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# Boundary Doubling Inequality and Nodal sets of Robin and Neumann eigenfunctions

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**Abstract**
We investigate the doubling inequality and upper bounds of nodal sets for Robin and Neumann eigenfunctions on the boundary and in the interior of the domain. Most efforts are devoted to the sharp boundary doubling inequality with new and novel quantitative global Carleman estimates. We are able to obtain the sharp upper bounds for boundary nodal sets of Neumann eigenfunctions.

**Keywords** Nodal sets · Doubling inequality · Carleman estimates · Robin eigenfunctions

**Mathematics Subject Classification (2010)** 35J05 · 58J50 · 35P15 · 35P20

## 1 Introduction

**Problem statement:** In this paper, we consider the Robin eigenfunctions with a possible large parameter $|\alpha|$

$$\begin{cases} -\triangle u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a smooth and compact domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, where $\nu$ is a unit outer normal and $n$ is the dimension of the space. In the case of $\alpha = 0$, the Eq. 1.1 is called the Neumann eigenvalue problem

$$\begin{cases} -\triangle u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In the case of $\alpha = \infty$, it can be considered as the Dirichlet eigenvalue problem

$$\begin{cases} -\triangle u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

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For any fixed constant $\alpha$, there exists a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \to \infty$. If $\alpha$ is negative, the first finite number of eigenvalues can be negative. Moreover, $\lambda_k \to -\infty$ as $\alpha \to -\infty$ for any fixed $k \geq 1$. If one considers the Robin eigenvalue problem (1.1) as an elliptic problem in a special case $\lambda = 0$, the Eq. 1.1 is reduced to the Steklov eigenvalue problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

One may regard $-\alpha$ as the eigenvalue for the Steklov eigenvalue problem (1.4). Hence, the model (1.1) includes general kind of Laplacian eigenvalue problems.

**Discussion of existing results:** We are mainly interested in the doubling inequality and nodal sets of eigenfunctions in Eqs. 1.1 and 1.2 on the boundary $\partial\Omega$. Doubling inequalities are inequalities with norms in a ball controlling the norms in a double ball. They are quantitative properties to control the growth of functions and quantitatively characterize the strong unique continuation property. Doubling inequalities imply the vanishing order of functions. Moreover, they are important tools to prove the measure of nodal sets. Nodal sets are the zero level sets of eigenfunctions. Due to the rich literature on nodal sets of eigenfunctions, let us first briefly review the history of this topic. For the eigenfunctions of Laplacian

$$\triangle u + \lambda u = 0 \tag{1.5}$$

on a compact smooth Riemannian manifold $\mathcal{M}$, Yau [43] conjectured that the Hausdorff measure of nodal sets can be controlled above and below by eigenvalues as

$$c\sqrt{\lambda} \leq H^{n-1}(\{x \in \mathcal{M} | u(x) = 0\}) \leq C\sqrt{\lambda}, \tag{1.6}$$

where $c$, $C$ depend on the manifold $\mathcal{M}$. For the real analytic manifolds, the conjecture was answered by Donnelly-Fefferman in their seminal paper [10]. A relatively simpler proof for the upper bound of general second order elliptic equations on the analytic domain was given by Lin [27] by a different approach.

For the smooth manifolds with $n = 2$, Donnelly-Fefferman [11] and Dong [9] independently showed the upper bound $H^1(\{x \in \mathcal{M}|u(x) = 0\}) \leq C\lambda^{\frac{3}{4}}$ by using different arguments. A slight improvement with upper bound $C\lambda^{\frac{3}{4}-\epsilon}$ was given by Logunov and Malinnikova [31]. For higher dimensions $n \geq 3$, Hardt and Simon [19] derived an exponential upper bound. Logunov in [29] obtained a polynomial upper bound $H^{n-1}(\{x \in \mathcal{M}|u(x) = 0\}) \leq C\lambda^{\beta}$ for some $\beta > \frac{1}{2}$. Very recently, Logunov, Malinnikova, Nadirashvili and Nazarov obtained the sharp upper bounds of nodal sets for analytical domains with $C^1$ boundary [33]. For the lower bound, Logunov [30] completely answered the Yau's conjecture and obtained the sharp lower bound for smooth manifolds in any dimensions. For $n = 2$, such sharp lower bound was obtained earlier by Brüning [6]. This breakthrough improved a polynomial lower bound obtained early by Colding and Minicozzi [7], Sogge and Zelditch [40]. See also other polynomial lower bounds by different methods, e.g. [20, 35, 41] and other related results on nodal geometry of eigenfunctions, e.g. [18, 34]. The recent breakthrough on nodal sets of eigenfunctions in [29, 31] and [30] is based on seminal work on new combinatorial arguments for doubling index and further exploration of frequency functions in [15] and [17].

The main goal of the paper is devoted to deriving doubling inequalities. As a consequence, we are able to obtain the upper bound of nodal sets on the boundary and in the interior. In order to prove the Yau's conjecture (1.6) for eigenfunctions in Eq. 1.5 on compact manifolds without boundary, the following sharp doubling inequality plays an essential role in [10]

$$\|u\|_{L^2(\mathbb{B}_{2r}(x))} \leq e^{C\sqrt{\lambda}} \|u\|_{L^2(\mathbb{B}_r(x))} \tag{1.7}$$

for any $\mathbb{B}_r(x) \subset \mathcal{M}$. In most aforementioned literature on the study of nodal sets, the doubling inequality (1.7) is important. To derive the results on the measure of nodal sets on the boundary, our main efforts are also devoted to obtaining the doubling inequalities for Neumann and Robin eigenfunctions on the boundary $\partial\Omega$. It is also challenging to obtain boundary doubling inequalities since there is only a derivative for the solutions on the boundary. Some partial results are only obtained for Steklov eigenfunctions, see e.g. [5, 38, 46]. The boundary doubling inequality relates to the quantitative boundary unique continuation property discussed in [1, 14] except the rough boundary regularity. Since the doubling inequality implies the vanishing order of solutions, it is also related to the local version of Landis' conjecture in [23] and [32].

**Statement of new results:** The following boundary doubling inequality seems to be new and sharp. It characterizes the growth of Robin eigenfunctions on the boundary with possible large values $|\alpha|$ and $|\lambda|$.

**Theorem 1** *Let $u$ be the Robin eigenfunction in* Eq. 1.1. *There exist positive constants $C$ and $r_0$ depending only on the smooth domain $\Omega$ such that*

$$\|u\|_{L^2(\mathbb{B}_{2r}(x))} \leq e^{C(|\alpha|+\sqrt{|\lambda|})} \|u\|_{L^2(\mathbb{B}_r(x))} \tag{1.8}$$

*for any $0 < r < r_0$ and any $\mathbb{B}_{2r}(x) \subset \partial\Omega$.*

For the Neumann and Robin eigenfunctions, it is likely that the nodal sets of eigenfunctions in $\Omega$ intersect the boundary $\partial\Omega$. More intuitively, the boundary nodal sets are where interior nodal sets touch the boundary. Thus, it is interesting to find out how large the upper bound of the measure of boundary nodal sets is and how the measure depends on $\alpha$ and $\lambda$. The nodal sets on the boundary are of co-dimension one. For the Neumann eigenfunctions, we can show that

**Theorem 2** *Let $u$ be the Neumann eigenfunction in* Eq. 1.2 *in the real analytic domain $\Omega$. There exists a positive constant $C$ that depends only on the domain $\Omega$ such that*

$$H^{n-2}(\{x \in \partial\Omega | u(x) = 0\}) \leq C\sqrt{\lambda}. \tag{1.9}$$

The upper bound of boundary nodal sets for Neumann eigenfunctions in the theorem is optimal. See the Remark 3 in Section 3. Among other interesting results in [42], Toth and Zelditch showed such upper bound for boundary nodal sets of Neumann eigenfunctions in planar analytic domains using a different method. Theorem 2 improves such result to general dimensions.

We are also interested in the measure of nodal sets in $\Omega$ for general eigenvalue problems of Laplacian (1.1). Especially, we want to find out how the upper bound of nodal sets depends on possible large parameter $|\alpha|$ on the boundary. Using the similar idea, we further study the upper bounds for the boundary nodal sets of Robin eigenfunctions.

**Corollary 1** *Let u be the Robin eigenfunction in* Eq. 1.1 *in the real analytic domain* $\Omega$*. There exists a positive constant C depending only on the domain* $\Omega$ *such that*

$$H^{n-2}(\{x \in \partial\Omega | u(x) = 0\}) \leq C(|\alpha| + \sqrt{|\lambda|}). \tag{1.10}$$

Since the model (1.1) includes the Steklov eigenvalue problem (1.4) as the special case with $\lambda = 0$, the upper bound in the corollary includes the sharp results for boundary nodal sets obtained by Zelditch [45] for Steklov eigenfunctions.

Furthermore, we want to know the role of $\alpha$ in the upper bound of interior nodal sets. For the interior nodal sets of Robin eigenfunctions in analytic domains, we can show that

**Theorem 3** *Let u be the Robin eigenfunction in* Eq. 1.1*. There exists a positive constant C depending only on the real analytic domain* $\Omega$ *such that*

$$H^{n-1}(\{x \in \Omega | u(x) = 0\}) \leq C(|\alpha| + \sqrt{|\lambda|}). \tag{1.11}$$

Let us give some comments on those aforementioned results.

*Remark 1* The results in Theorem 1, 3 and Corollary 1 actually hold for either $|\alpha|$ or $|\lambda|$ large. The general eigenvalue problem (1.1) includes Steklov eigenvalue problem as the special case. Thus, the results in Theorem 1, Corollary 1 and Theorem 3 hold for Steklov eigenfunctions. For the study of nodal sets and doubling estimates of Steklov eigenfunctions, see e.g. [5, 16, 37–39, 44–48], etc. Steklov eigenfunctions can be regarded as eigenfunctions of the Dirichlet-to-Neumann map on the boundary. Thus, global Fourier analysis techniques can be applied. However, those global analysis arguments seem not be used directly for the eigenvalue problem (1.1). Some new and novel global Carleman estimates are developed to obtain boundary doubling inequalities and boundary nodal sets. The conclusions in Theorem 1, 2, 3 also hold for Robin eigenfunctions of Laplace-Beltrami operator on any compact Riemannian manifolds.

*Remark 2* For Robin eigenvalue problems, the eigenvalue $\lambda$ depends on the parameter $\alpha$. It is interesting to study the asymptotic estimates of $\lambda$ with respect to $\alpha$. If $\alpha < 0$, it has been shown in [8] that

$$\lim_{\alpha \to -\infty} \frac{\lambda_k}{-\alpha^2} = 1$$

for every $k \geq 1$. Thus, the eigenvalue $|\lambda_k|$ and $\alpha^2$ grow at the same rate. In this case with $|\alpha|$ sufficiently large, we can replace the term $|\alpha|$ by $\sqrt{|\lambda|}$ in Theorem 1, 3 and Corollary 1. Furthermore, the growth rate of $|\lambda_k|$ and $\alpha^2$ seems to be sharp, because $|\alpha|$ is related to the first order differential operator on the boundary and $\lambda$ is related to the second order differential operator. Thus, the quantities $|\alpha|$ and $\sqrt{|\lambda|}$ are the right quantities in Theorem 1, 3 and Corollary 1.

**Discussions of interior nodal sets in the smooth setting:** The interior nodal sets estimates for Dirichlet eigenvalue problem (1.3) and Neumann eigenvalue problem (1.2) in real analytic domains have been shown by Donnelly and Fefferman in [12] to be

$$H^{n-1}(\{x \in \Omega | u(x) = 0\}) \leq C\sqrt{\lambda}.$$

For the smooth manifold, one can obtain the polynomial upper bounds for the nodal sets of Robin eigenfunctions. One can construct a double manifold $\tilde{\Omega} = \Omega \cup \Omega$ to get ride of boundary. Then one can do an even extension for the Neumann eigenvalue problem or an odd extension for the Dirichlet eigenvalue problem on the domain to have second order elliptic equations with Lipschitz metrics. The following sharp doubling inequality on the double manifold

$$\|u\|_{L^2(\mathbb{B}_{2r}(x))} \leq e^{C\sqrt{\lambda}}\|u\|_{L^2(\mathbb{B}_r(x))} \tag{1.12}$$

can be deduced as [12] for the second order elliptic equations with Lipschitz coefficients. Applying the new combinatorial arguments in [29] for the aforementioned second order elliptic equations with Lipschitz coefficients (i.e. Equation 2.9) and doubling inequality (1.14), one can obtain the polynomial upper bound. For the interior nodal sets, one can show that

$$H^{n-1}(\{x \in \Omega | u(x) = 0\}) \leq C(|\alpha| + \sqrt{|\lambda|})^{\beta}, \tag{1.13}$$

where $\beta > 1$ depending only on the dimension $n \geq 3$. Note that Eq. 1.13 holds for Robin eigenfunctions on any compact Riemannian manifolds.

**Outline of the proof:** Let us we briefly sketch the proof of these Theorems. We first need to derive the sharp doubling inequality

$$\|u\|_{L^2(\mathbb{B}_{2r}(x))} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^2(\mathbb{B}_r(x))} \tag{1.14}$$

on the double manifold $\tilde{\Omega} = \Omega \cup \Omega$. We introduce an auxiliary function involving the distance to the boundary, then transform the Robin eigenvalue problem into second order elliptic equations with Neumann boundary conditions. We do an even reflection and obtain some quantitative Carleman estimates to show (1.14) on the double manifold. To obtain the boundary doubling inequality (1.8) in Theorem 1, we prove a new quantitative propagation of smallness lemma (i.e. Lemma 1) with possible large $|\alpha|$ or $|\lambda|$, which is based on a new and novel global quantitative Carleman estimates with boundary terms (i.e. Proposition 2). To obtain the upper bound for boundary nodal sets in Theorem 2, we combine the boundary doubling inequality (1.8) and a complex zero growth lemma (i.e. Lemma 2). To prove Theorem 3, we first find out the upper bounds for nodal sets for the regions in the neighborhood of the boundary, then obtain the nodal sets estimates for regions away from the boundary. The combination of the estimates in the two regions gives Theorem 3.

**Organization of the paper:** Section 2 is devoted to the transformation of the Robin eigenvalue problem to elliptic equations on the double manifold. The doubling inequalities in balls and half-balls are presented. Section 3 is devoted to the proof of the boundary doubling inequalities and nodal sets estimates on the boundary. In the Section 4, we show the upper bounds of interior nodal sets for Robin eigenfunctions. In Section 5, we derive a new type of global quantitative Carleman estimates with boundary terms. In the Appendix, we include the construction of polar coordinates for Lipschitz metrics and the proof of doubling inequalities in balls. The letters $C$, $C_i$ and $\hat{C}$ denote generic positive constants that do not depend on $u$, and may vary from line to line. In the paper, since we study the asymptotic properties for eigenfunctions, we assume that either $|\alpha|$ or $|\lambda|$ is sufficiently large.

## 2 Preliminary

In this section, we transform the Robin eigenvalue problem to elliptic equations with Neumann boundary conditions. We want to move the parameter $\alpha$ on the boundary into the coefficients in a second order elliptic equation. At first, we will transform the Robin eigenvalue problem to a Neumann boundary problem. Considering a small $\rho$-neighborhood of smooth $\partial\Omega$, let

$$\Omega_\rho = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) < \rho\},$$

where $\operatorname{dist}(x, \partial\Omega) = d(x)$ is the distance function to the boundary $\partial\Omega$. Since the domain $\Omega$ is smooth, there exists some small $\rho_0$ depending only on $\Omega$ such that the distance function $d(x) \in C^\infty$ in $\Omega_\rho$ for $0 < \rho < \rho_0$. If $x \in \partial\Omega$, it is known that

$$\nabla d(x) = -\nu(x), \tag{2.1}$$

where $\nu(x)$ is a unit outer normal at $x$. Inspired by the construction in [5] for Steklov eigenfunctions, we introduce the following auxiliary function

$$\bar{u}(x) = e^{-\alpha d(x)} u(x) \qquad \text{for } x \in \Omega_\rho \cup \partial\Omega. \tag{2.2}$$

It is easy to check that $\bar{u}(x)$ satisfies the following second order elliptic equations in a neighborhood of $\Omega$

$$\begin{cases} \triangle\bar{u} + 2\alpha\nabla d(x)\cdot\nabla\bar{u} + (\alpha\triangle d(x) + \alpha^2|\nabla d(x)|^2 + \lambda)\bar{u} = 0 & \text{in } \Omega_\rho, \\ \frac{\partial\bar{u}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We use Fermi coordinates near the boundary to flatten the boundary. Let $0 \in \partial\Omega$. We can find a small constant $\rho > 0$ so that there exists a map $(x', x_n) \in \partial\Omega \times [0, \rho) \to \Omega$ sending $(x', x_n)$ to the endpoint, $x \in \Omega$, with length $x_n$, which starts at $x' \in \partial\Omega$ and is perpendicular to $\partial\Omega$. Such map is a local diffeomorphism. Note that $d(x) = x_n$ in the coordinates and $x'$ is the geodesic normal coordinates of $\partial\Omega$. The metric takes the form

$$\sum_{i,j=1}^{n} g_{ij}dx^i dx^j = dx_n^2 + \sum_{i,j=1}^{n-1} g'_{ij}(x', x_n)dx^i dx^j,$$

where $g'_{ij}(x', x_n)$ is a Riemannian metric on $\partial\Omega$ depending smooth on $x_n \in [0, \rho)$. In a neighborhood of the boundary, the Laplace can be written as

$$\triangle = \sum_{i,j=1}^{n} g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} q_i(x)\frac{\partial}{\partial x_i} \tag{2.4}$$

using local coordinates for $\partial\Omega$, where $g^{ij}$ is the matrix with entries $(g^{ij})_{1<i\le j<n-1} = (g'_{ij})^{-1}$ and $g^{nn} = 1$ and $g^{nk} = g^{kn} = 0$ for $k \neq n$, and $q_i(x) \in C^\infty$.

In the local coordinates, we identify $\partial\Omega$ locally as $\{x_n = 0\}$. The Fermi distance function from 0 on a relatively open neighborhood 0 in $\Omega$ is defined by

$$\tilde{r} = \sqrt{x_1^2 + \cdots + x_{n-1}^2 + x_n^2}.$$

The Fermi exponential map at 0, $exp_0$, which gives the Fermi coordinate system, is defined on a half space of $\mathbb{R}^n_+$. We choose a Fermi half-ball $\tilde{\mathbb{B}}^+_\delta(0)$ centered at origin at $\{x_n = 0\}$ for $0 < \delta < 10\rho_0$. It is known that $\mathbb{B}_{\delta/2}(0) \cap \Omega \subset \tilde{\mathbb{B}}^+_\delta(0) \subset \mathbb{B}_{2\delta}(0) \cap \Omega$, where $\mathbb{B}_\delta(0)$ is the ball centered at origin with radius $\delta$ in the Euclidean space. See e.g. the Appendix A in [26]. For ease of notation, we still write $\tilde{\mathbb{B}}^+_\delta(0)$ as $\mathbb{B}^+_\delta(0)$ . Then it follows from Eq. 2.3 that $\bar{u}$ satisfies the following equation in a neighborhood of the boundary

$$\begin{cases} \triangle_g\bar{u} + \bar{b}(x)\cdot\nabla\bar{u} + \bar{c}(x)\bar{u} = 0 & \text{in } \mathbb{B}^+_\delta(0), \\ \frac{\partial\bar{u}}{\partial\nu} = 0 & \text{on } \mathbb{B}^+_\delta(0) \cap \{x_n = 0\}, \end{cases} \tag{2.5}$$

where $g = (g_{ij})_{n\times n}$ is smooth in $\mathbb{B}^+_\delta$, and $\bar{b}(x)$ and $\bar{c}(x)$ satisfy

$$\begin{cases} \|\bar{b}\|_{C^\infty(\mathbb{B}^+_\delta)} \le C(|\alpha| + 1), \\ \|\bar{c}\|_{C^\infty(\mathbb{B}^+_\delta)} \le C(\alpha^2 + |\lambda|) \end{cases} \tag{2.6}$$

with $C$ depending only on $\partial\Omega$.

We also want to consider the eigenfunction globally on $\Omega$. As it is discussed that the distance function $d(x) = dist(x, \ \partial\Omega)$ is smooth to the boundary $\partial\Omega$ in a small neighborhood $\Omega_\rho$ for some small $\rho$, we make a smooth extension for $d(x)$ in the whole $\Omega$. Then we introduce a smooth function $l(x)$ such that $\varrho(x)$ defined as

$$\varrho(x) = \begin{cases} d(x) & x \in \Omega_\rho, \\ l(x) & x \in \Omega\backslash\Omega_\rho \end{cases}$$

is a smooth function in the whole $\Omega$. Performing the similar procedure as before, we first transform the Robin eigenvalue problem to a Neumann boundary problem. Let

$$\bar{u}(x) = e^{-\alpha\varrho(x)}u(x) \qquad \text{for } x \in \Omega. \tag{2.7}$$

Then $\bar{u}(x)$ satisfies the following Neumann boundary problem

$$\begin{cases} \triangle\bar{u} + 2\alpha\nabla\varrho(x)\cdot\nabla\bar{u} + (\alpha\triangle\varrho(x) + \alpha^2|\nabla\varrho(x)|^2 + \lambda)\bar{u} = 0 & \text{in } \Omega, \\ \frac{\partial\bar{u}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

We want to get rid of the boundary $\partial\Omega$ as well. We define a global double manifold $\tilde{\Omega} = \Omega\cup\Omega$. To extend $\bar{u}$ to be on the double manifold $\tilde{\Omega}$, we consider an even extension, that is

$$\bar{u}\circ\pi = \bar{u},$$

where $\pi : \tilde{\Omega} \to \tilde{\Omega}$ is a cononical involutive isometry which interchanges the two copies of $\tilde{\Omega}$. Near the boundary $\partial\Omega$, the new metric $\tilde{g}$ on the double manifold $\tilde{\Omega}$ is Lipschitz continuous. To explain the metric $\tilde{g}$ is only Lipschitz near the boundary, we use Fermi coordinates with respect to the boundary as before. The differential structure of $\tilde{\Omega}$ near $\partial\Omega$ uses the Fermi coordinates in $g_{ij}$. So $x_n > 0$ and $x_n < 0$ define the two copies of $\Omega$. In these coordinates, $g^{nk} = 0$ for $k \neq n$, there are no cross terms between $\partial_n$ and $\partial_{x_i}$. The metric $g_{ij}(x', |x_n|)$ is symmetric under $x_n \to -x_n$. Thus, it is Lipschitz continuous across $\partial\Omega$. Under the new metric $\tilde{g}$ on the double manifold, from the Eq. 2.8, the new solution $\bar{u}$ satisfies second order elliptic equations

$$\triangle_{\tilde{g}}\bar{u} + \tilde{b}(x)\cdot\nabla\bar{u} + \tilde{c}(x)\bar{u} = 0 \quad \text{in } \tilde{\Omega}, \tag{2.9}$$

where $\tilde{b}$ and $\tilde{c}$ satisfy

$$\begin{cases} \|\tilde{b}\|_{W^{1,\infty}} \leq C(|\alpha| + 1), \\ \|\tilde{c}\|_{W^{1,\infty}} \leq C(\alpha^2 + |\lambda|). \end{cases} \tag{2.10}$$

Now we deal with the second order elliptic equations with Lipschitz continuous coefficients. In order to apply Carleman estimates which are efficient tools to prove doubling inequalities, we want to use polar coordinates. Following the strategy on the regularization for Lipschitz metric in [3] by Aronszajn, Krzywicki and Szarski, we are still able to introduce a suitable geodesic normal coordinates with a conformal metric. For completeness of presentations, we construct such conformal metric in the Appendix. We are able to prove the doubling inequality for $\bar{u}$ on the double manifold. We include the proof in the Appendix as well.

**Proposition 1** *Let $\bar{u}$ be the solution of* Eq. 2.9 *satisfying the conditions* (2.10)*. There exists a positive constant C depending only on $\tilde{\Omega}$ such that the doubling inequality holds*

$$\|\bar{u}\|_{L^2(\mathbb{B}_{2r}(x))} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_r(x))} \tag{2.11}$$

*for any $x \in \tilde{\Omega}$.*

For the latter sections on the study of nodal sets and the doubling inequality on the boundary, we want to show the doubling inequality in the half ball $\mathbb{B}_r^+(0)$. Since we did an even extension across the boundary $\{x_n = 0\}$, the estimates (2.11) also holds in the half balls. Thus, there exist positive constants $C, r_0$ depending only on $\Omega$ such that the doubling inequality holds

$$\|\bar{u}\|_{L^2(\mathbb{B}_{2r}^+)} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_r^+)} \tag{2.12}$$

for $0 < r < r_0$.

## 3 Boundary doubling inequality and nodal sets

In this section, we prove new quantitative propagation smallness results for the second order elliptic (2.5) in the half ball. By rescaling, we may consider the equations in $\mathbb{B}_{1/2}^+$. To present the results in a general setting, we may consider the second order uniformly elliptic equations

$$-a_{ij}D_{ij}u + b_i(y)D_iu + c(y)u = 0 \quad in\ \mathbb{B}_{1/2}^+, \tag{3.1}$$

where $a_{ij}$ is $C^1$, and $b(y)$ and $c(y)$ satisfy

$$\begin{cases} \|b\|_{W^{1,\infty}(\mathbb{B}_{1/2}^+)} \leq C(|\alpha|+1), \\ \|c\|_{W^{1,\infty}(\mathbb{B}_{1/2}^+)} \leq C(\alpha^2+|\lambda|). \end{cases} \tag{3.2}$$

We are able to show the following quantitative two half-ball and one lower dimensional ball type result.

**Lemma 1** *Let $u \in C_0^\infty(\mathbb{B}_{1/2}^+)$ be a solution of* Eq. 3.1. *Denote the lower dimensional ball*

$$\boldsymbol{B}_{1/3} = \{(y',\ 0) \in \mathbb{R}^n | y' \in \mathbb{R}^{n-1},\ |y'| < \frac{1}{3}\}.$$

*Assume that*

$$\|u\|_{H^1(\boldsymbol{B}_{1/3})} + \|\frac{\partial u}{\partial \nu}\|_{L^2(\boldsymbol{B}_{1/3})} \leq \epsilon << 1 \tag{3.3}$$

*and $\|u\|_{L^2(\mathbb{B}_{1/2}^+)} \leq 1$. There exist positive constants $C$ and $\beta$ such that*

$$\|u\|_{L^2(\frac{1}{256}\mathbb{B}_1^+)} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\epsilon^\beta. \tag{3.4}$$

*More precisely, we can show that there exists $0 < \kappa < 1$ such that*

$$\|u\|_{L^2(\frac{1}{256}\mathbb{B}_1^+)} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|^{\kappa}_{L^2(\mathbb{B}_{1/2}^+)}\big(\|u\|_{H^1(\boldsymbol{B}_{1/3})} + \|\frac{\partial u}{\partial \nu}\|_{L^2(\boldsymbol{B}_{1/3})}\big)^{1-\kappa}. \tag{3.5}$$

Such estimates without considering the quantitative behavior of $\alpha$ and $\lambda$ have been established in [27]. To show the quantitative three-ball inequality in the lemma, we develop some novel quantitative global Carleman estimates involving the boundary. The weight function in Carleman estimates (3.6) is somewhat inspired by [24] and [21]. Such results play an important role not only in characterizing the doubling index in a cube in [29], but also in inverse problems, see [2].

The quantitative global Carleman estimates with boundary are stated in Proposition 2. We choose a weight function

$$\psi(y) = e^{sh(y)},$$

where

$$h(y)=-\frac{|y'|^2}{4}+\frac{y_n^2}{2}-y_n$$

and $s$ is a large parameter that will be determined later.

**Proposition 2** *Let $s$ be a fixed large constant. There exist positive constants $C_s$ and $C_0$ depending on $s$ such that for any $v\in C^\infty(\mathbb{B}^+_{1/2})$, and*

$$\tau>C_s(|\alpha|+\sqrt{|\lambda|}),$$

*one has*

$$\begin{aligned}&\|e^{\tau\psi}(-a_{ij}D_{ij}v+b_iD_iv+cv)\|_{L^2(\mathbb{B}^+_{1/2})}+\tau^{\frac{3}{2}}s^2\|\psi^{\frac{3}{2}}e^{\tau\psi}v\|_{L^2(\partial\mathbb{B}^+_{1/2})}+\tau^{\frac{1}{2}}s\|\psi^{\frac{1}{2}}e^{\tau\psi}\nabla v\|_{L^2(\partial\mathbb{B}^+_{1/2})}\\ \geq\ & C_0\tau^{\frac{3}{2}}s^2\|\psi^{\frac{3}{2}}e^{\tau\psi}v\|_{L^2(\mathbb{B}^+_{1/2})}+C_0\tau^{\frac{1}{2}}s\|\psi^{\frac{1}{2}}e^{\tau\psi}\nabla v\|_{L^2(\mathbb{B}^+_{1/2})}.\end{aligned}\tag{3.6}$$

Since the proof of Proposition 2 is lengthy, we postpone the proof in Section 4. Thanks to the Carleman estimates (3.6), we first show the proof of Lemma 1

*Proof of Lemma 1* Notice that the constant $s$ is fixed independent of $\alpha$ and $\lambda$. We also know $\psi$ is bounded below and above by some constant $C$. We obtain that

$$\begin{aligned}&\|e^{\tau\psi}(-a_{ij}D_{ij}v+b_iD_iv+cv)\|_{L^2(\mathbb{B}^+_{1/2})}+\tau^{\frac{3}{2}}\|e^{\tau\psi}v\|_{L^2(\partial\mathbb{B}^+_{1/2})}+\tau^{\frac{1}{2}}\|e^{\tau\psi}\nabla v\|_{L^2(\partial\mathbb{B}^+_{1/2})}\\ \geq\ & C\tau^{\frac{3}{2}}\|e^{\tau\psi}v\|_{L^2(\mathbb{B}^+_{1/2})}+C\tau^{\frac{1}{2}}\|e^{\tau\psi}\nabla v\|_{L^2(\mathbb{B}^+_{1/2})}.\end{aligned}\tag{3.7}$$

The following Caccioppolli inequality holds for the solutions of Eq. 3.1 in $\mathbb{B}^+_{1/2}$,

$$\|\nabla u\|_{L^2(\mathbb{B}^+_r)}\leq\frac{C(|\alpha|+\sqrt{|\lambda|})}{r}\big(\|u\|_{L^2(\mathbb{B}^+_{2r})}+\|\nabla u\|_{L^2(\mathbf{B}_{2r})}+\|u\|_{L^2(\mathbf{B}_{2r})}\big).\tag{3.8}$$

We select a smooth cut-off function $\eta$ such that $\eta(x)=1$ in $\mathbb{B}^+_{1/8}$ and $\eta(x)=0$ outside $\mathbb{B}^+_{1/4}$. Since $u\in C^\infty_0(\mathbb{B}^+_{1/2})$, substituting $v$ by $\eta u$ in the Carleman estimates (3.7) and then using the Eq. 3.1 yields that

$$\begin{aligned}\|e^{\tau\psi}(-a_{ij}D_{ij}\eta u-2a_{ij}D_i\eta D_ju+b_iD_i\eta u)\|_{L^2(\mathbb{B}^+_{1/2})}&+\tau^{\frac{3}{2}}\|e^{\tau\psi}\eta u\|_{L^2(\mathbf{B}_{1/4})}\\ &+\tau^{\frac{1}{2}}\|e^{\tau\psi}\eta\nabla u\|_{L^2(\mathbf{B}_{1/4})}\\ &\geq\ C\tau^{\frac{3}{2}}\|e^{\tau\psi}\eta u\|_{L^2(\mathbb{B}^+_{1/2})}.\end{aligned}\tag{3.9}$$

We want to find the maximum of $\psi$ in the first term on the left hand side of Eq. 3.9. Since $h(y)$ is negative in $\mathbb{B}^+_{1/2}$, then

$$\max_{\{\frac{1}{8}\leq|y|\leq\frac{1}{4}\}\cap\{y_n\geq 0\}}h(y)=\max_{\{\frac{1}{8}\leq|y|\leq\frac{1}{4}\}}-\frac{|y'|^2}{4}=-\frac{1}{256}.$$

We also need to find a lower bound of $\psi$ for the term on the right hand side of Eq. 3.9 such that

$$-\min_{|y|<a}h(y)-\frac{1}{256}<0$$

for some $0 < a < \frac{1}{2}$. Since $h(y)$ decreases with respect to $y'$ and $y_n$, then the minimum of $h(y)$ is $\hat{h}(a)$ for $|y| < a$, where

$$\hat{h}(a) = -\frac{a^2}{4} + \frac{a^2}{2} - a = \frac{a^2}{4} - a.$$

Solving the inequality $-\hat{h}(a) < \frac{1}{256}$, we have one solution $a = \frac{1}{256}$. Set

$$\psi_0 = e^{-\frac{s}{256}} - e^{s\hat{h}(\frac{1}{256})},$$

then $\psi_0 < 0$. Define

$$\psi_1 = 1 - e^{s\hat{h}(\frac{1}{256})}.$$

Since $\hat{h}(\frac{1}{256}) < 0$, then $\psi_1 > 0$.

Applying the Caccioppolli inequality (3.8), we arrive at

$$\begin{aligned} &\exp\{\tau e^{\frac{-s}{256}}\}\|u\|_{L^2(\mathbb{B}^+_{1/2})} + e^{\tau}\|u\|_{L^2(\mathbf{B}_{1/3})} + e^{\tau}\|\nabla u\|_{L^2(\mathbf{B}_{1/3})} \\ \geq\ & C\tau\exp\{\tau e^{s\hat{h}(\frac{1}{256})}\}\|u\|_{L^2(\mathbb{B}^+_{1/256})}. \end{aligned} \tag{3.10}$$

Let

$$\begin{aligned} B_1 &= \|u\|_{L^2(\mathbb{B}^+_{1/2})}, \\ B_2 &= \|u\|_{L^2(\mathbf{B}_{1/3})} + \|\nabla u\|_{L^2(\mathbf{B}_{1/3})}, \\ B_3 &= \|u\|_{L^2(\mathbb{B}^+_{1/256})}. \end{aligned}$$

Multiplying both sides of the last inequality with $\exp\{-\tau e^{s\hat{h}(\frac{1}{256})}\}$ leads to

$$e^{\tau\psi_0}B_1 + e^{\tau\psi_1}B_2 \geq CB_3. \tag{3.11}$$

We want to incorporate the first term on the left hand side of Eq. 3.11 into the right hand side. Let

$$e^{\tau\psi_0}B_1 \leq \frac{1}{2}CB_3.$$

Thus, we need to have

$$\tau \geq \frac{1}{\psi_0}\ln\frac{CB_3}{2B_1}.$$

Therefore, for such $\tau$,

$$e^{\tau\psi_1}B_2 \geq \frac{C}{2}B_3. \tag{3.12}$$

Recall that the assumption

$$\tau \geq C(|\alpha| + \sqrt{|\lambda|})$$

in Proposition 2. We assume that

$$\tau = C(|\alpha| + \sqrt{|\lambda|}) + \frac{1}{\psi_0}\ln\frac{CB_3}{2B_1}. \tag{3.13}$$

Note that $\psi_0$ and $\psi_1$ are constants. Substituting such $\tau$ in Eq. 3.12 yields that

$$e^{C(|\alpha|+\sqrt{|\lambda|})}B_1^{\frac{\psi_1}{\psi_1-\psi_0}}B_2^{\frac{-\psi_0}{\psi_1-\psi_0}} \geq CB_3. \tag{3.14}$$

Let $\kappa = \frac{\psi_1}{\psi_1-\psi_0}$. Then the following three-ball type inequality follows as

$$\|u\|_{L^2(\frac{1}{256}\mathbb{B}^+_1)} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|^{\kappa}_{L^2(\mathbb{B}^+_{1/2})}\left(\|u\|_{L^2(\mathbf{B}_{1/3})} + \|\nabla u\|_{L^2(\mathbf{B}_{1/3})}\right)^{1-\kappa}. \tag{3.15}$$

Since $u \in C^2(\mathbb{B}^+_{1/2})$ and $\nabla u = \nabla' u + \frac{\partial u}{\partial \nu}$ on the boundary $\mathbb{B}^+_{1/2} \cap \{y_n = 0\}$, the inequality (3.15) implies the desired estimates (3.5). The estimate (3.4) is a consequence of Eq. 3.5. Therefore, the lemma is finished. □

We are in the position to prove the boundary doubling inequality in Theorem 1 with some inspirations from [27].

*Proof of Theorem 1* We consider the solution $\bar{u}$ in the Eq. 2.5 with conditions (2.6). We argue on scale of order one. We may normalize $\bar{u}$ as

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/2})} = 1. \tag{3.16}$$

We claim that there exists a positive constant $C > 0$ such that the following lower bound holds on the boundary

$$\|\bar{u}\|_{H^1(\mathbf{B}_{1/6})} \geq e^{-C(|\alpha|+\sqrt{|\lambda|})}. \tag{3.17}$$

We will need to use the quantitative three-ball inequality (3.5) on the half balls. Note that $\frac{\partial \bar{u}}{\partial \nu} = 0$ on the boundary $\{x_n = 0\}$. We may normalize the inequality (3.5) as

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/512})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|^{\kappa}_{L^2(\mathbb{B}^+_{1/4})} \|\bar{u}\|^{1-\kappa}_{H^1(\mathbf{B}_{1/6})}. \tag{3.18}$$

We prove the claim by contradiction. If the claim is not true, from Eq. 3.18, for any constant $\hat{C} > 0$, we have

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/512})} \leq C e^{-\hat{C}(|\alpha|+\sqrt{|\lambda|})}. \tag{3.19}$$

Since the doubling estimate on the half ball has been shown in Eq. 2.12, using the doubling inequality finitely many times, we obtain that

$$\begin{aligned}\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/512})} &\geq e^{-C(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|_{L^2(\mathbb{B}^+_{1/2})} \\ &\geq e^{-C(|\alpha|+\sqrt{|\lambda|})},\end{aligned} \tag{3.20}$$

which contradicts the condition (3.19) since $\hat{C}$ is an arbitrary constant that can be chosen to be sufficiently large. Thus, the condition (3.17) holds.

Next we claim that there exists a constant $C$ such that

$$\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})} \geq e^{-C(|\alpha|+\sqrt{|\lambda|})}. \tag{3.21}$$

We recall the following interpolation inequality in [38] or [5]. For any small constant $0 < \epsilon < 1$, there holds

$$\|\nabla' w\|_{L^2(\mathbb{R}^{n-1})} \leq \epsilon^{\frac{3}{2}} (\|\nabla\nabla' w\|_{L^2(\mathbb{R}^n_+)} + \|w\|_{L^2(\mathbb{R}^n_+)}) + \epsilon^{-\frac{1}{3}} \|w\|_{L^2(\mathbb{R}^{n-1})}, \tag{3.22}$$

where $\nabla'$ is the derivative for first $n-1$ variables. We choose $w$ to be $\bar{u}\eta$, where $\eta$ is a radial cut-off function such that $\eta = 1$ in $\mathbb{B}^+_{1/6}$ and vanishes outside $\mathbb{B}^+_{1/5}$. Substituting $w = \bar{u}\eta$ in the interpolation inequality (3.22) gives that

$$\|\nabla'(\bar{u}\eta)\|_{L^2(\mathbb{R}^{n-1})} \leq \epsilon^{\frac{3}{2}} \left(\|\nabla\nabla'(\bar{u}\eta)\|_{L^2(\mathbb{B}^+_{1/5})} + \|\bar{u}\eta\|_{L^2(\mathbb{B}^+_{1/5})}\right) + \epsilon^{-\frac{1}{3}} \|\bar{u}\eta\|_{L^2(\mathbf{B}_{1/5})}. \tag{3.23}$$

Using the fact that $g_{in} = 0$ for $i \neq n$ and $\frac{\partial \bar{u}}{\partial \nu} = 0$ on $\{x_n = 0\}$, the following Caccioppolli inequality holds,

$$\|\nabla \bar{u}\|_{L^2(\mathbb{B}^+_r)} \leq \frac{C(|\alpha| + \sqrt{|\lambda|})}{r} \|\bar{u}\|_{L^2(\mathbb{B}^+_{2r})}. \tag{3.24}$$

By the elliptic estimates, it is true that

$$\|\nabla\nabla'\bar{u}\|_{L^2(\mathbb{B}^+_{1/5})} \le C(|\alpha| + \sqrt{|\lambda|})^2\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/2})}. \tag{3.25}$$

Applying the estimates (3.24) and (3.25) for $\bar{u}$ in Eq. 3.23, we derive that

$$\|\nabla'\bar{u}\|_{L^2(\mathbf{B}_{1/6})} \le C_0\epsilon^{\frac{3}{2}}(|\alpha| + \sqrt{|\lambda|})^2 + \epsilon^{-\frac{1}{3}}\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})}, \tag{3.26}$$

where we have used Eq. 3.16. Adding $\|\bar{u}\|_{L^2(\mathbf{B}_{1/6})}$ to both sides of the last inequality yields that

$$\|\bar{u}\|_{H^1(\mathbf{B}_{1/6})} \le C_0\epsilon^{\frac{3}{2}}(|\alpha| + \sqrt{|\lambda|})^2 + 2\epsilon^{-\frac{1}{3}}\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})}. \tag{3.27}$$

To incorporate the first term on the right hand side of the last inequality into the left hand side, we choose $\epsilon$ such that

$$C_0\epsilon^{\frac{3}{2}}(|\alpha| + \sqrt{|\lambda|})^2 = \frac{1}{2}\|\bar{u}\|_{H^1(\mathbf{B}_{1/6})}. \tag{3.28}$$

That is,

$$\epsilon = \Big(\frac{\|\bar{u}\|_{H^1(\mathbf{B}_{1/6})}}{2C_0(|\alpha| + \sqrt{|\lambda|})^2}\Big)^{2/3}.$$

Therefore, (3.27) turns into

$$\|\bar{u}\|^{11/9}_{H^1(\mathbf{B}_{1/6})} \le C(|\alpha| + \sqrt{|\lambda|})^{4/9}\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})}. \tag{3.29}$$

Because of Eq. 3.17, we infer that

$$\|\bar{u}\|_{H^1(\mathbf{B}_{1/6})} \le e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})}. \tag{3.30}$$

From (3.17) again, it also follows that

$$\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})} \ge e^{-C(|\alpha|+\sqrt{|\lambda|})}, \tag{3.31}$$

which verifies the claim (3.21).

Let $\bar{\eta}$ be a cut-off function such that $\bar{\eta}(y) = 1$ for $|y| \le \frac{1}{4}$ and vanishes for $|y| \ge \frac{1}{3}$. By the Hardy trace inequality and elliptic estimates (3.24), it follows that

$$\begin{aligned}\|\bar{u}\|_{L^2(\mathbf{B}_{1/4})} &\le \|\bar{\eta}\bar{u}\|_{L^2(\mathbf{B}_{1/4})} \le \|\nabla(\bar{\eta}\bar{u})\|_{L^2(\mathbb{R}^n_+)} \\ &\le C\|\nabla\bar{u}\|_{L^2(\mathbb{B}^+_{1/3})} + C\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/3})} \\ &\le C(|\alpha| + \sqrt{|\lambda|})\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/2})} \\ &\le C(|\alpha| + \sqrt{|\lambda|}).\end{aligned} \tag{3.32}$$

Combining the established estimates (3.31) and (3.32), we have

$$\|\bar{u}\|_{L^2(\mathbf{B}_{1/4})} \le e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbf{B}_{1/5})}. \tag{3.33}$$

Notice that $u = \bar{u}$ on $\mathbf{B}_{1/2}$. By rescaling and diffeomorphism of Fermi exponential map, we arrive at

$$\|u\|_{L^2(\mathbb{B}_{2r}(x_0))} \le e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^2(\mathbb{B}_r(x_0))} \tag{3.34}$$

for any $x_0 \in \partial\Omega$, $\mathbb{B}_{2r}(x_0) \subset \partial\Omega$, and $r < r_0$ for some $r_0$ depending only on $\partial\Omega$. This completes the proof of Theorem 1. □

We will show the upper bounds of nodal sets for Neumann and Robin eigenfunction on the analytic boundary. To achieve it, we need a quantitative inequality on the relation of $L^2$ norm of eigenfunctions on the boundary and on the half balls. We argue on scale with

$\delta = 1$ for Eq. 2.5 with the conditions (2.6). Applying quantitative two half-ball and one lower dimensional ball in Eq. 3.5 by replacing $u$ by $\bar{u}$, and the doubling inequalities in the half ball in Eq. 2.12 finitely many times, we have

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/2})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|^{\kappa}_{L^2(\mathbb{B}^+_{1/2})}\|\bar{u}\|^{1-\kappa}_{H^1(\mathbf{B}_{1/3})}. \tag{3.35}$$

Thus, we obtain that

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/8})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{H^1(\mathbf{B}_{1/3})}. \tag{3.36}$$

By the arguments in deriving the estimates (3.30), we can improve (3.36) as

$$\|\bar{u}\|_{L^2(\mathbb{B}^+_{1/8})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbf{B}_{2/5})}. \tag{3.37}$$

For the upper bounds of nodal sets, we need a lemma concerning the growth of a complex analytic function with the number of zeros, see [10] or Lemma 2.3.2 in [17].

**Lemma 2** *Suppose $f : \mathcal{B}_1(0) \subset \mathbb{C} \to \mathbb{C}$ is an analytic function satisfying*

$$f(0) = 1 \quad \textit{and} \quad \sup_{\mathcal{B}_1(0)} |f| \leq 2^N$$

*for some positive constant $N$. Then for any $r \in (0, 1)$, there holds*

$$\sharp\{z \in \mathcal{B}_r(0) : f(z) = 0\} \leq cN$$

*where $c$ depends on $r$. Especially, for $r = \frac{1}{2}$, there holds*

$$\sharp\{z \in \mathcal{B}_{1/2}(0) : f(z) = 0\} \leq N.$$

We are ready to provide the proof of upper bounds for the boundary nodal sets. See the pioneering work in [10] and [27] for interior nodal sets. For conveniences of the presentations, we first show the proof of Neumann eigenfunctions.

*Proof of theorem 2* To get the measure estimates of nodal sets of Neumann eigenfunctions on the boundary, we perform a standard lifting argument. Let

$$\hat{w}(x, t) = e^{\sqrt{\lambda}t}u(x). \tag{3.38}$$

Then $\hat{w}(x, t)$ satisfies the following equation

$$\begin{cases} -\hat{\triangle}\hat{w} = 0 \text{ in } \Omega \times (-\infty,\ \infty), \\ \frac{\partial \hat{w}}{\partial \nu} = 0 \text{ on } \partial\Omega \times (-\infty,\ \infty) \end{cases} \tag{3.39}$$

with $\hat{\triangle} = \triangle + \partial_t^2$. By straightening the boundary $\partial\Omega$ locally, rescaling and translation. we may assume that $(p, 0, t) \in \big(\partial\mathbb{B}^+_{1/16} \cap \{x_n = 0\}\big) \times (-\frac{1}{16}, \frac{1}{16})$ with $p \in \mathbb{R}^{n-1}$. From elliptic estimates in Lemma 2.3 in [36], we obtain that

$$|\frac{\nabla'^{\bar{\alpha}}\hat{w}(p, 0, 0)}{\bar{\alpha}!}| \leq C\hat{C}^k\|\hat{w}\|_{L^\infty\left(\mathbb{B}^+_{1/8}\times(-\frac{1}{8},\frac{1}{8})\right)}, \tag{3.40}$$

where $|\bar{\alpha}| = k$, the derivative is taken with respect to $x'$ on $\partial\mathbb{B}^+_{1/16} \cap \{x_n = 0\}$, and $\hat{C} > 1$ depends on $\Omega$. By the definition of $\hat{w}$, we have that

$$|\frac{\nabla'^{\bar{\alpha}}u(p, 0)}{\bar{\alpha}!}| \leq C\hat{C}^k e^{C\sqrt{\lambda}}\|u\|_{L^\infty(\mathbb{B}^+_{1/8})}. \tag{3.41}$$

Then $u(p,0)$ is real analytic for any $(p,0)\in\partial\mathbb{B}^+_{1/16}\cap\{x_n=0\}$. We may consider $p$ as the origin in $\mathbb{R}^{n-1}$. Summing up a geometric series gives a holomorphic extension of $u$ with

$$\sup_{|z|\le\frac{1}{2(n-1)\hat{C}}}|u(z)|\le e^{C\sqrt{\lambda}}\|u\|_{L^\infty(\mathbb{B}^+_{1/8})},\tag{3.42}$$

where $\frac{1}{2(n-1)\hat{C}}<\frac{1}{8}$ and $z\in\mathbb{C}^{n-1}$. The estimates (3.37) also hold for Neumann boundary conditions with $\alpha=0$. Hence, it follows that

$$\|u\|_{L^2(\mathbb{B}^+_{1/8})}\le e^{C\sqrt{\lambda}}\|u\|_{L^2(\mathbf{B}_{2/5})}.\tag{3.43}$$

Note that $\mathbf{B}_r$ is denoted as the ball in $\mathbb{R}^{n-1}$ with radius $r$. Taking the boundary doubling inequality (3.33) with $\alpha=0$, Eq. 3.42 and elliptic estimates into consideration, by finite steps of iterations, we conclude that

$$\sup_{|z|\le\frac{1}{2(n-1)\hat{C}}}|u(z)|\le e^{C\sqrt{\lambda}}\sup_{x\in\mathbf{B}_{\frac{1}{4(n-1)\hat{C}}}}|u(x)|.\tag{3.44}$$

By rescaling arguments, we derive that

$$\sup_{|z|\le 2r}|u(z)|\le e^{C\sqrt{\lambda}}\sup_{x\in\mathbf{B}_r}|u(x)|,\tag{3.45}$$

where $0<r<\hat{r}_0$ and $\hat{r}_0$, $C$ depends on $\Omega$.

Thanks to the doubling inequality (3.45) and the growth control lemma for zeros, i.e. Lemma 2, we are ready to give the proof of Theorem 2. Since $r$ does not depend on $\lambda$, we can argue on scales of $r=1$. Let $p\in\mathbf{B}_{1/4}\subset\mathbb{R}^{n-1}$ be the point where the supremum of $|u|$ is achieved. After rescaling, we assume that $|u(p)|=1$. By the doubling inequality (3.45), we have

$$\begin{aligned}\sup_{|z|\le 1}|u(z)| &\le e^{C\sqrt{\lambda}}\sup_{|x|\le\frac{1}{4}}|u(x)|\\ &\le e^{C\sqrt{\lambda}}.\end{aligned}\tag{3.46}$$

Applying (3.45) to the translation of $u$, we obtain that

$$\begin{aligned}\sup_{|z-p|\le 1}|u(z)| &\le e^{C\sqrt{\lambda}}\sup_{|x-p|\le\frac{1}{4}}|u(x)|\\ &\le e^{C\sqrt{\lambda}}\sup_{|x|\le\frac{1}{2}}|u(x)|\\ &\le e^{C\sqrt{\lambda}}.\end{aligned}\tag{3.47}$$

For each direction $\omega\in S^{n-2}$, we consider the function

$$u_\omega(z)=u(p+z\omega),\qquad z\in\mathcal{B}_1(0)\subset\mathbb{C}.$$

Denote $N(\omega)=\sharp\{z\in\mathcal{B}_{1/2}(0)\subset\mathbb{C}|e_\omega(z)=0\}$. By the doubling inequality (3.45) and Lemma 2, we obtain that

$$\begin{aligned}&\sharp\{x\in\mathbf{B}_{1/2}(p)\subset\mathbb{R}^{n-1}|x-p \text{ is parallel to } \omega \text{ and } u(x)=0\}\\ \le\ &\sharp\{z\in\mathcal{B}_{1/2}(0)\subset\mathbb{C}|u_\omega(z)=0\}\\ =\ &N(\omega)\\ \le\ &C\sqrt{\lambda}.\end{aligned}\tag{3.48}$$

By the integral geometry estimates, we further derive that

$$\begin{aligned} H^{n-2}\{x \in \mathbf{B}_{1/2}(p)|u(x)=0\} &\leq c(n)\int_{S^{n-2}} N(\omega)d\omega \\ &\leq \int_{S^{n-2}} C\sqrt{\lambda}\,d\omega \\ &\leq C\sqrt{\lambda}. \end{aligned} \tag{3.49}$$

Thus, we show the upper bound of nodal sets

$$H^{n-2}\{x \in \mathbf{B}_{1/4}(0)|u(x)=0\} \leq C\sqrt{\lambda}. \tag{3.50}$$

By rescaling, it also implies that

$$H^{n-2}\{\mathbb{B}_{r_0}(p) \subset \partial\Omega|u(x)=0\} \leq C\sqrt{\lambda} \tag{3.51}$$

for some $r_0$ depending only on $\Omega$ and for any $p \in \Omega$. Since the boundary $\partial\Omega$ is compact, by finite number of coverings, the theorem is arrived. □

By the strategy in the proof of Theorem 2, we consider the boundary nodal sets of Robin eigenfunctions (1.1).

*Proof of Corollary 1* To get rid of $\alpha$ on the boundary, we introduce the following lifting argument. Let

$$\hat{v}(x,t) = e^{\alpha t}u(x).$$

Then $\hat{v}(x,t)$ satisfies the equation

$$\begin{cases} \triangle\hat{v} + \partial_t^2\hat{v} - \alpha^2\hat{v} + \lambda\hat{v} = 0 \ \text{ in } \Omega \times (-\infty,\ -\infty), \\ \frac{\partial\hat{v}}{\partial\nu} + \frac{\partial\hat{v}}{\partial t} = 0 \ \text{ on } \partial\Omega \times (-\infty,\ -\infty). \end{cases} \tag{3.52}$$

Using the lifting arguments, we get rid of $\lambda$ and $\alpha$. Let

$$\hat{w}(x,t,s) = e^{(|\alpha|i+\sqrt{\lambda})s}\hat{v}(x,t).$$

If $\lambda < 0$, then $\sqrt{\lambda}$ is considered as the imaginary number. Then we have

$$\begin{cases} \triangle\hat{w} + \partial_t^2\hat{w} + \partial_s^2\hat{w} = 0 \ \text{ in } \Omega \times (-\infty,\ -\infty) \times (-\infty,\ -\infty), \\ \frac{\partial\hat{w}}{\partial\nu} + \frac{\partial\hat{w}}{\partial t} = 0 \ \text{ on } \partial\Omega \times (-\infty,\ -\infty) \times (-\infty,\ -\infty). \end{cases} \tag{3.53}$$

Note that Eq. 3.53 is a uniformly elliptic equation with oblique boundary conditions. The similar lifting argument has been used in [28]. We introduce the cube with unequal radius as

$$\Omega_{R,\delta} = \{(x,t,s) \in \mathbb{R}^{n+2}||x_i| < R \text{ when } i < n, |x_n| < \delta R,\ |t| < R,\ |s| < R\}$$

and half-cube

$$\Omega^+_{R,\delta} = \{(x,t,s) \in \mathbb{R}^{n+2}||x_i| < R \text{ when } i < n, 0 \leq x_n < \delta R,\ |t| < R,\ |s| < R\}.$$

By rescaling, we may consider the function $\hat{w}(x,t,s)$ locally in the cube with the flatten boundary using the Fermi coordinates in Section 2. Thus, $\hat{w}(x,t,s)$ satisfies

$$\begin{cases} \triangle\hat{w} + \partial_t^2\hat{w} + \partial_s^2\hat{w} = 0 \ \text{ in } \Omega^+_{2,1}, \\ \frac{\partial\hat{w}}{\partial x_n} + \frac{\partial\hat{w}}{\partial t} = 0 \qquad \text{ on } \Omega^+_{2,1} \cap \{x_n = 0\}. \end{cases} \tag{3.54}$$

By the analytical results in [36], we can extend $\hat{w}(x,t,s)$ to the region $\Omega_{1,\delta}$, where $\delta$ depends only on $\partial\Omega$. Moreover, we have

$$\|\hat{w}\|_{L^\infty(\Omega_{1,\delta})} \leq C_2(\Omega)\|\hat{w}\|_{L^\infty(\Omega^+_{2,1})}. \tag{3.55}$$

Let us assume $0 < \delta < \frac{1}{24}$. Choose any point $(p, 0) \in \partial\mathbb{B}^{+}_{\frac{1}{16}} \cap \{x_n = 0\}$ with $p \in \mathbb{R}^{n-1}$. By elliptic estimates in $\Omega_{1,\delta}$, we have

$$
\begin{aligned}
|\frac{\nabla'^{\bar{\alpha}}\hat{w}(p,0,0,0)}{\bar{\alpha}!}| &\le C\hat{C}^k\|\hat{w}\|_{L^\infty\left(\mathbb{B}_\delta(p,0)\times(-\delta,\delta)\times(-\delta,\delta)\right)} \\
&\le C\hat{C}^k\|\hat{w}\|_{L^\infty\left(\mathbb{B}^{+}_{\frac{1}{8}}(p,0)\times(-\frac{1}{8},\frac{1}{8})\times(-\frac{1}{8},\frac{1}{8})\right)},
\end{aligned}
\tag{3.56}
$$

where $|\bar{\alpha}| = k$, the derivative is taken with respect to $x'$ in $\partial\mathbb{B}^{+}_{1/16} \cap \{x_n = 0\}$, and $\hat{C} > 1$ depends on $\Omega$. In the second inequality of Eq. 3.56, we applied the growth control estimate (3.55). From the definition of $\hat{w}(x, t, s)$, it holds that

$$
|\frac{\nabla'^{\bar{\alpha}}u(p,0)}{\bar{\alpha}!}| \le C\hat{C}^k e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^\infty(\mathbb{B}^{+}_{\frac{1}{8}}(p,0))}. \tag{3.57}
$$

The estimate (3.57) is the same as Eq. 3.41 in the proof of Theorem 2. The rest of the proof follows from Theorem 2 by using the boundary doubling inequality (3.33), and (3.37), and Lemma 2. Thus, we can show the upper bounds of boundary nodal sets in the Corollary. □

At last, we show the sharpness of the upper bounds of boundary nodal sets for Neumann eigenfunctions.

*Remark 3* The upper bound for boundary nodal sets of Neumann eigenfunctions in Eq. 1.9 is sharp.

For $n = 2$, the sharp example was already constructed in [42] in a disc with radius one in $\mathbb{R}^2$.

For $n \ge 3$, we consider the Neumann eigenvalue problem in a ball with radius one. By separation of variables, let $u(x) = R(r)\Phi(\omega)$, Then $R(r)$ satisfies the equations

$$
R''(r) + \frac{n-1}{r}R'(r) + (\lambda - \frac{\gamma}{r^2})R(r) = 0 \tag{3.58}
$$

and

$$
-\triangle_\omega \Phi = \gamma\Phi \qquad \text{on } S^{n-1}, \tag{3.59}
$$

where $\gamma = k(k+n-2)$ is the eigenvalue for the spherical harmonics on $S^{n-1}$. By a standard scaling, let $W(r) = r^{\frac{n-2}{2}}R(r)$. Equation 3.58 is reduced to the equation

$$
W''(r) + \frac{1}{r}W(r) + \left(\lambda - (\gamma + \frac{(n-2)^2}{4})/r^2\right)W(r) = 0. \tag{3.60}
$$

We can write the solutions as

$$
W(r) = J_{\sqrt{\gamma+\frac{(n-2)^2}{4}}}(\sqrt{\lambda}r),
$$

where $J_{\sqrt{\gamma+\frac{(n-2)^2}{4}}}(y)$ is the Bessel function. Recall that the $k$-th Bessel function $J_k(y)$ is the solution for

$$
y^2J''(y) + yJ'(y) + (y^2 - k^2)J(y) = 0. \tag{3.61}
$$

That is,

$$
R(r) = \frac{J_{\sqrt{\gamma+\frac{(n-2)^2}{4}}}(\sqrt{\lambda}r)}{r^{\frac{n-2}{2}}}.
$$

The Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial (R(r)\Phi(\omega))}{\partial r} = 0$$

on $r = 1$ implies that

$$-\frac{n-2}{2} J_{\sqrt{\gamma + \frac{(n-2)^2}{4}}}(\sqrt{\lambda}) + J'_{\sqrt{\gamma + \frac{(n-2)^2}{4}}}(\sqrt{\lambda})\sqrt{\lambda} = 0. \tag{3.62}$$

The measure of nodal sets for spherical harmonics $\Phi(\omega)$ is known in [10] as

$$c\sqrt{\gamma} \leq H^{n-2}\{\omega \in S^{n-1} | \Phi(\omega) = 0\} \leq C\sqrt{\gamma}. \tag{3.63}$$

Let $C_{\sqrt{\gamma + \frac{(n-2)^2}{4}},\, k}$ be the $k$th positive zeros of the solution in Eq. 3.62, it is shown in Theorem 2.1 and Theorem 2.3 in [13] that

$$C^2_{\sqrt{\gamma + \frac{(n-2)^2}{4}},\, 1} \approx \gamma$$

as $\gamma$ is large. Let $\sqrt{\lambda} = C_{\sqrt{\gamma + \frac{(n-2)^2}{4}},\, 1}$. It follows from Eq. 3.63 that the conclusion in the Theorem 2 is optimal for $n \geq 3$.

## 4 Interior nodal sets on real analytic domains

In this section, we prove the upper bounds for interior nodal sets of Robin eigenfunctions using again the doubling inequality and complex zero growth lemma. Assume that $\Omega$ is a real analytic domain. If $\lambda = 0$, the Robin eigenvalues problem can be reduced to the Steklov eigenvalue problem as discussed in Remark 1. The measure of interior nodal sets for analytic domains has been obtained in [48]. Hence, we assume $\lambda \neq 0$ in the section. We first show the upper bounds of nodal sets in the neighborhood close to boundary, then show the upper bounds of nodal sets away from the boundary $\partial\Omega$. To deal with the nodal sets close to the boundary, we do an analytical extension across the boundary using lifting arguments and analyticity results.

*Proof of Theorem 3* By the lifting argument, we introduce

$$\hat{w}(x, t, s) = e^{(|\alpha| i + \sqrt{\lambda})s} e^{\alpha t} u(x) \tag{4.1}$$

as in the proof of Corollary 1. Then $\hat{w}$ satisfies (3.53) and $\hat{w}$ can be analytically extended to the region $\Omega_{1,\delta}$ whose definition is given in the proof of Corollary 1. Choose any point $p \in \partial B^+_{1/16}(0) \cap \{x_n = 0\}$ with $p \in \mathbb{R}^n$. By the elliptic estimates in [36], we have

$$\begin{aligned} \left|\frac{\nabla_x^{\bar{\alpha}} \hat{w}(p, 0, 0)}{\bar{\alpha}!}\right| &\leq C\hat{C}^k \|\hat{w}\|_{L^\infty\left(\mathbb{B}_\delta(p)\times(-\delta,\delta)\times(-\delta,\delta)\right)} \\ &\leq C\hat{C}^k \|\hat{w}\|_{L^\infty\left(\mathbb{B}^+_{\frac{1}{8}}(p)\times(-\frac{1}{8},\frac{1}{8})\times(-\frac{1}{8},\frac{1}{8})\right)}, \end{aligned} \tag{4.2}$$

where $|\bar{\alpha}| = k$, the derivative $\nabla_x^{\bar{\alpha}}$ is taken with respect to $x \in \mathbb{R}^n$ and $\hat{C} > 1$ depends on $\Omega$. From the definition of $\hat{w}(x, t, s)$, we obtain that

$$\left|\frac{\nabla_x^{\bar{\alpha}} u(p)}{\bar{\alpha}!}\right| \leq C\hat{C}^k e^{C(|\alpha| + \sqrt{|\lambda|})} \|u\|_{L^\infty(\mathbb{B}^+_{\frac{1}{8}}(p))}. \tag{4.3}$$

By the uniqueness of the analytic continuation, it also holds that

$$-\triangle u=\lambda u \qquad \text{in } \widehat{\Omega}_1, \tag{4.4}$$

where $\widehat{\Omega}_1=\{x\in\mathbb{R}^n|\operatorname{dist}\{x,\Omega\}\le\delta\}$. We may regard $p$ as the origin. If summing up a geometric series, we can extend $u(x)$ to be a holomorphic function $u(z)$ with $z\in\mathbb{C}^n$. Moreover, we derive that

$$\sup_{|z|\le\frac{1}{2n\hat{C}}}|u(z)|\le e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^\infty(\mathbb{B}^+_{\frac{1}{8}})} \tag{4.5}$$

with $C>1$. From the relations of $\bar{u}$ and $u$, the doubling inequality (2.12) also holds for $u$ as

$$\|u\|_{L^2(\mathbb{B}^+_{2r})}\le e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^2(\mathbb{B}^+_{r})} \tag{4.6}$$

for $0<r<r_0$. Applying the doubling inequality (4.6) in $L^\infty$ norm finitely many times in the half balls, we have

$$\begin{aligned}\sup_{|z|\le\frac{1}{2n\hat{C}}}|u(z)| &\le e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^\infty(\mathbb{B}^+_{\frac{1}{4n\hat{C}}})}\\ &\le e^{C(|\alpha|+\sqrt{|\lambda|})}\|u\|_{L^\infty(\mathbb{B}_{\frac{1}{4n\hat{C}}})}.\end{aligned} \tag{4.7}$$

From the rescaling argument, we conclude that

$$\sup_{|z|\le 2r}|u(z)|\le e^{C(|\alpha|+\sqrt{|\lambda|})}\sup_{|x|\le r}|u(x)| \tag{4.8}$$

for $0<r<r_0$ with $r_0$ depending on $\Omega$ and $C$ depends on $\Omega$.

In the first step, we prove the nodal sets in a neighborhood $\Omega_{\frac{\delta}{8}}$. Since $\partial\Omega$ is compact, we can choose a sequence of finite number of balls centered on $\partial\Omega$ such that those balls cover $\Omega_{\frac{\delta}{8}}$. By rescaling and translation, we can argue on scales of order one and choose balls centered at origin. Let $p\in\mathbb{B}_{1/4}$ be the point where the maximum of $|u|$ in $\mathbb{B}_{1/4}$ is attained. After a rescaling, we may assume that $u(p)=1$. By the doubling inequality (4.8), we have

$$\begin{aligned}\sup_{|z|\le 1}|u(z)| &\le e^{C(|\alpha|+\sqrt{|\lambda|})}\sup_{|x|\le\frac{1}{4}}|u(x)|\\ &\le e^{C(|\alpha|+\sqrt{|\lambda|})}.\end{aligned} \tag{4.9}$$

Applying (4.8) to the translation of $u$, we obtain that

$$\begin{aligned}\sup_{|z-p|\le 1}|u(z)| &\le e^{C(|\alpha|+\sqrt{|\lambda|})}\sup_{|x-p|\le\frac{1}{4}}|u(x)|\\ &\le e^{C(|\alpha|+\sqrt{|\lambda|})}\sup_{|x|\le\frac{1}{2}}|u(x)|\\ &\le e^{C(|\alpha|+\sqrt{|\lambda|})}.\end{aligned} \tag{4.10}$$

For each direction $\omega\in S^{n-1}$, set $u_\omega(z)=u(p+z\omega)$ in $z\in\mathcal{B}_1(0)\subset\mathbb{C}$. By the doubling property (4.10) and Lemma 2, we can show that

$$\begin{aligned}&\sharp\{x\in\mathbb{B}_{1/2}(p)\ \mid\ x-p \text{ is parallel to } \omega \text{ and } u(x)=0\}\\ &\quad\le\sharp\{z\in\mathcal{B}_{1/2}(0)\subset\mathbb{C}|u_\omega(z)=0\}\\ &\quad\le C(|\alpha|+\sqrt{|\lambda|}).\end{aligned} \tag{4.11}$$

Recall that $N(\omega) = \sharp\{z \in \mathcal{B}_{1/2}(0) \subset \mathbb{C} | u_\omega(z) = 0\}$. With aid of integral geometry estimates, it yields that

$$\begin{aligned} H^{n-1}\{x \in \mathbb{B}_{1/2}(p) | u(x) = 0\} &\leq c(n) \int_{S^{n-1}} N(\omega)\, d\omega \\ &\leq \int_{S^{n-1}} C(|\alpha| + \sqrt{|\lambda|})\, d\omega \\ &= C(|\alpha| + \sqrt{\lambda}). \end{aligned} \tag{4.12}$$

Since $\mathbb{B}_{1/4}(0) \subset \mathbb{B}_{1/2}(p)$, we obtain the upper bound estimates

$$H^{n-1}\{x \in \mathbb{B}_{1/4}(0) | u(x) = 0\} \leq C(|\alpha| + \sqrt{\lambda}). \tag{4.13}$$

Covering the domain $\Omega_{\frac{\delta}{8}} \subset \widehat{\Omega}_1$ with a finite number of balls centered at $\partial\Omega$, we obtain that

$$H^{n-1}\{x \in \Omega_{\frac{\delta}{8}} | u(x) = 0\} \leq C(|\alpha| + \sqrt{|\lambda|}). \tag{4.14}$$

In the second step, we deal with the measure of nodal sets in $\Omega \backslash \Omega_{\frac{\delta}{8}}$. Recall that we have obtained the doubling inequality in the interior of the domain in Proposition 1, i.e.

$$\|\bar{u}\|_{L^\infty(\mathbb{B}_{2r}(p))} \leq e^{C(|\alpha| + \sqrt{|\lambda|})} \|\bar{u}\|_{L^\infty(\mathbb{B}_r(p))}.$$

Since $\bar{u}(x) = u(x) \exp\{-\alpha \varrho(x)\}$ and $-C_0 < \varrho(x) \leq C_0$ for some constant $C_0$ depending on $\Omega$ in Eq. 2.7, it follows that

$$\|u\|_{L^\infty(\mathbb{B}_{2r}(p))} \leq e^{C(|\alpha| + \sqrt{\lambda})} \|u\|_{L^\infty(\mathbb{B}_r(p))} \tag{4.15}$$

holds for $p \in \Omega \backslash \Omega_{\frac{\delta}{8}}$ and $0 < r \leq r_0 \leq \frac{\delta}{8}$. We can similarly extend $u(x)$ locally as a holomorphic function in $\mathbb{C}^n$. Choose any point $p \in \Omega \backslash \Omega_{\frac{\delta}{8}}$. By the elliptic estimates, we have

$$\begin{aligned} \left|\frac{\nabla_x^{\bar{\alpha}} \hat{w}(p, 0, 0)}{\bar{\alpha}!}\right| &\leq C \hat{C}^k \|\hat{w}\|_{L^\infty\left(\mathbb{B}_{\frac{\delta}{24}}(p) \times (-\delta, \delta) \times (-\delta, \delta)\right)} \\ &\leq C \hat{C}^k \|\hat{w}\|_{L^\infty\left(\mathbb{B}_{\frac{\delta}{16}(p)} \times (-\frac{1}{8}, \frac{1}{8}) \times (-\frac{1}{8}, \frac{1}{8})\right)}, \end{aligned} \tag{4.16}$$

where $|\bar{\alpha}| = k$, the derivative $\nabla_x^{\bar{\alpha}}$ is taken with respect to $x \in \mathbb{R}^n$ and $\hat{C} > 1$ depends on $\Omega$. From the definition of $\hat{w}(x, t, s)$, we obtain that

$$\left|\frac{\nabla_x^{\bar{\alpha}} u(p)}{\bar{\alpha}!}\right| \leq C \hat{C}^k e^{C(\alpha + \sqrt{|\lambda|})} \|u\|_{L^\infty(\mathbb{B}_{\frac{\delta}{16}})}. \tag{4.17}$$

Summing up a geometric series, we can extend $u(x)$ to be a holomorphic function $u(z)$ with $z \in \mathbb{C}^n$. Moreover, we derive that

$$\sup_{|z| \leq \frac{1}{2n\hat{C}}} |u(z)| \leq e^{C(\alpha + \sqrt{|\lambda|})} \|u\|_{L^\infty(\mathbb{B}_{\frac{\delta}{16}})} \tag{4.18}$$

with $C > 1$. By iterating the doubling inequality (4.15) finite number steps and rescaling arguments, we arrive at

$$\sup_{|z| \leq 2r} |u(z)| \leq e^{C(|\alpha| + \sqrt{|\lambda|})} \sup_{|x| \leq r} |u(x)| \tag{4.19}$$

for $0 < r < r_0$ with $r_0$ depending on $\Omega$ and $C$ independent of $\lambda$ and $r$.

Following the same procedure in the neighborhood of the boundary, and making use of Lemma 2 and the doubling inequality (4.19), we can obtain the upper bounds of the interior nodal sets

$$H^{n-1}\{x \in \mathbb{B}_{1/2}(p)|u(x) = 0\} \le C(|\alpha| + \sqrt{|\lambda|}). \tag{4.20}$$

Covering the domain $\Omega\backslash\Omega_{\frac{\delta}{8}}$ using finite number of balls gives that

$$H^{n-1}\{x \in \Omega\backslash\Omega_{\frac{\delta}{8}}|u(x) = 0\} \le C(|\alpha| + \sqrt{|\lambda|}). \tag{4.21}$$

Combining the results in (4.14) and (4.21), we arrive at the conclusion in Theorem 3. □

## 5 Global Carleman estimates

In this section, we prove the quantitative global Carleman estimates in Proposition 2. Interested readers may refer to the survey [22] and [25] for more exhaustive literature for local and global Carleman estimates. We will use the integration by parts arguments repeatedly to get the desired estimates. Recall that the weight function

$$\psi(y) = e^{sh(y)}$$

with

$$h(y) = -\frac{|y'|^2}{4} + \frac{y_n^2}{2} - y_n.$$

Actually, the weight function can be chosen as any $h \in C^2$ such that $|\nabla h| \neq 0$ in $\mathbb{B}^+_{1/2}$ to have the Carleman estimates in Proposition 2. Recall the assumptions about $b(y)$ and $c(y)$ are

$$\begin{cases} \|b\|_{W^{1,\infty}(\mathbb{B}^+_{1/2})} \le C(|\alpha| + 1), \\ \|c\|_{W^{1,\infty}(\mathbb{B}^+_{1/2})} \le C(\alpha^2 + |\lambda|). \end{cases} \tag{5.1}$$

*Proof of Proposition 2* Choose

$$w(y) = e^{\tau\psi(y)}v(y). \tag{5.2}$$

Since $v(y) \in C^\infty(\mathbb{B}^+_{1/2})$, then $w(y) \in C^\infty(\mathbb{B}^+_{1/2})$. We introduce a second order elliptic operator

$$P_0 = -a_{ij}D_{ij} + b_i(y)D_i + c(y).$$

Define the conjugate operator as

$$P_\tau w = e^{\tau\psi(y)}P_0(e^{-\tau\psi(y)}w).$$

Direct calculations show that

$$\begin{aligned} P_\tau w &= -a_{ij}D_{ij}w + 2\tau a_{ij}D_i\psi D_jw + \tau a_{ij}D_{ij}\psi w \\ &\quad -\tau^2 a_{ij}D_i\psi D_j\psi w - \tau b_i(y)D_i\psi w + b_i(y)D_iw + c(y)w \\ &= -a_{ij}D_{ij}w + 2\tau s\psi a_{ij}D_ihD_jw - \tau^2 s^2\psi^2\beta(y)w + \tau\psi a(y,s)w \\ &\quad -\tau s\psi b_i(y)D_ihw + b_i(y)D_iw + c(y)w, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \beta(y) &= a_{ij}D_ihD_jh, \\ a(y,s) &= s^2\beta(y) + sa_{ij}D_{ij}h. \end{aligned} \tag{5.4}$$

Note that $\beta(y) \geq C$ for some positive constant $C$ on $\mathbb{B}^+_{1/2}$ by the uniform ellipticity. We split the expression $P_\tau w$ into the sum of two expressions $P_1 w$ and $P_2 w$, where

$$P_1 w = -a_{ij} D_{ij} w - \tau^2 s^2 \psi^2 \beta(y) w - \tau s \psi b_i(y) D_i h w + c(y) w,$$

$$P_2 w = 2\tau s \psi a_{ij} D_i h D_j w + b_i(y) D_i w.$$

Then

$$P_\tau w = P_1 w + P_2 w + \tau \psi a(y, s) w. \tag{5.5}$$

We compute the $L^2$ norm of $P_\tau w$. By the triangle inequality, we have

$$\begin{aligned} \|P_\tau w\|^2 &= \|P_1 w + P_2 w + \tau \psi a(y, s) w\|^2 \\ &\geq \|P_1 w\|^2 + \|P_2 w\|^2 + 2\langle P_1 w,\ P_2 w\rangle - \|\tau \psi a(y, s) w\|^2. \end{aligned} \tag{5.6}$$

Later on, we will absorb the term $\|\tau \psi a(y, s) w\|^2$. Now we are going to derive a lower bound for the inner product in Eq. 5.6. Let's write

$$\langle P_1 w,\ P_2 w\rangle = \sum_{k=1}^{4} I_k + \sum_{k=1}^{4} J_k, \tag{5.7}$$

where

$$\begin{aligned} I_1 &= \langle -a_{ij} D_{ij} w,\ 2\tau s \psi a_{ij} D_i h D_j w\rangle, \\ I_2 &= \langle -\tau^2 s^2 \psi^2 \beta(y) w,\ 2\tau s \psi a_{ij} D_i h D_j w\rangle, \\ I_3 &= \langle -\tau s \psi b_i(y) D_i h w,\ 2\tau s \psi a_{ij} D_i h D_j w\rangle, \\ I_4 &= \langle c(y) w,\ 2\tau s \psi a_{ij} D_i h D_j w\rangle, \\ J_1 &= \langle -a_{ij} D_{ij} w,\ b_i(y) D_i w\rangle, \\ J_2 &= \langle -\tau^2 s^2 \psi^2 \beta(y) w,\ b_i(y) D_i w\rangle, \\ J_3 &= \langle -\tau s \psi b_i(y) D_i h w,\ b_i(y) D_i w\rangle, \\ J_4 &= \langle c(y) w,\ b_i(y) D_i w\rangle. \end{aligned} \tag{5.8}$$

We will estimate each term on the right hand side of (5.7). Performing the integration by parts shows that

$$\begin{aligned} I_1 &= 2\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi a_{ij} D_i w D_j h a_{kl} D_k h D_l w \, dy + 2\tau s \int_{\mathbb{B}^+_{1/2}} D_j(a_{ij} a_{kl} D_k h) \psi D_i w D_l w \, dy \\ &\quad + 2\tau s \int_{\mathbb{B}^+_{1/2}} \psi a_{ij} D_i w a_{kl} D_k h D_{lj} w \, dy - 2\tau s \int_{\partial \mathbb{B}^+_{1/2}} \psi a_{kl} D_k h D_l w a_{ij} D_i w \nu_j \, dS \\ &= I_1^1 + I_1^2 + I_1^3 + I_1^4. \end{aligned} \tag{5.9}$$

The first term $I_1^1$ can be controlled as

$$\begin{aligned} I_1^1 &= 2\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi |a_{ij} D_i w D_j h|^2 \, dy \\ &\geq 0. \end{aligned} \tag{5.10}$$

Applying the integration by parts argument, the third term $I_1^3$ can be computed as

$$
\begin{aligned}
I_1^3 &= \tau s \int_{\mathbb{B}^+_{1/2}} \psi a_{ij} a_{kl} D_k h D_l (D_i w D_j w)\, dy \\
&= -\tau s \int_{\mathbb{B}^+_{1/2}} \psi D_l (a_{ij} a_{kl} D_k h) D_i w D_j w\, dy - \tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta(y) a_{ij} D_i w D_j w\, dy \\
&\quad + \tau s \int_{\partial\mathbb{B}^+_{1/2}} \psi a_{ij} D_i w D_j w a_{kl} D_k h \nu_l dS. \qquad (5.11)
\end{aligned}
$$

Combining (5.9), (5.10) and (5.11), and using the fact that $\|a_{ij}\|_{C^1}$ is bounded, we can estimate $I_1$ from below

$$
\begin{aligned}
I_1 &\geq 2\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi |a_{ij} D_i w D_j h|^2\, dy - \tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta(y) a_{ij} D_i w D_j w\, dy \\
&\quad - C\tau s \int_{\mathbb{B}^+_{1/2}} \psi |\nabla w|^2\, dy - C\tau s \int_{\partial\mathbb{B}^+_{1/2}} \psi |\nabla w|^2\, dy. \qquad (5.12)
\end{aligned}
$$

Thus,

$$
I_1 \geq -\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta a_{ij} D_i w D_j w\, dy - C\tau s \int_{\mathbb{B}^+_{1/2}} \psi |\nabla w|^2\, dy - C\tau s \int_{\partial\mathbb{B}^+_{1/2}} \psi |\nabla w|^2\, dy. \qquad (5.13)
$$

Now we compute the term $I_2$ using integration by parts argument,

$$
\begin{aligned}
I_2 &= -\tau^3 s^3 \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta(y) a_{ij} D_i h D_j w^2\, dy \\
&= 3\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta(y) a_{ij} D_i h D_j h w^2\, dy + \tau^3 s^3 \int_{\mathbb{B}^+_{1/2}} \psi^3 D_j (\beta a_{ij} D_i h) w^2\, dy \\
&\quad - \tau^3 s^3 \int_{\partial\mathbb{B}^+_{1/2}} \psi^3 \beta(y) w^2 a_{ij} D_i h \nu_j\, dS \\
&\geq 3\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta^2(y) w^2\, dy - C\tau^3 s^3 \int_{\mathbb{B}^+_{1/2}} \psi^3 w^2\, dy - C\tau^3 s^3 \int_{\partial\mathbb{B}^+_{1/2}} \psi^3 w^2\, dS. \qquad (5.14)
\end{aligned}
$$

Choosing $s$ large enough and noting that $\beta(y) \geq C$, we deduce that

$$
I_2 \geq \frac{17}{6} \tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta^2 w^2\, dy - C\tau^3 s^3 \int_{\partial\mathbb{B}^+_{1/2}} \psi^3 w^2\, dS. \qquad (5.15)
$$

For the term $I_3$, using the integration by parts argument leads to

$$
\begin{aligned}
I_3 &= -\tau^2 s^2 \int_{\mathbb{B}^+_{1/2}} \psi^2 b_k(y) D_k h a_{ij} D_i h D_j w^2\, dy \\
&= \tau^2 s^2 \int_{\mathbb{B}^+_{1/2}} D_j (\psi^2 b_k(y) D_k h a_{ij} D_i h) w^2\, dy - \tau^2 s^2 \int_{\partial\mathbb{B}^+_{1/2}} \psi^2 w^2 b_k(y) D_k h a_{ij} D_i h \nu_j\, dS. \qquad (5.16)
\end{aligned}
$$

Making use of the assumption of Eq. 5.1 gives that

$$
I_3 \geq -C(|\alpha| + 1)\tau^2 s^3 \int_{\mathbb{B}^+_{1/2}} \psi^2 w^2\, dy - C(|\alpha| + 1)\tau^2 s^2 \int_{\partial\mathbb{B}^+_{1/2}} \psi^2 w^2\, dS. \qquad (5.17)
$$

We proceed to estimate the term $I_4$. Using integration by parts shows that

$$\begin{aligned}I_4 &= \tau s\int_{\mathbb{B}^+_{1/2}} c(y)\psi a_{ij}D_ihD_jw^2\,dy\\ &= \tau s\int_{\mathbb{B}^+_{1/2}} \psi a_{ij}D_jc(y)D_ihw^2\,dy-\tau s^2\int_{\mathbb{B}^+_{1/2}} c(y)\psi a_{ij}D_jhD_ihw^2\,dy\\ &\quad-\tau s\int_{\mathbb{B}^+_{1/2}} c(y)\psi D_j(a_{ij}D_ih)w^2\,dy+\tau s\int_{\partial\mathbb{B}^+_{1/2}} c(y)\psi w^2a_{ij}D_ih\nu_j\,dS. \qquad (5.18)\end{aligned}$$

Again, the assumptions of Eq. 5.1 leads to

$$I_4\geq -C\tau s^2(\alpha^2+|\lambda|)\int_{\mathbb{B}^+_{1/2}}\psi w^2\,dy-C\tau s(\alpha^2+|\lambda|)\int_{\partial\mathbb{B}^+_{1/2}}\psi w^2\,dy. \qquad (5.19)$$

Together with the estimates on each $I_k$ from Eqs. 5.13-5.19, using the assumption that $\tau>C_s(|\alpha|+\sqrt{|\lambda|})$ for some $C_s$ depending on $s$, we arrive at

$$\begin{aligned}\sum_{k=1}^4 I_k &\geq \frac{14}{5}\tau^3s^4\int_{\mathbb{B}^+_{1/2}}\psi^3\beta^2w^2\,dy-C\tau^3s^3\int_{\partial\mathbb{B}^+_{1/2}}\psi^3w^2\,dS-C\tau s\int_{\partial\mathbb{B}^+_{1/2}}\psi|\nabla w|^2\,dS\\ &\quad-\tau s^2\int_{\mathbb{B}^+_{1/2}}\psi\beta a_{ij}D_iwD_jw\,dy-C\tau s\int_{\mathbb{B}^+_{1/2}}\psi|\nabla w|^2\,dy. \qquad (5.20)\end{aligned}$$

Next we continue to estimate the integration about $J_k$ using the strategy of integration by parts. Direct computations show that

$$\begin{aligned}J_1 &= \int_{\mathbb{B}^+_{1/2}} D_j(a_{ij}b_k)D_iwD_kw\,dy+\frac{1}{2}\int_{\mathbb{B}^+_{1/2}} a_{ij}b_kD_k(D_iwD_jw)\,dy\\ &\quad-\int_{\partial\mathbb{B}^+_{1/2}} b_kD_kwa_{ij}D_iw\nu_j\,dS\\ &= \int_{\mathbb{B}^+_{1/2}} D_j(a_{ij}b_k)D_iwD_kw\,dy-\frac{1}{2}\int_{\mathbb{B}^+_{1/2}} D_k(a_{ij}b_k)D_iwD_jw\,dy\\ &\quad+\frac{1}{2}\int_{\partial\mathbb{B}^+_{1/2}} a_{ij}D_iwD_jwb_k\nu_k\,dS-\int_{\partial\mathbb{B}^+_{1/2}} b_kD_kwa_{ij}D_iw\nu_j\,dS. \qquad (5.21)\end{aligned}$$

Thus, from the assumption of $b_i$,

$$J_1\geq -C(|\alpha|+1)\int_{\mathbb{B}^+_{1/2}}|\nabla w|^2\,dy-C(|\alpha|+1)\int_{\partial\mathbb{B}^+_{1/2}}|\nabla w|^2\,dS. \qquad (5.22)$$

For the term $J_2$, integration by parts argument yields that

$$\begin{aligned}J_2 &= -\frac{1}{2}\tau^2s^2\int_{\mathbb{B}^+_{1/2}}\psi\beta(y)b_i(y)D_iw^2\,dy\\ &= \frac{\tau^2s^3}{2}\int_{\mathbb{B}^+_{1/2}}\psi\beta(y)b_iD_ihw^2\,dy+\frac{\tau^2s^2}{2}\int_{\mathbb{B}^+_{1/2}}\psi D_i\beta(y)b_i(y)w^2\,dy\\ &\quad+\frac{\tau^2s^2}{2}\int_{\mathbb{B}^+_{1/2}}\psi\beta(y)D_ib_i(y)w^2\,dy-\frac{\tau^2s^2}{2}\int_{\partial\mathbb{B}^+_{1/2}}\psi\beta(y)w^2b_i(y)\nu_j\,dS. \qquad (5.23)\end{aligned}$$

Therefore, we can show that

$$J_2 \geq -C\tau^2 s^3(|\alpha|+1)\int_{\mathbb{B}^+_{1/2}} \beta\psi w^2\,dy - C\tau^2 s^2(|\alpha|+1)\int_{\partial\mathbb{B}^+_{1/2}} \psi w^2\,dS. \quad (5.24)$$

In the same way, we can show that

$$\begin{aligned} J_3 &= -\frac{\tau s}{2}\int_{\mathbb{B}^+_{1/2}} \psi b_k D_k h b_i D_i w^2\,dy \\ &= \frac{\tau s}{2}\int_{\mathbb{B}^+_{1/2}} D_i(\psi b_k D_k h b_i) w^2\,dy - \frac{\tau s}{2}\int_{\partial\mathbb{B}^+_{1/2}} \psi w^2 b_k D_k h b_i \nu_j\,dS. \end{aligned} \quad (5.25)$$

We can control $J_3$ below as

$$J_3 \geq -C\tau s^2(|\alpha|+1)^2\int_{\mathbb{B}^+_{1/2}} \psi w^2\,dy - C\tau s(|\alpha|+1)^2\int_{\partial\mathbb{B}^+_{1/2}} \psi w^2\,dS. \quad (5.26)$$

Similarly, applying the integration by parts leads to

$$\begin{aligned} J_4 &= \frac{1}{2}\int_{\mathbb{B}^+_{1/2}} c(y) b_i D_i w^2\,dy \\ &= -\frac{1}{2}\int_{\mathbb{B}^+_{1/2}} D_i c(y) b_i w^2\,dy - \frac{1}{2}\int_{\mathbb{B}^+_{1/2}} c(y) D_i b_i w^2\,dy \\ &\quad +\frac{1}{2}\int_{\partial\mathbb{B}^+_{1/2}} c(y) w^2 b_i \nu_i\,dS. \end{aligned} \quad (5.27)$$

Then we can obtain that

$$J_4 \geq -C(\alpha^2+|\lambda|)(|\alpha|+1)\int_{\mathbb{B}^+_{1/2}} w^2\,dy - C(\alpha^2+|\lambda|)(|\alpha|+1)\int_{\partial\mathbb{B}^+_{1/2}} w^2\,dS. \quad (5.28)$$

Using the fact that $\tau > C_s(|\alpha| + \sqrt{|\lambda|})$ for $C_s$ depending on $s$, and summing up the estimates from Eqs. 5.22-5.28 gives that

$$\begin{aligned} \sum_{k=1}^{4} J_k \geq& -C(|\alpha|+1)\int_{\mathbb{B}^+_{1/2}} \psi|\nabla w|^2\,dy - C\tau^2(|\alpha|+1)s^3\int_{\mathbb{B}^+_{1/2}} \psi w^2\,dy \\ & -C\tau^2 s^2(|\alpha|+1)\int_{\partial\mathbb{B}^+_{1/2}} \psi w^2\,dS - C(|\alpha|+1)\int_{\partial\mathbb{B}^+_{1/2}} |\nabla w|^2\,dS. \end{aligned} \quad (5.29)$$

Recall the inner product (5.7). Combining (5.20), and (5.29) and using the the assumption $\tau > C_s(|\alpha| + \sqrt{|\lambda|})$ again, we derive that

$$\begin{aligned} \langle P_1 w,\ P_2 w\rangle \geq& \frac{11}{4}\tau^3 s^4\int_{\mathbb{B}^+_{1/2}} \psi^3\beta^2 w^2\,dy - \frac{5}{4}\tau s^2\int_{\mathbb{B}^+_{1/2}} \psi a_{ij} D_i w D_j w\,dy \\ & -C\tau^3 s^3\int_{\partial\mathbb{B}^+_{1/2}} \psi^3 w^2\,dS - C\tau s\int_{\partial\mathbb{B}^+_{1/2}} \psi|\nabla w|^2\,dS. \end{aligned} \quad (5.30)$$

We want to control the gradient term on the second term on the right hand side of Eq. 5.30. To this end, we consider the following inner product

$$\langle P_1 w,\ \tau s^2\psi\beta(y) w\rangle = \sum_{k=1}^{4} L_k, \quad (5.31)$$

where

$$L_1 = \langle -a_{ij} D_{ij} w,\ \tau s^2 \psi \beta w \rangle,$$

$$L_2 = \langle -\tau^2 s^2 \psi^2 \beta w, \tau s^2 \psi \beta w \rangle = -\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta^2 w^2 \, dy, \tag{5.32}$$

$$\begin{aligned} L_3 &= \langle -\tau s \psi b_i D_i h w,\ \tau s^2 \psi \beta w \rangle \\ &= -\tau^2 s^3 \int_{\mathbb{B}^+_{1/2}} \psi^2 \beta b_i D_i h w^2 \, dy \\ &= -C \tau^2 s^3 (|\alpha| + 1) \int_{\mathbb{B}^+_{1/2}} \psi^2 w^2 \, dy, \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} L_4 &= \langle c(y) w,\ \tau s^2 \psi \beta w \rangle \\ &= \tau s^2 \int_{\mathbb{B}^+_{1/2}} c(y) \psi \beta w^2 \, dy \\ &\geq -C \tau s^2 (\alpha^2 + |\lambda|) \int_{\mathbb{B}^+_{1/2}} \psi \beta w^2 \, dy. \end{aligned} \tag{5.34}$$

We want to find out a lower estimate for $L_1$ to include the gradient terms. It follows from integration by parts and Cauchy-Schwartz inequality that

$$\begin{aligned} L_1 &= \tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta a_{ij} D_i w D_j w \, dy + \tau s^2 \int_{\mathbb{B}^+_{1/2}} D_i (a_{ij} \psi \beta) w D_j w \, dy \\ &\quad - \tau s^2 \int_{\partial \mathbb{B}^+_{1/2}} \psi \beta w a_{ij} D_i w \nu_j \, dS \\ &\geq \tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta |a_{ij} D_i w D_j w| \, dy - C \tau s^3 \int_{\mathbb{B}^+_{1/2}} \psi |\nabla w| |w| \, dy \\ &\quad - \tau s \int_{\partial \mathbb{B}^+_{1/2}} \psi \beta w a_{ij} D_j w \nu_j \, dS \\ &\geq \frac{9}{10} \tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi \beta a_{ij} D_i w D_j w \, dy - C \tau s^{10} \int_{\mathbb{B}^+_{1/2}} \psi^2 w^2 \, dy - C \tau s^2 \int_{\partial \mathbb{B}^+_{1/2}} \psi w^2 \, dS \\ &\quad - C \tau s^2 \int_{\partial \mathbb{B}^+_{1/2}} \psi |\nabla w|^2 \, dS. \end{aligned} \tag{5.35}$$

Taking (5.31), (5.32), (5.33), (5.34), and (5.35), and $\tau > C_s(|\alpha| + \sqrt{|\lambda|})$ into account gives that

$$\begin{aligned} \langle P_1 w,\ \frac{5\tau s^2}{2} \psi \beta(y) w \rangle \geq \frac{9\tau s^2}{4} \int_{\mathbb{B}^+_{1/2}} \psi \beta a_{ij} D_i w D_j w \, dy - \frac{5\tau^3 s^4}{2} \int_{\mathbb{B}^+_{1/2}} \psi^3 \beta^2 w^2 \, dy \\ - C \tau s^2 \int_{\partial \mathbb{B}^+_{1/2}} \psi w^2 \, dS - C \tau s^2 \int_{\partial \mathbb{B}^+_{1/2}} \psi |\nabla w|^2 \, dS. \end{aligned} \tag{5.36}$$

Since

$$\| P_1 w \|^2 + \frac{25}{4} \| \tau s^2 \psi \beta w \|^2 \geq 2 \langle P_1 w, \frac{5\tau s^2}{2} \psi \beta w \rangle, \tag{5.37}$$

from Eq. 5.6, we obtain that

$$\|P_\tau w\|^2 + \frac{25}{4}\|\tau s^2\psi\beta w\|^2 \geq 2\langle P_1 w, \frac{5\tau s^2}{2}\tau s^2\psi\beta w\rangle + 2\langle P_1 w, P_2 w\rangle \\ -\|\tau\psi a(y,s)w\|^2. \tag{5.38}$$

From the expression of $a(y,s)$ in Eq. 5.4, we can absorb $\|\tau\psi a(y,s)w\|^2$ into the inner product $\langle P_1 w, \ P_2 w\rangle$ by the dominating term $\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3\beta^2 w^2\, dy$ in Eq. 5.30. We can absorb $\|\tau s^2\psi\beta w\|^2$ into the inner product $\langle P_1 w, \ P_2 w\rangle$ as well. Thanks to Eqs. 5.6, 5.30 and 5.36, using the assumption that $\tau > C_s(|\alpha| + \sqrt{|\lambda|})$ and $s$ is a fixed large constant, we arrive at

$$\|P_\tau w\|^2 + \tau^3 s^3 \int_{\partial\mathbb{B}^+_{1/2}} \psi^3 w^2\, dS + \tau s^2 \int_{\partial\mathbb{B}^+_{1/2}} \psi|\nabla w|^2\, dS \\ \geq C\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 w^2\, dy + C\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi|\nabla w|^2\, dy. \tag{5.39}$$

Recall (5.2) and the operator $P_0$. We derive the following Carleman estimates for $v$ as

$$\|e^{\tau\psi} P_0 v\|^2 + \tau^3 s^3 \int_{\partial\mathbb{B}^+_{1/2}} \psi^3 e^{2\tau\psi} v^2\, dS + \tau s^2 \int_{\partial\mathbb{B}^+_{1/2}} \psi e^{2\tau\psi}|\nabla v|^2\, dS \\ \geq C\tau^3 s^4 \int_{\mathbb{B}^+_{1/2}} \psi^3 e^{2\tau\psi} v^2\, dy + C\tau s^2 \int_{\mathbb{B}^+_{1/2}} \psi e^{2\tau\psi}|\nabla v|^2\, dy. \tag{5.40}$$

Thus, we arrive at the conclusion in the proposition. □

## Appendix

In the Appendix, we first construct polar coordinates for equations with Lipschitz metrics in Eq. 2.9, then we obtain the doubling inequalities on the double manifold. Most of the arguments in the Appendix are kind of known and scattered in the literature. We present the details for the conveniences of the readers. Without loss of generality, we consider the construction of normal coordinates at origin. Starting from a ball $\mathbb{B}_\delta$ in local coordinates, for the metric $\tilde{g}_{ij}$ in Eq. 2.9, we introduce a "radial" coordinate and a conformal change metric $\hat{g}_{ij}$. Let

$$r = r(x) = (\tilde{g}_{ij}(0)x_i x_j)^{\frac{1}{2}} \tag{5.41}$$

and

$$\hat{g}_{ij}(x) = \tilde{g}_{ij}(x)\hat{\psi}(x), \tag{5.42}$$

where

$$\hat{\psi}(x) = \tilde{g}^{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l} \tag{5.43}$$

for $x \neq 0$ and $(\tilde{g}^{ij}) = (\tilde{g}_{ij})^{-1}$ is the inverse matrix. In the whole paper, we adopt the Einstein notation. The summation over index is understood. We assume the uniform ellipticity condition holds in $\mathbb{B}_\delta$ for

$$\Lambda_1\|\xi\|^2 \leq \sum_{i,j=1}^{n} \tilde{g}_{ij}(x)\xi_i\xi_j \leq \Lambda_2\|\xi\|^2$$

for some positive constant $\Lambda_1$ and $\Lambda_2$ depending only on $\Omega$. Then $\hat{\psi}$ is bounded above and below satisfying

$$\frac{\Lambda_1}{\Lambda_2} \leq \hat{\psi} \leq \frac{\Lambda_2}{\Lambda_1}. \tag{5.44}$$

We can also see that $\hat{\psi}$ is Lipschitz continuous. With these auxiliary quantities, the following replacement of geodesic polar coordinates are constructed in [3]. In the geodesic ball $\hat{\mathbb{B}}_{\hat{r}_0} = \{x \in \tilde{\Omega} | r(x) \leq \hat{r}_0\}$, the following properties hold:

(i) $\hat{g}_{ij}(x)$ is Lipschitz continuous;

(ii) $\hat{g}_{ij}(x)$ is uniformly elliptic with $\frac{\Lambda_1^2}{\Lambda_2}\|\xi\|^2 \leq \hat{g}_{ij}(x)\xi_i\xi_j \leq \frac{\Lambda_2^2}{\Lambda_1}\|\xi\|^2$.

(iii) Let $\Sigma = \partial\hat{\mathbb{B}}_{\hat{r}_0}$. We can parametrize $\hat{\mathbb{B}}_{\hat{r}_0}\backslash\{0\}$ by the polar coordinate $r$ and $\theta$, with $r$ defined by Eq. 5.41 and $\theta = (\theta_1, \cdots \theta_{n-1})$ be the local coordinates on $\Sigma$. In these polar coordinates, the metric can be written as

$$\hat{g}_{ij}(x)dx^i dx^j = dr^2 + r^2\hat{\gamma}_{ij}d\theta^i d\theta^j \tag{5.45}$$

with $\hat{\gamma}_{ij} = \frac{1}{r^2}\hat{g}_{kl}(x)\frac{\partial x^k}{\partial \theta^i}\frac{\partial x^l}{\partial \theta^j}$.

(iv) There exists a positive constant $M$ depending on $\tilde{g}_{ij}$ such that for any tangent vector $\xi_j \in T_\theta(\Sigma)$,

$$|\frac{\partial \hat{\gamma}_{ij}(r,\theta)}{\partial r}\xi^i\xi^j| \leq M|\hat{\gamma}_{ij}(r,\theta)\xi^i\xi^j|. \tag{5.46}$$

Let $\hat{\gamma} = \det(\hat{\gamma}_{ij})$. Then Eq. 5.46 implies that

$$|\frac{\partial \ln\sqrt{\hat{\gamma}}}{\partial r}| \leq CM. \tag{5.47}$$

The existence of the coordinates $(r, \theta)$ allows us to pass to "geodesic polar coordinates". In particular, $r(x) = (\tilde{g}_{ij}(0)x_i x_j)^{\frac{1}{2}}$ is the geodesic distance to the origin in the metric $\hat{g}_{ij}$. In the new metric $\hat{g}_{ij}$, the Laplace-Beltrami operator is

$$\triangle_{\hat{g}} = \frac{1}{\sqrt{\hat{g}}}\frac{\partial}{\partial x_i}(\hat{g}^{ij}\sqrt{\hat{g}}\frac{\partial}{\partial x_j}),$$

where $\hat{g} = \det(\hat{g}_{ij})$. If $\bar{u}$ is a solution of Eq. 2.9, then $\bar{u}$ is locally the solution of the equation

$$\triangle_{\hat{g}}\bar{u} + \hat{b}(x)\cdot\nabla\bar{u} + \hat{c}(x)\bar{u} = 0 \quad \text{in } \hat{\mathbb{B}}_{\hat{r}_0}, \tag{5.48}$$

where

$$\begin{cases} \hat{b}_i = \frac{2-n}{2\hat{\psi}^2}\tilde{g}^{ij}\frac{\partial\hat{\psi}}{\partial x_j} + \frac{1}{\hat{\psi}}\tilde{b}_i, \\ \hat{c}(x) = \frac{\tilde{c}(x)}{\hat{\psi}}. \end{cases} \tag{5.49}$$

By the properties of $\hat{\psi}$, we can see $\hat{c}(x)$ is Lipschitz continuous. Since the term $\frac{2-n}{2\hat{\psi}^2}\tilde{g}^{ij}\frac{\partial\hat{\psi}}{\partial x_j}$ in $\hat{b}_i$ is only continuous and does not depend on either $\alpha$ or $\lambda$, it can be ignored in the future quantitative estimates for doubling inequality or nodal sets. The major term $\frac{1}{\hat{\psi}}\tilde{b}_i$ is Lipschitz continuous. From the conditions in Eq. 2.10, we still write the conditions for $\hat{b}$ and $\hat{c}$ as

$$\begin{cases} \|\hat{b}\|_{W^{1,\infty}(\hat{\mathbb{B}}_{\hat{r}_0})} \leq C(|\alpha|+1), \\ \|\hat{c}\|_{W^{1,\infty}(\hat{\mathbb{B}}_{\hat{r}_0})} \leq C(\alpha^2+|\lambda|). \end{cases} \tag{5.50}$$

For simplicity, we may write $\triangle_{\hat{g}}$ or $\triangle_g$ as $\triangle$ if the metric is understood. Since the geodesic balls or half balls under different metrics are comparable, we write all as $\mathbb{B}_r(x)$ or $\mathbb{B}_r^+(x)$

centered at $x$ with radius $r$. The rest of section is to show the doubling inequality on the double manifold. Let $r = r(y)$ be the Riemannian distance from origin to $y$. Our major tools to get the three-ball theorem and doubling inequality are the quantitative Carleman estimates. Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\psi}$, where $\psi$ usually satisfies some convex condition. We construct the weight function $\psi$ as follows. Set

$$\psi(y) = -g(\ln r(y)),$$

where $g(t) = t + \log t^2$ for $-\infty < t < T_0$, and $T_0$ is negative with $|T_0|$ large enough. One can check that

$$\lim_{t\to-\infty} -e^{-t}g''(t) = \infty \quad \text{and} \quad \lim_{t\to-\infty} g'(t) = 1. \tag{5.51}$$

Define

$$\psi_\tau(y) = e^{\tau\psi(y)}. \tag{5.52}$$

We state the following quantitative Carleman estimates. The similar Carleman estimates with lower bound of the parameter $\tau$ have been obtained in e.g. [4, 10, 48]. Interested readers may refer to them for the proof of the following proposition.

**Proposition 3** *There exist positive constants $C_1$, $C_0$ and small $r_0$, such that for $v \in C_0^\infty(\mathbb{B}_{r_0}\backslash\mathbb{B}_\rho)$, and*

$$\tau > C_1(1 + |\alpha| + \sqrt{|\lambda|}),$$

*one has*

$$C_0\|r^2\psi_\tau\big(\triangle v + \hat{b}(y)\cdot\nabla v + \hat{c}(y)v\big)\|^2 \geq \tau^3\|\psi_\tau(\log r)^{-1}v\|^2 + \tau\|r\psi_\tau(\log r)^{-1}\nabla v\|^2 + \tau\rho\|r^{-\frac{1}{2}}\psi_\tau v\|^2. \tag{5.53}$$

The $\|\cdot\|_r$ or $\|\cdot\|$ norm in the whole paper denotes the $L^2$ norm over $\mathbb{B}_r(0)$ if not explicitly stated. Specifically, $\|\cdot\|_{\mathbb{B}_r(y))}$ for short denotes the $L^2$ norm on the ball $\mathbb{B}_r(y)$. Thanks to the quantitative Carleman estimates, it is a standard way to derive a quantitative three-ball theorem. Let $\bar{u}$ be the solutions of the second order elliptic (5.48). We apply such Carleman estimates with $v = \eta\bar{u}$, where $\eta$ is an appropriate smooth cut-off function, and then select an appropriate choice of the parameter $\tau$. The statement of the quantitative three-ball theorem is as follows.

**Lemma 3** *There exist positive constants $\bar{r}_0$, $C$ which depend only on $\Omega$ and $0 < \beta < 1$ such that, for any $0 < R < \bar{r}_0$, the solutions $\bar{u}$ of* Eq. 5.48 *satisfy*

$$\|\bar{u}\|_{\mathbb{B}_{2R}(x_0)} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|^{\beta}_{\mathbb{B}_R(x_0)}\|\bar{u}\|^{1-\beta}_{\mathbb{B}_{3R}(x_0)} \tag{5.54}$$

*for any $x_0 \in \tilde{\Omega}$.*

Since the proof of three-ball theorem is kind of standard by the applications of quantitative Carleman estimates (5.53). We skip the details. The readers may also refer to Proposition 1 for similar proofs.

Let $\|u\|_{L^2(\Omega)} = 1$. Because of the even extension, we may write

$$\|\bar{u}\|_{L^2(\tilde{\Omega})} = 2.$$

Set $\bar{x}$ be the point where

$$\|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0}(\bar{x}))} = \max_{x\in\tilde{\Omega}} \|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0}(x))}$$

for some $0 < \hat{r}_0 < \frac{\bar{r}_0}{8}$. The compactness of $\tilde{\Omega}$ implies that

$$\|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0}(\bar{x}))} \geq C_{\hat{r}_0}$$

for some $C_{\hat{r}_0}$ depending on $\tilde{\Omega}$ and $\hat{r}_0$. From the quantitative three-ball inequality (5.54), at any point $x \in \Omega$, one has

$$\|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0/2}(x))} \geq e^{-C(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|^{\frac{1}{\beta}}_{L^2(\mathbb{B}_{\hat{r}_0}(x))}. \tag{5.55}$$

Let $l$ be a geodesic curve between $\hat{x}$ and $\bar{x}$, where $\hat{x}$ is any point in $\tilde{\Omega}$. Define $x_0 = \hat{x}, \cdots, x_m = \bar{x}$ such that $x_i \in l$ and $\mathbb{B}_{\frac{\hat{r}_0}{2}}(x_{i+1}) \subset \mathbb{B}_{\hat{r}_0}(x_i)$ for $i$ from 0 to $m-1$. The number of $m$ depends only on diam$(\tilde{\Omega})$ and $\hat{r}_0$. The properties of $(x_i)_{1\leq i\leq m}$ and the inequality (5.55) imply that

$$\|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0/2}(x_i))} \geq e^{-C(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|^{\frac{1}{\beta}}_{L^2(\mathbb{B}_{\hat{r}_0/2}(x_{i+1}))}. \tag{5.56}$$

Iterating the argument to get to $\bar{x}$, we obtain that

$$\begin{aligned}\|\bar{u}\|_{L^2(\mathbb{B}_{\hat{r}_0/2}(\hat{x}))} &\geq e^{-C_{\hat{r}_0}(|\alpha|+\sqrt{|\lambda|})} C_{\hat{r}_0}^{\frac{1}{\beta^m}} \\ &\geq e^{-C_{\hat{r}_0}(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|_{L^2(\tilde{\Omega})}.\end{aligned} \tag{5.57}$$

Let $A_{R,\,2R} = (\mathbb{B}_{2R}(x_0)\backslash\mathbb{B}_R(x_0))$ for any $x_0 \in \tilde{\Omega}$. Then there exists $\mathbb{B}_{\hat{r}_0/2}(\hat{x}) \subset A_{\hat{r}_0,\,2\hat{r}_0}$ for some $\hat{x} \in A_{2\hat{r}_0,\,\hat{r}_0}$. Thus, by Eq. 5.57,

$$\|\bar{u}\|_{L^2(A_{\hat{r}_0,\,2\hat{r}_0})} \geq e^{-C_{\hat{r}_0}(|\alpha|+\sqrt{|\lambda|})} \|\bar{u}\|_{L^2(\tilde{\Omega})}. \tag{5.58}$$

With aid of the quantitative Carleman estimates (5.53) and the inequality (5.58), using the argument as the proof of Lemma 3, we are ready to derive the doubling inequality as follows.

*Proof of Proposition 1* Let $R = \frac{\bar{r}_0}{8}$, where $\bar{r}_0$ is the fixed constant in the three-ball inequality in Eq. 5.54. Choose $0 < \rho < \frac{R}{24}$, which can be chosen to be arbitrarily small. Define a smooth cut-off function $0 < \eta < 1$ as follows,

- $\eta(r) = 0$ if $r(x) < \rho$ or $r(x) > 2R$,
- $\eta(r) = 1$ if $\frac{3\rho}{2} < r(x) < R$,
- $|\nabla\eta| \leq \frac{C}{\rho}$ if $\rho < r(x) < \frac{3\rho}{2}$,
- $|\nabla^2\eta| \leq C$ if $R < r(x) < 2R$.

We substitute $v = \eta\bar{u}$ into the Carleman estimates (5.53) and consider the elliptic (5.48). It follows that

$$\begin{aligned}\tau^{\frac{3}{2}}\|(\log r)^{-1}e^{\tau\psi}\eta\bar{u}\| + \tau^{\frac{1}{2}}\rho^{\frac{1}{2}}\|r^{-\frac{1}{2}}e^{\tau\psi}\eta\bar{u}\| &\le C\|r^2e^{\tau\psi}(\triangle_{\hat{g}}(\eta\bar{u}) + \hat{b}(x)\cdot\nabla(\eta\bar{u}) + \hat{c}(x)\eta\bar{u})\| \\ &\le C\|r^2e^{\tau\psi}(\triangle\eta\bar{u} + 2\nabla\eta\cdot\nabla v + \hat{b}\cdot\nabla\eta\bar{u})\|.\end{aligned}$$

Thanks to the properties of $\eta$ and the fact that $\tau > 1$, we get that

$$\begin{aligned}\|(\log r)^{-1}e^{\tau\psi}\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + \|e^{\tau\psi}\bar{u}\|_{\frac{3\rho}{2},4\rho} &\le C(\|e^{\tau\psi}\bar{u}\|_{\rho,\frac{3\rho}{2}} + \|e^{\tau\psi}\bar{u}\|_{R,2R}) \\ &\quad + C(\rho\|e^{\tau\psi}\nabla\bar{u}\|_{\rho,\frac{3\rho}{2}} + R\|e^{\tau\psi}\nabla\bar{u}\|_{R,2R}) \\ &\quad + C(|\alpha|+1)(\rho\|e^{\tau\psi}\bar{u}\|_{\rho,\frac{3\rho}{2}} + R\|e^{\tau\psi}\bar{u}\|_{R,2R}).\end{aligned}$$

Since $R < 1$ is a fixed constant and $\rho < 1$, we get that

$$\begin{aligned}\|e^{\tau\psi}\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + \|e^{\tau\psi}\bar{u}\|_{\frac{3\rho}{2},4\rho} &\le C(|\alpha|+1)(\|e^{\tau\psi}\bar{u}\|_{\rho,\frac{3\rho}{2}} + \|e^{\tau\psi}\bar{u}\|_{R,2R}) \\ &\quad + C(\delta\|e^{\tau\psi}\nabla\bar{u}\|_{\rho,\frac{3\rho}{2}} + R\|e^{\tau\psi}\nabla\bar{u}\|_{R,2R}).\end{aligned}$$

Using the radial and decreasing property of $\psi$ yields that

$$\begin{aligned}e^{\tau\psi(\frac{2R}{3})}\|\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + e^{\tau\psi(4\rho)}\|\bar{u}\|_{\frac{3\rho}{2},4\rho} &\le C(|\alpha|+1)(e^{\tau\psi(\rho)}\|\bar{u}\|_{\rho,\frac{3\rho}{2}} + e^{\tau\psi(R)}\|\bar{u}\|_{R,2R}) \\ &\quad + C(\rho e^{\tau\psi(\rho)}\|\nabla\bar{u}\|_{\rho,\frac{3\rho}{2}} + Re^{\tau\psi(R)}\|\nabla\bar{u}\|_{R,2R}).\end{aligned}$$

For the Eq. 5.48, it is known that the Caccioppoli type inequality

$$\|\nabla\bar{u}\|_{(1-a)r} \le \frac{C(|\alpha| + \sqrt{|\lambda|})}{r}\|\bar{u}\|_r \tag{5.59}$$

holds with any $0 < a < 1$. With the help of the Caccioppoli type inequality (5.59), we have

$$e^{\tau\psi(\frac{2R}{3})}\|\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + e^{\tau\psi(4\rho)}\|\bar{u}\|_{\frac{3\rho}{2},4\rho} \le C(|\alpha| + \sqrt{|\lambda|})(e^{\tau\psi(\rho)}\|\bar{u}\|_{2\rho} + e^{\tau\psi(R)}\|\bar{u}\|_{3R}). \tag{5.60}$$

Adding the term $e^{\tau\psi(4\rho)}\|\bar{u}\|_{\frac{3\rho}{2}}$ to both sides of last inequality and taking $\psi(\rho) > \psi(4\rho)$ into account yields that

$$e^{\tau\psi(\frac{2R}{3})}\|\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + e^{\tau\psi(4\rho)}\|\bar{u}\|_{4\rho} \le C(|\alpha| + \sqrt{|\lambda|})(e^{\tau\psi(\rho)}\|\bar{u}\|_{2\rho} + e^{\tau\psi(R)}\|\bar{u}\|_{3R}). \tag{5.61}$$

We choose $\tau$ such that

$$C(|\alpha| + \sqrt{|\lambda|})e^{\tau\psi(R)}\|\bar{u}\|_{3R} \le \frac{1}{2}e^{\tau\psi(\frac{2R}{3})}\|\bar{u}\|_{\frac{R}{2},\frac{2R}{3}}.$$

To achieve it, we need to have

$$\tau \ge \frac{1}{\psi(\frac{2R}{3}) - \psi(R)}\ln\frac{2C(|\alpha| + \sqrt{|\lambda|})\|\bar{u}\|_{3R}}{\|\bar{u}\|_{\frac{R}{2},\frac{3R}{2}}}.$$

Then, we can absorb the second term on the right hand side of Eq. 5.60 into the left hand side,

$$e^{\tau\psi(\frac{2R}{3})}\|\bar{u}\|_{\frac{R}{2},\frac{2R}{3}} + e^{\tau\psi(4\rho)}\|\bar{u}\|_{4\rho} \le C(|\alpha| + \sqrt{|\lambda|})e^{\tau\psi(\rho)}\|\bar{u}\|_{2\rho}. \tag{5.62}$$

To apply the Carleman estimates (5.53), we have assumed that $\tau \geq C(|\alpha|+\sqrt{|\lambda|})$. Therefore, to have such $\tau$, we select

$$\tau = C(|\alpha|+\sqrt{|\lambda|}) + \frac{1}{\psi(\frac{2R}{3})-\psi(R)} \ln \frac{2C(|\alpha|+\sqrt{|\lambda|})\|\bar{u}\|_{3R}}{\|\bar{u}\|_{\frac{R}{2},\frac{3R}{2}}}.$$

Dropping the first term in (5.62), we get that

$$\begin{aligned} \|\bar{u}\|_{4\rho} &\leq C(|\alpha|+\sqrt{|\lambda|}) \exp\{\Big(\frac{1}{\psi(\frac{2R}{3})-\psi(R)} \ln \frac{2C(|\alpha|+\sqrt{|\lambda|})\|\bar{u}\|_{3R}}{\|\bar{u}\|_{\frac{R}{2},\frac{3R}{2}}}\Big)\big(\psi(\rho)-\psi(4\rho)\big) \\ &\quad + C(|\alpha|+\sqrt{|\lambda|})\}\|\bar{u}\|_{2\rho} \\ &\leq e^{C(|\alpha|+\sqrt{|\lambda|})}\Big(\frac{\|\bar{u}\|_{3R}}{\|\bar{u}\|_{\frac{R}{2},\frac{3R}{2}}}\Big)^C \|\bar{u}\|_{2\rho}, \end{aligned} \tag{5.63}$$

where we have used the condition that

$$\beta_1^{-1} < \psi(\frac{2R}{3}) - \psi(R) < \beta_1,$$

$$\beta_2^{-1} < \psi(\rho) - \psi(4\rho) < \beta_2$$

for some positive constants $\beta_1$ and $\beta_2$ independent on $R$ or $\rho$.

Let $\hat{r}_0 = \frac{R}{2}$ be fixed in Eq. 5.58. With aid of Eq. 5.58, we derive that

$$\frac{\|\bar{u}\|_{L^2(\mathbb{B}_{3R})}}{\|\bar{u}\|_{L^2(A_{\frac{R}{2},\frac{3R}{2}})}} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}.$$

Therefore, it follows from Eq. 5.63 that

$$\|\bar{u}\|_{L^2(\mathbb{B}_{4\rho})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_{2\rho})}.$$

Choosing $\rho = \frac{r}{2}$, we get the doubling inequality

$$\|\bar{u}\|_{L^2(\mathbb{B}_{2r})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_{r})} \tag{5.64}$$

for $r \leq \frac{R}{12}$. If $r \geq \frac{R}{12}$, from Eq. 5.57,

$$\begin{aligned} \|\bar{u}\|_{L^2(\mathbb{B}_r)} &\geq \|\bar{u}\|_{L^2(\mathbb{B}_{\frac{R}{12}})} \\ &\geq e^{-C_R(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\Omega)} \\ &\geq e^{-C_R(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_{2r})}. \end{aligned} \tag{5.65}$$

Together with Eqs. 5.64 and 5.65, we obtain the doubling estimates

$$\|\bar{u}\|_{L^2(\mathbb{B}_{2r})} \leq e^{C(|\alpha|+\sqrt{|\lambda|})}\|\bar{u}\|_{L^2(\mathbb{B}_{r})} \tag{5.66}$$

for any $r > 0$, where $C$ only depends on the double manifold $\tilde{\Omega}$. By the translation invariant of the arguments, the proof of Eq. 2.11 is derived. □

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