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Doubling inequalities and nodal sets in periodic elliptic homogenization

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ABSTRACT

We prove explicit doubling inequalities and obtain uniform upper bounds (under $(d-1)$ -dimensional Hausdorff measure) of nodal sets of weak solutions for a family of linear elliptic equations with rapidly oscillating periodic coefficients. The doubling inequalities, explicitly depending on the doubling index, are proved at different scales by a combination of convergence rates, a three-ball inequality from certain “analyticity,” and a monotonicity formula of a frequency function. The upper bounds of nodal sets are shown by using the doubling inequalities, approximations by harmonic functions and an iteration argument.

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1. Introduction

The paper is concerned with doubling inequalities and upper bounds of nodal sets of solutions in periodic elliptic homogenization. We consider a family of elliptic operators in divergence form with rapidly oscillating periodic coefficients

$$\mathcal{L}_\varepsilon = -\nabla \cdot (A(x/\varepsilon) \nabla), \quad (1.1)$$

where $\varepsilon > 0$, and $A(y) = (a_{ij}(y))$ is a symmetric $d \times d$ matrix-valued function in \mathbb{R}^d with dimension $d \geq 2$. Assume that $A(y)$ satisfies the following assumptions:

- Strong ellipticity: there is $\Lambda > 0$ such that

$$\Lambda |\xi|^2 \leq \langle A(y) \xi, \xi \rangle \leq |\xi|^2, \quad \text{for any } y \in \mathbb{R}^d, \xi \in \mathbb{R}^d. \quad (1.2)$$

- Periodicity:

$$A(y+z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ for any } z \in \mathbb{Z}^d. \quad (1.3)$$

- Lipschitz continuity: There exists a constant $\gamma \geq 0$ such that

$$|A(x) - A(y)| \leq \gamma |x - y|, \quad \text{for any } x, y \in \mathbb{R}^d. \quad (1.4)$$

The doubling inequality describes quantitative behavior to characterize the strong unique continuation property, which has important applications in inverse problems,

control theory and the study of nodal sets of eigenfunctions. For harmonic functions or solutions of general elliptic equations in divergence form with Lipschitz coefficients, the doubling inequality is a consequence of a monotonicity formula or Carleman estimates; see [1–4]. In periodic elliptic homogenization, the first doubling inequality was obtained recently by Lin and Shen [5] with an implicit dependence on the doubling index. Precisely, they proved that if u_ε is a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B_2 = B_2(0)$ and

$$\int_{B_2} u_\varepsilon^2 \leq N \int_{B_\Lambda} u_\varepsilon^2, \quad (1.5)$$

then for any $r \in (0, 1)$,

$$\int_{B_r} u_\varepsilon^2 \leq C(N) \int_{B_{r/2}} u_\varepsilon^2, \quad (1.6)$$

where $C(N)$ depends only on d, Λ, γ and N . The point here is that the constant $C(N)$ is independent of the small parameter ε . This cannot be derived directly from the classical doubling inequality as the Lipschitz constant of the coefficients blows up as ε approaches zero. However, it is not known that how the constant $C(N)$ in (1.6) depends on N , because (1.6) was proved by a compactness argument. We mention that if $\varepsilon = 1$, the classical doubling inequality shows that $C(N) = CN^K$ for some $C, K \geq 1$; also see [Lemma 3.2](#).

On the other hand, the Hadamard three-ball inequality also describes the quantitative unique continuation property. In periodic elliptic homogenization, two different versions of the three-ball inequality with error terms were discovered in [6] and [7]. In general, the three-ball inequalities with errors are weaker than the doubling inequalities, as they alone do not imply the strong unique continuation.

Our first goal of this paper is to find an explicit estimate for the constant $C(N)$ in the doubling inequality in periodic elliptic homogenization. The explicit doubling inequality not only provides more clear quantitative information for the solutions (such as the vanishing order), but also has more applications. We state the result as follows.

Theorem 1.1. *Assume that $A = A(y)$ satisfies the conditions (1.2)–(1.4). Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_1 .*

- i. *For $d \geq 3$ and every $\tau > 0$, there exist $\theta \in (0, 1/2)$ and $C > 1$, depending only on d, τ, Λ and γ , such that if u_ε satisfies*

$$\int_{B_1} u_\varepsilon^2 \leq N \int_{B_\theta} u_\varepsilon^2, \quad (1.7)$$

then for every $r \in (0, 1)$,

$$\int_{B_r} u_\varepsilon^2 \leq \exp(\exp(CN^\tau)) \int_{B_{\theta r}} u_\varepsilon^2. \quad (1.8)$$

- ii. *For $d = 2$, there exists a constant $C > 1$ depending only on Λ and γ such that if u_ε satisfies*

$$\int_{B_1} u_\varepsilon^2 \leq N \int_{B_{\frac{N}{2}}} u_\varepsilon^2, \quad (1.9)$$

then for every $r \in (0, 1)$,

$$\int_{B_r} u_\varepsilon^2 \leq \exp(C(\ln N)^2) \int_{B_{\frac{r}{2}}} u_\varepsilon^2. \quad (1.10)$$

The double exponential growth $\exp(\exp(CN^\tau))$ for $d \geq 3$ in (1.8) and sub-exponential growth $\exp(C(\ln N)^2) = N^{C \ln N}$ for $d=2$ in (1.10) seem to be the best we can obtain from our method; see [Remark 4.5](#). Our ultimate hope is for an estimate of the form $C(N) = N^{C(\ln \ln N)^p}$, for some $p > 0$ depending on d, Λ and γ . Such an estimate would have very important consequences for the study of long-standing open problems regarding the spectral properties of second order elliptic operators with periodic coefficients and their quantitative unique continuation properties (see for instance [Conjecture 6.13](#), [Theorem 6.15](#) and [Conjecture 6.16](#) in [\[8\]](#)). This connection between the conjectured optimal doubling estimates and [Conjecture 6.13](#) in [\[8\]](#) was observed by the first author, D. Mendelson and C. Smart in the fall of 2019. This motivated the current work.

As a straightforward corollary, [Theorem 1.1](#) implies that the vanishing order of u_ε at the origin does not exceed $\exp(CN^\tau)$ for $d \geq 3$ and $C(\ln N)^2$ for $d=2$. [Theorem 1.1](#) also implies a three-ball inequality without an error term, in contrast to the results in [\[6\]](#) and [\[7\]](#), namely (e.g., for $d \geq 3$),

$$\int_{B_{\theta r}} u_\varepsilon^2 \leq \exp(\exp(CN^\tau)) \left(\int_{B_{\theta^2 r}} u_\varepsilon^2 \right)^{\tau_1} \left(\int_{B_r} u_\varepsilon^2 \right)^{1-\tau_1} \quad (1.11)$$

for any $0 < \tau_1 < 1$.

The Proof of [Theorem 1.1](#) breaks down into three steps:

- Step 1: $\varepsilon/r \leq N^{-5}$. In this case, we take advantage of the convergence rate in homogenization theory and use the precise three-ball inequality of harmonic functions. The smoothness of the coefficients is not needed in this step.
- Step 2: In this step, we need to use “analyticity,” which distinguishes between $d \geq 3$ and $d=2$. For $d \geq 3$, we let $N^{-5} \leq \varepsilon/r \leq N^{-\frac{1}{2}\tau}$, and use a three-ball inequality with a sharp exponential error term proved recently in [\[6\]](#) by Armstrong, Kuusi and Smart, which is a consequence of the “large-scale analyticity” from periodic homogenization. This will lead to a nontrivial improvement on the exponent so that $\tau > 0$ in [Theorem 1.1](#) can be arbitrarily small. Again in this case, the periodic structure will play a role; but the smoothness of coefficients is still not required. For $d=2$, we let $N^{-5} \leq \varepsilon/r \leq 1$, and apply a doubling inequality derived from quasi-regular mappings [\[9\]](#) (related to complex analyticity), which requires no smoothness or periodicity on the coefficients. Unfortunately, this method works only in two dimensions.
- Step 3: $\varepsilon/r \geq N^{-\frac{1}{2}\tau}$ for $d \geq 3$ or $\varepsilon/r \geq 1$ for $d=2$. In this case, the classical doubling inequality for elliptic operators with Lipschitz coefficients can be handled

by a monotonicity formula for the frequency function. If $d \geq 3$, the Lipschitz constant of the coefficients turns out to be $O(N^{\frac{1}{2}\tau})$ after rescaling. A careful calculation shows that the constant in (1.8) is at least $\exp(\exp(CN^\tau))$, if the periodicity is not used. If $d=2$, the Lipschitz constant of the coefficients after rescaling is bounded by C , independent of ε and N . This allows us to obtain a much better estimate in two dimensions.

For $d \geq 3$, one will see in the proof that the estimate in Step 3 leads to the double-exponential growth of the constant in (1.8). What happens when $\varepsilon/r \geq N^{-\frac{1}{2}\tau}$? To gain some intuition, consider a typical harmonic function $w_k = r^k \cos(k\alpha)$ in \mathbb{R}^2 (see [1] or [10]). Note that

$$\frac{\int_{B_1} w_k^2}{\int_{B_\theta} w_k^2} = C\theta^{-2k}.$$

By setting $N = C\theta^{-2k}$, we see that the intrinsic frequency of w_k (i.e., the number of times that w_k changes signs) is approximately $\ln N/(-\ln \theta)$. Now, let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ whose limit is w_k (the homogenized solution) as $\varepsilon \rightarrow 0$. In view of the interior first-order approximation $u_\varepsilon \sim w_k + \varepsilon \chi(x/\varepsilon) \nabla w_k$, the intrinsic frequency of w_k will interact with the frequency of oscillation of the corrector $\chi(x/\varepsilon)$. Particularly, under rescaling, if $r/\varepsilon \approx \ln N$, the frequency of oscillation of the rescaled coefficients $A(rx/\varepsilon)$ (or correctors) is comparable to the intrinsic frequency of w_k . Note that the intrinsic frequency does not change under rescaling. It seems that the resonance between these two frequencies causes the failure of the arguments in Step 1 and Step 2 when $\varepsilon/r \approx (\ln N)^{-1} \geq N^{-\frac{1}{2}\tau}$ (note that τ can be arbitrarily small and thus $N^{-\frac{1}{2}\tau}$ is close to the resonant situation), and we do not have a tool to handle this situation (except for $d=2$). We believe that an effective argument should take advantage of both the periodicity and the Lipschitz continuity of the coefficients.

Our second goal is to obtain an upper bound for the nodal sets of solutions in periodic elliptic homogenization. The study of the $(d-1)$ -dimensional Hausdorff measure of nodal sets centers around Yau's conjecture for Laplace eigenfunctions on smooth manifolds:

$$-\Delta_g \phi_\lambda = \lambda \phi_\lambda, \quad \text{on } \mathcal{M}, \quad (1.12)$$

where \mathcal{M} is a compact smooth Riemannian manifold without boundary. It was conjectured in [11] that the bounds of nodal sets of eigenfunctions in (1.12) are controlled by

$$c\sqrt{\lambda} \leq H^{d-1}(\{x \in \mathcal{M} | \phi_\lambda(x) = 0\}) \leq C\sqrt{\lambda} \quad (1.13)$$

where C, c depend only on the manifold \mathcal{M} and H^{d-1} denotes the $(d-1)$ -dimensional Hausdorff measure. The conjecture (1.13) was shown for real analytic manifolds by Donnelly-Fefferman in [12]. Lin [13] also proved the upper bound for the analytic case, using an approach by frequency functions. We should mention that, by a lifting argument, Yau's conjecture can be reduced to studying the nodal sets of harmonic functions on smooth manifolds. In recent years, there was an important breakthrough made by Logunov and Malinnikova [14,15] and [16]. A polynomial upper bound was given in [14] and the sharp lower bound in the conjecture was shown in [16]. We are interested

in the upper bound of nodal sets for $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ with rapidly oscillating periodic coefficients. The study of nodal sets in homogenization was initiated by Lin and Shen [5], where an implicit upper bound depending on the doubling index was shown. We are able to provide an explicit upper bound.

Theorem 1.2. *Assume that $A = A(y)$ satisfies the conditions (1.2)–(1.4). Let u_ε be a non-zero weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ satisfying (1.5).*

i. *If $d \geq 3$, then for any $\alpha > 8$, it holds that*

$$H^{d-1}(\{x \in B_{\frac{\Lambda}{16}} | u_\varepsilon(x) = 0\}) \leq \exp(CN^\alpha), \quad (1.14)$$

where C depends only on d, Λ, γ and α .

ii. *If $d = 2$, then it holds that*

$$H^1(\{x \in B_{\frac{\Lambda}{16}} | u_\varepsilon(x) = 0\}) \leq \exp(C(\ln N)^2), \quad (1.15)$$

where C depends only on Λ and γ .

The strategy of the proof is as follows. For relatively large ε , we adapt a blow-up argument to obtain the upper bounds of nodal sets. For small ε , the solution u_ε can be approximated by a harmonic function u_0 , and thus the nodal set of u_ε is a small perturbation of the nodal set of u_0 . We then derive a quantitative estimate for the nodal set of u_ε by carefully studying the small perturbations near the nodal set and critical set of u_0 , which has its root in the analogous qualitative estimates obtained in [5]. By iterating such quantitative estimate, we are able to show the upper bound for the nodal sets of u_ε . The restriction $\alpha > 8$ for $d \geq 3$ arises from the doubling inequality (5.2) for $\beta \in (\frac{3}{4}, 1)$. If we consider N to be $\exp(CM)$ for some large constant M , which is the case for the doubling inequality of eigenfunctions, the upper bounds of nodal sets are double exponential functions $\exp(\exp(CM))$. In this sense, the restriction $\alpha > 8$ only affects the constant C in such upper bounds, which does not play an important role. For $d = 2$, we point out that there is no misprint in the exponential (compared to $d \geq 3$). We still have the exponential, because in this situation, instead of the doubling inequality in (1.10), the suboptimal quantitative stratification of the critical set of harmonic functions [17] dominates the upper bound.

Remark 1.3. Several months after the first version of our paper appeared on arXiv, Armstrong, Kuusi and Smart [18] proved the almost optimal doubling inequality at large scales by a more accurate use of the large-scale analyticity, and therefore improved the results in Theorem 1.1. We point out that for $d = 2$, by using an intermediate estimate (4.4) in [18] and the technique of quasi-regular mappings in Section 4 (see Remark 4.6), we can obtain

$$\int_{B_r} u_\varepsilon^2 \leq CN^{\ln \ln N} \int_{B_{\theta r}} u_\varepsilon^2,$$

for all $r > \varepsilon$ if (1.4) is not assumed, or for all $r > 0$ if (1.4) is assumed.

The paper is organized as follows. Section 2 is devoted to a doubling inequality at relatively large scales by the homogenization theory. In section 3, we derive the doubling inequality, using frequency functions, and show how it depends on the large

Lipschitz constant of the coefficients. Then, [Theorem 1.1](#) is proved in [section 4](#) and [Theorem 1.2](#) is proved in [section 5](#). Throughout the paper, the letters $c, C, \hat{C}, \tilde{C}, C_b, c_i$ denote positive constants that do not depend on ε or u_ε , and they may vary from line to line.

2. Homogenization

In this section, we deal with the case $\varepsilon/r \geq N^{-\frac{1}{\beta-\frac{3}{4}}}$ for all dimensions. Indeed, we will prove a quantitative version of [\[5, Theorem 3.1\]](#).

Let $\mathcal{L}_0 = -\nabla \cdot (\hat{A} \nabla)$ be the homogenized operator and \hat{A} be the homogenized coefficient matrix of A (see, e.g., [\[19\]](#) for the general theory of periodic elliptic homogenization). Define the ellipsoid

$$E_r = \{x \in \mathbb{R}^d : \langle (\hat{A})^{-1} x, x \rangle < r^2\}.$$

The following is the main theorem of this section.

Theorem 2.1. *Let $\theta \in (0, 1/2]$ and A satisfy conditions (1.2)–(1.4). There exists $C > 0$ depending only on d and Λ such that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in E_1 and*

$$\int_{E_1} u_\varepsilon^2 \leq N \int_{E_\theta} u_\varepsilon^2,$$

then for any $CN^{\frac{1}{\beta-\frac{3}{4}}} \varepsilon < r < 1 - \sqrt{\varepsilon}$, we have

$$\int_{E_r} u_\varepsilon^2 \leq 2N \int_{E_{\theta r}} u_\varepsilon^2.$$

This follows from [Lemmas 2.2](#) and [2.3](#).

Lemma 2.2. *Let $\theta \in (0, 1/2]$. Suppose u_ε is a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in E_1 satisfying*

$$\int_{E_1} u_\varepsilon^2 \leq N \int_{E_\theta} u_\varepsilon^2.$$

For any $\beta \in (3/4, 1)$, there exist $c, C > 0$, depending only on d, Λ and β , such that if $\varepsilon < cN^{\frac{1}{\beta-\frac{3}{4}}}$, then for any $r \in [\theta, 1 - \sqrt{\varepsilon}]$

$$\int_{E_r} u_\varepsilon^2 \leq N(1 + CN\varepsilon^{\beta-\frac{3}{4}}) \int_{E_{\theta r}} u_\varepsilon^2. \quad (2.1)$$

Proof. Let $t > 0$, to be determined. Since $\text{dist}(\partial E_{1-t}, \partial E_1) \leq Ct$, by the Caccioppoli inequality, we have

$$\int_{E_{1-t}} |\nabla u_\varepsilon|^2 + \int_{E_{1-t}} u_\varepsilon^2 \leq \frac{C}{t^2} \int_{E_1} u_\varepsilon^2 \leq \frac{CN}{t^2} \int_{E_0} u_\varepsilon^2.$$

By the co-area formula, we can find some $c_0 \in (1, 2)$ so that

$$\int_{\partial E_{1-c_0 t}} |\nabla u_\varepsilon|^2 + \int_{\partial E_{1-c_0 t}} u_\varepsilon^2 \leq \frac{CN}{t^3} \int_{E_\theta} u_\varepsilon^2.$$

Without loss of generality, let us simply assume $c_0 = 1$. Hence $u_\varepsilon|_{\partial E_{1-t}} \in H^1(\partial E_{1-t})$. By

[20, Theorem 1.1],

$$\int_{E_{1-t}} (u_\varepsilon - u_0)^2 \leq C\varepsilon^{2\beta} \|u_\varepsilon\|_{H^1(\partial E_{1-t})}^2 \leq \frac{CN\varepsilon^{2\beta}}{t^3} \int_{E_\theta} u_\varepsilon^2, \quad (2.2)$$

where u_0 is the solution of $\mathcal{L}_0(u_0) = 0$ and $u_0 = u_\varepsilon$ on ∂E_{1-t} and $\beta \in (0, 1)$ is arbitrary.

As a result, we have

$$\begin{aligned} \|u_0\|_{L^2(E_{1-t})} &\leq \|u_\varepsilon\|_{L^2(E_1)} + \|u_\varepsilon - u_0\|_{L^2(E_{1-t})} \\ &\leq \sqrt{N}(1 + C\varepsilon^\beta t^{-3/2}) \|u_\varepsilon\|_{L^2(E_\theta)}. \end{aligned} \quad (2.3)$$

Also,

$$\|u_\varepsilon\|_{L^2(E_\theta)} \leq \|u_0\|_{L^2(E_\theta)} + C\sqrt{N}\varepsilon^\beta t^{-3/2} \|u_\varepsilon\|_{L^2(E_\theta)}.$$

We will choose $t < 1$ so that $C\sqrt{N}\varepsilon^\beta t^{-3/2} < 1/2$. Consequently,

$$\|u_\varepsilon\|_{L^2(E_\theta)} \leq (1 - C\sqrt{N}\varepsilon^\beta t^{-3/2})^{-1} \|u_0\|_{L^2(E_\theta)} \leq (1 + C\sqrt{N}\varepsilon^\beta t^{-3/2}) \|u_0\|_{L^2(E_\theta)}. \quad (2.4)$$

Inserting this into (2.3), we have

$$\begin{aligned} \|u_0\|_{L^2(E_{1-t})} &\leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^\beta t^{-3/2})(1 + C\varepsilon^\beta t^{-3/2}) \|u_0\|_{L^2(E_\theta)} \\ &\leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^\beta t^{-3/2}) \|u_0\|_{L^2(E_\theta)}, \end{aligned} \quad (2.5)$$

where we have used the simple fact that $(1 + a)^2 \leq 1 + 3a$ for $a \in [0, 1]$ and enlarged the constant C in the last inequality.

Next, by the interior L^∞ estimate for \hat{A} -harmonic functions, we have

$$\begin{aligned} \|u_0\|_{L^2(E_\theta)} &\leq \|u_0\|_{L^2(E_{\theta(1-t)})} + \|u_0\|_{L^2(E_\theta \setminus E_{\theta(1-t)})} \\ &\leq \|u_0\|_{L^2(E_{\theta(1-t)})} + C\sqrt{\theta t} \|u_0\|_{L^2(E_{1-t})} \end{aligned}$$

Inserting this into (2.5) and choosing t sufficiently small so that $C\sqrt{N}\sqrt{\theta t} < 1/2$, we obtain

$$\|u_0\|_{L^2(E_{1-t})} \leq \sqrt{N}(1 + \sqrt{N}C\varepsilon^\beta t^{-3/2})(1 + C\sqrt{N}\sqrt{\theta t}) \|u_0\|_{L^2(E_{\theta(1-t)})}.$$

Choose $t = \sqrt{\varepsilon}$. We arrive at

$$\|u_0\|_{L^2(E_{1-t})} \leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4}) \|u_0\|_{L^2(E_{\theta(1-t)})}. \quad (2.6)$$

Note that the above calculation goes through only if $\sqrt{N}C\varepsilon^\beta t^{-3/2} < 1/2$ and $C\sqrt{N}\sqrt{t} < 1/2$. This implies that we require

$$\varepsilon \leq C^{-1} N^{\frac{-1}{2(\beta-3/4)}},$$

for some large constant C .

Recall that u_0 is a weak solution of $\mathcal{L}_0(u_0) = 0$ in $E(1 - \sqrt{\varepsilon})$. Let $w_0(x) = u_0(\hat{A}^{\frac{1}{2}}x)$. Then $\Delta w_0 = 0$ in $B_{1-\sqrt{\varepsilon}}$ and (2.6) is equivalent to

$$\|w_0\|_{L^2(B_{1-\sqrt{\varepsilon}})} \leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4}) \|w_0\|_{L^2(B_{\theta(1-\sqrt{\varepsilon})})}. \quad (2.7)$$

Now, as a consequence of the well-known three-sphere theorem for harmonic functions,

$$\varphi(r) = \log_2 \int_{B_{2r}} w_0^2 \quad (2.8)$$

is a convex function in $(-\infty, 0]$ and therefore $\varphi(t) - \varphi(t - c)$ is a nondecreasing function in t , for any fixed $c > 0$. Hence, we obtain from (2.7) that for any $r \in (0, 1 - \sqrt{\varepsilon})$,

$$\|w_0\|_{L^2(B_r)} \leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|w_0\|_{L^2(B_{\theta r})}.$$

(The doubling index with θ is an increasing function of radius.) Again, this is equivalent to

$$\|u_0\|_{L^2(E_r)} \leq \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_0\|_{L^2(E_{\theta r})} \quad (2.9)$$

for any $r \in (0, 1 - \sqrt{\varepsilon})$.

Now, let $r \in [\theta, 1 - \sqrt{\varepsilon})$. It follows by (2.9) that

$$\begin{aligned} \|u_\varepsilon\|_{L^2(E_r)} &\leq \|u_\varepsilon - u_0\|_{L^2(E_r)} + \|u_0\|_{L^2(E_r)} \\ &\leq C\sqrt{N}\varepsilon^{\beta-3/4}\|u_\varepsilon\|_{L^2(E_\theta)} + \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_0\|_{L^2(E_{\theta r})} \\ &\leq C\sqrt{N}\varepsilon^{\beta-3/4}\|u_\varepsilon\|_{L^2(E_r)} + \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_0 - u_\varepsilon\|_{L^2(E_{\theta r})} \\ &\quad + \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_\varepsilon\|_{L^2(E_{\theta r})} \\ &\leq CN\varepsilon^{\beta-3/4}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_\varepsilon\|_{L^2(E_r)} + \sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})\|u_\varepsilon\|_{L^2(E_{\theta r})}, \end{aligned}$$

where we have used the fact $E_\theta \subset E_r$ in the third inequality and (2.2) in the second and last inequalities. Assume further that $\varepsilon \leq cN^{-1/(\beta-3/4)}$. Then

$$\begin{aligned} \|u_\varepsilon\|_{L^2(E_r)} &\leq \frac{\sqrt{N}(1 + C\sqrt{N}\varepsilon^{\beta-3/4})}{1 - CN\varepsilon^{\beta-3/4}}\|u_\varepsilon\|_{L^2(E_{\theta r})} \\ &\leq \sqrt{N}(1 + CN\varepsilon^{\beta-3/4})\|u_\varepsilon\|_{L^2(E_{\theta r})}. \end{aligned}$$

This proves the lemma. \square

Now, if $\varepsilon < cN^{-1/(\beta-3/4)}$, the above lemma allows us to iterate (2.1) down to $r = c^{-1}N^{1/(\beta-3/4)}\varepsilon$. Precisely, if $r = \theta^k > CN^{1/(\beta-3/4)}\varepsilon$ and

$$\int_{E_r} u_\varepsilon^2 \leq A_k \int_{E_{\theta r}} u_\varepsilon^2,$$

with $A_0 = N$, then

$$\int_{E_{\theta r}} u_\varepsilon^2 \leq A_{k+1} \int_{E_{\theta^2 r}} u_\varepsilon^2,$$

where

$$A_{k+1} = A_k(1 + CA_k(\theta^{-k}\varepsilon)^{\beta-3/4}),$$

provided $A_k(\theta^{-k}\varepsilon)^{\beta-3/4} < c$.

Lemma 2.3. *For all $k \leq k_0$ with $\theta^{-k_0}\varepsilon \simeq c_1N^{-1/(\beta-3/4)}$ and $c_1 > 0$ sufficiently small, one has*

$$A_k \leq 2N.$$

Proof. Define $B_k = A_k/N$. Then $B_0 = 1$ and

$$B_{k+1} = B_k(1 + CB_k\theta^{-k(\beta-3/4)}\varepsilon^{\beta-3/4}N) \leq B_k(1 + c_1CB_k\theta^{(k_0-k)(\beta-3/4)}).$$

It follows that

$$B_{k+1} - B_k \leq c_1CB_k^2\delta^{k_0-k},$$

where $\delta = \theta^{\beta-3/4} \leq (1/2)^{\beta-3/4} < 1$. The above inequality yields

$$B_{k+1} \leq B_0 + \sum_{j=0}^k c_1CB_j^2\delta^{k_0-j}. \quad (2.10)$$

We prove by induction that if c_1 is sufficiently small, then $B_k \leq 2$ and $A_k(\theta^{-k}\varepsilon)^{\beta-3/4} \leq 2c_1$ for all $k \leq k_0$. Actually, if

$$c_1 \leq \left(4C \sum_{j=0}^{\infty} \delta^j\right)^{-1} = \frac{1-\delta}{4C},$$

and $B_j \leq 2$ for all $1 \leq j \leq k$, then it is easy to see from (2.10) that $B_{k+1} \leq 2$ and $A_k(\theta^{-k}\varepsilon)^{\beta-3/4} \leq 2N(\theta^{-k_0}\varepsilon)^{\beta-3/4} \leq 2c_1$. This proves the desired estimate. \square

Remark 2.4. Observe that in the above proof, the smoothness of the coefficients has not been used explicitly, except for (2.2) by [20, Theorem 1.1]. But this actually can be replaced by, e.g., [21, Theorem 1.4] with $m=1$, which does not require any smoothness.

Remark 2.5. It is not difficult to see that (2.1) implies the following three-ball inequality with an error term

$$\|u_\varepsilon\|_{L^2(E_{\theta r})} \leq \|u_\varepsilon\|_{L^2(E_r)}^{\frac{1}{2}} \|u_\varepsilon\|_{L^2(E_{\theta^2 r})}^{\frac{1}{2}} + C\left(\frac{\varepsilon}{r}\right)^{\frac{1}{2}(\beta-\frac{3}{4})} \|u_\varepsilon\|_{L^2(E_r)}, \quad (2.11)$$

for any $\theta \in (0, \frac{1}{2}]$ and $\beta \in (\frac{3}{4}, 1)$. Compared to the three-ball inequalities in [6] (see Theorem 4.1 below) and [7], our major term on the right-hand side of (2.11) is sharp. In particular, if $\varepsilon \rightarrow 0$, (2.11) recovers precisely the three-ball inequality for \hat{A} -harmonic functions.

Theorem 2.6. Given arbitrary $\theta \in (0, \Lambda/2]$, there exists $C > 0$ depending only on d and Λ such that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_1 and

$$\int_{B_1} u_\varepsilon^2 \leq N \int_{B_\theta} u_\varepsilon^2, \quad (2.12)$$

then for any $CN^{\frac{1}{\beta-\frac{3}{4}}}\varepsilon < r < 1$, we have

$$\int_{B_r} u_\varepsilon^2 \leq 8N^3 \int_{B_{\theta r}} u_\varepsilon^2.$$

Proof. This is deduced from Theorem 2.1 and the fact

$$B_{\sqrt{\Lambda}r} \subset E_r \subset B_r. \quad (2.13)$$

Indeed, (2.12) and (2.13) imply

$$\int_{E_1} u_\varepsilon^2 \leq N \int_{B_\theta} u_\varepsilon^2 \leq N \int_{E_{\theta/\sqrt{\Lambda}}} u_\varepsilon^2.$$

Note that $\theta \in (0, \Lambda/2]$ implies $\theta/\sqrt{\Lambda} \in (0, \sqrt{\Lambda}/2] \subset (0, 1/2]$. Now, if $r \in \left(CN^{\beta-\frac{3}{4}}\varepsilon, 1 - \sqrt{\varepsilon}\right)$, we may apply [Theorem 2.1](#) (three times) with $\theta' = \theta/\sqrt{\Lambda}$ and obtain

$$\int_{B_r} u_\varepsilon^2 \leq \int_{E_{r/\sqrt{\Lambda}}} u_\varepsilon^2 \leq (2N)^3 \int_{E_{\theta^3\Lambda^{-2}r}} u_\varepsilon^2 \leq 8N^3 \int_{E_{\theta r}} u_\varepsilon^2 \leq 8N^3 \int_{B_{\theta r}} u_\varepsilon^2.$$

For $r \in [1 - \sqrt{\varepsilon}, 1]$ (without loss of generality, assume $\varepsilon < 1/4$), we may apply [Theorem 2.1](#) once to obtain

$$\int_{B_r} u_\varepsilon^2 \leq \int_{B_1} u_\varepsilon^2 \leq N \int_{B_\theta} u_\varepsilon^2 \leq 2N^2 \int_{B_{\theta^2\Lambda^{-1}}} u_\varepsilon^2 \leq 2N^2 \int_{B_{\theta r}} u_\varepsilon^2.$$

This ends the proof. \square

3. Dependence on the lipschitz constant

In this section, we derive the doubling inequality with a large Lipschitz constant, which will be used in the Step 3 of the Proof of [Theorem 1.1](#). We aim to show how the Lipschitz character of the coefficients plays a role in quantitative unique continuation, which seems to be largely unexplored. Assume that

$$\mathcal{L}_1(u) = -\nabla \cdot (A(x)\nabla u) = 0, \quad (3.1)$$

where $A(x)$ satisfies (1.2) and

$$|A(x) - A(y)| \leq L|x - y| \quad (3.2)$$

for some large positive constant $L > 1$. We emphasize that throughout this section, the constant C will never depend on L . Since the L^∞ norm and the L^2 norm of u are comparable, parallel to the assumption (1.7), we may assume the following

$$\|u\|_{L^\infty(B_1)} \leq M\|u\|_{L^\infty(B_\theta)} \quad (3.3)$$

for some large constant $M > 1$.

In order to define the frequency function later, we need to construct the geodesic polar coordinates. The construction of polar coordinates has been obtained in [\[22\]](#). We adopt a slightly different construction of the metric from [\[10, Chapter 3.1\]](#). We follow the construction with an eye on the explicit dependence of the Lipschitz constant L . For $d \geq 3$, we define the Lipschitz metric $\hat{g} = \hat{g}_{ij}(x)dx_i \otimes dx_j$ as follows

$$\hat{g}_{ij}(x) = a^{ij}(x)\det(A(x))^{\frac{1}{d-2}}, \quad (3.4)$$

where $a^{ij}(x)$ is the entry of $A^{-1}(x)$. The case $d=2$ will be discussed in [Remark 3.3](#). Note that \hat{g} is Lipschitz continuous and satisfies

$$|\hat{g}(x) - \hat{g}(y)| \leq CL|x - y|. \quad (3.5)$$

Define

$$r^2 = r^2(x) = \hat{g}_{ij}(0)x_i x_j \quad (3.6)$$

and

$$\psi(x) = \hat{g}^{kl}(x) \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_l}.$$

From (3.6), we can also write

$$\psi(x) = \frac{1}{r^2} \hat{g}^{kl}(x) \hat{g}_{ik}(0) \hat{g}_{jl}(0) x_i x_j.$$

Thus, we can check that $\psi(x)$ is a non-negative Lipschitz function satisfying

$$|\psi(x) - \psi(y)| \leq CL|x - y|, \quad (3.7)$$

where C depends only on d and Λ . We introduce a new metric $g = g_{ij}(x)dx_i \otimes dx_j$ by setting

$$g_{ij}(x) = \psi(x) \hat{g}_{ij}(x). \quad (3.8)$$

We can write the metric g in terms of the intrinsic geodesic polar coordinates $(r, \sigma_1, \dots, \sigma_{d-1})$,

$$g = dr \otimes dr + r^2 b_{ij}(r, \sigma) d\sigma_i \otimes d\sigma_j, \quad (3.9)$$

where b_{ij} satisfies

$$|\partial_r b_{ij}(r, \sigma)| \leq CL, \quad \text{for } i, j = 1, \dots, d-1, \quad (3.10)$$

and C depends only on d and Λ .

The existence of the geodesic polar coordinates (r, σ) allows us to consider geodesic balls. Denote by \mathbb{B}_r the geodesic ball in the metric g of radius r and centered at the origin. In particular, from (3.6) and (3.9), $r(x) = \sqrt{\hat{g}_{ij}(0)x_i x_j}$ is the geodesic distance from x to the origin in the new metric g . Thus, it is conformal to the usual Euclidean ball. For convenience of presentation, we may assume that the geodesic balls coincide with the Euclidean balls, i.e., $\hat{g}_{ij}(0) = \delta_{ij}$.

Let

$$\eta(x) = \psi^{-\frac{d-2}{2}}. \quad (3.11)$$

Obviously, $\eta(x)$ is a Lipschitz function satisfying

$$C_1 \leq \eta(x) \leq C_2, \quad (3.12)$$

where C_1 and C_2 depend on d and Λ . In the polar coordinates,

$$|\partial_r \eta(r, \sigma)| \leq CL. \quad (3.13)$$

In this new metric g , Eq. (3.1) can be written as

$$-\nabla_g \cdot (\eta(x) \nabla_g u(x)) = 0 \quad \text{in } B_1. \quad (3.14)$$

Let

$$D(r) = \int_{B_r} \eta |\nabla_g u|^2 dV_g \quad (3.15)$$

and

$$H(r) = \int_{\partial B_r} \eta u^2 dS_g, \quad (3.16)$$

where dS_g represents the area element of ∂B_r under the metric g . We define the frequency function by

$$\mathcal{N}(r) = \frac{rD(r)}{H(r)}. \quad (3.17)$$

For future application, we will also use the notation $\mathcal{N}(p, r)$ to specify the center of the ball $B_r(p)$ in the definition of frequency function.

Lemma 3.1. *Let $u \in H^1(B_1)$ be a nontrivial solution of (3.1). There exists a positive constant C depending on d and Λ such that*

$$\bar{\mathcal{N}}(r) = \exp(CLr)\mathcal{N}(r) \quad (3.18)$$

is a non-decreasing function of $r \in (0, 1)$.

Proof. The proof of the lemma is essentially contained in [1]. Since we want to show the explicit dependence of the Lipschitz constant L in the estimates, we sketch the proof by considering the role of L . Taking derivative with respect to r for \mathcal{N} , we have

$$\frac{\mathcal{N}'(r)}{\mathcal{N}(r)} = \left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \right). \quad (3.19)$$

In order to prove the lemma, it suffices to show

$$\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \geq -CL. \quad (3.20)$$

Thus, we consider the derivatives of $H(r)$ and $D(r)$, respectively. Setting $b(r, \sigma) = |\det(b_{ij}(r, \sigma))|$. Note that $dS_g = r^{d-1} \sqrt{b(r, \sigma)} d\sigma$. We write $H(r)$ as

$$H(r) = r^{d-1} \int_{\partial B_1} \eta(r, \sigma) u^2(r, \sigma) \sqrt{b(r, \sigma)} d\sigma. \quad (3.21)$$

Taking derivative with respect to r , one has

$$H'(r) = \frac{d-1}{r} H(r) + \int_{\partial B_r} \frac{1}{\sqrt{b}} \partial_r (\eta \sqrt{b}) u^2 dS_g + 2 \int_{\partial B_r} \eta u \partial_r u dS_g, \quad (3.22)$$

where $\partial_r u = \langle \nabla_g u, \frac{x}{r} \rangle$ on ∂B_r . By (3.10), (3.12) and (3.13), we have

$$H'(r) = \left(\frac{d-1}{r} + O(L) \right) H(r) + 2 \int_{\partial B_r} \eta u \partial_r u dS_g. \quad (3.23)$$

Multiplying both sides of (3.14) by u and performing the integration by parts give that

$$D(r) = \int_{B_r} \eta |\nabla_g u|^2 dV_g = \int_{\partial B_r} \eta u \partial_r u dS_g. \quad (3.24)$$

It follows that

$$H'(r) = \left(\frac{d-1}{r} + O(L) \right) H(r) + 2D(r). \quad (3.25)$$

Similarly, we may compute the derivative of $D(r)$ as in [1] and obtain

$$D'(r) = \left(\frac{d-2}{r} + O(L) \right) D(r) + 2 \int_{\partial B_r} \eta (\partial_r u)^2 dS_g. \quad (3.26)$$

Combining the estimates (3.25) and (3.26), and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} &= O(L) + 2 \frac{\int_{\partial B_r} \eta (\partial_r u)^2 dS_g}{\int_{\partial B_r} \eta u \partial_r u dS_g} - 2 \frac{\int_{\partial B_r} \eta u \partial_r u dS_g}{\int_{\partial B_r} \eta u^2 dS_g} \\ &\geq O(L). \end{aligned}$$

This proves (3.20) and thus the lemma. \square

Next we derive the doubling inequality with an explicit dependence on L .

Lemma 3.2. *Let u be a solution of (3.1) satisfying (3.2) and (3.3). For a fixed constant $0 < \theta \leq \frac{1}{2}$, we have*

$$\|u\|_{L^2(B_r)} \leq M^{C_1 e^{C_2 L}} \|u\|_{L^2(B_{\theta r})} \quad (3.27)$$

for $0 < r \leq \frac{1}{2}$, where C_1 depends on θ , and C_2 depends on d, Λ .

Proof. From (3.25) and the definition of $\bar{\mathcal{N}}(r)$, we have

$$\left(\ln \frac{H(r)}{r^{d-1}} \right)' = O(L) + \frac{2}{r} \bar{\mathcal{N}}(r) \exp(-CLr). \quad (3.28)$$

Note that here $O(L)$ is a function in r satisfying $-CL \leq O(L) \leq CL$. We would like to obtain an upper bound and a lower bound for the quotient $H(r_2)/H(r_1)$ with $0 < r_1 < r_2$. To find the upper bound, we integrate the equality (3.28) from r_1 to r_2 and use the monotonicity of $\bar{\mathcal{N}}(r)$ to obtain

$$\ln \frac{H(r_2)}{r_2^{d-1}} - \ln \frac{H(r_1)}{r_1^{d-1}} \leq CL(r_2 - r_1) + 2\bar{\mathcal{N}}(r_2) \ln \left(\frac{r_2}{r_1} \right) \exp(-CLr_1). \quad (3.29)$$

Taking the exponential of both sides gives the upper bound

$$\frac{H(r_2)}{H(r_1)} \leq e^{CL(r_2 - r_1)} \left(\frac{r_2}{r_1} \right)^{2\bar{\mathcal{N}}(r_2) \exp(-CLr_1) + d - 1}. \quad (3.30)$$

To see the lower bound, we integrate (3.28) from r_1 to r_2 and apply the monotonicity of $\bar{\mathcal{N}}(r)$ again to obtain

$$\ln \frac{H(r_2)}{r_2^{d-1}} - \ln \frac{H(r_1)}{r_1^{d-1}} \geq -CL(r_2 - r_1) + 2\bar{\mathcal{N}}(r_1) \exp(-CLr_2) \ln \left(\frac{r_2}{r_1} \right). \quad (3.31)$$

Raising to the exponential form, we have

$$\frac{H(r_2)}{H(r_1)} \geq e^{-CL(r_2-r_1)} \left(\frac{r_2}{r_1} \right)^{2\bar{\mathcal{N}}(r_1) \exp(-CLr_2) + d-1}. \quad (3.32)$$

Combining (3.30) and (3.32), we arrive at

$$e^{-CL(r_2-r_1)} \left(\frac{r_2}{r_1} \right)^{2\bar{\mathcal{N}}(r_1) \exp(-CLr_2) + d-1} \leq \frac{H(r_2)}{H(r_1)} \leq e^{CL(r_2-r_1)} \left(\frac{r_2}{r_1} \right)^{2\bar{\mathcal{N}}(r_2) \exp(-CLr_1) + d-1}. \quad (3.33)$$

Next we want to show an upper bound for $\bar{\mathcal{N}}(\frac{3}{4})$. Let $r_2 = \frac{3}{4}$ and $0 < r_1 = r < \frac{3}{4}$. From the estimate (3.32), we have

$$e^{-CL(\frac{3}{4}-r)} \left(\frac{\frac{3}{4}}{r} \right)^{d-1} \leq \frac{H(\frac{3}{4})}{H(r)}. \quad (3.34)$$

Using the fact that $0 < \theta \leq \frac{1}{2}$, we have

$$\begin{aligned} \|u\|_{L^\infty(B_\theta)}^2 &\leq C \int_{B_{\frac{3}{4}}} u^2 dV_g \leq C \int_0^{\frac{3}{4}} H(r) dr \\ &\leq C \int_0^{\frac{3}{4}} r^{d-1} e^{CL(\frac{3}{4}-r)} H\left(\frac{3}{4}\right) dr \\ &\leq C e^{CL} H\left(\frac{3}{4}\right), \end{aligned} \quad (3.35)$$

where C depends on d and Λ . Obviously,

$$\|u\|_{L^\infty(B_1)}^2 \geq C \int_{\partial B_1} u^2 dS_g. \quad (3.36)$$

Therefore, from (3.3), (3.32) and (3.35), we have

$$\begin{aligned} M^2 &\geq \frac{\|u\|_{L^\infty(B_1)}^2}{\|u\|_{L^\infty(B_\theta)}^2} \geq \frac{CH(1)}{Ce^{CL} H\left(\frac{3}{4}\right)} \\ &\geq e^{-CL} \left(\frac{4}{3}\right)^{d-1+2\bar{\mathcal{N}}\left(\frac{3}{4}\right) e^{-CL}}. \end{aligned} \quad (3.37)$$

Thus, we can get an upper bound for $\bar{\mathcal{N}}(\frac{3}{4})$ as

$$\bar{\mathcal{N}}\left(\frac{3}{4}\right) \leq Ce^{CL} \ln M, \quad (3.38)$$

where $M > 1$ is a large constant. Choosing any $r \leq \frac{1}{2}$, we integrate (3.28) from θr to r , by the monotonicity of $\bar{\mathcal{N}}$, we derive that

$$\begin{aligned} \ln \frac{H(r)}{r^{d-1}} - \ln \frac{H(\theta r)}{(\theta r)^{d-1}} &\leq CLr + 2\bar{\mathcal{N}}\left(\frac{3}{4}\right) \ln \frac{1}{\theta} \\ &\leq CLr + Ce^{CL} \ln M \ln \frac{1}{\theta}. \end{aligned} \quad (3.39)$$

Thus, we obtain that

$$\begin{aligned} H(r) &\leq \exp \left(CLr + e^{CL} \ln M \ln \frac{1}{\theta} \right) H(\theta r) \\ &\leq \theta^{1-d} M^{-(\ln \theta)e^{CL}} H(\theta r), \end{aligned} \quad (3.40)$$

where $M > 1$ is large. By further integrations, we can also obtain that

$$\|u\|_{L^2(B_r)} \leq \theta^{\frac{-d}{2}} M^{-(\ln \theta)e^{CL}} \|u\|_{L^2(B_{\theta r})} \quad (3.41)$$

for $0 < r \leq \frac{1}{2}$, where C depends only on d and Λ . \square

Remark 3.3. For the case $d = 2$, we introduce a new variable to apply a lifting argument. Let $v(x, t) = e^t u(x)$. Then the new function $v(x, t)$ satisfies the equation

$$-\nabla \cdot (\tilde{A}(x, t) \nabla v) + v = 0 \quad \text{in } \hat{B}_1, \quad (3.42)$$

where

$$\tilde{A}(x, t) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}$$

and \hat{B}_1 is the ball with radius 1 in \mathbb{R}^3 . It is easy to see that \tilde{A} satisfies the conditions (1.2) and (3.2). Following the procedure performed as $d \geq 3$, we are able to introduce the new metric g and geodesic polar coordinates. Thus, in the metric g as (3.8) and η as (3.13), we have

$$-\nabla_g \cdot (\eta(x) \nabla_g v) + c_g v = 0 \quad \text{in } \hat{B}_1, \quad (3.43)$$

where $c_g = \frac{1}{\sqrt{\det g}}$. As before, we could make use of the monotonicity of the frequency function to obtain the doubling inequality. Precisely, we may define

$$D(r) = \int_{\hat{B}_r} \eta |\nabla_g v|^2 + c_g v^2 dV_g \quad (3.44)$$

and

$$H(r) = \int_{\partial \hat{B}_r} \eta v^2 dS_g. \quad (3.45)$$

Then the frequency function is defined as

$$\mathcal{N}(r) = \frac{rD(r)}{H(r)}. \quad (3.46)$$

Following the proof of Lemma 3.1 and [10, Theorem 3.2.1], we can obtain the almost monotonicity of $\mathcal{N}(r)$. That is, for any $r_0 \in (0, 1)$, it holds that

$$\exp(CLr)\mathcal{N}(r) \leq \exp(CLr_0) + \exp(CLr_0)\mathcal{N}(r_0) \quad (3.47)$$

for any $r \in (0, r_0)$ where C depends on Λ . By mimicking the argument in the proof of Lemma 3.2, we can obtain the doubling inequality for v in \hat{B}_r . This also leads to the doubling inequality for u as (3.27) in B_r .

Remark 3.4. For a better estimate when $d=2$, see [Remark 4.6](#).

4. Proof of Theorem 1.1

This section is devoted to the Proof of [Theorem 1.1](#). Step 1 and Step 3 of the proof have been handled in [Section 2](#) and [Section 3](#), respectively. For convenience of presentation, we choose β such that $\frac{1}{\beta-\frac{3}{4}} = 5$. Our argument works for any $\beta \in (\frac{3}{4}, 1)$. To handle the case $N^{-5} \leq \varepsilon/r \leq N^{-\frac{1}{2\tau}}$ in Step 2, we will use particular doubling properties, obtained from some sort of “analyticity,” as a transition in order to improve our estimates. Indeed, for $d \geq 3$, we will employ the three-ball inequality with a sharp exponential error term obtained in [\[6\]](#); for $d=2$, we will use quasi-regular mappings [\[9\]](#) which provides a much better doubling estimate.

We first introduce a three-ball inequality for all dimensions $d \geq 2$. For convenience, we define the normalized L^2 norm by

$$\|u\|_{\underline{L}^2(B_t)} = \left(\int_{B_t} u^2 \right)^{1/2}.$$

The following theorem is essentially taken from [\[6, Theorem 1.4\]](#), which is a corollary of the “large-scale analyticity” in periodic homogenization. This result relies on the periodic structure of the coefficients, but does not depend on the smoothness of coefficients.

Theorem 4.1. *For each $\hat{\tau} \in (0, 1/2)$, there exist $c = c(d, \Lambda) > 0$ and $\theta = \theta(\hat{\tau}, d, \Lambda) \in (0, 1/2]$ such that if u is a weak solution of $\mathcal{L}_1(u) = 0$ in B_R with $\theta^2 R > 2$, then*

$$\|u\|_{\underline{L}^2(B_{\theta R})} \leq \|u\|_{\underline{L}^2(B_{\theta^2 R})} \|u\|_{\underline{L}^2(B_R)}^{1-\hat{\tau}} + \exp(-c\theta^2 R) \|u\|_{\underline{L}^2(B_R)}. \quad (4.1)$$

As a simple corollary, we have

Corollary 4.2. *Let u be a weak solution of $\mathcal{L}_1(u) = 0$ in B_R . For every $\alpha_1 > 0$, there exist $C > 0$ and $\theta \in (0, 1/2)$ such that if*

$$N > C \quad \text{and} \quad \theta^2 R \geq C \ln N, \quad (4.2)$$

and

$$\|u\|_{\underline{L}^2(B_R)} \leq N \|u\|_{\underline{L}^2(B_{\theta R})}, \quad (4.3)$$

then

$$\|u\|_{\underline{L}^2(B_{\theta R})} \leq CN^{1+\alpha_1} \|u\|_{\underline{L}^2(B_{\theta^2 R})}. \quad (4.4)$$

The sharp exponential tail in (4.1) is crucial for our purpose which is related to the condition (4.2). The lower bound $\ln N$ in (4.2) allows us to iterate the estimate down to a scale at which the classical theory in [Section 3](#) may apply.

Next, for the case $d=2$, we introduce a stronger doubling property using quasi-regular mappings (related to complex analyticity). We briefly give some background on

quasi-regular mappings. For a detailed account of this topic, please refer to the presentation in [9, 23, Chapter II.6] and references therein.

Let u be a weak solution of the equation $\mathcal{L}_1(u) = 0$ in B_R with only bounded measurable coefficients satisfying (1.2). Let $z = x + iy$ for $x, y \in \mathbb{R}$. Define

$$\partial_z f = \frac{1}{2}(\partial_x f + i\partial_y f), \quad \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f - i\partial_y f). \quad (4.5)$$

We introduce a stream function (the generalized harmonic conjugate) associated with u as

$$\nabla v = JA\nabla u,$$

where J is the rotation matrix in the plane

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $f = u + iv$. Then we have $f \in H_{\text{loc}}^1(B_R)$ and satisfies

$$\partial_z f = \mu \partial_z f + \nu \overline{\partial_z f},$$

where the complex valued function μ and ν can be explicitly written in term of A and

$$|\mu| + |\nu| \leq \frac{1 - \Lambda}{1 + \Lambda} < 1.$$

Hence, $f : B_R \rightarrow \mathbb{C}$ is a $\frac{1}{\Lambda}$ -quasi-regular mapping. Moreover, it can be written as $f = F \circ \hat{\chi}$, where F is holomorphic and $\hat{\chi} : B_R \rightarrow B_R$ is a $\frac{1}{\Lambda}$ -quasiconformal homeomorphism satisfying $\hat{\chi}(0) = 0$ and $\hat{\chi}(1) = 1$. Define

$$\mathcal{B}_{\hat{r}} = \{z \in B_R : |\hat{\chi}(z)| < \hat{r}\}.$$

The quasi-balls $\mathcal{B}_{\hat{r}}$ are comparable to the standard Euclidean balls in the sense

$$\mathcal{B}_R = B_R, \quad \text{and} \quad B_{R(\frac{\hat{r}}{CR})^{\frac{1}{2}}} \subset \mathcal{B}_{\hat{r}} \subset B_{R(\frac{C\hat{r}}{R})^{\alpha}}, \quad \text{for } r < R, \quad (4.6)$$

where $C \geq 1$ and $0 < \alpha < 1$ depend only on Λ . Observe that $\mathcal{B}_{\hat{r}}$ tends to be singular if $\hat{r} \ll R$, which fortunately is not too restrictive as we only use it in the transition at intermediate scales.

From the fact that F is a holomorphic function, the following doubling property holds [9].

Lemma 4.3. *If $u \in H_{\text{loc}}^1(B_R)$ is a nonzero weak solution of $\mathcal{L}_1(u) = 0$ in B_R , then*

$$\frac{\|u\|_{L^\infty(\mathcal{B}_{\hat{r}})}}{\|u\|_{L^\infty(\mathcal{B}_{\frac{\hat{r}}{2}})}} \leq C \frac{\|u\|_{L^\infty(\mathcal{B}_R)}}{\|u\|_{L^\infty(\mathcal{B}_{\frac{R}{4}})}}, \quad \text{for } 0 < \hat{r} \leq R. \quad (4.7)$$

Remark 4.4. Note that Lemma 4.3 does not use periodicity, and it is also true for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$, with a constant independent of ε . This gives “almost monotonicity of the doubling constant,” a statement stronger than that of Theorem 1.1, but for quasi-balls, as opposed to the usual balls. As we pointed out above, quasi-balls are difficult to manage as the radius r goes to zero. Because of this, we still need to use the periodicity assumption and Step 1 below, when $d = 2$, to cover the range $r \geq CN^5\varepsilon$. We then apply

Lemma 4.3 in the range $C\varepsilon < r < CN^5\varepsilon$, using (4.6) since r is not too small. Finally, the case $0 < r < C\varepsilon$ is handled by scaling and **Lemma 3.2**. The details are below.

Equipped with **Corollary 4.2** and **Lemma 4.3**, we are ready to prove **Theorem 1.1**.

Proof of Theorem 1.1. According to the relationship between ε and N , one needs to consider three cases based on the comparison of ε with N^{-5} and $N^{-\frac{5}{2}}$ (or 1 for $d=2$). Without loss of generality, we may just consider the most complicated case $\varepsilon \leq N^{-5}$, since all the three steps listed in the introduction will be involved as r approaches 0. Hence, we fix ε and N so that $CN^5\varepsilon < 1$, and then discuss the different ranges of r .

Step 1: $CN^5\varepsilon < r < 1$. Under either (1.7) or (1.9), **Theorem 2.6** implies

$$\int_{B_r} u_\varepsilon^2 \leq 8N^3 \int_{B_{\theta r}} u_\varepsilon^2, \quad (4.8)$$

for any given $\theta \in (0, \frac{1}{2}]$. This estimate holds for all dimensions $d \geq 2$.

Step 2: In this step, we need to treat the cases $d \geq 3$ and $d = 2$ separately.

Case 1: $d \geq 3$ and $C\varepsilon N^{\frac{5}{2}} < r < CN^5\varepsilon$ for any fixed $\tau > 0$. Let m be the smallest integer so that $\theta^{-m}r > CN^5\varepsilon$. If N is bounded by some absolute constant, then Step 2 is not needed. Since $r > C\varepsilon N^{\frac{5}{2}}$, for sufficiently large N , m satisfies

$$m \leq \frac{6 \ln N}{-\ln \theta}. \quad (4.9)$$

Because of (4.8), we have

$$\int_{B_{\theta^{-m}r}} u_\varepsilon^2 \leq 8N^3 \int_{B_{\theta^{-m+1}r}} u_\varepsilon^2. \quad (4.10)$$

Let $M_0 = 8N^3$ and M_j be the constant such that

$$\int_{B_{\theta^{-m+j}r}} u_\varepsilon^2 \leq M_j \int_{B_{\theta^{-m+j+1}r}} u_\varepsilon^2. \quad (4.11)$$

The goal is to estimate M_m with m comparable to the bound in (4.9).

Thanks to **Corollary 4.2**, and by rescaling, we know that for a given $\alpha_1 > 0$ with θ small enough, we have

$$M_j = CM_{j-1}^{1+\alpha_1}. \quad (4.12)$$

Note that the left-end restriction $r > C\varepsilon \ln M_j$ is needed in order to apply **Corollary 4.2**, due to (4.2). This can be guaranteed if we eventually show $M_j \leq M_m < C \exp(N^{\frac{5}{2}})$.

We now proceed to estimate M_j . Using the initial condition $M_0 = 8N^3$, one can show explicitly that

$$M_j = \exp(-\ln C/\alpha_1) \exp \left[(1 + \alpha_1)^j (3 \ln N + \ln(8C^{1/\alpha_1})) \right]. \quad (4.13)$$

It follows from (4.9) that

$$M_m \leq C \exp \left[\exp (\ln (1 + \alpha_1) (-\ln \theta)^{-1} 6 \ln N) \cdot (3 \ln N + \ln (8C^{1/\alpha_1})) \right]. \quad (4.14)$$

Note that τ is any given positive constant. Then, we may choose α_1 small enough (hence θ is also small), so that

$$\frac{\tau}{3} \geq 6 \ln (1 + \alpha_1) (-\ln \theta)^{-1}. \quad (4.15)$$

Thus, if N is large enough,

$$M_m \leq C \exp (N^{\frac{1}{2}\tau}). \quad (4.16)$$

This implies that for any $CN^{\frac{1}{2}\tau} \varepsilon < r < CN^5 \varepsilon$, we have

$$\int_{B_r} u_\varepsilon^2 \leq C \exp (N^{\frac{1}{2}\tau}) \int_{B_{\theta r}} u_\varepsilon^2. \quad (4.17)$$

Case 2: $d=2$ and $C\varepsilon < r < CN^5 \varepsilon$. From (4.8) with $\theta = \frac{\Lambda}{2}$ in Step 1, for $R \simeq C\varepsilon N^5$,

$$\int_{B_R} u_\varepsilon^2 \leq 8N^3 \int_{B_{\frac{\Lambda R}{2}}} u_\varepsilon^2.$$

By the L^∞ norm estimates, it follows that

$$\|u_\varepsilon\|_{L^\infty(B_R)} \leq CN^{\frac{3}{2}} \|u_\varepsilon\|_{L^\infty(B_{\frac{R}{2}})}. \quad (4.18)$$

We would like to apply [Lemma 4.3](#) to u_ε . From the relation (4.6) of quasi-balls $\mathcal{B}_{\hat{r}}$ and the standard balls, as well as the iteration of the doubling inequality (4.18), we have

$$\frac{\|u_\varepsilon\|_{L^\infty(\mathcal{B}_R)}}{\|u_\varepsilon\|_{L^\infty(\mathcal{B}_{\frac{R}{4}})}} \leq \frac{\|u_\varepsilon\|_{L^\infty(B_R)}}{\|u_\varepsilon\|_{L^\infty(B_{R(4C)^{-\frac{1}{2}}})}} \leq CN^k, \quad (4.19)$$

where k depends only on Λ . Thus, (4.7) implies that for any $0 < \hat{r} < R$

$$\frac{\|u_\varepsilon\|_{L^\infty(\mathcal{B}_{\hat{r}})}}{\|u_\varepsilon\|_{L^\infty(B_{\frac{R}{2}})}} \leq CN^k. \quad (4.20)$$

In order to establish a doubling inequality at small scale on standard Euclidean balls, we iterate the above doubling inequality m times to obtain

$$\|u_\varepsilon\|_{L^\infty(\mathcal{B}_{\hat{r}})} \leq (CN^k)^m \|u_\varepsilon\|_{L^\infty(B_{\frac{\hat{r}}{2^m}})}. \quad (4.21)$$

By the relation (4.6),

$$\|u_\varepsilon\|_{L^\infty(B_{R(\frac{\hat{r}}{CR})^{\frac{1}{2}}})} \leq (CN^k)^m \|u_\varepsilon\|_{L^\infty(B_{R(\frac{C\hat{r}}{2^m R})^\alpha})}.$$

We choose $m > 0$ to be the smallest integer so that

$$R \left(\frac{C\hat{r}}{2^m R} \right)^\alpha \leq \frac{1}{2} R \left(\frac{\hat{r}}{CR} \right)^{\frac{1}{\alpha}}. \quad (4.22)$$

Consequently,

$$\|u_\varepsilon\|_{L^\infty\left(B_{R(\frac{\hat{r}}{CR})^{\frac{1}{2}}}\right)} \leq (CN^k)^m \|u_\varepsilon\|_{L^\infty\left(B_{\frac{1}{2}R(\frac{\hat{r}}{CR})^{\frac{1}{2}}}\right)}. \quad (4.23)$$

Note that \hat{r} , satisfying $0 < \hat{r} < R \simeq C\varepsilon N^5$, is arbitrary and m is chosen depending on \hat{r} . We now assume $\hat{r} \geq C\varepsilon^\alpha R^{1-\alpha} \simeq C\varepsilon N^{5(1-\alpha)}$. Hence, $r := R(\frac{\hat{r}}{CR})^{\frac{1}{2}} \geq C\varepsilon$. Moreover, from (4.22), we have $m \leq C \ln N$, where C depends only on Λ . Thus, it follows from (4.23) that

$$\|u_\varepsilon\|_{L^\infty(B_r)} \leq CN^{C \ln N} \|u_\varepsilon\|_{L^\infty(B_{\frac{r}{2}})} \quad (4.24)$$

for all $C\varepsilon < r < R \simeq C\varepsilon N^5$. Since L^∞ norm can be replaced by L^2 norm in the above inequality, we derive the desired estimate for the case $d=2$.

Step 3: For $r < C\varepsilon N^{\frac{1}{2}\tau}$ (or $r < C\varepsilon$ for $d=2$), by rescaling, the equation may be reduced to the case in which the Lipschitz constant of coefficients is bounded by $CN^{\frac{1}{2}}$ (bounded by C for $d=2$). It follows from (3.27) and (4.17) that for $d \geq 3$ and any $0 < r < C\varepsilon N^{\frac{1}{2}\tau}$,

$$\begin{aligned} \int_{B_r} u_\varepsilon^2 &\leq C \left[\exp(N^{\frac{1}{2}\tau}) \right]^{C \exp(N^{\frac{1}{2}\tau})} \int_{B_{\theta r}} u_\varepsilon^2 \\ &\leq \exp(\exp(CN^\tau)) \int_{B_{\theta r}} u_\varepsilon^2. \end{aligned}$$

For $d=2$, it follows from (3.27) and (4.24) that for any $0 < r < C\varepsilon$,

$$\int_{B_r} u_\varepsilon^2 \leq C [N^{C \ln N}]^C \int_{B_{\frac{r}{2}}} u_\varepsilon^2 \leq CN^{C \ln N} \int_{B_{\frac{r}{2}}} u_\varepsilon^2. \quad (4.25)$$

Note that $N^{C \ln N} = \exp(C(\ln N)^2)$. This completes the Proof of [Theorem 1.1](#). \square

Remark 4.5. It was shown in [6] that the exponential tail in (4.1) is sharp (up to the end point $\hat{\tau} = \frac{1}{2}$), without any smoothness assumption on the coefficients. If the critical $\hat{\tau} = 1/2$ in (4.1) can also be achieved (which seems like a very difficult task), then [Corollary 4.2](#) with $\alpha_1 = 0$ would follow. By the argument in Step 2, this would yield the estimate

$$\int_{B_r} u_\varepsilon^2 \leq CN^k \int_{B_{\theta r}} u_\varepsilon^2 \quad (4.26)$$

for $C\varepsilon \ln N \leq r \leq C\varepsilon N^5$. If we then apply (3.27) as in Step 3, with Lipschitz constant $C \ln N$, we would obtain the bound $C(N) = \exp(CN^C)$ for $0 < r \leq \frac{1}{2}$ (for the range $0 < r < C\varepsilon$, (3.27) does give the optimal bound). On the other hand, the estimate (3.27) in term of the large Lipschitz constant L may not be sharp. This is a well-known difficult issue in quantitative unique continuation, for which none of the currently known methods apply. Any improvement here would have many consequences. Alternatively, in the range $C\varepsilon \leq r \leq C\varepsilon \ln N$, one could try to use a method taking advantage of both periodicity and smoothness. No such method is available at the moment.

Remark 4.6. If we consider, when $d=2$, elliptic operators with Lipschitz coefficients, with Lipschitz constant $L > 1$ (and no periodicity assumption), we can obtain the improved bound $M^{C_1 \ln L}$ in [Lemma 3.2](#). To show this, we break down the scales into $\frac{1}{L} \leq r < 1$ and $0 < r < \frac{1}{L}$. For the case $\frac{1}{L} \leq r < 1$, we use [Lemma 4.3](#), (4.6) and the

argument from (4.19) to (4.23). For the case $0 < r < \frac{1}{L}$, we scale to reduce to the case $L=1$ and then apply [Lemma 3.2](#) as it stands. This may suggest that the bound in [Lemma 3.2](#) is not optimal, also for $d \geq 3$.

Remark 4.7. The disadvantage of [Theorem 4.1](#) for $d \geq 3$ is that θ may be very small. If we do not apply [Theorem 4.1](#) to improve the exponent τ , Step 1 and 3 in the Proof of [Theorem 1.1](#) allows θ to be any number in $(0, \Lambda/2]$. In particular, under (1.5), for any $\beta \in (\frac{3}{4}, 1)$, we have

$$\int_{B_r} u_\varepsilon^2 \leq \exp \left(\exp \left(CN^{\frac{1}{\beta-4}} \right) \right) \int_{B_{\Lambda r/2}} u_\varepsilon^2. \quad (4.27)$$

For convenience, we will use this doubling inequality (4.27), instead of (1.8), in estimating the upper bound of nodal sets in the next section. The price is that α has to be larger than 8 in (1.14).

5. Upper bounds of nodal sets

In this section, we study of the upper bounds of nodal sets for u_ε , where u_ε is a non-zero solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ satisfying (1.5). We will focus on the general treatment for all dimensions $d \geq 2$ and with an eye toward $d=2$ in the end. Throughout this section, up to a change of variable, we assume $\mathcal{L}_0 = -\Delta$. Note that in this case, E_r 's are just balls, and in view of [Theorem 2.1](#), the assumption (1.5) can be replaced by

$$\int_{B_2} u_\varepsilon^2 \leq N \int_{B_1} u_\varepsilon^2, \quad (5.1)$$

and (4.27) holds with $\Lambda = 1$.

5.1. Small scales

We first show that a doubling inequality centered at 0 implies the doubling inequality with shifted centers.

Lemma 5.1. *Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_2 satisfying (5.1). Then for any $x \in B_{1/3}$ and $B_{2r}(x) \subset B_2$, we have*

$$\int_{B_{2r}(x)} u_\varepsilon^2 \leq \exp \left(\exp \left(CN^{\frac{2}{\beta-4}} \right) \right) \int_{B_r(x)} u_\varepsilon^2. \quad (5.2)$$

Proof. Let us first assume $C\varepsilon N^{\frac{1}{\beta-4}} < 1$ for some large C . In this case, by [Theorem 2.1](#) with $\theta = 1/2$, we have

$$\int_{B_2} u_\varepsilon^2 \leq N \int_{B_1} u_\varepsilon^2 \leq 2N^2 \int_{B_{1/2}} u_\varepsilon^2. \quad (5.3)$$

Now, for any $x \in B_{1/3}$, note that $B_{1/2} \subset B_{5/6}(x)$ and $B_{5/6}(x) \subset B_2$. It follows from (5.3) that

$$\int_{B_{5/3}(x)} u_\varepsilon^2 \leq \int_{B_2} u_\varepsilon^2 \leq 2N^2 \int_{B_{1/2}} u_\varepsilon^2 \leq 2N^2 \int_{B_{5/6}(x)} u_\varepsilon^2. \quad (5.4)$$

Since [Theorem 2.1](#) and (4.27) are invariant under translation, we can apply them in $B_{5/3}(x)$ with N replaced by $2N^2$. Thus, for all $r \in (0, 5/6)$,

$$\int_{B_{2r}(x)} u_\varepsilon^2 \leq \exp(\exp(CN^{\frac{2}{\beta-4}})) \int_{B_r(x)} u_\varepsilon^2.$$

To handle the case $C\varepsilon N^{\frac{1}{\beta-4}} \geq 1$, we use (4.27) directly and obtain

$$\int_{B_2} u_\varepsilon^2 \leq N \int_{B_1} u_\varepsilon^2 \leq \exp(\exp(CN^{\frac{1}{\beta-4}})) \int_{B_{1/2}} u_\varepsilon^2.$$

Then the desired estimate follows from the same idea as the first case and a blow up argument as in Step 3 in the Proof of [Theorem 1.1](#). \square

Let us define the nodal sets as

$$Z(u_\varepsilon) = \{x \in B_2 | u_\varepsilon = 0\} \quad (5.5)$$

and the density function of nodal sets as

$$E_\varepsilon(y, r) = \frac{H^{d-1}(Z(u_\varepsilon) \cap B_r(y))}{r^{d-1}}. \quad (5.6)$$

Based on [Lemma 5.1](#) and a blow up argument, we can estimate the Hausdorff measure of the nodal set of u_ε in small balls.

Lemma 5.2. *For any $0 < r < 1/3$ and $x_0 \in B_{1/3}$ such that $B_r(x_0) \subset B_{1/3}$,*

$$E_\varepsilon(x_0, r) \leq \left(1 + \frac{r}{\varepsilon}\right) \exp(CN^{\frac{2}{\beta-4}}), \quad (5.7)$$

where C depends on d, Λ, β and γ .

Proof. First of all, we consider the case $0 < r \leq \varepsilon$ and $B_r(x_0) \subset B_{1/3}$. Let $v(x) = u_\varepsilon(x_0 + rx)$ and $A_{x_0}^{\varepsilon, r}(x) = A(\varepsilon^{-1}(x_0 + rx))$. Then

$$\nabla(A_{x_0}^{\varepsilon, r}(x) \nabla v(x)) = 0. \quad (5.8)$$

By (1.4),

$$|A_{x_0}^{\varepsilon, r}(x) - A_{x_0}^{\varepsilon, r}(y)| \leq \gamma r \varepsilon^{-1} |x - y| \leq \gamma |x - y| \quad (5.9)$$

for $x, y \in B_2$. Therefore, in this case, the coefficient matrix has a uniform Lipschitz constant independent of ε and N . Then, a change of variable and the doubling inequality in [Lemma 5.1](#) give that

$$\begin{aligned}
\int_{B_2} v^2 \, dx &= \int_{B_{2r}(x_0)} u_\varepsilon^2 \, dx \\
&\leq \exp \left(\exp \left(CN^{\frac{2}{\beta-\frac{3}{4}}} \right) \right) \int_{B_r(x_0)} u_\varepsilon^2 \, dx \\
&\leq \exp \left(\exp \left(CN^{\frac{2}{\beta-\frac{3}{4}}} \right) \right) \int_{B_1} v^2 \, dx.
\end{aligned} \tag{5.10}$$

By the upper bound of nodal sets in [14], there exists a constant $\beta_0 > \frac{1}{2}$ so that

$$H^{d-1}(Z(v) \cap B_1) \leq \left[\exp \left(CN^{\frac{2}{\beta-\frac{3}{4}}} \right) \right]^{\beta_0} \leq \exp \left(C_1 N^{\frac{2}{\beta-\frac{3}{4}}} \right), \tag{5.11}$$

which implies, by rescaling,

$$H^{d-1}(Z(u_\varepsilon) \cap B_r(x_0)) \leq \exp \left(C_1 N^{\frac{2}{\beta-\frac{3}{4}}} \right) r^{d-1}$$

for any $r \in (0, \varepsilon]$ and $B_r(x_0) \subset B_{1/3}$.

Next, to deal with the case $r > \varepsilon$, we simply use a covering argument. Let $x_0 \in B_{1/3}$ and $r > \varepsilon$. There exists a family of balls $B_\varepsilon(x_i)$, $i = 1, 2, \dots, M$, that covers $B_r(x_0)$ with a finite number of overlaps depending only on d . Note that $M \approx (r/\varepsilon)^d$. Consequently,

$$\begin{aligned}
H^{d-1}(Z(u_\varepsilon) \cap B_r(x_0)) &\leq \sum_{i=1}^M H^{d-1}(Z(u_\varepsilon) \cap B_\varepsilon(x_i)) \\
&\leq M \exp \left(C_1 N^{\frac{2}{\beta-\frac{3}{4}}} \right) \varepsilon^{d-1} \\
&\leq C r^d \varepsilon^{-1} \exp \left(C_1 N^{\frac{2}{\beta-\frac{3}{4}}} \right).
\end{aligned}$$

We obtain the desired estimate by enlarging the constant C_1 . \square

Remark 5.3. The above lemma does not rely on the periodicity of the coefficients. Actually, its proof also gives how the estimate depends on the Lipschitz constant of the coefficients. Precisely, if v is a solution of $\nabla \cdot (A(x) \nabla v) = 0$ in B_2 . In addition to the ellipticity condition (1.2), we assume

$$|(A(x) - A(y))| \leq L|x - y|. \tag{5.12}$$

Then

$$E_\varepsilon(x_0, r) \leq C(1 + Lr)N(v, Q)^{\beta_0},$$

for $B_r(x_0) \subset Q$, where the definition of $N(v, Q)$ is given below in (5.27).

5.2. Large scales

To deal with the nodal sets at large scales, we need to use the homogenization theory. Precisely, in the following, we find an approximate solution u_0 , close to u_ε under L^∞ norm, and satisfying a doubling inequality.

Lemma 5.4. Suppose $r > 3C\sqrt{N}\varepsilon$ for some large C . Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_{2r} satisfying

$$\int_{B_{2r}} u_\varepsilon^2 \leq N \int_{B_r} u_\varepsilon^2. \quad (5.13)$$

Then there exists u_0 satisfying $\mathcal{L}_0(u_0) = 0$ in $B_{\frac{r}{4}}$ such that

$$\|u_\varepsilon - u_0\|_{L^\infty(B_{\frac{3r}{2}})} \leq \frac{C\varepsilon}{r} \|u_\varepsilon\|_{L^2(B_{2r})}, \quad (5.14)$$

and

$$\int_{B_r} u_0^2 \leq 16N^2 \int_{B_{r/2}} u_0^2, \quad (5.15)$$

where C depends on d, Λ and γ .

Proof. By rescaling, we may assume $r = 1$. The construction of such locally homogenized solution u_0 and the estimate (5.14) can be found in [5, Theorem 2.3]. Note that it is not necessary that $u_\varepsilon = u_0$ on $\partial B_{\frac{7}{4}}$. Then, it suffices to show (5.15). By (5.13) and (5.14), we have

$$\|u_\varepsilon - u_0\|_{L^\infty(B_{\frac{3}{2}})}^2 \leq C\varepsilon^2 \int_{B_2} u_\varepsilon^2 \leq C\varepsilon^2 N \int_{B_1} u_\varepsilon^2. \quad (5.16)$$

We now establish estimates to compare the norms of u_ε and u_0 . Thanks to (5.16),

$$\begin{aligned} \|u_0\|_{L^2(B_{\frac{3}{2}})} &\leq \|u_\varepsilon\|_{L^2(B_2)} + \|u_\varepsilon - u_0\|_{L^2(B_{\frac{3}{2}})} \\ &\leq \sqrt{N} \|u_\varepsilon\|_{L^2(B_1)} + C\sqrt{N}\varepsilon \|u_\varepsilon\|_{L^2(B_1)} \\ &= \sqrt{N}(1 + C\varepsilon) \|u_\varepsilon\|_{L^2(B_1)}. \end{aligned} \quad (5.17)$$

By the same strategy, using (5.16), we obtain that

$$\begin{aligned} \|u_\varepsilon\|_{L^2(B_1)} &\leq \|u_\varepsilon - u_0\|_{L^2(B_1)} + \|u_0\|_{L^2(B_1)} \\ &\leq C\sqrt{N}\varepsilon \|u_\varepsilon\|_{L^2(B_1)} + \|u_0\|_{L^2(B_1)}. \end{aligned} \quad (5.18)$$

Since $C\sqrt{N}\varepsilon \leq \frac{1}{3}$, the above estimate yields

$$\|u_\varepsilon\|_{L^2(B_1)} \leq (1 + C\sqrt{N}\varepsilon) \|u_0\|_{L^2(B_1)}. \quad (5.19)$$

Combining (5.17) and (5.19) together yields that

$$\begin{aligned} \|u_0\|_{L^2(B_{\frac{3}{2}})} &\leq \sqrt{N}(1 + C\sqrt{N}\varepsilon) \|u_0\|_{L^2(B_1)} \\ &\leq 2\sqrt{N} \|u_0\|_{L^2(B_1)}. \end{aligned} \quad (5.20)$$

Now, we use the fact that

$$\varphi(s) = \log_2 \int_{B_{2s}} u_0^2 \quad (5.21)$$

is a convex function with respect to s . Then $f(s) = \varphi(s) - \varphi(s - c)$ is nondecreasing for any $c > 0$. This implies

$$\begin{aligned}
\|u_0\|_{L^2(B_1)} &\leq 2\sqrt{N}\|u_0\|_{L^2(B_{\frac{1}{2}})} \\
&\leq 4N\|u_0\|_{L^2(B_{\frac{1}{2}})} \\
&\leq 4N\|u_0\|_{L^2(B_1)}.
\end{aligned} \tag{5.22}$$

This proves (5.15) and the lemma. \square

Remark 5.5. We would like to point out that the advantage of Lemma 5.4, compared to (2.2), is that it provides an L^∞ (or pointwise) error estimate which is much stronger than the L^2 error estimate in (2.2). This L^∞ estimate will play an essential role in the estimation of nodal sets.

Let B be a ball and u_0 be a C^1 function in $2B$. In order to show some quantitative stratification results for u_0 and ∇u_0 , we introduce the doubling index:

$$N(u_0, B) = \log_2 \frac{\sup_{2B} |u_0|}{\sup_B |u_0|} \tag{5.23}$$

and

$$N(\nabla u_0, B) = \log_2 \frac{\sup_{2B} |\nabla u_0|}{\sup_B |\nabla u_0|}. \tag{5.24}$$

If u_0 is a weak solution of the equation $\mathcal{L}_0(u_0) = 0$, the doubling index for $|u_0|$ and $|\nabla u_0|$ are monotonic in the sense that

$$N(u_0, tB) \leq CN(u_0, B) \tag{5.25}$$

and

$$N(\nabla u_0, tB) \leq CN(\nabla u_0, B), \tag{5.26}$$

for $t \leq \frac{1}{2}$ and C depending only on d . This follows from (2.8) and the line after it.

We also define a variant of the above doubling index for cubes. For a cube Q , denote by $s(Q)$ the side length of Q . Define the doubling index in the cube Q by

$$N(u_0, Q) = \sup_{x \in Q, r \leq s(Q)} N(u_0, B_r(x)) \tag{5.27}$$

and

$$N(\nabla u_0, Q) = \sup_{x \in Q, r \leq s(Q)} N(\nabla u_0, B_r(x)). \tag{5.28}$$

The doubling index defined in cubes is convenient in the sense that if a cube q is a subset of Q , then $N(u_0, q) \leq N(u_0, Q)$. Let q be a subcube of Q and $K = \frac{s(Q)}{s(q)} \geq 2$. Then

$$\sup_q |u_0| \geq K^{-CN(u_0, Q)} \sup_Q |u_0|, \tag{5.29}$$

where C depends only on d . Similarly, it also holds

$$\sup_q |\nabla u_0| \geq K^{-CN(\nabla u_0, Q)} \sup_Q |\nabla u_0|. \tag{5.30}$$

The following quantitative stratification for ∇u_0 is the key ingredient of this section. The idea of the proof originates from Lemmas 3.5 and 5.2 in [24].

Lemma 5.6. *Assume that u_0 is harmonic in $5Q$. Suppose $N(\nabla u_0, Q) \leq \hat{N}_0$. If $0 < \hat{\delta} \leq e^{-C\hat{N}_0^3}$ for some $C > 0$ depending on d , there exists a finite sequence of balls $\{B_{i_j}(x_j)\}_{j=1}^{\hat{m}}$ such that*

$$\hat{G}_{\hat{\delta}} = \left\{ x \in \frac{1}{2}Q : |\nabla u_0(x)| < \hat{\delta} \sup_Q |\nabla u_0(x)| \right\} \subset \bigcup_{j=1}^{\hat{m}} B_{i_j}(x_j) \quad (5.31)$$

and

$$\sum_{j=1}^{\hat{m}} i_j^{d-1} \leq \frac{1}{4} \left(\frac{s(Q)}{4} \right)^{d-1}. \quad (5.32)$$

Proof. In the following proof, all the constants C, C_d, C_1, C_2, \dots , depend only on d , and \hat{N}_0 is a large constant. We divide the cube $\frac{1}{2}Q$ into K_1^d subcubes with side length $\frac{s(Q)}{2K_1}$. The size of K_1 , depending on $\hat{\delta}$, will be chosen later. The cube q_j is called bad if

$$\inf_{q_j} |\nabla u_0| \leq c \sup_{2q_j} |\nabla u_0| \quad (5.33)$$

for some small c depending only on d . We claim that the number of bad cubes q_j is not greater than $e^{C_d \hat{N}_0^2} K_1^{d-2}$, where C_d depends on d .

To show the above claim, we need to use [17, Theorem 1.1]. Recall the effective critical set is defined as

$$\mathcal{C}_r(u_0) = \left\{ x \in Q : \inf_{B_r(x)} r^2 |\nabla u_0|^2 \leq \frac{d}{16} \int_{\partial B_{2r}(x)} (u - u(x))^2 \right\}.$$

Let $B_r(\mathcal{C}_r(u_0))$ be the r -neighborhood of $\mathcal{C}_r(u_0)$, namely, $B_r(\mathcal{C}_r(u_0)) = \{x \in Q : \text{dist}(x, \mathcal{C}_r(u_0)) < r\}$. Then [17, Theorem 1.1] implies

$$|B_r(\mathcal{C}_r(u_0)) \cap B_s| \leq C^{(\tilde{N}(u_0, B_{2s}))^2} \left(\frac{r}{s} \right)^2 |B_s|, \quad (5.34)$$

where B_s, B_{2s} are concentric balls such that $B_{4s} \subset Q$ and \tilde{N} is the modified frequency function defined by

$$\tilde{N}(u_0, B_{2s}) := \frac{2s \int_{B_{2s}} |\nabla u_0|^2}{\int_{\partial B_{2s}} (u_0 - u_0(z))^2},$$

where z is the center of B_s . By [10, Corollary 2.2.6] and the mean value property of harmonic functions, we have

$$\tilde{N}(u_0, B_{2s}) \leq C \log_2 \frac{\int_{B_{4s}} (u_0 - u_0(z))^2}{\int_{B_{2s}} (u_0 - u_0(z))^2} \leq C \log_2 \frac{\sup_{B_{4s}} |\nabla u_0|}{\sup_{B_s} |\nabla u_0|} \leq CN(\nabla u_0, Q) \leq C\hat{N}_0,$$

where we have also used a gradient estimate for harmonic functions in the second inequality. Hence, (5.34) implies

$$|B_r(\mathcal{C}_r(u_0)) \cap B_s| \leq C^{\hat{N}_0^2} \left(\frac{r}{s} \right)^2 |B_s|. \quad (5.35)$$

Next, we show that if q_j is a bad cube with sufficiently small c , then $q_j \cap \mathcal{C}_r(u_0) \neq \emptyset$. Actually if q_j is bad and x_j is the point in q_j so that $|\nabla u_0(x_j)| = \inf_{q_j} |\nabla u_0|$, then

$$\inf_{B_r(x_j)} |\nabla u_0| \leq |\nabla u_0(x_j)| = \inf_{q_j} |\nabla u_0| \leq c \sup_{2q_j} |\nabla u_0|,$$

where we used the condition (5.33) in the last inequality. Fix $r = 2\sqrt{ds}(q_j)$. Then $2q_j \subset B_r(x_j)$. It follows from the gradient estimate and the Caccioppoli inequality that

$$\begin{aligned} \inf_{B_r(x_j)} r |\nabla u_0| &\leq c r \sup_{B_r(x_j)} |\nabla u_0| \leq c C_d r \left(\int_{B_{\frac{r}{2}}(x_j)} |\nabla u_0|^2 \right)^{1/2} \\ &\leq c C_d^2 \left(\int_{B_{2r}(x_j)} |u_0 - u_0(x_j)|^2 \right)^{1/2}. \end{aligned}$$

In view of [10, Corollary 2.2.7], we have

$$\begin{aligned} \inf_{B_r(x_j)} r |\nabla u_0| &\leq \frac{c C_d^2}{d} \left(\int_{\partial B_{2r}(x_j)} |u_0 - u_0(x_j)|^2 \right)^{1/2} \\ &\leq \sqrt{\frac{d}{16}} \left(\int_{\partial B_{2r}(x_j)} |u_0 - u_0(x_j)|^2 \right)^{1/2}, \end{aligned}$$

where in the last inequality, we choose c small so that $c C_d^2 / d < \sqrt{d/16}$. This implies that $x_j \in \mathcal{C}_r(u_0)$ and $q_j \cap \mathcal{C}_r(u_0) \neq \emptyset$. Because $r = 2\sqrt{ds}(q_j)$, we have $q_j \subset B_r(\mathcal{C}_r(u_0))$. This means that all the bad cubes q_j are contained in $B_r(\mathcal{C}_r(u_0))$. Finally, let s be comparable to $s(Q)$ and note that $\frac{1}{2}Q$ can be covered by finitely many, depending only on d , B_s with $B_{4s} \subset Q$. Then, by (5.35), the total volume of bad cubes in $\frac{1}{2}Q$ is bounded by $C^{\hat{N}_0^2} (s(q_j)/s(Q))^2 |Q| \leq C^{\hat{N}_0^2} K_1^{-2} |Q|$. Hence, the number of bad cubes is not greater than $C^{\hat{N}_0^2} K_1^{d-2}$. The claim has been proved.

Now, for any q_j , the monotonicity of the doubling index of ∇u_0 in cubes in (5.30) shows that

$$\sup_{q_j} |\nabla u_0| \geq C_2 K_1^{-C_1 \hat{N}_0} \sup_Q |\nabla u_0|. \quad (5.36)$$

If q_j is not bad, the reverse inequality of (5.33) yields

$$\inf_{q_j} |\nabla u_0| > C_3 K_1^{-C_1 \hat{N}_0} \sup_Q |\nabla u_0|. \quad (5.37)$$

Given $\hat{\delta}$, small enough (to be quantified later), we want to estimate the set $\hat{G}_{\hat{\delta}}$ defined in (5.31). If q_j is not bad and we choose K_1 to be the smallest integer such that

$$C_3 (K_1 + 1)^{-C_1 \hat{N}_0} < \hat{\delta}, \quad (5.38)$$

then (5.37) gives

$$\inf_{q_j} |\nabla u_0| \geq C_3 K_1^{-C_1 \hat{N}_0} \sup_Q |\nabla u_0| > \hat{\delta} \sup_Q |\nabla u_0|.$$

This implies that q_j does not intersect $\hat{G}_{\hat{\delta}}$. It also shows that $K_1 \approx \hat{\delta}^{\frac{-1}{C_{10} \hat{N}_0}}$. Thus, the set $\hat{G}_{\hat{\delta}}$ is covered by the union of bad cubes of size $\frac{s(Q)}{2K_1}$. Again, we may now replace bad q_j

by $B_{\hat{t}_j}(x_j)$ with the same center and $\hat{t}_j = \frac{s(q_j)\sqrt{d}}{2}$. Let \hat{m} be the number of bad cubes and recall that $e^{C_d \hat{N}_0^2} K_1^{d-2} \geq \hat{m}$. It follows that

$$\begin{aligned} \sum_{j=1}^{\hat{m}} \hat{t}_j^{d-1} &= C\hat{m} \cdot s(q_j)^{d-1} \leq Ce^{C_d \hat{N}_0^2} K_1^{d-2} \left(\frac{s(Q)}{2K_1} \right)^{d-1} \\ &\leq \left(\frac{1}{2} \right)^{d-1} s^{d-1}(Q) Ce^{C_d \hat{N}_0^2} K_1^{-1} \\ &\leq \left(\frac{1}{2} \right)^{d-1} s^{d-1}(Q) Ce^{C_d \hat{N}_0^2} \hat{\delta}^{\frac{1}{C_4 \hat{N}_0}} \\ &\leq \left(\frac{1}{4} \right)^d s^{d-1}(Q) \end{aligned} \tag{5.39}$$

where we have chosen $\hat{\delta} \leq e^{-CN_0^3}$ in the last inequality for some $C > 0$ depending only on d . This completes the proof. \square

Next we estimate the density function $E_\varepsilon(y, r)$ of nodal sets, which is the initial step for an iterative argument to obtain [Theorem 1.2](#). The following lemma is a quantitative version of [\[5, Lemma 4.5\]](#). Without loss of generality, we may identify Q in [Lemma 5.6](#) by $B_{\frac{1}{8}}$.

Lemma 5.7. *Let $\beta \in (\frac{3}{4}, 1)$ and (5.1) hold. If $\varepsilon \leq \exp(-C(\ln N)^3)$, then there exists a finite sequence of balls $\{B_{\hat{t}_j}(y_j) : j = 1 \dots, \hat{m}\}$ such that $y_j \in B_{\frac{1}{16}}$, $\hat{t}_j \in (0, \frac{1}{128})$ and*

$$E_\varepsilon\left(0, \frac{1}{16}\right) \leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_{1 \leq j \leq \hat{m}} E_\varepsilon(y_j, \hat{t}_j), \tag{5.40}$$

where c and C depend on d , Λ and γ .

Proof. We assume $E_\varepsilon(0, \frac{1}{16}) > 0$. Otherwise, (5.40) is trivial. By normalization, we may assume that $\int_{B_2} u_\varepsilon^2 = 1$. We will make use of the approximation estimate by a harmonic function in [Lemma 5.4](#). Using (5.14) with $r = 1$, we have

$$\|u_\varepsilon - u_0\|_{L^\infty(B_{\frac{1}{8}})} \leq \hat{C}\varepsilon \|u_\varepsilon\|_{L^2(B_2)} = \hat{C}\varepsilon. \tag{5.41}$$

We would like to estimate the doubling indices for u_0 and ∇u_0 . From (5.20) and convexity of φ in (5.21), we have

$$\|u_0\|_{L^2(B_{\frac{3t}{2}})} \leq 2\sqrt{N} \|u_0\|_{L^2(B_t)} \tag{5.42}$$

for all $0 < t < 1$. By elliptic estimates and using (5.42) twice,

$$\begin{aligned} \|u_0\|_{L^\infty(B_1)} + \|\nabla u_0\|_{L^\infty(B_1)} &\leq C \left(\int_{B_{\frac{9}{8}}} u_0^2 \right)^{\frac{1}{2}} \\ &\leq C\sqrt{N} \left(\int_{B_{\frac{3}{4}}} u_0^2 \right)^{\frac{1}{2}} \leq CN \left(\int_{B_{\frac{1}{2}}} u_0^2 \right)^{\frac{1}{2}} \leq CN \|u_0\|_{L^\infty(B_{\frac{1}{2}})}. \end{aligned} \tag{5.43}$$

The above estimate includes

$$\|u_0\|_{L^\infty(B_1)} \leq CN\|u_0\|_{L^\infty(B_{\frac{1}{2}})}. \quad (5.44)$$

Thus, $N(u_0, B_{\frac{1}{2}}) \leq \log_2(CN)$. Similarly to the frequency function in [Section 3](#), one can define the frequency function introduced for the harmonic function u_0 as

$$\mathcal{N}(x, r) = \frac{r \int_{B_r(x)} |\nabla u_0|^2}{\int_{\partial B_r(x)} u_0^2}.$$

It is known that the doubling index and the frequency function are comparable. Indeed, it follows from [\[14, Lemma 7.1\]](#) that

$$\mathcal{N}\left(0, \frac{1}{2}\right) \leq CN(u_0, B_{\frac{1}{2}}) \leq C \log_2(CN), \quad (5.45)$$

where C depends on d . By [Theorem 2.2.8](#) in [\[10\]](#), it holds that

$$\mathcal{N}\left(x, \frac{1}{2}\left(\frac{1}{2} - R\right)\right) \leq C \mathcal{N}\left(0, \frac{1}{2}\right) \quad (5.46)$$

for any $x \in B_R$ and $0 < R < \frac{1}{2}$. From [\(5.46\)](#) and [\(5.45\)](#), let $R = \frac{1}{8}$, we obtain that

$$\mathcal{N}\left(x, \frac{3}{16}\right) \leq C \log_2(CN).$$

Using [\[14, Lemma 7.1\]](#) again, we see that

$$N(u_0, B_r(x)) \leq C \mathcal{N}\left(x, \frac{3}{16}\right) \leq C \log_2(CN) \quad (5.47)$$

for any $0 < r \leq \frac{1}{8}$ and $x \in B_{\frac{1}{8}}$.

Next, we claim that there are zeros for u_0 in $B_{\frac{1}{8}}$. In fact, from [\(5.41\)](#) and [Theorem 2.1](#) with $\theta = 1/2$, we have

$$\|u_\varepsilon - u_0\|_{L^2(B_{\frac{1}{8}})} \leq CN^2 \varepsilon \|u_\varepsilon\|_{L^2(B_{\frac{1}{8}})}. \quad (5.48)$$

Then

$$\begin{aligned} \|u_0\|_{L^2(B_{\frac{1}{8}})} &\geq (1 - CN^2 \varepsilon) \|u_\varepsilon\|_{L^2(B_{\frac{1}{8}})} \\ &\geq \frac{(1 - CN^2 \varepsilon)}{CN^2} \|u_\varepsilon\|_{L^2(B_2)}. \end{aligned} \quad (5.49)$$

It follows that $\|u_0\|_{L^2(B_{\frac{1}{8}})} \geq \frac{1}{CN^2}$ if $\varepsilon < cN^{-2}$. Hence, [\(5.15\)](#) implies

$$\frac{1}{CN^2} \leq \|u_0\|_{L^2(B_{\frac{1}{8}})} \leq CN \|u_0\|_{L^2(B_{\frac{1}{16}})}. \quad (5.50)$$

Now, let us assume that u_0 has no zeros in $B_{\frac{1}{8}}$ and therefore does not change signs in $B_{\frac{1}{8}}$. Without loss of generality, we may assume that u_0 is positive. By the Harnack inequality and [\(5.50\)](#),

$$\inf_{B_{\frac{1}{16}}} |u_0| \geq C \sup_{B_{\frac{1}{16}}} |u_0| \geq \frac{1}{CN^3}. \quad (5.51)$$

From [\(5.41\)](#), for $x \in B_{\frac{1}{16}}$, we get

$$\frac{1}{CN^3} - \hat{C}\varepsilon \leq u_\varepsilon(x). \quad (5.52)$$

Since $\varepsilon < CN^{-3}$, then $u_\varepsilon(x) > 0$ for $x \in B_{\frac{1}{16}}$. This contradicts our assumption that $E_\varepsilon(0, \frac{1}{16}) > 0$. Thus, the claim has been shown.

Now, since u_0 has zeros in $B_{\frac{1}{8}}$ (and hence in $B_{\frac{1}{2}}$), we obtain from (5.43) and the mean value theorem that

$$\sup_{B_1} |\nabla u_0| \leq CN \sup_{B_{\frac{1}{2}}} |\nabla u_0|. \quad (5.53)$$

Again, by the relation of the frequency function for ∇u_0 (namely, the frequency function with $u - u(0)$) and the doubling index, we can argue as the derivation of (5.47) that (5.53) implies that

$$N(\nabla u_0, B_r(x)) \leq C \log_2(CN) \quad (5.54)$$

for any $0 < r \leq \frac{1}{8}$ and any $x \in B_{\frac{1}{8}}$.

From the definition of $N(\nabla u_0, Q)$ and (5.54), we know that $N(\nabla u_0, Q) \leq \hat{N}_0 := C \log N$ for $Q \subset B_{\frac{1}{8}}$ and $N > C$. Since we may choose sequence of $Q \subset B_{\frac{1}{8}}$, comparable to $B_{\frac{1}{16}}$, to cover $B_{\frac{1}{16}}$, in the following argument, we will work on $B_{\frac{1}{16}}$ for convenience. In order to apply [Lemma 5.6](#), we assume that $\frac{\hat{\delta}}{2} \approx \frac{1}{2} e^{-C(\log N)^3}$. With the aid of (5.41), we have

$$\begin{aligned} Z(u_\varepsilon) \cap B_{\frac{1}{16}} &\subset Z(u_\varepsilon) \cap \left\{ x \in B_{\frac{1}{16}} : |u_0(x)| \leq \hat{C}\varepsilon \right\} \\ &\subset Z(u_\varepsilon) \cap \left\{ x \in B_{\frac{1}{16}} : |u_0(x)| \leq \hat{C}\varepsilon \text{ and } |\nabla u_0(x)| \geq \frac{\hat{\delta}}{2} \sup_{B_{\frac{1}{8}}} |\nabla u_0| \right\} \\ &\quad \cup Z(u_\varepsilon) \cap \left\{ x \in B_{\frac{1}{16}} : |u_0(x)| \leq \hat{C}\varepsilon \text{ and } |\nabla u_0(x)| \leq \frac{\hat{\delta}}{2} \sup_{B_{\frac{1}{8}}} |\nabla u_0| \right\} \\ &\subset (Z(u_\varepsilon) \cap G) \cup (\bigcup_{j=1}^{\hat{m}} Z(u_\varepsilon) \cap B_{\hat{t}_j}(y_j)), \end{aligned} \quad (5.55)$$

where

$$G = \left\{ x \in B_{\frac{1}{8}} : |u_0(x)| \leq \hat{C}\varepsilon \text{ and } |\nabla u_0(x)| \geq \frac{\hat{\delta}}{2} \sup_{B_{\frac{1}{8}}} |\nabla u_0| \right\}, \quad (5.56)$$

and $B_{\hat{t}_j}(y_j)$ are given by [Lemma 5.6](#). Thus, it follows from [Lemma 5.6](#) that

$$\begin{aligned} H^{d-1}(Z(u_\varepsilon) \cap B_{\frac{1}{16}}) &\leq H^{d-1}(Z(u_\varepsilon) \cap G) + \sum_{j=1}^{\hat{m}} H^{d-1}(Z(u_\varepsilon) \cap B_{\hat{t}_j}(y_j)) \\ &\leq H^{d-1}(Z(u_\varepsilon) \cap G) + \sup_j E_\varepsilon(y_j, \hat{t}_j) \sum_{j=1}^{\hat{m}} \hat{t}_j^{d-1} \\ &\leq H^{d-1}(Z(u_\varepsilon) \cap G) + \frac{1}{4} \left(\frac{1}{16} \right)^{d-1} \sup_j E_\varepsilon(y_j, \hat{t}_j). \end{aligned} \quad (5.57)$$

Since N is large, by the decomposition in [Lemma 5.6](#), we may assume $0 < \hat{t}_j < \frac{1}{128}$.

Next we estimate the upper bound for $H^{d-1}(Z(u_\varepsilon) \cap G)$. We will cover G by balls with radius comparable to ε and estimate the number of these balls so that [Lemma 5.2](#) can apply. First, note that (5.50), the fact that u_0 has zeros in $B_{\frac{1}{8}}$ and the definition of G imply

$$|\nabla u_0(x)| \geq \frac{\hat{\delta} \tilde{C}}{2N^2}, \quad \text{and} \quad |u_0(x)| \leq \hat{C}\varepsilon \quad (5.58)$$

for any point $x \in G$. For $k = 1, 2, \dots, d$, define

$$F_k^\pm = \left\{ x \in G \mid |u_0(x)| \leq \hat{C}\varepsilon, \pm \frac{\partial u_0}{\partial x_k}(x) \geq \frac{\hat{\delta}c}{2dN^2} \right\}.$$

Then (5.58) implies that G is contained in $\cup_{k=1}^d (F_k^+ \cup F_k^-)$. Without loss of generality, it suffices to estimate F_k^+ . By the C^2 regularity of u_0 , for any $x_0 \in F_k^+$, there exists a cylinder $\mathcal{C}(x_0)$ centered at x_0 , whose base is a square perpendicular to e_k with side length $C\varepsilon$, such that the height of $\mathcal{C}(x_0)$ is $\frac{\hat{\delta}c_1}{2dN^2}$ and

$$\frac{\partial u_0}{\partial x_k}(x) \geq \frac{\hat{\delta}c_1}{2dN^2}, \quad \text{for any } x \in \mathcal{C}(x_0), \quad (5.59)$$

where $c_1 > 0$ is a constant smaller than c .

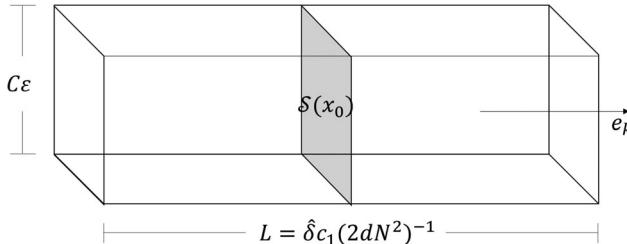


Figure 1. The cylinder $\mathcal{C}(x_0)$.

We would like to show that $\mathcal{C}(x_0) \cap F_k^+$ can be covered by m_1 balls with radius $C\varepsilon$, where $m_1 \leq C_2 2dN^2 \hat{\delta}$. Let $\mathcal{S}(x_0)$ be the cross section containing x_0 of the cylinder $\mathcal{C}(x_0)$ which is perpendicular to e_k . Since $|\nabla u_0| \leq C$ and $|u_0(x_0)| \leq \hat{C}\varepsilon$, we see that $|u_0(y)| \leq C_1\varepsilon$ for any $y \in \mathcal{S}(x_0)$. Next, because of (5.59), for any $y \in \mathcal{S}(x_0)$ and $t > 0$,

$$u_0(y + te_k) \geq t \frac{\hat{\delta}c_1}{2dN^2} - C_1\varepsilon.$$

This implies that $y + te_k \notin F_k^+$ if $t > \frac{(C_1 + \hat{C})\varepsilon 2dN^2}{\hat{\delta}c_1}$. Similarly, $y + te_k \notin F_k^+$ if $t < -\frac{(C_1 + \hat{C})\varepsilon 2dN^2}{\hat{\delta}c_1}$. This implies that

$$F_k^+ \cap \mathcal{C}(x_0) \subset \left\{ y + te_k \mid y \in \mathcal{S}(x_0), |t| \leq \frac{(C_1 + \hat{C})\varepsilon 2dN^2}{\hat{\delta}c_1} \right\}.$$

Consequently, $F_k^+ \cap \mathcal{C}(x_0)$ can be covered by m_1 balls with radius $C\varepsilon$ and $m_1 \leq \frac{C_2 2dN^2}{\hat{\delta}}$.

Now, because $F_k^+ \subset G$ can be covered by m_2 cylinders, with $m_2 = \frac{CdN^2}{\varepsilon^{d-1}\hat{\delta}}$, such that the $\mathcal{C}(x_0)$'s have finite overlaps, then F_k^+ can be covered by m balls with radius $C\varepsilon$ (denoted by $\{B_{C\varepsilon}(z_{k,\ell}^+) : \ell = 1, 2, \dots, m\}$), where

$$m = m_1 m_2 \leq \frac{Cd^2 N^4}{\varepsilon^{d-1} \hat{\delta}^2}.$$

Note that the same estimate holds for F_k^- as well for each $k = 1, 2, \dots, d$.

Hence, by [Lemma 5.2](#), we derive that

$$\begin{aligned} H^{d-1}(Z(u_\varepsilon) \cap G) &\leq \sum_{k=1}^d \sum_{l=1}^m H^{d-1}(Z(u_\varepsilon) \cap B_{C\varepsilon}(z_{k,l}^\pm)) \\ &\leq C \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) m \varepsilon^{d-1} \\ &\leq CN^4 \hat{\delta}^{-2} \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) \\ &\leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right), \end{aligned} \tag{5.60}$$

where in the last inequality, we have used the fact $\hat{\delta}^{-1} \approx \exp(C(\ln N)^3)$ and enlarged the constant C . Note that here $\beta \in (\frac{3}{4}, 1)$ can be arbitrary. Thus, (5.60) and (5.57) together lead to the conclusion (5.40). \square

5.3. Proof of Theorem 1.2

Thanks to [Lemma 5.7](#), we are able to show the upper bound of the nodal sets of u_ε in the interior domain.

Proof of Theorem 1.2 ($d \geq 3$). We first consider the case $\varepsilon \leq \exp(-C(\ln N)^3)$. Recall from (5.4) that

$$\int_{B_{5/3}(x)} u_\varepsilon^2 \leq 2N^2 \int_{B_{5/6}(x)} u_\varepsilon^2 \tag{5.61}$$

for any $x \in B_{1/3}$. By [Theorem 2.1](#), it follows that

$$\int_{B_{2r}(x)} u_\varepsilon^2 \leq 4N^2 \int_{B_r(x)} u_\varepsilon^2 \tag{5.62}$$

for $CN^{\frac{1}{\beta-\frac{3}{4}}}\varepsilon < r < \frac{5}{6}$ and any $x \in B_{1/3}$. By (5.62) and [Lemma 5.4](#), we derive that

$$\int_{B_r(x)} u_0^2 \leq CN^4 \int_{B_{\frac{r}{2}}(x)} u_0^2, \tag{5.63}$$

as in (5.15) for $x \in B_{\frac{1}{3}}$ and $CN^{\frac{1}{\beta-\frac{3}{4}}}\varepsilon < r < \frac{5}{6}$. By examining the proof of [Lemma 5.7](#), the estimates (5.62) and (5.63) guarantee that the arguments in the [Lemma 5.7](#) hold for $E(x_0, s)$ for $x_0 \in B_{\frac{1}{8}}$ and $C\varepsilon \exp(C(\ln N)^3) < s \leq \frac{r}{16} \cdot \frac{1}{1}$.

Let $v(x) = u_\varepsilon(x_0 + tx)$ for any t satisfying $CN^{\frac{1}{\beta-\frac{3}{4}}}\varepsilon \leq C\varepsilon \exp(C(\ln N)^3) < t < \frac{5}{6}$ and $x_0 \in B_{1/8}$. Then $v(x)$ satisfies

$$\nabla \cdot (A_{x_0}^{\varepsilon, t}(x) \nabla v(x)) = 0 \quad \text{in } B_2, \tag{5.64}$$

where $A_{x_0}^{\varepsilon, t}(x) = A(\varepsilon^{-1}(x_0 + tx))$. By [Lemma 5.7](#), we have

$$\frac{H^{d-1}(Z(v) \cap B_{\frac{1}{16}}(0))}{(\frac{1}{16})^{d-1}} \leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_j \frac{H^{d-1}(Z(v) \cap B_{\tilde{s}_j}(y_j))}{(\tilde{s}_j)^{d-1}}, \tag{5.65}$$

where $\tilde{s}_j \in (0, \frac{1}{16 \times 8})$ and $y_j \in B_{\frac{1}{16}}(0)$. By rescaling, we reduce the estimate to u_ε and obtain that

$$\frac{H^{d-1}(Z(u_\varepsilon) \cap B_{\frac{1}{16}}(x_0))}{\left(\frac{t}{16}\right)^{d-1}} \leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_j \frac{H^{d-1}(Z(u_\varepsilon) \cap B_{t\tilde{s}_j}(x_0 + ty_j))}{(t\tilde{s}_j)^{d-1}}. \quad (5.66)$$

Let $\tau = \frac{t}{16}$, then $C\varepsilon \exp(C(\ln N)^3) < \tau < \frac{5}{96}$. Thus,

$$E_\varepsilon(x_0, \tau) \leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_j E_\varepsilon(\hat{y}_j, \hat{s}_j). \quad (5.67)$$

where $\hat{y}_j = x_0 + ty_j \in B_\tau(x_0)$, $\hat{s}_j = t\tilde{s}_j \in (0, \frac{\tau}{8})$. Note that $B_{\hat{s}_j}(\hat{y}_j)$ may not be fully contained in $B_\tau(x_0)$, since \hat{y}_j may be the centers of subcubes which intersect the boundary of $B_\tau(x_0)$ (we identify the ball $B_\tau(x_0)$ as a cube when we perform the subcubes decomposition). However, $B_{\hat{s}_j}(\hat{y}_j) \subset B_{\tau+\frac{\tau}{8}}(x_0)$ since $\hat{s}_j \in (0, \frac{\tau}{8})$. If we iterate (5.67), $B_{\hat{s}_j}(\hat{y}_j)$ still stays close to $B_\tau(x_0)$. Actually, $B_{\hat{s}_j}(\hat{y}_j) \subset B_{\hat{\tau}}(x_0)$ for any large j , where $\hat{\tau} = \sum_{j=1}^{\infty} \frac{\tau}{8^{j-1}} = \frac{8\tau}{7}$.

Now, we iterate (5.67) to obtain the desired estimate. The estimate (5.40) yields the initial step of the iteration,

$$E_\varepsilon\left(0, \frac{1}{16}\right) \leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_{1 \leq j \leq \hat{m}} E_\varepsilon(y_j, \hat{t}_j). \quad (5.68)$$

Assume that $\sup_{1 \leq j \leq \hat{m}} E_\varepsilon(y_j, \hat{t}_j)$ is achieved at some $E_\varepsilon(y_{j_0}, \hat{t}_{j_0})$ with $|y_{j_0}| < \frac{1}{16}$ and $|\hat{t}_{j_0}| < \frac{1}{128}$. Let $x_0 = y_{j_0}$ and $\hat{t}_{j_0} = \tau$. Since $\hat{s}_j < \frac{\tau}{8}$, we apply (5.67) to $E_\varepsilon(y_{j_0}, \hat{t}_{j_0})$ to get to the estimates of nodal sets at a smaller scale, that is,

$$E_\varepsilon\left(0, \frac{1}{16}\right) \leq \left(1 + \frac{1}{4}\right) \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \frac{1}{4} \sup_j E_\varepsilon(\hat{y}_j, \hat{s}_j). \quad (5.69)$$

We apply (5.67) repeatedly down to the case $r \approx \hat{C}\varepsilon \exp(C(\ln N)^3)$ or the case that $E_\varepsilon(y, r)$ is empty. Note that $B_{\hat{s}_j}(\hat{y}_j) \subset B_{\frac{1}{16} + \frac{8}{7 \times 128}}(0) \subset B_{\frac{1}{12}}(0)$. Thus, we derive that

$$\begin{aligned} E_\varepsilon\left(0, \frac{1}{16}\right) &\leq \sum_{i=0}^{\infty} 4^{-i} \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + \sup_{y \in B_{\frac{1}{12}}(0)} \{E_\varepsilon(y, r) : 0 < r \leq \hat{C}\varepsilon \exp(C(\ln N)^3)\} \\ &\leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) + (1 + \hat{C} \exp(C(\ln N)^3)) \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right) \\ &\leq \exp\left(CN^{\frac{2}{\beta-\frac{3}{4}}}\right), \end{aligned} \quad (5.70)$$

where we have used (5.7) in the second inequality. This proves the desired estimate for the case $\varepsilon \leq \exp(-C(\ln N)^3)$.

Finally, for the case $\varepsilon \geq \exp(-C(\ln N)^3)$, the desired estimate follows directly from (5.7). Since $\beta \in (\frac{3}{4}, 1)$ is arbitrary, so (1.14) holds for any $\alpha > 8$. This ends the proof of the theorem. \square

Following the above Proof of [Theorem 1.2](#) for $d \geq 3$, we sketch the proof of upper bounds of nodal sets in $d=2$.

Proof of Theorem 1.3 ($d=2$). Since the proof is parallel to $d \geq 3$, we only present the changes for $d=2$. Thus, we only present the changes for $d=2$. By the argument in

Lemma 5.2 and the doubling inequality (1.10), we can obtain for $0 < r < 1/3$,

$$E_\varepsilon(x_0, r) \leq C \left(1 + \frac{r}{\varepsilon}\right) (\ln N)^2. \quad (5.71)$$

On the other hand, the statements of (5.31) and (5.32) still hold for u_0 with a better bound for $\hat{\delta}$ for $d=2$. Precisely, [17, Theorem B.1] implies

$$|B_r(\mathcal{C}_r(u_0)) \cap B_s| \leq C^{(\tilde{N}(u_0, B_{2s}))} \left(\frac{r}{s}\right)^2 |B_s|. \quad (5.72)$$

By mimicking the argument of Lemma 5.6, we can show (5.31) and (5.32) with $0 < \hat{\delta} < e^{-C\hat{N}_0^2}$. Following the proof of Lemma 5.7, we may show that if $0 < \varepsilon < \exp(-C(\ln N)^2)$, then

$$E_\varepsilon\left(0, \frac{1}{16}\right) \leq \exp(C(\ln N)^2) + \frac{1}{4} \sup_{1 \leq j \leq \hat{m}} E_\varepsilon(y_j, \hat{t}_j), \quad (5.73)$$

with $y_j \in B_{\frac{1}{16}}, \hat{t}_j \in (0, \frac{1}{128})$. Observe that the quantitative stratification of critical sets ∇u_0 , instead of the doubling inequality, plays the dominant role in the estimate (5.73). We iterate (5.73), as in the proof for $d \geq 3$, to get

$$\begin{aligned} E_\varepsilon\left(0, \frac{1}{16}\right) &\leq \sum_{i=0}^{\infty} 4^{-i} \exp(C(\ln N)^2) + \sup_{y \in B_{\frac{1}{12}}(0)} \{E_\varepsilon(y, r) : 0 < r \leq \hat{C}\varepsilon \exp(C(\ln N)^2)\} \\ &\leq \exp(C(\ln N)^2) + C(1 + \hat{C} \exp(C(\ln N)^2))(\ln N)^2 \\ &\leq \exp(C(\ln N)^2). \end{aligned}$$

This provides the desired estimate (1.15) for $0 < \varepsilon < \exp(-C(\ln N)^2)$. For $\varepsilon \geq \exp(-C(\ln N)^2)$, (5.71) yields the desired estimate directly. This ends the proof. \square

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