellation eliminated a would-be highly exotic behavior, in which the coupling constant for OF pairing increases with increasing T.

To summarize, inclusion of vertex corrections with proper treatment of the thermal terms makes OF superconductivity possible. However, the condition $\lambda > \lambda_c^o = O(1)$ is required, and λ_c^o is large if the vertex corrections are weak. The conditions for OF pairing are easier to fulfill in a quantum-critical regime, where the coupling is large.

IV. INTERPLAY BETWEEN EF AND OF PAIRING AT STRONG REPULSION

A. EF superconductivity and its suppression by static repulsion

As we said in the Introduction, a particle-particle interaction of the form of Eq. (2) also allows for a conventional superconductivity with even-frequency gap function $\Delta_e(\omega)$. Below we set $\Delta_e(\omega)$ to be real (we recall that we label by ω a continuous Matsubara frequency at $T = 0$). For a nonzero Hubbard repulsion f and $\Lambda \gg 1$, such $\Delta_e(\omega)$ necessarily has a node [22,38]. A representative $\Delta_e(\omega)$ at $T = 0$ is shown in Fig. 6(b).

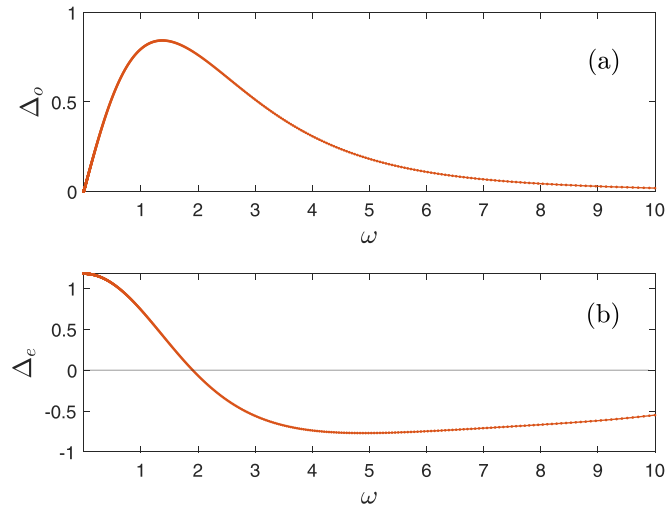

FIG. 6. A comparison of OF (a) and EF (b) gap functions at $T = 0$, obtained by solving the gap equation without the fermionic self-energy. We set $\lambda = 1.1$, $f = 1.5$, and $\Lambda = 10$.

For generic $f \leqslant 1$, Δ_e is much larger than Δ_o because of the Cooper logarithm, and for $\lambda < \lambda_c^o$ is the only superconducting solution at $T = 0$. When the Hubbard repulsion f increases, Δ_e is suppressed. At weak coupling, a phase transition into the normal state occurs when f reaches a critical value f_c. For $\Lambda \gg 1$ and $\alpha = 0$, this critical value is given by [28]

$$f_c \simeq \frac{1}{1 - 2\lambda \log(\Lambda)} + O(\lambda) \quad \text{for} \quad \Lambda \gg 1. \tag{18}$$

An exemplary (T-f) phase diagram for EF superconductivity is shown in Fig. 7. It is obtained by solving the gap equation (14) for infinitesimally small $\Delta_e(\omega)$ for two values of α. As expected, the critical temperature for EF pairing

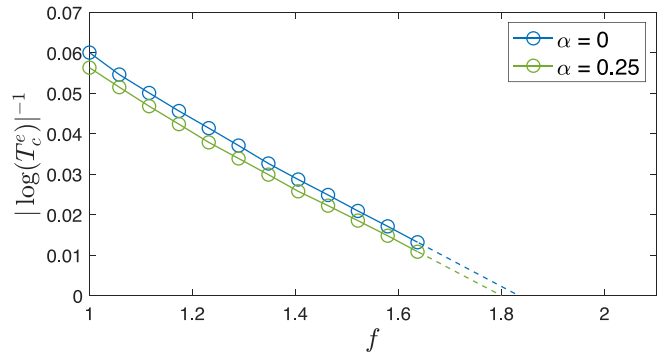

FIG. 7. $|\log(T_c^e)|^{-1}$ as a function of f for two values of α, which measures the strength of pair-breaking effect due to fermionic self-energy (at $\alpha = 0$ there is no effect from the self-energy). One can see that $|\log(T_c^e)|^{-1}$ scales linearly with f, meaning that $T_c^e \sim \exp[1/(f_c - f)]$. We set $\Lambda = 10$, $\lambda = 0.12$. A nonlinear frequency grid was used to reach the required exponentially small temperatures. We have checked that the details of the discretization have no impact on T_c^e near where it vanishes. Very close to f_c, the critical temperatures become too small to be numerically accessible; an extrapolation of T_c^e is shown with dashed lines.

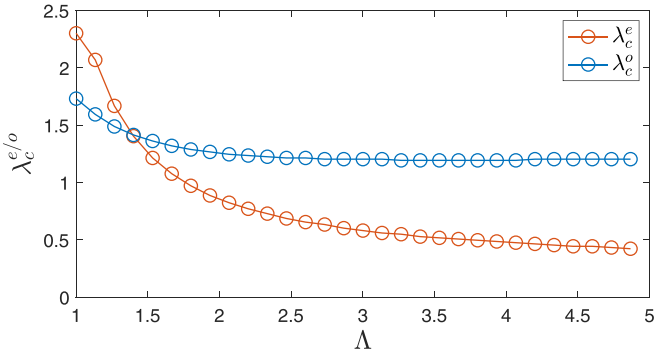

FIG. 8. Comparison of λ_c^e (red) and λ_c^o (blue) as a function of Λ for $\alpha = 0.2$.

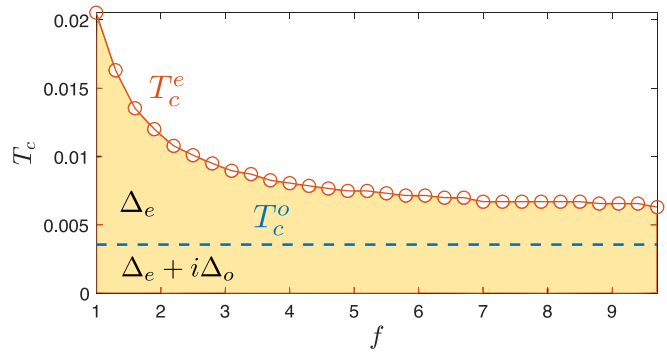

FIG. 9. Numerical results for T_c^e (circles) and T_c^o (dashed horizontal line) as a function of f. We set $\Lambda = 2, \lambda = 1.5, \alpha = 0.3$. Yellow shading denotes the superconducting region. The label $\Delta_e + i\Delta_o$ indicates the superconducting state with broken time-reversal symmetry.

T_c^e, vanishes at $f > f_c$. As seen in the figure, an inclusion of a finite self-energy reduces T_c^e at $f < f_c$, but hardly impacts the value of f_c itself. To understand this, we note that the self-energy in the even-frequency case still contains the odd component of the dynamical interaction χ_o, see Eq. (4). Near $f = f_c$, the transition temperature T_c^e is determined by fermions with small frequencies due the IR-divergent Cooper logarithm. Because $\chi_o(\omega, \omega')$ vanishes at $\omega = \omega' = 0$, it does not affect the critical f_c.

At a first glance the vanishing of T_c^e at $f > f_c$ implies that for such f EF superconductivity is not a competitor to OF superconductivity, and to get OF pairing one just needs to find a way to increase f. However, the actual situation is more complex. The reason is that, as we found earlier, OF superconductivity only holds when $\lambda > \lambda_c^o$, and λ_c^o is at least $O(1)$, while Eq. (18) for f_c only holds for small λ. Once we increase λ, we find that there is another critical value

$$\lambda_c^e \simeq \frac{1}{2\log(\Lambda)} \quad \text{for} \quad \Lambda \gg 1, \tag{19}$$

at which f_c diverges. We have verified Eq. (19) numerically and show the results in Appendix C. For $\lambda > \lambda_c^e$, the EF gap function is nonzero at $T = 0$ for all values of f. The node in the corresponding $\Delta_e(\omega)$ is placed in such a way that $f \int d\omega' \frac{\Delta_e(\omega')}{\omega'} \to$ const. when $f \to \infty$.

B. Critical couplings and temperatures for EF and OF superconductivity

We see from Eq. (19) that at large Λ and $\alpha = 0$, $\lambda_c^e \sim 1/\log(\Lambda)$ while $\lambda_c^o \simeq 0.88$ is a constant. Then $\lambda_c^e < \lambda_c^o$. This is also true at a finite α as λ_c^o increases and λ_c^e remains almost the same. In this situation it is natural to expect that EF superconductivity prevents the development of OF superconductivity, because when $\Delta_e(\omega)$ becomes nonzero, it reduces the strength of the pairing kernel on the OF channel. This is the last of the three obstacles for OF pairing that we listed in the Introduction.

This obstacle becomes less drastic when $\Lambda = O(1)$, like in low-density materials, e.g., $SrTiO_3$ [23,25,32], Bi [33], and half-Heusler compounds [34]. In this situation λ_c^e and λ_c^o become comparable, as we show in Fig. 8. Correspondingly, T_c^e and T_c^o also become comparable. There is even a window of Λ in Fig. 8 where OF superconductivity develops first. For small α, the requirement on Λ is even less restrictive. In Fig. 6 we show Δ_o and Δ_e, obtained independently by solving the gap equation at $T = 0$ at $\alpha = 0$, i.e., without the self-energy term, for $\lambda = 1.1$, $f = 1.5$, and Λ as large as 10. We see that even for such large Λ the magnitudes of Δ_o and Δ_e are comparable.

These observations suggest that at $T = 0$ both EF and OF gap functions may be nonzero. Our next goal is to find such a mixed state and determine the relative phase factor between the $U(1)$ order parameters Δ_o and Δ_e.

V. SPONTANEOUS BREAKING OF TIME-REVERSAL SYMMETRY

We show an exemplary phase diagram in the (T, f) plane in Fig. 9 by choosing a Λ for which EF superconductivity develops first below T_c^e, but T_c^o is close, and while it is reduced by a finite Δ_e, the OF component still develops below a finite T.

As we will demonstrate below, the gap function in the mixed state at $T = 0$ is of the form

$$\Delta(\omega) = \Delta_e(\omega) \pm i\Delta_o(\omega). \tag{20}$$

This agrees with Ref. [29], where a mixed state with spin-singlet EF and spin-triplet OF order parameter with the relative phase $\pm\pi/2$ has been found numerically (for a cupratelike Fermi surface and d-wave spatial symmetry of both gap functions).

A superconducting state with $\Delta(\omega)$ from Eq. (20) has a special property: it *spontaneously* breaks time-reversal symmetry, despite that separately spin-singlet Δ_e and spin-triplet Δ_o are invariant under time reversal. We show in Appendix D that time-reversal $\mathcal{T}$ acts on the gap function along the Matsubara axis simply as a complex conjugation:

$$(\mathcal{T} \circ \Delta)(\omega) = [\Delta(\omega)]^*. \tag{21}$$

Taken separately, $\Delta_e(\omega)$ and $\Delta_o(\omega)$ are time-reversal invariant (we recall that Δ_o is odd under time permutation, but even under time reversal [3,39]). However, $\Delta(\omega)$ from (20) does not remain invariant under time reversal.

To see that the relative phase between Δ_e and Δ_o is $\pm\pi/2$, we consider the gap equation at $T = 0$ without self-energy

correction and assume that the gap function is

$$\Delta(\omega) = \Delta_e(\omega) + \exp(i\phi)\Delta_o(\omega), \tag{22}$$

and that Δ_o is smaller than Δ_e. To leading order in Δ_o, the gap equation in the OF channel then takes the form

$$\begin{aligned}\exp(i\phi)\Delta_o(\omega) = 2\int_0^\Lambda d\omega' \frac{\chi_o(\omega-\omega')\Delta_o(\omega')}{\sqrt{(\omega')^2+\Delta_e^2(\omega')}} \\ \times\left(\exp(i\phi) - \frac{\cos(\phi)\Delta_e^2(\omega')}{(\omega')^2+\Delta_e^2(\omega')}\right) \\ + O\big[\Delta_o^2(\omega')\big].\end{aligned} \tag{23}$$

One can easily verify that ϕ can be either zero or $\pm\pi/2$. For $\phi = 0$, the expression in parentheses reduces to $1 - \Delta_e^2/[(\omega')^2 + \Delta_e^2]$, which reduces χ_o. For $\phi = \pm\pi/2$, the expression in parentheses becomes $\pm i$, in which case there is no suppression. We conclude therefore that the mixed state with $\phi = \pm\pi/2$ is indeed preferential. We note in passing that the state $\Delta = \Delta_e \pm i\Delta_o$ is also realized when the time-reversal symmetry is broken explicitly by applying a magnetic field, as shown in Ref. [14].

A spontaneous breaking of time-reversal symmetry can be detected experimentally via muon spin relaxation or Kerr rotation [40] and such states have been intensively discussed in recent years, but chiefly for non-s-wave spatial symmetry, or for multiband s-wave superconductors [41]. In our case a superconducting state with broken time-reversal symmetry emerges in a one-band s-wave superconductor.

VI. CONCLUSION AND OUTLOOK

In this work we considered OF superconductivity in a model of fermions with an interaction potential which contains a static Hubbard repulsion and a dynamical phonon-mediated attraction. We critically reexamined the three foes which usually prevent OF superconductivity: The necessity for strong coupling, the self-energy effect, and the suppression by EF superconductivity. We have found that the strong coupling requirement cannot be avoided, but there are ways to overcome the other two obstacles. The self-energy does prevent OF superconductivity in the Eliashberg approximation, if the same interaction determines the pairing and the fermionic self-energy. We argued that vertex corrections change this balance and make OF pairing possible. At the same time, the self-energy cannot be simply neglected as with and without vertex corrections it leads to cancellation of the thermal terms in the gap equation. Consequently, we find an OF state, which is stable below T_c^o down to zero temperature, and not the reentrant behavior observed in previous works.

The suppression of OF pairing by pre-existing EF superconductivity remains a problem when the Fermi energy is much larger than the typical phonon energy scale. However, when these scales become comparable, the critical temperatures for EF and OF orders are comparable, and OF superconducting order can coexist with EF superconductivity. We have shown that a mixed state with the gap function $\Delta_e(\omega) \pm i\Delta_o(\omega)$ can be realized in this case. This state spontaneously breaks the time-reversal invariance.

It has been argued that induced OF superconducting state may exhibit a paramagnetic Meissner effect (see Ref. [42] and references therein). However, as shown in Refs. [4,5,43] and also discussed in Ref. [3], for spontaneous OF superconductivity in the bulk, induced by a retarded interaction, the Meissner effect is diamagnetic. This can be seen explicitly by computing the superfluid density n_s (see Appendix E), which is manifestly positive. While this result has been questioned in the literature due to possible issues with spontaneous $U(1)$ symmetry breaking [44], in our understanding all ambiguities can be avoided by consistently working in the functional-integral formalism (and avoiding a Hamiltonian description). The fact that a diamagnetic Meissner effect is "conventional" can also be seen by reformulating the OF theory in terms of $D(\omega) = \Delta_o(\omega)/\omega$, which is an even function of frequency, like $\Delta_e(\omega)$ for EF superconductivity. Using the description of OF superconductivity in terms of $D(\omega)$, one straightforwardly obtains conventional electromagnetic response of the superconducting state. A *paramagnetic* Meissner effect can develop for induced OF superconductivity at a boundary of a system, but this is a different setup from the one considered in this work.

Our analysis of the OF state was performed by studying gap functions on the Matsubara axis. On the other hand, the measurable physical properties of the system are determined by the gap function on the real axis. Since the OF gap vanishes at $\omega = 0$, the density of states of an OF superconductor is qualitatively similar to that of an EF gapless superconductor in the presence of magnetic impurities [45]. However, we expect crucial differences in, say, the phase winding of the gap function and in the behavior of low-energy collective modes, which could be fruitful objects for future studies.

Note added. Shortly after this paper was posted, a work appeared [46] which contains a more general version of the "No-go-theorem" for OF superconductivity within the Eliashberg approximation due to the self-energy effects.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF THAT Δ_o VANISHES WITHIN THE ELIASHBERG APPROXIMATION

We introduce a function $D(\omega) = \Delta_o(\omega)/\omega$. In terms of D, the linearized gap equation (3) at finite T reads

$$D(\omega_n) = \sum_m K^T_{n,m}[D(\omega_m) - D(\omega_n)], \tag{A1}$$

where $K^T_{n,m} = K(\omega_m, \omega_n)$ is the transpose of the OF kernel defined in Eq. (10).

By bringing $D(\omega_n)$ to the left side, we get

$$D(\omega_n) = \sum_m \frac{K^T_{n,m}}{1 + \sum_m K^T_{n,m}} D(\omega_m). \tag{A2}$$

Now we assume that $D(\omega_n)$ is finite for all n and look at $\max_n |D(\omega_n)|$:

$$\begin{aligned}\max_n |D(\omega_n)| &= \max_n \left| \frac{\sum_m K^T_{n,m}}{1+\sum_m K^T_{n,m}} D(\omega_m) \right| \\ &\leqslant \max_n \left| \frac{\sum_m K^T_{n,m}}{1+\sum_m K^T_{n,m}} \right| |D(\omega_m)| \\ &\leqslant \max_n \left| \frac{\sum_m K^T_{n,m}}{1+\sum_m K^T_{n,m}} \right| \max_l |D(\omega_l)|. \end{aligned} \tag{A3}$$

Since χ_o is attractive, all entries of K are positive. Then we can rewrite Eq. (A3) as

$$\max_n |D(\omega_n)| \leqslant \frac{\sum_m K^T_{n,m}}{1+\sum_m K^T_{n,m}} \max_l |D(\omega_l)|.$$

The first factor in the second inequality is smaller than 1 for any n. Assuming that $\max_n |D(\omega_n)| > 0$, we obtain the strict inequalitiy

$$\max_n |D(\omega_n)| < \max_n |D(\omega_n)|, \tag{A4}$$

which is a contradiction. Therefore, we must have $D = 0 \Leftrightarrow \Delta_o = 0$.

APPENDIX B: VERTEX CORRECTIONS

1. Evaluation at $T = 0$

Including the second-order diagrams of Fig. 4 in the main text, the linearized Eliashberg equation at $T = 0$ can be written down as

$$\begin{aligned}\Phi(k) &= -\int_p \Phi(p)G(p)G(-p)V(k-p)[1+2\Gamma(k,p)], \\ \Sigma(k) &= -i\int_p G(p)V(k-p)[1+\Gamma(k,p)], \end{aligned} \tag{B1}$$

where

$$\Gamma(k,p) = -\int_l G(l)G(p+l-k)V(k-l), \tag{B2}$$

and we use notations $k = (\omega, \boldsymbol{k})$, $p = (\omega', \boldsymbol{p})$, $l = (\tilde{\omega}, \boldsymbol{l})$ and the conventions

$$G(k) = (i\omega - \xi(\boldsymbol{k}) + i\Sigma(k))^{-1}, \tag{B3}$$

$$\int_k = \int \frac{d\omega d\boldsymbol{k}}{(2\pi)^d}, \tag{B4}$$

with $\xi(\boldsymbol{k})$ the electron dispersion, which can be linearized near the Fermi surface. We will focus on $d = 2$ for concreteness, and comment on the analogous 3D results along the way.

Without the vertex corrections, $\Gamma = 0$, the gap equation (3) directly follows from (B1) by rewriting the definition of the gap $\Delta(\omega) = \Phi(\omega)/[1 + \Sigma(\omega)/\omega]$ as

$$\Delta(\omega) = \Phi(\omega) - \frac{\Delta(\omega)}{\omega}\Sigma(\omega), \tag{B5}$$

and evaluating the momentum integrals in (B1).

We now evaluate the vertex correction Γ, using bare Green's functions (no Σ), and working in the limit $\Lambda \to \infty$ for simplicity. Shifting the integration variables, Γ can be written as

$$\begin{aligned}\Gamma(k,p) &= (-1)\lambda \int \frac{d\boldsymbol{l}d\tilde{\omega}}{(2\pi)^3}\frac{2}{\rho}\left(f - \frac{1}{1+(\tilde{\omega})^2}\right) \\ &\quad \times \frac{1}{i(\tilde{\omega}+\omega') - \xi_+}\frac{1}{i(\tilde{\omega}+\omega) - \xi_-}, \\ \xi_+ &= \xi\left(\boldsymbol{l} + \frac{1}{2}(\boldsymbol{p}-\boldsymbol{k})\right), \quad \xi_- = \xi\left(\boldsymbol{l} - \frac{1}{2}(\boldsymbol{p}-\boldsymbol{k})\right). \end{aligned} \tag{B6}$$

We expand the dispersion as

$$\begin{aligned}&\xi_+ = \xi(\boldsymbol{l}) + \delta q, \quad \xi_- = \xi(\boldsymbol{l}) - \delta q, \\ &\delta q = \tfrac{1}{2}v_F|\boldsymbol{q}|\cos(\phi), \quad \boldsymbol{q} = |\boldsymbol{p}-\boldsymbol{k}|, \quad \phi = \measuredangle(\boldsymbol{q},\boldsymbol{l}), \end{aligned} \tag{B7}$$

and integrate over $\xi(\boldsymbol{l})$ in infinite limits. Such an expansion is legitimate as for $\Omega \equiv \omega' - \omega \ll E_F$ the relevant contributions come from small angle scattering where $|\boldsymbol{q}| \ll |\boldsymbol{k}|, |\boldsymbol{p}|$. We call Γ_1 the part $\propto f$ and the remainder Γ_2. To compute Γ_1, we need to perform the frequency integral first, since the integral is not absolutely convergent. The computation is standard; it is the same as for the polarization function, since the part $\sim f$ is short range. We obtain

$$\Gamma_1 = 2\lambda f\left(1 - \frac{|\Omega|}{\sqrt{\Omega^2 + (v_F|\boldsymbol{q}|)^2}}\right), \tag{B8}$$

The second term in Γ_1 contains $|\boldsymbol{q}|$. To find the renormalization of the interactions which enter the Eliashberg equations, we can take the s-wave part of this term. That is we write

$$|\boldsymbol{q}|^2 = |\boldsymbol{p}-\boldsymbol{k}|^2 = 4k_F^2\sin^2(\theta/2), \quad \theta = \measuredangle(\boldsymbol{p},\boldsymbol{k}), \tag{B9}$$

where $\boldsymbol{k}$ and $\boldsymbol{p}$ are on the Fermi surface. We then average

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{|\Omega|}{\sqrt{\Omega^2 + 16E_F^2\sin^2(\theta/2)}} \simeq \frac{|\Omega|}{2\pi E_F}\log\left(\frac{E_F}{|\Omega|}\right). \tag{B10}$$

In $d = 3$, one obtains a correction $\sim|\Omega|/E_F$ without the logarithm. In the limit $E_F \gg 1$, the dynamical correction is small, in accordance with Migdal's theorem [47].

To compute Γ_2, it is easier to perform the integral over dispersion $\xi(\boldsymbol{l})$ first. This is allowed because the extra frequency dependence renders the integral absolutely convergent. The result is

$$\begin{aligned}\Gamma_2 &= 2\lambda f \,\mathrm{sgn}(\Omega)\frac{1}{\sqrt{(v_F|\boldsymbol{q}|)^2 + \Omega^2}} \\ &\quad \times \left[\arctan\left(\frac{1}{\omega}\right) - \arctan\left(\frac{1}{\omega'}\right)\right]. \end{aligned} \tag{B11}$$

This term depends on ω and ω' separately. That is it depends on both $\omega - \omega'$ and $\omega + \omega'$. However, after taking the s-wave part it will scale as $|\Omega|/E_F$ similar to (B10).

Collecting the results, the vertex correction reads

$$\Gamma = \Gamma_1 + \Gamma_2 = 2\lambda f + |\omega - \omega'|O\left(\frac{1}{E_F}\right), \tag{B12}$$

The static part reads $2\lambda f$, as stated in the main text.

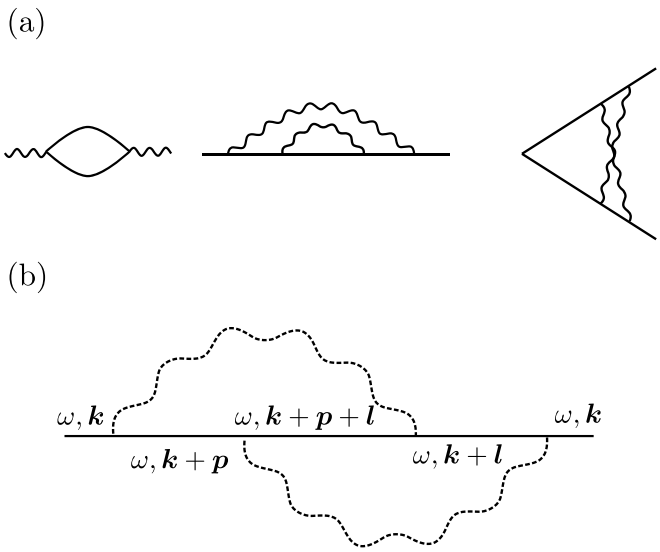

FIG. 10. (a) Additional second-order diagrams not considered in the main text. (b) Self-energy contribution where both interaction lines are thermal.

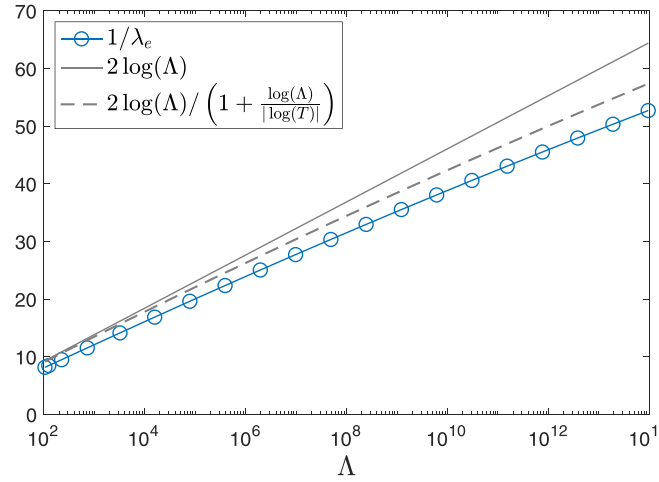

FIG. 11. Numerical estimation of λ_c^e (blue line), without self-energy correction. Gray lines show the analytical estimates (19) and (C1).

2. Additional diagrams

In Fig. 10(a) we show additional second-order diagrams not considered so far. The first diagram renormalizes both χ_{pp} and χ_{ph} alike, thus it cannot lead to a nonzero OF solution. The second "rainbow" diagram is already contained in the self-consistent Eliashberg equation. The third diagram can contribute in principle, but it depends on $\omega+\omega'$ even for frequency-independent interactions, and its contribution does not vanish for $\omega=\omega'$. Thus, it cannot be treated in the Eliashberg framework.

3. Cancellation of the thermal terms

As discussed in the main text, at a finite temperature we can isolate two thermal contributions from the second-order self-energy diagram. However, we need to subtract the contribution shown in Fig. 10(b), where the frequency transfer on both lines is zero. But this contribution vanishes: To see this, we can expand the fermionic dispersion around the external momentum $\boldsymbol{k}$ as

$$\xi(\boldsymbol{k}+\boldsymbol{p}) \simeq \xi(\boldsymbol{k}) + v_F p_\parallel + \boldsymbol{p}_\perp^2/(2m), \tag{B13}$$

where $p_\parallel$, $\boldsymbol{p}_\perp$ are the components of $\boldsymbol{p}$ parallel and perpendicular to $\boldsymbol{k}$, respectively, and m is an effective mass. Expanding $\xi(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{l})$ in the same way, for instance, the integral over $p_\parallel$ vanishes for zero frequency transfer: if the integral is evaluated by contour integration in infinite limits, both poles are in the same half-plane. Note that such an argument does not work if only one of the transferred frequencies is nonzero, while the other frequency is integrated over, since in this case the additional frequency integral must be evaluated before the momentum integrals, yielding nonzero.

As a result, the thermal contributions to Φ and Σ in Eq. (B1) read (with $\omega=\omega'$ and after evaluating momentum integrals):

$$\begin{aligned}
\Phi_{\rm th}(\omega) &= -\frac{\Phi(\omega)}{|\omega+\Sigma(\omega)|}\rho\pi TV(0)[1+4\lambda f] \\
&= -\frac{\Delta(\omega)}{|\omega|}\rho\pi TV(0)[1+4\lambda f], \\
\Sigma_{\rm th}(\omega) &= -{\rm sgn}(\omega)\rho\pi TV(0)[1+4\lambda f], \\
\Delta_{\rm th}(\omega) &= \Phi_{\rm th}(\omega) - \frac{\Delta(\omega)}{\omega}\Sigma_{\rm th}(\omega) = 0.
\end{aligned} \tag{B14}$$

APPENDIX C: BEHAVIOR OF λ_c^e

In Fig. 11 we numerically check the behavior of $\lambda_c^e \simeq 1/[2\log(\Lambda)]$ at large Λ by plotting $1/\lambda_c^e$. Apart from the nearly constant offset, which is expected since the formula only holds with "logarithmic accuracy," at very large Λ the numerical result for $1/\lambda_c^e$ decreases compared to the asymptotic expression. This is expected since in the numerics a nonzero, though very small temperature T is used, while λ_c^e is defined as the critical coupling at zero temperature. Adapting the evaluation in Ref. [28] [see Eq. (15) within], one can provide an estimate for λ_c^e at a finite temperature T if one assumes that T only serves as an IR cutoff, similar to $\Delta_e(0)$ in Ref. [28]. One finds

$$\lambda_c^e = \frac{1}{2\log(\Lambda)}\left(1+\frac{\log\Lambda}{|\log(T)|}\right). \tag{C1}$$

As seen in Fig. 11, this formula correctly reproduces the numerics up to the constant offset.

APPENDIX D: TIME-REVERSAL TRANSFORMATION OF THE GAP FUNCTION

To derive the action of the time-reversal transformation $\mathcal{T}$ on gap functions on the Matsubara axis, we first derive the action on real-frequency gap functions. We work with retarded and advanced gap functions in the time domain at

zero temperature, which are defined as

$$\Delta^R(t) = -i\theta(t)\,\langle 0|\hat{M}(t)|0\rangle\,,$$
$$\Delta^A(t) = +i\theta(-t)\,\langle 0|\hat{M}(t)|0\rangle\,, \tag{D1}$$

with $|0\rangle$ the interacting ground state, and

$$\hat{M}(t) = \{c_\alpha(t), c_\beta(0)\}[i\sigma_y(d_0\mathbb{1} + d_z\cdot\sigma_z)]_{\alpha\beta}, \tag{D2}$$

where the spin term describes both singlet ($d_0 = 1, d_z = 0$) and mixed triplet ($d_0 = 0, d_z = 1$). $\mathcal{T}$ acts on state vectors $|\psi\rangle$ and operators $\hat{O}$ as

$$|\mathcal{T}\psi\rangle = T\,|\psi\rangle\,,$$
$$\mathcal{T}\circ\hat{O} = T\hat{O}T^{-1}, \tag{D3}$$

where T is an antiunitary matrix. Furthermore, expectation values fulfill

$$\langle\mathcal{T}\psi|\mathcal{T}\phi\rangle = \langle\psi|T^\dagger T\phi\rangle = \langle\psi|\phi\rangle^*\,. \tag{D4}$$

Using these properties, Δ^R transforms as

$$\begin{aligned}(\mathcal{T}\circ\Delta^R)(t) &= i\theta(-t)\,\langle\mathcal{T}0|\mathcal{T}\circ\hat{M}(t)|\mathcal{T}0\rangle\\ &= i\theta(-t)\,\langle 0|T^\dagger T\hat{M}(t)T^{-1}T|0\rangle\\ &= i\theta(-t)\,\langle 0|\hat{M}(t)|0\rangle^* = -[\Delta^A(t)]^*.\end{aligned} \tag{D5}$$

Fourier transforming, we obtain

$$\begin{aligned}\big(\mathcal{T}\circ\Delta^R\big)(\Omega) &= \mathcal{T}\circ\int dt\,\exp(i\Omega t)\Delta^R(t)\\ &= -\int dt\,\exp[-i\Omega(-t)][\Delta^A(t)]^*\\ &= -[\Delta^A(-\Omega)]^*.\end{aligned} \tag{D6}$$

Note that $\mathcal{T}$ does not act on the measure dt, as can be checked considering the inverse Fourier transform. With Eq. (D6) at hand, we can infer the action of $\mathcal{T}$ on the Matsubara gap function $\Delta(\omega)$ from Cauchy's theorem. For concreteness, we focus on Δ_o and assume that $\Delta_o \in \mathbb{R}$ as in the main text. For $\omega > 0$, Δ_o is related to Δ_o^R and Δ_o^A as

$$\Delta_o(\omega) = \frac{1}{2\pi i}\int d\Omega\frac{\Delta_o^R(\omega)}{\Omega - i\omega}, \tag{D7}$$

$$\Delta_o(-\omega) = -\frac{1}{2\pi i}\int d\Omega\frac{\Delta_o^A(\Omega)}{\Omega + i\omega}. \tag{D8}$$

By writing $\Delta_o^{R/A}(\Omega) = \Delta_1^{R/A}(\Omega) + i\Delta_2^{R/A}(\Omega)$, one can check that the condition $\Delta_o \in \mathbb{R}$ leads to

$$\Delta_1^{R/A}(\Omega) = \Delta_1^{R/A}(-\Omega),$$
$$\Delta_2^{R/A}(\Omega) = -\Delta_2^{R/A}(-\Omega), \tag{D9}$$

In addition, from $\Delta_o(\omega) = -\Delta_o(-\omega)$ we obtain

$$\Delta_o^A(\Omega) = -\Delta_o^R(\Omega)^*. \tag{D10}$$

Now we can compute $(\mathcal{T}\circ\Delta_o)(\omega)$ for $\omega > 0$:

$$\begin{aligned}(\mathcal{T}\circ\Delta_o)(\omega) &\overset{(D7),(D6)}{=} -\frac{1}{2\pi i}\int d\Omega\frac{-\big[\Delta_o^A(-\Omega)\big]^*}{\Omega + i\omega}\\ &\overset{(D9)}{=} \frac{1}{2\pi i}\int d\Omega\frac{\Delta_o^A(\Omega)}{\Omega + i\omega}\\ &\overset{(D8)}{=} -\Delta_o(-\omega) = \Delta_o(\omega) = \Delta_o(\omega)^*.\end{aligned} \tag{D11}$$

Proceeding analogously for $\Delta_e \in \mathbb{R}$, one also finds $(\mathcal{T}\circ\Delta_e)(\omega) = \Delta_e(\omega)^*$. Due to the antilinearity of $\mathcal{T}$, $(\mathcal{T}\circ\Delta)(\omega) = \Delta(\omega)^*$ then holds for arbitrary $\Delta(\omega)$ of the form $\Delta(\omega) = \Delta_e(\omega) + \exp(i\phi)\Delta_o(\omega)$, as required in Sec. V.

APPENDIX E: MEISSNER EFFECT

The magnetic response of a superconductor is determined by the energy cost of phase fluctuations. Given a spatially homogeneous mean-field solution Δ with fluctuating phase $\theta(\boldsymbol{x})$, the momentum-space action for θ is

$$S_\theta \sim \int d\boldsymbol{q}\; n_s|\boldsymbol{q}|^2\theta(\boldsymbol{q})\theta(-\boldsymbol{q}), \tag{E1}$$

where we neglected temporal fluctuations of θ. As shown in previous works [3–5,43,44], the superfluid density n_s, which enters Eq. (E4), takes the same form for both EF and OF bulk superconductivity. We explicitly verified this result and confirmed it. At $T = 0$, the superfluid density n_s, normalized to the normal-state density of electrons, is

$$n_s = \frac{1}{2}\int d\omega\frac{|\Delta(\omega)|^2}{[|\Delta(\omega)|^2 + \omega^2]^{3/2}}. \tag{E2}$$

This is a manifestly positive expression. Therefore, the Meissner effect is diamagnetic in both EF and OF cases.

If we simply evaluate Eq. (E4) in the OF state at $T = 0$, we run into a problem: for $\Delta(\omega) \sim \omega$ as $\omega \to 0$ (see Sec. III A), the integral is logarithmically divergent at small frequencies, as already observed in Ref. [44]. Note that this is the case only if the gap function is linear in ω; for a general gap function which scales as $\Delta(\omega) \sim \omega^a$ with $a \neq 1$, the integrand scales as

$$\begin{aligned}&\omega^{2a-3} \quad a > 1,\\ &\omega^{-a} \quad a < 1,\end{aligned} \tag{E3}$$

which leads to a convergent result.

For the given ω-linear gap function, the logarithmic singularity of n_s will be cut off by finite momenta $\boldsymbol{q}$. Therefore, the action for the phase field (E4) is modified to

$$S_\theta \sim \int d\boldsymbol{q}\; n_s(\boldsymbol{q})|\boldsymbol{q}|^2\theta(\boldsymbol{q})\theta(-\boldsymbol{q}), \quad n_s(\boldsymbol{q}) \sim \log(k_F/|\boldsymbol{q}|). \tag{E4}$$

The Meissner response can be obtained by coupling the system to an electromagnetic field $\boldsymbol{A}$. In the conventional case of Eq. (E4), the constant n_s acts a a mass term for $\boldsymbol{A}$, which leads to a penetration depth $\lambda \sim 1/\sqrt{n_s}$. Taken at face value, the logarithmic $n_s(\boldsymbol{q})$ obtained in (E4) then implies a superexponential decay of an external magnetic field in a superconductor, $B(x) \sim \exp[-x\log(x)]$. However, we do not regard this as an observable effect, but rather as an artifact of the mean-field approximation: the logarithmic divergence of $n_s(\boldsymbol{q})$ likely signals that in the full theory with fluctuations included, the gap function $\Delta(\omega)$ scales as ω^a, with $a \neq 1$. As discussed above, in this case n_s is finite, and the Meissner response is a conventional.

[1] V. L. Berezinskii, JETP Lett. **20**, 287 (1974).
[2] A. Balatsky and E. Abrahams, Phys. Rev. B **45**, 13125 (1992).
[3] J. Linder and A. V. Balatsky, Rev. Mod. Phys. **91**, 045005 (2019).
[4] D. Solenov, I. Martin, and D. Mozyrsky, Phys. Rev. B **79**, 132502 (2009).
[5] H. Kusunose, Y. Fuseya, and K. Miyake, J. Phys. Soc. Jpn. **80**, 054702 (2011).
[6] V. J. Emery and S. Kivelson, Phys. Rev. B **46**, 10812 (1992).
[7] T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **66**, 1533 (1991).
[8] V. A. Zyuzin and A. M. Finkel'stein, arXiv:1912.04258.
[9] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Rev. Mod. Phys. **77**, 1321 (2005).
[10] A. A. Golubov, Y. Tanaka, Y. Asano, and Y. Tanuma, J. Phys.: Condens. Matter **21**, 164208 (2009).
[11] A. Di Bernardo, S. Diesch, Y. Gu, J. Linder, G. Divitini, C. Ducati, E. Scheer, M. G. Blamire, and J. W. A. Robinson, Nat. Commun. **6**, 8053 (2015).
[12] A. Di Bernardo, Z. Salman, X. L. Wang, M. Amado, M. Egilmez, M. G. Flokstra, A. Suter, S. L. Lee, J. H. Zhao, T. Prokscha, E. Morenzoni, M. G. Blamire, J. Linder, and J. W. A. Robinson, Phys. Rev. X **5**, 041021 (2015).
[13] Y. Tanaka, M. Sato, and N. Nagaosa, J. Phys. Soc. Jpn. **81**, 011013 (2012).
[14] M. Matsumoto, M. Koga, and H. Kusunose, J. Phys. Soc. Jpn. **81**, 033702 (2012).
[15] A. Aperis, P. Maldonado, and P. M. Oppeneer, Phys. Rev. B **92**, 054516 (2015).
[16] K. Shigeta, S. Onari, and Y. Tanaka, Phys. Rev. B **85**, 224509 (2012).
[17] A. Aperis, E. V. Morooka, and P. M. Oppeneer, Ann. Phys. **417**, 168095 (2020), Eliashberg theory at 60: Strong-coupling superconductivity and beyond.
[18] A. Tsvelik, Proc. Natl. Acad. Sci. USA **116**, 12729 (2019).
[19] E. Abrahams, A. Balatsky, J. R. Schrieffer, and P. B. Allen, Phys. Rev. B **47**, 513 (1993).
[20] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).
[21] H. Rietschel and L. J. Sham, Phys. Rev. B **28**, 5100 (1983).
[22] A. Chubukov, N. V. Prokof'ev, and B. V. Svistunov, Phys. Rev. B **100**, 064513 (2019).
[23] J. Ruhman and P. A. Lee, Phys. Rev. B **94**, 224515 (2016).
[24] J. Ruhman and P. A. Lee, Phys. Rev. B **96**, 235107 (2017).
[25] P. Wölfle and A. V. Balatsky, Phys. Rev. B **98**, 104505 (2018).
[26] H. Kusunose, Y. Fuseya, and K. Miyake, J. Phys. Soc. Jpn. **80**, 044711 (2011).
[27] D. Phan and A. V. Chubukov, Phys. Rev. B **105**, 064518 (2022).
[28] D. Pimenov and A. V. Chubukov, npj Quantum Mater. **7**, 45 (2022).
[29] F. Schrodi, A. Aperis, and P. M. Oppeneer, Phys. Rev. B **104**, 174518 (2021).
[30] F. Schrodi, P. M. Oppeneer, and A. Aperis, Phys. Rev. B **102**, 024503 (2020).
[31] P. O. Sukhachov, V. Juričić, and A. V. Balatsky, Phys. Rev. B **100**, 180502(R) (2019).
[32] M. N. Gastiasoro, J. Ruhman, and R. M. Fernandes, Ann. Phys. **417**, 168107 (2020), Eliashberg theory at 60: Strong-coupling superconductivity and beyond.
[33] O. Prakash, A. Kumar, A. Thamizhavel, and S. Ramakrishnan, Science **355**, 52 (2017).
[34] Y. Nakajima, R. Hu, K. Kirshenbaum, A. Hughes, P. Syers, X. Wang, K. Wang, R. Wang, S. R. Saha, D. Pratt, J. W. Lynn, and J. Paglione, Sci. Adv. **1**, e1500242 (2015).
[35] A. Abanov and A. V. Chubukov, Phys. Rev. B **102**, 024524 (2020).
[36] Y. Fuseya, H. Kohno, and K. Miyake, J. Phys. Soc. Jpn. **72**, 2914 (2003).
[37] Y.-M. Wu, S.-S. Zhang, A. Abanov, and A. V. Chubukov, Phys. Rev. B **106**, 094506 (2022).
[38] M. H. Christensen and A. V. Chubukov, Phys. Rev. B **104**, L140501 (2021).
[39] D. Kuzmanovski and A. M. Black-Schaffer, Phys. Rev. B **96**, 174509 (2017).
[40] S. K. Ghosh, M. Smidman, T. Shang, J. F. Annett, A. D. Hillier, J. Quintanilla, and H. Yuan, J. Phys.: Condens. Matter **33**, 033001 (2021).
[41] S. Maiti and A. V. Chubukov, Phys. Rev. B **87**, 144511 (2013).
[42] D. L. Cox and A. Zawadowski, Adv. Phys. **47**, 599 (1998).
[43] D. Belitz and T. R. Kirkpatrick, Phys. Rev. B **60**, 3485 (1999).
[44] Y. V. Fominov, Y. Tanaka, Y. Asano, and M. Eschrig, Phys. Rev. B **91**, 144514 (2015).
[45] A. V. Balatsky, I. Vekhter, and J.-X. Zhu, Rev. Mod. Phys. **78**, 373 (2006).
[46] E. Langmann, C. Hainzl, R. Seiringer, and A. V. Balatsky, arXiv:2207.01825.
[47] A. B. Migdal, Sov. Phys. JETP **7**, 996 (1958).