Polar Coded Repetition

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***Abstract*—Constructing efficient low-rate error-correcting codes with low-complexity encoding and decoding has become increasingly important for applications involving ultra-low-power devices such as Internet-of-Things (IoT). To this end, schemes based on concatenating the state-of-the-art codes at moderate rates with repetition codes have emerged as practical solutions deployed in various standards. In this paper, we propose a novel mechanism for concatenating outer polar codes with inner repetition codes which we refer to as *polar coded repetition*. More specifically, we propose to transmit a slightly modified polar codeword by deviating from Arıkan's standard 2×2 Kernel in a certain number of polarization recursions at each repetition block. We show how this modification can improve the asymptotic achievable rate of the standard polar-repetition scheme, while ensuring that the overall encoding and decoding complexity is kept almost the same. The achievable rate is analyzed for the binary erasure channel (BEC) and additive white Gaussian noise (AWGN) channel. Moreover, we show that the finite-length performance of the polar coded repetition scheme under cyclic redundancy check (CRC) aided successive cancellation list (SCL) decoder over AWGN channel is better than the *uncoded* polar-repetition scheme at the cost of a slight increase in decoding complexity. We also compare the proposed scheme, in terms of performance and complexity, with other low-rate solution based on polar codes in the literature.**

I. Introduction

Recently, the Third Generation Partnership Project (3GPP) has introduced various features including Narrow-Band Internet of Things (NB-IoT) and enhanced Machine-Type Communications (eMTC) into the cellular standard in order to address the diverse requirements of massive IoT networks including low-power and wide-area (LPWA) cellular connectivity [4].

In general, devices in IoT networks have strict limitations on their total available power and are not equipped with advanced transceivers due to cost constraints. Consequently, they often need to operate at very low signal-to-noise ratio (SNR) necessitating ultra-low-rate error-correcting codes for reliable communications. For instance, the SNR of -13 dB is translated to capacity being 0.03 bits per transmission. The solution adopted in the 3GPP standard is to use the legacy turbo codes or convolutional codes at moderate rates, e.g., the turbo code of rate $1/3$, together with up to 2048 repetitions to support effective code rates as low as 1.6×10^{-4}. Although this repetition leads to efficient implementations with reduced computational complexity, repeating a high-rate code to enable low-rate communication will result in rate loss and mediocre performance. As a result, studying efficient channel coding strategies for reliable communication in this low SNR regime is necessary [1].

The fundamental non-asymptotic laws for channel coding in the low-capacity regimes have been recently studied in [1]. Furthermore, the optimal number of repetitions with negligible rate loss, in terms of the code block length and the underlying channel capacity, is characterized in [1]. It is also shown in [1] that the state-of-the-art polar codes, proposed by Arıkan [2], naturally invoke this optimal number of repetitions when constructed for low-capacity channels. In another related work, low-rate codes for binary symmetric channels are constructed by concatenating high-rate i.e., rate close to 1, polar codes with repetitions [5]. *Weakly-coded* modulation scheme which is the concatenation of a binary low-density parity-check (LDPC)-type code with a polar code has been introduced in [6] and [7]. Tight lower and upper bounds are obtained for the bit error rate (BER) of this scheme at any SNR and it is shown that the proposed scheme outperforms uncoded modulation over high noise memoryless channels at the cost of the increase in decoding complexity.

In a recent work [8], in parallel to this work, we proposed *hybrid non-binary multiplicative repetition* code as an alternative mechanism for the repetition concatenation scheme. In this scheme, the outer code is a hybrid polar code constructed in two stages, one with a binary kernel and another also with a binary kernel but applied over a binary extension field. The inner code is a non-binary multiplicative repetition code. The proposed scheme benefits from the multiplicative repetition over an extension field while keeping the complexity of the encoder/decoder almost the same as that of the polar-repetition. Simulation results demonstrated that the proposed scheme outperforms the polar-repetition scheme with comparable decoding complexity under cyclic redundancy check (CRC) aided successive cancellation list (SCL) decoder over additive white Gaussian noise (AWGN).

In this paper, we propose another alternative mechanism called *coded repetition*, for the binary repetition concatenation scheme. In particular, a slightly modified codeword in each repetition block is transmitted instead of identical codewords in all repetition blocks. The goal is to reduce the rate loss due to the repetition at the cost of a *slight* increase in the decoding complexity. In particular, we consider polar codes as the outer code. In the proposed polar coded repetition scheme, a slightly

This paper was presented in part at *IEEE Information Theory Workshop (ITW), Italy, April, 2021, [32].*

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This work is supported in part by the National Science Foundation under grants CCF–1763348, CCF–1909771, and CCF—1941633, and by the Australian Research Council through the Discovery Project under Grant DP200100731.

modified polar codeword is transmitted in each repetition block by deviating from Arıkan's standard 2×2 kernel in a certain number of polarization recursions at each repetition block. We show that our proposed scheme outperforms the standard polar-repetition scheme, in terms of the asymptotic achievable rate, for any given number of repetitions over the binary erasure channel (BEC). A similar result is shown empirically for the additive white Gaussian noise (AWGN) channel. The proposed polar coded repetition has almost the same encoding and decoding complexity as the polar-repetition scheme. Furthermore, simulations results demonstrate that the finite-length performance of the polar-coded repetition scheme under cyclic redundancy check (CRC) aided successive cancellation list (SCL) decoder over AWGN channel outperforms the straightforward polar-repetition scheme, also referred to as *uncoded* polar-repetition scheme. The overall decoding complexity of both schemes scale similarly with the block length, assuming the number of repetitions is a constant, but in terms of the actual number of operations, the polar-coded repetition scheme has a slightly higher decoding complexity. The main contributions of the paper are summarized as follows.

- Considering the polar-repetition scheme as our point of departure in the low-rate regime, we propose to use a slightly modified codeword in each repetition block instead of identical codewords in all of them.
- We develop methods to search for and find the best modified codeword in each repetition scheme. To this end, we first consider all the possible cases with two and four repetitions, constructed with *regular* and *irregular* polar coding approaches, over binary erasure channel (BEC). Then, we calculate the asymptotic achievable rate for all such cases and pick the one that maximizes this.
- Inspired by solutions to the cases with two and four repetitions, we propose a certain regular pattern given any arbitrary repetition r. Then, we prove that the asymptotic achievable rate of the proposed pattern is better than that of the straightforward repetition scheme for any given number of repetitions over BEC and AWGN channel.
- Finally, we demonstrate, through simulations, that the proposed polar-coded repetition scheme outperforms the straightforward polar-repetition scheme at the cost of a slight increase in the decoding complexity. The scheme is also compared with that of [6] in terms of performance and complexity.

The rest of this paper is organized as follows. In Section II, we review the basic concepts of the polarization of the polar codes along with the repetition codes. In Section III, we explain the proposed scheme first with some examples for 2 and 4 repetitions, then we generalize it to an arbitrary r repetitions of power 2. The numerical results for BEC and AWGN channel are discussed in Section IV. Finaly, Section V concludes the paper.

II. Background

In this section, we provide a brief background about the *channel polarization* of the polar codes and repetitions codes as follow.

A. Channel Polarization of Polar Codes

Consider two copies of a binary discrete memoryless channel (B-DMC) $W : \mathcal{X} \to \mathcal{Y}$ with binary inputs $x_1, x_2 \in \mathcal{X}$ and outputs $y_1, y_2 \in \mathcal{Y}$. The transformation $G_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is applied on the inputs of these two channels and u_1 and u_2 are generated. Then, x_1 and x_2 are transmitted through the independent copies of W. At the decoder side, u_1 is decoded by using two observations y_1, y_2 and then u_2 is decoded by using the decoded sequence, $\hat{u}_1$, and the observations y_1, y_2. The transformation G_2 along with this successive decoding, referred to as *successive cancellation (SC)*, transforms the two copies of the channel W into two synthetic channels $W^0 : W \boxplus W : \mathcal{X} \to \mathcal{Y}^2$ and $W^1 : W \circledast W : \mathcal{X} \to \mathcal{Y}^2 \times \mathcal{X}$ as follows:

$$\begin{aligned} W \boxplus W(y_1, y_2|u_1) &= \sum_{u_2 \in \mathcal{X}} \frac{1}{2} W(y_1|u_1 + u_2) W(y_2|u_2), \\ W \circledast W(y_1, y_2, u_1|u_2) &= \frac{1}{2} W(y_1|u_1 + u_2) W(y_2|u_2). \end{aligned} \tag{1}$$

Here, the channel W^0 is *weaker* (i.e., less reliable) compared to W, while the channel W^1 is *stronger* (i.e., more reliable) compared to the channel W. The quality of a channel is measured by a reliability metric such as the Bhattacharyya parameter defined as

$$Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}, \tag{2}$$

which is equal to the erasure probability for BECs, i.e., for BEC(ϵ), $Z(W) = \epsilon$. The Bathacharyya parameters of the synthetic channels follow the properties

$$\begin{aligned} Z(W^1) &= Z(W)^2, \\ Z(W^0) &\leq 2Z(W) - Z(W)^2, \end{aligned} \tag{3}$$

with equality in (3) iff W is a BEC.

If we continue applying the transformation G_2 recursively m times, we will obtain $n = 2^m$ synthetic channels $\{W_m^{(i)}\}_{i \in \{0,1,\ldots,n-1\}}$. More specifically, if we let $\{i_1, i_2, ..., i_m\}$ be the binary expansion of $i = \{0, 1, ..., n-1\}$ over m bits, where i_1 is the most significant bit and i_m is the least significant one, then we define the synthetic channels $\{W_m^{(i)}\}_{i \in \{0,\ldots,n-1\}}$ as

$$W_m^{(i)} = (((W^{i_1})^{i_2})^{\cdots})^{i_m}. \tag{4}$$

Arıkan in his seminal paper, [2], showed that as $m \to \infty$, these 2^m synthetic channels are either purely noiseless or purely noisy channels. Thus, on the encoder side, using k entries of the input vector u_0^{n-1} as the information bits and setting the remaining entries to zero (frozen bits) will provide almost error-free communication. Hence, an $(n = 2^m, k)$ polar code is a linear block code generated by k rows of $G_n = G_2^{\otimes m}$, which correspond to the best k synthetic channels. Here, $.^{\otimes m}$ is the m-times Kronecker product of a matrix with itself.

Polar codes have attracted the attention from both academia and industry in the past decade. They have been successfully applied to a wide range of problems including data compression [17], [18], broadcast channels [19], [20], multiple access

channels [21], [22], physical layer security [23] –[26], secret key agreement [27], and coded modulation [28], [29]. On the other hand, they were chosen as a channel coding scheme by the fifth generation (5G) standardization process of the 3rd generation partnership project (3GPP), only seven years after the invention of the polar codes [16]. Note that according to the 3GPP technical report 38.913, there are three main 5G usage scenarios: enhanced mobile broadband (eMBB), ultra-reliable and low latency communications (URLLC), and massive machine type communications (mMTC). Polar codes have been adopted as channel coding for uplink and downlink control information for the eMBB communication service which requires codes with short lengths and low rates [16]. Moreover, polar codes are among the possible coding schemes for two other frameworks of 5G, URLLC and mMTC.

B. Repetition Codes

Repetition coding is a simple way of designing a practical low-rate code. Let r denote the number of the repetitions and N, the length of the code. For constructing the repetition code, first, one needs to design a smaller outer code (e.g. polar codes) of length $n = N/r$ for channel W^r and then repeat each of its code bits r times. Consequently, the length of the final code will be $n \times r = N$. This is equivalent to transmitting an input bit over the r-repetition channel W^r and outputs an r tuple. For example, if W is BEC(ϵ), then its corresponding r-repetition channel is $W^r = BEC(\epsilon^r)$. The main advantage of this concatenation scheme is that the decoding complexity is essentially reduced to that of the outer code making it appealing to low-power applications. This comes at the expense of loss in the asymptotic achievable rate especially if the number of the repetitions is large. Suppose that $C(W)$ is the capacity of the channel W and $NC(W)$ is the capacity corresponding to N channel transmissions. With repetition coding, since we transmit n times over the channel W^r, the capacity will be reduced to $nC(W^r)$. Note that, in general, we have $nC(W^r) \leq NC(W)$ and the ratio vanishes with growing r. Let's consider BEC(ϵ) as an example with $r = 2$. If $\epsilon = 0.5$, then $\frac{1}{2}C(W^2) = 0.375$ whereas $C(W) = 0.5$. However, when ϵ is close to 1, $C(W^2) = 1-\epsilon^2$ is very close to $2C(W) = 2(1-\epsilon)$.

III. Proposed Scheme

In this section, the proposed polar coded repetition scheme is discussed. It is shown how to improve the performance of the polar-repetition scheme in the low-rate regime, while keeping the computational complexity as low as possible.

Consider an outer polar code with $r = 2^t$ repetitions and let c denote a polar codeword of length $n = 2^m$ designed for transmission over a channel W, r times. Owing to the recursive structure of the polar codes, one can write the polarization transform matrix as $G_n = G'_{r'} \otimes G_2^{\otimes(m-t')}$, where $G'_{r'}$ is an $r' \times r'$ binary matrix with $r' = 2^{t'}$. In our proposed scheme, we consider a different $G'_{r'}$ in each repetition block, while keeping $G_2^{\otimes(m-t')}$ the same in all of them. In other words, the first t' recursions of Arıkan's polarization transform are modified in each repetition while the rest of $m - t'$ recursions are kept the same. Note that if one chooses $r' = n$, i.e., the transmission in each block being different, then the channel capacity $C(W)$ can be achieved. However, we choose $r' = r$ to have a comparable complexity with the straightforward repetition scheme. The complexity of the simple polar-repetition and the proposed modified polar-repetition schemes will be provided in subsection IV-B.

We illustrate the idea through some examples with two and four repetitions and constructed with *regular* and *irregular* polar coding approaches. Then, we generalize the regular scheme to accommodate an arbitrary repetition r and we discuss the encoder and decoder algorithms.

A. Examples for two and four repetitions

In this subsection, we provide three examples for two and four repetitions as follows.

Example 1 (Two repetitions): Consider an outer polar code with two repetitions. Hence, the polar codeword c needs to be designed for $W^2 = W \circledast W$. The recursive structure of polar codes implies that codeword $c = (c_1 \oplus c_2, c_2)$ is constructed from the generator matrix $G_n = G'_2 \otimes G_2^{\otimes(m-1)}$, where $G'_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and c_1 and c_2 are polar codewords of length $n/2$ generated from $G_2^{\otimes(m-1)}$.

Now, we consider an alternative scheme where in each repetition, we transmit different combinations of c_1 and c_2 by choosing different G'_2 in each of them. Let $G'^{(i)}_2$ be a lower triangular matrix[1] $G'^{(i)}_2 = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$, where $e \in \mathcal{F}_2$ and $i = \{1, 2\}$ is the index of the transmission (see TABLE. I for two possible matrices). There are three possible cases for two

Table I: Two possible matrices for two repetitions

Pattern no.	$G'^{(i)}_2$
$P_2^{(0)}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$P_2^{(1)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

transmissions as follows.

1) $G'^{(1)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: In this case, $(c_1 \oplus c_2, c_2)$ and $(c_1 \oplus c_2, c_2)$ are transmitted in each repetition. By considering both transmissions, one concludes that codeword c_1 is implicitly designed for the effective channel that the sub-block of length $n/2$ observes, i.e., for $W^2 \circledast W^2$ and c_2 is designed for $W^2 \circledast W^2$. As a result, the capacity per channel use per transmission for this case and specifically for BEC will be
$$\begin{aligned} C_2^{(1)} &= (C(W^2 \circledast W^2) + C(W^2 \circledast W^2))/4 \\ &= (1-\epsilon^2)/2. \end{aligned}$$

2) $G'^{(1)}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: For this case, $(c_1 \oplus c_2, c_2)$ and (c_1, c_2) are transmitted in the first and second repetitions. Codeword c_1 is designed for

[1][12] showed that the column permutations and the one-directional row operations can always transform a non-singular kernel $G'^{(i)}_2$ to a lower triangular kernel G'' with the same polarization behavior.

Table II: All possible cases for four repetitions

Pattern no.	$G'^{(i)}_{4R}$
$P^{(0)}_{4R}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\1&1\end{pmatrix}$
$P^{(1)}_{4R}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$P^{(2)}_{4R}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\1&1\end{pmatrix}$
$P^{(3)}_{4R}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$

the effective channel that the sub-block of length $n/2$ observes, i.e., for $(W \boxplus W^2) \circledast W$, and c_2 is designed for $W^2 \circledast W$. As a result, the capacity per channel use per transmission for this case is

$$\begin{aligned} C_2^{(2)} &= (C((W \boxplus W^2) \circledast W) + C(W^2 \circledast W))/4 \\ &= (2 - \epsilon^2 - 2\epsilon^3 + \epsilon^4)/4. \end{aligned}$$

3) $G'^{(1)}_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $G'^{(2)}_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$: In the first and second repetitions, (c_1, c_2) and (c_1, c_2) are transmitted. Both Codewords c_1 and c_2 are designed for the effective channel that the sub-block of length $n/2$ observes, i.e., for W^2. As a result, the capacity for this case will be

$$\begin{aligned} C_2^{(3)} &= (C(W^2) + C(W^2))/4 \\ &= (1 - \epsilon^2)/2. \end{aligned}$$

It can be observed that for $0 < \epsilon < 1$, the capacity of case 2 is larger than the capacities of both cases 1 and 3, which are simple repetition schemes. In other words,

$$C((W \boxplus W^2) \circledast W) + C(W^3) > 2C(W^2), \tag{5}$$

where the right hand side of (5) is the capacity for the straightforward repetition scheme and the left hand side of (5) is the capacity of case 2.

In the proposed modified approach, which we refer to as coded repetition scheme, we consider case 2. This modified scheme has the same encoding/decoding complexity compared to a simple repetition scheme.

Example 2 (Four repetitions with regular polar codes): Consider an outer polar code with four repetitions. Since we intend to keep the complexity of the proposed scheme the same as the complexity of the simple repetition one, let's consider all possible Kronecker products of the patterns $P_2^{(0)}$ and $P_2^{(1)}$ for $G'^{(i)}_{4R}$, $i = \{1, 2, 3, 4\}$ as the ones depicted in Table II. We call these patterns *regular* polar codes. Then, for four transmissions, we try all 35 multi-subsets of size 4 from the set $\{P^{(0)}_{4R}, P^{(1)}_{4R}, P^{(2)}_{4R}, P^{(3)}_{4R}\}$ to find the best one in terms of the capacity. The channel that each codeword c_i observes follows the recursive structure shown in Fig. 1. With a simple search among these 35 multi-subsets, it is found that the pattern $(P^{(0)}_{4R}, P^{(3)}_{4R}, P^{(3)}_{4R}, P^{(3)}_{4R})$ has the largest capacity.

In this modified repetition scheme, $(c_1 \oplus c_2 \oplus c_3 \oplus c_4, c_2 \oplus c_4, c_3 \oplus c_4, c_4), (c_1, c_2, c_3, c_4), (c_1, c_2, c_3, c_4)$ and (c_1, c_2, c_3, c_4) are transmitted in the first, second, third and fourth transmissions, respectively. Codeword c_1 is constructed for the effective channel that the first sub-block of length $n/4$ observes, i.e., for $W_1 = ((W \boxplus W^2) \boxplus (W \boxplus W^2)^2) \circledast W^3$, c_2 for $W_2 = (W \boxplus W^2)^2 \circledast W^3$, c_3 for $W_3 = (W^2 \boxplus W^4) \circledast W^3$ and c_4 for $W_4 = W^4 \circledast W^3$. For BEC W, the capacity of the modified scheme is larger than that of the repetition scheme for $0 < \epsilon < 1$:

$$C_{4\text{R}} = C(W_1) + C(W_2) + C(W_3) + C(W_4) > 4C(W^4). \tag{6}$$

Example 3 (Four repetitions with irregular polar codes[2]): We consider an alternative type of patterns for 4 repetitions, referred to as *irregular* polar codes, which have the same computational complexity as the simple repetition scheme.

Polar codes can be considered as a special class of generalized concatenated codes (GCCs), proposed in [9] and [10], with the outer and inner codes of polar codes (see Fig. 2) [31]. GCCs are based on a family of (l, K_i) outer polar codes, $1 \le i \le \frac{r}{l}$ with the generator matrix $G'_{l,i}$, and a family of nested inner polar codes with generator matrix $G'_{\frac{r}{l}}$. The binary input bits $u_0, u_1, \ldots, u_{r-1}$ are first encoded with the outer polar codes to generate $x_{0,j}, x_{1,j}, \ldots, x_{l-1,j}$, $j = \{1, 2, \ldots, \frac{r}{l}\}$. Then for each $i = \{0, 1, \ldots, \frac{r}{l} - 1\}$, $x_{i,1}, x_{i,2}, \ldots, x_{i,\frac{r}{l}}$ are encoded with the inner polar code to obtain codewords $z_0, z_1, \ldots, z_{r-1}$. This results in a $(r, \sum_{i=1}^{\frac{r}{l}} K_i)$ linear binary polar code with generator matrix

$$G'_r = \begin{pmatrix} G'_{\frac{r}{l}}(1,1)G'_{l,1} & G'_{\frac{r}{l}}(1,2)G'_{l,1} & \cdots & G'_{\frac{r}{l}}(1,\frac{r}{l})G'_{l,1} \\ G'_{\frac{r}{l}}(2,1)G'_{l,2} & G'_{\frac{r}{l}}(2,2)G'_{l,2} & \cdots & G'_{\frac{r}{l}}(2,\frac{r}{l})G'_{l,2} \\ \vdots & \vdots & \cdots & \vdots \\ G'_{\frac{r}{l}}(\frac{r}{l},1)G'_{l,\frac{r}{l}} & G'_{\frac{r}{l}}(\frac{r}{l},2)G'_{l,\frac{r}{l}} & \cdots & G'_{\frac{r}{l}}(\frac{r}{l},\frac{r}{l})G'_{l,\frac{r}{l}} \end{pmatrix},$$

where $G'_{\frac{r}{l}}(i,j)$ is the (i,j)-th element of the matrix $G'_{\frac{r}{l}}$.

In regular polar codes, all matrices $G'_{l,i}$ are the same for all $i = \{1, 2, \ldots, \frac{r}{l}\}$. However, in irregular polar codes, one can choose these matrices different from each other. For 4 repetitions ($l = 2$, $r = 4$), there are 8 irregular patterns $G'^{(i)}_{4I}$, $i = \{1, 2, \ldots, 8\}$ where G'_2, $G'_{2,1}$ and $G'_{2,2}$ are one of the patterns $P_2^{(j)}$ $j = \{0, 1\}$ (see Fig. 3). With a simple search among all 330 multi-subsets of size 4 from the set $\{P^{(k)}_{4I}\}_{k=0}^{7}$, it is found that the pattern $(P^{(2)}_{4I}, P^{(5)}_{4I}, P^{(7)}_{4I}, P^{(7)}_{4I})$ has the largest capacity. The channel that each codeword c_i observes follows the recursive structure shown in Fig. 3. In this scheme, $(c_1 \oplus c_3 \oplus c_4, c_2 \oplus c_4, c_3 \oplus c_4, c_4)$, $(c_1 \oplus c_2, c_2, c_3, c_4)$, (c_1, c_2, c_3, c_4) and (c_1, c_2, c_3, c_4) are transmitted in the first, second, third and fourth transmissions, respectively. Codewords c_1, c_2, c_3, c_4 are constructed for the effective channels W_1, W_2, W_3, W_4 as follows.

$$\begin{aligned} W_1 &= (W \boxplus W^2) \circledast (W \boxplus W^2) \circledast W \circledast W, \\ W_2 &= (W \boxplus W^2) \circledast W^2 \circledast W \circledast W, \\ W_3 &= (W^2 \boxplus W^4) \circledast W \circledast W \circledast W, \\ W_4 &= W^4 \circledast W \circledast W \circledast W. \end{aligned}$$

For BEC W, the capacity of the modified scheme with irregular polar codes is larger than the one with regular polar codes for $0 < \epsilon < 1$. In other words,

$$C_{4\text{I}} = C(W_1) + C(W_2) + C(W_3) + C(W_4) > C_{4\text{R}}. \tag{7}$$

[2]Note that regular scheme is a special case of the irregular scheme.

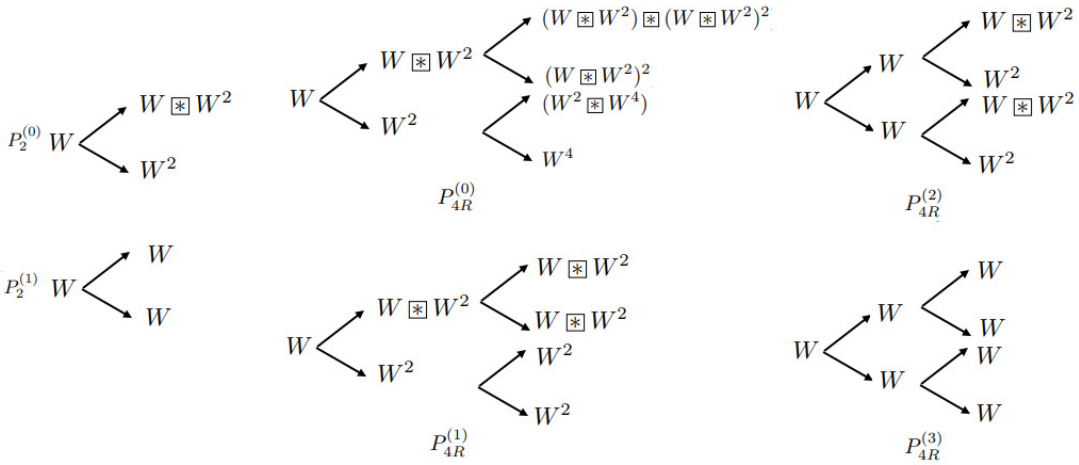

Figure 1: The recursive structure of the channels that each codeword c_i observes for two and four transmissions.

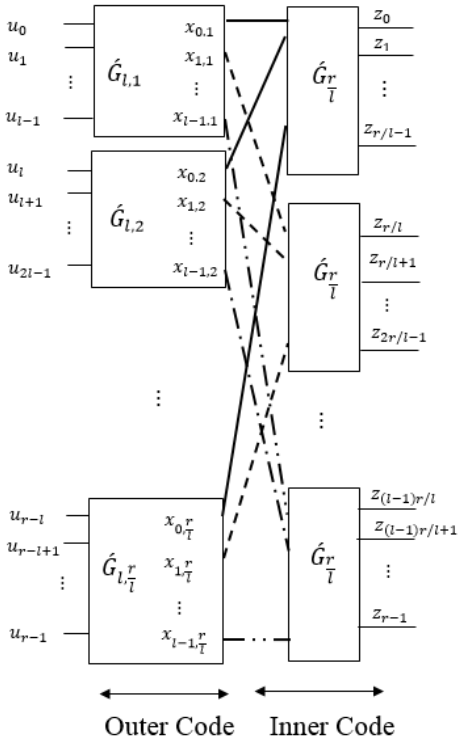

Figure 2: GCCs based on polar codes

B. General case for regular polar codes

For the general case of $r = 2^t$ repetitions with regular polar codes, we consider all r possible t times Kronecker products of the patterns $P_2^{(0)}$ and $P_2^{(1)}$, as $P_r^{(i)}$, $i = 0, 1, \ldots, r-1$. In the proposed scheme, we use $P_r^{(0)} = (P_2^{(0)})^{\otimes t}$ for the first transmission and $P_r^{(r-1)} = (P_2^{(1)})^{\otimes t}$ for the rest $r-1$ ones. For BEC W with an erasure probability ϵ, let's define $Z_{P_r^{(i)}}(W_r^{(k)}) \triangleq Z_{(i_1,\ldots,i_t)}(W_r^{(k)})$ as the erasure probabilities of the channels that each codeword c_k, $k = \{1, 2, \ldots, r\}$ for pattern $P_r^{(i)}$ observes and $\{i_1, i_2, \ldots i_t\}$ as the t-bit binary expansion of i. Then, the recursive formula for computing $Z_{P_r^{(i)}}(W_r^{(k)})$ can be written as

$$\begin{aligned} Z_{(i_1,\ldots,i_t)}(W_r^{(2j-1)}) &= Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)}) \times \\ &\quad [1 + Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)}) - Z^2_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})]^{(1-i_t)}, \\ Z_{(i_1,\ldots,i_t)}(W_r^{(2j)}) &= Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)}) \times \\ &\quad [Z_{(i_1,\ldots,i_{t-1})}(W_{\frac{r}{2}}^{(j)})]^{(1-i_t)}, \end{aligned} \tag{8}$$

where $Z(W_1^{(1)}) = \epsilon$ and $j = 1, 2, \ldots, \frac{r}{2}$. Hence, the capacity for the proposed scheme will be

$$C_{rR} = \frac{r - \sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) \times (Z_{P_r^{(r-1)}}(W_r^{(k)}))^{r-1}}{r^2}. \tag{9}$$

Since $Z_{P_r^{(r-1)}}(W_r^{(k)}) = \epsilon$, for all $k = 1, 2, \ldots, r$, we will have

$$C_{rR} = \frac{r - \sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) \times \epsilon^{r-1}}{r^2}. \tag{10}$$

Next, we show that $C_{rR} > \frac{C(W^r)}{r}$ for any r repetitions and $0 < \epsilon < 1$. In other words,

$$\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) < r\epsilon. \tag{11}$$

To this end, we first prove that $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) - r\epsilon$ has zeros at $\epsilon = 0$ and $\epsilon = 1$.

Theorem 1. *$Z_{P_r^{(0)}}(W_r^{(k)}) = 0$ at $\epsilon = 0$ and $Z_{P_r^{(0)}}(W_r^{(k)}) = 1$ at $\epsilon = 1$ for all $k = \{1, 2, \ldots, r\}$.*

Proof. Let us write the recursive formula for erasure probability as $Z_{P_r^{(0)}}(W_r^{(k)}) = f_{k_1}(f_{k_2}(\ldots f_{k_t}(\epsilon)))$, where $k_i = \{0, 1\}$, $i = \{1, 2, \ldots, t\}$ and $f_0(a) = a + a^2 - a^3$, $f_1(a) = a^2$, $\forall k = \{1, 2, \ldots, r\}$.

Since $f_{k_i}(a)|_{a=1} = 1$ and $f_{k_i}(a)|_{a=0} = 0$, by using recursion, we conclude $Z_{P_r^{(0)}}(W_r^{(k)}) = 1$ at $\epsilon = 1$ and $Z_{P_r^{(0)}}(W_r^{(k)}) = 0$ at $\epsilon = 0$ $\forall k = \{1, 2, \ldots r\}$. ■

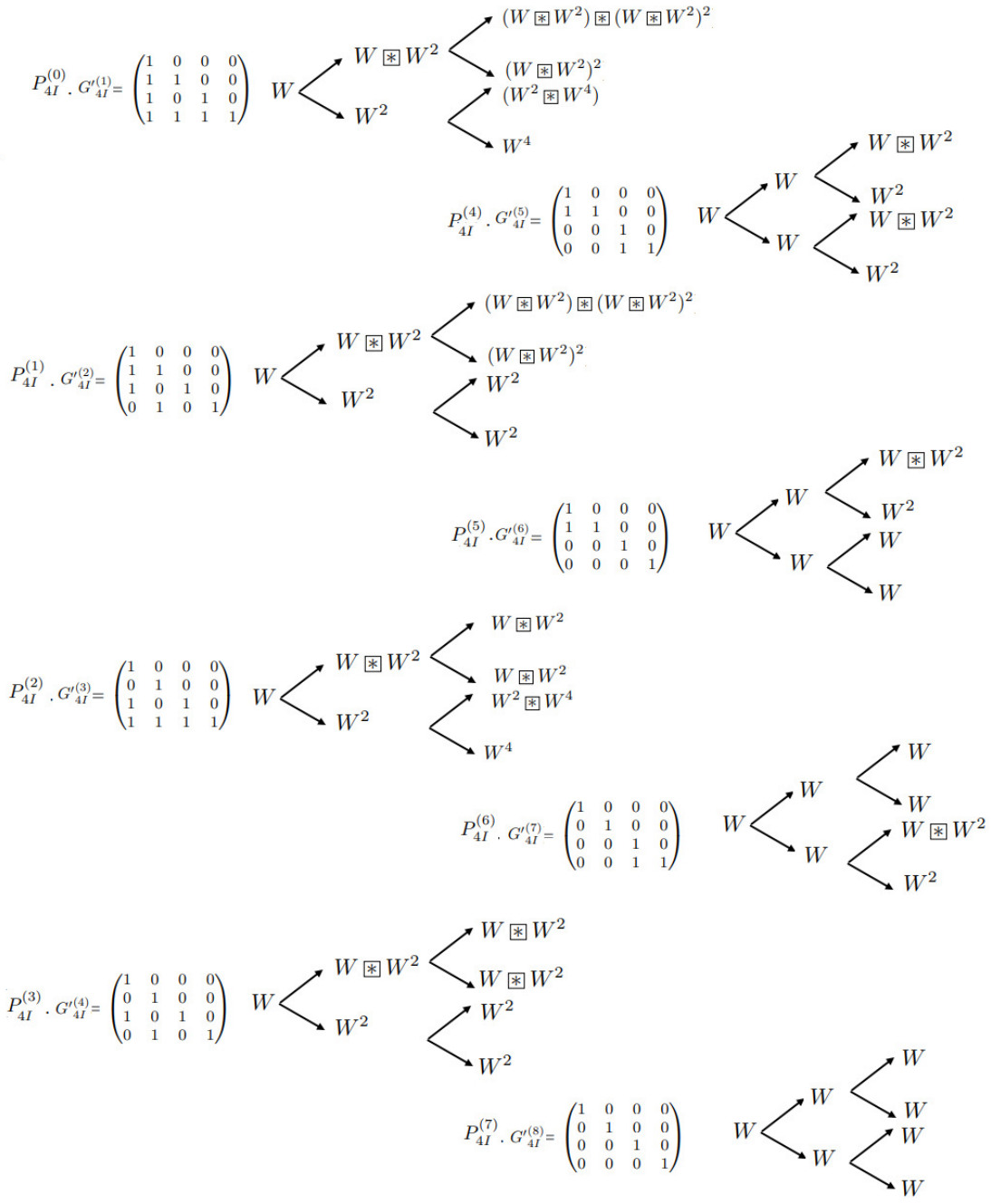

Figure 3: All 8 possible irregular kernels $G_{4I}'^{(i)}$ for 4 transmissions and the corresponding recursive structure of the channels that each codeword c_i observes.

Then, one can use Sturm algorithm[3] [13] to show that $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)}) - r\epsilon$ does not have any root in $\epsilon = (0, 1)$. Finally, one can choose an ϵ in the interval $(0, 1)$ and compare the values of $\sum_{k=1}^{r} Z_{P_r^{(0)}}(W_r^{(k)})$ and $r\epsilon$ at that point to see that the capacity of proposed modified scheme is greater than the repetition one for r number of repetitions. Fig. (4) shows the left and the right sides of eq. (11) for $r = 4$.

[3]Although Sturm's theorem is a complete solution for finding the number of the real roots of the polynomials, when the degree of the polynomial increases, it isn't efficient in terms of implementation. The algorithm proposed in [14], [15] is more efficient for higher degrees.

C. Encoder of the polar coded repetition scheme

To encode the polar coded repetition scheme, in the first transmission, the binary input bits u_0^{n-1} are encoded with the polarization matrix $G_2^{\otimes m} = {P_2^{(0)}}^{\otimes t} \otimes G_2^{\otimes(m-t)}$ and the code $C_1 = \{z_0^{n-1}\}$ is generated. Then, the input u_0^{n-1} are encoded with the polarization matrix ${P_2^{(1)}}^{\otimes t} \otimes G_2^{\otimes(m-t)}$, the codewords

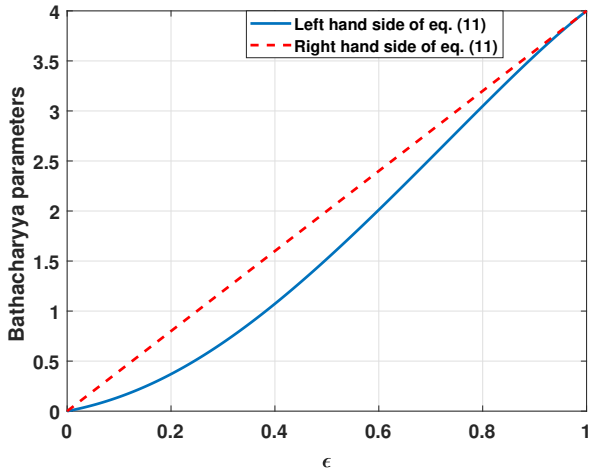

Figure 4: Comparison between the left and the right hand sides of eq. (11).

x_0^{n-1} are generated, and repeated $r-1$ times. The resulting code C_r, $r>1$ is:

$$C_r = \{c_0^{rn-1} | c_{(r-1)n+v} = x_v \quad \text{for} \quad v = \{0,\dots,n-1\}, \quad c_0^{(r-1)n-1} \in C_{r-1}\}. \tag{12}$$

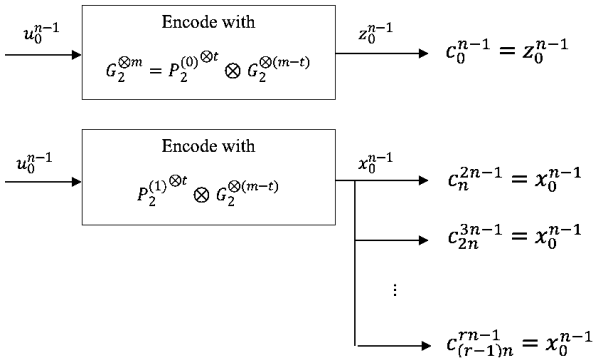

Figure 5: Encoder of the polar coded repetition scheme.

Algorithm 1 shows the process of encoding. The inputs to this algorithm are input bits u_0^{n-1}, n and r. The output is the codeword c_0^{rn-1}.

D. CRC-aided SCL decoder of the polar coded repetition scheme

The coded bits c_i from Algorithm 1 are transmitted over AWGN channel using a binary modulation scheme. The received signals for this channel is given by

$$y_i = (2c_i - 1) + w_i, \quad i = 0, 1, \dots, N-1$$

where w_i is a zero mean Gaussian noise, $w_i \sim \mathcal{N}(0, \sigma^2)$.

For decoding the proposed polar coded repetition scheme, we use the CRC-aided log likelihood ratio (LLR)-based SCL decoder. The only difference between the algorithm of the CRC-aided SCL decoder of the polar codes constructed with

Algorithm 1: Encoding of the Proposed Polar Coded Repetition

input : u_0^{n-1}, n, r
output: Codeword c_0^{rn-1}
1 $m = \log_2 n$; $t = log_2 r$;
2 Define List Z $= \{\}$
3 Z $\leftarrow u_0^{n-1}$ // Init.
4 **for** $i = 0$ **to** $m-1$ **do**
5 $B_1 = 2^{(m-i)}$; $B_2 = 2^i$;
6 **for** $j = 0$ **to** $B_2 - 1$ **do**
7 $base = jB_1$;
8 **for** $l = 0$ **to** $B_1/2 - 1$ **do**
9 $x(base + l) = mod(Z[base + l] + Z[base + B_1/2 + l], 2)$;
10 $x(base + B_1/2 + l) = Z[base + B_1/2 + l]$;
11 **end**
12 **end**
13 **if** $i == m - t - 1$ **then**
14 $c_n^{rn-1} \leftarrow$ repeat x_0^{n-1} for $r-1$ times $(r > 1)$
// Encoding matrix $P_2^{(1)\otimes t} \otimes G_2^{\otimes(m-t)}$
15 **end**
16 $Z = \{\}$ // empty List Z
17 $Z \leftarrow x_0^{n-1}$
18 **end**
19 $c_0^{n-1} \leftarrow Z$ // Encoding matrix $P_2^{(0)\otimes t} \otimes G_2^{\otimes(m-t)}$
20 **return** c_0^{rn-1}

Arıkan's kernel and the polar coded repetition scheme is in updating the LLRs. Algorithm 2 shows the details of the process. This algorithm is the same as the SCL Algorithms of [30] with the differences highlighted in red.

In the implementation of CRC-aided SCL decoder, one needs to calculate the following LLRs.

Initial LLRs: The initial LLRs of the i-th bit, $i = 0, 1, \dots, rn-1$, for AWGN channel is defined as

$$S_{in}[i] = \ln \frac{W(y_i|-1)}{W(y_i|1)}, \tag{13}$$

Intermediate LLRs: For updating the intermediate LLRs of the matrix G_2 at each decoding stage, the following f-function and g-function are used.

$$f(L_1, L_2) = \ln\left(\frac{1+\exp(-L_1-L_2)}{\exp(-L_1)+\exp(-L_2)}\right), \qquad g(L_1, L_2, b_0) = L_2 + (-1)^{b_0} L_1, \tag{14}$$

where L_1 and L_2 are the input LLRs and b_0 is the partial sum calculated with the bits that have been previously decoded. One can use the LogSumExp approximation, $\ln(\sum_i e^{-f_i}) \approx -\min_i(f_i)$, to simplify (14) as follows:

$$f(L_1, L_2) = \min(L_1, L_2) - \min(0, L_1 + L_2), \qquad g(L_1, L_2, b_0) = L_2 + (-1)^{b_0} L_1, \tag{15}$$

Note that the output LLRs of the $P_2^{(1)\otimes t}$ matrix is the same as its input LLRs (lines 27-29 and 32-34 of Algorithm 2).

Example: Figure 6 shows an example of decoding $\hat{u}_1$ for $n = 8$ and $r = 4$. Here, LLR1[3][0 : 7] $= S_{in}[0:7]$, LLR2[1][0] $= S_{in}[9] + S_{in}[17] + S_{in}[25]$ and LLR2[1][1] $= S_{in}[13] + S_{in}[21] + S_{in}[29]$.

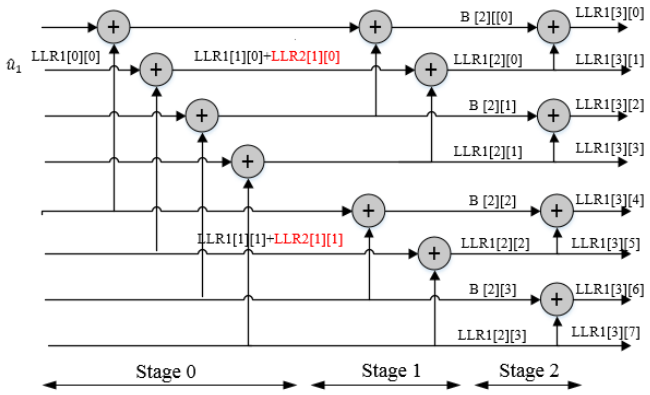

Figure 6: Example: LLR updates of decoding $\hat{u}_1$ for $m = 3$, $r = 4$.

IV. Analysis and Numerical Results

In this section, we first analyze the numerical result of the proposed scheme over BEC and AWGN channel and compare it with the polar-repetition scheme. Then, we provide complexity analysis of both of these schemes.

A. Numerical Analysis

Here, we first provide the asymptotic achievable rate comparison. Then we compare the finite-length performance of the aforementioned schemes.

1) Asymptotic Achievable Rate: In this subsection, we provide numerical results for the achievable rate of the proposed polar coded repetition scheme for different numbers of repetitions over BEC and AWGN channel and compare them with the capacity of the simple repetition scheme and the Shannon bound.

Fig. (7) illustrates the capacities of the proposed schemes for 2, 4 and 8 repetitions over BEC. It can be observed that the proposed scheme outperforms the simple repetition scheme for all of these repetitions. The irregular scheme also slightly outperforms the regular one for 4 repetitions. On the other hand, as the number of repetitions increases, the gap to the Shannon bound increases as expected.

Fig. (8) compares the capacity of the proposed scheme for 2, 4 and 8 repetitions with that of the repetition scheme over AWGN channel. It can be observed that the proposed scheme outperforms the repetition scheme for all of these repetitions. To calculate the capacity, the method proposed in [30] is used.

2) Finite-Length Performance: We provide numerical results for the proposed polar coded repetition scheme. The communication is assumed over an AWGN channel with binary modulation. The decoding is performed under the SC and CRC-aided SCL decoding algorithms. Then, we compare the performance of the proposed polar coded repetition scheme with that of the uncoded polar-repetition scheme, where the

Algorithm 2: List Decoding Algorithm for the Proposed Polar Coded Repetition Scheme

input : List size L, n, r and y_0^{rn-1}
output: The estimated bits $\hat{u}_0^{n-1}$
$m = \log_2 n$; $t = log_2 r$;
LLR1[0:m][0:n-1]=zeros[0:m][0:n-1]; LLR2[0:m][0:n-1]=zeros[0:m][0:n-1]; S_{r-1}=zeros[0:n-1]; B[0:m-1][0:n-1]=zeros[0:m-1][0:n-1]; // Init.
S_{in} = Calculate the initial LLRs with eq. (13) by using y_0^{rn-1}.
LLR1[m][0:n-1]$\leftarrow S_{in}[0:n-1]$ // Init.: Pick the first n elements of S_{in}.
for $j = 2$ **to** r **do**
    $S_{r-1} = S_{r-1} + S_{in,i}[(j-1)n : jn-1]$. // LLRs of $r-1$ rep. sch. with $P_2^{(1)^{\otimes t}} \otimes G_2^{\otimes(m-t)}$.
end
LLR2[m][0:n-1] $\leftarrow S_{r-1}$;
for $k = 0$ **to** $n-1$ **do**
    LLR1[0][0] $\leftarrow$ UpdateLLR $(k, m, t, B,$ LLR1, LLR2)
    Calculate the PM for LLR1[0][0] according to [30] for each of the L paths.
    B $\leftarrow$ Update bits according to [2].
end
CRC-aided SCL decoder with list size L chooses the path with the smallest PM which passes the CRC and outputs $\hat{u}_0^{n-1}$.
return $\hat{u}_0^{n-1}$
subroutine UpdateLLR$(k, m, t, B,$ *LLR1, LLR2*$)$**:**
    i=BitReverse[k]; // Finding the bit-reverse of i
    if *i==0* **then**
        // Finding the last stage to update i
        LastStage=m;
    else
        LastStage=index of the first $'1'$ from MSB(i);
    end
    for l =*LastStage-1* **to** 0 **by** -1 **do**
        // Updating the intermidiate LLRs of Stage l
        for *indx* = 0 **to** $2^l - 1$ **do**
            if l ==*LastStage-1* && $k \neq 0$ **then**
                LLR1[l][indx] $\leftarrow$ g (B[l][indx], LLR1[$l+1$][2$\times$ indx], LLR1[$l+1$][2$\times$ indx+1]);
                if $l \geq m - t$ **then**
                    // Updating the LLRs of the matrix $P_2^{(1)^{\otimes t}}$
                    LLR2[l][indx] $\leftarrow$ LLR2[$l+1$][2$\times$ indx+1];
                end
            else
                LLR1[l][indx] $\leftarrow$ f (LLR1[$l+1$][2$\times$ indx], LLR1[$l+1$][2$\times$ indx+1]);
                if $l \geq m - t$ **then**
                    LLR2[l][indx] $\leftarrow$ LLR2[$l+1$][2$\times$ indx];
                end
            end
        end
        if $l == m - t$ **then**
            for *indx* $= 0$ **to** $2^l - 1$ **do**
                LLR1[l][indx] = LLR1[l][indx]+LLR2[l][indx];
            end
        end
    end
    return LLR1[0][0]

generator matrix in all the transmissions is $P_2^{(1)^{\otimes t}} \otimes G_2^{\otimes(m-t)}$,[4] as well as the scheme proposed in [6]. Note that the SCL decoder is implemented using the randomized order statistic algorithm for the selection of the L most likely paths in each stage [33], which has the complexity $O(L)$, where L is the list size. Moreover, the construction for the schemes is based on Monte-Carlo simulation at an optimized design SNR, denoted by γ, to be specified separately for each case.

[4]In the uncoded polar-repetition scheme $P_r^{(i)} = (P_2^{(1)})^{\otimes t}$, $i = 0, 1, \ldots, r-1$, is considered for all r transmissions, while for the polar-coded repetition scheme $P_r^{(0)} = (P_2^{(0)})^{\otimes t}$ is used for the first transmission and $P_r^{(r-1)} = (P_2^{(1)})^{\otimes t}$ for the remaining $r-1$ ones.

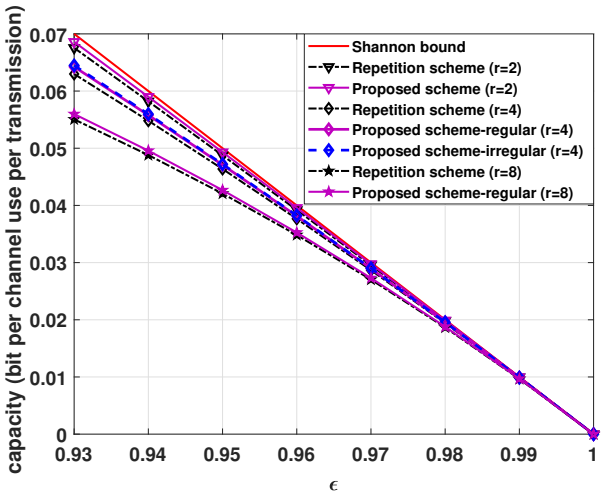

Figure 7: Capacity of the proposed scheme compared with the capacity of the repetition scheme for $r = 2, 4, 8$ over BEC.

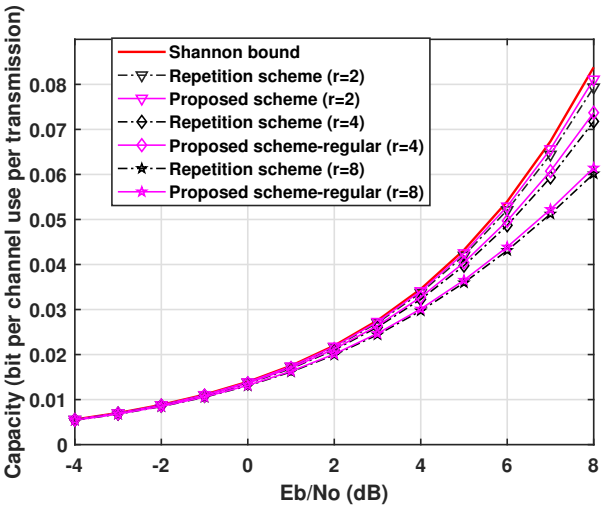

Figure 8: Capacity of the proposed scheme compared with the capacity of the repetition scheme for $r = 2, 4$ and 8 repetitions over AWGN channel.

Figures 9 and 10 compare the performance of the proposed polar coded repetition scheme for $N = 8192$, $k = 80$ with that of the uncoded polar-repetition scheme over AWGN channel for two different values $r = 2, 4$. Here, $R = k/N = 0.0098$ is the rate of the code. The optimized design SNR γ for each simulated case is specified in the legend of the corresponding plot. It can be seen that the proposed polar coded repetition scheme outperforms the uncoded polar-repetition scheme, under SC and CRC-aided SCL decoder with the same list size and 6-bit CRC.

For comparison, the performance of the scheme proposed in [6] for low-SNR regime is also considered. It can be seen that for $r = 2$, the proposed scheme under CRC-aided SCL with $L = 32$ outperforms the scheme in [6].

Finally, for the purpose of comparison in the figures, we use the finite-length analysis of [35] to demonstrate the approximation of the limit in the finite block-length regime (up

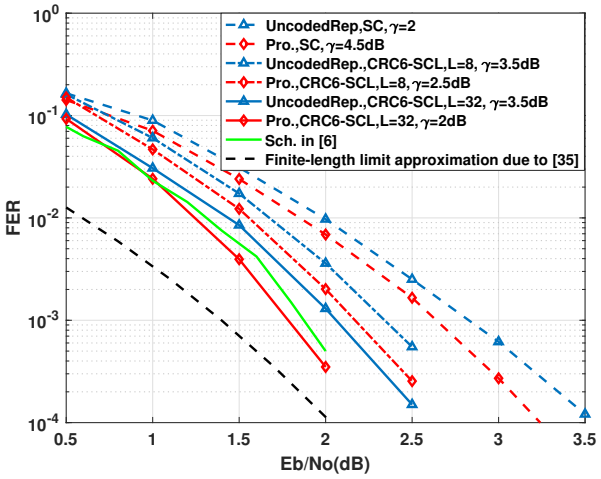

Figure 9: Performance comparison of the polar coded repetition scheme with the uncoded polar-repetition scheme and the scheme in [6] over AWGN channel for $m = 12$, $r = 2$, $k = 80$, $R = 0.0098$

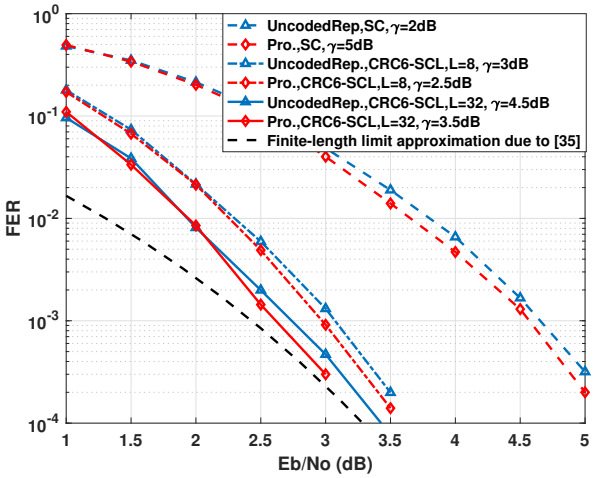

Figure 10: Performance comparison of the polar coded repetition scheme with the uncoded polar-repetition scheme over AWGN channel for $m = 11$, $r = 4$, $k = 80$, $R = 0.0098$.

to the third order) on the required SNR for each block error probability in the range of interest.

B. Complexity Analysis

In this subsection, we count the exact number of the operations needed for calculating the LLRs of the polar coded repetition scheme, uncoded polar-repetition, and the scheme in [6] to show the performance-complexity trade-off. For this purpose, we used the total number of the summation and comparison operations for calculating the LLRs of the SC decoder, eq. (15).

Uncoded polar-repetition scheme: To update the LLRs of the uncoded polar-repetition code, one needs to compute

- $n(r-1)$ operations for decoding the repetition code of size $r = 2^t$;
- $(\frac{n}{2} \times 4 + \frac{n}{2} \times 1) \times (m-t)$ operations for decoding the matrix $G_2^{\otimes(m-t)}$.

Polar coded repetition scheme: The complexity of the proposed polar coded repetition scheme consists of the complexity of the three stages.

- $(\frac{n}{2} \times 4 + \frac{n}{2} \times 1) \times t$ for decoding the matrix G'_r of size $r = 2^t$;
- $n(r-1)$ operations for decoding the repetition code of size r;
- $(\frac{n}{2} \times 4 + \frac{n}{2} \times 1) \times (m-t)$ operations for decoding the matrix $G_2^{\otimes(m-t)}$.

Scheme proposed in [6]: This scheme is the concatenation of the $(n_l, k_l) = (8192, 128)$ LDPC with $(n_p, k_p) = (128, 80)$ polar code. In this scheme, one needs $8192(8\log(128)+32) = 720896$ operations for decoding the LDPC code and 110469 operations for decoding the polar code under SCL decoder with list size $L = 32$. As a result the total number of operations for this scheme is 831365.

The complexity comparison of the three discussed schemes is shown in Table III for $N = 8192$, $m = 11, 12$ and two different values $r = 4, 2$. It can be observed that with the same list size L, the complexity of the polar coded repetition scheme is slightly greater than the complexity of the uncoded polar-repetition.

We conclude that the complexity order of the total decoding process of the uncoded polar-repetition scheme and polar coded repetition are $O(nr + n\log\frac{n}{r})$ and $O(nr + n\log r + n\log\frac{n}{r}) = O(nr + n\log n)$, respectively. If r is a constant, then the complexity of the both of the schemes is $O(n\log n)$. Moreover, the complexity of the scheme proposed in [6] is $O(n_l(\log n_p + L) + Ln_p\log n_p)$.

Remark. Note that one could use the simplification method proposed in [34] to reduce the decoding complexity of our proposed scheme. Since the underlying polar code is low rate, the binary tree representation of the decoder structure is very sparse. Hence, one expects that the resulting complexity reduction will be significant without affecting the performance. Furthermore, when compared with the scheme proposed in [6], our scheme is using only one decoder core, while the scheme in [6], concatenation of the polar and LDPC codes, needs to use two different decoder cores adding to the associated hardware complexity and costs making it less favorable for low-cost low-complexity IoT devices as one of the main applications of low-rate codes. Note that such an argument could be further expanded and elaborated using a space complexity analysis tied with actual considerations in the hardware implementation, which is beyond the scope of our work.

V. Conclusion

In this paper, we proposed a modified approach for the repetition scheme. In this scheme, we used polar codes as the outer code and proposed to transmit slightly modified codeword in each repetition. We showed that the proposed scheme outperforms the polar-repetition scheme, in terms of the asymptotic achievable rate, over BEC and AWGN channel while it keeps the decoding complexity almost the same as the polar-repetition scheme. Moreover, it was demonstrated that the polar-coded repetition scheme has better performance than the uncoded polar-repetition under CRC-aided SCL decoder over AWGN channel in the finite-length regime at the cost of the slightly increase in the decoding complexity.

In this paper, we used computer search to find the best pattern in terms of the asymptotic achievable rate for repetitions $r = 2, 4, 8$. Then, we generalized the pattern to an arbitrary number of repetitions r and proved that the proposed scheme with this pattern is performing better than the uncoded polar-repetition scheme. Future research will focus on exploring a low-complexity method to find the best pattern for each repetition. Moreover, generalizing the irregular polar coded repetition scheme to an arbitrary r repetitions is another direction. Another interesting direction for future work is to study similar problems when other families of codes, other than polar codes, are concatenated with repetition and to study methods on how to improve their performance while keeping the decoder core and its complexity almost the same.

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Table III: Complexity comparison of different schemes

Parameters	Uncoded Polar-Repetition Scheme				
	Rep. Sch. $n(r-1)$	Outer polar code $2.5 \times n \times (m-t)$	Total SC	Total SCL $L=8$	Total SCL $L=32$
$m=12$, $n=2^{12}$, $t=1$, $r=2^1$	4096	112640	116736	910864	3639088
$m=11$, 2^{11}, $t=2$, $r=2^2$	6144	46080	52224	380433	1511301

Parameters	Polar Coded Repetition Scheme					
	Rep. Sch. $n(r-1)$	Matrix $G_2^{\otimes(m-t)}$ $2.5 \times n \times (m-t)$	Matrix G'_r $2.5 \times n \times t$	Total SC	Total SCL $L=8$	Total SCL $L=32$
$m=12$, $n=2^{12}$, $t=1$, $r=2^1$	4096	112640	10240	126976	1021458	4093983
$m=11$, $n=2^{11}$, $t=2$, $r=2^2$	6144	46080	10240	62464	505366	2029368

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