A REMARK ON QUANTUM HOCHSCHILD HOMOLOGY

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Abstract. Beliakova-Putyra-Wehrli studied various kinds of traces, in relation to annular Khovanov homology [2]. In particular, to a graded algebra and a graded bimodule over it, they associate a quantum Hochschild homology of the algebra with coefficients in the bimodule, and use this to obtain a deformation of the annular Khovanov homology of a link. A spectral refinement of the resulting invariant was recently given by Akhmechet-Krushkal-Willis [1].

In this short note we observe that quantum Hochschild homology is a composition of two familiar operations, and give a short proof that it gives an invariant of annular links, in some generality. Much of this is implicit in Beliakova-Putyra-Wehrli [2].

Definition 1. *[2, Section 3.8.5] Let A be a graded ring, M a chain complex of graded A-bimodules (so M is bigraded), and $q \in A$ an invertible central element with grading 0. The* quantum Hochschild complex *of A with coefficients in M and parameter q has chain groups $qCH_n(A;M) = M \otimes_{\mathbb{Z}} A^{\otimes_{\mathbb{Z}} n}$ and differential*

$$\partial(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1}(-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n q^{-|a_n|} a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The homology of this complex is the quantum Hochschild homology $qHH_\bullet(A;M)$ *of A with coefficients in M and parameter q.*

The goal of this note is to reformulate this operation and deduce that it often leads to annular link invariants. The data of A and q specifies a ring homomorphism $f_q\colon A \to A$ defined on homogeneous elements a of A by

$$f_q(a) = q^{-|a|}a,$$

where $|a|$ denotes the grading of a. We can twist the left action on the A-bimodule M by f_q to obtain a new bimodule ${}_{f_q}M$ which is equal to M as a right A-module and has left action given by the composition $A \otimes_{\mathbb{Z}} {}_{f_q}M \xrightarrow{f_q \otimes \mathbb{I}} A \otimes_{\mathbb{Z}} M \xrightarrow{m} M = {}_{f_q}M$. This operation is a special case of tensor product:

$${}_{f_q}M \cong {}_{f_q}A \otimes_A M$$

(compare [2, Section 3.8.3]).

Our first observation is:

Proposition 2. *The quantum Hochschild homology of A with coefficients in M is isomorphic to the ordinary Hochschild homology of A with coefficients in ${}_{f_q}M$.*

Proof. This is immediate from the definitions. □

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Call a chain complex of graded A-bimodules M *weakly central* if for any graded A-bimodule N there is a quasi-isomorphism $M \otimes^L_A N \simeq N \otimes^L_A M$.

Lemma 3. *The bimodule ${}_{f_q}A$ is weakly central.*

Proof. The isomorphism $M \otimes_A {}_{f_q}A \to {}_{f_q}A \otimes M$ sends m to $q^{-|m|}m$. ☐

We turn next to annular link invariants. Consider the category Tan with one object for each even integer and $\mathrm{Hom}(2m, 2n)$ given by the set of isotopy classes of $(2m, 2n)$-tangles (embedded in $D^2 \times [0,1]$). Given a (graded) algebra A, a *very weak action* of Tan on the category of A-modules is a choice of quasi-isomorphism class of chain complex of (graded) A-bimodules $C(T)$ for each $T \in \mathrm{Hom}(2m, 2n)$ so that $C(T_2 \circ T_1)$ is quasi-isomorphic to $C(T_2) \otimes^L_A C(T_1)$. For example, if we take A to be the direct sum of the Khovanov arc algebras [4] then Khovanov defined a very weak action of Tan on ${}_A\mathsf{Mod}$, and if we define A to be the direct sum of the Chen-Khovanov algebras [3] then Chen-Khovanov defined a very weak action of Tan on ${}_A\mathsf{Mod}$. (In fact, in both cases, they did more; cf. Remark 6.)

Any $(2n, 2n)$-tangle $T \subset D^2 \times [0,1]$ has an *annular closure* in $D^2 \times S^1$.

Proposition 4. *Fix a very weak action of* Tan *on* ${}_A\mathsf{Mod}$ *and a weakly central A-bimodule P. Then for any $(2n, 2n)$-tangle T, the isomorphism class of $HH_*(A; C(T) \otimes^L_A P)$ is an invariant of the annular closure of T.*

(Compare [2, Corollary 3.23].)

Proof. This is immediate from the definitions and the trace property of Hochschild homology, i.e., that given A-bimodules M and N,

$$HH_*(A; M \otimes^L_A N) \cong HH_*(A; N \otimes^L_A M).$$ ☐

The following is part of Beliakova-Putyra-Wehrli's Theorem B [2]:

Corollary 5. *Up to isomorphism, the quantum Hochschild homology of the Chen-Khovanov bimodule associated to a $(2n, 2n)$-tangle T is an invariant of the annular closure of T.*

Proof. This is immediate from Lemma 3, Proposition 4, and the fact that the Chen-Khovanov bimodules induce a very weak action of Tan [3]. ☐

Remark 6. To keep this note short, we have not discussed functoriality of these annular link invariants under annular cobordisms. To do so, one replaces Tan by an appropriate 2-category of tangles and weak centrality by a notion keeping track of the isomorphisms. See Beliakova-Putyra-Wehrli [2] for further discussion.

References

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